QM III
Phys 653

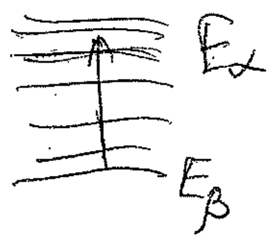
# Lecture #6



## Fermi's golden rule

Consider a transition between two continuum states $\alpha$ and $\beta$ $\Rightarrow$ the transition probability ($\alpha \neq \beta$, 1st order) $\Rightarrow$

$$\mathcal{P}_{\beta \to \alpha} = \frac{1}{\hbar^2} \int_{B(E)} \left| \int_0^t \tilde{V}_{\alpha\beta}(t') e^{i\omega_{\alpha\beta} t'} dt' \right|^2 \rho_\alpha(E)\, dE$$




Consider a "constant" perturbation,

i.e. $V(\vec{r}, t) = \begin{cases} V(\vec{r}), & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$

Then $\tilde{V}_{\alpha\beta}(t') = \int \psi_\alpha^* V(\vec{r}) \psi_\beta \, dV \equiv V_{\alpha\beta}$

$$\mathcal{P}_{\beta \to \alpha} = \frac{1}{\hbar^2} \int_{B(E)} |V_{\alpha\beta}|^2 \rho_\alpha(E) \underbrace{\left| \int_0^T e^{i\omega_{\alpha\beta} t'} dt' \right|^2}_{= \frac{\sin^2 \frac{\omega_{\alpha\beta} T}{2}}{\left(\frac{\omega_{\alpha\beta}}{2}\right)^2}} dE =$$

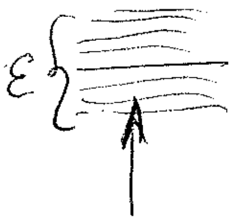
$$= \frac{1}{\hbar^2} \int_{B(E)} |V_{\alpha\beta}|^2 \rho_\alpha(E) \frac{\sin^2 \frac{\omega_{\alpha\beta} T}{2}}{\left(\frac{\omega_{\alpha\beta}}{2}\right)^2} dE \quad (7.1)$$

Note that if $\rho_\alpha(E) = \delta(E - E_f) \Rightarrow$ Eq. (7.1)

$\Downarrow$

discrete states $\Rightarrow$ Eq. on p. 2, Lec. #4

Let $B(E) = \left[E_f - \frac{\varepsilon}{2}, E_f + \frac{\varepsilon}{2}\right]$, $\varepsilon \to 0$




$\Downarrow$

$\rho_\alpha(E)$, $V_{\alpha\beta}(E)$ are practically E-independent

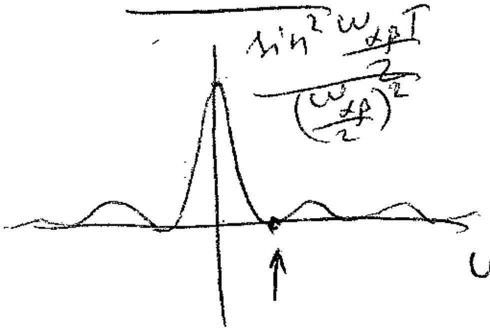
$\Downarrow$

Then, Eq. (7.1) $\Rightarrow$

$$\mathcal{P}_{\beta \to B(E)} = \frac{1}{\hbar^2} |V_{\alpha\beta}|^2 \rho_\alpha(E_f) \int_{E_f - \frac{\varepsilon}{2} < E < E_f + \frac{\varepsilon}{2}} \frac{\sin^2 \frac{\omega_{\alpha\beta}}{2} T}{(\omega_{\alpha\beta}/2)^2} dE$$

$$\omega_{\alpha\beta} = \frac{E - E_\beta}{\hbar}$$

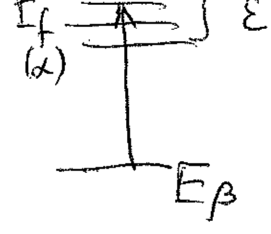
Recall:




