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6.4.5b

Find all the left and right cosets of $\langle 3 \rangle$ in U(8).

Recall $U(8) = \{1, 3, 5, 7\}$ and we have that $\langle 3 \rangle = 1, 3$. Thus we have that:

$$\begin{aligned}
1\langle 3 \rangle &= \langle 3 \rangle 1 = \{1, 3\} \\
3\langle 3 \rangle &= \langle 3 \rangle 3 = \{3, 1\} \\
5\langle 3 \rangle &= \langle 3 \rangle 5 = \{5, 7\} \\
7\langle 3 \rangle &= \langle 3 \rangle 7 = \{7, 5\}
\end{aligned}$$

Thus we have that the left cosets and right cosets are the same... i.e. $L_H = R_H = \{\{1, 3\}, \{5, 7\}\}$ which we can see is a partition of U(8) as expected.

6.4.5h

Find all the left and right cosets of $H = \{(1), (123), (132)\}$ in S_4.

Recall that S_4 is defined as:

$$\begin{aligned}
S_4 = \{&(1), (12), (13), (14), (23), (24), \\
&(34), (12)(34), (13)(24), (14)(23), (123), (124), \\
&(132), (134), (142), (143), (234), (243), \\
&(1234), (1243), (1324), (1342), (1423), (1432)\}
\end{aligned}$$

Now we need to look at gH where g is in S_4. Note that we know both the left and right cosets must have the same number of elements and the index H in S_4 is $24/3 = 8$ Thus we can stop once we get 8 unique cosets.

$$\begin{aligned}
(1)H &= \{(1)(1), (1)(123), (1)(132)\} \\
&= \{(1), (123), (132)\} \\
(12)H &= \{(12)(1), (12)(123), (12)(132)\} \\
&= \{(12), (23), (12)\} \\
(13)H &= \{(13)(1), (13)(123), (13)(132)\} \\
&= \{(13), (12), (23)\} \\
(14)H &= \{(14)(1), (14)(123), (14)(132)\} \\
&= \{(14), (1234), (1324)\} \\
(23)H &= \{(23)(1), (23)(123), (23)(132)\} \\
&= \{(23), (13), (12)\} \\
(24)H &= \{(24)(1), (24)(123), (24)(132)\} \\
&= \{(24), (1423), (1342)\} \\
(34)H &= \{(34)(1), (34)(123), (34)(132)\} \\
&= \{(34), (1243), (1432)\} \\
(12)(34)H &= \{(12)(34)(1), (12)(34)(123), (12)(34)(132)\} \\
&= \{(12)(34), (243), (143)\} \\
(13)(24)H &= \{(13)(24)(1), (13)(24)(123), (13)(24)(132)\} \\
&= \{(13)(24), (142), (234)\} \\
(14)(23)H &= \{(14)(23)(1), (14)(23)(123), (14)(23)(132)\} \\
&= \{(14)(23), (134), (124)\}
\end{aligned}$$

Thus we have found all of the left cosets. They form a partition of S_4:

$$\begin{aligned}
L_H = \{&\{(1), (123), (132)\} \\
&\{(12), (23), (12)\} \\
&\{(14), (1234), (1324)\} \\
&\{(24), (1423), (1342)\} \\
&\{(24), (1423), (1342)\} \\
&\{(12)(34), (243), (143)\} \\
&\{(13)(24), (142), (234)\} \\
&\{(14)(23), (134), (124)\}\}
\end{aligned}$$

t Now we will do the same for the right cosets although we will find the partition is not the same as that created by L_H.

$$\begin{aligned}
H(1) &= \{(1)(1), (123)(1), (132)(1)\} \\
&= \{(1), (123), (132)\} \\
H(12) &= \{(1)(12), (123)(12), (132)(12)\} \\
&= \{(12), (13), (23)\} \\
H(13) &= \{(1)(13), (123)(13), (132)(13)\} \\
&= \{(13), (23), (12)\} \\
H(14) &= \{(1)(14), (123)(14), (132)(14)\} \\
&= \{(14), (1423), (1432)\} \\
H(23) &= \{(1)(23), (123)(23), (132)(23)\} \\
&= \{(23), (12), (13)\} \\
H(24) &= \{(1)(24), (123)(24), (132)(24)\} \\
&= \{(24), (1243), (1324)\} \\
H(34) &= \{(1)(34), (123)(34), (132)(34)\} \\
&= \{(34), (1234), (1342)\} \\
H(12)(34) &= \{(1)(12)(34), (123)(12)(34), (132)(12)(34)\} \\
&= \{(12)(34), (341), (234)\} \\
H(13)(24) &= \{(1)(13)(24), (123)(13)(24), (132)(13)(24)\} \\
&= \{(13)(24), (243), (124)\} \\
H(14)(23) &= \{(1)(14)(23), (123)(14)(23), (132)(14)(23)\} \\
&= \{(14)(23), (142), (143)\}
\end{aligned}$$

Thus we have found the right cosets of H in S_4. They form the partition:

$$\begin{aligned}
R_H = \{&\{(1), (123), (132)\} \\
&\{(12), (13), (23)\} \\
&\{(14), (1423), (1432)\} \\
&\{(24), (1243), (1324)\} \\
&\{(34), (1234), (1342)\} \\
&\{(12)(34), (341), (234)\} \\
&\{(13)(24), (243), (124)\} \\
&\{(14)(23), (142), (143)\}\}
\end{aligned}$$

6.4.14

given $g^n = e$ prove the order of g divides n.

