Geodesics

What is a straight line?

we can think of a straight line as a curve whose tangent vector always points in the same direction. e.g.

if $\vec{r}(\lambda) = x(\lambda)\hat{x} + y(\lambda)\hat{y}$

then $\frac{d}{d\lambda}\vec{r} = \vec{V}(\lambda) = \text{const}$

$$\Leftrightarrow \frac{d^2}{d\lambda^2}\vec{r} = \vec{a}(\lambda) = 0$$

$$\Leftrightarrow \ddot{x}(\lambda) = 0 \quad \underline{\text{AND}} \quad \ddot{y}(\lambda) = 0$$

how about in polar coordinates?

$$d\vec{r} = d(\vec{r}) = d(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}}$$

What is $d(\hat{r})$? , $d(\hat{\phi})$

$$d\hat{e}_j = \omega^i{}_j \hat{e}_i \quad , \quad \hat{e}_i \in \{\hat{r}, \hat{\phi}\}$$

$$d\hat{r} = \omega^r{}_r \hat{r} + \omega^\phi{}_r \hat{\phi}$$
$$d\hat{\phi} = \omega^r{}_\phi \hat{r} + \omega^\phi{}_\phi \hat{\phi}$$

and $$\omega^i{}_j + \omega^j{}_i = 0$$
$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$\Rightarrow \quad \omega^r{}_r = \omega^\varphi{}_\varphi = 0$$

$$d(dr) + \omega^r{}_\phi \wedge r\,d\phi = 0$$
$$d(r\,d\phi) + \omega^\phi{}_r \wedge dr = 0$$

$$\omega^r{}_\phi \wedge r\,d\phi = 0 \longrightarrow \boxed{\omega^r{}_\phi = \Gamma^r_{\phi\phi}\, r\,d\phi}$$
$$dr \wedge d\phi - \omega^r{}_\phi \wedge dr = 0$$
$$\longrightarrow \quad dr \wedge d\phi + dr \wedge \Gamma^r_{\phi\phi}\, r\,d\phi = 0$$

$$\Gamma^r_{\phi\phi}\, r = -1 \longrightarrow \Gamma^r_{\phi\phi} = \frac{-1}{r} \Rightarrow \omega^r{}_\phi = -d\phi$$

$$\Rightarrow \quad d\hat{r} = \omega^\phi{}_r\,\hat{\phi} = d\phi\,\hat{\phi}$$
$$d\hat{\phi} = \omega^r{}_\phi\,\hat{r} = -d\phi\,\hat{r}$$

So that, we have

$$\boxed{\frac{d\hat{r}}{d\lambda} = \dot{\phi}\,\hat{\phi} \qquad \frac{d\hat{\phi}}{d\lambda} = -\dot{\phi}\,\hat{r}}$$

$$\longrightarrow \quad \vec{V} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}$$

we want the acceleration to be 0.
Therefore

$$\vec{0} = \frac{d\vec{v}}{d\lambda} = \dot{r}\dot{\hat{r}} + \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}}$$

$$= \dot{r}\dot{\phi}\hat{\phi} + \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r(\dot{\phi})^2\hat{r}$$

$$\vec{0} = (\ddot{r} - r(\dot{\phi})^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$$

$$\Longrightarrow \boxed{\ddot{r} - r(\dot{\phi})^2 = 0 \qquad 2\dot{r}\dot{\phi} + r\ddot{\phi} = 0}$$

Can we do this for the sphere?

$$d\vec{r} = r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \qquad (r \text{ is const.} \rightarrow dr = 0)$$

what are $\omega^i{}_j$ for spherical coordinates

answer: you should get great circles!

Review: | Geodesic Equations

$\ddot{x} = \ddot{y} = 0$ plane-rectangular coords

$\ddot{r} - r\dot{\phi}^2 = 0$ plane-polar coords
$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$

$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$ sphere
$\sin\theta\,\ddot{\phi} + 2\cos\dot{\theta}\,\dot{\phi} = 0$

(4) General Case

$$\frac{d\vec{r}}{d\lambda} = \vec{v} = v^i \hat{e}_i$$

(The numerator gives equality of vector-valued 1-forms)

$$\vec{v}\,d\lambda = d\vec{r} = \sigma^i \hat{e}_i$$

↳ basis 1-forms

NOTE: $v^i d\lambda = \sigma^i$

($\hat{\phi}$) Polar: $v^\phi d\lambda = \sigma^i = r\,d\phi$

$\rightarrow v^\phi = r\frac{d\phi}{d\lambda} = r\dot{\phi}$

Where do connection 1-forms come in?

Geodesic Equation: $\boxed{d\vec{v}=0 \quad \underline{\text{or}} \quad \dot{\vec{v}}=0}$

$$\vec{v} = v^i \hat{e}_i \Longrightarrow d\vec{v} = dv^j \hat{e}_j + v^i d\hat{e}_i$$

$$= dv^j \hat{e}_j + v^i \omega^j_i \hat{e}_j$$

$$d\vec{v} = (dv^j + \omega^j{}_i v^i)\, \hat{e}_j$$

so geodesic $\Longleftrightarrow$ $dv^j + \omega^j{}_i v^i = 0$

$$dv^j + \Gamma^j_{ik} \sigma^k v^i = 0$$

but $v^i d\lambda = \sigma^i$ (all 1 forms on curve are multiples of $d\lambda$.)

"pullback to the curve"

$\rightarrow$ $dv^j + \Gamma^j_{ik} v^i v^k d\lambda = 0$

$\Longleftrightarrow$ $\boxed{\dot{v}^j + \Gamma^j_{ik} v^i v^k = 0}$

first order diff eq on V's. so
2nd order w.r.t. coordinates.

Coupled system of n 2nd order ODE's
in n coordinates.

(1) Work out $\omega^i{}_j$
(2) write Γ^i_{kj}
(3) Geodesic Equations.

Notice symmetry in i and k!

ODE trick: separate and then integrate

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

$$\frac{\ddot{\phi}}{\dot{\phi}} = -2\frac{\dot{r}}{r}$$

$$\rightarrow \int \frac{1}{\dot{\phi}} d\dot{\phi} = -2\int \frac{1}{r} dr$$

$$\rightarrow \ln(\dot{\phi}) = -2\ln(r) + c$$

$$\rightarrow \boxed{\dot{\phi} = C r^{-2}}$$

Now $\qquad \vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y}$

$$\dot{x} = \hat{x} \cdot \vec{v} = \text{const}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

"linear momentum"

$$r^2\dot{\phi} = r\hat{\phi} \cdot \vec{v} = \text{const}$$

"angular momenta"

almost but w.o. mass

Think Noether's theorem

Orthogonal coordinates

$$ds^2 = h_1^2 {dx^1}^2 + h_2^2 {dx^2}^2 + \dots$$

Theorem: if $\dfrac{\partial h_i}{\partial x^j} = 0 \ \forall i$ with j fixed

then, $h_i \hat{e}^i$ (no sum) satisfies

$$d(h_i \hat{e}^i) \cdot d\vec{r} = 0$$

$\Downarrow$

Killing vectors