Homework 5 **John Waczak**
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1 Imagine a situation in which there are three particles and only three states a, b, and c available to them. What is the total number of allowed, distinct configurations for the following systems:

(a) Labeled (i.e. distinguishable) particles

If we consider 3 distinguishable particles $\{1, 2, 3\}$ and three possible states $\{|a\rangle, |b\rangle, |c\rangle\}$ then each particle can occupy any of the three states so that there are

$$3^3 = 27 \tag{1}$$

possible states in total. If we insist that states of the particles must be different, then there are

$$3! = 6 \tag{2}$$

possible states.

(b) identical bosons

If instead the particles are bosons, multiple particles may occupy the same state and the state vector describing the system must by *symmetric* under exchange. That is,

$$\hat{P}_{ij} |\psi_A\rangle = |\psi_A\rangle \tag{3}$$

where $\hat{P}_{ij}$ denotes the permutation operator which exchanges the states of the i^{th} and j^{th} particles. We will begin counting by first examining states for which each particle is in the same state and then will work up to the case for which each particle occupies a different state.

Hereafter, we assume the notation $|a; b; c\rangle$ for the combined state vector which indicates particle one is in state a, particle two is in state b, and so forth. With this convention, we can easily see that there are only three symmetric cases for which each particle is in the same state, namely

$$|a; a; a\rangle \qquad |b; b; b\rangle \qquad |c; c; c\rangle \tag{4}$$

Next, let us consider the possible symmetric state vectors for the case where exactly two particles occupy the same state. To account for all possible permutations, we must have

$$|a; a; b\rangle + |a; b; a\rangle + |b; a; a\rangle \tag{5}$$
$$|b; b; a\rangle + |b; a; b\rangle + |a; b; b\rangle \tag{6}$$

$$|c; c; a\rangle + |c; a; c\rangle + |a; c; c\rangle \tag{7}$$
$$|a; a; c\rangle + |a; c; a\rangle + |c; a; a\rangle \tag{8}$$

$$|b; b; c\rangle + |b; c; b\rangle + |c; b; b\rangle \tag{9}$$
$$|c; c; b\rangle + |c; b; c\rangle + |b; c; c\rangle \tag{10}$$

Where we can think of this as taking a state (for example $|a\rangle$) and multiplying by the a symmetric combination of that state with another (e.g. $|ab\rangle + |ba\rangle$).

Finally, we have the case for which every particle in a different state. The symmetric state vector in this configuration is given by

$$|a;b;c\rangle + |a;c;b\rangle + |b;c;a\rangle + |b;a;c\rangle + |c;b;a\rangle + |c;a;b\rangle \tag{11}$$

Therefore, in total we have 10 possible states.

Again, if we insist that the particles must occupy separate states, then the only option is equation (11) so that there is 1 symmetric state vector.

(c) identical fermions

For the case of three fermions, we expect that the system will behave altogether as a fermion and therefore require that the state vector describing this system is antisymmetric. That is

$$\hat{P}_{ij} |\psi_A\rangle = - |\psi_A\rangle \tag{12}$$

No state vector for which each particle occupies the same state can be antisymmetric. To see if there are any possible antisymmetric states where two particles occupy the same sate, we can construct the so called Slater determinant

$$\begin{aligned}
|\psi_{A,2}\rangle &= \begin{vmatrix} |a\rangle & |a\rangle & |b\rangle \\ |a\rangle & |a\rangle & |b\rangle \\ |a\rangle & |a\rangle & |b\rangle \end{vmatrix} && (13)\\
&= |a\rangle \Big(|a;b\rangle - |b;a\rangle \Big) - |a\rangle \Big(|a;b\rangle - |b;a\rangle \Big) + |b\rangle \Big(|a;a\rangle - |a;a\rangle \Big) && (14)\\
&= 0 && (15)
\end{aligned}$$

therefore we can clearly see that there is no way we can make an antisymmetric state where two particles occupy the same state. This result is simply a restatement of the Pauli Exclusion Principle. Thus, the only possible way to construct an antisymmetric state for three particle with three possible states is by taking the following Slater determinant.

$$\begin{aligned}
|\psi_A\rangle &= \begin{vmatrix} |a\rangle & |b\rangle & |c\rangle \\ |a\rangle & |b\rangle & |c\rangle \\ |a\rangle & |b\rangle & |c\rangle \end{vmatrix} && (16)\\
&= |a\rangle \Big(|b;c\rangle - |c;b\rangle \Big) - |b\rangle \Big(|a;c\rangle - |c;a\rangle \Big) + |c\rangle \Big(|a;b\rangle - |b;a\rangle \Big) && (17)\\
&= |a;b;c\rangle - |a;c;b\rangle - |b;a;c\rangle + |b;c;a\rangle + |c;a;b\rangle - |c;b;a\rangle && (18)
\end{aligned}$$

Thus, we conclude that there is exactly one antisymmetric state for a three particle, three state system.

2 Two non-interacting particles, with the same mass m, are in a 1D box of length $2a$.

(a) What are the values of the three lowest energies of the system?

