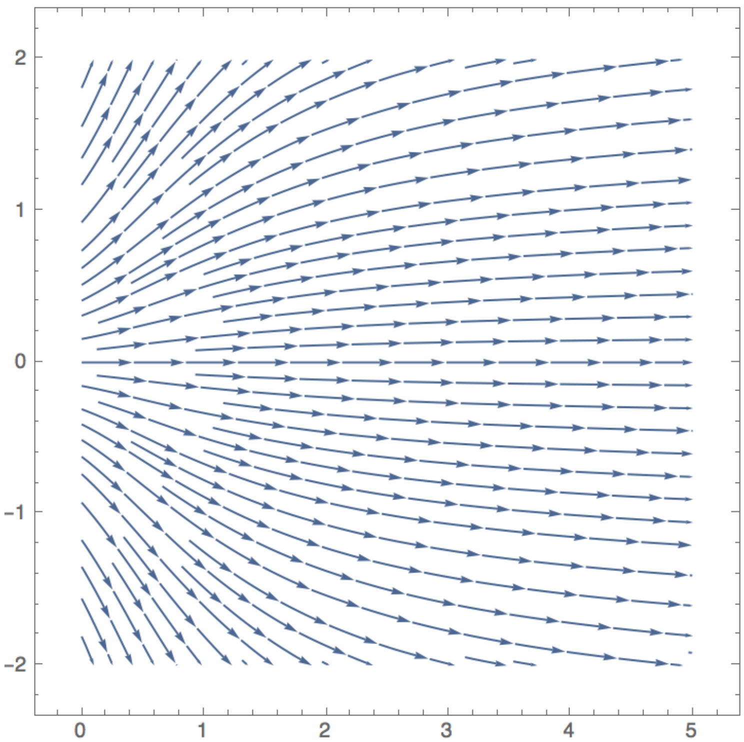
Week 1 Wrap-up

Slope Fields

Suppose we have $y'(t) = \dfrac{y(t)}{1+t^2}$.

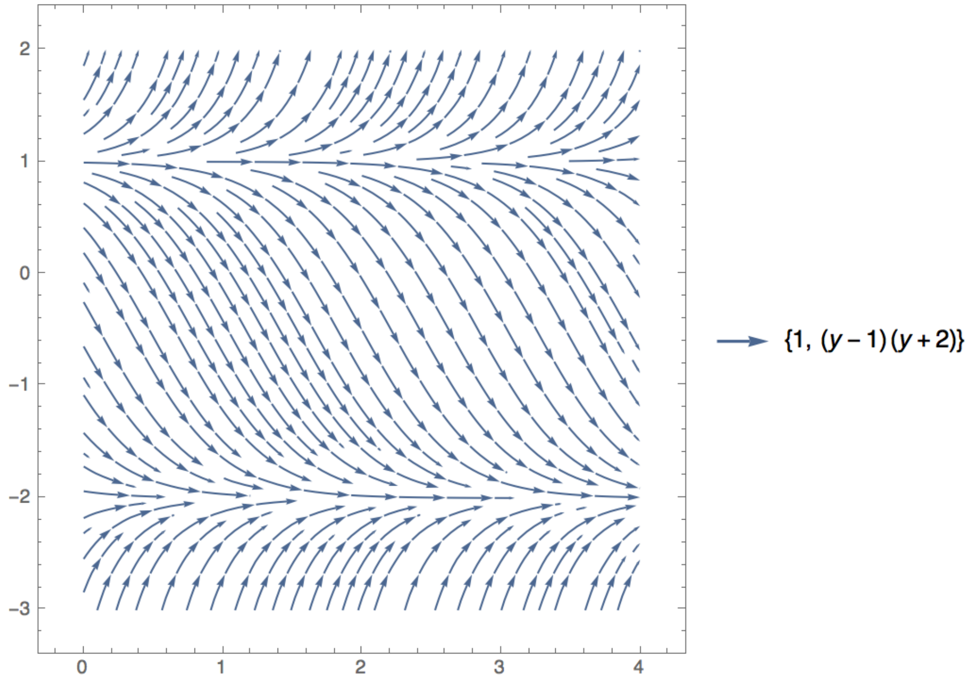
The left hand side of the differential equation is the slope of the solution. The right hand side gave us a function of two variables to evaluate the slope at various points in the ty-plane.



$\longrightarrow \left\{1, \frac{y}{1+t^2}\right\}$

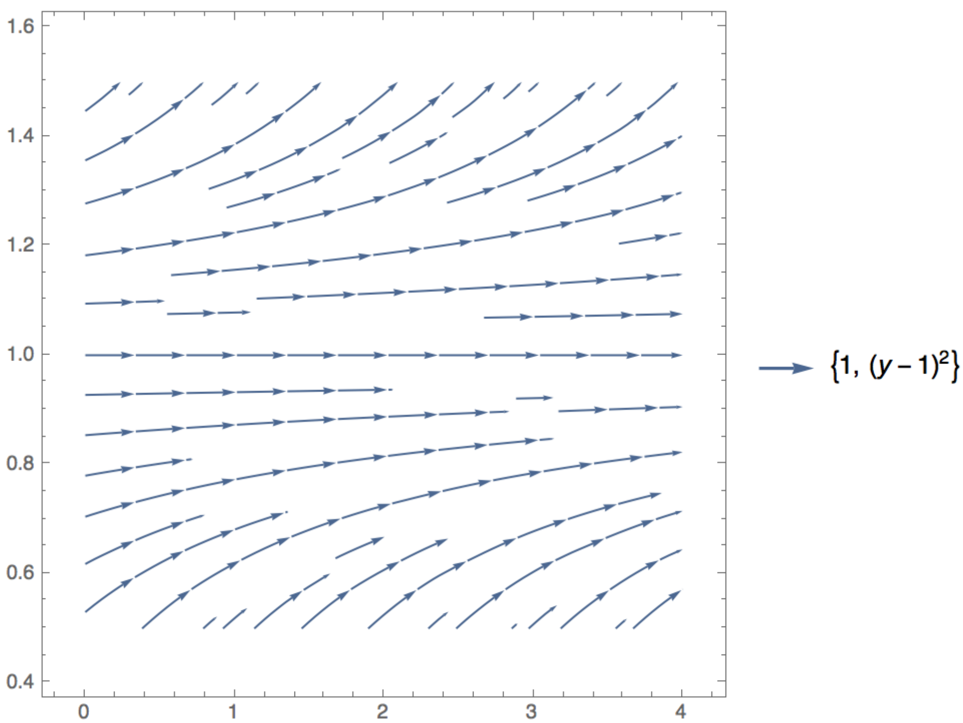
We can then see how solutions of our equation will flow through the ty-plane once we choose a point in space we want our solution to pass through. We will find a solution of this equation **next week (https://oregonstate.instructure.com/courses/1589230/pages/week-2-wrap-up)** .

Consider the next equation $y'(t) = (y-1)(y+2)$.



We can see by inspection that the constant solutions, when $y'(t) = 0$, are given by $y(t) = 1$ or $y(t) = -2$. These constant solutions are the equilibrium solutions of the differential equation. We can then classify them as being stable, semi-stable, or unstable. In this case $y(t) = -2$ is stable since it attracts solutions from above and below and $y(t) = 1$ is unstable because the slope fields repels solutions that approach it.

To get a semi-stable solution we could use $y'(t) = (y(t) - 1)^2$. Here $y'(t) \geq 0$ and thus $y(t) = 1$ is semi-stable.



Linear Equation With Constant Coefficients

The final differential equation we studied was of the form

$$\frac{dy}{dt} = ay(t) - b$$

and the solutions of such an equation are

$$y(t) = Ce^{at} + \frac{b}{a}.$$

You can verify that the solutions satisfy the differential equation. The equilibrium solution is $y = \frac{b}{a}$.

Differential Equations

We saw how to find solutions of equations of the form

$$\frac{dy}{dt} = f(t, y(t))$$

where $y(t)$ is our unknown function. The simplest cases were handled throughout MTH 252,

$$\frac{dy}{dt} = f(t)$$

then $y = F(t) + c$.

A differential equation is linear if it can be written in the form

$$y'(t) + p(t)y(t) = g(t)$$

otherwise it is nonlinear. If p and g are constant then the differential equation is linear, first order, with constant coefficients. It is important to be able to determine whether a differential equation is linear or not. A differential equation can have a sine term and still be linear,

$$y'(t) + \sin(t)y(t) = \ln(t)$$

is linear. However,

$$y'(t) + \sin(y(t)) = \ln(t)$$

and

$$y'(t)y^2(t) = \sin(t)$$

are nonlinear.

The order of a differential equation is simply the order of the highest derivative term. Here

$$y'''(t) = y''(t) + y(t)$$

is third order and linear. We can also have partial differential equations,

$$u_{tt} = u_{xx} + u_{yy} = \nabla \cdot \nabla u$$

or

$$u_t = \nabla \cdot \nabla u$$

where u is some vector valued function.

