Homework 1B **John Waczak**

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Dr. Escher

Total Time: 2 hours

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(a) *Which of the following are regular curves?*

1. $\alpha(\theta) = (\cos(\theta), 1 - \cos(\theta) - \sin(\theta), -\sin(\theta))$

   Recall that a curve is regular if it's speed is never zero. Thus we need to find $|\alpha'(\theta)|$

$$\begin{aligned}
\alpha'(\theta) &= (-\sin(\theta), \sin(\theta) - \cos(\theta), -\cos(\theta)) \\
|\alpha'(\theta)| &= (\sin^2(\theta) + (\sin(\theta) - \cos(\theta))^2 + \cos^2(\theta))^{1/2} \\
&= \sqrt{2 - 2\sin(\theta)\cos(\theta)}
\end{aligned}$$

   Thus because sine and cosine are never 1 at the same time, this function will never take on the value of zero. Therefore, $\alpha(\theta)$ is a regular curve.

2. $\beta(\theta) = (2\sin^2(\theta), 2\sin^2(\theta)\tan(\theta), 0)$

$$\begin{aligned}
\beta'(\theta) &= (4\cos(\theta)\sin(\theta), 4\sin^2(\theta) + 2\tan^2(\theta), 0) \\
|\beta'(\theta)| &= \sqrt{16\cos^2(\theta)\sin^2(\theta) + (4\sin^2(\theta) + 12\tan^2(\theta))^2} \\
|\beta'(0)| &= \sqrt{0^2 + (0 + 0^2)} = 0
\end{aligned}$$

   Thus because $|\beta'(\theta)|$ does take the value of zero, the curve is *not* regular.

3. $\gamma(\theta) = (\cos(\theta), \cos^2(\theta), \sin(\theta))$

$$\begin{aligned}
\gamma'(\theta) &= (-\sin(\theta), -2\sin(\theta)\cos(\theta), \cos(\theta)) \\
|\gamma'(\theta)| &= \sqrt{\cos^2(\theta) + \sin^2(\theta) + 4\cos^2(\theta)\sin^2(\theta)} \\
&= \sqrt{1 + 4\cos^2(\theta)\sin^2(\theta)} \\
&= \sqrt{\sin^2(2\theta) + 1} \\
0 &= \sin^2(2\theta) + 1 \\
\Rightarrow \sin^2(2\theta) &= -1
\end{aligned}$$

   Because this last statement can not be true without letting θ be complex, there is no way for $\gamma(\theta)$ to be zero and therefore it is a regular curve.

(b) Find the tangent line to each curve at $\theta = \frac{\pi}{4}$.

In general construct the tangent line for a parametrized curve $f(t) = (f_x(t), f_y(t), f_z(t))$ at a point t_0 we need to evaluate the function and it's derivative. i.e.

$$T_{t_0}(t) = f(t_0) + f'(t_0)t$$

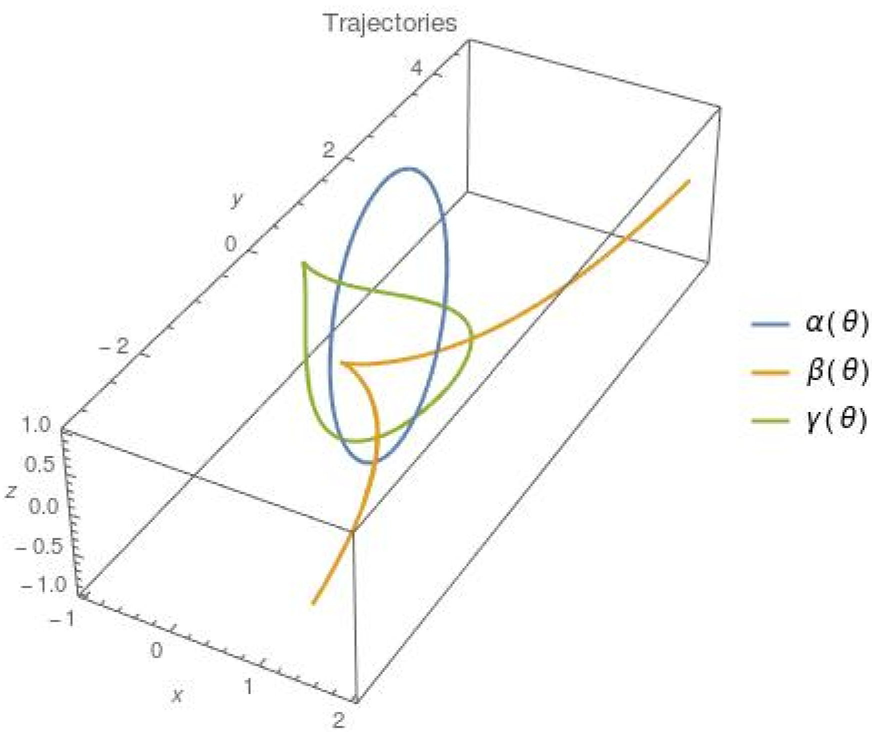
Thus we will make use of the following vectors:

