Complex Analysis - MTH 483 **John Waczak**
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Polar Coordinates

If we have a complex number $z = x + iy$ we can also write this as $z = re^{i\phi}$. What are the r and the ϕ? r is the distance to the origin $r = \sqrt{x^2 + y^2}$ and $e^{i\phi} = \cos\phi + i\sin\phi$. We call ϕ the argument.

Some nice properties:

$$
\begin{aligned}
e^{i(\phi_1+\phi_2)} &= e^{i\phi_1} \cdot e^{i\phi_2} && (1)\\
e^{i\cdot 0} &= 1 && (2)\\
e^{-i\phi} &= \frac{1}{e^{i\phi}} && (3)\\
|e^{i\phi}| &= 1 && (4)\\
e^{i\phi+2\pi n} &= e^{i\phi} && (5)\\
\frac{d}{d\phi}e^{i\phi} &= ie^{i\phi} && (6)
\end{aligned}
$$

Most can be proven by sine and cosine properties. (3) follows from (1) and (2). Others are pretty easy. Let's focus on (1):

$$
\begin{aligned}
e^{i(\phi_1+\phi_2)} &= cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)\\
&= \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 + i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)\\
&= (\cos\phi_1 + i\sin\phi_1)\cdot(\cos\phi_2 + i\sin\phi_2)\\
&\equiv e^{i\phi_1}e^{i\phi_2} \qquad \square
\end{aligned}
$$

The more interesting way is to go in reverse to deduce the trig identities using Taylor expansions for our functions:

$$
\begin{aligned}
e^{iz}e^{iw} &= \sum_{k=0}^{\infty}\frac{z^k}{k!}\sum_{n=0}^{\infty}\frac{w^k}{k!}\\
&= \sum_{\ell=0}^{\infty}\frac{(z+w)^\ell}{\ell!}\\
&\text{Take: } z = i\phi_1, w = i\phi_2
\end{aligned}
$$

This is more logical because we don't have a definition of exponent of a complex number algebraically... that doesn't make much sense. So if we accept that Taylor series over the field $\mathbb{C}$ then we get these properties as a result.

Now let's check (6) with the fact that derivatives apply linearly to real and imaginary part.

$$\begin{aligned}
\frac{d}{d\phi}e^{i\phi} &= \frac{d}{d\phi}(\cos\phi + i\sin\phi) \\
&= -\sin\phi + i\cos\phi \\
&= i(\cos\phi - \frac{1}{i}\sin\phi) \\
&= i(\cos\phi + i\sin\phi) \\
&= ie^{i\phi}
\end{aligned}$$

Euler's Identity

When you define e^z as a power series you can prove Euler's Identity using Taylor expansions. In general the fancy formula is:

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{7}$$

The very famous version is take $\phi = \pi$.

$$e^{i\pi} + 1 = 0 \tag{8}$$

What this is really saying is that the sum $\sum_{n=0}^{\infty}(i\pi)^n/n! = -1$. You can check this with a computer.

Transferring between coordinate systems

Exmaple: *Write $3e^{i\pi/4}$ in cart. coords.*

$$\begin{aligned}
3e^{i\pi/4} &= 3(\cos(\phi/4) + i\sin(\pi/4)) \\
&\frac{3\sqrt{2}}{2}(1+i)
\end{aligned}$$

Example: *Write $z = 1 + i\sqrt{3}$ in polar coords.*

$$\begin{aligned}
1 + i\sqrt{3} &= 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\
&= 2(\cos(\pi/3) + i\sin(\pi/3)) \\
&= 2e^{i\pi/3}
\end{aligned}$$

Polar coordinates are super handy when we want to raise some z to a power.

$$\begin{aligned}
(1 + i\sqrt{3})^{10} &\text{ don't use binomial theorem!} \\
&= (2e^{i\pi/3})^1 0 \\
&= 2^{10}e^{i\frac{10\pi}{3}} \\
&= 2^{10}e^{i\frac{4\pi}{3}} \\
&= 2^{10}(-\frac{1}{2} - \frac{i\sqrt{3}}{2}) \\
&= -2^9(-1 - i\sqrt{3})
\end{aligned}$$

Roots of Unity

Definition: A *n-root* of unity is a number $\in \mathbb{C}$ is a number ζ satisfying:

$$\zeta^n = 1$$

Where n is a positive integer. If n is the smallest number for which $\zeta^n = 1$ then we say ζ is a *primitive* root of unity.

Example: $1^1 = 1$ so 1 is prim first root of unity. $(-1)^2 = 1$ so -1 is a 2nd primitive root of unity. $i^4 = 1$ so i is a primitive fourth root of unity (so is -i). These lie on the unit circle and alway form a perfect n-gon.

Since $1^4 = 1$ and $(-1)^4 = 1$ and $(\pm i)^4 = 1$ are the fourth roots of unity (only $\pm i$ are primitive fourth roots). In general the n-th roots of unity are precisely the roots of the polynomial $f(x) = x^n - 1$. By **Fundamental theorem of Algebra** there are at most n roots for a polynomial over any field i.e. the degree of the polynomial is maximum number of roots. Thus there are at most n roots of unity.

We would like to know if there are *exactly* n roots... So far it seems like there are always n roots. In fact, using polar coordinates we can show that there are always exactly n nth-roots of unity namely:

$$u_n = e^{i\frac{2\pi k}{n}} \quad k \in [1, ..., n] \tag{9}$$

Note that 1 is *always* an nth-root of unity. These numbers form a regular n-gon around unit circle.

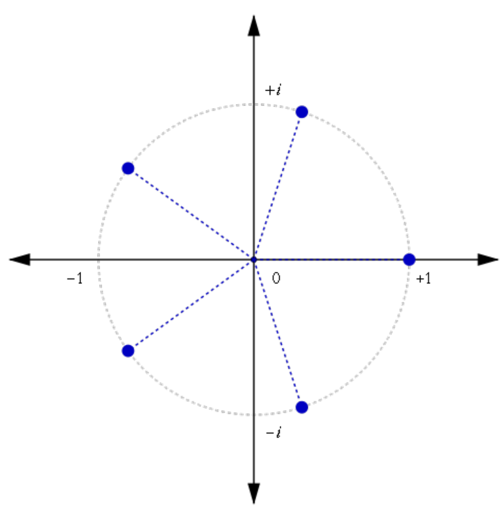

Figure 1: 5th roots of unity (regular pentagon)