Mth 435 Assignment #4. John Wacr

(1) $\sigma(u,v) = (\, f(v)\cos(u),\, f(v)\sin(u),\, g(v)\,)$

(a) Compute christoffel symbols

$$\sigma_u = (\, -f(v)\sin(u),\ f(v)\cos(u),\ 0\,)$$

$$\sigma_v = (\, f'(v)\cos(u),\ f'(v)\sin(u),\ g'(v)\,)$$

$$E = \langle \sigma_u, \sigma_u \rangle = (f(v))^2$$

$$F = \langle \sigma_u, \sigma_v \rangle = 0$$

$$G = \langle \sigma_v, \sigma_v \rangle = f'(v)^2 + g'(v)^2$$

$$\alpha = 2EG - F^2 = 2f(v)^2[\, f'(v)^2 + g'(v)^2 \,]$$

Now we are free to assume $\gamma(v) = (\, f(v), 0, g(v)\,)$ is ~~g~~ parametrized by arc length and thus has speed $=1$ $\Longrightarrow$ $G = 1$, $\alpha = 2f(v)^2$.

(a) $\alpha\Gamma^1_{11} = GE_u - 2FF_u + FE_v$ $\qquad$ $\alpha\Gamma^2_{11} = 2EF_u - EE_v - FE_u$

$2f(v)^2\Gamma^1_{11} = 1(0) - 0 + 0$ $\qquad$ $2f(v)^2\Gamma^2_{11} = 0 - f(v)^2 2f(v)f'(v) + 0$

$\Longrightarrow \Gamma^1_{11} = 0$ $\qquad$ $\Gamma^2_{11} = -f(v)f'(v)$

(b) $\alpha\Gamma^1_{12} = GE_v - FG_u$ $\qquad$ $\alpha\Gamma^2_{12} = EG_u - FE_v$

$2f(v)^2\Gamma^1_{12} = 2f(v)f'(v)$ $\qquad$ $2f(v)^2\Gamma^2_{12} = 0$

$\Longrightarrow$ $\Gamma^1_{12} = \dfrac{f'(v)}{f(v)}$ $\qquad$ $\Gamma^2_{12} = 0$

This becomes:

$$\left\{ \begin{array}{l} \Gamma^1_{22} = 0 \\ \\ \Gamma^2_{22} = 0 \end{array} \right\} \text{since} \quad \begin{array}{l} u = \text{const} \\ v = t \end{array}$$

which is true as we found that for surface of revolution $\Gamma^1_{22} = \Gamma^2_{22} = 0$

$\Longrightarrow$ longitudes are geodesics.

(c) Show latitudes are geodesics iff $f' = 0$

Recall christoffel's are

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = -f(v)f'(v)$$

$$\Gamma^1_{12} = \frac{f'(v)}{f(v)} \qquad \Gamma^2_{12} = 0$$

$$\Gamma^1_{22} = 0 \qquad \Gamma^2_{22} = 0$$

For latitudes we have that $u(t) = t$, $v(t) = v_0$

$\longrightarrow f' = 0 \Longrightarrow \Gamma^k_{ij} = 0 \quad \forall\, i,j,k \in \{1,2\}$

* ask about how to deal with u'' & v'' terms *

$u'' = 0$, $v'' = 0$ $\qquad u(t) = t \quad v(t) = v_0 = \text{const}$

$\longrightarrow$ upheld since

(2)

(c) $\alpha\Gamma^1_{22} = 2G\overset{0}{\cancel{F_v}} - G\overset{0}{\cancel{G_u}} - \overset{0}{\cancel{F}}\overset{0}{\cancel{G_v}}$ $\quad \alpha\Gamma^2_{22} = E\overset{0}{\cancel{G_v}} - 2\overset{b-0}{\cancel{F}}\overset{0}{\cancel{F_v}} + \overset{0}{\cancel{F}}\overset{0}{\cancel{G_u}}$

