Central Forces Homework 4

Due 5/21/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

REQUIRED:

1. Consider the frictionless motion of a hockey puck of mass m on a perfectly circular bowl-shaped ice rink with radius a. The central region of the bowl ($r < 0.8a$) is perfectly flat and the sides of the ice bowl smoothly rise to a height h at $r = a$.

   (a) Draw a sketch of the potential energy for this system. Set the zero of potential energy at the top of the sides of the bowl.

   (b) Situation 1: the puck is initially moving radially outward from the exact center of the rink. What minimum velocity does the puck need to escape the rink?

   (c) Situation 2: a stationary puck, at a distance $\frac{a}{2}$ from the center of the rink, is hit in such a way that it's initial velocity $\vec{v}_0$ is perpendicular to its position vector as measured from the center of the rink. What is the total energy of the puck immediately after it is struck?

   (d) In situation 2, what is the angular momentum of the puck immediately after it is struck?

   (e) Draw a sketch of the effective potential for situation 2.

   (f) In situation 2, for what minimum value of $\vec{v}_0$ does the puck just escape the rink?

2. In a solid, a free electron doesn't "see" a bare nuclear charge since the nucleus is surrounded by a cloud of other electrons. The nucleus will look like the Coulomb potential close-up, but be "screened" from far away. A common model for such problems is described by the Yukawa or screened potential:

$$U(r) = -\frac{k}{r}e^{-\frac{r}{\alpha}}$$

   (a) Graph the potential, with and without the exponential term. Describe how the Yukawa potential approximates the "real" situation. In particular, describe the role of the parameter α.

   (b) Draw the effective potential for the two choices $\alpha = 10$ and $\alpha = 0.1$ with $k = 1$ and $\ell = 1$. For which value(s) of α is there the possibility of stable circular orbits?

3. (Challenge Question) Consider a fictitious mass μ subject to a conservative central force $\vec{F} = -\vec{\nabla}U(r)$. In the lecture, we showed using Newtonian Mechanics that the angular momentum and total energy of μ are conserved. In what follows, you will use an alternative approach, namely the Lagrangian formalism, to show that both angular momentum and total energy for the same fictitious mass μ are conserved.

   (a) In polar coordinates, express the Lagrangian of μ in terms of two generalized coordinates r and ϕ.

   (b) You may recall that some generalized coordinates can be ignorable or cyclic. Please identify the cyclic generalized coordinate(s) in the Lagrangian of μ.

   (c) Show that both angular momentum and total energy for μ are conserved.

4. **Circular Orbits**

   (a) Write down the total energy E for a general Kepler orbit with the conservative attractive potential $V(r) = -\frac{k}{r}$.

   (b) How would the above equation change for a circular orbit? Write the total energy E for a circular orbit.

   (c) Solve the above equation to determine the radius r of the circular orbit and show that for a circular orbit

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} = 0.$$