Week 2 Wrap-up

First Order Linear Equations

To begin this chapter we start with any linear, first order differential equation,

$$y'(t) + p(t)y(t) = g(t).$$

We introduce an auxiliary function $\mu(t)$ such that

$$\frac{d}{dt}\mu(t)y(t) = \mu(t)y'(t) + \mu'(t)y(t)$$

where

$$\mu'(t) = p(t)\mu(t).$$

We will focus on finding $\mu(t)$ in a moment but first let's multiply our original equation by $\mu(t)$,

$$\begin{aligned} \mu(t)y'(t) + \mu(t)p(t)y(t) &= \mu(t)g(t) \\ \mu(t)y'(t) + \mu'(t)y(t) &= \\ \frac{d}{dt}\mu(t)y(t) &= \end{aligned}.$$

If we integrate both sides we find

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)\ dt.$$

Now to find $\mu(t)$, the integrating factor, we compute

$$\mu(t) = e^{\int p(t)\ dt}.$$

For this process we only concern ourselves with the constant of integration with regards to the solution of the equation and not with regards to finding the integrating factor.

If we have $y'(t) + 2y(t) = t^2$ then $p(t) = 2$ and $\mu(t) = e^{2t}$. This gives us a solution

$$y(t) = \frac{1}{e^{2t}}\int t^2e^{2t}\ dt$$

and to compute the integral we need to invoke integration by parts to find

$$\begin{aligned} y(t) &= \frac{1}{e^{2t}}\left(t^2e^{2t} - 2t\frac{1}{2}e^{2t} + 2\frac{1}{8}e^{2t} + C\right) \\ &= t^2 - 2t + \frac{1}{4} + Ce^{-2t} \end{aligned}.$$

Let's review the process of integration by parts.

Integration by Parts

In this course we encounter integrals of the form

$$\int u\ dv = uv - \int v\ du$$

$$u = f(x)$$
$$dv = g'(x)\ dx$$

or the product of two functions. With this method there is an art for picking the u's and dv's. Here is how I prioritize my choice of dv

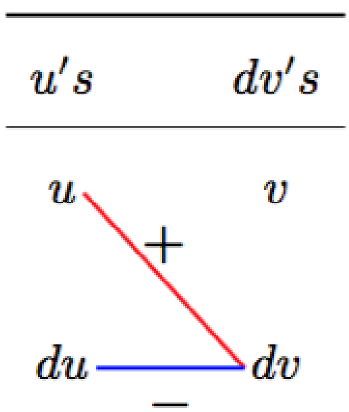
1. Trigonometric or an exponential function. If both types appear in the expression I will be invoking integration by parts twice and doing some algebra as shown below.
2. Polynomials.
3. Logarithms or a function just appearing by itself. If the function just appears by itself you can let dv = 1, a polynomial, and u be the function you are trying to integrate.

Tabular Integration by Parts

We can summarize the integration by formula

$$\int u\ dv = uv - \int v\ du$$

by the table



The diagonal connected entries represent multiplication, u dv, and the entries connected by items in the same row represent the integral of v du. The addition and subtraction alternates as we move down the table.

When applying integration by parts is sometimes necessary to apply the technique multiple times. In this situation the algebra can be difficult to keep track of after multiple substitutions and the alternating signs

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$
$$= f(x)g(x) - f'(x)G(x) + \int f''(x)G(x)\ dx.$$

and this can be represented by the table below.

$u's$		$dv's$
$f(x)$	$+$	$g'(x)$
$f'(x)$	$-$	$g(x)$
$f''(x)$	$+$	$G(x)$

Integration by Parts Applied Twice

In lecture we saw that to compute

$$\int \sin(x)e^{2x}\ dx$$

it was helpful to apply integration by parts two times and recognize that we have the some multiple of the above integral on the other side of our equation. The tabular method can help us with that computation too.

$u's$		$dv's$
$\sin(x)$	$+$	e^{2x}
$\cos(x)$	$-$	$\frac{1}{2}e^{2x}$
$-\sin(x)$	$+$	$\frac{1}{4}e^{2x}$

Then

$$\begin{aligned}
\int \sin(x)e^{2x}\ dx &= \sin(x)\frac{1}{2}e^{2x} - \cos(x)\frac{1}{4}e^{2x} - \int \sin(x)\frac{1}{4}e^{2x}\ dx \\
\int \sin(x)e^{2x}\ dx + \frac{1}{4}\int \sin(x)e^{2x}\ dx &= \sin(x)\frac{1}{2}e^{2x} - \cos(x)\frac{1}{4}e^{2x} + C \\
\frac{5}{4}\int \sin(x)e^{2x}\ dx &= \sin(x)\frac{1}{2}e^{2x} - \cos(x)\frac{1}{4}e^{2x} + C \\
\int \sin(x)e^{2x}\ dx &= \frac{4}{5}\left(\sin(x)\frac{1}{2}e^{2x} - \cos(x)\frac{1}{4}e^{2x}\right) + C'.
\end{aligned}$$

Integration by Parts Applied Many Times

In situations where integration by parts has to be applied multiple times, for instance a trig function multiplied by a polynomial, the tabular method of integration by parts is very useful.

Find

$$\int x^2 e^{2x}\ dx$$

then our table is

$u's$		$dv's$
x^2	$+$	e^{2x}
$2x$	$-$	$\frac{1}{2}e^{2x}$
2	$+$	$\frac{1}{4}e^{2x}$
0		$\frac{1}{8}e^{2x}$

and therefore

$$\int x^2 e^{2x}\ dx = x^2 e^{2x} - 2x\frac{1}{2}e^{2x} + 2\frac{1}{8}e^{2x} + C.$$

This is the very formula we used in our integrating factor example,

$$y'(t) + 2y(t) = t^2$$

which had solution

$$y(t) = \frac{1}{e^{2t}} \int t^2 e^{2t}\ dt.$$

Separable Equations

The differential equation

$$\frac{dy}{dt} = \frac{y}{1+t^2}$$

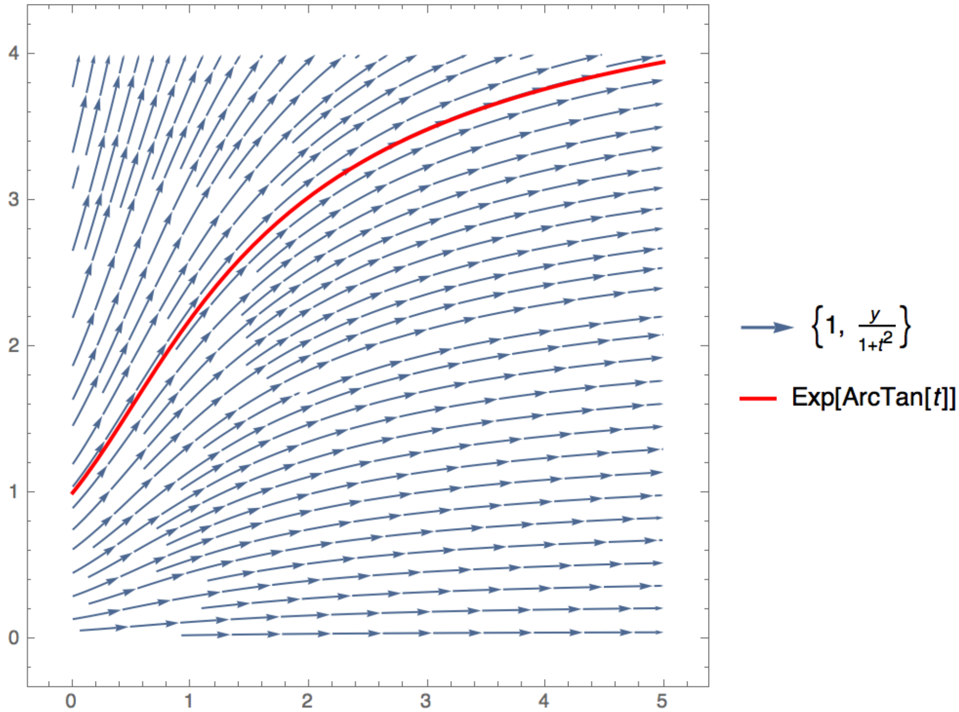
can be separated into

$$\frac{dy}{y} = \frac{dt}{1+t^2}$$
$$\int \frac{dy}{y} = \int \frac{dt}{1+t^2}.$$

Once we integrate both sides we get

$$\ln(y) = \arctan(t) + C_1$$
$$y(t) = e^{\arctan(t)+C_1}$$
$$y(t) = Ce^{\arctan(t)}$$

as our solution to the differential equation. Here is our solution that passes through the point (0, 1).



