PH 431 Review for Midterm

Electrostatics in Vacuum

Coulomb's law for the force on a test charge Q due to a single point charge q is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{d^2}\hat{\mathbf{d}} \tag{1}$$

Where we use $\mathbf{d} = \mathbf{r} - \mathbf{r}'$. The Factoring out the test charge Q allows us to define the electric field which depends only on the charged object.

$$\mathbf{F} = Q\mathbf{E} \tag{2}$$

We can define the Electric field for a collection of charges q_i or can extend the concept to continuous distributions:

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{d_i^2}\hat{\mathbf{d}}_\mathbf{i} && (3)\\
&= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{d^2}\hat{\mathbf{d}}dl' && (4)\\
&= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{d^2}\hat{\mathbf{d}}da' && (5)\\
&= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{d^2}\hat{\mathbf{d}}d\tau' && (6)
\end{aligned}$$

Now let's right down the vector identities i.e. Maxwell's equations for electrostatics:

$$\begin{aligned}
\oint \mathbf{E}\cdot d\mathbf{a} &= \frac{Q_{enc}}{\epsilon_0} && (7)\\
\Rightarrow \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho && (8)\\
\oint \mathbf{E}\cdot d\mathbf{l} &= 0 && (9)\\
\Rightarrow \nabla\times\mathbf{E} &= 0 && (10)
\end{aligned}$$

Since the curl of $\mathbf{E}$ is zero, we can define a scalar potential ϕ such that

$$\mathbf{E} = -\nabla\phi \tag{11}$$

$$\Delta\phi = -\int_O^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} \tag{12}$$

The divergence and curl of $\mathbf{E}$ in terms of ϕ become Laplace and Poisson's equations.

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{13}$$

$$\nabla^2\phi = 0 \tag{14}$$

The second equation is Laplace's equation which applies for a region where there is zero free charge. From our definition of $\Delta\phi$ we can deduce definitions for the Electric potential.

$$\begin{aligned}
\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{d_i} && (15)\\
&= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{d}d\tau' && (16)
\end{aligned}$$

Boundary Conditions

The electric field $\mathbf{E}$ always undergoes a discontinuity across a surface charge σ. The amount of this jump can be found using the electric field for a infinite surface and a Gaussian pillbox. The tangential component must always be continuous because $\oint \mathbf{E} \cdot d\mathbf{l} = 0$. Here I use a for above and b for below.

$$\mathbf{E}_a^{\perp} - \mathbf{E}_a^{\perp} = \frac{1}{\epsilon_0}\sigma \tag{17}$$

$$\mathbf{E}_a^{\parallel} = \mathbf{E}_b^{\parallel} \tag{18}$$

In general for any surface, these become the following in terms of the potential (n refers to the 'normal' direction i.e. $\frac{\partial \phi}{\partial n} = \nabla\phi \cdot \hat{n}$)

$$\frac{\partial \phi_a}{\partial n} - \frac{\partial \phi_b}{\partial n} = -\frac{\sigma}{\epsilon_0} \tag{19}$$

First Uniqueness Theorem: The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S.

Image Charge Method

The image charge method is the process of using a combination of fake point charges to simulate the potential of a given surface charge distribution. You need to satisfy Laplace's equation in a particular region ($\nabla^2\phi_{ind} = 0$). Also $\phi_T = \phi_Q + \phi_{ind}$. Apply boundary conditions and insure image charges satisfy (ex: $\phi_{ind} \to 0$ as $r \to \infty$)

Separation of Variables

Given a problem with azimuthal symmetry (i.e. symmetric about z axis) we can immediately write solution to Laplace's equation as:

$$\sum_{L=0}^{\infty} \Big(A_L r^L + \frac{B_L}{r^{L+1}}\Big) P_L(\cos(\theta)) \tag{20}$$

The first 3 Legendre Polynomials of $\cos(\theta)$ are:

$$P_1 = 1 \tag{21}$$

$$P_2 = \cos(\theta) \tag{22}$$

$$P_3 = \frac{1}{2}\Big(3\cos^2(\theta) - 1\Big) \tag{23}$$

In general you just need to find A_L, B_L that satisfy b.c.'s. For example if $\phi_{in} \to 0$ as $r \to \infty$ then $A_L = 0 \quad \forall L$. If you need ϕ_{in} to be defined at $r = 0$ then $B_L = 0 \quad \forall L$.

