Abstract Surfaces **John Waczak**
MTH 435 Date: May 25, 2018

Recap

Goal: Free ourselves from $\mathbb{R}^3$. So far all of our surfaces were sitting in $\mathbb{R}^3$ i.e. $S \subseteq \mathbb{R}^3$. We also defined the tangent space for all points $p \in S$ as T_pS. Then we developed the differential geometry around a point p as a study of T_pS.

We have seen that all notions of *intrinsic geometry* such as K the Gaussian curvature, geodesics, intrinsic distance, completeness, etc... were all defined in terms of the first fundamental form $\mathbb{I}$. The first fundamental form was really just a *choice* of an inner product on each T_pS.

Abstractions

We want to define S surfaces abstractly without any reference to $\mathbb{R}^3$ such that differentiable sets on S make sense and we can extend the intrinsic geometry to such sets.

This was very, very difficult to develop

Definition. *An Abstract Surface (a.k.a differentiable, smooth 2−manifold) is a set S together with a family of injective maps (continuous) $x_\alpha : U_\alpha \to S$ of open sets $U_\alpha \subseteq \mathbb{R}^2$ (i.e.* ***Surface Charts****) into S such that*

1. $\cup_\alpha x_\alpha(U_\alpha) = S$

2. *For each α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$. We have that $x_\alpha^{-1}(W)$, $x_\beta^{-1}(W)$ are open subsets of $\mathbb{R}^2$ and $x_\beta^{-1} \circ x_\alpha$, $x_\alpha^{-1} \circ x_\beta$ are differentiable. $W = x_\alpha(u_\alpha) \cap x_\beta(U_\beta)$.*

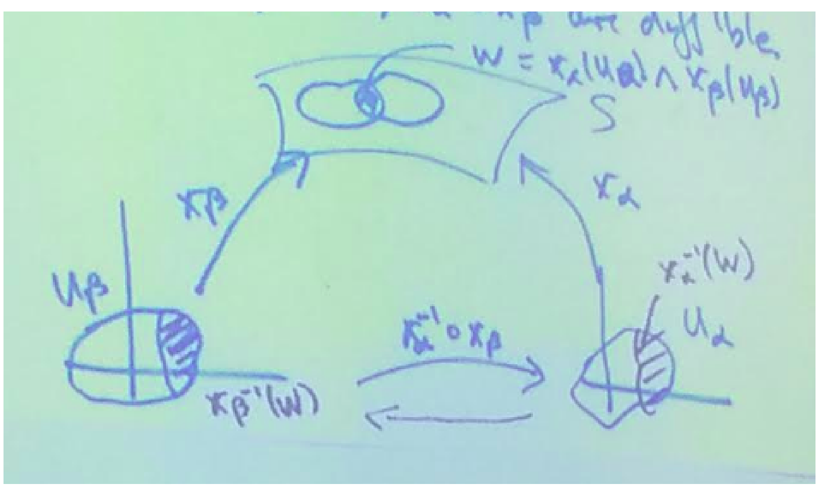


Figure 1: Pictorial representation of (2)

(u_α, x_α) with $p \in x_\alpha(U_\alpha)$: parameterization or a coordinate chart of S around p. $q = x_\alpha(u_\alpha, v_\alpha)$ is how you map the point $(u_\alpha, v_\alpha) \in \mathbb{R}^2$ to the surface S. I.e. those arguments are the coordinates.

Definition. *A differentiable structure for S is a family $\{(U_\alpha, x_\alpha)\}$*

From (2) we have that a change of parameters $x_\beta^{-1} \circ x_\alpha : x_\alpha^{-1}(W) \to x_\beta^{-1}(W)$ is a diffeomorphism.

NOTE Sometimes it is convenient to have further conditions (differs from book to book).

1. Differentiable structure $\{(U_\alpha, x_\alpha)\}$ should be maximal relative to conditions (1) and (2) above. ie.e any of the family satisfy (1) and (2) is already contained in $\{(U_\alpha, x_\alpha)\}$
2. We may want Haussdorff, second countable $\to$ topology.

