Multivariable Calculus 5c

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Tapp 3.1

let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation
$\forall p \in \mathbb{R}^m$ prove that $df_p = f$

let $p, v \in \mathbb{R}^m$. Then by definition 3.3, we have

$$df_p(v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

because f is a linear operator we may distribute it across $p + tv$. Note t is a scalar. Thus

$$df_p(v) = \lim_{t \to 0} \frac{f(p) + t f(v) - f(p)}{t}$$

$$= \lim_{t \to 0} \frac{t f(v)}{t}$$

$$= \lim_{t \to 0} f(v) = f(v)$$

thus $df_p(v) = f(v)$

Therefore, because p was arbitrary

$$df_p = f \quad \forall p \in \mathbb{R}^m$$

Tapp 3.2

If $A \in O(3)$ is an orthogonal matrix, $v \in \mathbb{R}^3$, f is the rigid motion $T_v \circ L_A : \mathbb{R}^3 \to \mathbb{R}^3$ prove $\forall p$ that $df_p = L_A$.

Let $p, q \in \mathbb{R}^3$. Then by def 3.3, we have

$$df_p q = \lim_{t \to 0} \frac{f(p+tq) - f(p)}{t}$$

because f is a composition of L_A w/ translation by v we have

$$= \lim_{t \to 0} \frac{T_v \circ L_A(p+tq) - T_v \circ L_A(p)}{t}$$

Since $A \in O(3)$, L_A is a linear operator we can distribute so that

$$= \lim_{t \to 0} \frac{T_v \circ (L_A(p) + tL_A(q)) - T_v \circ L_A(p)}{t}$$

Now T_v just translates by v so we have

$$= \lim_{t \to 0} \frac{(L_A(p) + tL_A(q)) + v - (L_A(p) + v)}{t}$$

$$= \lim_{t \to 0} \frac{L_A(p) + tL_A(q) - L_A(p) + v - v}{t}$$

$$= \lim_{t \to 0} \frac{tL_A(q)}{t}$$

$$= \lim_{t \to 0} L_A(q) = L_A(q)$$

Therefore, because p, q were arbitrary we have shown $df_p = L_A \quad \forall p \in \mathbb{R}^3$

Tapp 3.3

$$f(x,y) = (5x^2y^3, 2x+y^2, x^2-y^2)$$

$$J \in \mathcal{M}_{3\times 2} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} 10xy^3 & 2 & 2x \\ 15x^2y^2 & 2y & -2y \end{pmatrix}$$

Now evaluate $J(1,-1)$

$$J(1,-1) = \begin{pmatrix} -10 & 2 & 2 \\ 15 & -2 & 2 \end{pmatrix}$$

The rank of this matrix is 2 as we see

$$\det\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = 4+4 = 8 \neq 0$$

Tapp 3.4. is $f(x,y) = (x^3, y)$ a diffeomorphism?

Recall Definition 3.13:

a diffeomorphism is a smoot bijective function whose inverse is also smooth.

Yes $f^{-1} = (x^{1/3}, y)$

and all functions involved are smooth

i.e. f^{-1} is C^r $\forall r \in \mathbb{Z}^+$