Central Forces Homework 9

Due 6/6/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

REQUIRED:

1. Show that the angular momentum operators L^2 and L_z commute with the central force Hamiltonian H, where

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$L_z = -i\hbar \frac{\partial}{\partial\phi}$$

$$H = -\frac{\hbar^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] + V(r)$$

2. Write out the first 9 terms in the sum:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell,m} Y_{\ell,m}$$

   Describe the energy degeneracy of the rigid rotor system, i.e. give the number of eigenstates that all have the same energy.

3. Consider the normalized function:

$$f(\theta,\phi) = \begin{cases} N\left(\frac{\pi^2}{4} - \theta^2\right) & 0 < \theta < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \theta < \pi \end{cases}$$

   where

$$N = \frac{1}{\sqrt{\frac{\pi^5}{8} + 2\pi^3 - 24\pi^2 + 48\pi}}$$

   (a) Find the $|\ell, m\rangle = |0,0\rangle$, $|1,-1\rangle$, $|1,0\rangle$, and $|1,1\rangle$ terms in a spherical harmonics expansion of $f(\theta,\phi)$.

   (b) If a quantum particle, confined to the surface of a sphere, is in the state above, what is the probability that a measurement of the square of the total angular momentum will yield $2\hbar^2$? $4\hbar^2$?

(c) If a quantum particle, confined to the surface of a sphere, is in the state above, what is the probability that the particle can be found in the region $0 < \theta < \frac{\pi}{6}$ and $0 < \phi < \frac{\pi}{6}$? Repeat the question for the region $\frac{5\pi}{6} < \theta < \pi$ and $0 < \phi < \frac{\pi}{6}$. Plot your approximation from part (a) above and check to see if your answers seem reasonable.

4. Make a table, similar to the one you made for a particle confined to a ring, showing the different representations of the physical quantities associated with the rigid rotor. Include information about the operators $\hat{H}$, $\hat{L}_z$, and $\hat{L}^2$.

Particle on a Ring

	Ket Representation	Wave Function Representation	Matrix Representation
Hamiltonian	$\hat{H}$	$-\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}$	$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & E_1 & 0 & 0 & \cdots \\ \cdots & 0 & E_0 & 0 & \cdots \\ \cdots & 0 & 0 & E_{-1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \hbar^2/2I & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \hbar^2/2I & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
Eigenvalues of Hamiltonian	$E_m = \frac{\hbar^2}{2I}m^2$	$E_m = \frac{\hbar^2}{2I}m^2$	$E_m = \frac{\hbar^2}{2I}m^2$
Normalized Eigenstates of Hamiltonian	$\lvert m\rangle$	$\Phi_m(\phi) = \sqrt{\frac{1}{2\pi r_0}}e^{im\phi}$	$\ldots, \begin{pmatrix} \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \quad \ldots$
Coefficient of m^{th} energy eigenstate	$c_m = \langle m \vert \Phi\rangle$	$c_m = \int_0^{2\pi} \sqrt{\frac{1}{2\pi r_0}} e^{-im\phi}\,\Phi(\phi)\, r_0 d\phi$	$\begin{pmatrix} \cdots & 1 & \cdots & 0 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ c_m \\ \vdots \\ c_0 \\ \vdots \end{pmatrix}$
Probability of measuring E_m	$P(E_m) = \lvert c_{+m}\rvert^2 + \lvert c_{-m}\rvert^2$ $= \lvert\langle +m \vert \Phi\rangle\rvert^2 + \lvert\langle -m \vert \Phi\rangle\rvert^2$	$P(E_m) = \left\lvert \int_0^{2\pi} \sqrt{\frac{1}{2\pi r_0}} e^{-im\phi}\,\Phi(\phi)\, r_0 d\phi \right\rvert^2$ $+ \left\lvert \int_0^{2\pi} \sqrt{\frac{1}{2\pi r_0}} e^{im\phi}\,\Phi(\phi)\, r_0 d\phi \right\rvert^2$	$P(E_m) = \left\lvert \begin{pmatrix} \cdots & 1 & \cdots & 0 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ c_m \\ \vdots \\ c_0 \\ \vdots \end{pmatrix} \right\rvert^2 + \left\lvert \begin{pmatrix} \cdots & 0 & \cdots & 1 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ c_0 \\ \vdots \\ c_{-m} \\ \vdots \end{pmatrix} \right\rvert^2$
Expectation value of Hamiltonian	$\langle \Phi \vert H \vert \Phi\rangle = \sum_m \lvert c_m\rvert^2 E_m$	$\langle \Phi \vert H \vert \Phi\rangle = \int_0^{2\pi} \Phi^*(\phi)\ \hat{H}\ \Phi(\phi)\, r_0 d\phi$	$\langle \Phi \vert H \vert \Phi\rangle = \begin{pmatrix} \cdots & c_1^* & c_0^* & c_{-1}^* & \cdots \end{pmatrix} \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & E_1 & 0 & 0 & \cdots \\ \cdots & 0 & E_0 & 0 & \cdots \\ \cdots & 0 & 0 & E_{-1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ c_1 \\ c_0 \\ c_{-1} \\ \vdots \end{pmatrix}$