The process works for any differential equation of the form

$$\frac{dy}{dt} = \frac{f(t)}{g(y)}$$

or

$$\int g(y)\, dy = \int f(t)\, dt.$$

Homogeneous Nonlinear Equations

If we have

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

then the equation is homogeneous. Then we can use the change of variables $v = \frac{y}{x}$ and thus $\frac{dy}{dx} = v + x\frac{dv}{dx}$ by the product rule. This yields a new separable differential equation

$$\begin{gathered} v + x\frac{dv}{dx} = f(v) \\ \frac{dv}{f(v) - v} = \frac{dx}{x} \end{gathered}.$$

If

$$y'(x) = \csc\left(\frac{y}{x}\right) + \frac{y}{x}$$

then

$$\frac{dv}{\csc(v)+v-v} = \frac{dx}{x}$$
$$\sin(v)\ dv = \frac{dx}{x}$$
$$-\cos(v) = \ln(x) + C$$

and therefore in our original variables y and x the solution is

$$-\cos\left(\frac{y}{x}\right) = \ln(x) + C.$$

Solving Nonlinear Equations

Suppose we wanted to find an explicit solution to the equation

$$-\cos\left(\frac{y}{x}\right) = \ln(x) + C$$

which passed through the ordered pair $x = \frac{3}{\pi}, y = -1$. Then we can adjust C such that this occurs,

$$-\cos\left(-\frac{\pi}{3}\right) = \ln\left(\frac{3}{\pi}\right) + C$$
$$-\frac{1}{2} - \ln\left(\frac{3}{\pi}\right) = C$$

Then we could try to express the above relationship, or curve, as a function of x,

$$y = x\arccos\left(-\ln(x) + \frac{1}{2} + \ln\left(\frac{3}{\pi}\right)\right).$$

But then if $x = \frac{3}{\pi}$ we see $y = 1 \neq -1$. What went wrong?

The trap is that $\arccos(x)$ returns a reference angle θ and that angle has to be in $[0, \pi]$, in particular $\arccos(\cos(\theta)) \neq \theta$ for all θ, only $\theta \in [0, \pi]$. Therefore to solve the equation

$$\cos(\theta) = x$$

the solutions are $\theta = \arccos(x) + 2\pi k$ for angles in quadrant I or II and $\theta = -\arccos(x) + 2\pi k$ for angles in quadrant III or IV. This is true because we know that cosine is even and thus $\cos(\theta) = \cos(-\theta)$.

Since our ordered pair is in IV we have to use the function

$$y = -x\arccos\left(-\ln(x) + \frac{1}{2} + \ln\left(\frac{3}{\pi}\right)\right).$$

Week 3 Wrap-up

Application Problems

We took a break from developing new methods to solve differential equations and began looking at problems that can be modeled by first order equations.

Consider a mixing tank that has initial volume V_0 and a mixture of water flowing in at a_{in} liters per minute with a concentration of salt of $s_{in}(t)$ kgs per liter. At the other end of the tank the mixture exits at a rate of a_{out} liters per minute. Then the change in salt in the tank, $Q'(t)$, is given by

$$Q'(t) = a_{in}s_{in}(t) - a_{out}s_{out}(t).$$

We need to uncover the the value of $s_{out}(t)$, the concentration of salt in the tank at time t. The concentration of salt in the tank is

$$s_{out}(t) = \frac{Q(t)}{V(t)} = \frac{Q(t)}{V_0 + (a_{in} - a_{out})t}$$

since the tank's volume is changing a constant rate of $a_{in} - a_{out}$ liters per minute. If we assume that the tank starts out with Q_0 kgs of salt then our linear initial value problem is

$$\begin{aligned} Q'(t) + a_{out}\frac{Q(t)}{V_0 + (a_{in} - a_{out})t} &= a_{in}s_{in}(t) \\ Q(0) &= Q_0 \end{aligned}$$

which has an integrating factor

$$\begin{aligned} \mu &= e^{\int \frac{a_{out}}{V_0 + (a_{in} - a_{out})t}\ dt} \\ &= (V_0 + (a_{in} - a_{out})t)^{\frac{a_{out}}{a_{in} - a_{out}}} \end{aligned}.$$

If $V_0 = 40$, $s_{in}(t) = t$, $a_{in} = 3$, and $a_{out} = 1$ we get

$Q(t) = \frac{2}{5}(t + 20)(3t - 40) + \frac{C}{\sqrt{20 + t}}$. Naturally, we could extend this situation to having variable a_{in} and a_{out} at the cost of making the integrating factor harder to compute.

Existence and Uniqueness

We have two statements of a our existence and uniqueness result.

Let

$$\begin{aligned} y'(t) &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}$$

then if f is continuous near (t_0, y_0) we have a differentiable solution. If f_y is continuous near (t_0, y_0) then we have a unique solution. A special case of this theorem handles the linear case

$$\begin{aligned} y'(t) + p(t)y(t) &= g(t) \\ y(t_0) &= y_0 \end{aligned}.$$

If p and g are continuous near t_0 then we have a unique solution that is differentiable. Note that are theorems are based on our differential equation being in a standard form where $y'(t)$ is isolated from the other terms

$$ty'(t) + \sin(t)y(t) = 3$$

looks like it has no problems at all because every term is continuous. However, we need to study the related equation

$$y'(t) + \frac{\sin(t)}{t}y(t) = \frac{3}{t}$$

which is undefined for $t = 0$. Therefore, if we want $y(2) = \pi$ then our solution will be valid for $t > 0$. Likewise, for $y(-1) = 7$ our solution is valid for $t < 0$. The existence and uniqueness result is harder to study for the nonlinear differential equations since the issue of continuity in several variables is tricky when there is division by zero. Observe that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2} \quad \text{d.n.e}$$

since

$$\frac{xy}{x^2+3y^2} \to \frac{1}{4}$$

along the line $y = x$ and

$$\frac{xy}{x^2+3y^2} \to 0$$

along the line $x = 0$. In practice, we rarely encounter this problem in MTH 256 and mainly worry about taking square roots of negative numbers and division by zero.

Bernoulli Equations

If we have the equation

$$y'(t) + p(t)y(t) = q(t)y^n(t)$$

we can invoke the change of variables $u = y^{1-n}$ and transform the differential equation into

.

This change of variables allows us to solve the original nonlinear equation using standard methods we developed earlier. For example,

$$y'(t) - \frac{1}{2t}y(t) = y^3$$

transforms to

$u'(t) + \frac{1}{t}u(t) = -2$. From here it is trivial to show that $u(t) = -t + \frac{C}{t} = \frac{1}{y^2(t)}$.

Week 4 Wrap-up

Autonomous Differential Equations

We studied equations that had no direct dependency on t

$$y'(t) = f(y).$$

For the most part, this discussion was a review of the slope field material we saw in **week 1 (https://oregonstate.instructure.com/courses/1589230/pages/week-1-wrap-up)** . One new step we made was in discovering that a differential equation can lead to higher order differential equations,

$$y'(t) = \cos(y(t))$$

implies

$$y''(t) = -\sin(y(t))y'(t) = -\sin(y(t))\cos(y(t)).$$

Therefore we can determine that if $y(t) \in (0, \pi/2)$ then our solution is concave down, $y''(t) < 0$.

Logistics Equation

If we have the equation

$$y' = ry\ln\left(\frac{k}{y}\right)$$

then our equilibrium solutions are $y = 0$ and $y = k$. In our model we assume that $r, k > 0$. If we make a change of variables $u = \ln\left(\frac{y}{k}\right)$ and $u' = \frac{y'}{y}$, transforms our equation to

$$u' = -ru$$

and this lead to the general solution

$$y(t) = ke^{e^{-rt+c}}.$$

Thus all solutions will tend towards the equilibrium solution $y = k$ provided $y(0) > 0$. The value of k gives us the carrying capacity of the system and r provides the growth rate.

Exact Differential Equations

Let's assume that the curve $F(t, y) = 0$ defines y as differentiable with respect to t. You learned in MTH 254 that we can perform implicit differentiation using the following result,

$$y'(t) = -\frac{F_t}{F_y}.$$

If

$$t^3y + 3ty^2 - 4 = F(t, y)$$

then

$$y'(t) = -\frac{6t^2y + 3y^2}{t^3 + 6ty}.$$

Therefore if we have some nonlinear differential equation and we recognize that the ratio of the terms correspond to a function $F(t, y)$'s partial derivatives then it is trivial for us to recover the function F. We call such differential equations that satisfy this condition exact. (Please note that the notation that I use is consistent with the theorem from MTH 254 and not with the notation that appears in Boyce Di-Prima or Trench.)