$\Rightarrow 2f(v)^2\Gamma^1_{22} = 0 \qquad 2f(v)^2\Gamma^2_{22} = 0$

$\Rightarrow \Gamma^1_{22} = 0 \qquad \Gamma^2_{22} = 0$

thus
$$\begin{cases} \Gamma^1_{11} = 0 & \Gamma^2_{11} = -f(v)f'(v) \\ \Gamma^1_{12} = \dfrac{f'(v)}{f(v)} & \Gamma^2_{12} = 0 \\ \Gamma^1_{22} = 0 & \Gamma^2_{22} = 0 \end{cases}$$

(b) Show that longitudes parametrized by arc length are geodesics.

$\text{long}(v) = \sigma(u_0, v) = (f(v)\cos(u_0), f(v)\sin(u_0), g(v))$

geodesic equations for curve $\gamma(t) = \sigma(u(t), v(t))$
thus for our longitudes we have
$u(t) = u_0 \quad v(t) = t \quad \forall t$

$\Rightarrow h(t) = (f(t)\cos(u_0), f(t)\sin(u_0), g(t))$

$\Rightarrow u' = 0, \ u'' = 0 \qquad v' = 1, \ v'' = 0$

geodesic equations
$$\begin{cases} u'' + (u')^2\Gamma^1_{11} + 2(u'v')\Gamma^1_{12} + (v')^2\Gamma^1_{22} = 0 \\ v'' + (u')^2\Gamma^2_{11} + 2(u'v')\Gamma^2_{12} + (v')^2\Gamma^2_{22} = 0 \end{cases}$$

$\Leftarrow$ assume latitudes geodesics

show $f'(v)=0$.

$\Rightarrow$ must obey geodesic equations with

$u(t)=t \qquad u'=1 \quad u''=0$

$v(t)=v_0 \qquad v'=0 \quad v''=0$

$\Longrightarrow \qquad \Gamma_{11}^{1}=0$ and $\Gamma_{11}^{1}=0$ already

$\Gamma_{11}^{2}=0$

so we need $\Gamma_{11}^{2}=-f(v)f'(v)=0$

except $f(v)>0$ by def surf of rev

$\Longrightarrow \quad f'(v)=0 \qquad \square$

(d) Prove Clairaut's theorem

$\beta : I \to S$ unit speed curve in S^1.

$\beta(s) = \sigma(u(s), v(s))$. $\quad \rho(s) = f(v(s))$.

$\psi(s) =$ angle between $\beta'(s)$ and longitudinal curve through $\beta(s)$.

(e) $\gamma(s) = (u(s), v(s))$ geodesic parametrized by arc length neither latitude nor long.

show first diffeq is $\quad f^2 u' = \text{const} = c \neq 0$.

Also that $1 = f^2\left(\frac{du}{ds}\right)^2 + \left((f')^2 + (g')^2\right)\left(\frac{dv}{ds}\right)^2$

and together w/ $\quad f^2 u' = c$ is equivalent to 2nd differential equation for a geodesic.

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recall from earlier we had for surface of Revolution, that

$$E = f(v)^2 \qquad F = 0 \qquad G = \underbrace{f'(v)^2 + g'(v)^2 = 1}_{\text{parametrized by arc length.}}$$

thus $$1 = E\left(\frac{du}{ds}\right)^2 + 2F\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + G\left(\frac{dv}{ds}\right)^2$$

and $\quad \Gamma^2_{11} = -f(v)f'(v), \quad \Gamma^1_{12} = \dfrac{f'(v)}{f(v)}$, rest $= 0$

thus first d.e. becomes:

$$u'' + 2u'v'\Gamma^1_{12} = 0$$

$$\Rightarrow \quad u'' + 2u'v'\frac{f'(v)}{f(v)} = 0$$

something happened.

(2) Connected Surface S such that $\forall$ p, q $\exists$ minimal geodesic γ connecting p, q however S not geodesically complete.