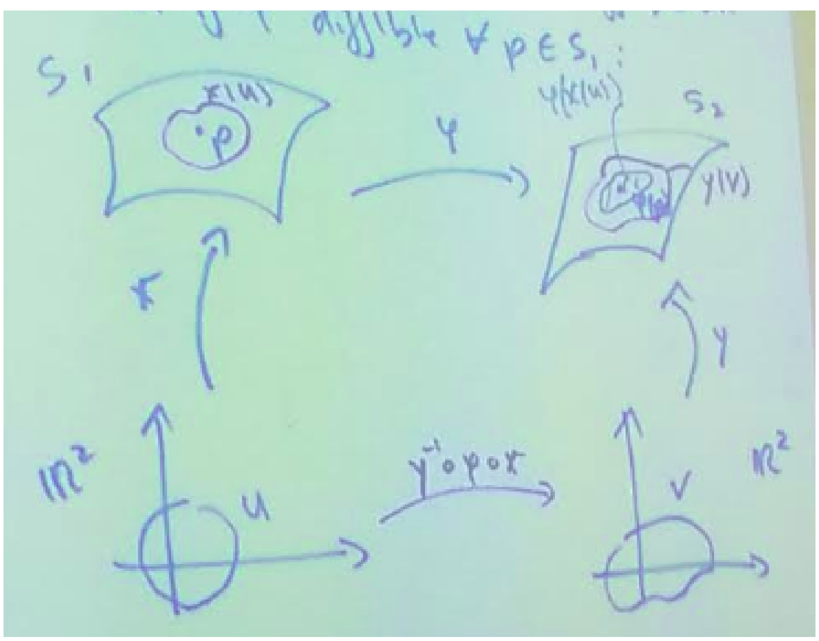
These are conditions we use to define higher dimensional manifolds.

When we compare this to the old definition, the big difference is that (2) condition.

What is a differentiable map?

Remember: we do not have $\mathbb{R}^3$ at our fingertips so we need to be precise.

Definition. *If we have S_1, S_2 abstract surfaces and $\phi : S_1 \to S_2$ is Differentiable at $p \in S_1$ if given a parametrization $y : V \subset \mathbb{R}^2 \to S_2$ around $\phi(p)$ $\exists$ a parametrization $x : U \subset \mathbb{R}^2 \to S_1$ around p such that $\phi(x(U)) \subset y(V)$ we want the image it land in y. And the map $y^{-1} \circ \phi \circ x : U \to \mathbb{R}^2$ is differentiable at $x^{-1}(p)$. We say ϕ is differentiable on S_1 if ϕ is differentiable for all $p \in S_1$.*



NOTE: *By condition (2) this does not depend on choice of parametrization. $y^{-1} \circ \phi \circ x$ is the expression of ϕ in the parametrizations x and y.*

Example: Real Projective Plane

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and define

$$\begin{aligned} A : S^2 &\to S^2 \quad \text{antipodal map} \\ (x, y, z) &\to (-x, -y, -z) \end{aligned}$$

Define $\mathbb{R}P^2 = \frac{S^2}{N}$ where $p \cong q$ for $p, q \in S^2$ iff $q = A(p)$. This means we get a map (the natural projection)

$$\begin{aligned} \pi : S^2 &\to \mathbb{R}P^2 \\ p &\mapsto [p] = \{p, A(p)\} \end{aligned}$$

Is this an abstract surface?

Cover S^2 with coordinate charts $x_\alpha : U_\alpha \to S^2$ such that $x_\alpha(U_\alpha) \cap A \circ x_\alpha(U_\alpha) = \emptyset$

Now: S^2 is a regular surface, A is a diffeomorphism $\Rightarrow \mathbb{R}P^2$ with $\{U_\alpha, \pi \circ x_\alpha\}$ is an abstract surface. We need to check that $\pi \circ x_\alpha$ is injective.

Injectivity:

$$\begin{aligned} &\pi(x_\alpha(x)) = \pi(x_\alpha(y)) \\ &\Rightarrow [x_\alpha(x)] = [x_\alpha(y)] \\ &\Rightarrow x_\alpha(x) = x_\alpha(y) \text{ or } A(x_\alpha(y)) \end{aligned}$$

but $A(x_\alpha(y))$ is not in $x_\alpha(U_\alpha) \to x_\alpha(x) = x_\alpha(y)$ which implies $x = y$ since x_α was injective already.