Given

$$y'(t) = -\frac{y^2+t}{1+2ty}$$

we see that

$$\int y^2 + t \; dt = y^2 t + \frac{t^2}{2} + C(y)$$

and then

$$\frac{\partial}{\partial y}\left(y^2 t + \frac{t^2}{2} + C(y)\right) = 2yt + C'(y).$$

If we let $C'(y) = 1$ then we find that $C(y) = y + c$. Therefore the curve

$$F(t, y) = y^2 t + \frac{t^2}{2} + y + c$$

satisfies

$$y'(t) = -\frac{F_t}{F_y}.$$

In general, the equation

$$y' = -\frac{M(x,y)}{N(x,y)}$$

is exact iff

$$M_y = N_x.$$

An Exact Equation Simplified isn't Exact

In some cases our terms F_t and F_y have some common factors that can be removed when we simplify

$$y'(t) = -\frac{F_t}{F_y}.$$

Consider

$$F(t, y) = y^3 t + \frac{t^2}{2} y^2$$

then

$$y'(t) = -\frac{ty^2 + y^3}{t^2 y + 3ty^2} = -\frac{ty + y^2}{t^2 + 3ty}$$

since there is a common factor of y in the ratio. Now if we try to integrate the simplified numerator in t and then differentiate in y we discover

$$\begin{aligned}\frac{\partial}{\partial y}\int yt+y^2\ dt &= \frac{\partial}{\partial y}\left(\frac{t^2}{2}y+y^2t+C(y)\right)\\ &= \frac{t^2}{2}+2yt+C'(y)\\ &\neq t^2+3y(t).\end{aligned}$$

Therefore it would appear that our simplified expression for $y'(t)$ is not exact even though we constructed it using the theorem for implicit differentiation. The question then becomes, can we recover any terms lost to the simplification process to give us back an exact differential equation?

Surprisingly, the answer to this question relies on us to construct an auxiliary differential equation. We used this very same process when we solved the linear differential equation using an unknown integrating factor μ where $\mu'(t) = \mu(t)p(t)$.

Integrating Factors

To recover an exact equation from its simplified form we introduce a special function μ where

$$y' = -\frac{\mu F_t}{\mu F_y}$$

is exact and this occurs iff

$$(\mu F_t)_y = (\mu F_y)_t.$$

If we expand these terms using the product rule we get

$$\mu_y F_t + \mu F_{ty} = \mu_t F_y + \mu F_{yt}$$

and then we make a decision that μ only depends on either t or y, but not both. Let

$$\mu_y = 0$$

then

$$\mu_t F_y + (F_{yt} - F_{ty})\mu = 0$$

is a linear differential equation for μ. Thus we solve

$$\mu_t + \frac{(F_{yt}-F_{ty})}{F_y}\mu = 0$$

provided that the term

$$\frac{(F_{yt}-F_{ty})}{F_y}$$

only depends on t. If we instead assume that

$$\mu_t = 0$$

we solve

$$\mu_y + \frac{(F_{ty}-F_{yt})}{F_t}\mu = 0$$

provided

$$\frac{(F_{ty}-F_{yt})}{F_t}$$

only depends on y.

In the case where

$$y'(t) = -\frac{y}{2t - ye^y} = -\frac{F_t}{F_y}$$

we have

$$\begin{aligned} \mu_y + \frac{(F_{ty} - F_{yt})}{F_t}\mu = 0 \\ \mu_y + \left(\frac{1-2}{y}\right)\mu = \\ \mu_y - \left(\frac{1}{y}\right)\mu = \end{aligned}$$

and therefore $\mu(y) = y$. Thus

$$y'(t) = -\frac{y^2}{2ty - y^2e^y}$$

is exact since

$$\frac{\partial}{\partial y}y^2 = 2y = \frac{\partial}{\partial t}\left(2ty - y^2e^y\right).$$

It is not hard to show that the general solution to the exact equation is

$$F = ty^2 - y^2e^y + 2ye^y - 2e^y + C = 0.$$

Week 5 Wrap-up

2nd Order, Homogeneous, with Constant Coefficients

If we have

$$\begin{aligned} y''(t) + by'(t) + cy(t) &= 0 \\ y(t_0) &= y_0 \\ y'(t_0) &= y'_0 \end{aligned}$$

then there is an associated characteristic polynomial

$$c(r) = r^2 + br + c.$$

The roots of $c(r)$ are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

and if $b^2 - 4c > 0$ the roots are distinct. If the roots are distinct then they help form the general solution to the differential equation

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where

$c(r_1) = c(r_2) = 0$ and $r_1 \neq r_2$.

If $y'' - 4y' - 5y = 0$ then $c(r) = r^2 - 4r - 5 = (r-5)(r-1)$ and our roots of the characteristic polynomial are $r = 1, 5$. Thus our general solution is

$$y(t) = C_1 e^t + C_2 e^{5t}.$$

Now suppose that $y(0) = 2$ and $y'(0) = -3$ then we need to pick C_1 and C_2 to fit the initial conditions. First we need to compute the derivative of our solution, $y'(t) = C_1 e^t + 5e^{5t}$

and therefore

$$\begin{aligned} y(0) = C_1 e^0 + C_2 e^0 &= C_1 + C_2 = 2 \\ y'(0) = C_1 e^0 + 5C_2 e^0 &= C1 + 5C_2 = -3 \end{aligned}.$$

We can then solve this system to find $C_1 = \dfrac{13}{4}$ and $C_2 = -\dfrac{5}{4}$.

Existence and Uniqueness for 2nd Order Linear Equations

If

$$\begin{aligned} y''(t) + p(t)y'(t) + q(t)y(t) &= g(t) \\ y(t_0) &= y_0 \\ y'(t_0) &= y'_0 \end{aligned}$$

then we have existence and uniqueness provided $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval containing t_0. This theorem is a natural extension of what we saw in Section 2.4 and it generalizes to all higher order, linear equations.

Wronskian

We want to introduce a function that can determine if the initial conditions can be satisfied by a linear combination of solutions to the differential equation. The Wronskian determinant is defined by

$$W(f(x), g(x)) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - f'(x)g(x)$$

and it follows the same determinant rules seen in MTH 254 and MTH 306.

Then $W(y_1(t_0), y_2(t_0)) \neq 0$ iff there exists C_1 and C_2 such that $y(t) = C_1 y_1(t) + C_2 y_2(t)$ satisfy the homogenous IVP

$$\begin{aligned} y''(t) + p(t)y'(t) + q(t)y(t) &= 0 \\ y(t_0) &= y_0 \\ y'(t_0) &= y_0' \end{aligned}$$

In this case we say that y_1 and y_2 are fundamental solutions of the differential equation, or they form the general solution of the differential equation. The Wronskian's job is to determine if our two solutions are linearly independent, that is, can the two functions generate all possible solutions by having the vectors

$$\begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \quad \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$$

not be multiples of each other.

Let $y_1(t) = e^{3t}$ and $y_2 = 4e^{3t}$, then $W(y_1, y_2) = 3e^{3t}4e^{3t} - 12e^{3t}e^{3t} = 0$ for all t. Thus $y(t) = C_1 e^3 t + C_2 4e^{3t}$ cannot be the general solution of any 2nd order, linear and homogenous differential equation. This shouldn't be a surprise because $C_1 e^3 t + C_2 4e^{3t} = Ce^{3t}$ or the functions are multiples of each other.

Abel's Theorem

If

$$y'' + p(t)y' + q(t)y = 0$$

then

$$W = Ce^{\int -p(t)\, dt}$$

provided that $p(t)$ and $q(t)$ are continuous.

Suppose $y''(t) + ty'(t) - \sin(t)y = 0$ then $p(t) = t$ and thus $W = Ce^{-1/2t^2}$.

If we know that $W = t^2 \sin(t)$ and $y_1 = t$ is a solution then we can recover y_2 by a direct computation.

$W(y_1, y_2) = ty_2' - 1y_2 = t^2 \sin(t)$. This is a 1st order linear differential equation for y_2 and we can use an integrating factor to show that $y_2(t) = Ct - t\cos(t)$.

Week 6 Wrap-up

Complex Roots

Let $c(r) = ar^2 + br + c$ where $b^2 - 4ac < 0$. Then the characteristic polynomial has complex roots, $r = \lambda \pm iu$, and a fundamental solution set of the differential equation $ay'' + by' + cy = 0$ is given by

$$y_1(t) = e^{(\lambda+ui)t} \quad y_2(t) = e^{(\lambda-ui)t}$$

and a straightforward calculation shows that $W\left(e^{(\lambda+ui)t}, e^{(\lambda-ui)t}\right) \neq 0$ for $u \neq 0$.

Now to remove the dependency on i we compute a new fundamental solution set

$$\frac{y_1 + y_2}{2} = e^{\lambda t}\cos(ut)$$
$$\frac{y_1 - y_2}{2i} = e^{\lambda t}\sin(ut)$$

and this gives us the general solution

$$y(t) = e^{\lambda t}\left(C_1\cos(ut) + C_2\sin(ut)\right).$$

To find solutions to the associated IVP we have to compute

$$y'(t) = \lambda e^{\lambda t}C_1\cos(ut) + e^{\lambda t}C_1 u(-\sin(ut)) + \lambda e^{\lambda t}C_2\sin(ut) + e^{\lambda t}C_2 u\cos(ut)$$

in order to find a system of equations for C_1 and C_2.

