Lecture # 11

Spontaneous emission of radiation

From last week;
transition rate for absorption ($\vec{A} \parallel Oz, \vec{k} \parallel Oy$) $\Rightarrow$

$$P_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2 |A_0|^2 |\langle f| e^{iky} p_z |i\rangle|^2 \delta(E_f - E_i - \hbar\omega)$$

$$\Rightarrow \quad P_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2 |A_0|^2 |\langle f| e^{i\vec{k}\cdot\vec{r}} \vec{\epsilon}\cdot\vec{p} |i\rangle|^2 \delta(E_f - E_i - \hbar\omega)$$

generalize to an arbitrary direction of $\vec{k}$ and light polarization $\vec{\epsilon}$

Recall:
$$V(t) = \underbrace{V_0 e^{-i\omega t}}_{\text{absorption}} + \underbrace{V_0^* e^{i\omega t}}_{\text{emission}}$$

Transition rate for emission $\Rightarrow$

$$P_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2 |A_0|^2 |\langle f| e^{-i\vec{k}\cdot\vec{r}} \vec{\epsilon}^*\cdot\vec{p} |i\rangle|^2 \delta(E_f - E_i + \hbar\omega)$$

So, the absorption process occurs when the atom receives a photon from the radiation, and the emission occurs when the radiation gains a photon from the decaying atom. Note that this is stimulated emission $\Rightarrow$

no emission if $\vec{A} = 0$!! $\Rightarrow$ use it for light amplification by stimulated emission of radiation (LASER) $\Rightarrow$ if a large number of atoms are in the same excited state, and one photon is incident $\Rightarrow$ cause chain reaction, as the atoms release photons of the same ω within a very short time

What happens if $\vec{A} = 0$? Does it mean that the atoms will stay in the excited state forever? $\Rightarrow$

nope! $\Rightarrow$ Spontaneous emission $\Rightarrow$ cannot be described by classical treatment of the EM field (as we did so far, in the case of absorption and stimulated emission) $\Rightarrow$ need QM treatment of EM radiation

$\Downarrow$

Second quantization $\Rightarrow$ replace fields (such as $\vec{A}, \vec{E}, \vec{B}$) by operators expressed in terms of a^+, a $\Rightarrow$ $\hat{A}_{\lambda,\vec{k}} = \sqrt{\frac{2\pi\hbar c^2}{\omega_k}}\, a_{\lambda,\vec{k}}$

$$H = \sum_{\vec{k}} \sum_{\lambda=1}^{2} \hbar\omega_k \left(a^+_{\lambda,\vec{k}} a_{\lambda,\vec{k}} + \frac{1}{2}\right) \Leftarrow \text{like in harmonic oscillator}$$

$\vec{k}$ ↑ wave number; λ ↑ polarization (2 components in the plane $\perp \vec{k}$)

$a^+_{\lambda,\vec{k}} \to$ creates a photon of wave number $\vec{k}$ and polarization λ

$n_{\lambda,\vec{k}} = 0, 1, 2, \ldots$

eigenvalues of $N_{\lambda,\vec{k}}$ ← number operator

$$\underbrace{|n_{\lambda,\vec{k}}\rangle}_{\text{state with}} = \frac{1}{\sqrt{n_{\lambda,\vec{k}}!}} \left(a^+_{\lambda,\vec{k}}\right)^{n_{\lambda,\vec{k}}} \underbrace{|0\rangle}_{\text{state with no photons ("vacuum state")}}$$

state with $n_{\lambda,\vec{k}}$ photons with wave vector $\vec{k}$ and polarization λ

"occupation number"

$$a_{\lambda,\vec{k}} |n_{\lambda,\vec{k}}\rangle = \sqrt{n_{\lambda,\vec{k}}}\, |n_{\lambda,\vec{k}} - 1\rangle$$

$$a^+_{\lambda,\vec{k}} |n_{\lambda,\vec{k}}\rangle = \sqrt{n_{\lambda,\vec{k}} + 1}\, |n_{\lambda,\vec{k}} + 1\rangle$$

Eigenstates of $H \Rightarrow |\underbrace{n_{\lambda_1\vec{k}_1}, n_{\lambda_2\vec{k}_2}, n_{\lambda_3\vec{k}_3}, \ldots}\rangle$

$$E = \sum_{\vec{k}} \sum_{\lambda} \hbar\omega_k \left(n_{\lambda\vec{k}} + \frac{1}{2}\right)$$

energy

EM field with $n_{\lambda_1\vec{k}_1}$ photons in the mode $(\lambda_1, \vec{k}_1)$ etc.

