Homework 6 Mth 311

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2) Define $f(x) \equiv \begin{cases} x & \text{if } x \in [0,1] \text{ and } x \in \mathbb{Q} \\ -x & \text{if } x \in [0,1] \text{ and } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

prove $f: [0,1] \to \mathbb{R}$ is not integrable.

Because both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in the ~~rationals~~ reals $\forall$ interval $[\alpha, \beta]$ w/ $\alpha, \beta \in \mathbb{R}$ contains points $\in \mathbb{Q}$ and points in $\mathbb{R}\setminus\mathbb{Q}$.

Hence $\forall$ partition $\{x_i\}$ of $[0,1]$ $\forall$ width $\delta > 0$ we can choose x_i' to be such that $x_i' \in \mathbb{Q}$ $\forall i$ or $x_i' \in \mathbb{R}\setminus\mathbb{Q}$ $\forall i$.

Let $\{x_i\}$ be a partition for $[0,1]$. Then we define

$$S_1 = \sum_{i=1}^{N} f(x_i^{1\prime})(x_i - x_{i-1}) \quad \text{and}$$

$$S_2 = \sum_{i=1}^{N} f(x_i^{2\prime})(x_i - x_{i-1})$$

such that $x_i^{1\prime} \in \mathbb{Q}$ $\forall i$

$x_i^{2\prime} \in \mathbb{R}\setminus\mathbb{Q}$ $\forall i$

Then $\forall \delta > 0$ (width of partition), we have

$\cancel{|S_1 - S_2|}$

$|S_1 - S_2| > 0$ since S_2 must be negative

since $f(x_i^{2\prime}) \leq 0$ $\forall i$ by def

so $\exists \varepsilon$ s.t. $|S_1 - S_2| \geq \varepsilon$ thus f is not integrable

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9) suppose $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ continuous
prove $\int_a^b |f+g| \le \int_a^b |f| + \int_a^b |g|$

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observe that because f,g continuous on $[a,b]$ $\exists \int_a^b f$, $\int_a^b g$. By linearity of integration we have that $\exists \int_a^b f+g$.

Now observe that $\forall \alpha,\beta \in \mathbb{R}$ $\alpha\ne\beta$ by triangle inequality we know

$$|\alpha+\beta| \le |\alpha| + |\beta|$$

Since f and g are real valued we can say $|f+g| \le |f| + |g|$ $\forall x \in [a,b]$

all the absolute value does is make the functions strictly positive so $|f+g|, |f|, |g|$ are continuous and therefore intlable on $[a,b]$.

$\therefore$ by corollary 1 (page 117) we have that

$|f+g|$ and $|f|+|g|$ are intlable on $[a,b]$ ~~so~~
and $|f+g| \le |f|+|g|$ so

$$\int_a^b |f+g| \le \int_a^b |f|+|g| = \int_a^b |f| + \int_a^b |g|$$

$\blacksquare$

3. Compute $\int_0^1 x\,dx$ directly from definition assuming only that the integral exists

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(3)

Since the integral exists we know
$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $\{X_i\}$ is a partition of width $< \delta$ then

$|S - A| < \varepsilon$ where S is Reimann sum and A is $\int_a^b f(x)\,dx$

let $\{X_i\}$ be a regular partition of width $\frac{b-a}{N} = \frac{1}{N}$ i.e. $X_i = 0 + \frac{i}{N} = \frac{i}{N}$

we can make the width arbitrarily small by controlling N so by the above def:

$$\int_0^1 x\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} x_i \left(\frac{1}{N}\right)$$

choosing to use right end points gives

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N}\left(\frac{1}{N}\right) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} i$$

$$= \lim_{N\to\infty} \frac{1}{N^2}\,\frac{N(N+1)}{2} = \lim_{N\to\infty} \frac{1}{2} + \frac{1}{2N} = \frac{1}{2}$$

thus $\int_0^1 x\,dx = \frac{1}{2}$ which is the same answer we get if we use the FTC to calculate the integral instead!

4. Let $f:[a,b]\to\mathbb{R}$ be strictly increasing function. Show $\int_a^b f(x)dx$ exists

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Recall $f:[a,b]\to\mathbb{R}$ is int'able iff $\forall \varepsilon>0$ $\exists$ step functions f_1, f_2 on $[a,b]$ s.t.

$$f_1(x)\le f(x)\le f_2(x) \quad \forall x\in[a,b]$$

and

$$\int_a^b f_2(x)-f_1(x)\,dx < \varepsilon$$

Let's define f_2 to be a step function defined as

$$f_2(x) \equiv \sum_{i=1}^{N} f(x_i)\,\mathbb{1}_{(x_{i-1},x_i)}(x) \quad \text{(right ~~hand~~ end pt.)}$$

and f_1 to be

$$f_1(x) \equiv \sum_{i=1}^{N} f(x_{i-1})\,\mathbb{1}_{(x_{i-1},x_i)}(x) \quad \text{(left end pt)}$$

then $\forall x\in[a,b]$ it is true that

$$f_1(x)\le f(x)\le f_2(x)$$

Now WTS

$$\int_a^b f_2(x)-f_1(x)\,dx < \varepsilon$$

by linearity we have then that

$$\int_a^b f_2(x)dx-\int_a^b f_1(x)dx < \varepsilon$$

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4. continued...

Recall that for any step function $f : [a,b] \to \mathbb{R}$

$$\int_a^b f \, dx = \sum_{i=1}^{N} c_i (x_i - x_{i-1})$$

So we can write

$$\int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx = \sum_{i=1}^{N} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(x_{i-1})(x_i - x_{i-1})$$

$$= \sum_{i=1}^{N} \left(f(x_i) - f(x_{i-1}) \right)(x_i - x_{i-1})$$

because f is strictly increasing $f(x_i) - f(x_{i-1}) > 0 \;\; \forall i$ so since we can make $(x_i - x_{i-1})$ arbitrarily small, we can make $\sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$ arbitrarily small and thus

$$\int_a^b f_2(x) - f_1(x)\,dx < \varepsilon.$$

Therefore if f is strictly increasing on $[a,b]$ it is integrable.

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