Homework 4B Total Time: 2 hr

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Math 434

Tapp 1.70 prove that the inverse of an orthogonal matrix is an orthogonal matrix and that the product of two orthogonal matrices is orthogonal.

From prop. 1.55 we have that a matrix $A \in M_{n \times n}$ is orthogonal if

$$A^T A = I$$

Suppose that $A \in M_{n \times n}$ is an invertible orthogonal matrix. Then we have

$$A^T A = I$$
$$A^T A A^{-1} = I A^{-1}$$
$$A^T I = I A^{-1}$$
$$A^T = A^{-1}$$

So an orthogonal matrix is a matrix such that its inverse is its transpose. Now taking the inverse of both sides yields

$$(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$$

and thus by the above definition, The inverse to an orthogonal matrix is itself an orthogonal matrix.

$\square$

Now we WTS if $A, B \in O(n)$
then $AB \in O(n)$

To show this we must demonstrate that

$$(AB)^T \cdot (AB) = I$$

Recall that $(AB)^T = B^T A^T$
Thus we have that

$$\begin{aligned}(AB)^T \cdot (AB) &= B^T A^T A B \\ &= B^T (A^T A) B \\ &= B^T (I) B \\ &= B^T B \\ &= I\end{aligned}$$

Thus we have shown $(AB)^T \cdot AB = I$
therefore $AB \in O(n)$. W.L.O.G. we could switch the orders of A, B to show that $BA \in O(n)$ by the same proof. $\square$

Tapp. 1.71

(1) if $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then $\mathcal{L}_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by the angle θ about the origin.

a general vector $\vec{v} \in \mathbb{R}^2$ can be defined as $\vec{v} = \begin{pmatrix} R\cos(\varphi) \\ R\sin(\varphi) \end{pmatrix}$ w/ $R, \varphi \in \mathbb{R}$.

If A is the above matrix then

then $\mathcal{L}_A \vec{v} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \end{pmatrix}$

$$= R \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$= R \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi \end{pmatrix}$$

$$= R \begin{pmatrix} \cos(\theta + \varphi) \\ \sin(\theta + \varphi) \end{pmatrix}$$

$$= \begin{pmatrix} R\cos(\theta+\varphi) \\ R\sin(\theta+\varphi) \end{pmatrix}$$

Thus we have shown that if A is the above matrix then $\mathcal{L}_A : \mathbb{R}^2 \to \mathbb{R}^2$ is infact a rotation about the origin by the angle θ

(2) if $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ then $\mathcal{L}_A : \mathbb{R}^2 \to \mathbb{R}^2$

is a reflection over the line through the origin making an angle of $\theta/2$ w/ the x axis.

We can show this is the correct matrix by simply composing 3 seperate linear operations:

1 Rotate line at $\theta/2$ to x-axis (inverse of 3)
2 Reflect about x axis
3 Rotate Back to $\theta/2$

Since matrices apply from right to left, this operation can be encoded by

$$A = \Theta \circ R \circ \Theta^{-1}$$

where $\Theta \doteq \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$

$$R \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Theta^{-1} \doteq \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$\Theta R \Theta^{-1} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta/2 - \sin^2\theta/2 & 2\cos\theta/2\sin\theta/2 \\ 2\cos\theta/2\sin\theta/2 & -(\cos^2\theta/2 - \sin^2\theta/2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = A$$ 5

thus we have shown A is a reflection about the line that makes an angle $\theta/2$ with the x-axis.

Tapp 1.74 Let $B \in O(n)$ w $\det(B) = -1$ prove that every member of $O(n)$ w/ negative determinant can be written as $A \cdot B$ for some $A \in O(n)$ w/ $\det A = 1$.

Fix $B \in O(n)$ w/ $\det(B) = -1$ we want to show that $\forall\, C \in O(n)$ w/ $\det(C) = -1$ $\exists\, A \in O(n)$ w/ $\det(A) = 1$ s.t.

$$C = AB$$

Recall that $\det(XY) = \det(X)\det(Y)$ so that

$$\begin{aligned} \det(AB) &= \det(A)\det(B) \\ &= 1 \cdot (-1) \\ &= -1 \\ &= \det(C) \end{aligned}$$

Now from problem 1.70 we have that since $A, B \in O(n)$, $AB \in O(n)$. Thus all that remains is to construct A. Observe that

$$\begin{aligned} C &= AB \\ CB^{-1} &= ABB^{-1} \\ CB^{T} &= A \end{aligned}$$

Now since $\det(X) = \det(X^T)$ we have $\det(A) = \det(CB^T) = (-1)^2 = 1$

thus given B we can always construct A such that $C = AB$

$\square$