1. The potential due to a ring of charge is given by:

$$V(s,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{s^2+R^2-2sR\cos(\phi-\phi')+z^2}}$$

Expand this potential in a power series to fourth order, in the plane of the ring, for $s < R$. Warning: Make sure you keep **all** of the terms up to fourth order and none of the terms of higher order. This is tricky to do and is the most important lesson from this homework problem.

Solution:

To expand this potential in a power series, it would be nice to save some effort and use the series we have already memorized (Quiz 1). Recall,

$$(1+u)^p = 1 + pu + \frac{p(p-1)}{2!}u^2 + \frac{p(p-1)(p-2)}{3!}u^3 + \frac{p(p-1)(p-2)(p-3)}{4!}u^4 + \cdots \tag{1}$$

We are looking at the potential in the plane of the ring, so $z = 0$. We can also rewrite the square root as a power.

$$V(s,\phi,z=0) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\Big(s^2+R^2-2sR\cos(\phi-\phi')\Big)^{-1/2}d\phi' \tag{2}$$

The integrand is almost the same as equation (1) but we need it to match exactly for the power series to be valid. **Remember:** equation (1) is valid only for $|u| < 1$. We are interested in finding the potential where $s < R$, or in other words, $s/R < 1$ is a small quantity and we can *pull out* R^2 from the expression. That is,

$$\Big(s^2+R^2-2sR\cos(\phi-\phi')\Big)^{-1/2} = \left[R^2\Big(1+\frac{s^2}{R^2}-\frac{2s}{R}\cos(\phi-\phi')\Big)\right]^{-1/2} \tag{3}$$

$$= \frac{1}{R}\Big(1+\frac{s^2}{R^2}-\frac{2s}{R}\cos(\phi-\phi')\Big)^{-1/2} \tag{4}$$

$$u \equiv \frac{s^2}{R^2}-\frac{2s}{R}\cos(\phi-\phi'), \qquad p = -1/2 \tag{5}$$

where in the final line I have identified our u and p for the series expansion.

Now, we need to expand the powers of u in order to find all of the fourth order terms in $\frac{s}{R}$. Yes, this is a lot of algebra.

$$p(u) = \left(\frac{s}{R}\right)\cos(\phi-\phi') - \frac{1}{2}\left(\frac{s}{R}\right)^2 \tag{6}$$

$$\frac{p(p-1)}{2!}u^2 = \frac{5}{2}\left(\frac{s}{R}\right)^2\cos^2(\phi-\phi') - \frac{3}{2}\left(\frac{s}{R}\right)^3\cos(\phi-\phi') + \frac{3}{8}\left(\frac{s}{R}\right)^4 \tag{7}$$

$$\frac{p(p-1)(p-2)}{3!}u^3 = \frac{5}{2}\left(\frac{s}{R}\right)^3\cos^3(\phi-\phi') - \frac{15}{4}\left(\frac{s}{R}\right)^4\cos^2(\phi-\phi/) + ... \tag{8}$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}u^4 = \frac{35}{8}\left(\frac{s}{R}\right)^4\cos^4(\phi-\phi') + ... \tag{9}$$

Using this to combine all terms with like powers in s/R, the integral reduces to

$$
\begin{aligned}
V(s,\phi) \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{R}\int_0^{2\pi}\Bigg\{ & 1 + \cos(\phi-\phi')\left(\frac{s}{R}\right) \\
& + \left[\frac{3}{2}\cos^2(\phi-\phi') - \frac{1}{2}\right]\left(\frac{s}{R}\right)^2 \\
& + \left[\frac{5}{2}\cos^3(\phi-\phi') - \frac{3}{2}\cos(\phi-\phi')\right]\left(\frac{s}{R}\right)^3 \\
& + \left[\frac{3}{8} - \frac{15}{4}\cos^2(\phi-\phi') + \frac{35}{8}\cos^4(\phi-\phi')\right]\left(\frac{s}{R}\right)^4 \Bigg\} d\phi'
\end{aligned}
\tag{10}
$$

The original equation for the potential can not be integrated analytically. Now that we have expanded the integrand, we have reduced the problem to a bunch of integrals of $\cos^n(\phi-\phi')$ which we can solve by brute force. If you wish to do the integrals by hand, take advantage of the exponential form of cosine. Otherwise, Mathematica is a great option for this sort of integral.

In[12]:= $u = \left(\frac{s}{R}\right)^2 - 2 * \left(\frac{s}{R}\right) * \text{Cos}[\phi - \phi_0]$

Out[12]= $\frac{s^2}{R^2} - \frac{2\, s\, \text{Cos}[\phi - \phi_0]}{R}$

Find terms in expansion up to u^4

In[13]:= $p = -(1/2);$

In[14]:= $\text{Expand}\left[\frac{p}{1} * u\right]$

$\text{Expand}\left[\frac{p\,(p-1)}{2\,!} * u\wedge 2\right]$

$\text{Expand}\left[\frac{p\,(p-1)\,(p-2)}{3\,!} * u\wedge 3\right]$

$\text{Expand}\left[\frac{p\,(p-1)\,(p-2)\,(p-3)}{4\,!} * u\wedge 4\right]$

Out[14]= $-\frac{s^2}{2\,R^2} + \frac{s\,\text{Cos}[\phi - \phi_0]}{R}$

Out[15]= $\frac{3\,s^4}{8\,R^4} - \frac{3\,s^3\,\text{Cos}[\phi - \phi_0]}{2\,R^3} + \frac{3\,s^2\,\text{Cos}[\phi - \phi_0]^2}{2\,R^2}$

