Homework 3 **John Waczak**
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Dr. Schmidt Worked with: Garrett Jepson

1.) Let $S, T : V \to V$ be linear operators.

a) Suppose that V is finite dimensional. If $S \circ T$ is invertible, prove that both S and T are invertible.

Proof. Let $S, T : V \to V$ such that the above assumptions hold. Then because $S \circ T$ is invertible, it is a bijection and is therefore onto. Thus, we have that

$$\dim(V) = \dim(S \circ T(V))$$

Now by composition of functions, we apply T and then S. Therefore $T : V \to T(V) \subseteq V$ and then $S : T(V) \to V$ in other words, S must map from the image of T back to V. Then by the rank-nullity theorem, we have that

$$\begin{aligned}
\dim(V) &= \dim(T(V)) + \dim(\ker(T)) \\
\dim(T(V)) &= \dim(S(T(V)))) + \dim(\ker(S)) \\
\Rightarrow \dim(S \circ T(V)) &= \dim(T(V)) + \dim(\ker(T)) \\
&= \dim(S(T(V))) + \dim(\ker(S)) + \dim(\ker(T)) \\
\Rightarrow 0 &= \dim(\ker(S)) + \dim(\ker(T))
\end{aligned}$$

Where in the last line we observe that $\dim(S \circ T(V)) = \dim(S(T(V)))$. Since the $\dim(\ker(A)) \geq 0$ $\forall$ linear transformations A, we have that $\Rightarrow \dim\ker(S)) = \dim\ker(T) = 0$. Therefore, it follows that

$$\begin{aligned}
\dim(V) &= \dim(T(V)) = \text{rank}(T) \\
\dim(V) &= \dim(S(T(V))) = \text{rank}(S)
\end{aligned}$$

Therefore, by theorem 2.5 (page 71) both S and T are one to one and onto. This is equivalent to being a bijection and thus, we conclude that they are both invertible. □

4.) Let n and m be positive integers and $\mathbb{F}$ a field. Let $l_1, ..., l_m$ be linear functionals on $\mathbb{F}^n$.
a) Show that the mapping

$$\begin{aligned}
T :& \mathbb{F}^n \to F^m \\
& v \mapsto (l_1(v), ..., l_m(v))
\end{aligned}$$

is a linear transformation.

Proof. Let $\lambda \in \mathbb{F}$ and $v_1, v_2 \in \mathbb{F}^n$. Then we have that

$$T(\lambda v_1 + v_2) = (l_1(\lambda v_1 + v_2), ..., l_m(\lambda v_1 + v_2))$$

Because linear functionals are linear transformations, we have that

$$\begin{aligned} &= (\lambda l_1(v_1) + l_1(v_2), ..., \lambda l_m(v_1) + l_m(v_2)) \\ &= \lambda T(v_1) + T(v_2) \quad \text{by vector addition in } \mathbb{F}^m \end{aligned}$$

Thus we have shown that T is a linear transformation.

□

b) Show that every linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ is of the above form.

Proof. (Contrapositive) We will show that if a transformation cannot be expressed in the above form, then it can not be a linear transformation.

If T can not be expressed in terms of m linear functionals, then it must be true that for some i, l_i is *not* a linear functional. Thus we have that $\forall \lambda \in \mathbb{F}, v_1, v_2 \in \mathbb{F}^n$,

$$\begin{aligned} T(\lambda v_1 + v_2) &= (l_1(\lambda v_1 + v_2), ...l_i(\lambda v_1 + v_2), ..., l_m(\lambda v_1 + v_2)) \\ &= (\lambda l_1(v_1) + l_1(v_2), ..., l_i(\lambda v_1 + v_2), ..., \lambda l_m(v_1) + l_m(v_2)) \\ &\neq \lambda T(v_1) + T(v_2) \end{aligned}$$

Therefore we conclude that if we cannot represent as in the hypothesis, then T is not a linear transformation. It follows from this contrapositive that every linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ can be represented as taking $v \in F^n$ to the vector in F^m whose components are given by m linear functionals acting on v. □