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1.3.15

Prove $A \cap (B\ C) = (A \cap B)\ (A \cap C)$

Recall $R \cap S = \{x : x \in R, x \in S\}$ and $R \setminus S = \{x : x \in R, x \notin S\}$. Thus:

$$\begin{aligned}
A \cap (B \setminus C) &= \{x : x \in A, x \in B \setminus C\} \\
&= \{x : x \in A, x \in B, x \notin C\} \\
&= \{x : x \in A \cap B, x \notin C\} \\
x \notin c \to x \notin A \cap C &\quad \text{since} \quad A \cap C \subseteq C \\
\text{Thus:} &= \{x : x \in A \cap B, x \notin A \cap C\} \\
&= (A \cap B) \cap (A \cap C)' \\
&= (A \cap B) \setminus (A \cap C) \quad \square
\end{aligned}$$

1.3.19

Given $f : A \mapsto B$ *and* $g : B \mapsto C$ *are invertible, show* $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$

Recall that the composition of mappings is associative i.e. $h \circ (g \circ f) = (h \circ g) \circ f$. Thus:

$$\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\
&= g \circ Id_B \circ g^{-1} \\
&= g \circ g^{-1} \\
&= Id_C \\
(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ g) \circ f \\
&= f^{-1} \circ Id_B \circ f \\
&= f^{-1} \circ f \\
&= Id_c \\
\text{Thus:} \qquad (g \circ f)^{-1} &= (f^{-1} \circ g^{-1}) \quad \square
\end{aligned}$$

1.3.26

define $(a, b) \sim (c, d)$ *if* $a^2 + b^2 \leq c^2 + d^2$. *Show* $\sim$ *is reflexive and transitive but not symmetric.*

$$\begin{aligned} \text{Reflexive:} \quad & \text{w.t.s} \quad (a,b) \sim (a,b) \\ \text{observe that:} \quad & a^2+b^2 = a^2+b^2 \\ & \text{Thus } \sim \text{ is reflexive} \end{aligned}$$

$$\begin{aligned} \text{Transitive:} \quad & (a,b) \sim (c,d), (c,d) \sim (e,f) \Rightarrow (a,b) \sim (e,f) \\ \text{observe that:} \quad & (a,b) \sim (c,d) \Rightarrow a^2+b^2 \leq c^2+d^2 \\ & (c,d) \sim (e,f) \Rightarrow c^2+d^2 \leq e^2+f^2 \\ & \text{thus} \quad a^2+b^2 \leq c^2+d^2 \leq e^2+f^2 \\ & \Rightarrow a^2+b^2 \leq e^2+f^2 \\ & \text{and so } \sim \text{ is transitive} \\ \text{Not symmetric:} \quad & (a,b) \sim (c,d) \Rightarrow (c,d) \sim (a,b) \\ & (a,b) \sim (c,d) \Rightarrow a^2+b^2 \leq c^2+d^2 \\ & (c,d) \sim (a,b) \Rightarrow c^2+b^2 \leq a^2+d^2 \\ & a^2+b^2 \leq c^2+d^2 \Rightarrow\Leftarrow c^2+d^2 \leq a^2+b^2 \end{aligned}$$

thus we have a contradiction and so $\sim$ is not symmetric □

2.3.6

prove $4 \cdot 10^{2n} + 0 \cdot 10^{2n-1} + 5$ *is divisible by 99* $\forall n \in \mathbb{N}$
Proof by mathematical induction:

$$\begin{aligned} \text{let} \quad n = 1, \text{then} \quad & 4 \cdot 10^2 + 9 \cdot 10 + 5 = \\ & = 400 + 90 + 5 \\ & = 495 \\ & = 5 \cdot 99 \end{aligned}$$

Thus the base step is true. Now assuming n=k is true, w.t.s. that n=k+1 is true.

