Complex Analysis - MTH 483 **John Waczak**
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Mobius transformations

Def: A *Mobius transformation* is a function of the form:

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in mathbbC$ and $ad - bc \neq 0$. (Sometimes called linear fractional transformation)

If $c = 0$ then $f(z) = \frac{a}{d}z + \frac{b}{d}$ which is entire since $f(z)$ is a linear polynomial. If $c \neq 0$ then $f(z)$ is holomorphic everywhere except at $-\frac{d}{c}$. Wherever this is holomorphic, we have that

$$f'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

The composition of $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{a'z+b'}{c'z+d'}$ is yet another Mobius transformation.

$$\begin{aligned} f(g(z)) &= \frac{a\left(\frac{a'z+b'}{c'z+d'}\right) + b}{c\left(\frac{a'z+b'}{c'z+d'}\right) + d} \\ &= \frac{a(a'z+b') + b(c'z+d')}{c(a'z+b') + d(c'z+d')} \\ &= \frac{(aa'+bc')z + (ab'+bd')}{(ca'+dc')z + (cb'+dd')} \end{aligned}$$

all that remains is to show that we have the nonzero requirement.

$$\begin{pmatrix} a & b \\ c & c \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb\text{''}+dd' \end{pmatrix}$$

since the determinant multiplies and each of the two matrices has nonzero matrix we have that the determinant of the product is nonzero and therefore the composition of Mobius transformations is a Mobius transformation and furthermore we can encode Mobius transformations as M_{2x2}. Algebraically, the set of all Mobius transformations is a group with $(\circ)$ function composition and identity $f(z) = z$. The inverse of $f(z)$ is the corresponding *inverse matrix...*

$$\begin{aligned} \text{if} \quad f(z) &= \frac{az+b}{cz+d} \\ \text{then} \quad f^{-1}(z) &= \frac{dz-b}{-cz+a} \\ \Rightarrow f(f^{-1}(z)) &= \frac{a\left(\frac{dz-b}{-cz+a}\right) + b}{c\left(\frac{dz-b}{-cz+a}\right) + d} \\ &= \frac{a(dz-b) + b(-zc+a)}{(dz-b) + d(-cz+a)} \\ &= \frac{(ad-bc)z - 0}{0z + ad - bc} = z \quad \checkmark \end{aligned}$$

but why don't we need to "divide by determinant"? Multiply a Mobius transformation by constant r would have no effect since it would scale numerator and denominator the same. So really dividing by determinant doesn't change anything.

$$r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

This implies that

$$rf(z) = \frac{raz + rb}{rcz + rd} = \frac{az + b}{cz + d} = f(z)$$

Special types

- **Translations** $f(z) = z + b \quad (b \in \mathbb{C})$
- **Dilations** $f(z) = az \quad (a \in \mathbb{C}, a \neq 0)$
  For dilation $f(z) = az$, in polar form we have $a = re^{i\phi}$ so that f "stretches" by r and "rotates" by angle of ϕ
- **Inversion** $f(z) = \frac{1}{z}$

Proposition All Mobius transformations can be expressed as a composition of translations, dilations, and inversions.

pf. If $c = 0$ then $f(z) = \frac{a}{d}z + \frac{b}{d}$ which is dilation composed with a translation. If $c \neq 0$ then $f(z) = \left(\frac{bc-ad}{c^2}\right)\left(\frac{1}{z+\frac{d}{c}}\right) + \frac{a}{c}$ Which we can see is a combination of all three. $\square$

Theorem If $S \subseteq \mathbb{C}$ is either a circle or a line and $f(z)$ is a Mobius transformation, then $f(S)$ is either a circle or a line.

Example: $f(z) = \frac{z-1}{iz+i}$ takes circle $x^2 + y^2 = 1$ to the real line $y = 0$.

pf. Let $|z| = 1$. Then $z = e^{i\phi}$ Then $f(z) = \frac{e^{i\phi}-1}{ie^{i\phi}+i}$. This is just $= \frac{(e^{i\phi}-1)(e^{i\phi}+1)}{i(e^{i\phi}+1)(e^{i\phi}+1)}$. This means that $= \frac{(e^{i\phi}-1)(\overline{e^{i\phi}}+1)}{i|e^{i\phi}+1|^2}$ which further simplifies to $\frac{(e^{i\phi}\overline{e^{i\phi}}+e^{i\phi}-\overline{e^{i\phi}}-1)}{i|e^{i\phi}+1|^2} = \frac{i2\sin\phi}{i|e^{i\phi}+1|^2} = \frac{2\sin\phi}{|e^{\phi}+1|^2} \in \mathbb{R}$ $\square$