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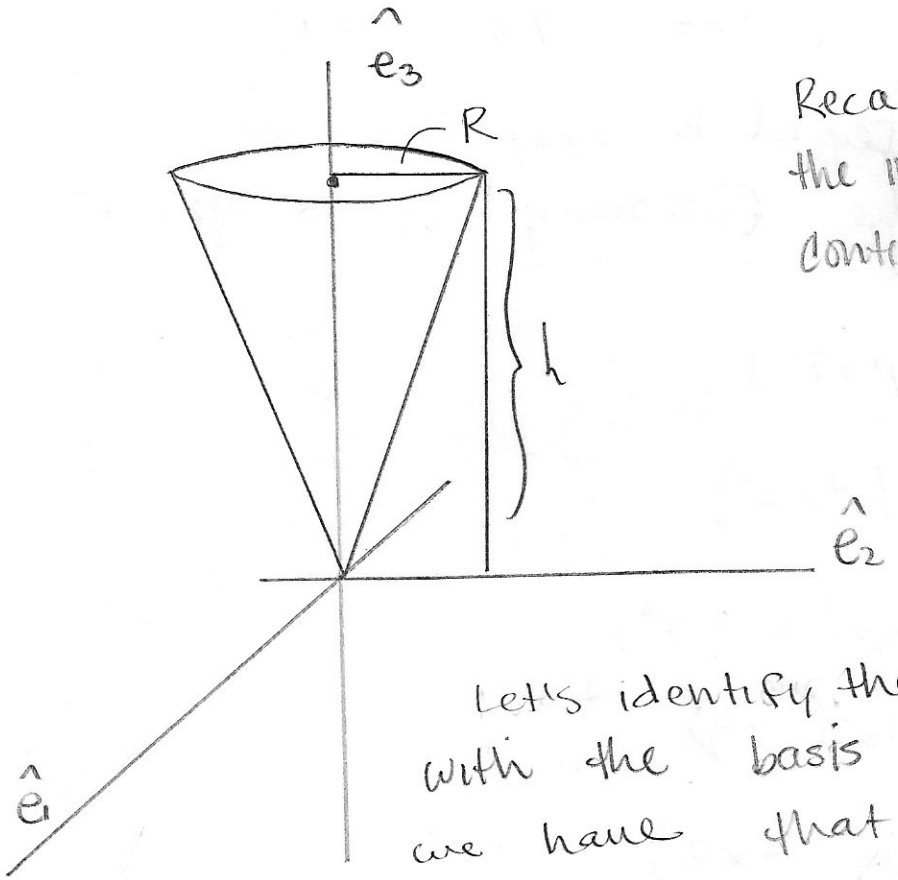
①

(1) Homogenous circular cone of height h and base radius R. Calculate inertial tensor in the following basis.



Recall the equation for the inertial tensor in continuum

$$I_{ij} = \int d^3\vec{x}\, \rho(\vec{x}) (\delta_{ij} x^2 - x_i x_j)$$

where $x^2 = x_1^2 + x_2^2 + x_3^2$

Let's identify the coordinates x_1, x_2, x_3 with the basis vectors $\{\hat{e}_i\}$ then we have that the matrix formed

by $(\delta_{ij} x^2 - x_i x_j) = \begin{pmatrix} x_1^2+x_2^2+x_3^2-x_1^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2+x_2^2+x_3^2-x_2^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2+x_2^2+x_3^2-x_3^2 \end{pmatrix}$

$$= \begin{pmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2+x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2+x_2^2 \end{pmatrix}$$

to make things nicer lets let

$x = x_1 \quad y = x_2 \quad z = x_3$ then

the I.T. is

$$I = \rho \int_V dxdydz \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

and so because the integral is symmetric we need to perform the following 6 integrals

1) $I_{11} = \rho \int_V dxdy\,dz\,(y^2+z^2)$

2) $I_{22} = \rho \int_V dxdydz\,(x^2+z^2)$

3) $I_{33} = \rho \int_V dxdydz(x^2+y^2)$

4) $I_{12} = \rho \int_V dxdy\,dz\,(-xy)$

5) $I_{13} = \rho \int_V dxdy\,dz(-xz)$

6) $I_{23} = \rho \int_V dx\,dy\,dz\,(-yz)$

we also have that since the cone is homogenous that

$$\rho = \frac{M}{V_{cone}} = \boxed{\frac{M}{\frac{1}{3}\pi R^2 h} = \rho}$$

Now all we need to do is get the limits of integration and we can have mathematica crank it out.