Out[16]= $-\frac{5\,s^6}{16\,R^6} + \frac{15\,s^5\,\text{Cos}[\phi - \phi_0]}{8\,R^5} - \frac{15\,s^4\,\text{Cos}[\phi - \phi_0]^2}{4\,R^4} + \frac{5\,s^3\,\text{Cos}[\phi - \phi_0]^3}{2\,R^3}$

Out[17]= $\frac{35\,s^8}{128\,R^8} - \frac{35\,s^7\,\text{Cos}[\phi - \phi_0]}{16\,R^7} + \frac{105\,s^6\,\text{Cos}[\phi - \phi_0]^2}{16\,R^6} - \frac{35\,s^5\,\text{Cos}[\phi - \phi_0]^3}{4\,R^5} + \frac{35\,s^4\,\text{Cos}[\phi - \phi_0]^4}{8\,R^4}$

Collect like terms of $\left(\frac{s}{R}\right)^k$

In[18]:=

```
Integrand1 = s Cos[ϕ - ϕ0] / R
Integrand2 = - s^2/(2 R^2) + 3 s^2 Cos[ϕ - ϕ0]^2/(2 R^2)
Intergrand3 = - 3 s^3 Cos[ϕ - ϕ0]/(2 R^3) + 5 s^3 Cos[ϕ - ϕ0]^3/(2 R^3)
Integrand4 = 3 s^4/(8 R^4) - 15 s^4 Cos[ϕ - ϕ0]^2/(4 R^4) + 35 s^4 Cos[ϕ - ϕ0]^4/(8 R^4)
```

$$\text{Integrand1} = \frac{s \operatorname{Cos}[\phi - \phi_0]}{R}$$

$$\text{Integrand2} = -\frac{s^2}{2 R^2} + \frac{3 s^2 \operatorname{Cos}[\phi - \phi_0]^2}{2 R^2}$$

$$\text{Intergrand3} = -\frac{3 s^3 \operatorname{Cos}[\phi - \phi_0]}{2 R^3} + \frac{5 s^3 \operatorname{Cos}[\phi - \phi_0]^3}{2 R^3}$$

$$\text{Integrand4} = \frac{3 s^4}{8 R^4} - \frac{15 s^4 \operatorname{Cos}[\phi - \phi_0]^2}{4 R^4} + \frac{35 s^4 \operatorname{Cos}[\phi - \phi_0]^4}{8 R^4}$$

Out[18]= $$\frac{s \operatorname{Cos}[\phi - \phi_0]}{R}$$

Out[19]= $$-\frac{s^2}{2 R^2} + \frac{3 s^2 \operatorname{Cos}[\phi - \phi_0]^2}{2 R^2}$$

Out[20]= $$-\frac{3 s^3 \operatorname{Cos}[\phi - \phi_0]}{2 R^3} + \frac{5 s^3 \operatorname{Cos}[\phi - \phi_0]^3}{2 R^3}$$

Out[21]= $$\frac{3 s^4}{8 R^4} - \frac{15 s^4 \operatorname{Cos}[\phi - \phi_0]^2}{4 R^4} + \frac{35 s^4 \operatorname{Cos}[\phi - \phi_0]^4}{8 R^4}$$

Calculate the nasty integral

In[22]:=

$$\text{Int} = \text{Expand}\Big[\int_0^{2\pi} \Big(1 + \operatorname{Cos}[\phi - \phi_0] * \left(\frac{s}{R}\right) + \left(\frac{3}{2} \operatorname{Cos}[\phi - \phi_0]^2 - \frac{1}{2}\right) * \left(\frac{s}{R}\right)^2 + \left(\frac{5}{2} \operatorname{Cos}[\phi - \phi_0]^3 - \frac{3}{2} \operatorname{Cos}[\phi - \phi_0]\right) * \left(\frac{s}{R}\right)^3 + \left(\frac{3}{8} - \frac{15}{4} \operatorname{Cos}[\phi - \phi_0]^2 + \frac{35}{8} \operatorname{Cos}[\phi - \phi_0]^4\right) * \left(\frac{s}{R}\right)^4\Big) \, d\phi_0\Big]$$

Out[22]= $$2\pi + \frac{\pi s^2}{2 R^2} + \frac{9 \pi s^4}{32 R^4}$$

Multiply by all of the constants to get the potential --> V= $\frac{Q}{4\pi\epsilon_0} * \frac{1}{2\pi}$ *Integral

In[23]:=

In[24]:= $$V = \frac{Q}{4\pi\epsilon_0} * \text{Expand}\left[\frac{1}{2\pi} * \text{Int}\right]$$

Out[24]= $$\frac{Q\left(1 + \frac{s^2}{4 R^2} + \frac{9 s^4}{64 R^4}\right)}{4\pi\epsilon_0}$$

Therefore, our solution for the electric potential in the plane of the ring to fourth order in s is

$$V(s,\phi,z=0) \approx \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{R} + \frac{1}{4}\frac{s^2}{R^3} + \frac{9}{64}\frac{s^4}{R^5}\right\} \tag{11}$$

NOTE: Our solution does not depend on ϕ *and* is an even function in s. Why is that?

CHECK: Does our solution agree with the original integral equation at the origin?