$$\alpha(\pi/4) = \Big(\frac{1}{\sqrt{2}}, 1-\sqrt{2}, -\frac{1}{\sqrt{2}}\Big)$$
$$\alpha'(\pi/4) = \Big(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\Big)$$
$$\beta(\pi/4) = (1,1,0)$$
$$\beta'(\pi/4) = (2,4,0)$$
$$\gamma(\pi/4) = \Big(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{2}}\Big)$$
$$\gamma'(\pi/4) = \Big(-\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}\Big)$$

Using this information, the tangent lines are:

$$T_\alpha(\theta) = \Big(\frac{1}{\sqrt{2}} - \frac{\theta}{\sqrt{2}}, 1-\sqrt{2}, -\frac{1}{\sqrt{2}} - \frac{\theta}{\sqrt{2}}\Big)$$
$$T_\beta(\theta) = \Big(1+2\theta, 1+4\theta, 0\Big)$$
$$T_\gamma(\theta) = \Big(\frac{1}{\sqrt{2}} - \frac{\theta}{\sqrt{2}}, \frac{1}{2} - \theta, \frac{1}{\sqrt{2}} + \frac{\theta}{\sqrt{2}}\Big)$$

(c) graph α, β, γ.



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1.17: Let $\mathbf{x} = (0,2)$ and $\mathbf{y} = (3,4)$. Find the component and projection of $\mathbf{x}$ in the direction of $\mathbf{y}$. Write $\mathbf{x}$ as the sum of vectors, one parallel to $\mathbf{y}$ and the other orthogonal to $\mathbf{y}$.

Recall the familiar inner product identity $\langle \mathbf{u}, mathbf{v} \rangle = uv\cos(\theta)$ where θ is the angle between the two vectors. If we manipulate this equation we can identify the component of u in the direction of v as $u\cos(\theta) \equiv \frac{\langle \mathbf{uv} \rangle}{|v|}$. From this component we define the projection by multiplying by the unit vector in the y direction. In $\mathbb{R}^2$ our inner product is the simple dot product and so:

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= (0,2) \cdot (3,4) = 8 \\
\text{comp}_x y &= \frac{8}{\sqrt{3^2+4^2}} = \frac{8}{5} \\
\text{proj}_x y &= \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2}\mathbf{y} = \frac{8}{25}\mathbf{y} = \frac{8}{25}(3,4)
\end{aligned}$$

Now to find the perpendicular component we have a 2x2 system:

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}^{\parallel} + \mathbf{x}^{\perp} \\
(0,2) &= \frac{8}{25}(3,4) + (a,b) \\
0 &= \frac{8}{25} \cdot 3 + a \\
2 &= \frac{8}{25} \cdot 4 + b \\
\Rightarrow a &= -\frac{24}{25} \\
b &= \frac{18}{25} \\
\Rightarrow \mathbf{x}^{\perp} &= \frac{3}{25}(-8,6) \\
\mathbf{x} &= \frac{8}{25}(3,4) + \frac{3}{25}(-8,6)
\end{aligned}$$

Thus we have written $\mathbf{x}$ in terms of its projection.

1.26 What can be said about a space filling curve of constant acceleration?

Given a space filling curve $\gamma(t)$ with constant acceleration, integrating once allows us to find a formula for the velocity:

$$\begin{aligned}
\gamma''(t) &= \vec{a} \\
\Rightarrow \gamma'(t) &= \mathbf{a}t + \mathbf{b}
\end{aligned}$$

Where $\mathbf{b}$ is a vector of initial velocities. From this equation we can define a third vector, $\mathbf{n}$ which is necessarily perpendicular to $\gamma'(t)$.

$$\begin{aligned}
&\mathbf{n} = \mathbf{a} \times \mathbf{b} \\
&\Rightarrow \mathbf{n} \cdot \mathbf{a} = 0, \quad \mathbf{n} \cdot \mathbf{b} = 0
\end{aligned}$$

