Homework 1 **John Waczak**
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The line element for a Schwarzschild black hole takes the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

where the mass m and the radius r are measured in the same units

1. PROPER DISTANCE BETWEEN SHELLS

(a) Use the line element to find a formula for the proper distance between *nearby* spherical shells (surfaces with $r =$ constant). That is, find an expression for the infinitesimal distance between nearby shells, assuming that only the radius changes (and that $r > 2m$).

Given that the only coordinate changing between shells is r, our simplified line element is

$$ds^2 = \frac{1}{1 - \frac{2m}{r}}dr^2 \quad (2)$$

so that the infinitesimal distance is given by

$$d\ell = \sqrt{ds^2} = \frac{1}{\sqrt{1 - \frac{2m}{r}}}dr \quad (3)$$

where I have used $d\ell$ to make it clear that this is a distance and to emphasize that it is greater than zero and not equal to dr.

(b) As you approach the horizon ($r \to 2m^+$), what happens to your expression? How far away do you think the horizon is? Do you think that you can ever get to the horizon?

As we take the above limit, $d\ell \to \infty$. This is interesting because we would have a separate singularity if $r \to 0$. I am a little confused about the wording "How far away do you think the horizon is". As we have yet to define *horizon*, it appears that a reasonable definition would be to chose

$$r_s = 2m \quad (4)$$

I am unsure of how to answer the question if the intention is what would I see as an observer. Certainly, if I am moving towards the black hole, we can not neglect the dt term, so I will assume the question is asking for a definition of the horizon of a black hole based on the Schwarzschild metric we were given.

Given that the equation blows up as $r \to 2m$, it seems unlikely that the horizon can be reached, however I'm not sure how to consider this from the perspective of various observers yet...

2. EARTH DISTANCE

The mass m of a particular black hole is 5 km, a little more than three times that of our Sun. Two concentric spherical shells surround this black hole. The inner shell has circumference $2\pi r$, and the outer shell has a circumference $2\pi(r + \Delta r)$, where $\Delta r = 100$ cm. Use your expression for the infinitesimal distance between nearby shells to estimate the radial distance between the shells in each of the cases bellow. *Explicitly state any approximations you make.*

(**a**) $r = 50$

(**b**) $r = 15$

(**c**) $r = 10.5$

First, note that 100 cm = 0.001 km. Given this fact, I will use the value given for r as my estimate for the radius along the path. Our approximation to each distance is therefore given by

$$\Delta s \approx \frac{1}{\sqrt{1 - \frac{2m}{r}}}\Delta r \tag{5}$$

Plugging in the values for each case yields the following

$$(a) \qquad \Delta s \approx \frac{1}{\sqrt{1 - \frac{10}{50}}}0.001 \approx 0.00112 \text{ km} \tag{6}$$

$$(b) \qquad \Delta s \approx \frac{1}{\sqrt{1 - \frac{10}{15}}}0.001 \approx 0.00173 \text{ km} \tag{7}$$

$$(a) \qquad \Delta s \approx \frac{1}{\sqrt{1 - \frac{10}{10.5}}}0.001 \approx 0.00458 \text{ km} \tag{8}$$

$$\tag{9}$$

Here we can clearly see how a change in radial coordinate by Δr corresponds to an increasingly larger change in distance Δs as the segment Δr is moved towards the black hole.

3. EXACT PROPER DISTANCE

(**a**) Use your expression for the infinitesimal distance between nearby shells to determine the exact (radial) distance traveled between two spherical shells of arbitrary circumference (but outside the horizon, that is, with $r > 2m$).

In order to find the exact distance, we must integrate the line element. If the radii of the two shells are R_1, and R_2, then the integral becomes

$$\text{dist} = \int_{R_1}^{R_2} \frac{dr}{\sqrt{1 - \frac{2m}{r}}} \tag{10}$$

This is a challenging integral for which I used mathematica to simplify. The attached code shows the evaluation of the indefinite version of (10) which we can then use to evaluate at our radii. Alternatively, we can use the substitution $\cosh\alpha = \sqrt{\frac{r}{2m}}$ but this requires good knowledge of hyperbolic trig identities which I do knot yet have.

In[1]:= `$Assumptions = {m ∈ Reals, r ∈ Reals}`

Out[1]= $\{m \in \mathbb{R}, r \in \mathbb{R}\}$

In[8]:= $$\int \frac{1}{\sqrt{1 - \frac{2*m}{r}}} \, dr$$

Out[8]= $$r \sqrt{\frac{-2 m + r}{r}} + m \operatorname{Log}\left[-m + r + r \sqrt{\frac{-2 m + r}{r}}\right]$$

Therefore, the fully evaluated integral is given by the following

$$\text{dist} = R_2\sqrt{1-\frac{2m}{R_2}} - R_1\sqrt{1-\frac{2m}{R_1}} + m\ln\left(\frac{R_2 - m + R_2\sqrt{1-\frac{2m}{R_2}}}{R_1 - m + R_1\sqrt{1-\frac{2m}{R_1}}}\right) \tag{11}$$

If we use this equation to find the exact values for the previous problem, the values become

$$(a) \qquad R_1 = 50 \qquad R_2 = 50 + 0.001 \qquad \text{dist} = 0.001118 \tag{12}$$
$$(b) \qquad R_1 = 15 \qquad R_2 = 15 + 0.001 \qquad \text{dist} = 0.001731 \tag{13}$$
$$(c) \qquad R_1 = 10.5 \qquad R_2 = 10.5 + 0.001 \qquad \text{dist} = 0.004580 \tag{14}$$

Interestingly enough, these values do not deviate significantly from our approximation in the previous problem. This is likely due to the fact that $\Delta r = 0.001$ km is such a small fraction of the radii used.

(b) Use your result to decide whether the radial distance to the horizon is finite or infinite.

To evaluate whether or not the distance to the horizon is finite, we can simply take the limit as $R_1 \to 2m$ in equation (11). Doing so results in the expression

$$\text{dist} = R_2\sqrt{1-\frac{2m}{R_2}} + m\ln\left(R_2 - m + R_2\sqrt{1-\frac{2m}{R_2}}\right) - m\ln(m) \tag{15}$$

Clearly, this result is finite and therefore, for finite R_2 the distance to the horizon is finite. This result is definitely counter intuitive to what I thought by simply examining the *blow up* of ds at $r = 2m$.