QM III
hep 653

Lecture # 4

Time-dependent perturbation: special cases

Last time: if the system is in some initial state $|i\rangle$, a perturbation $\hat{V}(t)$ is turned on at $t=0$, the probability that the system will make a transition to state $|f\rangle$ after time t:

$$\mathcal{P}_{i\to f}(t) = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{fi}t'}\,V_{fi}(t')\,dt'\right|^2,$$

to the 1st order; $i\neq f$

$$\omega_{fi} = \frac{E_f - E_i}{\hbar}$$

$$V_{fi} = \langle f|\hat{V}(t)|i\rangle$$

Special cases:

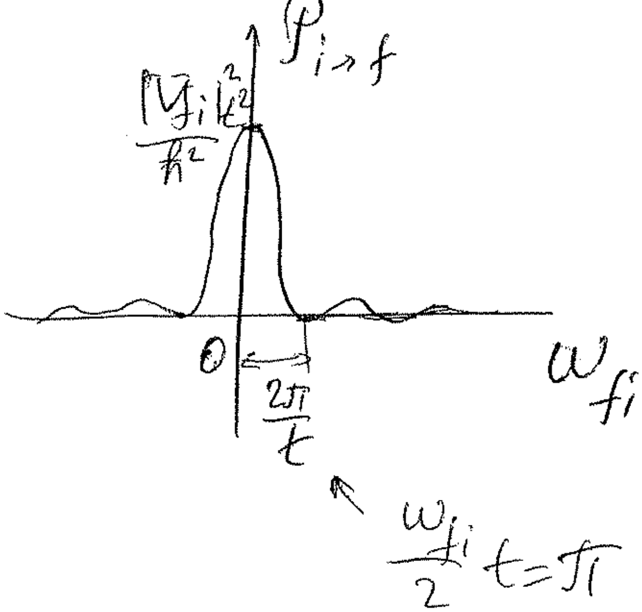
(a) $\hat{V}(t) \neq$ function of time (so, at $t=0$, some time-independent perturbation is applied)

but can be function of $\vec{X}, \vec{P}, \vec{S}, \ldots$

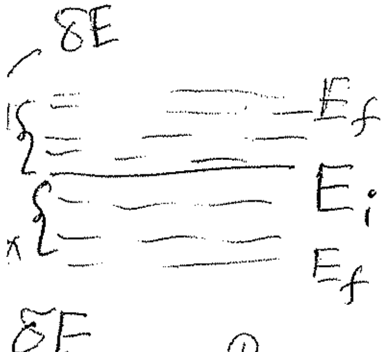
$$\mathcal{P}_{i\to f}(t) = \frac{1}{\hbar^2}|V_{fi}|^2\cdot\underbrace{\left|\int_0^t e^{i\omega_{fi}t'}dt'\right|^2}_{\frac{1}{i\omega_{fi}}(e^{i\omega_{fi}t}-1)} =$$

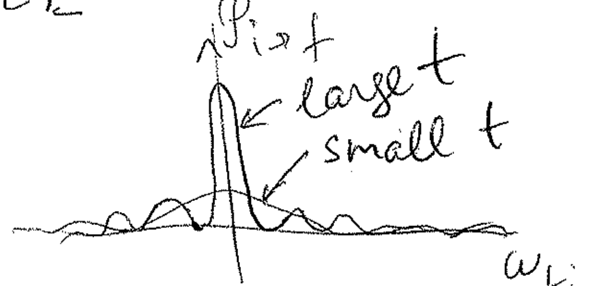
$$= \frac{4|V_{fi}|^2}{\hbar^2 \omega_{fi}^2} \sin^2 \frac{\omega_{fi}}{2} t = \frac{|V_{fi}|^2}{\hbar^2} \left(\frac{\sin \frac{\omega_{fi}}{2} t}{\frac{\omega_{fi}}{2}} \right)^2$$

<u>Analysis</u> for a fixed t $\Longrightarrow$



So, the largest probability of transitions is for states with $|\omega_{fi}| < \frac{2\pi}{t}$, i.e. transitions will be made preferentially to states whose energy is situated in a band of width $\delta E \simeq \frac{2\pi\hbar}{t}$ about the energy of the initial state





At small t $\Rightarrow$ more chances of finding the system at some state with E_f very different from E_i

At large t $\rightarrow$ the function acts as a $\delta(\omega_{fi})$ $\Rightarrow$ the most likely outcome is transitions between degenerate levels ($E_f \simeq E_i$) $\leftarrow$ "energy conservation"

At $\omega_{fi}=0 \Rightarrow \mathcal{P}_{i\to f} = \frac{|V_{fi}|^2}{\hbar^2}t^2$

Problem: $t\to\infty \Rightarrow \mathcal{P}_{i\to f}\to\infty \Rightarrow >1$!

$\Downarrow$

1st-order approximation is valid at

$$t \ll \frac{\hbar}{|V_{fi}|}$$

At fixed $\omega_{fi}\neq 0 \Rightarrow \mathcal{P}_{i\to f} = \frac{4|V_{fi}|^2}{\hbar^2\omega_{fi}^2}\sin^2\frac{\omega_{fi}t}{2}$

$\Downarrow$

oscillates between 0 and $\frac{4|V_{fi}|^2}{\hbar^2\omega_{fi}^2}$

as $\omega_{fi}\uparrow$ (i.e. $|E_f-E_i|\gg 0$) $\Leftrightarrow$

$\Downarrow$

amplitude of oscillations $\downarrow$

(б) $V(t) = V_0 \sin\omega t$

$\uparrow$

time-independent observable

$$\mathcal{P}_{i\to f}(t) = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{fi}t'}\left(e^{i\omega t'}-e^{-i\omega t'}\right)dt'\right|^2\cdot\frac{|V_{0fi}|^2}{4}$$