our equation for the cone is given by

$$R^2 z^2 = h^2 x^2 + h^2 y^2$$

i.e. when $z=0$ we get $r=0$ and when $z=h$ we get $r=R$. We can solve this for z to find

$$z = \frac{h}{R}\sqrt{x^2+y^2},$$

Since we have a maximum height of h it must be true that

$$\frac{h}{R}\sqrt{x^2+y^2} \leq z \leq h$$

Now looking down from above the cone, the x-y projection is just a circle of radius R at height $z=h$ so that we have

$$x^2+y^2 \leq R^2$$

if we solve this for y we have that $-\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2}$ and finally by inspection $-R \leq x \leq R$. Thus our integration order is $dz\,dy\,dx$ giving the general integral

$$I_{ij} = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{\frac{h}{R}\sqrt{x^2+y^2}}^{h} \rho(\delta_{ij} r^2 - x_i x_j)\, dz\,dy\,dx$$

See attached mathematica code for the evaluation of the integrals

Thus we have that the off diagonal elements are zero and the full tensor is

$$I = \begin{pmatrix} \frac{3}{20}M(4h^2+R^2) & 0 & 0 \\ 0 & \frac{3}{20}M(4h^2+R^2) & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

this makes sense as we know rotations around x and y axis should have same I by symmetry. This verifies that $\{\hat{e}_i\}$ given in the problem form the principal axes for the cone as well as an orthonormal basis

In[40]:= `$Assumptions = Element[R, Reals] && Element[M, Reals] && Element[h, Reals] && R > 0`

$$\rho = \frac{M}{\frac{1}{3}\pi * R^2 * h}$$

Out[40]= R ∈ Reals && M ∈ Reals && h ∈ Reals && R > 0

Out[41]= $$\frac{3\,M}{h\,\pi\,R^2}$$

In[45]:= $$I_{11} = \rho * \int_{-R}^{R}\int_{-\sqrt{R\verb|^|2-x\verb|^|2}}^{\sqrt{R\verb|^|2-x\verb|^|2}}\int_{\frac{h*\sqrt{x\verb|^|2+y\verb|^|2}}{R}}^{h}\left(y\verb|^|2+z\verb|^|2\right)\,dz\,dy\,dx$$

Out[45]= $$\frac{3}{20}\,M\left(4\,h^2+R^2\right)$$

In[46]:= $$I_{22} = \rho * \int_{-R}^{R}\int_{-\sqrt{R\verb|^|2-x\verb|^|2}}^{\sqrt{R\verb|^|2-x\verb|^|2}}\int_{\frac{h*\sqrt{x\verb|^|2+y\verb|^|2}}{R}}^{h}\left(x\verb|^|2+z\verb|^|2\right)\,dz\,dy\,dx$$

Out[46]= $$\frac{3}{20}\,M\left(4\,h^2+R^2\right)$$

In[47]:= $$I_{33} = \rho * \int_{-R}^{R}\int_{-\sqrt{R\verb|^|2-x\verb|^|2}}^{\sqrt{R\verb|^|2-x\verb|^|2}}\int_{\frac{h*\sqrt{x\verb|^|2+y\verb|^|2}}{R}}^{h}\left(x\verb|^|2+y\verb|^|2\right)\,dz\,dy\,dx$$

Out[47]= $$\frac{3\,M\,R^2}{10}$$

In[48]:= $$I_{12} = \rho * \int_{-R}^{R}\int_{-\sqrt{R\verb|^|2-x\verb|^|2}}^{\sqrt{R\verb|^|2-x\verb|^|2}}\int_{\frac{h*\sqrt{x\verb|^|2+y\verb|^|2}}{R}}^{h}(-x*y)\,dz\,dy\,dx$$

Out[48]= 0

In[49]:= $$I_{13} = \rho * \int_{-R}^{R}\int_{-\sqrt{R\verb|^|2-x\verb|^|2}}^{\sqrt{R\verb|^|2-x\verb|^|2}}\int_{\frac{h*\sqrt{x\verb|^|2+y\verb|^|2}}{R}}^{h}(-x*z)\,dz\,dy\,dx$$

Out[49]= 0

In[51]:= $$I_{23} = \rho * \int_{-R}^{R}\int_{-\sqrt{R\verb|^|2-x\verb|^|2}}^{\sqrt{R\verb|^|2-x\verb|^|2}}\int_{\frac{h*\sqrt{x\verb|^|2+y\verb|^|2}}{R}}^{h}(-y*z)\,dz\,dy\,dx$$

