Math 443/543, HW #2; due Friday, Oct. 12

1.) a.) Clearly state under what conditions the range and null space of a linear transformation T are the same set.

b.) Prove your assertion.

c.) Give an example.

2.) Let V and W be finite dimensional vector spaces, and suppose that U is a vector subspace of V. Prove that there exists a surjective linear transformation from V to W whose nullspace is U if and only if $\dim U = \dim V - \dim W$.

3.) (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformations. What is the rank of the composition, UT ? (That is, list all possibilities, and prove that your list is complete and correct.)

(b) (543) Generalize this.

4.) Given $T : V \to V$ a linear transformation and W a subspace of V, we say that W is T-*invariant* if $Tw \in W$ for all $w \in W$.

a.) Prove that the range, R_T, and the nullspace, N_T, are T-invariant.

b.) Suppose that W is a k-dimensional T-invariant subspace of V. Show that there is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ has the form $\begin{pmatrix} A & B \\ O & D \end{pmatrix}$, where $A \in \mathcal{M}_{k,k}(\mathbb{F})$ and O is the $(n-k) \times k$ zero matrix.