Time dependent potentials

Recall: $H_s = H_0 + V_s(t)$

Schrödinger's Equation

$$i\hbar \frac{\partial}{\partial t} |\alpha\, t_0; t\rangle = H_s |\alpha\, t_0; t\rangle \Longrightarrow |\alpha\, t_0; t\rangle_I = e^{\frac{i}{\hbar} H_0 t} |\alpha\, t_0; t\rangle$$

$$V_I = e^{\frac{i}{\hbar} H_0 t} V_s e^{-i/\hbar H_0 t} \Longrightarrow \boxed{i\hbar \frac{\partial}{\partial t} |\alpha\, t_0; t\rangle = V_I |\alpha\, t_0; t\rangle_I}$$

Dirac Picture & time dependent potentials
"Interaction Picture"

both the state and operators are time dependent

Let's take this a little further

$H_0 |n\rangle = E_n |n\rangle$ (given)

initial state $|i\rangle \xrightarrow{H = H_0 + V(t)} |f\rangle$ final state

$|\alpha\, t_0; t\rangle_I = \sum_n c_n(t) |n\rangle$ time dependence is in the coefficents

if $|f\rangle = |\alpha\, t_0\, t\rangle = \sum_n c_n(t) |n\rangle$

then $\longrightarrow P(|f\rangle) = \sum_n |c_n(t)|^2$

bracket w/ $\langle n|$

$$\langle n| \left(i\hbar \frac{\partial}{\partial t} |\alpha\, t_0\, t\rangle \right) = \langle n| \left(V_I |\alpha\, t_0\, t\rangle \right)$$

$\cancel{i\hbar}$ $$i\hbar \frac{\partial}{\partial t} \underbrace{\langle n|\alpha\, t_0\, t\rangle}_{c_n(t)} = \sum_m \underbrace{\langle n|V_I|m\rangle}_{\text{matrix elem}} \underbrace{\langle m|\alpha\, t_0\, t\rangle}_{c_m(t)}$$

$$\langle n|V_I|m\rangle = \langle n| e^{\frac{i}{\hbar}H_0 t} V_S(t)\, e^{-i/\hbar H_0 t} |m\rangle$$

$$= e^{\frac{i}{\hbar}(E_n - E_m)t} \langle n| V_m(t) |m\rangle$$

$$= e^{\frac{i}{\hbar}(E_n - E_m)t}\, V_{nm}$$

back into original equation gives

$$i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t)$$

"coupled differential equations"

$$i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & \cdots \\ V_{21}e^{i\omega_{21}t} & V_{22} & \\ \vdots & & V_{33} \\ & & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

How do we solve if n runs to infinity?

Consider a 2 level system

$$H_0|n\rangle = E_n|n\rangle \qquad n=1,2$$

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$$

$$\doteq \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

apply time dependent field

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$$

$\gamma, \omega \in \mathbb{R}$

@ $t=0$ only level 1 is populated

$$\rightarrow \begin{cases} i\hbar \frac{\partial}{\partial t} c_1(t) = \gamma e^{i(\omega+\omega_{12})t} c_2(t) \\ i\hbar \frac{\partial}{\partial t} c_2(t) = \gamma e^{i(\omega-\omega_{12})t} c_1(t) \\ \\ c_1(0) = 1 \qquad c_2(0) = 0 \end{cases}$$

Methods to solve: go to second derivative

$$i\hbar \ddot{c}_1 = \underbrace{i(\omega+\omega_{12})\gamma e^{i(\omega+\omega_{12})t} c_2}_{i\hbar \frac{dc_1}{dt}} + \gamma e^{i(\omega+\omega_{12})t}\dot{c}_2$$

$$\frac{1}{i\hbar}\gamma e^{i(\omega+\omega_{12})t} c_1(t)$$

$$\longrightarrow \quad \ddot{c}_1 - i(\omega+\omega_{12})\dot{c}_1 + \overbrace{\cancel{\quad}}^{\frac{\gamma^2}{\hbar^2}} c_1 = 0$$

$$\ddot{c}_2 + i(\omega+\omega_{12})\dot{c}_2 + \frac{\gamma^2}{\hbar^2} c_2 = 0$$

damped oscillator!

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\frac{\gamma^2}{\hbar^2}+(\omega+\omega_{12})^2/4} \sin^2\left(\sqrt{\frac{\gamma^2}{\hbar^2}+\frac{(\omega+\omega_{12})^2}{4}}\, t\right)$$

$$|c_1(t)|^2 = \quad '' \quad \cos^2\left(\quad '' \quad \right)$$

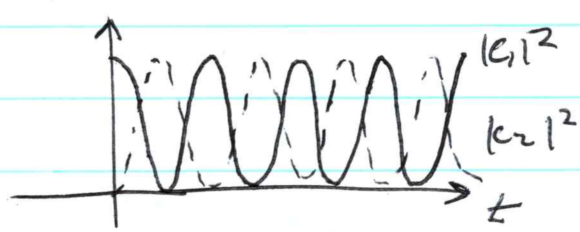
because $|c_1|^2 + |c_2|^2 = 1$ always

$$\boxed{\Omega \equiv \sqrt{\frac{\gamma^2}{\hbar^2}+\frac{(\omega+\omega_{12})^2}{4}}}$$ Rabi frequency!

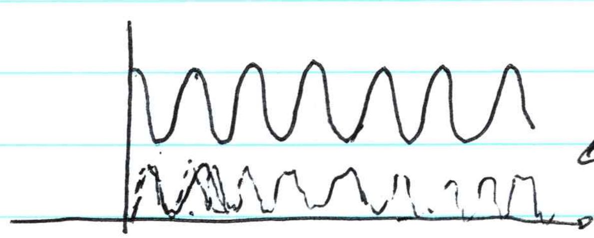
if $\omega+\omega_{12}=0 \Rightarrow$ Resonance!

What happens when you are on Resonance?

$$\Omega_{res} = \frac{\gamma}{\hbar}$$



Off resonance:



still sum to 1 but

What do we do if we have more than 2 levels?

TIME DEPENDENT PERTURBATION THEORY

$H_0|n\rangle = E_n|n\rangle$; $H = H_0 + V(t)$ $\quad |i\rangle \xrightarrow{?} |f\rangle$

$$i\hbar \frac{d}{dt} C_n(t) = \lambda \sum_k V_{nk}(t) e^{i\omega_{nk}t} C_k(t)$$

to keep track of order of expansions.

Let $\quad C_n(t) = c_n^0(t) + \lambda C_n^1(t) + \lambda^2 C_n^2(t) + \ldots$

λ^0: $\quad i\hbar \dot{c}_n^0(t) = 0 \qquad$ (potential already has λ)

$\hookrightarrow \quad c_n^0 = \delta_{ni}$ initial condition

λ^1: $\quad i\hbar \dot{c}_n^1(t) = \sum_k V_{nk}(t) e^{i\omega_{nk}t} \overbrace{c_k^0}^{\delta_{ki}} = V_{ni} e^{i\omega_{ni}t}$

$\vdots$

λ^r: $\quad i\hbar \dot{c}_n^{(r)}(t) = \sum_k V_{nk}(t) e^{i\omega_{nk}t} c_k^{(r-1)}$

$$\boxed{c_n^{(1)}(t) = \frac{1}{i\hbar} \int_0^t V_{ni}(t') e^{i\omega_{ni}t'} dt'}$$