Complex Analysis: Assignment 1 **John Waczak**
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1.1abc

Let $z = 1 + 2i$ and $w = 2 - i$. Compute the following:

(a) $z + 3w$

$$\begin{aligned} z + 3w &= (1 + 2i) + 3(2 - i) \\ &= 1 + 6 + 2i - 3i \\ &= 7 - i \end{aligned}$$

(b) $\overline{w} - z$

$$\begin{aligned} \overline{w} - z &= (2 - i)^* - (1 + 2i) \\ &= (2 + i) - (1 + 2i) \\ &= 2 + i - 1 - 2i \\ &= 1 - i \end{aligned}$$

(c) z^3

$$\begin{aligned} z^3 &= (1 + 2i)^3 \\ &= (1 + 2i)(1 + 2i)(1 + 2i) \\ &= (1 + 2i)(-3 + 4i) \\ &= -3 - 6i + 4i - 8 \\ &= -11 - 2i \end{aligned}$$

1.2ab

Find the real and imaginary parts of the following:

(a) $\frac{z-a}{z+a}$ with $a \in \mathbb{R}$

let $z = x + iy$. Then:

$$\begin{aligned} \frac{z - a}{z + a} &= \frac{x + iy - a}{x + iy + a} \\ &= \frac{(x - a) + iy}{(x + a) + iy} \\ &= \frac{[(x - a) + iy][(x + a) - iy]}{(x + a)^2 + y^2} \\ &= \frac{x^2 + a^2 + y^2 + 2ayi}{(x + a)^2 + y^2} \end{aligned}$$

Thus identifying the imaginary and real parts gives:

$$\operatorname{Re}\left(\frac{z-a}{z+a}\right) = \frac{\operatorname{Re}(z)^2 + a^2 + \operatorname{Im}(z)^2}{(\operatorname{Re}(z)+a)^2 + \operatorname{Im}(z)^2}$$
$$\operatorname{Im}\left(\frac{z-a}{z+a}\right) = \frac{2a\operatorname{Im}(z)}{(\operatorname{Re}(z)+a)^2 + \operatorname{Im}(z)^2}$$

(b) $z = \frac{3+5i}{7i+1}$

$$\begin{aligned}\frac{3+5i}{7i+1} &= \frac{3+5i}{1+7i}\frac{1-7i}{1-7i} \\ &= \frac{(3+5i)(1-7i)}{1+49} \\ &= \frac{1}{50}(38-16i) \\ &= \frac{19}{25} - \frac{8i}{25}\end{aligned}$$

$$\begin{aligned}\text{thus:}\ \ \operatorname{Re}(z) &= \frac{19}{25} \\ \operatorname{Im}(z) &= -\frac{8}{25}\end{aligned}$$

1.3abd

Find the absolute value and conjugate of the following:

(a) $z = -2 + i$

$$\begin{aligned}|z| &= |-2+i| \\ &= \sqrt{4+1} \\ &= \sqrt{5} \\ \overline{z} &= -2 - i\end{aligned}$$

(b) $z = (2+i)(4+3i)$

$$\begin{aligned}z &= 8 - 3 + 4i + 6i \\ &= 5 + 10i \\ \Rightarrow |z| &= \sqrt{125} = 5\sqrt{5} \\ \overline{z} &= 5 - 10i\end{aligned}$$

(d) $z = (1+i)^6$

$$\begin{aligned} z &= (\sqrt{2})^6\Big(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\Big)^6 \\ &= (\sqrt{2})^6(e^{i\pi/4})^6 \\ &= 2^3 e^{i3\pi/2} \\ &= 8e^{i3\pi/2} \\ &= -8i \\ \Rightarrow |z| &= 8 \\ \text{and} \quad \overline{z} &= -8i \end{aligned}$$

1.4ace

Write the following in polar form.

(a) $z = 2i$

$$z = 2i = 2e^{i\pi/2}$$

(c) $z = -3 + \sqrt{3}i$

$$\begin{aligned} r &= \sqrt{9+3} = \sqrt{12} = 2\sqrt{3} \\ \Rightarrow 2\sqrt{3}\cos\theta &= -3 \\ \cos\theta &= -\frac{\sqrt{3}}{2} \\ \text{and} \quad 2\sqrt{3}\sin\theta &= \sqrt{3} \\ \sin\theta &= \frac{1}{2} \\ \Rightarrow \theta &= 5\pi/6 \\ \text{thus} \quad z &= 2\sqrt{3}e^{i5\pi/6} \end{aligned}$$

(e) $z = (2-i)^2$

$$\begin{aligned} \text{first} \quad 2 - i \Rightarrow r &= \sqrt{5} \\ \theta &= \arctan(-1/2) \\ \text{thus} \quad z &= (\sqrt{5}e^{i\arctan(-1/2)})^2 \\ &= 5e^{2i\arctan(-1/2)} \end{aligned}$$

1.5ab

Write the following in their Cartesian representation.

(a) $z = \sqrt{2}e^{i3\pi/4}$

$$\begin{aligned} z &= \sqrt{2}e^{i3\pi/4} \\ &= \sqrt{2}\cos(3\pi/4) + i\sqrt{2}\sin(3\pi/4) \\ &= -\sqrt{2}\frac{\sqrt{2}}{2} + i\sqrt{2}\frac{\sqrt{2}}{2} \\ &= -1 + i \end{aligned}$$

(b) $z = 34e^{i\pi/2}$

$$z = 34\cos\pi/2 + i34\sin\pi/2 = 34i$$

1.8b

Use the quadratic formula to solve the following.