Multi-pole expansion

The multi-pole expansion is the expansion of the potential for any distribution into powers of $1/r$. Note that:

$$\frac{1}{d} = \frac{1}{r}\sum_{n=0}^{\infty}\Big(\frac{r'}{r}\Big)^n P_n(\cos(\alpha)) \tag{24}$$

Plugging this into our definition of ϕ gives us the multi-expansion:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int(\mathbf{r}')^n P_n)(\cos(\alpha))\rho(\mathbf{r}')d\tau' \tag{25}$$

So that you can use the famous physics method of **Guess It**, here are the monopole and dipole terms at large r:

$$\phi(\mathbf{r})_{mon} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \tag{26}$$

$$\phi(\mathbf{r})_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \tag{27}$$

for a physical dipole, $\mathbf{p} = q\mathbf{d}$ where d goes from negative to positive.

Electric Fields In Matter

A dipole in a uniform electric field experiences a torque $\mathbf{N} = \mathbf{p}\times\mathbf{E}$. The force on the dipole can be written as: $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$.

For polarizable materials we will consider collections of dipoles that we can describe with a dipole moment density. We will denote this as $\mathbf{P}$ which has dimensions of dipole per volume. Now if we want to construct the potential for this distribution, we can take equation (27).

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{d}}}{d^2}$$

This formulation is equivalent to a group of surface bound charges and volume bound charges:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b da'}{d} + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b d\tau'}{d} \tag{28}$$

Where $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ and $\rho_b = -\nabla\cdot\mathbf{P}$.

If we assume materials to be linearly polarizable, we can define some useful fields with corresponding differential vector equations.

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \tag{29}$$

$$\nabla\cdot\mathbf{D} = \nabla\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho_f \tag{30}$$

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \tag{31}$$

Dielectric Boundary Conditions

The boundary conditions we have from the homework are:

$$\phi_{in} = \phi_{out}, \quad r = R \tag{32}$$

$$\epsilon_1\frac{\partial\phi_{in}}{\partial r} = \epsilon_2\frac{\partial\phi_{out}}{\partial r} \tag{33}$$

The boundary equations essentially are that ϕ is continuous across the interface, $\epsilon\partial_\perp\phi$ is continuous across interface and $\partial_\parallel\phi$ is continuous across the interface.

Strategies for polarizable materials: Solve for the displacement field $\mathbf{D}$ then we want to go from ρ_f to our $\mathbf{D}$ field. To do that we need *symmetry*. Then you want to take $\mathbf{D}$ and derive **E, P** i.e. use $\nabla \cdot \mathbf{D} = \rho_f$. We assume that the free charge ρ_f will be given. Now for linearly polarizable materials we have $\mathbf{D} = \epsilon \mathbf{E}$ and $\epsilon = \epsilon_0 \epsilon_r$. Now to get $\mathbf{P}$, you use $\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{D}$.

At a boundary, we derived that $\Delta \mathbf{E} = \frac{\sigma_b}{\epsilon_0}$... Katy wanted me to add this... so here it is.

Energy in a Dielectric System

The energy stored in a dielectric configuration is equivalent to the work required to form the system. This is given by:

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau \tag{34}$$

Also the energy of a capacitor is given by $U = \frac{1}{2}CV^2$

Magnetostatics

The magnetic force on a particle moving through magnetic and electric fields is given by the Lorentz force law (taken to be true based off of observation). It is:

$$\mathbf{F}_{E/M} = Q)[\mathbf{E} + \vec{v} \times \mathbf{B}] \tag{35}$$

Notice that because the magnetic force always points PERPENDICULAR to velocity, **the magnetic field does NO work**.

