1) Show $K = \dfrac{-1}{2\sqrt{EG}} \left\{ \left(\dfrac{E_v}{\sqrt{EG}}\right)_v + \left(\dfrac{G_u}{\sqrt{EG}}\right)_u \right\}$

when $F = 0$

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + 0 - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & 0 - \frac{1}{2}E_v \\ -\frac{1}{2}G_u & E & 0 \\ \frac{1}{2}G_v & 0 & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & 0 \\ \frac{1}{2}G_u & 0 & G \end{vmatrix}}{(EG - 0)^2}$$

$$\left(-\tfrac{1}{2}E_{vv} - \tfrac{1}{2}G_{uu}\right)EG - \left(\tfrac{1}{2}E_u\right)\left(-\tfrac{1}{2}G_u G\right) + \tfrac{1}{2}E_v \tfrac{1}{2}G_v E - \left(-\tfrac{1}{2}E_v \tfrac{1}{2}E_v G + \tfrac{1}{2}G_u\left(-\tfrac{1}{2}G_u E\right)\right) \Big/ (EG)^2$$

$$\frac{-\frac{1}{2}E_{vv}EG - \frac{1}{2}G_{uu}EG + \frac{1}{4}E_u G_u G + \frac{1}{4}E_v G_v E + \frac{1}{4}E_v E_v G + \frac{1}{4}G_u G_u E}{(EG)^2}$$

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u \right)$$

$$\frac{\partial}{\partial v}\left(E_v (EG)^{-1/2}\right) = \frac{E_{vv}}{\sqrt{EG}} - \frac{1}{2}\frac{E_v E_v G}{(EG)^{3/2}} - \frac{1}{2}\frac{E_v G_v E}{(EG)^{3/2}}$$

$$\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) = \frac{\partial}{\partial u} G_u (EG)^{-1/2} = \frac{G_{uu}}{\sqrt{EG}} - \frac{1}{2}\frac{G_u E_u G}{(EG)^{3/2}} - \frac{1}{2}\frac{G_u G_u E}{(EG)^{3/2}}$$

$$K = -\frac{1}{2}\frac{E_{vv}}{EG} + \frac{1}{4}\frac{E_v E_v G}{(EG)^2} + \frac{1}{4}\frac{E_v G_v E}{(EG)^2} - \frac{1}{2}\frac{G_{uu}}{(EG)^2} + \frac{1}{4}\frac{G_u E_u G}{(EG)^2} + \frac{1}{4}\frac{G_u G_u E}{(EG)^2}$$

$$= \frac{-\frac{1}{2}E_{vv}EG + \frac{1}{4}E_v E_v G + \frac{1}{4}E_v G_v E - \frac{1}{2}G_{uu}EG + \frac{1}{4}G_u E_u G + \frac{1}{4}G_u G_u E}{(EG)^2}$$

2. $\sigma(u,v) = (u\cos(v), u\sin(v), \ln(u))$ · Catenoid

$\bar{\sigma}(u,v) = (u\cos v, u\sin(v), v)$ – Helicoid

$\sigma_u = (\cos(v), \sin(v), \frac{1}{u})$ $\sigma_v = (-u\sin(v), u\cos(v), 0)$

$E = 1 + \frac{1}{u^2}$, $F = 0$, $G = u^2$

$\bar{\sigma}_u = (\cos(v), \sin(v), 0)$ $\bar{\sigma}_v = (-u\sin(v), u\sin(v), 1)$

$\bar{E} = 1$ $F = 0$ $G = u^2+1$

from $$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

$E_v = 0$ $G_u = 2u$ $EG = u^2+1$

$$\rightarrow K = -\frac{1}{2\sqrt{u^2+1}}\left(\frac{2u}{\sqrt{u^2+1}}\right)_u$$

$\bar{E}_v = 0$ $\bar{G}_u = 2u$ $\bar{E}\bar{G} = u^2+1$

$$\bar{K} = -\frac{1}{2\sqrt{u^2+1}}\left(\frac{2u}{\sqrt{u^2+1}}\right)_v$$ thus the curvatures are the same

Now show $\bar{\sigma} \circ \sigma^{-1}$ is not an isometry.

first, to be an isometry $f = \bar{\sigma} \circ \sigma^{-1}$ must be a diffeomorphism. We know that the helicoid and catenoid are locally isometric which means they have the same first fundamental form. The global isometry must fail because $\bar{\sigma} \circ \sigma^{-1}$ is not a diffeomorphism.

