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1 VECTORS IN MINKOWSKI SPACE

Show that a timelike vector cannot be orthogonal to a null vector or to another timelike vector. Show that two null vectors are orthogonal if and only if they are parallel. (Assume these vectors are nonzero.)

Try to do this in 4 spacetime dimensions rather than 2. A convenient notation is to view a 4-vector $\mathbf{u}$ *as consisting of a timelike component* u^t *and spacelike components making up an ordinary 3-vector* $\vec{\boldsymbol{u}}$*; one often writes*

$$\mathbf{u} = \begin{pmatrix} u^t \\ \vec{\boldsymbol{u}} \end{pmatrix} \tag{1}$$

Let $\mathbf{u} \in \mathbb{M}^4$ with $|\mathbf{u}| > 0$ be a timelike vector representing the motion of some object; that is, $(u^t)^2 > |\vec{\boldsymbol{u}}|^2$. Assume without loss of generality that $\vec{\boldsymbol{u}} = |\vec{\boldsymbol{u}}|\hat{\mathbf{e}}_i$. We can always choose an appropriate hyperbolic rotation by some angle α so that in the object's rest frame, $\mathbf{u}$ is given by

$$\mathbf{u} = \begin{pmatrix} u^t \\ \vec{\mathbf{0}} \end{pmatrix} \tag{2}$$

Now consider a second vector $\mathbf{v}$ with $|\mathbf{v}| > 0$. In general, if $\mathbf{v}$ is timelike we may write

$$\mathbf{v} = \pm|\mathbf{v}| \begin{pmatrix} \cosh\beta \\ \sinh\beta\hat{\boldsymbol{v}} \end{pmatrix} \tag{3}$$

in order to make the timelike quality explicit.

Taking the inner product yields

$$\mathbf{u} \cdot \mathbf{v} = \mp|v|\cosh\beta u^t \neq 0 \tag{4}$$

because $\cosh\beta > 0\ \forall\beta$.

If instead $\mathbf{v}$ is a null vector, then it may be written in as

$$\mathbf{v} = a \begin{pmatrix} 1 \\ \hat{\boldsymbol{v}} \end{pmatrix} \tag{5}$$

for any $a \neq 0$. The inner product with $\mathbf{u}$ yields

$$\mathbf{u} \cdot \mathbf{v} = au^t \neq 0 \tag{6}$$

Therefore, we conclude that a timelike vector in $\mathbb{M}^4$ cannot be orthogonal to a timelike vector or a null vector.

For the second part of the question, let $\mathbf{u}, \mathbf{v} \in \mathbb{M}^4$ be null vectors. That is,

$$\mathbf{u} = a\begin{pmatrix} 1 \\ \hat{\boldsymbol{u}} \end{pmatrix} \qquad \mathbf{v} = b\begin{pmatrix} 1 \\ \hat{\boldsymbol{v}} \end{pmatrix} \tag{7}$$

Let's do the forward direction ($\rightarrow$) first. Assume that $\mathbf{u} \parallel \mathbf{v}$. In other words $\hat{\boldsymbol{v}} = \hat{\boldsymbol{u}}$. Then taking the inner product yields

$$\mathbf{u} \cdot \mathbf{v} = ab(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} - 1) = 0 \tag{8}$$

Therefore, $\mathbf{u} \parallel \mathbf{v}$ implies that $\mathbf{u} \cdot \mathbf{v} = 0$.

For the reverse direction ($\leftarrow$), assume that $\mathbf{u} \cdot \mathbf{v} = 0$. Then the inner product gives

$$0 = ab(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} - 1) \tag{9}$$

$$\Rightarrow \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} = 1 \tag{10}$$

which is precisely the condition for $\mathbf{u} \parallel \mathbf{v}$. Therefore we conclude that two null vectors are orthogonal if and only if they are parallel.

2 EARTH DISTANCE

Corvallis is located at approximately $(44.55°N, 123.25°W)$, that is, 44.55° north of the equator (latitude), and 123.25° west of the prime meridian (longitude). Tangent is located at approximately $(44.55°N, 123.1°W)$, and Eugene is at approximately $(44.05°N, 123.1°W)$. Gresham, OR, is located at approximately $(45.50°N, 122.4W)$, Millbrae, CA, is located at approximately $(37.60°N, 122.4°W)$, and Richmond, VA, is at approximately $(37.60°N, 77.50°W)$.

Assume that the Earth is a perfect sphere with radius $r = 3959$ mi with line element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{11}$$

(a) Approximate the following distances using the Pythagorean Theorem:
Corvallis to Tangent; Tangent to Eugene; Corvallis to Eugene.

Figure 1 shows a sphere with an infinitesimal triangle whose side lengths are given by the and hypotenuse are given by the equation (11) for the line element. In order to approximate these distances, we can find $\Delta\theta$ and $\Delta\phi$ from the given information and then pick a value for θ to use. **NOTE** the latitude is measured in degrees north of the equator but we must instead use the compliment for θ because the spherical inclination is measured from the pole. This is not a problem for $\Delta\theta$ and $\Delta\phi$ as all that matters for these is the difference in angle.

