Homework 3C $\boxed{\text{Total Time 1.5 hours}}$ John Waczak
Mth 434

1 (a) Derive the Frenet frame, curvature, and torsion for a unit speed curve

We are given that $|V(t)|=1$ so by prop 1.13 on page 14 this implies that $a \perp V$ and therefore $a^{\perp} \equiv a$. Thus our curvature is:

$$K(t) = \frac{|a^{\perp}(t)|}{|V(t)|^2}$$

$$K(s) = \frac{|a^{\perp}(s)|}{1} = |a|$$

$$\Longrightarrow \boxed{K(s) = |a|}$$

Now for the Frenet frame:

$$T(t) = \frac{V(t)}{|V(t)|} \Rightarrow T(s) = \frac{V(s)}{1} = V(s)$$

Similarly $N(T) = \frac{a^{\perp}}{|a^{\perp}|} \Rightarrow N(s) = \frac{a(s)}{|a(s)|} = \frac{1}{K(s)} a(s)$

$B(t) = T(t) \times N(t) \Rightarrow B(s) = T(s) \times N(s)$

Thus we have an orthonormal basis

$$\{T, N, B\} = \left\{ V(s), \frac{1}{K(s)} a(s), V(s) \times \frac{1}{K(s)} a(s) \right\}$$

To determine the torsion we need to find $B'(s)$ and then $\langle B'(s), N(s) \rangle$

From orthonormality we know

$$|N|^2 = N \cdot N = 1$$

$$\frac{d}{ds}\left(N(s) \cdot N(s) \right) = 0$$

$$N'(s) \cdot N(s) + N(s) \cdot N'(s) = 0$$

$$2\left(N'(s) \cdot N(s) \right) = 0$$

$$\rightarrow \; N'(s) \cdot N(s) = 0$$

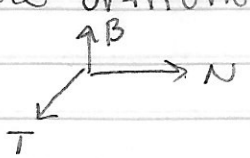
Therefore $N'(s) \perp N(s)$ Now we compute $B'(s)$

$$\begin{aligned}
B'(s) &= \frac{d}{ds}\left(T(s) \times N(s) \right) \\
&= T'(s) \times N(s) + T(s) \times N'(s) \\
&= \alpha(s) \times N(s) + T(s) \times N'(s) \\
&= K(s) N(s) \times N(s) + T(s) \times N'(s) \\
&= 0 + T(s) \times N'(s)
\end{aligned}$$

Because $N' \perp N$ we may write
$N' = \alpha T + \beta B$ since $N \cdot T = N \cdot B = 0$
and so we maintain orthogonality. this means

$$\begin{aligned}
B'(s) &= T(s) \times N'(s) \\
&= T(s) \times (\alpha T(s) + \beta B(s)) \\
&= T(s) \times \alpha T(s) + T(s) \times \beta B(s) \\
&= 0 + T(s) \times \beta B(s) \\
&= \beta (T(s) \times B(s))
\end{aligned}$$

we know $B = T \times N$ because they are orthonormal we therefore have



$$\Rightarrow \; T(s) \times B(s) = -N(s)$$

Thus $B'(s) = -\beta N(s)$

So we have shown $B' = -\beta N(s)$ for some $\beta = \beta(s)$. Now we have everything to calculate the torsion

$$\tau = \frac{-\langle B'(s), N(s)\rangle}{|v(s)|}$$

$$= -\langle B'(s), N(s)\rangle$$

$$= -\langle -\beta N(s), N(s)\rangle$$

$$= \beta \langle N(s), N(s)\rangle$$ and because $N(s)$ is normalized this inner product must be unity. Therefore

$\tau = \beta(s)$ which means we can write the simplified Frenet frame as

$$\boxed{\begin{aligned} &|v(s)| = 1 \\ &K(s) = |a(s)| \\ &T(s) = v(s) \\ &N(s) = \frac{1}{K(s)} a(s) \\ &B(s) = T(s) \times N(s) \\ &B'(s) = -\tau(s) N(s) \end{aligned}}$$

(b) $\beta(s) = \left(\frac{4}{5}\cos(s),\ 1-\sin(s),\ -3/5\cos(s)\right)$

Determine the curvature, Torsion, and Frenet frame

$V(s) = \beta'(s) = \left(-\frac{4}{5}\sin(s),\ -\cos(s),\ 3/5\sin(s)\right)$

$|V(s)| = \sqrt{\frac{16}{25}\sin^2(s) + \cos^2(s) + \frac{9}{25}\sin^2(s)} = \sqrt{1} = 1$

thus the curve is parametrized by arc length. $\Rightarrow a^{\perp} = a$ and so the curvature is given by $K(s) = |a(s)|$

$$a(s) = \beta''(s) = \left(-\frac{4}{5}\cos(s),\ \sin(s),\ 3/5\cos(s)\right)$$

$$\Rightarrow K(s) = \sqrt{\frac{16}{25}\cos^2(s) + \sin^2(s) + \frac{9}{25}\cos^2(s)} = \sqrt{1} = 1$$

Thus $\boxed{K(s) = 1}$

Now $T(s) = \frac{V(s)}{|V(s)|} = V(s) = \left(-\frac{4}{5}\sin(s), -\cos(s), 3/5\sin(s)\right)$

$$N(s) = \frac{a^{\perp}}{|a^{\perp}|} = \frac{1}{K}a(s) = a(s) = \left(-\frac{4}{5}\cos(s), \sin(s), 3/5\cos(s)\right)$$

$$B(s) = T(s) \times N(s) = \begin{vmatrix} i & j & k \\ -\frac{4}{5}\sin(s) & -\cos(s) & \frac{3}{5}\sin(s) \\ -\frac{4}{5}\cos(s) & \sin(s) & \frac{3}{5}\cos(s) \end{vmatrix} =$$

$$= \left(-\frac{3}{5}\cos^2(s) - \frac{3}{5}\sin^2(s),\ -\left(-\frac{12}{25}\sin(s)\cos(s) + \frac{12}{25}\sin(s)\cos(s)\right),\ -\frac{4}{5}\sin^2(s) - \frac{4}{5}\cos^2(s)\right)$$

$$= \left(-\frac{3}{5},\ 0,\ -4/5\right)$$

$\Rightarrow$ $B(s)$ = constant and therefore the torsion is zero. This suggest that this curve represents some kind of circle (I'm guessing from the components) that is in a plane normal to $\begin{pmatrix} -3/5 \\ 0 \\ 4/5 \end{pmatrix}$