If $c(r) = r^2 - 2r - 15$ then the complex roots are $r = 1 \pm 4i$. If we have initial conditions $y(0) = 2$ and $y'(0) = 3$ we get

$$\begin{aligned} C_1 &= 2 \\ C_1 + 4C_2 &= 3 \end{aligned}$$

and therefore $C_1 = 2$ and $C_2 = \frac{1}{4}$. Our particular solution to

$$\begin{aligned} y''(t) - 2y'(t) - 15y(t) &= 0 \\ y(t_0) &= 2 \\ y'(t_0) &= 3 \end{aligned}$$

is given by

$y(t) = e^t\left(2\cos(4t) + \frac{1}{4}\sin(4t)\right)$. This process is usually very straightforward and follows the patterns established in section 3.1.

Repeated Roots

If we are looking at differential equations of the form

$$y'' - 2\lambda y' + \lambda^2 y = 0$$

then the associated characteristic polynomial is

$$c(r) = (r - \lambda)^2.$$

The repeated roots pose a challenge because we only get one solution of the form $y_1(t) = e^{\lambda t}$. To find another solution we invoke a reduction of order to generate a solution of the form $y_2(t) = v(t)e^{\lambda t}$. If we place y_2 into the differential equation we get

$y_2'' - 2\lambda y_2' + \lambda^2 y_2 = v''(t)e^{\lambda t} = 0$. Thus $v''(t) = 0$, $v'(t) = C_1$, and $v(t) = C_1 t + C_2$.

Putting this all together we see that general solution of the differential equation is

$$y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}.$$

If we have $y'' - 16y' + 64y = 0$ then the characteristic polynomial is $c(r) = (r-8)^2$. Then our repeated root is $r = 8$ and we get the general solution

$$y(t) = C_1 e^{8t} + C_2 t e^{8t}$$

Suppose we want $y(0) = 4$ and $y'(0) = 2$ then

$$y(0) = C_1 = 4$$

$$y'(0) = 8C_1 + C_2 = 2$$

gives us $C_1 = 4$ and $C_2 = -30$. Then our particular solution to the IVP is

$$y(t) = 4e^{8t} - 30te^{8t}.$$

Reduction of Order

Suppose we have

$$t^2 y'' + 3ty' + y = 0$$

and we are given that $y_1 = \frac{1}{t}$ is a solution. Let us try to invoke the same reduction of order method we used earlier. Let $y_2 = v(t)\frac{1}{t}$. Then

$$y_2' = v'(t)\frac{1}{t} - v(t)\frac{1}{t^2}$$

$$y_2'' = v''(t)\frac{1}{t} - 2v'(t)\frac{1}{t^2} + v(t)\frac{2}{t^3}$$

and if we place these into the differential equation we get

$$t^2 y_2'' + 3ty_2' + y_2 = v''(t) + tv'(t) = 0,$$

or a new differential equation for $v'(t)$. By our methods from section 2.1 we can see that

$$v' = \mu \int 0\mu \; dt = \frac{1}{t}\int 0 \; dt = \frac{C_1}{t}.$$

Thus $v(t) = C_1 \log(t) + C_2$. Therefore our other solution is $y_2 = C_1 \frac{\log(t)}{t} + \frac{C_2}{t}$

and since y_1 is included in that solution we can find that our general solution is

$$y(t) = C_1 \frac{\log(t)}{t} + \frac{C_2}{t}.$$

In each reduction of order problem the constants of integration found in y_2 include the same function as y_1 and thus finding y_2 is equivalent to finding the general solution.

Euler's Equation

Suppose we have

$$t^2 y''(t) + \alpha t y'(t) + y(t) = 0.$$

Then the change of variables $x = \log(t)$ gives us

$$\frac{dt}{dx} = t$$
$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = t\frac{dy}{dt}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(t\frac{dy}{dt}\right) = \frac{dt}{dx}\frac{dy}{dt} + t\frac{d}{dx}\frac{dy}{dt} = \frac{dy}{dx} + t\left(t\frac{d^2y}{dt^2}\right)$$

by the chain rule.
Thus

$$t\frac{dy}{dt} = \frac{dy}{dx}$$
$$t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

transforms our differential equation

$$t^2y''(t) + \alpha ty'(t) + y(t) = 0$$

in the variable t to

$$y''(x) + (\alpha - 1)y'(x) + y(x) = 0,$$

a differential equation, with constant coefficients, in the variable x.
If $\alpha = 3$, we get a characteristic polynomial, with repeated roots, $c(r) = (r+1)^2$. As we outlined above, the general solution in the variable x is

$$y(x) = C_1e^{-x} + C_2xe^{-x}$$

since we have repeated roots and our differential equation has constant coefficients. Therefore the solution to the related differential equation

$$t^2y''(t) + 3ty'(t) + y(t) = 0$$

is

$$y(t) = C_1e^{-\log(t)} + C_2\log(t)e^{-\log(t)} = \frac{C_1}{t} + C_2\frac{\log(t)}{t}.$$

You may notice that this is the same solution we found in the last example of the reduction of order section. We have seen a similar change of variable technique before with regards to first order differential equations. The Bernoulli equation

$$y'(t) + p(t)y(t) = q(t)y^n(t)$$

is equivalent to
$$u'(t) + (1-n)p(t)u(t) = (1-n)q(t)$$

where
$$u(t) = y^{(1-n)}(t).$$

Likewise, a homogeneous equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

transforms to
$$\frac{dv}{dx} = \frac{f(v) - v}{x}$$
if $v = \frac{y}{x}$.

Week 7 Wrap-up

Method of Undetermined Coefficients

If we have the differential equation

$$y'' + by' + cy = g(t)$$

we can construct a solution of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 solve the associated homogenous equation. Then based on the terms that appear in $g(t)$, y_1 and y_2 we can construct our solution of $Y(t)$.

Suppose $y_1(t) = e^{3t}$, $y_2(t) = e^{7t}$ and we the differential equation

$$y'' - 10y' + 21y = \sin(t)$$

then our guess for $Y(t)$ is a linear combination of sine and cosine

$$Y(t) = A\cos(t) + B\sin(t).$$

We compute

$$Y'(t) = -A\sin(t) + B\cos(t)$$
$$Y''(t) = -A\cos(t) - B\sin(t)$$

to get

$$Y'' - 10Y' + 21Y = (20A - 10B)\cos(t) + (A + 2B)\sin(t) = \sin(t).$$

This leads to the system of equations

$$20A - 10B = 0$$
$$A + 2B = 1$$

which has solution $A = \frac{1}{50}$ and $B = \frac{1}{25}$. Thus our general solution is

$$y(t) = C_1 e^{3t} + C_2 e^{7t} + \frac{1}{50}\cos(t) + \frac{1}{25}\sin(t).$$

Our guess for the term $Y(t)$ can be discovered from the following table.

If our terms in g(t) share no common terms with are homogenous solutions then the process is pretty straight forward.

If

$$g(t) = t^3$$

then

$$Y(t) = A_3 t^3 + A_2 t^2 + A_1 t + A_0,$$

a generic degree 3 polynomial.

If

$$g(t) = t^2 e^{2t}$$

then

$$Y(t) = (A_2t^2 + A_1t + A_0)e^{2t},$$

a generic degree 2 polynomial multiplied by an exponential function.

We then place $Y(t)$ into the differential equation and find a system of equations for the unknowns.

Common terms with g(t)

Now suppose $y_1(t) = e^{3t}, y_2(t) = e^{7t}$ and our differential equation is

$$y'' - 10y' + 21y = 3e^{7t}.$$

Then $g(t)$ and $y_2(t)$ share common terms. This means that

$$Y(t) = Ae^{7t}$$

cannot generate the terms is $g(t)$ because it will be a homogeneous solution. To correct for this we do something that is similar to the repeated roots in case we saw last week,

$$Y(t) = Ate^{7t}.$$

We then need to generate Y' and Y'' using the product rule and then solve a linear equation for A.

Likewise if our general solution to the homogeneous equation was

$$y_c(t) = C_1e^{7t} + C_2te^{7t}$$

and we had

$$y'' - 14y' + 49y = e^{7t}$$

then we would pick

$$Y(t) = At^2e^{7t}.$$

Once the guess for the term Y(t) is correctly made then the process for solving for the unknown terms follows the same process that we established above.

Variation of Parameters

If we have

$$y'' + p(t)y' + q(t)y = g(t)$$

where y_1 and y_2 solve the associated homogenous equation then we can introduce two auxiliary functions u_1 and u_2 such that

$$Y(t) = u_1y_1 + u_2y_2.$$

If we assume $u_1'y_1 + u_2'y_2 = 0$ and place $Y(t)$ into the differential equation we find

$u_1'y_1' + u_2'y_2' = g(t)$. The solution to this system of equations for u_1' and u_2' is

$$u_1' = -\frac{y_2g(t)}{W(y_1, y_2)}$$

$$u_2' = \frac{y_1g(t)}{W(y_1, y_2)}$$

and therefore

$$u_1 = \int -\frac{y_2 g(t)}{W(y_1, y_2)}\, dt$$
$$u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$
.

