Homework 7 **John Waczak**
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1 SPHERICAL COORDINATES II Consider the sphere of radius r, in spherical coordinates (θ, ϕ), with line element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

(**a**) Find the connection 1-forms ω_{ij} in this basis.

Because we are confined to the surface of the sphere, $d\vec{\boldsymbol{r}}$ vector will not aid in our calculation. However, The metric compatibility requirement

$$\omega_{ij} + \omega_{ji} = 0 \tag{2}$$

Implies that every ω_{ii} is identically zero. Furthermore, because r is constant on the sphere, there are only 4 connection 1-forms to calculate. The above asymmetry argument implies we need only explicitly calculate 1 of them. To do this we will consider the torsion free requirement which states

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0 \tag{3}$$

The leads to the following system of linear equations

$$\omega^\theta{}_\phi \wedge r\sin\theta d\phi = 0 \tag{4}$$
$$r\cos\theta d\theta \wedge d\phi + r d\theta \wedge \omega^\theta{}_\phi = 0 \tag{5}$$

where in equation (4) we note that $dr = 0$ for the sphere. Recalling that $\omega^i{}_j$ are 1-forms and are expanded in the usual basis as

$$\omega^i{}_j = \Gamma^i{}_{jk}\,\sigma^k \tag{6}$$

equation (4) implies that $\Gamma^\theta{}_{\phi\theta} = 0$ so that $\omega^\theta{}_\phi$ has only a $d\phi$ component. With this, equation (5) becomes

$$\begin{aligned} 0 &= r\cos\theta d\theta \wedge d\phi + r d\theta \wedge \Gamma^\theta{}_{\phi\phi} d\phi \\ &= r\cos\theta d\theta \wedge d\phi + r\Gamma^\theta{}_{\phi\phi} d\theta \wedge d\phi \\ \Rightarrow \quad & \boxed{\Gamma^\theta{}_{\phi\phi} = -\cos\theta} \end{aligned} \tag{7}$$

Where I have done the calculation in a coordinate basis for simplicity. Thus, we can conclude that the connection 1-forms for $\mathbb{S}^2$ (in the orthonormal basis) are

$$\boxed{\begin{aligned} &\omega_{\theta\theta} = 0 && \omega_{\theta\phi} = -\cos\theta\ d\phi = -\frac{\cot\theta}{r} r\sin\theta\ d\phi \\ &\omega_{\phi\theta} = \cos\theta\ d\phi = \frac{\cot\theta}{r} r\sin\theta\ d\phi && \omega_{\phi\phi} = 0 \end{aligned}} \tag{8, 9}$$

(**b**) Compute $\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ for $i, j = 1, 2$ (and where there is an implicit sum over k)

As in the last homework set, equation (2) combined with the fact that $d\alpha = 0$ for any $\alpha \in \bigwedge^1$ implies $\Omega_{ii} = 0$ for all i. We also have that

$$\Omega_{ji} = -d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\Omega_{ij} \tag{10}$$

Therefore, we need only explicitly calculate one of the curvature 2-forms. For ease of calculation, I will begin in the coordinate basis and then convert back into the orthonormal basis to make the Gauss curvature obvious.

$$\begin{aligned}
\Omega_{\theta\phi} &= d\omega_{\theta\phi} + \omega_{\theta\ k} \wedge \omega_{k\phi} && (11)\\
&= d\omega_{\theta\phi} + \omega_{\theta\theta} \wedge \omega_{\theta\phi} + \omega_{\theta\phi} \wedge \omega_{\phi\phi} && (12)\\
&= d\omega_{\theta\phi} && (13)\\
&= d(-\cos\theta\ d\phi) && (14)\\
&= \sin\theta\ d\theta \wedge d\phi && (15)\\
&= \frac{\sin\theta}{r^2 \sin\theta}\ r d\theta \wedge r\sin\theta\ d\phi && (16)\\
&= \frac{1}{r^2}\ r d\theta \wedge r\sin\theta\ d\phi && (17)\\
&= \frac{1}{r^2}\omega && (18)
\end{aligned}$$

where ω is the orientation of $\mathbb{S}^2$.

In summary, we have found the following

$$\boxed{\begin{array}{ll}
\Omega_{\theta\theta} = 0 & \Omega_{\theta\phi} = \frac{1}{r^2}\omega \\
\Omega_{\phi\theta} = -\frac{1}{r^2}\omega & \Omega_{\phi\phi} = 0
\end{array}} \qquad \begin{array}{r}(19)\\(20)\end{array}$$

From class, we know that the Gaussian curvature of a two dimensional surface is related to these curvature 2-forms by

$$\Omega^1{}_2 = K\omega \tag{21}$$

and $\frac{1}{r^2}$ is precisely the Gaussian curvature of $\mathbb{S}^2$

(**c**) (Optional) Compare your answers (and your computations) with those from the previous homework assignment.

Looking at the previous homework, we see that the connection 1-forms are exactly the same for our computation in $\mathbb{E}^3$ despite the fact that the computation on $\mathbb{S}^2$ does not include any dr components. It is interesting to note that the dr components of the 1-forms in the first structure equation from $d\sigma^i$ exactly cancel out the dr components of the 1-forms from $\omega^i{}_j \wedge \sigma^j$ leaving the ω_{ij} unchanged as we go from $\mathbb{E}^3$ to $\mathbb{S}^2$.

The exact opposite happens for the curvature 2-forms for which the dr components of the 2-forms in the second structure equation exactly cancel in 3-dimensions to give zero for each Ω_{ij}. On the sphere, $dr = 0$ which removes these canceling terms results in non-zero curvature 2-forms.

This whole process illustrates how we can derive the curvature for surfaces in $\mathbb{E}^3$ by considering curvilinear coordinate systems and then setting a particular basis 1-form to 0.