Spacetime Diagrams

vertical line ⟷ standing still
horizontal line ⟷ snapshot of an instant while standing still

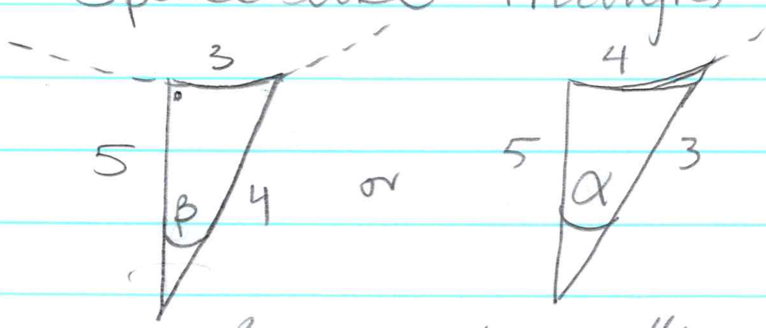
lines w/ slope ± 1 ⟷ light
lines w/ $|slope| > 1$ ⟷ moving observers (inertial
lines w/ $|slope| < 1$ ⟷ moving (inertial) snapshot

Hyperbolic Right triangle (3,4,5)

$$\underbrace{\Delta x^2 - \Delta t^2 = \Delta s^2}$$

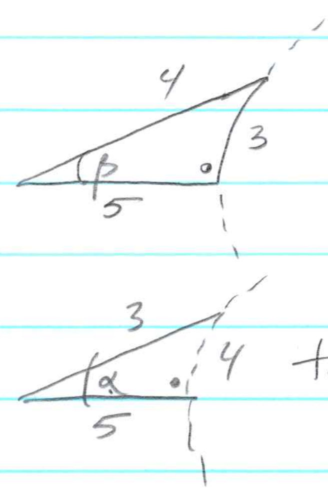
the hypotenuse is <u>not</u> the longest side but it must be ~~the~~ positive

Space-like Triangles



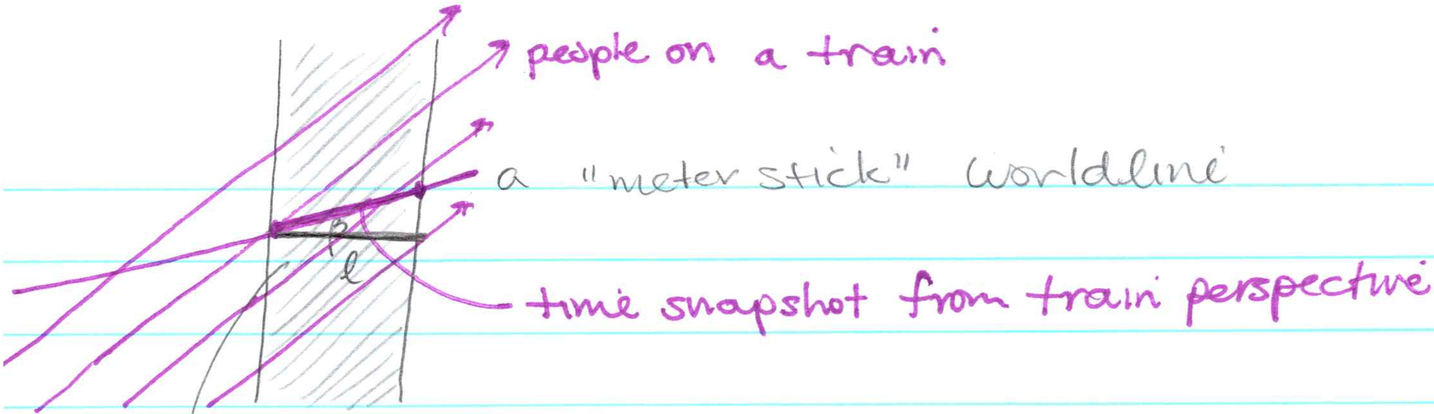
$\tan\beta = \frac{3}{5}$ $\tan\beta = 4/5$

