Regular Surfaces 7c

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Tapp 3.23 (a torus)

let $\gamma(t) = (2+\cos t, 0, \sin t)$ Then to verify example 3.25, we need to show that

$$R_\theta \gamma(t)^T = \sigma(\theta, t).$$

where $\sigma(\theta,t)$ is the surface patch for the for the Torus and $R_\theta$ is the rotation matrix for arbitrary rotation about the $z$ axis. i.e.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore we compute $R_\theta \gamma(t)^T$ as follows

$$R_\theta \gamma(t)^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2+\cos t \\ 0 \\ \sin t \end{pmatrix}$$

$$= ((2+\cos t)\cos\theta, (2+\cos t)\sin\theta, \sin t)^T = \sigma(\theta, t)$$

Thus we have shown that $R_\theta \gamma(t)^T = \sigma(\theta, t)$ in accordance with example 3.25. Now we want to show $S$ is the $F=0$ level surface for

$$F(x,y,z) = (x^2+z^2+y^2+3)^2 - 16(x^2+y^2)$$

from $\sigma(\theta, t)$, we identify the following:

$$\left.\begin{aligned} x &= (2+\cos t)\cos\theta \\ y &= (2+\cos t)\sin\theta \\ z &= \sin t \end{aligned}\right\rangle \quad \begin{aligned} x^2 &= (4+4\cos t+\cos^2 t)\cos^2\theta \\ y^2 &= (4+4\cos t+\cos^2 t)\sin^2\theta \\ z^2 &= \sin^2 t \end{aligned}$$

therefore when $F=0$ we have

$$(x^2+y^2+z^2+3)^2 = 16(x^2+y^2)$$

we will work with the left hand side

$$(x^2+y^2+z^2+3)^2 = (4+4\cos t+\cos^2 t+\sin^2 t+3)^2$$

$$=(4+4\cos t+4)^2$$

$$=(8+4\cos t)^2$$

$$=(64+16\cdot 4\cos t+16\cos^2 t)$$

$$=16(4+4\cos t+\cos^2 t)$$

$$=16(x^2+y^2)$$

To make sure this is a level ~~set~~ surface all we need to do now is verify the Jacobian of $F$ is not the zero matrix.

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) =$$

using mathematica to differentiate

$$=\left(4x(-15x^2-15y^2+z^2+3),\ 4y(-15x^2-15y^2+z^2+3),\ 4z(x^2+y^2+z^2+3)\right)$$

Since this is $\underline{\underline{\text{not}}}$ identically zero, we have that $S$ is a level surface and not just a level set $\square$

Tapp 3.26 which of the following are parametrized surfaces?

(1) $\sigma(u,v) = (u^2, v^2, u^2+v^2+u+v)$

$$d\sigma(u,v) = \begin{pmatrix} 2u & 0 \\ 0 & 2v \\ 2u+1 & 2v+1 \end{pmatrix}$$

notice if $u=v=0$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$ and so for this choice rank$(d\sigma) \neq 2$.

therefore $\sigma$ is <u>not</u> a parametrized surface.

(2) $\sigma(u,v) = (u, u^2, v^3)$

$$d\sigma(u,v) = \begin{pmatrix} 1 & 0 \\ 2u & 0 \\ 0 & 3v^2 \end{pmatrix}$$

again if we let $u=v=0$ then we have

$$d\sigma(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$ which does <u>not</u> have Rank 2

and so again $\sigma$ is <u>not</u> a parametrized surface.

(3) $\sigma(u,v) = (\cos u, \sin v, \sin(u+v))$

$$d\sigma(u,v) = \begin{pmatrix} -\sin u & 0 \\ 0 & \cos v \\ \cos(u+v) & \cos(u+v) \end{pmatrix}$$

if we let $u = 0$, $v = \pi/2$ we get

$$d\sigma(0, \pi/2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which does <u>not</u> have rank 2 so <u>again</u> $\sigma$ is <u>not</u> a parametrized surface. See attached mathematica for plots.

Tapp 3.27

verify that $\sigma(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$ is a parametrized surface

$$d\sigma(u,v) = \begin{pmatrix} 1-u^2+v^2 & 2uv \\ 2vu & 1-v^2+u^2 \\ 2u & -2v \end{pmatrix}$$

Now notice that $\det \begin{pmatrix} 1-u^2+v^2 & 2uv \\ 2uv & 1-v^2+u^2 \end{pmatrix} =$

$$= (1-u^2+v^2)(1-v^2+u^2) - 4u^2v^2$$

$$= 1 \cancel{-v^2} \cancel{+u^2} \cancel{-u^2} + u^2v^2 - u^4 \cancel{+v^2} - v^4 + u^2v^2 - 4u^2v^2$$

$$= 1 - u^4 - v^4 - 2u^2v^2 \neq 0 \quad \forall u,v$$

Thus $d\sigma(u,v)$ has rank 2 $\forall u,v$ and so $\sigma(u,v)$ is a parametrized surface

$\square$