Now because the time dependence of $\gamma'(t)$ serves to simply stretch and compress the $\mathbf{a}$ vector, $\mathbf{n}$ will *always* be orthogonal to the velocity $\gamma'(t)$. This can be summarized by the following statement:

$$\begin{aligned}
&\gamma'(t) \cdot \mathbf{n} = 0 \\
&x'(t)n_x + y'(t)n_y + z'(t)n_z = 0
\end{aligned}$$

Integrating once more, we find that:

$$x(t)n_x + y(t)n_y + z(t)n_z = k$$

Where k is a constant. Choosing an initial time t_0 allows us to make the following simplification.

$$\begin{aligned}
&x(t_0)n_x + y(t_0)n_y + z(t_0)n_z = k \\
&\Rightarrow x(t)n_z + y(t)n_y + z(t)n_z - (x(t_0)n_x + y(t_0)n_y + z(t_0)n_z) = 0 \\
&(x(t) - x(t_0))n_z + (y(t) - y(t_0))n_y + (z(t) - z(t_0))n_z = 0 \\
&(\gamma(t) - \gamma(t_0)) \cdot \mathbf{n} = 0
\end{aligned}$$

The final equation, we see, is the definition for a plane. Thus we can say that for a curve with constant acceleration, integrating once gives a velocity from which we can define a normal vector that is perpendicular to the position function $\gamma(t)$ for every value of t. Integrating again we can show that $\gamma(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ meaning that the curve is a quadratic curve. Therefore, as in exercise 1.6 in page 7, the points of the quadratic curve all lie in a plane like we have shown.

1.29 *Verify that $\tilde{\gamma}$ is a reparametrization of γ. Hint:* $t = \tan(s/2)$

$$\begin{aligned}
t &= tan(s/2) \\
\frac{1-t^2}{1+t^2} &= \frac{1-\tan^2(s/2)}{1+\tan^2(s/t)} \\
&= \frac{2 - sec^2(s/2t)}{sec^2(s/2)} \\
&= \frac{2 - \frac{1}{\cos^2(s/2)}}{\frac{1}{\cos^2(s/2)}} \\
&= 2\cos^2(s/2) - 1 \\
&= \cos(s)
\end{aligned}$$

$$\begin{aligned}
\frac{2t}{1+t^2} &= \frac{2\tan(s/2)}{1+\tan^2(s/2)} \\
&= \frac{2\tan(s/2)}{\sec^2(s/2)} \\
&= 2\sin(s/2)\cos(s/2) \\
&= \sin(s)
\end{aligned}$$

Thus $\tilde{\gamma}$ is a reparametrization for $\gamma(t)$.

3

Reparametrize $\alpha(t) = (e^t \cos(t), e^t \sin(t), e^t)$ by arc length.

$$\begin{aligned}
\alpha'(t) &= (e^t \cos(t) - e^t \sin(t), e^t \cos(t) + e^t \sin(t), e^t) \\
|\alpha'(t)| &= \sqrt{(e^t \cos(t) - e^t \sin(t))^2 + (e^t \cos(t) + e^t \sin(t))^2 + e^{2t}} \\
&= \sqrt{2e^{2t}(\cos^2(t) + \sin^2(t)) + e^{2t}} \\
&= \sqrt{3e^{2t}} \\
&= \sqrt{3}e^t
\end{aligned}$$

Now if we allow t to start at $t = 0$ then we can find the arc length parametrization as follows:

$$\begin{aligned}
s &= \int_0^t \sqrt{3}e^{t'} dt' \\
&= \sqrt{3}(e^t - e^0) \\
&= \sqrt{3}(e^t - 1) \\
\Rightarrow t &= \ln\left(\frac{s}{\sqrt{3}} + 1\right)
\end{aligned}$$

And so the completely reparametrized function is:

$$\begin{aligned}
\alpha(s) &= \left(e^{\ln(\frac{s}{\sqrt{3}}+1)} \cos\left(\ln(\frac{s}{\sqrt{3}} + 1)\right), e^{\ln(\frac{s}{\sqrt{3}}+1)} \sin\left(\ln(\frac{s}{\sqrt{3}} + 1)\right), e^{\ln(\frac{s}{\sqrt{3}}+1)}\right) \\
&= \left((\frac{s}{\sqrt{3}} + 1) \cos\left(\ln(\frac{s}{\sqrt{3}} + 1)\right), (\frac{s}{\sqrt{3}} + 1) \sin\left(\ln(\frac{s}{\sqrt{3}} + 1)\right), (\frac{s}{\sqrt{3}} + 1)\right)
\end{aligned}$$