

## 1. INDEX GYMNASTICS

In a coordinate basis  $\{dx^i\}$  of 1-forms, the components  $g_{ij}$  of the metric are defined by  $ds^2 = g_{ij}dx^i dx^j$ . The dual basis  $\{\vec{e}_i\}$  of vectors satisfies  $\vec{e}_i \cdot \vec{e}_j = g_{ij}$ .

*These bases are **not** necessarily orthogonal. It is however still true that  $d\vec{r} = dx^i \vec{e}_i$*

- (a) Determine an expression for  $\vec{e}_i \cdot \vec{\nabla} f$  in terms of partial derivatives.

Expanding the total derivative of  $f$ , we have

$$df = \frac{\partial f}{\partial x^i} dx^i = \vec{\nabla} f \cdot d\vec{r} \quad (1)$$

$$= \vec{\nabla} f \cdot dx^i \vec{e}_i \quad (2)$$

$$= (\vec{e}_i \cdot \vec{\nabla} f) dx^i \quad (3)$$

$$\Rightarrow \vec{e}_i \cdot \vec{\nabla} f = \frac{\partial f}{\partial x^i} \quad (4)$$

- (b) Acting on 1-forms  $F = \vec{F} \cdot d\vec{r}$ ,  $G = \vec{G} \cdot d\vec{r}$ , the metric satisfies  $g(F, G) = \vec{F} \cdot \vec{G}$  for any vectors  $\vec{F}, \vec{G}$ . Express the components  $g^{ij} = g(dx^i, dx^j)$  in terms of the components  $g_{ij}$ .

*A derivation in 2 dimensions is acceptable if you don't see how to handle the general case.*

In order to expand  $g^{ij} = g(dx^i, dx^j)$ , we need to find the fields related to  $dx^i$  so that we can take advantage of the above fact. To do this, consider the following inner products

$$\vec{e}_i \cdot d\vec{r} = \vec{e}_i \cdot dx^k \vec{e}_k \quad (5)$$

$$= dx^k g_{ik} \quad (6)$$

$$\vec{e}_j \cdot d\vec{r} = \vec{e}_j \cdot dx^\ell \vec{e}_\ell \quad (7)$$

$$= dx^\ell g_{j\ell} \quad (8)$$

This suggest that we should look at the inner product on these two fields.

$$g(dx^k g_{ik}, dx^\ell g_{j\ell}) = \vec{e}_i \cdot \vec{e}_j = g_{ij} \quad (9)$$

$$\Rightarrow g_{ik} g_{j\ell} g(dx^k, dx^\ell) = g_{ij} \quad (10)$$

$$g_{ik} g_{j\ell} g^{k\ell} = g_{ij} \quad (11)$$

Now we note that this index equation may be reinterpreted as a matrix equation. If we take

$$(g_{ij}) \equiv G \quad (g^{ij}) \equiv \tilde{G} \quad (12)$$

Then equation 11 reinterpreted in terms of these matrices says

$$G(G\tilde{G}) = G \quad (13)$$

Such an equation has a nontrivial solution only if

$$G\tilde{G} = \mathcal{I} \quad (14)$$

where  $\mathcal{I}$  is the identity matrix. This further implies that we must have  $\tilde{G} = G^{-1}$ . Therefore each  $g^{ij}$  are the elements of the inverse matrix to  $G$  corresponding to all of the  $g_{ij}$ . There is no simple way to write this out as an inverse matrix is the adjugate matrix divided by the determinant. We would therefore expect each  $g^{ij}$  depends the determinant of many submatrices of  $G$  resulting in 16 linear equations that we can, in principle, solve.

## 2. DOUBLE-NUL COORDINATES

In 2-dimensional Minkowski space, let  $u = t - x$ ,  $v = t + x$ .

- (a) Express the line element  $ds^2 = -dt^2 + dx^2$  in terms of the coordinate basis  $\{du, dv\}$ .

Taking the exterior derivative of the above, we find

$$du = dt - dx \quad dv = dt + dx \quad (15)$$

so that

$$dt = \frac{1}{2}(dv + du) \quad (16)$$

$$dx = \frac{1}{2}(dv - du) \quad (17)$$

Therefore, the re-expressed line element is

$$ds^2 = -dt^2 + dx^2 \quad (18)$$

$$= -\left[\frac{1}{2}(dv + du)\right]^2 + \left[\frac{1}{2}(dv - du)\right]^2 \quad (19)$$

$$= -\frac{1}{2}dv \, du - \frac{1}{2}du \, dv \quad (20)$$

I chose to leave equation (20) as it is to make identifying elements of the metric easier.