Time like Triangles

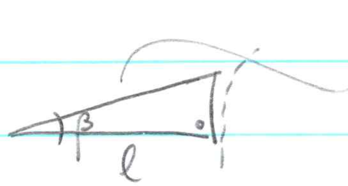


$\tan\beta = \frac{3}{5}$

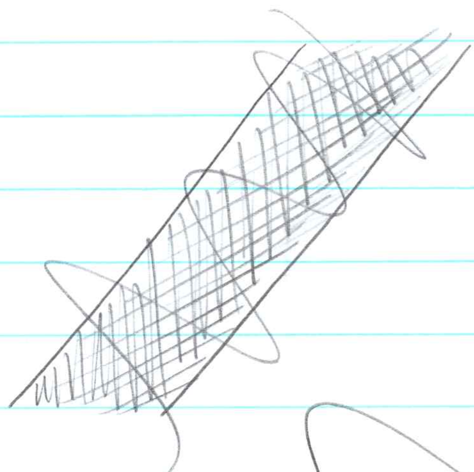
$\tan\alpha = \frac{4}{5}$

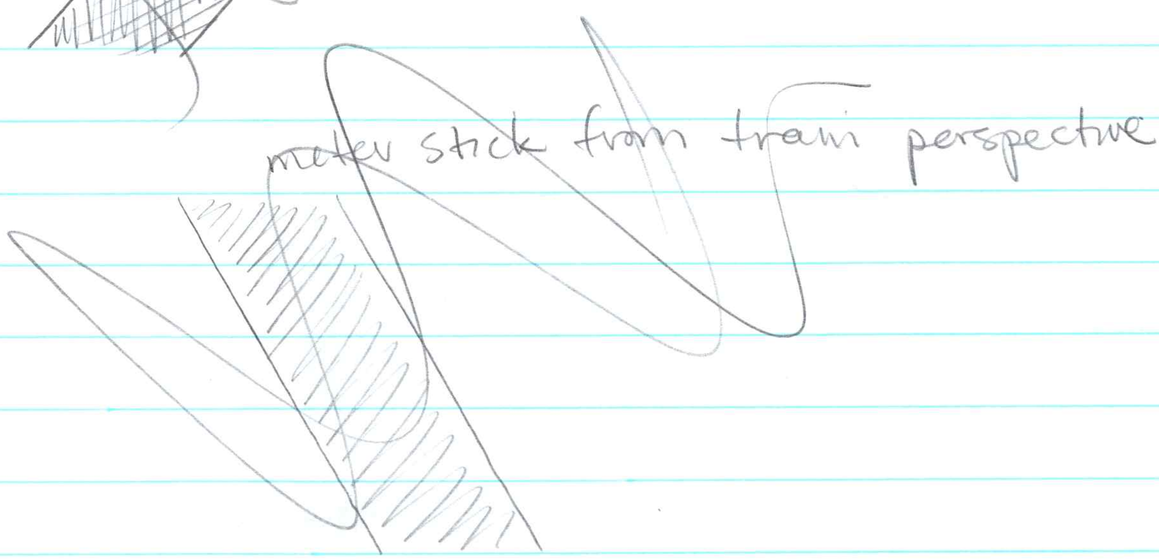


what is width of suitcase on a moving train from



$l/\cos\beta$ } length contraction





Hyperbola Trig/Geometry

Trig functions

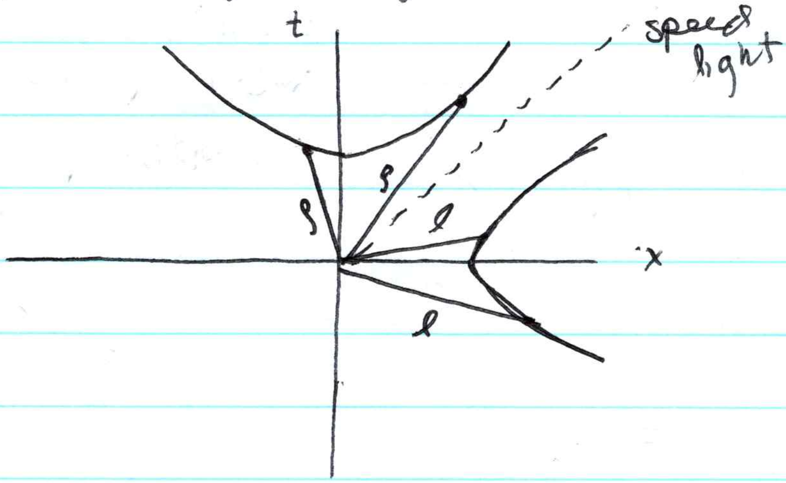
$$\cosh\beta = \frac{e^{\beta} + e^{-\beta}}{2} \qquad \sinh\beta = \frac{e^{\beta} - e^{-\beta}}{2}$$

$$\tanh\beta = \frac{\sinh\beta}{\cosh\beta}$$

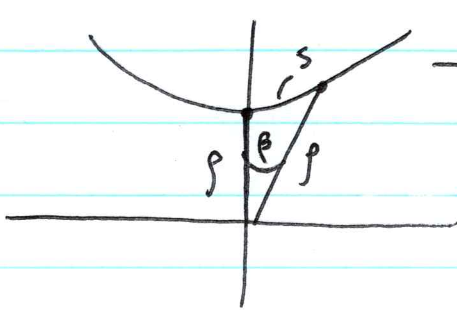
Distance: $\rho^2 = x^2 - y^2$

~~cases~~ "circles" are ~~are~~ points of constant distance.