Particle on a Ring

Operator for z-component of angular momentum	L_z	$-i\hbar\dfrac{\partial}{\partial\phi}$	$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & 1\hbar & 0 & 0 & \cdots \\ \cdots & 0 & 0\hbar & 0 & \cdots \\ \cdots & 0 & 0 & -1\hbar & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
Eigenvalues of L_z	$m\hbar$	$m\hbar$	$m\hbar$
Normalized Eigenstates of L_z	$\lvert m\rangle$	$\Phi_m(\phi)=\sqrt{\dfrac{1}{2\pi r_0}}e^{im\phi}$	$\ldots,\begin{pmatrix}\vdots\\1\\0\\0\\\vdots\end{pmatrix},\begin{pmatrix}\vdots\\0\\1\\0\\\vdots\end{pmatrix},\begin{pmatrix}\vdots\\0\\0\\1\\\vdots\end{pmatrix}\ \ldots$
Coefficient of m^{th} eigenstates of L_z	$c_m=\langle m\lvert\Phi\rangle$	$c_m=\int_0^{2\pi}\sqrt{\frac{1}{2\pi r_0}}e^{-im\phi}\,\Phi(\phi)\,r_0 d\phi$	$\begin{pmatrix}\cdots & 1 & \cdots & 0 & \cdots\end{pmatrix}\begin{pmatrix}\vdots\\c_m\\\vdots\\c_0\\\vdots\end{pmatrix}$
Probability of measuring $m\hbar$ for z-component of angular momentum	$P(\hbar m)=\lvert c_m\rvert^2=\lvert\langle m\lvert\Phi\rangle\rvert^2$	$P(\hbar m)=\lvert c_m\rvert^2=\left\lvert\int_0^{2\pi}\sqrt{\frac{1}{2\pi r_0}}e^{-im\phi}\,\Phi(\phi)\,r_0 d\phi\right\rvert^2$	$P(m\hbar)=\lvert c_m\rvert^2=\left\lvert\begin{pmatrix}\cdots & 1 & \cdots & 0 & \cdots\end{pmatrix}\begin{pmatrix}\vdots\\c_m\\\vdots\\c_0\\\vdots\end{pmatrix}\right\rvert^2$
Expectation value of z-component of angular momentum	$\langle\Phi\lvert\hat{L}_z\rvert\Phi\rangle=\sum_m\lvert c_m\rvert^2\,m\hbar$	$\langle\Phi\lvert\hat{L}_z\rvert\Phi\rangle=\int_0^{2\pi}\Phi^*(\phi)\,\left(-i\hbar\dfrac{\partial}{\partial\phi}\right)\Phi(\phi)\,r_0 d\phi$	$\langle\Phi\lvert L_z\rvert\Phi\rangle=\begin{pmatrix}\cdots & c_1^* & c_0^* & c_{-1}^* & \cdots\end{pmatrix}\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & 1\hbar & 0 & 0 & \cdots \\ \cdots & 0 & 0\hbar & 0 & \cdots \\ \cdots & 0 & 0 & -1\hbar & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix}\vdots\\c_1\\c_0\\c_{-1}\\\vdots\end{pmatrix}$

5 (Challenge Problem) Let $\mathbf{J}$ be an angular momentum with a set of three observables J_x, J_y, and J_z that satisfy:

$$
\begin{aligned}
[J_x, J_y] &= i\hbar J_z \\
[J_y, J_z] &= i\hbar J_x \\
[J_z, J_x] &= i\hbar J_y
\end{aligned}
$$

$\mathbf{J}^2$, J_+, and J_- are three operators that are defined as following:

$$
\begin{aligned}
\mathbf{J}^2 &= J_x^2 + J_y^2 + J_z^2 \\
J_+ &= J_x + iJ_y \\
J_- &= J_x - iJ_y
\end{aligned}
$$

Show that the operators J_+, J_-, J_z, and $\mathbf{J}^2$ satisfy the following commutation relations:

$$
\begin{aligned}
&[\mathbf{J}^2, J_z] = [\mathbf{J}^2, J_+] = [\mathbf{J}^2, J_-] = 0 \\
&[J_z, J_+] = +\hbar J_+ \\
&[J_z, J_-] = -\hbar J_- \\
&[J_+, J_-] = 2\hbar J_z
\end{aligned}
$$