(assume EM in a box) volume

$$\vec{A}(\vec{r},t) = \sum_{\vec{k}} \sum_{\lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_k V}} \left[a_{\lambda,\vec{k}}\, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)}\, \vec{\epsilon}_\lambda + a^+_{\lambda,\vec{k}}\, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\, \vec{\epsilon}^{\,*}_\lambda \right] \Rightarrow$$

$$V(t) = \frac{e}{m}\sqrt{\frac{2\pi\hbar}{V}} \sum_{\vec{k}} \sum_{\lambda} \frac{1}{\sqrt{\omega_k}} \Big[a_{\lambda\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} \vec{\epsilon}_\lambda \cdot \vec{p} \cdot$$

$$\cdot e^{i\omega_k t} + a^{+}_{\lambda,\vec{k}}\, e^{-i\vec{k}\cdot\vec{r}} \vec{\epsilon}^{\,*}_\lambda \cdot \vec{p}\, e^{-i\omega_k t} \Big] =$$

$$= \sum_{\vec{k}} \sum_{\lambda} \Big(V_{0_{\lambda,\vec{k}}}\, e^{i\omega_k t} + V_{0_{\lambda,\vec{k}}}\, e^{-i\omega_k t} \Big)$$

absorption (annihilation) emission (creation)

as in the classical case, QM description has the structure of a <u>harmonic</u> perturbation

<u>Absorption</u> $\Rightarrow$ initial state $|\varphi_i\rangle = |\psi_i\rangle |n_{\lambda,\vec{k}}\rangle$

atom radiation

final state $|\varphi_f\rangle = |\psi_f\rangle |n_{\lambda,k} - 1\rangle$

$$\langle \varphi_f | V_{0_{\lambda,\vec{k}}} | \varphi_i \rangle = \frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega_k V}} \sqrt{n_{\lambda,\vec{k}}} \langle \psi_f | e^{i\vec{k}\cdot\vec{r}} \cdot$$

$$\cdot \vec{\epsilon}_\lambda \cdot \vec{p} \,|\psi_i\rangle$$

$$a_{\lambda,\vec{k}} |n_{\lambda,\vec{k}}\rangle = \sqrt{n_{\lambda,\vec{k}}}\, |n_{\lambda,\vec{k}} - 1\rangle$$

$$\Downarrow$$

$$P_{i\to f} = \frac{4\pi^2 e^2}{m^2 \omega_k V}\, n_{\lambda,\vec{k}} \left| \langle \psi_f | e^{i\vec{k}\cdot\vec{r}} \vec{\epsilon}_\lambda \cdot \vec{p} \,| \psi_i \rangle \right|^2 \cdot$$

$$\cdot\, \delta(E_f - E_i - \hbar\omega_k)$$

absorption of a photon of energy $\hbar\omega_k = \hbar c k$, wave number $\vec{k}$ and polaris. λ

Emission $\Rightarrow$ $|\Phi_f\rangle = |\Psi_f\rangle |n_{\lambda,\vec{k}}+1\rangle$

$$\langle \Phi_f | V^{+}_{0\,\lambda,\vec{k}} | \Phi_i\rangle = \frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega_k V}}\sqrt{n_{\lambda,\vec{k}}+1}\,\cdot$$

$$\cdot \langle \Psi_f | e^{-i\vec{k}\cdot\vec{r}}\vec{\epsilon}^{\,*}_{\lambda}\cdot\vec{p}\,|\Psi_i\rangle \Rightarrow$$

$$P_{i\to f} = \frac{4\pi^2 e^2}{m^2\omega_k V}\,(\underbrace{n_{\lambda,\vec{k}}}_{\text{stimulated}} + \underbrace{1}_{\text{spontaneous}})\,|\langle\Psi_f|e^{-i\vec{k}\cdot\vec{r}}\vec{\epsilon}^{\,*}_{\lambda}\cdot\vec{p}|\Psi_i\rangle|^2 \cdot \delta(E_f - E_i + \hbar\omega_k)$$

spontaneous emission is there even if $\vec{A}=0$ (i.e. $n_{\lambda,\vec{k}}=0$)

Stimulated

Consider spontaneous emission in the electric-dipole approx.

$\Downarrow$

$n_{\lambda,\vec{k}}=0$; $e^{-i\vec{k}\cdot\vec{r}} \simeq 1$

$\Downarrow$ WS#10 $\langle\Psi_f|(-e\,\vec{r})|\Psi_i\rangle$

$$\underbrace{P_{i\to f}}_{\substack{\text{probability of transition per unit time}\\ \text{with spontaneous emission of a photon } \hbar\omega_k}} = \frac{4\pi^2\omega_{fi}^2}{\omega_k V}\,|\vec{\epsilon}^{\,*}_{\lambda}\cdot\underbrace{\vec{d}_{fi}}_{\text{dipole moment matrix element}}|^2\,\delta(E_f - E_i + \hbar\omega_k)$$

Note: spontaneous emission is typically much weaker (slower) than stimulated ($n_{\lambda,\vec{k}} \gg 1$) when radiation is present

Now... The final states of the system is a product of a discrete atomic state and a continuum of photonic states $\Rightarrow$ need to integrate $P_{i \to f}$ with $\rho(E)\,dE$ to find a total transition rate.