If the two particles-in-a-box are non-interacting, the Hamiltonian is separable in terms of each individual particle so that we may write:

$$H = H_1 + H_2 \tag{19}$$

This leads to energies given by

$$E_{n_1,n_2} = \frac{\pi^2\hbar^2}{8ma^2}\left(n_1^2 + n_2^2\right) \tag{20}$$

The ground state of such a system occurs for $n_1 = n_2 = 0$ with energy

$$E_{gs} = E_{11} = \frac{1}{4}\frac{\pi^2\hbar^2}{ma^2} \tag{21}$$

The first excited state then occurs for either $n_1 = 1\ n_2 = 2$ *or* $n_1 = 2\ n_2 = 1$ with energy

$$E_{12} = E_{21} = \frac{5}{8}\frac{\pi^2\hbar^2}{ma^2} \tag{22}$$

The second excited state occurs when $n_1 = n_2 = 2$ with energy

$$E_{22} = 1\frac{\pi^2\hbar^2}{ma^2} \tag{23}$$

As a sanity check, the state $n_1 = 1, n_2 = 3$ has energy

$$E_{13} = E_{31} = \frac{5}{4}\frac{\pi^2\hbar^2}{ma^2} > E_{22} \tag{24}$$

Thus, we have identified the three lowest energy levels.

(**b**) What are the degeneracies of these energy levels if the two particles are:

(**i**) identical, with spin 1/2;

A system of two identical fermions obeys Fermi-Dirac statistics and must therefore have an antisymmetric overall state vector under particle exchange. We must construct all possible antisymmetric state vectors for each of the energy levels identified in part (a). The total state vector is the tensor product of the spatial state (relating to the n quantum number) with the spin state (relating to the m_s quantum number).

Therefore either the spacial or the spin part must be antiymmetric while the opposite is symmetric as discussed in McIntyre (pg 413 eq 13.10). To that end, we adopt the notation

$$|n_1, n_2; m_{s1}, m_{s2}\rangle \tag{25}$$

for the combined, two-particle state.

In the ground state $n_1 = n_2$ and therefore, we must have an antisymmetric spin component. There is only one way to do this

$$|1,1;+,-\rangle - |1,1;-,+\rangle \tag{26}$$

For the first excited state we have degeneracy in the energy and spin so that we have either $|n_1, n_2\rangle_S \, |m_{s1}, m_{s2}\rangle_A$ or $|n_1, n_2\rangle_A \, |m_{s1}, m_{s2}\rangle_S$. This results in four possible states

$$\big(\,|1,2\rangle - |2,1\rangle\,\big)\,|{+}{+}\rangle = |1,2;{+}{+}\rangle - |2,1;{+}{+}\rangle \tag{27}$$

$$\big(\,|1,2\rangle - |2,1\rangle\,\big)\,|{-}{-}\rangle = |1,2;{-}{-}\rangle - |2,1;{-}{-}\rangle \tag{28}$$

$$|1,2\rangle\,\big(\,|+,-\rangle - |-,+\rangle\,\big) = |1,2;+,-\rangle - |1,2;{-}{+}\rangle \tag{29}$$

$$|2,1\rangle\,\big(\,|+,-\rangle - |-,+\rangle\,\big) = |2,1;{+}{-}\rangle - |2,1;{-}{+}\rangle \tag{30}$$

For the second excited state, we again have $n_1 = n_2$ so that there is only one way to make the overall state antisymmetric:

$$|2,2;+,-\rangle - |2,2;-,+\rangle \tag{31}$$

(ii) not identical, but both have spin 1/2;

As discussed in class (see page 8 of the lecture 11 notes) a system of two non-identical fermions (like a hydrogen atom) behaves as a boson. Therefore, we require that the overall state vector must be symmetric under particle exchange and consequently, both the spatial and spin components of the state must be symmetric.

For the ground state where $n_1 = n_2$, the spacial component is necessarily symmetric. There are 3 ways to make a symmetric spin state and therefore, we have 3 total states possible

$$|1,1;+,+\rangle \tag{32}$$

$$|1,1;-,-\rangle \tag{33}$$

$$|1,1\rangle\left(\,|+,-\rangle + |-,+\rangle\,\right) = |1,1;+,-\rangle + |1,1;-,+\rangle \tag{34}$$

For the first excited state we have exactly one way to make a symmetric spatial state and (again) three ways to make a symmetric spin state. Thus, the options are

$$\left(\,|1,2\rangle + |2,1\rangle\,\right)|{+}{+}\rangle = |1,2;{+}{+}\rangle + |2,1;{+}{+}\rangle \tag{35}$$

$$\left(\,|1,2\rangle + |2,1\rangle\,\right)|{-}{-}\rangle = |1,2;{-}{-}\rangle + |2,1;{-}{-}\rangle \tag{36}$$

$$\begin{aligned}\left(\,|1,2\rangle + |2,1\rangle\,\right)\left(\,|+,-\rangle + |-,+\rangle\,\right) = {}& |1,2;+,-\rangle + |1,2;-,+\rangle \\ & + |2,1;+,-\rangle + |2,1;-,+\rangle\end{aligned} \tag{37}$$

