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Notation Comments

Remark. *The notation $L_A : \mathbb{F}^n \to F^m$ when $A \in M_n(\mathbb{F})$ has the letter L to indicate left multiplication by A on column vectors.*

Remark. *Given any linear operator $T : V \to V$, and finite ordered bases B, C for V. The matrix of T with respect to B and C is denoted $[T]_B^C$. In particular,*

$$[Id_v]_C^B = \left([Id_v]_b^C\right)^{-1} \tag{1}$$

From this,

$$[T]_C^C = [Id_v]_B^C \; [T]_B^B \; [Id_v]_C^B \tag{2}$$
$$= Q^{-1} \; [T]_B^C \; Q \tag{3}$$

where $Q = [Id_v]_C^B$ is the change of basis matrix.

Cosets

If U is a subspace of V and $v \in V$ then the left coset of U in V represented by v is

$$v + U = \{v + u | u \in U\} \tag{4}$$

The **set of left cosets** of U in V is

$$V/U = \{v + U | v \in V\} \tag{5}$$

Note that if $v \in V$ and $u \in U$ then $v + U = (v + u) + U$. Naively, we could hope that

$$V/U \times V/U \to V/U$$
$$(v_1 + U, v_2 + U) \mapsto (v_1 + v_2) + U$$

actually defines a function. We have to be sure that when you choose some v_1', v_2' that the resulting coset is the same... i.e. that we need to check that this really is a function for which inputs have exactly one output. That is,

$$(v_1 + U, v_2 + U) \mapsto v_1 + v_2 + U$$

is well-defined, in the sense that the right hand side value is independent of choice of coset representatives of the initial cosets. Here, if $v_1' = v_1 + u_1$, $v_2' = v_2 + u_2$ with $u_i \in U$. Now

$$v_1' + v_2' + U = \Big[(v_1 + u_1) + (v_2 + u_2)\Big] + U$$

Thus,

$$\begin{aligned} v_1' + v_2' + U &= (v_1 + u_1 + v_2) + (u_2 + U) \\ &= (v_1 + v_2 + u_1) + U \\ &= v_1 + v_2 + U \end{aligned}$$

That is, since addition on V is Abelian, every subgroup U is normal and thus the naive formula does give a well-defined function. We now check if

$$\mathbb{F} \times V/U \to V/U$$
$$(\lambda, v + U) \mapsto \lambda v + U$$

is a well-defined function (**IT IS**). So the family of cosets of subspace U in vector space V is itself a vector space over $\mathbb{F}$.

Lemma. *Suppose U is a vector subspace of V, and B is a basis of U. Let $C = B \cup B'$ be any basis of V extending B. Then, $\{v + U | v \in B'\}$ is a basis of our quotient vector space V/U.*

Proof. Suppose $\sum_i \lambda_i (v_i + U) = 0_{V/U}$ for some $\lambda_1, ..., \lambda_n \in \mathbb{F}$. and $v_1, ..., v_n \in B'$. Since $0_{V/U} = 0_v + U = U$ is our zero vector, thus

$$\Big(\sum_i^n \lambda_i v_i \Big) + U = U$$

This holds if and only if

$$\sum_i^n \lambda_i v_i \in U$$

However, the $v_i \in B'$ and hence are linearly independent of the $sp(B)$. Therefore, this linear combination can only be $0_V \in U$. But C is a basis and thus all of the $\lambda_i = 0$. Note if $U = V$ then V/U is only $\{0_v + U\}$ and one uses logical statements. □

More on cosets

Recall that given U is a subspace of V, we let V/U be the quotient vector space, whose elements are cosets, thus of the form $v + U = \{v + u | u \in U\}$. We call v a coset representative of $v + U$; in general, cosets have many representative. We checked that

$$V/U \times V/U \to V/U$$
$$(v_1 + U, v_2 + U) \mapsto (v_1 + v_2) + U$$
$$\mathbb{F} \times V/U \to V/U$$
$$(\lambda, v + U) \mapsto \lambda v + U$$

are functions. (We needed to check that their values were independent of coset representatives). You check that V/U is then an $\mathbb{F}$-vectorspace.

Recall further that if B is a basis for U and B' is such that $C = B \cup B'$ is a basis of V extending B, then $\{v + U | v \in B'\}$ is a basis for V/U.

Lemma. *Let U be a subspace of V and define*

$$\begin{aligned} \pi : V &\to V/U \\ v &\mapsto v + U \end{aligned}$$

Then π is a surjective linear transformation whose kernel is U.

sketch. We check linearity:

$$\begin{aligned} \pi(\lambda v_1 + v_2) &= (\lambda v_1 + v_2) + U \\ &= (\lambda v_1 + U) + (v_2 + U) \\ &= \lambda(v_1 + U) + (v_2 + U) \\ &= \lambda \pi(v_1) + \pi(v_2) \end{aligned}$$

$\forall \lambda \in \mathbb{F}$, $\forall v_1, v_2 \in V$. Thus the function π is a linear transformation. Surjectivity is clear. Now we check the kernel. If $v \in \ker(\pi)$, then $\pi(v) = 0_{V/U} = 0_v + U$ that is $v + U = U$. Hence $v \in U$. The "other direction" holds equally well; $\ker(\pi) = U$. □

Lemma. *If $T : V \to W$ is a linear transformation, let $\bar{T} : V/\ker(T) \to W$ be given by*

$$\bar{T}(v + \ker(T)) = T(v)$$

$\forall v \in V$. Then, $\bar{T}$ is a linear transformation, which is injective.

Proof. We must first check that $\bar{T}$ is a well-defined function. Suppose $v_1, v_2 \in V$ are such that $v_1 + \ker(T) = v_2 + \ker(T)$. In particular, this means

$$v_1 - v_2 \in \ker(T)$$

Hence $T(v_1 - v_2) = 0_W$. By the linearity of T, $T(v_1) - T(v_2) = 0_w$; that is $T(v_1) = T(v_2)$. Thus $\bar{T}$ is well-defined. Now $\forall \lambda \in \mathbb{F}$, $v_1, v_2 \in V$,

$$\begin{aligned} \bar{T}(\lambda(v_1 + \ker(T)) + (v_2 + \ker(T))) &= \bar{T}(\lambda v_1 + v_2 + \ker(T)) \\ &= T(\lambda v_1 + v_2) \\ &= \lambda T(v_1) + T(v_2) \\ &= \lambda \bar{T}(v_1 + \ker(T)) + \bar{T}(v_2 + \ker(T)) \end{aligned}$$

Thus, $\bar{T}$ is a linear transformation. Recall any linear transformation is injective if and only if its kernel is trivial ($\{0_V\}$).

Suppose $\bar{T}(v + \ker(T)) = 0_W$. Then $T(v) = 0_W$. Hence $v \in \ker(T)$. Thus, $v + \ker(T) = \ker(T) = 0_{V/U}$. Therefore, $\bar{T}$ is injective □

Corollary. *Given a linear transformation $T : V \to W$, we have $V/\ker(T)$ iso $T(V)$*

Proof. By the lemma, $\bar{T} : V/\ker(V) \to W$ is injective. Its range, $R_T = \{T_v | v \in V\}$ Certainly, any linear transformation □