- (b) Determine the components  $g^{ij}$  in this basis.

Recall from problem 1 that  $ds^2 = g_{ij}dx^i dx^j$ . Therefore, by inspection, we have that

$$\begin{cases} g_{uu} = g_{vv} = 0 \\ g_{uv} = g_{vu} = -\frac{1}{2} \end{cases} \quad (21)$$

so that as a matrix, the metric is

$$(g_{ij}) = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad (22)$$

Now we wish to find all of the  $g^{ij}$ . Recall from problem 1 that  $g^{ij} \equiv g(dx^i, dx^j)$ . Therefore,

$$g^{uu} = g(du, du) = g(dt - dx, dt - dx) \quad (23)$$

$$= g(dt, dt) + g(dx, dx) \quad (24)$$

$$= -1 + 1 = 0 \quad (25)$$

$$g^{uv} = g(du, dv) = g(dt - dx, dt + dx) \quad (26)$$

$$= g(dt, dt) - g(dx, dx) \quad (27)$$

$$= -1 - 1 = -2 \quad (28)$$

$$g^{vu} = g(dv, du) = g(dt + dx, dt - dx) \quad (29)$$

$$= g(dt, dt) - g(dx, dx) \quad (30)$$

$$= -1 - 1 = -2 \quad (31)$$

$$g^{vv} = g(dv, dv) = g(dt + dx, dt + dx) \quad (32)$$

$$= g(dt, dt) + g(dx, dx) \quad (33)$$

$$= -1 + 1 = 0 \quad (34)$$

To summarize, the matrix corresponding to  $g^{ij}$  is

$$(g^{ij}) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \quad (35)$$

which is in fact the inverse matrix to  $(g_{ij})$  as we predicted in problem 1.

(c) Compute  $g_{ij}g^{jk}$ . What sort of beast is it the components of?

There are four terms we must compute. Using our results from (b), they are

$$g_{uu}g^{uu} + g_{uv}g^{vu} = 1 = \delta_u^u \quad (36)$$

$$g_{uu}g^{vu} + g_{uv}g^{vv} = 0 = \delta_u^v \quad (37)$$

$$g_{vu}g^{uu} + g_{vv}g^{uv} = 0 = \delta_v^u \quad (38)$$

$$g_{vu}g^{uv} + g_{vv}g^{vv} = 1 = \delta_v^v \quad (39)$$

Apparently,  $g_{ij}g^{jk} = \delta_i^k$  but we know this to be the case from problem 1 where we concluded that this kind of “beast” corresponds to an identity matrix. In other words,

$$(g_{ij}g^{jk}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (40)$$

### 3. TRACES

Suppose that two vector-valued 1-forms  $\vec{G} = G^i_j \sigma^j \vec{e}_i$  and  $\vec{R} = R^i_j \sigma^j \vec{e}_i$  have components that are related by

$$G^i_j = R^i_j - \frac{1}{2} \delta^i_j R, \quad (41)$$

where  $R = R^i_i$ . Find an expression for the trace  $G = G^i_i$  of  $\vec{G}$  in terms of  $R$ .

$R$  is called the **trace** of  $\vec{R}$ ; more precisely, it is the trace of the matrix of components  $(R^i_j)$ . You may assume if desired that the underlying geometry is 4-dimensional, with signature 1.

To find the trace, we simply let  $j = i$ . Doing this yields

$$G^i_i = R^i_i - \frac{1}{2}\delta^i_i R \quad (42)$$

$$= R^i_i - \frac{1}{2} \left( \sum_i 1 \right) R \quad (43)$$

$$= R^i_i - \frac{N}{2} R = R \left( 1 - \frac{N}{2} \right) \quad (44)$$

where  $N$  is the dimensionality of the space. For the particular case of 4-dimensional spacetime with signature 1, we have

$$G^i_i = R(1 - 2) = -R \quad (45)$$

which a quick internet search confirms. Apparently, equation (45) is the reason why  $\vec{\mathbf{G}}$ , the Einstein tensor, is also known as the trace-reversed Ricci tensor.