In hyperbola geometry, these are hyperbolas



$$\boxed{\beta \equiv \frac{s}{\rho}}$$



→ (x, y) on hyperbola yields

$$\cosh\beta = \frac{x}{\rho}$$

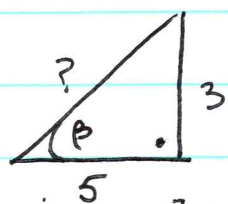
$$\sinh\beta = \frac{y}{\rho}$$

$$(x, y) \longrightarrow (\rho\sinh\beta,\ \rho\cosh\beta)$$

For a unit circle (hyperbola) $\rho = 1$
we see that

$\cos\beta \geq 1$ since the ~~mapping~~ $\min(x) = \rho$

Ex: $\tanh\beta = \frac{3}{5}$. what is $\cosh\beta$?
soln: draw any hyperbolic triangle



$\rho^2 = x^2 - t^2$ since $\rho^2 > 0$ we must
therefore have that the larger leg
is the positive piece in the distance function.
If you know $\tanh\beta$ and hypotenuse then projections are $\rho\sinh\beta$ and $\rho\cosh\beta$

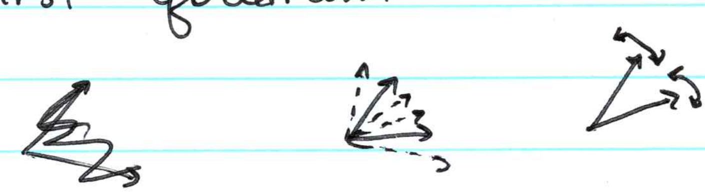
Rotations

Hyperbolic rotations are defined in the same way as Euclidean, but using hyperbolic functions.

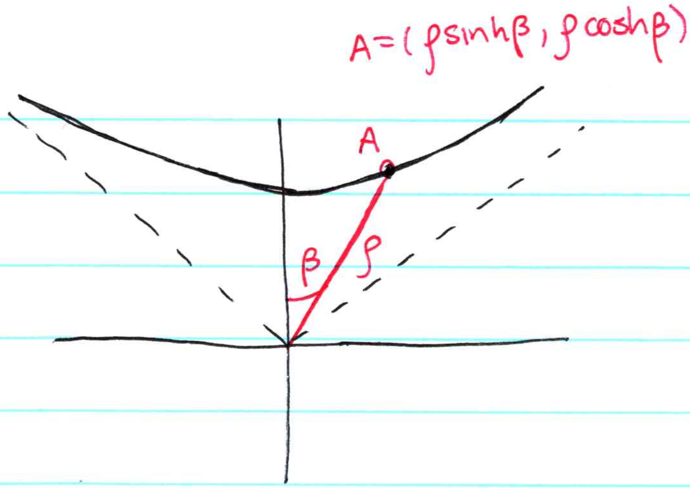
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The difference though is a hyperbolic rotation rotates vectors in and out of the first quadrant

e.g.



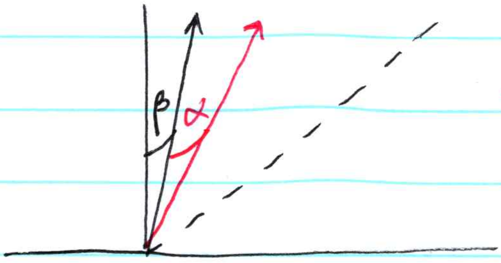
<u>addition</u>



slope is given by $\frac{\Delta y}{\Delta x}$ and therefore the equation of the line through A is

$$y = \frac{\cosh\beta}{\sinh\beta}x \longrightarrow \boxed{x = y\tanh\beta}$$

because hyperbolic trig functions have an exponential representation, we also have that



red line: $x = y\tanh(\alpha+\beta)$

the inverse slope relates to the speed of objects. That is: $\boxed{\frac{Y}{X} = \frac{1}{\tanh\beta}}$

this equation has an asymptote at 45° corresponding to the speed of light.

