Complex Analysis - MTH 483 **John Waczak**

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More on integration

If $f : \Omega \to \mathbb{C}$ is a function and $\gamma : [a,b] \to \Omega$ is a curve, then $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$

Definition we say two curves are *equivalent* if

$$\gamma_1 : [a,b] \to \mathbb{C}$$
$$\gamma_2 : [c,d] \to \mathbb{C}$$

and there exists a continuous function $u : [a,b] \to [c,d]$ with $u(a) = c$ and $u(b) = d$ differentiable with $u'(t) > 0$ such that $\gamma_1(t) = \gamma_2(u(t))$. In this situation, γ_1 and γ_2 define the same curves, just parametrized differently.

Proposition if γ_1, γ_2 are equivalent, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

Definition the length of a curve γ is given by:

$$\text{length}\gamma = \int_a^b |\gamma'(t)|dt$$

Example $\gamma : [0,2] \to \mathbb{C}$ such that $\gamma(t) = (1+3i)t$ (a straight line).

$$\begin{aligned} \text{length}\gamma &= \int_0^2 |1+3i|dt \\ &= \sqrt{10}t\Big|_0^2 \\ &= 2\sqrt{10} \end{aligned}$$

Properties of path integrals

1. $\int_\gamma (af(z) + bg(z))dz = \int_\gamma af(z)dz + \int_\gamma bg(z)dz$ i.e. integration is linear
2. if $-\gamma$ denotes the path γ traversed in opposite direction, then $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$.
3. if $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ then if γ_3 is obtained by γ_1 then γ_2 we have:
$$\int_{\gamma_3} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$
4. $|\int_\gamma f(z)dz| \leq \max\{|f(z)| \; z \in \gamma\} \cdot \text{length}(\gamma)$. Think of bounding an integral by making a square with its maximum value i.e. $|\int_a^b f(x)dx| \leq M(b-a)$.