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1 SPHERICAL COORDINATES II Consider spherical coordinates $\{r, \theta, \phi\}$ and the adapted orthonormal basis

$$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} = \left\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right\} \tag{1}$$

The "infinitesimal displacement vector" $d\vec{\boldsymbol{r}}$ relates this basis to an orthonormal basis of 1-forms via

$$\vec{\boldsymbol{r}} = dr\ \hat{\boldsymbol{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin\theta d\phi\ \hat{\boldsymbol{\phi}} \tag{2}$$

WARNING: *these conventions imply* $\tan\phi = \frac{y}{x}$

(**a**) Determine the exterior derivative of each basis vector (not 1-form) above, that is, compute $d\hat{\boldsymbol{r}}, d\hat{\boldsymbol{\theta}}$, and $d\hat{\boldsymbol{\phi}}$.

We begin by recalling the definition of *connection* 1-forms given by

$$d\hat{\boldsymbol{e}}_j = \omega^i{}_j\ \hat{\boldsymbol{e}}_i \tag{3}$$

i.e. the 1-form coefficients of the expansion of $d\hat{\boldsymbol{e}}_j$ in the regular basis. In class, we also defined the metric compatibility and torsion free requirements in terms of connections as

$$\omega_{ij} + \omega_{ji} = 0 \tag{4}$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0 \tag{5}$$

where a connection with both indices downstairs is given by

$$\omega_{ij} = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j \tag{6}$$

Equipped with these definitions we now can solve for the connection 1-forms. To evade a brute force calculation, recall the line element for $\mathbb{E}^3$ in spherical coordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{7}$$

so that

$$d\vec{\boldsymbol{r}} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \tag{8}$$

however, we also know that

$$d\vec{\boldsymbol{r}} = d(r\hat{\boldsymbol{r}}) = dr\hat{\boldsymbol{r}} + rd\hat{\boldsymbol{r}} \tag{9}$$

so that direct comparison gives

$$\boxed{d\hat{\boldsymbol{r}} = d\theta\ \hat{\boldsymbol{\theta}} + \sin\theta d\phi\ \hat{\boldsymbol{\phi}}} \tag{10}$$

Knowing (10) significantly simplifies our job. For the other two derivatives we have

$$
\begin{aligned}
d\hat{\boldsymbol{\theta}} &= \omega^r{}_\theta\, \hat{\boldsymbol{r}} + \omega^\theta{}_\theta\, \hat{\boldsymbol{\theta}} + \omega^\phi{}_\theta\, \hat{\boldsymbol{\phi}} \\
d\hat{\boldsymbol{\phi}} &= \omega^r{}_\phi\, \hat{\boldsymbol{r}} + \omega^\theta{}_\phi\, \hat{\boldsymbol{\theta}} + \omega^\phi{}_\phi\, \hat{\boldsymbol{\phi}}
\end{aligned} \tag{11}
$$

However, equation (4) further simplifies by allowing us to remove the diagonal terms.

$$
\begin{aligned}
d\hat{\boldsymbol{\theta}} &= \omega^r{}_\theta\, \hat{\boldsymbol{r}} + \omega^\phi{}_\theta\, \hat{\boldsymbol{\phi}} \\
d\hat{\boldsymbol{\phi}} &= \omega^r{}_\phi\, \hat{\boldsymbol{r}} + \omega^\theta{}_\phi\, \hat{\boldsymbol{\theta}}
\end{aligned} \tag{12}
$$

The vector version of (4) states that

$$
d(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j) = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j + d\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = 0 \tag{13}
$$

Thus, equation (13) yields the following:

$$
\omega^r{}_\theta = -\hat{\boldsymbol{\theta}} \cdot d\hat{\boldsymbol{r}} = -d\theta \tag{14}
$$

$$
\omega^r{}_\phi = -\hat{\boldsymbol{\phi}} \cdot d\hat{\boldsymbol{r}} = -\sin\theta d\phi \tag{15}
$$

