4. Solution:

(**a**)

(**b**) For these parts we use mathematica to display the first five Legendre polynomials and then use the Rodrigues' formula (shown in class) to verify that the formula works.

Show the first 5 Legendre polynomials

In[23]:=
```
Do[Print[StringForm["P`` (z) = ``", n, LegendreP[n, z]]], {n, 0, 4}]
```

$P_0(z) = 1$

$P_1(z) = z$

$P_2(z) = \frac{1}{2}\left(-1 + 3 z^2\right)$

$P_3(z) = \frac{1}{2}\left(-3 z + 5 z^3\right)$

$P_4(z) = \frac{1}{8}\left(3 - 30 z^2 + 35 z^4\right)$

Use Rodrigques’ formula to calculate the first 5 polynomials.

$$P_l(z) = \frac{1}{2^l\, l!} \frac{d^l}{dz^l} (z^2 - 1)^l$$

In[32]:=
```
P_rodrigues[ℓ_, z_] := FullSimplify[1/(2^ℓ * ℓ!) * D[(z^2 - 1)^ℓ, {z, ℓ}]]
```

In[35]:=
```
Do[Print[StringForm["P`` (z) = ``", n, P_rodrigues[n, z]]], {n, 0, 4}]
```

$P_0(z) = 1$

$P_1(z) = z$

$P_2(z) = \frac{1}{2}\left(-1 + 3 z^2\right)$

$P_3(z) = \frac{1}{2} z \left(-3 + 5 z^2\right)$

$P_4(z) = \frac{1}{8}\left(3 - 30 z^2 + 35 z^4\right)$

(c) The recurrence relations for Legendre Polynomials (and other special functions) are **NOT** the recurrence relations for Legendre's equation. *Be careful.*

There are two important recurrence relations for the Legendre polynomials. They are

$$(\ell+1)P_{\ell+1} = (2\ell+1)zP_\ell - \ell P_{\ell-1} \tag{1}$$

$$\frac{z^2-1}{\ell}P'_\ell = zP_\ell - P_{\ell-1} \tag{2}$$

We are given that $P_0(z) = 1$ and $P_1(z) = z$. Using, the first recurrence relations, we find that

$$P_2 = \frac{3zP_1 - P_0}{2} = \frac{1}{2}(3z^2 - z) \tag{3}$$

$$P_3 = \frac{5zP_2 - 2P_1}{3} = \frac{5z\left(\frac{1}{2}(3z^2-z)\right) - 2z}{3} = \frac{1}{2}\left(5z^3 - 3z\right) \tag{4}$$

Now we can use the second recurrence relation to find $P'_3(z)$.

$$\begin{aligned}
P'_3(z) &= \frac{3}{z^2-1}\left(zP_3 - P_2\right) && (5)\\
&= \frac{3}{z^2-1}\left(z\frac{1}{2}\left(5z^3-3z\right) - \frac{1}{2}(3z^2-z)\right) && (6)\\
&= \frac{3}{2(z^2-1)} - \frac{9z^2}{(z^2-1)} + \frac{15z^4}{2(z^2-1)} && (7)\\
&= \frac{3}{2}\frac{(z^2-1)(5z^2-1)}{(z^2-1)} && (8)\\
&= \frac{3}{2}\left(5z^2-1\right) && (9)
\end{aligned}$$

Thus we have found expressions for $P_3(z)$ and $P'_3(z)$ using the two recurrence relations. You are encouraged to double check your solution for $P'_3(z)$ by differentiating your result for $P_3(z)$