Complex Analysis: Day 20 **John Waczak**
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more stuff

Theorem (Identity Principle)**.** *Let $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be holomorphic on a region Ω and suppose $\{\alpha_k\}$ is a sequence of distinct complex numbers in Ω converging to $\alpha \in \Omega$. Suppose that $f(\alpha_k) = g(\alpha_k)$ for all $k \geq 1$ Then $f(z) = g(z)$ for all $z \in \Omega$. This shows that holomorphic functions are very rigid.*

Proof. Define $h = f - g$. In other words $h(\alpha_k) = 0\ \forall k$. Now define two sets

$$X = \{a \in \Omega : \exists \text{ some } r > 0 \text{ for which} h(a) = 0 \quad \forall z \in D_R(a)\}$$
$$Y = \{a \in \Omega : \exists r > 0 \text{ s.t. } h(a) \neq 0 \quad \forall z \in D_R(a) \setminus \{a\}\}$$

Note that $X \cap Y = \emptyset$ and $X \cup Y = \Omega$. Also note that both X and Y are open. Given $a \in X$ or Y points close enough to a are also in X or Y. Thus Ω is the disjoint union of two open sets. By the definition of *connectedness* one of these sets must be $\emptyset$. We know that $\alpha = \lim_{k\to\infty} \alpha_k \in X$. Thus $X \neq \emptyset$. Therefore $\Omega = X$ and $h(z) = 0 \quad \forall z \in \Omega$. This implies that $f(z) = g(z) \quad \forall z \in \Omega$. $\square$

Corollary. *Let $\Omega_1 \subseteq \Omega_2$ be regions, and let $f : \Omega_1 \to \mathbb{C}$ be holomorphic. If $\exists$ holomorphic function $F : \Omega_2 \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in \Omega_1$ then F is unique.*

Proof. If $F, G : \Omega_2 \to \mathbb{C}$ are holomorphic and $F(z) = G(z) = f(z)$ for all $z \in \Omega_1$ then $F(z) = G(z)$ $\forall z \in \Omega_2$ by the identity principle. $\square$

Definition. *If such a $F : \Omega_2 \to \mathbb{C}$ exists it is called an analytic continuation of f.*

Example $\Omega_1 = \{z = x + iy : x > 1\}$

$$f : \Omega_1 \to \mathbb{C}$$
$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

It turns out that this function $f(z)$ has an analytic continuation $F : \Omega_2 \to \mathbb{C}$ where $\Omega_2 = \mathbb{C} \setminus \{1\}$. The *Riemann Hypothesis* says that all of the zeros of $F(z)$ in the strip $0 < x < 1$ occur on the line $x = 1/2$.