By definition of the order of an element g in the group G, the order is the smallest integer k such that $g^k = e$. Thus there are two cases we must consider: $n \neq k$ and $n = k$.

If $n = k$ then we have that n clearly divides itself. Thus the proposition is true for the first case. Now if $n \neq k$ then for some $q, r \in \mathbb{Z}$ the division algorithm tells us that $n = qk + r$. Thus the statement of the proposition becomes: $g^n = g^{qk+r} = g^{qk}g^r = e$. Now $g^{qk} = e$ as $g^{qk} = (g^k)^q = e^q = e$. Therefore in $r = 0$ and so then we have $n = qk$ which means that k, the order of g divides n. $\square$

6.4.19

Let H and K be subgroups of G. Prove that $gH \cap gK$ is a coset of $H \cap K$ in G.

Suppose $gH \cap gK \neq \emptyset$. Now let $f \in gH \cap gK$. Then by definition of the intersection of two sets, we have that:

$$f \in gH \quad \text{and} \quad f \in gK$$

This implies that $f = gh = gk$ for some $h \in H, k \in K$. Since G is a subgroup, $\exists g^{-1}$ such that:

$$\begin{aligned} g^{-1}f &= g^{-1}gh = g^{-1}gk \\ g^{-1}f &= h = k \\ \Rightarrow g^{-1}f &\in H \cap K \\ f &\in g(H \cap K) \end{aligned}$$

i.e. f is an element of $g(H \cap K)$ which is a coset of $H \cap K$ in G. $\square$

9.3.2

Prove that $\mathbb{C}^\star$ is isomorphic to the subgroup of $GL_2(\mathbb{R})$ consisting of matrices of the form: $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $\forall a, b \in \mathbb{R} \quad s.t. \quad a^2 + b^2 \neq 0$.

Recall that $\mathbb{C}^\star$ is $\{\mathbb{C} \setminus \{0\}, \cdot\}$. I claim that the mapping $\phi : \mathbb{C}^\star \to S$, the subgroup of $GL_2(\mathbf{R})$ defined by:

$$\phi(\gamma + i\delta) = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}$$

is an isomorphism between the two groups. Clearly this function is a bijection as the inverse can be seen to be:

$$\phi_{-1} \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix} = \gamma + i\delta \in \mathbb{C}^\star$$

Now all that is left to show is that for any $z_1, z_2 \in \mathbb{C}^\star$ we have that $\phi(z_1 \cdot z_2) = \phi(z_1) \cdot \phi(z_2)$. Let

$z_1 = \alpha + i\beta$ and $z_2 = a + ib$. We have that:

$$\begin{aligned}
\phi(z_1 \cdot z_2) &= \phi((\alpha + i\beta)(a + ib)) \\
&= \phi((\alpha a - \beta b) + i(\alpha b + a\beta) \\
&= \begin{pmatrix} \alpha a - \beta b & \alpha b + a\beta \\ -\alpha b - a\beta & \alpha a + \beta b \end{pmatrix} \\
\phi(z_1) \cdot \phi(z_2) &= \phi(\alpha + i\beta) \cdot \phi(a + ib) \\
&= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\
&= \begin{pmatrix} \alpha a + \beta(-b) & \alpha b + \beta a \\ -\beta a - \alpha b & -\beta b + \alpha a \end{pmatrix} \\
&= \begin{pmatrix} \alpha a - \beta b & \alpha b + a\beta \\ -\alpha b - a\beta & \alpha a - \beta b \end{pmatrix} \\
\Rightarrow \phi(z_1 \cdot z_2) &= \phi(z_1) \cdot \phi(z_2)
\end{aligned}$$

We have constructed a bijection ϕ that preserves group operations. Thus we have proved that $\mathbb{C}^\star \cong S$.

9.3.3

prove or disprove that $U(8) \cong \mathbb{Z}_4$

Suppose that there exists an isomorphism $\phi : U(8) \to \mathbb{Z}_4$. Then by theorem 9.6 there must exist an inverse mapping $\phi^{-1} : \mathbb{Z}_4 \to U(8)$ since ϕ is a bijection. Now again by theorem 9.6 because ϕ^{-1} is an isomorphism, if $\mathbb{Z}_4$ is cyclic then U(8) must be cyclic. We know $\mathbb{Z}_4$ is cyclic with $\langle 1 \rangle$ the generator. We can test this by examining the powers of each element in U(8):

$$\begin{aligned}
U(8) &= \{1, 3, 5, 7\} \\
1^n &= 1 \\
3^2 mod(8) &= 1 \\
5^2 mod(8) &= 1 \\
7^2 mod(8) &= 1
\end{aligned}$$

From this we can see that none of the elements in $U(8)$ generate $U(8)$. Therefore, it can *not* be cyclic and so we have a contradiction to our supposition. Therefore we conclude that $U(8) \ncong \mathbb{Z}_4$