

Homework 7

MTH 443

Dr. Schmidt

John Waczak

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Worked with: Garrett Jepson

1.) Let T be a linear operator defined on \mathbb{C}^4 by $Te_1 = 0$, $Te_2 = -5e_1$, $Te_3 = 5e_1$, and $Te_4 = 2e_2 + 5e_3$. Determine a Jordan basis for \mathbb{C}^4 with respect to T .

In order to find a Jordan basis with respect to T , let us consider the matrix representation of T in the standard basis

$$\mathcal{B} = \{e_1, e_2, e_3, e_4\}$$

From the definition of how T acts on each of these vectors we find

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we can see that the characteristic polynomial for T is

$$c_T[x] = x^4$$

This polynomial splits over \mathbb{C} and gives a single eigenvalue of $\lambda = 0$ with algebraic multiplicity 4. To find the corresponding eigenspace, we look for solutions to $[T]_{\mathcal{B}} - 0 \mathcal{I}_{\mathbb{C}^4} = 0$ i.e.

$$\begin{pmatrix} 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow x = \text{anything}$$
$$y = z$$
$$t = 0$$

From this we see that the eigenspace for $\lambda = 0$ is given by

$$\mathcal{E}_0 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

From class we know that $\mathcal{E}_0 \subseteq \mathcal{K}_0$ where \mathcal{K}_0 is the generalized eigenspace for $\lambda = 0$. Therefore, each of the basis vectors we have identified for \mathcal{E}_0 constitutes a Jordan chain with one element. Now we want to *grow* these chains so that we have 4 linearly independent vectors. Growing the chain means looking for vectors

that satisfy $([T]_{\mathcal{B}} - 0 \mathcal{I}_{\mathbb{C}^4})^2 v = 0$. Let's try a vector that lies in the range of T and not in the null space. An easy choice would be e_4 . Calculation yields

$$\begin{aligned} [T]_{\mathcal{B}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 2 \\ 5 \\ 0 \end{pmatrix} \\ [T]_{\mathcal{B}}^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= [T]_{\mathcal{B}} \begin{pmatrix} 0 \\ 2 \\ 5 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Therefore we have that $e_4 \in \ker(T^2)$ which gives us a new Jordan Chain of length 2:

$$C = \{2e_2 + 5e_3, e_4\}$$

Thus we conclude that the Jordan basis for \mathbb{C}^4 with respect to T is

$$\mathcal{J} = \{e_2 + e_3, e_1, 2e_2 + 5e_3, e_4\}$$

Writing \mathcal{J} as a matrix and taking the determinant yields $\det([\mathcal{J}]) = -3 \neq 0$ which confirms that these vectors are linearly independent and therefore do form a basis for \mathbb{C}^4 . If we compose this change of basis transformation with $[T]_{\mathcal{B}}$ we find

$$\mathcal{J}^{-1}[T]_{\mathcal{B}}\mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is in Jordan Canonical form as desired.

2.) Let V be a finite dimensional \mathbb{C} -vector space and let T be a linear operator on V . Assume that $c_T[x] = x^{10}$ and $N_T = R_T$. Does this information determine the Jordan canonical form of T ? If so, give this form and if not, give examples to this effect.

We see that the characteristic polynomial splits over \mathbb{C} into 10 factors of $(x - 0)$. Therefore, by theorem (Jordan Mania class notes) we have that there exists a Jordan basis for T and therefore the matrix of T is similar to a matrix in Jordan Canonical Form. Furthermore, because the degree of $c_T[x]$ is 10 we have that $\dim(V) = 10$. Therefore, by the Rank-Nullity theorem, we have that

$$\begin{aligned} 10 &= \dim(N_T) + \dim(R_T) \\ &= \dim(N_T) + \dim(N_T) \\ \Rightarrow \dim(N_T) &= \dim(R_T) = 5 \end{aligned}$$

$c_T[x]$ has a single root at 0 meaning that T has a single eigenvalue of $\lambda = 0$ with algebraic multiplicity 10. Based on the above calculation, we therefore have that the eigenspace $\mathcal{E}_0 = \ker(T - 0 \mathcal{I}) = N_T$ is spanned by 5 linearly independent vectors. Because each of these vectors is in the kernel of T , each constitutes a Jordan chain of one element. We now wish to extend these chains so that they have two elements. Therefore, we are looking for vectors b_i satisfying $b_i \in \ker(T - 0 \mathcal{I})^2$ or

$$T^2 b_i = T(Tb_i) = 0$$

From this we see that we must have $Tb_i \in \ker T$ and therefore we can write

$$Tb_i = \sum_i^5 \alpha_i v_i$$

At this point nothing prevents us from choosing the b_i such that the summation simplifies to something nice.

