

1 SPHERICAL COORDINATES II Consider the sphere of radius r , in spherical coordinates (θ, ϕ) , with line element

$$ds^2 = r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

(a) Find the connection 1-forms ω_{ij} in this basis.

Because we are confined to the surface of the sphere, $d\vec{r}$ vector will not aid in our calculation. However, The metric compatibility requirement

$$\omega_{ij} + \omega_{ji} = 0 \quad (2)$$

Implies that every ω_{ii} is identically zero. Furthermore, because r is constant on the sphere, there are only 4 connection 1-forms to calculate. The above asymmetry argument implies we need only explicitly calculate 1 of them. To do this we will consider the torsion free requirement which states

$$d\sigma^i + \omega^i_j \wedge \sigma^j = 0 \quad (3)$$

This leads to the following system of linear equations

$$\omega^\theta_\phi \wedge r \sin \theta d\phi = 0 \quad (4)$$

$$r \cos \theta d\theta \wedge d\phi + r d\theta \wedge \omega^\theta_\phi = 0 \quad (5)$$

where in equation (4) we note that $dr = 0$ for the sphere. Recalling that ω^i_j are 1-forms and are expanded in the usual basis as

$$\omega^i_j = \Gamma^i_{jk} \sigma^k \quad (6)$$

equation (4) implies that $\Gamma^\theta_{\phi\theta} = 0$ so that ω^θ_ϕ has only a $d\phi$ component. With this, equation (5) becomes

$$\begin{aligned} 0 &= r \cos \theta d\theta \wedge d\phi + r d\theta \wedge \Gamma^\theta_{\phi\phi} d\phi \\ &= r \cos \theta d\theta \wedge d\phi + r \Gamma^\theta_{\phi\phi} d\theta \wedge d\phi \\ \Rightarrow \quad &\boxed{\Gamma^\theta_{\phi\phi} = -\cos \theta} \end{aligned} \quad (7)$$

Where I have done the calculation in a coordinate basis for simplicity. Thus, we can conclude that the connection 1-forms for \mathbb{S}^2 (in the orthonormal basis) are

$\omega_{\theta\theta} = 0$	$\omega_{\theta\phi} = -\cos \theta d\phi = -\frac{\cot \theta}{r} r \sin \theta d\phi \quad (8)$
$\omega_{\phi\theta} = \cos \theta d\phi = \frac{\cot \theta}{r} r \sin \theta d\phi$	$\omega_{\phi\phi} = 0 \quad (9)$

- (b) Compute $\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ for $i, j = 1, 2$ (and where there is an implicit sum over k)

As in the last homework set, equation (2) combined with the fact that $d\alpha = 0$ for any $\alpha \in \bigwedge^1$ implies $\Omega_{ii} = 0$ for all i . We also have that

$$\Omega_{ji} = -d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\Omega_{ij} \quad (10)$$

Therefore, we need only explicitly calculate one of the curvature 2-forms. For ease of calculation, I will begin in the coordinate basis and then convert back into the orthonormal basis to make the Gauss curvature obvious.

$$\Omega_{\theta\phi} = d\omega_{\theta\phi} + \omega_{\theta k} \wedge \omega_{k\phi} \quad (11)$$

$$= d\omega_{\theta\phi} + \omega_{\theta\theta} \wedge \omega_{\theta\phi} + \omega_{\theta\phi} \wedge \omega_{\phi\phi} \quad (12)$$

$$= d\omega_{\theta\phi} \quad (13)$$

$$= d(-\cos\theta d\phi) \quad (14)$$

$$= \sin\theta d\theta \wedge d\phi \quad (15)$$

$$= \frac{\sin\theta}{r^2 \sin\theta} r d\theta \wedge r \sin\theta d\phi \quad (16)$$

$$= \frac{1}{r^2} r d\theta \wedge r \sin\theta d\phi \quad (17)$$

$$= \frac{1}{r^2} \omega \quad (18)$$

where ω is the orientation of \mathbb{S}^2 .

In summary, we have found the following

$$\boxed{\begin{array}{ll} \Omega_{\theta\theta} = 0 & \Omega_{\theta\phi} = \frac{1}{r^2} \omega \\ \Omega_{\phi\theta} = -\frac{1}{r^2} \omega & \Omega_{\phi\phi} = 0 \end{array}} \quad (19)$$

$$\boxed{\begin{array}{ll} \Omega_{\theta\theta} = 0 & \Omega_{\theta\phi} = \frac{1}{r^2} \omega \\ \Omega_{\phi\theta} = -\frac{1}{r^2} \omega & \Omega_{\phi\phi} = 0 \end{array}} \quad (20)$$

From class, we know that the Gaussian curvature of a two dimensional surface is related to these curvature 2-forms by

$$\Omega^1_2 = K\omega \quad (21)$$

and $\frac{1}{r^2}$ is precisely the Gaussian curvature of \mathbb{S}^2

- (c) (Optional) Compare your answers (and your computations) with those from the previous homework assignment.

Looking at the previous homework, we see that the connection 1-forms are exactly the same for our computation in \mathbb{E}^3 despite the fact that the computation on \mathbb{S}^2 does not include any dr components. It is interesting to note that the dr components of the 1-forms in the first structure equation from $d\sigma^i$ exactly cancel out the dr components of the 1-forms from $\omega^i_j \wedge \sigma^j$ leaving the ω_{ij} unchanged as we go from \mathbb{E}^3 to \mathbb{S}^2 .

The exact opposite happens for the curvature 2-forms for which the dr components of the 2-forms in the second structure equation exactly cancel in 3-dimensions to give zero for each Ω_{ij} . On the sphere, $dr = 0$ which removes these canceling terms results in non-zero curvature 2-forms.

This whole process illustrates how we can derive the curvature for surfaces in \mathbb{E}^3 by considering curvilinear coordinate systems and then setting a particular basis 1-form to 0.