Complex Analysis: Day 12 **John Waczak**
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Proof of Cauchy's theorem (out verison)

Thm: If $f : \Omega \to \mathbb{C}$ is a holomorphism, and γ_0, γ_1 are Ω-homotopic closed curves in Ω, then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Proof under additional hypotheses:

1. Assume f' is continuous
2. Assume that homotopy h has continuous second partial derivatives.

Recall that γ_0, γ_1 parametrized by interval $[0,1]$ and $h : [0,1] \times [0,1] \to \Omega$, the homotopy map is such that $h(t,0) = \gamma_0(t)$, $h(t,1) = \gamma_2(t)$ and $h(0,s) = h(1,s)$. Think of γ_s as the continuously varying family of curves.

Define $I(s) = \int_{\gamma_s} f$. So $I(0) = \int_{\gamma_0} f$ and $I(1) = \int_{\gamma_1} f$. So we want to show $I(0) = I(1)$. To show this, it suffices to show that $I'(s) = 0 \quad \forall s$.

$I'(s) = \frac{d}{ds}\int_0^1 f(\gamma_s(t))\gamma_s'(t)dt$. When we switch to use $h(t,s)$ instead, our derivatives become partials.

$$\begin{aligned}
I'(s) &= \frac{\partial}{\partial s}\int_0^1 f(h(t,s))\frac{\partial}{\partial t}h(t,s)dt \\
&= \int_0^1 \frac{\partial}{\partial s}\Big[f(h)\frac{\partial}{\partial t}h\Big]dt \\
&= \int_0^1 f'(h(t,s))\frac{\partial h}{\partial s}\frac{\partial h}{\partial t} + f(h(t,s))\frac{\partial^2 h}{\partial s \partial t}dt \\
&= \int_0^1 f'(h(t,s))\frac{\partial h}{\partial t}\frac{\partial h}{\partial s} + f(h(t,s))\frac{\partial^2 h}{\partial t \partial s}dt \\
&= \int_0^1 \frac{\partial}{\partial t}\Big[f(h(t,s))\frac{\partial h}{\partial s}\Big]dt \quad \text{product rule} \\
&= f(h(1,s))\frac{\partial h}{\partial s}(1,s) - f(h(0,s))\frac{\partial h}{\partial s}(0,s) \\
&= 0 \quad \text{since} \quad h(0,s) = h(1,s)\forall s \\
\text{thus } I'(s) &= 0 \Rightarrow I(s) = \text{ const } \forall s
\end{aligned}$$

Def we say γ is contractible (or null-homotopic) in Ω if γ is Ω-homotopic to a constant curve (i.e. a point).

Consequence: If γ is null-homotopic then,

$$\int_\gamma f = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 f(\gamma(t))0dt = 0$$

Think – integral of a point is always zero.

Ex: $\int_{|z-2|-1} Log(z)dz = 0$. Since $Log(z)$ is holomorphic on $\Omega = \mathbb{C}\backslash(-\infty, 0]$ and the curve $|z-2| = 1$ is null-homotopic in Ω by inspection.

Def if f is entire and γ is closed, then $\int_\gamma f = 0$.

p.f. Every closed curve is null-homotopic in $\mathbb{C}$. (Straight line homotopy) *Note:* if $\Omega \to \mathbb{C}$ is a region in which every closed curve is null-homotopic in Ω, we say Ω is **Simply connected** (no holes). $\mathbb{C}$ is simply connected. $\mathbb{C} \setminus \{0\}$ is not. Recall that we proved $\int_{|z|=1} \frac{1}{z} dz = 2\pi i \neq 0$ when $\Omega = \mathbb{C} \setminus \{0\}$ which does not agree with what Cauchy's theorem would give if we included the origin.

More generally, if Ω is simply-connected, then $\int_\gamma f = 0 \ \forall$ closed curves γ and holomporphisms f.

Ex: $\int_{|z|=1} \frac{1}{z^2-2z} dz = \int_{|z|=1} -\frac{1}{2z} + \frac{1}{2}\frac{1}{z-2} dz = -\frac{1}{2} 2\pi i$ Since we know the value of the first integral and the second is null-homotopic on the unit circle (the hole is at z=2). We did this using partial fraction decomposition.

Cauchy's Integral Formula

Theorem(Cauchy's Integral Formula): Let $\Omega \subseteq \mathbb{C}$ be a region and suppose the closed disc with center w and radius R, $D_R(w) \subseteq \Omega$, i.e. $\{z : |z-w| \leq R\} \subseteq \Omega$. Then if $f : \Omega \to \mathbb{C}$ is holomorphic, we have

$$f(w) = \frac{1}{2\pi i} \int\limits_{|z-w|=R} \frac{f(z)}{z-w} dz$$