$$Tb_i = v_i$$

The v_i are linearly independent and therefore each of the b_i are distinct because linear operators are functions and therefore send each input to exactly one output. We now have that the Jordan chains become

$$\mathcal{J}_i = \{v_i, b_i\}$$

The matrix representation of each of these distinct chains forms a Jordan block. These are

$$B_i = T\mathcal{J}_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Now we have everything we need to construct the JCF of T . We have 5, 2x2 Jordan blocks with the above form. If $\mathcal{B}_{\mathcal{J}} = \bigcup_i^5 \mathcal{J}_i$, then

$$[T]_{\mathcal{B}_{\mathcal{J}}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In finding this JCF we assumed that each chain could be grown to a chain with 2 elements. This is just an assumption and without more information it is not clear whether or not each chain can be grown or should stop at 2 elements. From class we know that the dimension of the eigenspace is the number of chains of a fixed eigenvalues. Thus it is conceivable that we could also find a JCF in which we have

three 3x3 blocks and a 1x1 block, or perhaps a 5x5 block, two 2x2 blocks and a 1x1 block. So long as there are 5 blocks and the total size is 10, the JCF is valid.

Therefore, we conclude that we are not given enough information to explicitly determine the JCF of the linear operator T .

- 3.) Determine the Jordan canonical form of the linear operator T on \mathbb{C}^n defined by $Te_i = \sum_{i=1}^n e_i$

If we take the basis for T to be the standards basis,

$$\mathcal{B} = \{e_1, e_2, \dots, e_n\},$$

then the above definition indicates that the matrix of T in the basis \mathcal{B} is the nxn matrix of ones. That is,

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

We want to find the Jordan Canonical Form of this matrix. To do this, we first consider the characteristic polynomial of T given by

$$\begin{aligned} c_T[x] &= \det([T]_{\mathcal{B}} - \lambda I) \\ &= \begin{vmatrix} 1-x & 1 & 1 & \cdots & 1 \\ 1 & 1-x & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1-x \end{vmatrix} \end{aligned}$$

For the case of $n=2$, we have $c_T[x] = (1-x)^2 - 1 = (x-2)x$. For the case of $n=3$, we have $c_T[x] = 3x^2 - x^3 = (3-x)x^2$. Therefore, by an induction, the characteristic polynomial for the $n=n$ cases is

$$c_T[x] = (x-n)x^{n-1}$$

Therefore, equating $c_T[x]$ to zero yields the eigenvalues $\lambda = n$ with algebraic multiplicity one, and $\lambda = 0$ with algebraic multiplicity $n-1$. Now consider the eigenvalue equation for the first eigenvalue. We have

$$Tv = nv$$

for an eigenvector v . If $v = \sum_i \alpha_i e_i$ then the above equation becomes

$$\sum_i \alpha_i = n\alpha_j$$

For each j . This is solved if we allow $\alpha_i = \alpha_j \forall i, j$. Therefore we have that the vector $v_1 = \sum_i e_i$ spans the eigenspace \mathcal{E}_n . For the other eigenvalue of $\lambda = 0$

we have that this reduces to finding the kernel of T . That is, we are looking for solutions to

$$\begin{aligned} Tv &= 0v \\ \Rightarrow \sum_i \alpha_i &= 0 \end{aligned}$$

This leads to the condition that

$$\sum_{i \neq j} \alpha_i = -\alpha_j$$

Which means that we have $n - 1$ free parameters. Therefore, the eigenspace \mathcal{E}_0 is spanned by $n - 1$ vectors. Each of these vectors constitutes a Jordan chain of size 1 so that overall we have a single chain from \mathcal{E}_n and $n - 1$ chains from \mathcal{E}_0 . Therefore if we take the ordering of our Jordan basis to be the $\lambda = 0$ eigenvectors first, our JCF of T will actually be diagonalized, i.e.

$$[T]_{\text{JCF}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

In terms of the original transformation, this is

$$\begin{cases} Tb_i = 0; & i < n \\ Tb_i = nb_i; & i = n \end{cases}$$

where the b_i are the elements of the Jordan basis described above.

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