Lecture # 12

Lifetimes, line intensities, widths, shapes

Back to Lectures # 2-3 ⇒ discrete states ⇒

$$|\Psi(t)\rangle = \sum_n C_n(t)\, e^{-\frac{i}{\hbar}E_n t}|n\rangle$$

$$i\hbar \frac{dC_n(t)}{dt} = \lambda \sum_k V_{nk}(t) C_k(t)\, e^{i\omega_{nk}t}$$

$$P_{i\to f}(t) = |C_f(t)|^2$$

Two-level system ⇒ $C_1(t)$, $C_2(t)$, (Lecture # 2)

$C_1(0) = 1$, $C_2(0) = 0$

As we've shown, under harmonic perturbation (in the resonance, i.e. $\omega = \omega_{21}$) the system oscillates $|C_1|^2 \sim \cos^2 \Omega t$, $|C_2|^2 \sim \sin^2 \Omega t$

$\Omega \sim \frac{V}{\hbar}$ ← strength of perturbation

Can we think of an atom that can made a spontaneous transition from 2 to 1 as a two-level system? ⇒ no, since the final state is actually a state of an atom together with that of a photon which is continuous. These final states are incoherent and cannot act cooperatively to build up the reverse transitions, so probability of finding the atom in state 2 decreases steadily with time. How do we describe it?

$$\underset{\substack{\uparrow \\ \text{probability} \\ \text{of finding the atom} \\ \text{in state 2 at } t+dt}}{P_2(t+dt)} = P_2(t)\underbrace{\left(1 - W_{21}^{em}\,dt\right)}_{\substack{\text{probability that} \\ \text{no transition from 2} \\ \text{to 1 has taken place} \\ \text{(due to spont. emission)}}}$$

$$P_2(t) = e^{-t/\tau}, \qquad \tau = \frac{1}{W_{21}^{em}} \rightarrow \text{lifetime}$$

$$\Downarrow$$

$$C_2(t) = e^{-t/2\tau}$$

↑ assume C_2 is real

$$\Psi_2(\vec{r},t) = C_2(t)\,\psi_2(\vec{r})\,e^{-\frac{i}{\hbar}E_2 t} =$$

$$= \psi_2(\vec{r})\,e^{-\frac{i}{\hbar}\left(E_2 - \frac{i\hbar}{2\tau}\right)t} \leftarrow \text{a state with } \underline{\text{complex energy}}!!$$

$$e^{-\frac{i}{\hbar}\left(E_2 - \frac{i\hbar}{2\tau}\right)t} = \frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{+\infty} a(E')\,e^{-\frac{i}{\hbar}E't}\,dE'$$

↑ decompose into energy eigenstates

$$a(E) = \frac{1}{(2\pi\hbar)^{1/2}} \int_0^\infty e^{-\frac{i}{\hbar}\left(E_2 - \frac{i\hbar}{2\tau}\right)t} e^{\frac{i}{\hbar}E't}\, dt =$$

consider $\Psi_2(t<0) = 0$

$$= \frac{1}{(2\pi\hbar)^{1/2}} \frac{-i\hbar}{E_2 - E - \frac{i\hbar}{2\tau}}$$

Then, probability to find the system in state 2, but with definite energy E is $\sim |a(E)|^2 =$

$$= \frac{\hbar}{2\pi} \frac{1}{(E_2 - E)^2 + \frac{\hbar^2}{4\tau^2}}$$

Conservation of energy (assuming that state 1 does not decay).

$\Downarrow$

$$E = E_1 + \hbar\omega$$

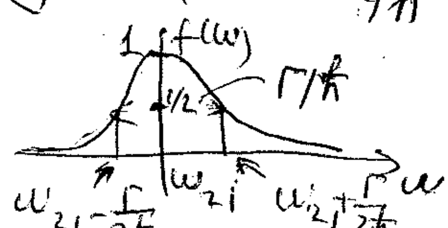
$\Downarrow$

$$|a(E)|^2 = \frac{\hbar}{2\pi} \frac{1}{(\underbrace{E_2 - E_1}_{\hbar\omega_{21}} - \hbar\omega)^2 + \frac{\hbar^2}{4\tau^2}} = \leftarrow \text{Lorentzian distribution}$$