Finally, for the second excited state, we have the same scenario as the ground state but with $n_1 = n_2 = 2$. Therefore, there are three more states

$$|2,2;+,+\rangle \tag{38}$$

$$|2,2;-,-\rangle \tag{39}$$

$$|2,2\rangle\left(\,|+,-\rangle + |-,+\rangle\,\right) = |2,2;+,-\rangle + |2,2;-,+\rangle \tag{40}$$

(iii) identical, with spin 1;

We now consider what happens if the two particles are identical bosons with spin 1. Because spin 1 particles may have z-projections of either $\hbar, 0$, or $-\hbar$, there are more ways to write a symmetric spin state vector. They are:

$$|1,1\rangle \tag{41}$$
$$|0,0\rangle \tag{42}$$
$$|-1,-1\rangle \tag{43}$$
$$|1,0\rangle + |0,1\rangle \tag{44}$$
$$|1,-1\rangle + |-1,1\rangle \tag{45}$$
$$|0,-1\rangle + |-1,0\rangle \tag{46}$$

For the ground state $n_1 = n_2 = 1$ so that the spatial component is already symmetric. Therefore, the possible overall symmetric state vectors are just

$$|1,1\rangle |1,1\rangle = |1,1;1,1\rangle \tag{47}$$
$$|1,1\rangle |0,0\rangle = |1,1;0,0\rangle \tag{48}$$
$$|1,1\rangle |-1,-1\rangle = |1,1;-1,-1\rangle \tag{49}$$
$$|1,1\rangle \Big(|1,0\rangle + |0,1\rangle \Big) = |1,1;1,0\rangle + |1,1;0,1\rangle \tag{50}$$
$$|1,1\rangle \Big(|1,-1\rangle + |-1,1\rangle \Big) = |1,1;1,-1\rangle + |1,1;-1,1\rangle \tag{51}$$
$$|1,1\rangle \Big(|0,-1\rangle + |-1,0\rangle \Big) = |1,1;0,-1\rangle + |1,1;-1,0\rangle \tag{52}$$

For the first excited state we have exactly one way to make a symmetric spatial state and (again) six ways to make a symmetric spin state. Therefore, the possible states are

$$\Big(|1,2\rangle + |2,1\rangle \Big) |1,1\rangle = |1,2;1,1\rangle + |2,1;1,1\rangle \tag{53}$$
$$\Big(|1,2\rangle + |2,1\rangle \Big) |0,0\rangle = |1,2;0,0\rangle + |2,1;0,0\rangle \tag{54}$$
$$\Big(|1,2\rangle + |2,1\rangle \Big) |-1,-1\rangle = |1,2;-1,-1\rangle + |2,1;-1,-1\rangle \tag{55}$$
$$\begin{aligned} \Big(|1,2\rangle + |2,1\rangle \Big)\Big(|1,0\rangle + |0,1\rangle \Big) &= |1,2;1,0\rangle + |2,1;1,0\rangle \\ &\quad + |1,2;0,1\rangle + |2,1;0,1\rangle \end{aligned} \tag{56}$$
$$\begin{aligned} \Big(|1,2\rangle + |2,1\rangle \Big)\Big(|1,-1\rangle + |-1,1\rangle \Big) &= |1,2;1,-1\rangle + |2,1;1,-1\rangle \\ &+ \quad |1,2;-1,1\rangle + |2,1;-1,1\rangle \end{aligned} \tag{57}$$
$$\begin{aligned} \Big(|1,2\rangle + |2,1\rangle \Big)\Big(|0,-1\rangle + |-1,0\rangle \Big) &= |1,2;0,-1\rangle + |2,1;0,-1\rangle \\ &+ \quad |1,2;-1,0\rangle + |2,1;-1,0\rangle \end{aligned} \tag{58}$$

Finally, the second excited state is the same as the ground state but with $n_1 = n_2 = 2$. This leads to

$$|2,2\rangle\,|1,1\rangle = |2,2;1,1\rangle \tag{59}$$

$$|2,2\rangle\,|0,0\rangle = |2,2;0,0\rangle \tag{60}$$

$$|2,2\rangle\,|-1,-1\rangle = |2,2;-1,-1\rangle \tag{61}$$

$$|2,2\rangle\,\Big(\,|1,0\rangle + |0,1\rangle\,\Big) = |2,2;1,0\rangle + |2,2;0,1\rangle \tag{62}$$

$$|2,2\rangle\,\Big(\,|1,-1\rangle + |-1,1\rangle\,\Big) = |2,2;1,-1\rangle + |2,2;-1,1\rangle \tag{63}$$

$$|2,2\rangle\,\Big(\,|0,-1\rangle + |-1,0\rangle\,\Big) = |2,2;0,-1\rangle + |2,2;-1,0\rangle \tag{64}$$