Thus our general solution to the differential equation is

$$y(t) = y_c(t) + Y(t) = c_1 y_1(t) + c_2 y(t) + u_1(t) y_1(t) + u_2(t) y_2(t).$$

If

$$y'' + 9y = 9\sec^2(3t)$$

then $y_1 = \cos(3t)$, , and

.

Thus

.

Week 8 Wrap-up

Springs

If we have a spring-mass system we can measure the displacement downwards by a function $u(t)$ where

$$u''(t) + \frac{\gamma}{m}u'(t) + \frac{k}{m}u(t) = \frac{f(t)}{m}.$$

In this equation we have m is the mass of the object, g is the gravitational constant, γ is the dampening coefficient, $k = \frac{mg}{L}$ is the spring constant from Hooke's Law, L is the spring's displacement caused by attached mass, and $f(t)$ is the external forcing function.

If we describe initial displacement $u(t_0) = u_0$ and the initial velocity $u'(t_0) = u'_0$ then we have an 2nd order initial value problem with constant coefficients. Thus we can use the methods from chapter 3 to find $y_c(t)$ and $Y(t)$ by variation of parameters or method of undetermined coefficients.

Classifying the Dampening, Quasi Frequency and Quasi Period

In our differential equation the constant coefficients are all positive. This means the our roots to the characteristic polynomial will have negative real components.

Based on the discriminant, $\gamma^2 - 4km$, and the size of γ our system's dampening can be classified. If

$$\gamma = 2\sqrt{km}$$

then the spring-mass system is critically dampened. Our characteristic polynomial's discriminant is zero and we have a repeated roots, $y(t) = C_1e^{rt} + C_2te^{rt}$.

If we have two distinct roots, r_1 and r_2, then our spring-mass system is over dampened. If we have complex roots, then our spring-mass system is under dampened. The role of the dampening classification is to describe how often the system approaches its equilibrium position and if it does so by oscillating about the equilibrium position.

If we have dampening in our spring-mass system the quasi frequency is given by

$$\mu = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}$$

and the quasi period is

$$\frac{2\pi}{\mu}.$$

If our system is critically dampened, then our quasi frequency tends to zero.

Suppose an 8lb mass stretches a spring by 3/24 feet, 1.5 inches, and our dampening coefficient is 5. Then $k = 8/(3/24) = 64$, $m = 8/32$, and our differential equation is

$$u''(t) + \frac{\gamma}{m}u'(t) + \frac{k}{m}u(t) = u''(t) + 20u'(t) + 256u(t) = 0.$$

Higher Order Differential Equations

All of our concepts from chapter 3's 2nd order linear equation theory can be expanded to higher order equations. If we have

$$y^{(4)}(t) - 6y'''(t) + 13y''(t) - 24y'(t) + 36y(t) = 0$$

then the characteristic polynomial is

$$c(r) = r^4 - 6r^3 + 13r^2 - 24r + 36 = (r^2+4)(r-3)^2.$$

Thus the general solution is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3 e^{3t} + c_4 t e^{3t}$$

because of our repeated root.

An nth order ODE will have n LI functions in its fundamental solution set. If the Wronskian of our solutions is not zero then we have a fundamental solution set,

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} \neq 0.$$

If we want to solve

$$y^{(4)}(t) - 6y'''(t) + 13y''(t) - 24y'(t) + 36y(t) = 4e^{3t}$$

then

$$Y(t) = At^2 e^{3t}$$

since e^{3t} and te^{3t} are already in our fundamental solution set. If we place $Y(t), Y'(t), Y''(t), Y'''(t), Y^{(4)}$ into our differential equation we find that $A = \frac{2}{13}$. Thus our general solution to the nonhomogenous equation is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + c_3 e^{3t} + c_4 t e^{3t} + \frac{2}{13} t^2 e^{3t}.$$

Systems of ODEs

If we have the differential equation

$$y^{(4)}(t) - 6y'''(t) + 13y''(t) - 24y'(t) + 36y(t) = 0$$

we can rewrite it as a system of equations where $y_1 = y(t)$, $y_2 = y_1' = y'$, $y_3 = y_2' = y''$, $y_4 = y_3' = y'''$, and $y_4' = y^{(4)}$. Then our system is

$$\begin{aligned} y_4' &= 6y_4 - 13y_3 + 24y_2 - 36y_1 \\ y_3' &= y_4 \\ y_2' &= y_3 \\ y_1' &= y_2 \end{aligned}$$

We can also use a similar process to rewrite a system of ODEs as a higher order differential equation.

If

$$\begin{aligned} u_1' &= 7u_1 - u_2 \\ u_2' &= 4u_1 - u_2 \end{aligned}$$

then the 2nd equation implies

$$u_1 = \frac{u_2' + u_2}{4}$$
$$u_1' = \frac{u_2'' + u_2'}{4}.$$

If we substitute each of these into the first equation we get a 2nd order equation for u_2

$$\frac{u_2'' + u_2'}{4} = 7\left(\frac{u_2' + u_2}{4}\right) - u_2.$$
$$u_2'' = 6u_2' + 3u_2$$

Week 9 Wrap-up

Laplace Transforms

The Laplace transform of the function $f(t)$ is defined by the following integral equation

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty f(t)e^{-st}\ dt.$$

The variable s acts as a frequency variable if t is time variable. If a function has a lot of oscillations then its Laplace transform will have large values for large values of s. If the function does not change then its Laplace transform will vanish for s large. You can think of the Laplace transform as spectral decomposition of the function and changes can be made in frequency space that have global changes in how the function behaves, just like adjusting a song with an equalizer on a stereo system.

The Laplace transform satisfies a few nice properties

1. Linearity

$$\mathcal{L}\left\{cf(t) + g(t)\right\} = c\mathcal{L}\left\{f(t)\right\} + \mathcal{L}\left\{g(t)\right\}$$

2. Zero function

$$\mathcal{L}\left\{0\right\} = 0$$

3. Derivatives in Laplace space

$$\begin{aligned}\mathcal{L}\left\{tf(t)\right\} &= -F'(s)\\ \mathcal{L}\left\{-tf(t)\right\} &= F'(s)\end{aligned}$$

4. Higher order derivatives in Laplace space

$$\begin{aligned}\mathcal{L}\left\{t^k f(t)\right\} &= (-1)^k\frac{d^k}{ds^k}F(s)\\ \mathcal{L}\left\{(-t)^k f(t)\right\} &= \frac{d^k}{ds^k}F(s)\end{aligned}$$

5. Derivatives in t

$$\begin{aligned}\mathcal{L}\left\{f'(t)\right\} &= sF(s) - f(0)\\ \mathcal{L}\left\{\frac{d^k}{dt^k}f'(t)\right\} &= s^kF(s) - s^{k-1}f(0) - s^{k-2}f'(0)\dots sf^{(k-1)}(0) - f^{(k)}(0)\\ &= -\left(-F(s), f(0), f'(0), \dots, f^{(k-1)}(0), f^{(k)}(0)\right)^T\\ &\quad\cdot\left(s^k, s^{k-1}, s^{k-2}, \dots s, 1\right)^T\end{aligned}$$

6. One-to-one

$$\mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{g(t)\right\} \text{ if and only if } f(t) = g(t).$$

We also developed two very important Laplace transform pairs

$$\begin{aligned}\mathcal{L}\left\{1\right\} &= \frac{1}{s}, s > 0\\ \mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a}, s > a\end{aligned}$$

If we invoke our results from above for multiplying by t then

$$\mathcal{L}\{t1\} = \frac{1}{s^2}, s > 0$$
$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2}, s > a.$$

This saves us from doing the computations directly from the definition and, in the process, invoking integration by parts.

Now on the final exam and on the homework it is useful to use the **attached table** **(https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** **(https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** **(https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** for identifying Laplace transform pairs. However, you should able to compute the Laplace transforms for many of these function by hand.

Inverse Laplace Transform

The inverse Laplace transform is defined, but it is beyond the scope of this course. However, we can use some clever thinking to find the inverse of $\mathcal{L}\{f(t)\}$ by invoking that the forward transform is one-to-one.

If $\mathcal{L}\{f(t)\} = 3/s^2$ then $f(t) = 3t$. You can check this is correct by looking at the right column in our **Laplace transform table** **(https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** **(https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** **(https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)**.

Partial Fraction Decomposition

If $\mathcal{L}\{f(t)\}$ is a proper rational function, the ratio of two polynomials where the denominator has degree greater than the numerator, then we can invoke partial fraction decomposition to find that $f(t)$ is a linear combination of terms of the form $t^k e^{at}$, $\cos(bt)e^{at}$, and $\sin(bt)e^{at}$.