Lorentz Transformations

Recall from class

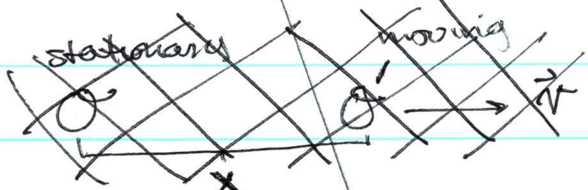
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

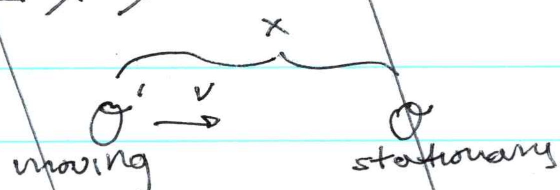
moving observer

$\Delta t = \gamma \Delta t'$ "moving clocks run slow"

station observer

$\Delta x = \frac{1}{\gamma} \Delta x'$ "moving rulers get smaller"





then the distance between as measured by O will be $x - vt$. however length contraction gives us that instead

$$\Delta x' = \gamma \Delta x$$

so that the distance x' for O' perspective is given by

$$x' = \gamma(x - vt)$$

However, physics is the same in all inertial frames. Therefore the primed coordinates

Lorentz transformations

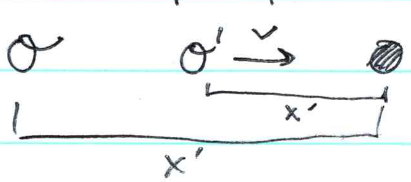
Recall from introductory special relativity that we have

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

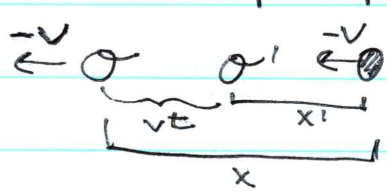
$$\left.\begin{aligned} \Delta t &= \gamma \Delta t' \\ \Delta x &= \tfrac{1}{\gamma} \Delta x' \end{aligned}\right\} \text{primed coordinates are for a moving observer}$$

Now consider an object, a stationary observer O and a moving observer O' (inertial)

O perspective



O' perspective



$\longrightarrow \quad x' = x - vt \qquad\qquad x = x' + vt'$

However w/ length contraction, these become

$$\boxed{x' = \gamma(x - vt) \qquad x = \gamma(x' - vt')}$$

we now wish to define a similar transformation
$t' = t'(x, t)$.

$$x' = \gamma(x - vt)$$
$$x = \gamma(x' + vt')$$

$$x = \gamma[\gamma(x - vt) + vt']$$

~~γ~~ WF solve for t'

$$\gamma v t' = x - \gamma^2 x + \gamma^2 vt$$
$$t' = x\frac{(1-\gamma^2)}{\gamma v} + \gamma t$$
$$t' = \gamma\left[x\frac{(1-\gamma^2)}{\gamma^2 v} + t\right]$$

Now, $$\frac{1-\gamma^2}{\gamma^2} = \frac{1 - \frac{1}{1-v^2/c^2}}{\frac{1}{1-v^2/c^2}}$$

$$= \left(1 - \frac{1}{1-v^2/c^2}\right)\left(1 - v^2/c^2\right)$$

$$= 1 - \frac{v^2}{c^2} - 1 = -\frac{v^2}{c^2}$$

$$\Longrightarrow \quad t' = \gamma\left(t - x\frac{v}{c^2}\right)$$

and by symmetry

$$t = \gamma\left(t' + x'\frac{v}{c^2}\right)$$

So in conclusion

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}}$$

Einstein addition formula

$$\begin{aligned} dx &= d\left[\gamma(x' + vt')\right] \\ &= \gamma(dx' + v\,dt') \\ dt &= d\left[\gamma\left(t' + x'\frac{v}{c^2}\right)\right] \\ &= \gamma\left(\frac{v}{c^2}dx' + dt'\right) \end{aligned}$$

$$\longrightarrow \quad \frac{dx}{dt} = \frac{dx' + v\,dt'}{\frac{v}{c^2}dx' + dt'}$$