Now only two connections remain. However, we have that $\omega^\phi{}_\theta = -\omega^\theta{}_\phi$. It is therefore sufficient to solve for either one alone. To do this, consider the torsion free condition (eq 5). We have that

$$
0 = d(rd\theta) + \omega^\theta{}_r \wedge dr + \omega^\theta{}_\theta \wedge rd\theta + \omega^\theta{}_\phi \wedge r\sin\theta d\phi \tag{16}
$$

$$
= dr \wedge d\theta + \omega^\theta{}_r \wedge dr + \omega^\theta{}_\phi \wedge r\sin\theta d\phi \tag{17}
$$

$$
= dr \wedge d\theta + d\theta \wedge dr + \omega^\theta{}_\phi \tag{18}
$$

$$
\Rightarrow \omega^\theta{}_\phi \wedge r\sin\theta d\phi = 0 \tag{19}
$$

Equation (19) tells us that $\omega^\theta{}_\phi$ must only include $d\phi$ and no other 1-forms in order for (19) to hold. Equation (5) also gives

$$
0 = d(r\sin\theta d\phi) + \omega^\phi{}_r \wedge dr + \omega^\phi{}_\theta \wedge rd\theta + \omega^\phi{}_\phi \wedge r\sin\theta d\phi \tag{20}
$$

$$
= \sin\theta dr \wedge d\phi + r\cos\theta d\phi \wedge d\phi + \omega^\phi{}_r \wedge dr + \omega^\phi{}_\theta \wedge rd\theta \tag{21}
$$

$$
= \sin\theta dr \wedge d\phi + \sin\theta d\phi \wedge dr + r\cos\theta d\theta \wedge d\phi + \omega^\phi{}_\theta \wedge rd\theta \tag{22}
$$

$$
= r\cos\theta d\theta \wedge d\phi - rd\theta \wedge \omega^\phi{}_\theta \tag{23}
$$

$$
\Rightarrow \omega^\phi{}_\theta = \cos\theta d\phi \tag{24}
$$

$$
\text{and } \omega^\theta{}_\phi = -\cos\theta d\phi \tag{25}
$$

...and that's all there is to it! In summary, we have

$$
\boxed{
\begin{aligned}
d\hat{\boldsymbol{r}} &= d\theta\, \hat{\boldsymbol{\theta}} + \sin\theta d\phi\, \hat{\boldsymbol{\phi}} \\
d\hat{\boldsymbol{\theta}} &= -d\theta\, \hat{\boldsymbol{r}} + \cos\theta d\phi\, \hat{\boldsymbol{\phi}} \\
d\hat{\boldsymbol{\phi}} &= -\sin\theta d\phi\, \hat{\boldsymbol{r}} - \cos\theta d\phi\, \hat{\boldsymbol{\theta}}
\end{aligned}
} \tag{26}
$$

(**b**) Compute $\omega_{ij} = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j$ for $i, j = 1, 2, 3$. *What sort of beast should you get?*

This question asks us to identify the connection 1-forms. We can easily read these off from our solution to part (a) by comparing with equation (11). They are

$$\boxed{\begin{array}{lll} \omega_{rr}=0 & \omega_{\theta r}=d\theta & \omega_{\phi r}=\sin\theta d\phi \\ \omega_{r\theta}=-d\theta & \omega_{\theta\theta}=0 & \omega_{\phi\theta}=\cos\theta d\phi \\ \omega_{r\phi}=-sin\theta d\phi & \omega_{\theta\phi}=-\cos\theta d\phi & \omega_{\phi\phi}=0 \end{array}} \tag{27}$$

(**c**) Compute $\Omega_{ij}=d\omega_{ij}+\omega_{ik}\wedge\omega_{kj}$ for $i,j=1,2,3$ (and where there is an implicit sum over k). *What sort of beast should you get?*

Inspection of the equation for each Ω_{ij} reveals some interesting structure given our solution to part (b) of the problem. Notice that the table in equation 27 in antisymmetric as a result of equation 4. Therefore, we have that.