$$= \frac{1}{h} \frac{1}{(\omega_{21} - \omega)^2 + \frac{\Gamma^2}{4\hbar^2}} \sim f(\omega) = \frac{\Gamma^2/4\hbar^2}{(\omega_{21} - \omega)^2 + \frac{\Gamma^2}{4\hbar^2}}$$

$$\Gamma = \frac{\hbar}{\tau}$$

natural width of the line



Generally $\Rightarrow$ $\Delta E \sim \Gamma$ ← uncertainty in energy

$\Delta t \sim \tau$ ← in time

$$\Delta E \, \Delta t \gtrsim \hbar$$

If the final state 1 is not stable $\Rightarrow$

$$\Gamma = \hbar \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right)$$

lifetimes of states 1 & 2

If state 2 can decay to more than one state

$$W^{em}_{2\to 1} \Rightarrow \sum_i W^{em}_{2\to i}$$

The natural width of atomic lines is very small $\Rightarrow$

(H) atom, 2p state $(E_{n=2} = -3.4\ eV)$ $\Rightarrow \Gamma = 4 \cdot 10^{-7}\ eV$

$$\frac{\Gamma}{|E_{n=2}|} \sim 10^{-7} \ ! \qquad \tau = 1.6\ ns$$

Typically, observed spectral lines are much wider $\Rightarrow$

- pressure broadening $\Rightarrow$ a.k.a. collisional broadening

$$W^{em}_{i\to f} \Rightarrow W^{total}_{i\to f}$$ ← include $W_c = n v \sigma$,

where n is the number density of atoms (5)
v is the relative velocity between pairs of atoms, σ is the collision cross-section.

· Mechanism: collision between atoms (especially relevant in gases) causes radiationless transitions. Since number of atoms participating in collisions (n) and their velocity (v) are functions of temperature and pressure of the gas $\Rightarrow$ measure spectral profiles and get this info (this is how we know these things about stellar atmospheres!!)

- Doppler broadening

wavelength of light emitted by a moving atom is shifted $\Rightarrow \lambda = \lambda_0 (1 \pm \frac{v}{c})$

going away from observer

approaching observer

emitted by a stationary atom

$$\omega = \omega_0 \left(1 \pm \frac{v}{c}\right)^{-1} \simeq \omega_0 \left(1 \mp \frac{v}{c}\right)$$

$$\omega = \frac{2\pi c}{\lambda}$$

$$\underbrace{dN}_{\uparrow} = N_0 \exp\left(-\frac{Mv^2}{2KT}\right) dv$$

atomic mass

Maxwell distribution

atoms

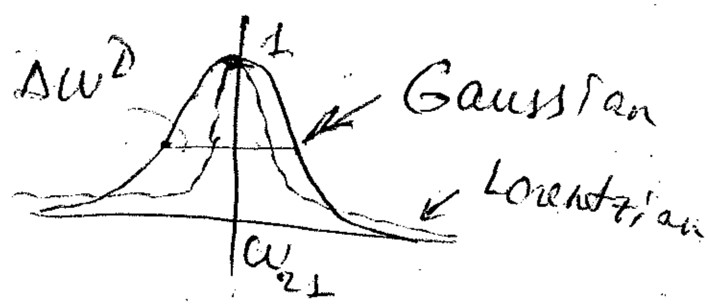
with velocities between v and $v+dv$

$$\underbrace{I(\omega)} = I(\omega_0) \exp\left[-\frac{Mc^2}{2KT}\left(\frac{\omega-\omega_0}{\omega_0}\right)^2\right]$$

ω_{21}

intensity of light emitted by atoms

Gaussian



Doppler width at half-max $\Rightarrow$

$$\Delta\omega^D = \frac{2\omega_0}{c}\left[\frac{2KT}{M}\ln 2\right]^{1/2} \Rightarrow$$

$\uparrow$ with temperature T and the frequency of the line ω_0

Typically observe a combination of Lorentzian and Gaussian $\Rightarrow$ Voigt profile

inhomogeneous broadening

homogeneous broadening