$$= \frac{dx' + v\,dt'}{\frac{v}{c^2}dx' + dt'} \cdot \frac{1/dt'}{1/dt'}$$

$$= \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}$$

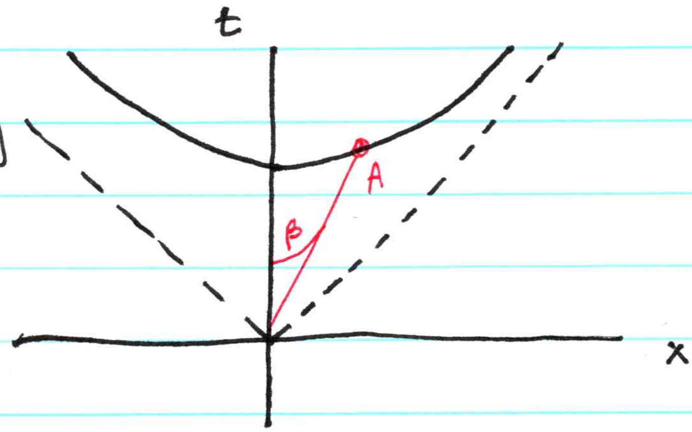
$$\Longrightarrow \quad \boxed{u = \frac{u' + v}{1 + \frac{v}{c^2}u'}}$$

for two observers (inertial) with relative velocity v

Lorentz transformations continued...

Now we wish to represent space and time in the same units

$$x = \gamma\left(x' + \frac{v}{c}\, ct'\right)$$
$$ct = \gamma\left(ct' + \frac{v}{c}\, x'\right)$$



Now define the following

$$ct = x\,\frac{1}{\tanh\beta}$$

$$\Rightarrow \tanh\beta = \frac{x}{t}\,\frac{1}{c} \equiv \frac{v}{c}$$

$$\boxed{\tanh\beta \equiv \frac{v}{c}}$$

then $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{1}{\sqrt{1 - \tanh^2\beta}} = \dfrac{1}{\sqrt{\operatorname{sech}^2\beta}} = \cosh\beta$

$$\longrightarrow \boxed{\gamma = \cosh\beta}$$

and thus we also have that

$$\frac{v}{c}\gamma = \tanh\beta\,\cosh\beta = \sinh\beta$$

$$\boxed{\sinh\beta = \frac{v}{c}\gamma}$$

from this we can establish a geometric transformation between (x,t) and (x',t').

$$x = \gamma\left(x' + \frac{v}{c}ct'\right) = \cosh\beta\, x' + \sinh\beta\, ct'$$
$$ct = \gamma\left(ct' + \frac{v}{c}x'\right) = \sinh\beta\, x' + \cosh\beta\, ct'$$

or

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}}_{\text{Hyperbolic rotation!}} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

Furthermore,

$$\begin{aligned} x^2 - c^2t^2 &= \cosh^2\beta\, x'^2 + \sinh^2\beta\, c^2t'^2 + \cancel{2\cosh\beta\sinh\beta\, x'ct'} \\ &\quad - \sinh^2\beta\, x'^2 - \cosh^2\beta\, c^2t'^2 - \cancel{2\cosh\beta\sinh\beta\, x'ct'} \\ &= \underbrace{(\cosh^2\beta - \sinh^2\beta)}_{1}(x'^2 - c^2t'^2) \\ &= x'^2 - c^2t'^2 \end{aligned}$$

in other words, the "interval"

$$\boxed{x^2 - c^2t^2 = x'^2 - c^2t'^2}$$

is invariant under hyperbolic rotation and can therefore be used to measure "distance"!

Hyperbola geometry is special relativity

Classification

$x^2 - c^2t^2 < 0$ timelike

$x^2 - c^2t^2 > 0$ spacelike

$x^2 - c^2t^2 = 0$ lightlike

"nonzero vector w/ length zero"

Dot Product

basis vectors $\{\hat{t}, \hat{x}_1, \hat{x}_2, \hat{x}_3\}$

Lorentzian dot product given by

$$\begin{cases} \hat{x}_i \cdot \hat{x}_j = \delta_{ij} \\ \hat{x}_i \cdot \hat{t} = 0 \\ \hat{t} \cdot \hat{t} = -1 \end{cases}$$

Let us consider then M^4. Vectors may be written as

$$\vec{r} = x_1 \hat{x}_1 + x_2 \hat{x}_2 + x_3 \hat{x}_3 + ct\,\hat{t}$$

$$|\vec{r}|^2 = x_1^2 + x_2^2 + x_3^2 - c^2t^2$$

Two vectors are orthogonal when $\vec{u} \cdot \vec{v} = 0$