$$\omega_{ik}\wedge\omega_{ki}=-\omega_{ik}\wedge\omega_{ik} \tag{28}$$

but each of our ω_{ik} are in $\bigwedge^1$ and therefore

$$\omega_{ik}\wedge\omega_{ik}=0 \quad \forall\, i,k \tag{29}$$

Thus, because $d(0)=0$ and because of (29) we have that $\Omega_{ii}=0\ \forall\ i$. If we zap equation (4) with d, we find that

$$d\omega_{ij}+d\omega_{ji}=0 \tag{30}$$

$$\Rightarrow d\omega_{ji}=-d\omega_{ij} \tag{31}$$

For the second half of the Ω_{ij} equation, we have that

$$\omega_{ik}\wedge\omega_{kj}=-\omega_{kj}\wedge\omega_{ik} \tag{32}$$

$$=\omega_{jk}\wedge\omega_{ik} \tag{33}$$

$$=-\omega_{jk}\wedge\omega_{ki} \tag{34}$$

Putting (31) together with (34) gives

$$\Omega_{ij}=-\Omega_{ji} \tag{35}$$

Therefore, it is sufficient to calculate elements with indices corresponding to the upper right half of the table in equation (27). Let's begin with $i=\theta$ and $j=r$.

$$d\omega_{\theta r}=d(d\theta)=0 \tag{36}$$

$$\omega_{\theta k}\wedge\omega_{kr}=\omega_{\theta r}\wedge\omega_{rr}+\omega_{\theta\theta}\wedge\omega_{\theta r}+\omega_{\theta\phi}\wedge\omega_{\phi r} \tag{37}$$

$$=0+0+\omega_{\theta\phi}\wedge\omega_{\phi r} \tag{38}$$

$$=-\cos\theta d\phi\wedge\sin\theta d\phi \tag{39}$$

$$=-\cos\theta\sin\theta d\phi\wedge d\phi \tag{40}$$

$$=0 \tag{41}$$

$$\Rightarrow \Omega_{\theta r}=0 \tag{42}$$

For the next pair, we have

$$
\begin{aligned}
d\omega_{\phi r} &= d(\sin\theta d\phi) = \cos\theta d\theta \wedge d\phi && (43)\\
\omega_{\phi k} \wedge \omega_{kr} &= \omega_{\phi r} \wedge \omega_{rr} + \omega_{\phi\theta} \wedge \omega_{\theta r} + \omega_{\phi\phi} \wedge \omega_{\phi r} && (44)\\
&= \omega_{\phi\theta} \wedge \omega_{\theta r} && (45)\\
&= \cos\theta d\phi \wedge d\theta && (46)\\
&= -\cos\theta d\theta \wedge d\phi && (47)\\
\Rightarrow \Omega_{\phi r} &= 0 && (48)
\end{aligned}
$$

finally,

$$
\begin{aligned}
d\omega_{\phi\theta} &= d(cos\theta d\phi) = -\sin\theta d\theta \wedge d\phi && (49)\\
\omega_{\phi k} \wedge \omega_{k\theta} &= \omega_{\phi r} \wedge \omega_{r\theta} + \omega_{phi\theta} \wedge \omega_{\theta\theta} + \wedge_{\phi\phi} \wedge \omega_{\theta\phi} && (50)\\
&= \omega_{\phi r} \wedge \omega_{r\theta} && (51)\\
&= \sin\theta d\phi \wedge -d\theta && (52)\\
&= \sin\theta d\theta \wedge d\phi && (53)\\
\Rightarrow \Omega_{\phi\theta} &= 0 && (54)
\end{aligned}
$$

Thus we conclude that every $\Omega_{ij} = 0$ for $\mathbb{E}^3$ described in spherical coordinates. The Ω_{ij} are related to curvature, and therefore, this makes sense as we are considering regular Euclidean space which is supposed to be flat.