Consider the rational function
$$f(s) = \frac{1}{(s^2+4s+9)(s+2)^2(s+1)}$$
which has one irreducible factor, a repeated root and a simple root in its denominator. The partial fraction decomposition of this function has the following structure
$$\begin{aligned}\frac{1}{(s^2+4s+9)(s+2)^2(s+1)} &= \frac{As+B}{s^2+4s+9} + \frac{C}{s+2} \\ &+ \frac{D}{(s+2)^2} + \frac{E}{s+1}.\end{aligned}$$
We have a degree one polynomial over the irreducible term because the irreducible is of degree 2. We have two terms, the one's involving C and D, for the repeated root. We have a single term, the one involving E, for the simple root.

When we put all of the terms in the PFD over a common denominator and equate the numerators we get the following
$$\begin{aligned}1 &= (As+B)(s+2)^2(s+1) + C(s^2+4s+9)(s+2)(s+1) \\ &\quad +D(s^2+4s+9)(s+1)) + E(s^2+4s+9)(s+2)^2\end{aligned}$$
and this yields the system of equations
$$\begin{aligned}1 &= 4B + 18C + 9D + 36E \\ 0s = 0 &= 4A + 8B + 35C + 13D + 52E \\ 0s^2 = 0 &= 8A + 5B + 23C + 5D + 29E \\ 0s^3 = 0 &= 5A + B + 7C + D + 8E \\ 0s^4 = 0 &= A + C + E\end{aligned}$$

because two polynomials are equal if and only if their coefficients are equal. We then use standard methods for solving systems of equations to discover
$$A = \frac{1}{30}, B = \frac{3}{30}, C = -\frac{1}{5}, D = -\frac{1}{5}, E = \frac{1}{6}.$$

Now that we have the partial fraction decomposition of $f(s)$ the real work can begin. The easiest part is to compute the inverse transform of the fractions with simple roots,

$$\frac{C}{s+2} = \mathcal{L}\left\{Ce^{-2t}\right\}, a = -2$$

$$\frac{E}{s+1} = \mathcal{L}\left\{Ee^{-t}\right\}, a = -1.$$

The next challenge is to handle the repeated root term,

$$\frac{D}{(s+2)^2} = \mathcal{L}\left\{Dte^{-2t}\right\}, a = -2, n = 1$$

and this is computed by **table entry 11 (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1) (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1). Now onto the very challenging term,

$$\frac{As+B}{s^2+4s+9}.$$

How do we find the inverse Laplace transform of a partial function with a degree 2 irreducible term in its denominator? Very, very carefully...

To begin we need to complete the square in the variable s, that is we want our denominator to be of the form $(s-a)^2+b^2$. Thus,

$$\begin{aligned}\frac{As+B}{s^2+4s+9} &= \frac{As+B}{s^2+4s+4+5}\\ &= \frac{As+B}{(s+2)^2+5}\\ &= \frac{As+B}{(s+2)^2+\left(\sqrt{5}\right)^2}.\end{aligned}$$

If that wasn't enough, the challenges continue even further as we try to massage our partial fraction into the form

$$C_1\frac{s+2}{(s+2)^2+\left(\sqrt{5}\right)^2}+C_2\frac{\sqrt{5}}{(s+2)^2+\left(\sqrt{5}\right)^2}.$$

Once this is complete we can invoke **table entries 9 and 10 (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1) (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1) to conclude

$$\begin{aligned}\frac{As+B}{s^2+4s+9} &= C_1\frac{s+2}{(s+2)^2+\left(\sqrt{5}\right)^2}+C_2\frac{\sqrt{5}}{(s+2)^2+\left(\sqrt{5}\right)^2}\\ &= C_1\mathcal{L}\left\{\cos(\sqrt{5}t)e^{-2t}\right\}+C_2\mathcal{L}\left\{\sin(\sqrt{5}t)e^{-2t}\right\}.\end{aligned}$$

Our final step is to compute C_1 and C_2. We need to solve the system of equations

$$As+B = C_1(s+2)+C_2\sqrt{5}$$

or

$$C_1 = A$$

$$C_2 = \frac{B-2A}{\sqrt{5}}.$$

Therefore the function with Laplace transform $F(s)$ is

$$f(t) = A\cos(\sqrt{5}t)e^{-2t}+\frac{B-2A}{\sqrt{5}}\sin(\sqrt{5}t)e^{-2t}+Ce^{-2t}+Dte^{-2t}+Ee^{-t}.$$

Initial Value Problems and Laplace Transforms

If we have a linear nth order differential equation then we can invoke Laplace transforms to turn the IVP into an algebra problem in Laplace space with the following three step process:

1. Rewrite the equation in Laplace space using the following result
$$\mathcal{L}\left\{y'(t)\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^k}{dt^k}y'(t)\right\} = s^kY(s) - s^{k-1}y(0) - s^{k-2}y'(0)\ldots sy^{(k-1)}(0) - y^{(k)}(0).$$
2. Solve algebraically for
$$\mathcal{L}\left\{y(t)\right\} = Y(s)$$
3. Compute the inverse Laplace transform of $Y(s)$ by a **table lookup (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1)** (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1) (https://oregonstate.instructure.com/courses/1589230/files/64740879/download?wrap=1) .

Let's solve

$$\begin{aligned} y'' + 2y &= 0 \\ y(0) &= 2. \\ y'(0) &= 1 \end{aligned}$$

Then

$$(s^2+2)\mathcal{L}\left\{y(t)\right\} - sy(0) - y'(0) = 0$$

and

$$\begin{aligned} \mathcal{L}\left\{y(t)\right\} &= \frac{2s+1}{s^2+2} \\ &= C_1\frac{s}{s^2+2} + C_2\frac{\sqrt{2}}{s^2+2} \\ &= 2\mathcal{L}\left\{\cos(\sqrt{2}t)\right\} + \frac{1}{\sqrt{2}}\mathcal{L}\left\{\sin(\sqrt{2}t)\right\} \end{aligned}$$

Therefore

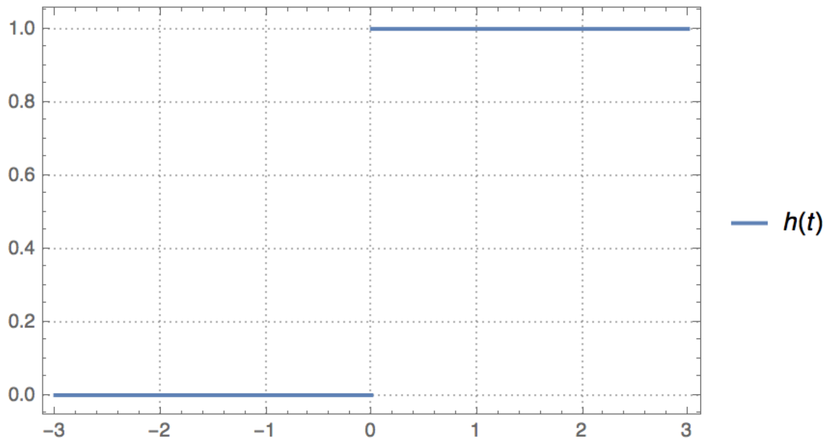
$$y(t) = 2\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t).$$

Week 10 Wrap-up

Step Functions

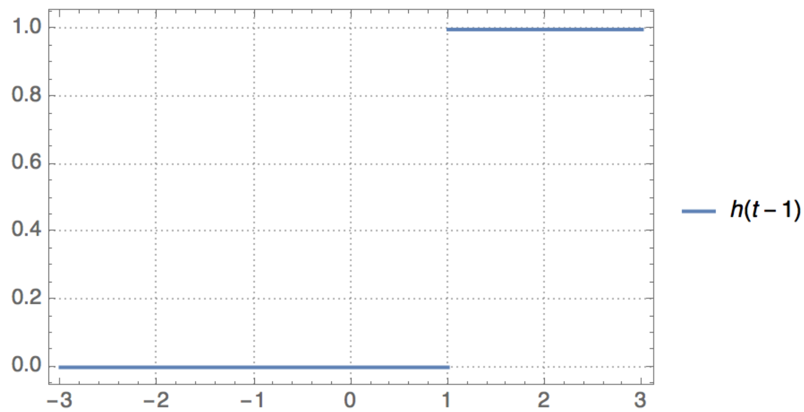
The heavy side function is defined by

$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



and the unit step function is given by

$$u_c(t) = h(t-c) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$

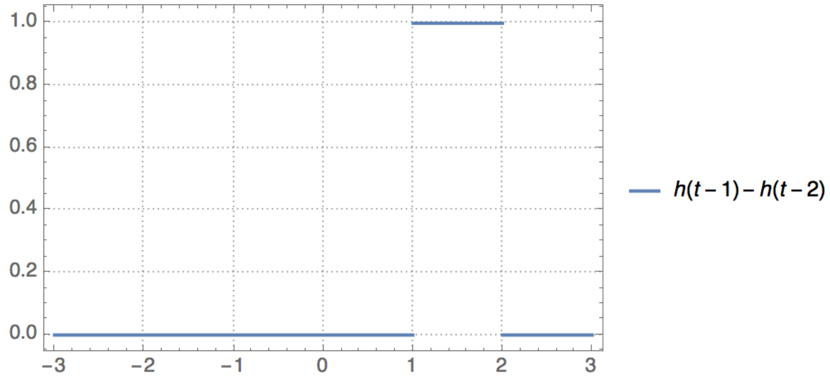


The unit step function acts a switch that turns on for all time $t > c$. We can create an off switch by taking the complement of $u_c(t)$

$$1 - u_c(t) = 1 - h(t-c) = \begin{cases} 1 & t < c \\ 0 & t \geq c \end{cases}.$$