Out[51]= 0

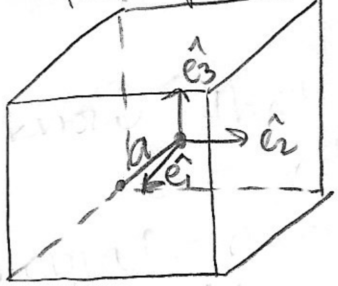
therefore our moment of inertia for the general rectangular prism is just

$$I = \begin{pmatrix} \frac{1}{3}M(l_1^2+l_3^2) & -\frac{1}{4}Ml_1l_2 & -\frac{1}{4}Ml_1l_3 \\ -\frac{1}{4}Ml_1l_2 & \frac{1}{3}M(l_1^2+l_3^2) & -\frac{1}{4}Ml_2l_3 \\ -\frac{1}{4}Ml_1l_3 & -\frac{1}{4}Ml_2l_3 & \frac{1}{3}M(l_1^2+l_2^2) \end{pmatrix}$$

For the cube where $l_1 = l_2 = l_3 = b$ this becomes

$$I = \begin{pmatrix} \frac{2}{3}mb^2 & -\frac{1}{4}mb^2 & -\frac{1}{4}mb^2 \\ -\frac{1}{4}mb^2 & \frac{2}{3}mb^2 & -\frac{1}{4}mb^2 \\ -\frac{1}{4}mb^2 & -\frac{1}{4}mb^2 & \frac{2}{3}mb^2 \end{pmatrix}$$

Now for (2) we want to compute I from the CM coordinates a distance $\vec{a}$ to the previous origin



This actually makes the integrals a bit nicer as our new bounds are simply:

$$-\frac{l_1}{2} \le x \le \frac{l_1}{2}$$

$$-\frac{l_2}{2} \le y \le \frac{l_2}{2}$$

$$-\frac{l_3}{2} \le z \le \frac{l_3}{2}$$

I will again only compute I_{11} and I_{12}) arguing the rest by symmetry

$$I_{11} = \rho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \int_{-\frac{l_3}{2}}^{\frac{l_3}{2}} (y^2 + z^2)\, dz\, dy\, dx$$

$$= \rho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} y^2 l_3 + \frac{1}{3} z^3 \Big|_{-\frac{l_3}{2}}^{\frac{l_3}{2}} dy\, dx$$

$$= \rho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} y^3 l_3 + \frac{1}{12} l_3^3 \, dy\, dx$$

$$\vdots$$

$$= \rho \left[\frac{1}{12} l_1 l_2^3 l_3 + \frac{1}{12} l_1 l_2 l_3^3 \right].$$

$$I_{11} = \frac{1}{12} M (l_2^2 + l_3^2)$$

and thus by symmetry

$$I_{22} = \frac{1}{12} M (l_1^2 + l_3^2)$$

$$I_{33} = \frac{1}{12} M (l_1^2 + l_2^2)$$

Now we will find I_{12}

$$I_{12} = \rho \int_{-l_3/2}^{l_3/2} \int_{-l_2/2}^{l_2/2} \int_{-l_1/2}^{l_1/2} (-xy)\, dx\, dy\, dz$$

$$= \rho \int_{-l_3/2}^{l_3/2} \int_{-l_2/2}^{l_2/2} -\frac{1}{2} x^2 y \Big|_{-l_1/2}^{l_1/2} dy\, dz = 0$$

therefore by symmetry $I_{13}, I_{23} = 0$ which makes sense as our axes fall along principal axes.

Therefore for the prism we have

$$I = \begin{pmatrix} \frac{1}{12}(l_2^2 + l_3^2) & 0 & 0 \\ 0 & \frac{1}{12}(l_1^2 + l_3^2) & 0 \\ 0 & 0 & \frac{1}{12}(l_1^2 + l_2^2) \end{pmatrix}$$

for the cube this simplifies to

$$I = \begin{pmatrix} \frac{1}{6} b^2 & 0 & 0 \\ 0 & \frac{1}{6} b^2 & 0 \\ 0 & 0 & \frac{1}{6} b^2 \end{pmatrix}$$

This makes sense as by the parallel axis theorem we can relate this to our first inertial tensor "$\tilde{I}$" by

$$\tilde{I}_{ij} = I_{ij} + M(a^2 \delta_{ij} - a_i a_j)$$

for $i \neq j$ this difference is $-\frac{M}{4} b^2$ and when $i = j$ we have $+M\left(\frac{3}{4} b^2 - \frac{1}{4} b^2\right) = \frac{Mb^2}{2}$ which gets us back the $\tilde{I}$ from part (1)!