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9.3.15

List all of the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Recall the definitions fo the following groups:

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}$$
$$\mathbb{Z}_2 = \{0, 1\}$$

Thus, we create $\mathbb{Z}_2 \times \mathbb{Z}_4$ via the Cartesian product of the two sets:

$$\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)\} \tag{1}$$

9.3.16.b

Find the order of $(6, 15, 4) \in \mathbb{Z}_{30} \times \mathbb{Z}_{45} \times \mathbb{Z}_{24}$.

Recall Corollary 9.18: For $(g_1 ... g_n) \in \Pi_i G_i$ if g_i has finite order r_i then the order of $(g_1 ... g_n)$ is the least common multiple of $r_1, ... r_n$.

So, we simply need to find the individual order of 6, 15, and 4 in order to determine the order of (6,15,4). Observe that:

$$\begin{aligned} 6 * 5 \quad &\text{mod } (30) = 0 \\ 15 * 3 \quad &\text{mod } (45) = 0 \\ 4 * 6 \quad &\text{mod } (24) = 0 \end{aligned}$$

So now that we have the order of each number in the tuple, the order of (6,15,4) is simply:

$$LCM(6, 15, 4) = 30 \tag{2}$$

Thus the order of (6, 15, 4) is 30 by Corollary 9.18. □

9.3.32

Prove that $U(5) \cong \mathbb{Z}_4$. *Can you generalize this for U(p) where p is prime?*

Recall that $U(5) = \{1, 2, 3, 4\}$ and $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. To prove that these two groups are isomorphic, we simply need to prove that U(5) is cyclic as both U(5) and $\mathbb{Z}_4$ have the same order (4).

$$\begin{aligned} 2 \quad &\text{mod } (5) = 2 \\ 2^2 \quad &\text{mod } (5) = 4 \\ 2^3 \quad &\text{mod } (5) = 8 \quad \text{mod } (5) = 3 \\ 2^4 \quad &\text{mod } (5) = 16 \quad \text{mod } (5) = 1 = e \end{aligned}$$

Thus 2 is a generator for U(5) and therefore U(5) is cyclic. U(5) is a cyclic group of order 4 and so by theorem 9.8, $U(5) \cong \mathbb{Z}_4$.

In order to extend this theorem to groups of the form $U(p)$ where p is prime, we need to be able to prove that $U(p)$ is cyclic so long a p is prime. First let's consider the type of elements in U(p). By definition this is all of the non-zero elements of $\mathbb{Z}_p$ that are relatively prime to p (i.e. k such that $gcd(k,p) = 1$). Since p itself is prime it's only divisors are 1 and itself. Thus U(p) is necessarily all of the integers from 1 up to p-1:

$$U(p) = \{1, 2, 3...p - 1\} \tag{3}$$

It can be shown that U(p) is cyclic and so by theorem 9.8 it must be isomorphic to $\mathbb{Z}_{p-1}$.

10.3.1.b

Determine whether $H = \{(1), (123), (132)\}$ is a normal subgroup of A_5.

Recall that A_5 is the subgroup of even permutations of S_5, the symmetric group of permutations on 5 letters. Observe that the cycle (134) is an element of A_5 as it is equivalent to (14)(13). Now for H to be a normal group we must have $gH = Hg \quad \forall g \in A_5$. The following will show that this is not true for the cycle in A_5 listed above:

$$\begin{aligned}
(134)H &= \{(134)(1), (134)(123), (134)(132)\} && (4)\\
&= \{(134), (124), (14)(23)\} && (5)\\
H(134) &= \{(1)(134), (123)(134), (132)(134)\} && (6)\\
&= \{(134), (234), (12)(34)\} && (7)
\end{aligned}$$

Thus $(134)H \neq H(134)$ and so H is not a normal subgroup of A_5.

10.3.4.c

Prove that U is normal in T

By definition we have the following:

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : ac \neq 0, \quad a, b \in \mathbb{R} \right\} \tag{8}$$

$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \tag{9}$$

By theorem 10.3 we need to show that $\forall m \in T$, $mUm^{-1} = U$. Let:

$$m = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$\Rightarrow m^{-1} = \frac{1}{ac} \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$$

Now with this we have:

$$\begin{aligned} mUm^{-1} &= \frac{1}{ac}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \\ &= \frac{1}{ac}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} c & -b+ax \\ 0 & ac \end{pmatrix} \\ &= \frac{1}{ac}\begin{pmatrix} ac & -ab+a^2x+ab \\ 0 & ac \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{a}{c}x \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now if we let $x' = \frac{a}{c}x$ Then we have shown:

$$mUm^{-1} = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \tag{10}$$

Thus since x' is arbitrary in $\mathbb{R}$ we have shown that $mUm^{-1} = U$ and thus U is normal in T. $\square$

10.3.6

If G is abelian, prove G/H is also abelian

Recall that G is abelian if $\forall a, b \in G$, $ab = ba$. By definition G/H is the group of cosets of H in G under the operation $(aH)(bH) = abH$. Now let $\alpha, \beta \in G$. By definition of G/H,

$$(\alpha H)(\beta H) = (\alpha\beta)H$$

Since we are given that G is abelian,

$$\begin{aligned} (\alpha\beta)H &= (\beta\alpha)H \\ &= (\beta)H(\alpha)H \\ \Rightarrow (\alpha)H(\beta)H &= (\beta)H(\alpha)H \end{aligned}$$

Therefore, because G is abelian, G/H must also be abelian. $\square$