$$\begin{aligned} 4 \cdot 10^{2(k+1)} + 9 \cdot 10^{2(k+1)-1} + 5 = & \\ & = 4 \cdot 10^{2k+2} + 0 \cdot 10^{2k+1} + 5 \\ & = 100(4 * 10^{2k}) + 100(9 * 10^{2k-1}) + 5 \\ & = 100(4 * 10^{2k} + 9 * 10^{2k-1}) + 5) - 500 + 5 \\ & = 100(4 * 10^{2k} + 9 * 10^{2k-1}) + 5) - 495 \\ & = 100(99 * a) - 495, a \in \mathbb{Z} \quad \text{because n=k is assumed true} \\ & = 100a \cdot 99 - 5 \cdot 99 \\ & = (100a - 5)99 \\ & = 99b, b \in \mathbb{Z} \end{aligned}$$

Thus by mathematical induction, the hypothesis $4 \cdot 10^{2n} + 0 \cdot 10^{2n-1} + 5$ is divisible by 99 $\forall n \in \mathbb{N}$. □

2.3.15

find r and s s.t. gcd(r, s) = ra + sb given a = 234 and b = 165

First, we need to find the gcd of a and b which we will do using the Euclidean algorithm. Then working backwards, we will determine r and s.

$$\begin{aligned}
234 &= 165 \cdot 1 + 69 \\
165 &= 69 \cdot 2 + 27 \\
27 &= 15 \cdot 1 + 12 \\
15 &= 12 \cdot 1 + 3 \\
12 &= 3 \cdot 4
\end{aligned}$$

Thus, the gcd(a,b) is 3. Now we will find $r, s \in \mathbb{Z}$

$$\begin{aligned}
3 &= 15 - 12 \\
&= (69 - (2)27) - (27 - 15) \\
&= 69 - (2)27 - 27 + 15 \\
&= 69 - (3)27 + 15 \\
&= 234 - 165 - (3)(165 - (2)69) + 69 - (2)27 \\
&= 234 - 165 - (3)165 + (6)69 + 69 - (2)27 \\
&= 234 - (4)165 + (7)69 - (2)27 \\
&= 234 - (4)165 + (7)(234 - 165) - (2)(165 - (2)69) \\
&= 234 - (4)165 + (7)234 - (7)165 - (2)(165 - (2)69) \\
&= 234 - (4)165 + (7)234 - (7)165 - (2)165 + (4)69 \\
&= 234 - (4)165 + (7)234 - (7)165 - (2)165 + (4)234 - 4(165) \\
&= (1 + 7 + 4)234 + (-4 - 7 - 2 - 4)165 \\
&= (12)234 + (-17)165 \\
&= 2808 - 2805 \\
&= 3 \Rightarrow r = 12, s = -17 \quad \square
\end{aligned}$$

2.3.19

Let $x, y \in \mathbb{N}$ be relatively prime. If xy is a perfect square, prove that x and y must be perfect sqaures

I will prove this proposition by first proving a **lemma:** *if n is a perfect square then each of the factors in its prime factorization must have an even power.*

Because n is a perfect square we can say $\exists m \in \mathbb{Z}_+$ such that $n = m^2$. By the fundamental theorem of arithmetic (FTA), both n and m have a unique prime factorization up to order of factors. Thus we can say:

$$m = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$$

Now since $n = m^2$ we have

$$n = m^2 = (p_1^{a_1} p_2^{a_2} ... p_k^{a_k})^2 = p_1^{2a_1} p_2^{2a_2} ... p_k^{2a_k} \tag{1}$$

And so by the uniqueness of the FTA, this must be *the* prime factorization of n. Now if $a_i, i \in [0, k]$ is odd then $2a_i$ is even. Similarly if a_i is even then $2a_i$ is also even. Thus all the factors in the prime factorization of n must have even power.

Now we w.t.s that given $gcd(a, b) = 1$ and xy is a perfect square $\Rightarrow$ x,y are perfect squares. Assume for contradiction that x is *not* a perfect square. Then $\exists$ some p_i in the prime factorization of x with an odd power. For xy to be a perfect square then p_i must also divide y so that p_i will have even power in the prime factorization of xy. This contradicts the assumption that $gcd(a, b) = 1$ and thus we say that x and y must *both* be perfect squares.