(b) $2z^2 + 2z + 5 = 0$

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 4\cdot 2\cdot 5}}{2\cdot 2} \\ &= \frac{-2 \pm \sqrt{-36}}{4} \\ &= \frac{-2 \pm 6i}{4} \\ &= -\frac{1}{2} \pm \frac{3}{2}i \end{aligned}$$

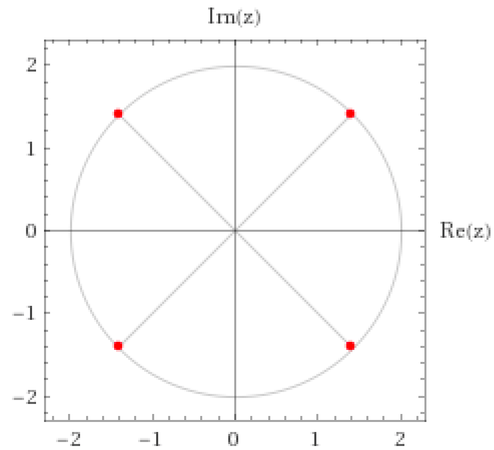
1.11bc

Find all solutions to the following equations:

(b) $z^4 = -16$

$$\begin{aligned} z^4 &= 16e^{i(\pi+2\pi n)} \\ z &= 16^{1/4}e^{i(\pi/4+n\pi/2)} \quad n \in [0,1,2,3] \\ &= 2e^{i(\pi/4+n\pi/2)} \\ &= \left\{2e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}\right\} \end{aligned}$$

These roots form a regular square



(c) $z^6 = -9$

$$
\begin{aligned}
z^6 &= 9e^{i(\pi+2\pi n)} \\
z &= 9^{1/6}e^{i(\pi/6+n\pi/3)} \quad n \in [0,1,2,3,4,5] \\
&= \left\{9^{1/6}e^{i\pi/6}, 9^{1/6}e^{i3\pi/6}, 9^{1/6}e^{i5\pi/6}, 9^{1/6}e^{i7\pi/6}, 9^{1/6}e^{i9\pi/6}, 9^{1/6}e^{i11\pi/6}\right\}
\end{aligned}
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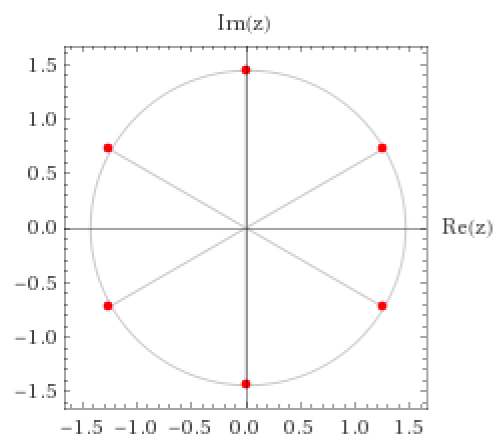
These roots form a regular hexagon



1.20

Use proposition 1.3 to derive the triple angle formulas.

$$\begin{aligned}
e^{i3\phi} &= \left(e^{i\phi}\right)^3 = \cos 3\phi + i \sin 3\phi \\
&= (\cos\phi + i\sin\phi)^3 \\
&= (\cos\phi + i\sin\phi)(\cos\phi + i\sin\phi)(\cos\phi + i\sin\phi) \\
&= (\cos\phi + i\sin\phi)(\cos^2\phi - \sin^2\phi + i2\sin\phi\cos\phi) \\
&= (\cos^3\phi - \cos\phi\sin^2\phi - 2\sin^2\phi\cos\phi) + i(2\sin\phi\cos^2\phi + \sin\phi\cos^{\phi} - \sin^3\phi) \\
\Rightarrow \cos 3\phi &= \cos^3\phi - 3\cos\phi\sin^2\phi \\
\sin 3\phi &= 3\sin\phi\cos^2\phi - \sin^3\phi
\end{aligned}$$

1.23ae

Sketch the following sets in the $\mathbb{C}$ plane.

(a) $\{z \in \mathbb{C} : |z - 1 + i| = 2\}$

From proposition 1.2 this set is all of the points in $\mathbb{C}$ that are of a constant distance 2 from the point $1 - i$. Thus this set is a circle of radius 2 centered around $1 - i$. See the following diagram.

(b) $\{z \in \mathbb{C} : |z| = |z+1|\}$

First let $z = x + iy$. Then from the definition of the set we have that:

$$\begin{aligned}
|x+iy| &= |x+iy1| \\
|x+iy|^2 &= |(x+1)+iy|^2 \\
\Rightarrow x^2+y^2 &= (x+1)^2+y^2 \\
x^2 &= (x+1)^2 \\
x^2 &= x^2+2x+1 \\
0 &= 2x+1 \\
x &= -\frac{1}{2}
\end{aligned}$$

Thus we can rewrite this set as $\{z = x+iy \in \mathbb{C} : x = -1/2\}$. Therefore the image of this set in the complex plane is a vertical line at $x = -1/2$