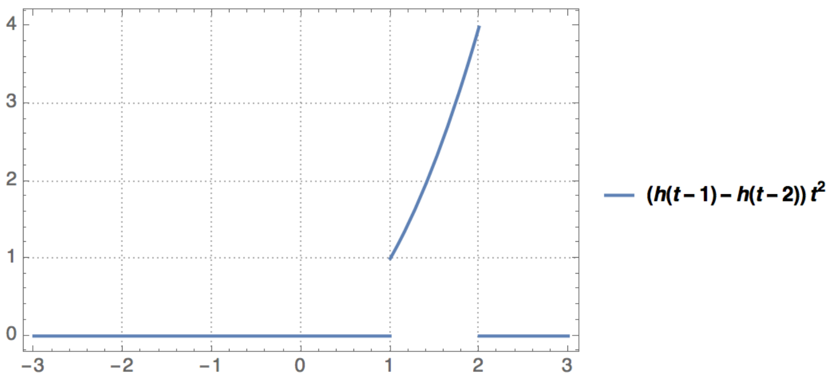
By combining our on and off switches we can make a switch that comes on for $t \in [a, b]$ with

$$u_a(t) - u_b(t) = h(t-a) - h(t-b) = \begin{cases} 1 & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}.$$

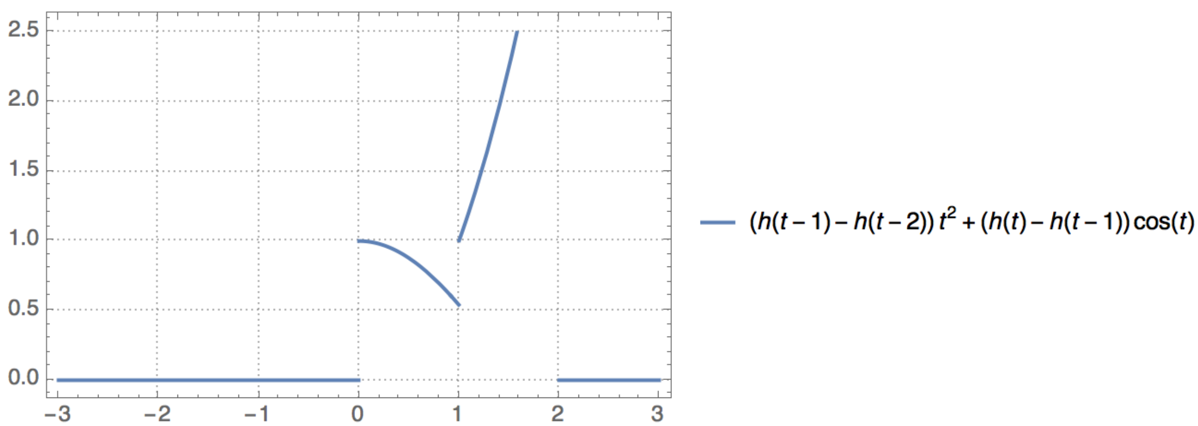


We can use these on/off switches to define a function piecewise

$$(u_a(t) - u_b(t))f(t) = (h(t-a) - h(t-b))f(t) = \begin{cases} f(t) & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}.$$



You can then extend this result to having multiple on/off switches



Second Shift Theorem

An elementary computation shows that

$$\mathcal{L}\left\{u_c(t)\right\} = \frac{e^{-cs}}{s}, \quad s > 0.$$

The Second Shift Theorem, or the time shift property, of the Laplace transform is given by

$$\begin{aligned} \mathcal{L}\left\{u_c(t)f(t-c)\right\} &= e^{-cs}\mathcal{L}\left\{f(t)\right\} = e^{-cs}F(s) \\ \mathcal{L}\left\{u_c(t)f(t)\right\} &= e^{-cs}\mathcal{L}\left\{f(t+c)\right\} \end{aligned}.$$

If your rational function is multiplied by an exponential function in Laplace space then you have a function in time space multiplied by a step function.

If we have

$$F(s) = e^{-2s}\frac{1}{s^2+1}$$

then we know

$$\mathcal{L}\left\{u_2(t)\sin(t-2)\right\} = e^{-2s}\frac{1}{s^2+1}.$$

Likewise,

$$\begin{aligned}\mathcal{L}\left\{(u_2(t)-u_3(t))(t-2)^2\right\} &= e^{-2s}\mathcal{L}\left\{((t+2)-2)^2\right\} - e^{-3s}\mathcal{L}\left\{((t+3)-2)^2\right\} \\ &= e^{-2s}\mathcal{L}\left\{t^2\right\} - e^{-3s}\mathcal{L}\left\{t^2+2t+1\right\} \\ &= e^{-2s}\frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)\end{aligned}$$

here we are invoking the shift theorem for $c = 2$ and $c = 3$.

Frequency Shift Theorem

The Frequency Shifty Theorem is given by

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c).$$

We have multiplication by exponential functions in time space corresponding to a shift in frequencies in Laplace space.

Now

$$\frac{1}{s-1} = F(s-1) \Rightarrow F(s) = \frac{1}{s}$$

then by the frequency shift theorem we have

$$\mathcal{L}\left\{e^t \times 1\right\} = \frac{1}{s+1}.$$

Shift Theorems and IVPs

If we want to solve the IVP

$$\begin{aligned}y'(t)+2y(t) &= u_3(t)t \\ y(0) &= 1\end{aligned}$$

then

$$\begin{aligned}(s+2)\mathcal{L}\left\{y(t)\right\} - 1 &= \mathcal{L}\left\{u_3(t)t\right\} \\ &= e^{-3s}\mathcal{L}\left\{t+3\right\} \\ &= e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right)\end{aligned}$$

implies

$$\begin{aligned}\mathcal{L}\left\{y(t)\right\} &= e^{-3s}\left(\frac{1}{s^2(s+2)}+\frac{3}{s(s+2)}\right)+\frac{1}{s+2} \\ &= e^{-3s}\left(\frac{3s+1}{s^2(s+2)}\right)+\frac{1}{s+2}\end{aligned}.$$

Now

$$\frac{1}{s+2} = F(s+2) \Rightarrow F(s) = \frac{1}{s}$$

then by the frequency shift theorem, or from a table look up, we have

$$\mathcal{L}\left\{e^{-2t} \times 1\right\} = \frac{1}{s+2}.$$

Now consider the terms associated with e^{-3t}. The partial fraction decomposition is

$$\frac{3s+1}{s^2(s+2)} = \frac{1}{2}\frac{1}{s^2} + \frac{5}{4}\frac{1}{s} - \frac{5}{4}\frac{1}{s+2}$$

and thus

$$\frac{3s+1}{s^2(s+2)} = \mathcal{L}\left\{\frac{t}{2} + \frac{5}{4} - \frac{5}{4}e^{-2t}\right\}.$$

Then

$$\begin{aligned} e^{-3t}\frac{3s+1}{s^2(s+2)} &= e^{-3t}\mathcal{L}\left\{\frac{t}{2} + \frac{5}{4} - \frac{5}{4}e^{-2t}\right\} \\ &= \mathcal{L}\left\{u_3(t)\left(\frac{t-3}{2} + \frac{5}{4} - \frac{5}{4}e^{-2(t-3)}\right)\right\}. \end{aligned}$$

Therefore

$$\mathcal{L}\left\{y(t)\right\} = \mathcal{L}\left\{u_3(t)\left(\frac{t-3}{2} + \frac{5}{4} - \frac{5}{4}e^{-2(t-3)}\right)\right\} + \mathcal{L}\left\{e^{-2t}\right\}$$

and

$$\begin{aligned} y(t) &= u_3(t)\left(\frac{t-3}{2} + \frac{5}{4} - \frac{5}{4}e^{-2(t-3)}\right) + e^{-2t} \\ &= \begin{cases} e^{-2t} & t \leq 3 \\ -\frac{5}{4}e^{6-2t} + e^{-2t} + \frac{t}{2} - \frac{1}{4} & t > 3 \end{cases}. \end{aligned}$$