The equivalent of charge densities to magnetostatics are current densities. They are:

$$\mathbf{I} = \lambda \vec{v} \tag{36}$$

$$\mathbf{K} = \sigma \vec{v} \tag{37}$$

$$\mathbf{J} = \rho \vec{v} \tag{38}$$

From the last density, we will write the general equation for the magnetic force as:

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B} d\tau \tag{39}$$

For steady current systems, the magnetic field can be calculated via the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \vec{d}}{d^3} d\tau' \tag{40}$$

The equivalent of Gauss's law for magnetostatics is called Ampere's law. It is:

$$\oint \mathbf{B} \cdot d\vec{l} = \mu_0 I_{enc} \tag{41}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{42}$$

Now we can summarize the equations into Maxwell's laws for vacuum:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{43}$$
$$\nabla \times \mathbf{E} = 0 \tag{44}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{45}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{46}$$

Also, since the divergence of the $\mathbf{B}$ field is zero, we can define a vector potential $\mathbf{A}$ since the divergence of the curl is always zero:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r'})}{d} d\tau' \tag{47}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{48}$$

The magnetic dipole moment, $\mathbf{m}$, is given by:

$$\mathbf{m} \equiv I \int d\mathbf{a} \tag{49}$$

Magnetic Fields in Matter

Similar to electric forces, magnetic dipoles in a magnetic field experience a torque defined by:

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \tag{50}$$

This torque is what accounts for paramagnetism since it aligns the dipole to be parallel with the magnetic field. The force for an infinitesimal loop around a dipole $\mathbf{m}$ in a field $\mathbf{B}$ is:

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{51}$$

As with electrically polarizable materials we can consider materials as collections of magnetic dipoles which we will account for by the magnetic dipole density $\mathbf{M}$ which has dimensions of magnetic dipole per volume. The magnetic vector potential for such a distribution is given by:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r'}) \times \hat{\mathbf{d}}}{d^2} d\tau' \tag{52}$$

We can show this is equivalent to a surface bound current $\mathbf{K}_b$ and a volume bound current $\mathbf{J}_b$ such that:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J_b}(\mathbf{r'})d^3\mathbf{r'}}{d} + \frac{\mu_0}{4\pi}\int \frac{\mathbf{K_b}(\mathbf{r'})d^2\mathbf{r'}}{d} \tag{53}$$

Where $\mathbf{J_b} = \nabla \times \mathbf{M}$ and $\mathbf{K_b} = \mathbf{M} \times \hat{\mathbf{n}}$

Now recall that for magnetostatics we know that:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{54}$$
$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b \tag{55}$$
$$= \mathbf{J}_f + \nabla \times \mathbf{M} \tag{56}$$
$$\Rightarrow \nabla \times (\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}) = \mathbf{J} - \mathbf{J_b} = \mathbf{J_f} \tag{57}$$
$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{58}$$

Now we have our analogous field to the displacement field with the vector identity that:

$$\nabla \times \mathbf{H} = \mathbf{J_f} \tag{59}$$

Thus the correspondence is that given $\mathbf{M}$ and $\mathbf{H}$ we can solve for $\mathbf{B}$ using (58). Then we have that $\mathbf{B} = \mu\mathbf{H}$ for linearly polarizable materials where $\mu = \mu_0(1+\chi_m)$. Then we can solve for the bound current densities using equations $\mathbf{J_b} = \nabla \times \mathbf{M}$ and $\mathbf{K_b} = \mathbf{M} \times \hat{\mathbf{n}}$.

usefull stuff...

the magnetic field for a naked wire is given by:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{60}$$

The magnetic field for a solenoid of n turns per length is given by:

$$\mathbf{B} = \mu_0 n I \hat{z} \tag{61}$$