Central Forces Homework 1

Due 5/9/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

PRACTICE:

1. In each of the following sums, shift the index $n \to n+2$. Don't forget to shift the limits of the sum as well. Then write out all of the terms in the sum (if the sum has a finite number of terms) or the first five terms in the sum (if the sum has an infinite number of terms) and convince yourself that the two different expressions for each sum are the same:

   (a)
   $$\sum_{n=0}^{3} n$$

   (b)
   $$\sum_{n=1}^{5} e^{in\phi}$$

   (c)
   $$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2}$$

REQUIRED:

1. Consider the differential equation $y'' - 2y' + y = 0$.

   (a) Use the power series method to find the first six terms in each of two independent solutions to this differential equation.

   Solution:

   Our answer at the end should have the form $y(x) \approx A(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5) + B(d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 x^5)$. There are two solutions because it is a second-order differential equation. Our task is to find the c and d coefficients.

   We start by assuming
   $$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$
   giving
   $$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Rearranging the terms in the sums gives us:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty}((n+2)(n+1)c_{n+2} - 2(n+1)c_{n+1} + c_n)(x^n) = 0$$

The parentheses must be equal to 0 for all n, so:
$c_{n+2} = \frac{2(n+1)c_{n+1} - c_n}{(n+1)(n+2)}$
We have the freedom to choose both c_0 and c_1. A first guess might be to try two solutions where these coefficients are each zero (with the other nonzero). This gives:

$$\begin{aligned}
x^0 &: c_2 = -c_0/2 \\
x^1 &: c_3 = 2c_2/3 = -c_0/3 \\
x^2 &: c_4 = c_3/2 - c_2/12 = -c_0/8 \\
x^3 &: c_5 = 2c_4/5 - c_3/20 = -c_0/30 \\
x^4 &: c_6 = c_5/3 - c_4/30 = -c_0/144
\end{aligned}$$

$y_0(x) \approx c_0(1 - x^2/2 - x^3/3 - x^4/8 - x^5/30)$ does not immediately look like a familiar power series. If you want a cleverer choice (see next part), choose $c_0 = c_1$. The other solution we try $c_0 = 0$:

$$x^0 : d_2 = d_1$$
$$x^1 : d_3 = d_1/2$$
$$x^2 : d_4 = d_3/2 - d_2/12 = d_1/6$$
$$x^3 : d_5 = 2d_4/5 - d_3/20 = d_1/24$$
$$x^4 : d_6 = d_5/3 - d_4/30 = d_1/120$$

This looks very familiar! $y_1(x) \approx d_1(x + x^2 + x^3/2 + x^4/6 + x^5/24) = d_1 x e^x$. The final answer puts these two solutions together as: $y(x) \approx c_0(1 - x^2/2 - x^3/3 - x^4/8 - x^5/30) + d_1(x + x^2 + x^3/2 + x^4/6 + x^5/24)$

(b) Challenge Problem - Solve this differential equation using a different method and show that your answers are the same as part a.

Solution:

This equation can also be solved using the method of constant coefficients. The characteristic polynomial is: $r^2 - 2r + 1 = (r-1)^2$. Since this equation has one repeated solution, an additional independent solution must be added in the form of $y(x) = Ae^x + Bxe^x$. The second term matches one of our solutions from before. The other term can be found from the series solution either by an appropriate superposition of the two solutions we found, or by going back to the series coefficients and choosing the initial condition of $c_1 = c_0$.

2. Consider the differential equation $y'' = \frac{2}{(1-x)^2}y$.

(a) Use a power series expanded about $x = 0$ to find the first six terms in each of two independent solutions to this differential equation.

Solution:

Our answer at the end should again have the form $y(x) \approx A(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5) + B(d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4 + d_5x^5)$. There are two solutions because it is a second-order differential equation. Our task is to find the c and d coefficients.

We start again by assuming

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

giving

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Rearranging the terms in the sums gives us:

$$(x^2 - 2x + 1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n(x^n - 2x^{n-1} + x^{n-2}) - \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n(n-1)c_n - 2(n+1)nc_{n+1} + (n+2)(n+1)c_{n+2})(x^n) - \sum_{n=0}^{\infty} 2c_n x^n = 0$$

We now write out the first several terms:

$$
\begin{aligned}
x^0 &: 2c_2 - 2c_0 = 0 c_2 = c_0 \\
x^1 &: -4c_2 + 6c_3 - 2c_1 = 0 c_3 = (c_1 + 2c_0)/3 \\
x^2 &: 2c_2 - 12c_3 + 12c_4 - 2c_2 = 0 c_4 = c_3 \\
x^3 &: 6c_3 - 24c_4 + 20c_5 - 2c_3 = 0 c_5 = c_4 \\
x^4 &: 12c_4 - 40c_5 + 30c_6 - 2c_4 = 0 c_6 = c_5
\end{aligned}
$$

We need two linearly independent solutions because the differential equation is second order. However, this power series looks like it may not converge. One way to force it to converge is to choose $c_3 = 0$, which results in all higher terms also being 0. This is known as truncating the series.
Solution 1: $c_3 = 0$ so $c_1 = -2c_0$, giving $y_1(x) = c_0 x^0 - 2c_0 x^1 + c_0 x^2 = c_0(x-1)^2$. The other solution can be found by observing that this power series has equal coefficients for all $n > 2$. By also choosing $d_1 = d_2 = d_3$, we find the power series

$$y_2(x) = \sum_{n=0}^{\infty} d_1 x^n$$

. This is the geometric series, which means $y_2(x) = \frac{C}{1-x} \approx 1 + x + x^2 + x^3 + x^4 + x^5$. The general solution is then a superposition of these two solutions: $y(x) = A(x-1)^2 + \frac{B}{1-x} \approx A(1 - 2x + x^2) + B(1 + x + x^2 + x^3 + x^4 + x^5)$.

(b) For what values of x does each of your power series solutions converge?

Solution:

The answer here should be a range of acceptable values for x. Note that while $y_1(x)$ converges for all x, $y_2(x)$ only converges for $|x| < 1$. The power series method centered about $x = 0$ will not produce a second solution to this differential equation outside of this range, due to the limits of convergence on power series.

(c) Suppose you were to subtract one of your two solutions from the other solution. Is the resulting function still a solution to the original differential equation? Explain.

Solution:

Any linear combination of the two solutions is also a solution, so subtracting them is automatically a solution. If you did not previously find the solution $y_2(x) = \frac{C}{1-x}$, you can arrived at it by finding the appropriate superposition of your solutions such that all the coefficients in the power series (namely the first three) have the same value.