Exercise 3.101 that f isometry iff $f \circ \sigma$ preserves first fund form. $f = \bar{\sigma} \circ \sigma^{-1}$, $f \circ \sigma = \bar{\sigma}$

and $F_1 \neq \bar{F}_1$ $(1+\frac{1}{u^2})du^2 + u^2dv^2 \neq du^2 + (u^2+1)dv^2$

3) Show $\nexists$ surface s.t. $E = G = 1$, $F = 0$

and $e = 1$, $g = -1$, $f = 0$.

$f = 0$ $e = 1 \Longrightarrow$ Proposition (IN)

$f_u = 0$ $e_v = 0$ $0 = \Gamma^1_{12} + \Gamma^2_{11}$

$K = \dfrac{eg - f^2}{EG - F^2}$ $\cancel{K}$ Corollary 5.55

$K = -\dfrac{(\sqrt{G})_{uu}}{\sqrt{G}}$

$= \dfrac{-1}{1} = -1$ $K = 0$

so contradiction X

(4) Compute Christoffel symbols for an open set of the plane.

$$\sigma(u,v) = (u, v, 0)$$

a)

$$\sigma_u = (1,0,0) \qquad \sigma_v = (0,1,0)$$

$$\sigma_{uu} = (0,0,0) \qquad \sigma_{vv} = (0,0,0)$$

$$\sigma_{uu} = 0\,\sigma_u + 0\,\sigma_v + 0N = \sigma_{vv}$$

thus $\Gamma_{ij}^{k} = 0$

Gauss (IV) $\rightarrow$ $EK = 0 \implies K = 0$ since $E = 1$

b)

$$\cancel{\sigma(r,\theta) = (r\cos\theta, r\sin\theta, 0)}$$

$$\cancel{\sigma_r = (\cos\theta, \sin\theta, 0)} \qquad \cancel{\sigma_\theta = (-r\sin\theta, r\cos\theta, 0)}$$

$$\cancel{\sigma_{rr} = (0,0,0)}$$

$$\cancel{\sigma_{r\theta} = (}$$

$$\sigma_r = (\cos\theta, \sin\theta, 0) \qquad \sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$E = 1 \qquad F = 0 \qquad G = r^2$$

$$\partial = 2EG - 2F^2 = 2r^2$$

(a) $\Gamma_{11}^{1} = 0 - 0 + 0 = 0$

(b) $\Gamma_{12}^{1} = 0$

(c) $\Gamma_{22}^{1} = \dfrac{-2r^3}{2r^2} = -r$

d) $0 = \Gamma_{11}^{2}$

e) $\Gamma_{12}^{2} = \dfrac{1}{r}$

f) $\Gamma_{22}^{2} = 0$

4) continued

$$\Gamma_{22}^{1} = -r \qquad \Gamma_{12}^{2} = \frac{1}{r}$$

(iv) $$EK = \cancel{(\Gamma_{11}^{2})_v} - \underset{u=r}{(\Gamma_{12}^{2})_u} + \cancel{\Gamma_{11}^{1}}\Gamma_{12}^{2} + \cancel{\Gamma_{11}^{2}}\Gamma_{22}^{2} - \cancel{\Gamma_{12}^{1}}\Gamma_{11}^{2} - (\Gamma_{12}^{2})^{2}$$

$$-\left(-\frac{1}{r^2}\right) \qquad + \qquad -\frac{1}{r^2}$$

$E \neq 0 \Rightarrow \boxed{K = 0}$

5) a) sphere
b) cylinder
c) $z = x^2 - y^2$

justify why not pairwise locally isometric.

$$\sigma_s(\phi, \theta) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

$$\sigma_c(u,v) = (u\cos v, u\sin v, u)$$

$$\sigma(u,v) = (u, v, u^2 - v^2)$$

~~E_s~~ $\cancel{\sigma_\phi^s = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0)}$

$\cancel{\sigma_\theta^s = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)}$

$\cancel{E_s =} \cos^2\theta, \quad F^s = 0, \quad G^s = 1$

$\cancel{K = \frac{-1}{2\sqrt{\cos\theta}}}$

unit sphere $K = 1$

cylinder $K = 0 \approx$ plane

recall $K(\text{graph}) = \dfrac{\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2}{(1+\varphi_x^2+\varphi_y^2)} = \dfrac{4 - 10}{(1+4x^2+4y^2)^2}$

thus by contrapositive of Theorema Egregium