Number of final photonic states within volume V, whose momenta are within the interval $[\vec{p}, \vec{p} + d\vec{p}]$, $\vec{p} = \hbar \vec{k}$ $\Rightarrow$

$$d^3 n = \underbrace{\frac{V}{(2\pi\hbar)^3}}_{\text{density of states in p-space}} d^3 p = \frac{V}{(2\pi\hbar)^3} \underbrace{p^2}_{= (\hbar \frac{\omega}{c})^2} dp\, d\Omega = \frac{V \omega^2}{(2\pi c)^3} \cdot d\omega\, d\Omega$$

Transition rate corresponding to the emission of a photon in the solid angle $d\Omega$ $\Rightarrow$

$$dW^{em}_{i \to f} = \frac{V}{(2\pi c)^3} d\Omega \int \omega^2 P_{i \to f}\, d\omega =$$

$$= \frac{V}{(2\pi c)^3} d\Omega \cdot \frac{4\pi^2 \omega_{fi}^2}{V} \int \omega\, \delta(E_f - E_i + \hbar\omega)\, d\omega \cdot |\vec{\epsilon}_\lambda^{\,*} \cdot \vec{d}_{fi}|^2 = \frac{\omega^3}{2\pi\hbar c^3} |\vec{\epsilon}_\lambda^{\,*} \cdot \vec{d}_{fi}|^2 d\Omega \qquad (12.1)$$

The transition rate (12.1) corresponds to a specific polarization. For any polarization ⇒ average over pol's.

arbitrary

$$\sum_{\lambda=1}^{2} |\vec{\epsilon}_\lambda^{\,*}\cdot \vec{d}_{fi}|^2 = |\vec{\epsilon}_1^{\,*}\cdot(d_{fi})_1|^2 + |\vec{\epsilon}_2^{\,*}(d_{fi})_2|^2 =$$

$$= |\vec{d}_{fi}|^2 - |(d_{fi})_3|^2 = |\vec{d}_{fi}|^2 - \tfrac{1}{3}|\vec{d}_{fi}|^2 =$$

$\parallel \vec{k}$

Since all directions of $\vec{d}_{fi}$ are equivalent ⇒

$$= \frac{2}{3}|\vec{d}_{fi}|^2 \Rightarrow$$

$$dW^{em}_{i\to f} = \frac{\omega^3}{3\pi\hbar c^3}|\vec{d}_{fi}|^2 d\Omega$$

Total transition rate associated with the emission of the photon ⇒ $\int d\Omega \Rightarrow 4\pi \Rightarrow$

$$W^{em}_{i\to f} = \frac{4}{3}\frac{\omega^3}{\hbar c^3}|\vec{d}_{fi}|^2 = \frac{4}{3}\frac{\omega^3 e^2}{\hbar c^3}|\langle\psi_f|\vec{r}|\psi_i\rangle|^2$$

$\omega = \frac{E_f - E_i}{\hbar}$; $\vec{d} = -e\vec{r}$ (for one-electron atoms)

See E&M!

Total power radiated

⇓

$$I_{i\to f} = \hbar\omega W^{em}_{i\to f} = \frac{4}{3}\frac{\omega^4 e^2}{c^3}|\langle\psi_f|\vec{r}|\psi_i\rangle|^2$$

The mean lifetime of an excited state $\Rightarrow$

$$\tau = \frac{1}{\sum_f W_{i\to f}} = \frac{1}{W}$$

<u>Example</u> A hydrogen atom is in 2p state. Find transition rate for $2p \to 1s$ transitions and the lifetime of the 2p state.

$$W_{2p\to 1s} = \frac{4}{3}\frac{e^2 \omega^3_{2p\to 1s}}{\hbar c^3} |\underbrace{\vec{r}_{fi}}_{\langle f|\vec{r}|i\rangle}|^2$$

$\langle f|\vec{r}|i\rangle$ $\Rightarrow$ need $\langle 21m| x, y, z |100\rangle$

$\uparrow$ 21m $\quad \uparrow$ 100

<u>HW:</u>

- obtain $|\vec{r}_{fi}|^2 = \text{const}\,(\delta_{m,1} + \delta_{m,-1} + \delta_{m,0})$
- if assume that all m-states equally contribute

$$\underbrace{W^{em}_{\substack{i\to f\\ 2p\ 1s}}}_{\Uparrow \text{ find it!}} = \frac{1}{3}\sum_{m=-1}^{1} W_{2pm\to 1s}$$

- Lifetime $\tau = \frac{1}{W_{2p\to 1s}}$