We will choose the greater of the inclination angles to use as θ. Thus, we have the following

$$\Delta\theta_{\text{corv, tang}} = 0° = 0 \quad \text{rad} \tag{12}$$

$$\Delta\phi_{\text{corv, tang}} = 123.25 - 123.1 = 0.15° \approx 0.0026 \quad \text{rad} \tag{13}$$

$$\theta = 90° - 44.55° = 44.45° \approx 0.775 \quad \text{rad} \tag{14}$$

$$\Delta s_{\text{corv, tang}} = \sqrt{3959^2(0^2 + \sin^2(0.775)0.0026^2)} = 7.2 \quad \text{mi} \tag{15}$$

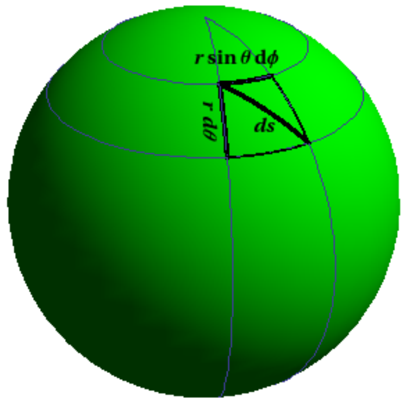

Figure 1: An infinitesimal triangle on the surface of the sphere. Image taken from [Dray 2015].

$$\Delta\theta_{\text{tang, eug}} = 44.55^\circ - 44.05^\circ \approx 0.0087 \quad \text{rad} \tag{16}$$
$$\Delta\phi_{\text{tang, eug}} = 123.1 - 123.1 \approx 0 \quad \text{rad} \tag{17}$$
$$\theta = 90^\circ - 44.05^\circ = 45.95^\circ \approx 0.802 \quad \text{rad} \tag{18}$$
$$\Delta s_{\text{tang, eug}} = \sqrt{3959^2(0.0087^2 + \sin^2(0.802)0^2)} = 34.44 \quad \text{mi} \tag{19}$$

$$\Delta\theta_{\text{corv, eug}} = 44.55^\circ - 44.05^\circ \approx 0.0087 \quad \text{rad} \tag{20}$$
$$\Delta\phi_{\text{corv, eug}} = 123.25 - 123.1 \approx 0.0026 \quad \text{rad} \tag{21}$$
$$\theta = 90^\circ - 44.05^\circ = 45.95^\circ \approx 0.802 \quad \text{rad} \tag{22}$$
$$\Delta s_{\text{corv, eug}} = \sqrt{3959^2(0.0087^2 + \sin^2(0.802)0.0026^2)} = 35.2 \quad \text{mi} \tag{23}$$

(**b**) How good are your approximations?

After a google search, it looks like the correct distances are 8.46 mi, 34.05 mi, and 37.06 mi... so not too bad!

(**c**) Approximate the following distances using the Pythagorean Theorem:
Gresham to Millbrae; Millbrae to Richmond; Gresham to Richmond.

$$\Delta\theta_{\text{gresh, mill}} = 44.50^\circ - 37.60^\circ \approx 0.1204 \quad \text{rad} \tag{24}$$
$$\Delta\phi_{\text{corv, eug}} = 122.4^\circ - 122.4^\circ = 0 \quad \text{rad} \tag{25}$$
$$\theta = 90^\circ - 37.60^\circ = 45.95^\circ \approx 0.9146 \quad \text{rad} \tag{26}$$
$$\Delta s_{\text{corv, eug}} = \sqrt{3959^2(0.1204^2 + \sin^2(0)0.9146^2)} = 476 \quad \text{mi} \tag{27}$$

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\begin{align}
\Delta\theta_{\text{mill, rich}} &= 37.60^\circ - 37.60^\circ = 0 \quad \text{rad} \tag{28}\\
\Delta\phi_{\text{mill, rich}} &= 122.4^\circ - 77.5^\circ = 0.7837 \quad \text{rad} \tag{29}\\
\theta &= 90^\circ - 37.60^\circ = 45.95^\circ \approx 0.9146 \quad \text{rad} \tag{30}\\
\Delta s_{\text{mill, rich}} &= \sqrt{3959^2(0^2 + \sin^2(0.7837)0.9146^2)} = 2556 \quad \text{mi} \tag{31}
\end{align}
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\begin{align}
\Delta\theta_{\text{gresh, rich}} &= 44.5^\circ - 37.60^\circ = 0.1204 \quad \text{rad} \tag{32}\\
\Delta\phi_{\text{gresh, rich}} &= 122.4^\circ - 77.5^\circ = 0.7837 \quad \text{rad} \tag{33}\\
\theta &= 90^\circ - 37.60^\circ = 45.95^\circ \approx 0.9146 \quad \text{rad} \tag{34}\\
\Delta s_{\text{gresh, rich}} &= \sqrt{3959^2(0.1204^2 + \sin^2(0.7837)0.9146^2)} = 2600 \quad \text{mi} \tag{35}
\end{align}
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(d) How good are your approximations

The correct distances are 545 mi, 2437 mi, and 2359 mi. Again, this is pretty impressive for not even integrating. I think if we wanted to be exact we would need to parametrize the great circle between the given cities and then integrate.