$=$

$$= \frac{|V_{0\,fi}|^2}{4\hbar^2}\left|\frac{e^{i(\omega_{fi}+\omega)t}-1}{i(\omega_{fi}+\omega)} - \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}\right|^2 =$$ (4)

$$= \frac{|V_{0\,fi}|^2}{4\hbar^2}\left| e^{i\frac{\omega_{fi}+\omega}{2}t}\,\frac{\sin\frac{\omega_{fi}+\omega}{2}t}{\frac{\omega_{fi}+\omega}{2}} - e^{i\frac{\omega_{fi}-\omega}{2}t}\cdot\frac{\sin\frac{\omega_{fi}-\omega}{2}t}{\frac{\omega_{fi}-\omega}{2}}\right|^2 \quad (4.1)$$

$\Rightarrow$ two terms with possible <u>resonant</u> behavior $\Rightarrow$

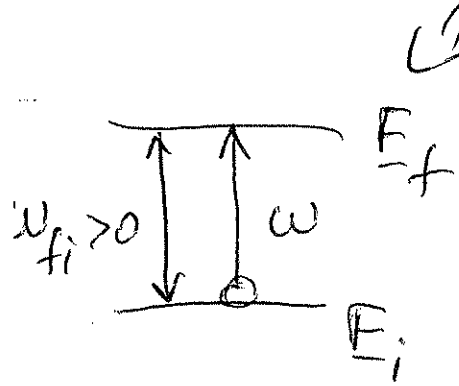
either at $\omega_{fi}+\omega=0$

or $\omega_{fi}-\omega=0$

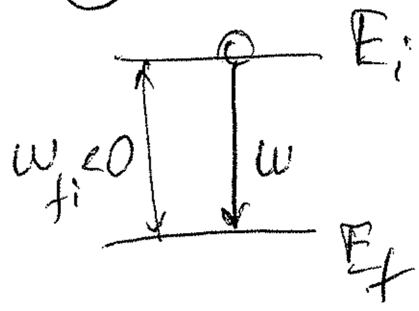
note that both terms can't be resonant at the same time

Let's specify that $\omega>0$. Then, the <u>resonant</u> conditions are $\omega=\omega_{fi}$ $(\omega_{fi}>0)$

$\omega=-\omega_{fi}$ $(\omega_{fi}<0)$



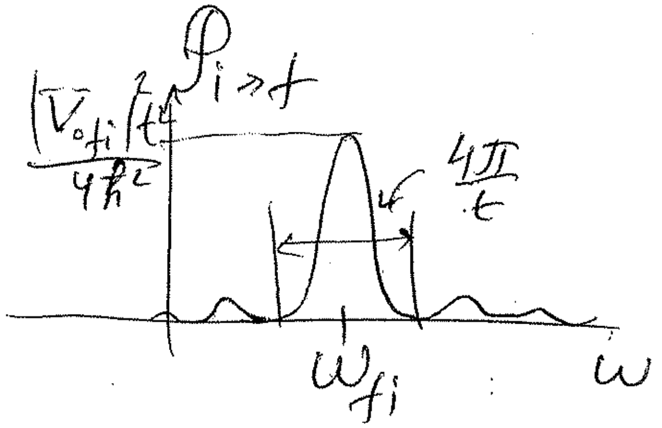
$\Rightarrow$ resonant absorption at $\omega\approx\omega_{fi}$



$\Rightarrow$ resonant emission at $\omega\approx-\omega_{fi}$

Consider resonant absorption $\Rightarrow$

$$P_{i\to f} = \frac{|V_{0fi}|^2}{4\hbar^2} \frac{\sin^2 \frac{\omega_{fi}-\omega}{2} t}{\left(\frac{\omega_{fi}-\omega}{2}\right)^2} \Rightarrow$$



very similar to constant perturbation (except $\omega_{fi} \to \omega_{fi}$ ...)

analysis is similar to the case of constant perturbation

most probable transitions are for $E_f - E_i \sim \hbar\omega$

$t << \frac{\hbar}{|V_{0fi}|}$, to keep the approximation valid

Another thing: since we neglected one of the terms in (4.1), we assumed that

$$\frac{1}{\frac{\omega_{fi}+\omega}{2}} << \frac{1}{\frac{\omega_{fi}-\omega}{2}} \Rightarrow \text{let's say } \omega_{fi} = \omega + \underset{\text{small!}}{\Delta\omega}$$

Then, $\omega_{fi} + \omega \simeq 2\omega \Rightarrow \underbrace{2\omega}_{\approx \omega_{fi}} >> \underbrace{\Delta\omega}_{\sim \frac{4\pi}{t}} \Rightarrow$

$$|\omega_{fi}| \gg \frac{2\pi}{t} \Rightarrow t \gg \frac{2\pi}{\omega} \quad (4.2)$$

So, overall, the result is valid if

$$\frac{\hbar}{|V_{0fi}|} \gg \frac{2\pi}{\omega_{fi}} \Rightarrow \underline{\hbar\omega_{fi} \gg |V_{0fi}|}$$

compare with the condition for validity of non-degen. time-independent perturbation theory!!

Note:

It is reasonable to expect the condition similar to (4.2) for validity of $P_{i\to f}$, since if $t < \frac{1}{\omega} \Rightarrow$ the perturbation $V_0 \sin\omega t$ would not have time to oscillate $\Rightarrow V_0 \sin\omega t \xrightarrow[t\to 0]{} V_0 \omega t$

$\Downarrow$

linear perturbation

$\Downarrow$

different $P_{i\to f}$!