largest probability

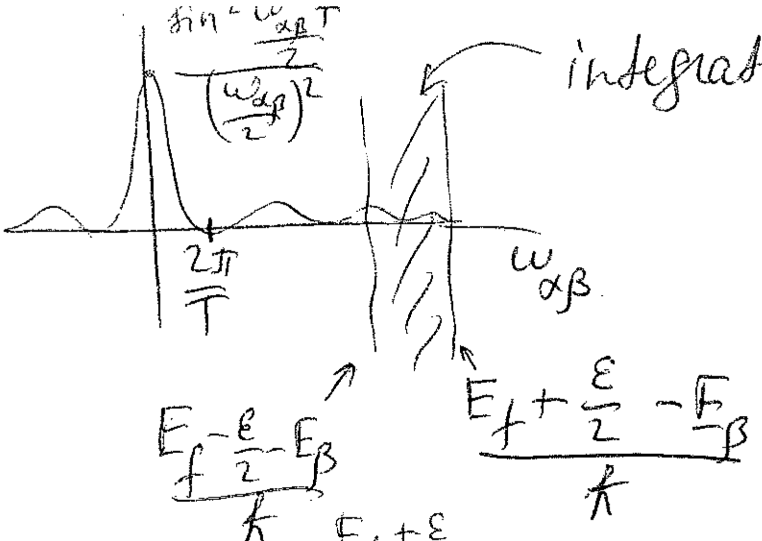
$$\frac{\omega_{\alpha\beta} T}{2} = \pi \Rightarrow |\omega_{\alpha\beta}| \leq \frac{2\pi}{T}$$

(a) $\hbar \omega_{\alpha\beta} \gg \varepsilon \gg \frac{2\pi\hbar}{T}$




$\Downarrow$

energy non-conserving transitions




integration over this region

$\Rightarrow$ in this case

$$\int_{B(E)} \frac{\sin^2\frac{\omega_{\alpha\beta}T}{2}}{\left(\frac{\omega_{\alpha\beta}}{2}\right)^2}\, dE \simeq$$

$$\simeq 2\hbar^2 \int_{E_f-\frac{\varepsilon}{2}}^{E_f+\frac{\varepsilon}{2}} \frac{dE}{(E-E_\beta)^2}$$

average

$$\overline{\frac{\sin^2\frac{\omega_{\alpha\beta}T}{2}}{\left(\frac{\omega_{\alpha\beta}}{2}\right)^2}}^{T} \simeq \frac{1}{2\left(\frac{\omega_{\alpha\beta}}{2}\right)^2}$$

Then, $\mathcal{P}_{\beta\to B(E)} = \frac{1}{\hbar^2}|V_{\alpha\beta}|^2\rho_\alpha(E_f)\cdot 2\hbar^2\cdot$

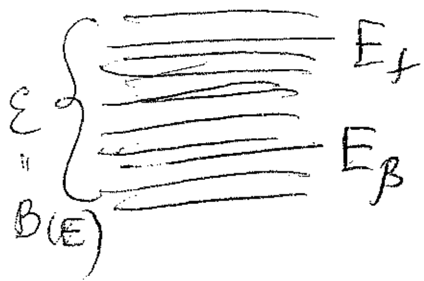
$$\cdot\int_{E_f-\frac{\varepsilon}{2}}^{E_f+\frac{\varepsilon}{2}} \frac{dE}{(E-E_\beta)^2} = 2|V_{\alpha\beta}|^2\rho_\alpha(E_f)\left[\frac{1}{E_f-\frac{\varepsilon}{2}-E_\beta} - \frac{1}{E_f+\frac{\varepsilon}{2}-E_\beta}\right]$$

$$= 2|V_{\alpha\beta}|^2\rho_\alpha(E_f)\frac{\varepsilon}{(E_f-E_\beta)^2-\frac{\varepsilon^2}{4}} \underset{\hbar\omega_{\alpha\beta}\gg\varepsilon}{\approx} \underbrace{\frac{2\varepsilon|V_{\alpha\beta}|^2\rho(E_f)}{(E_f-E_\beta)^2}}$$

does not depend on time

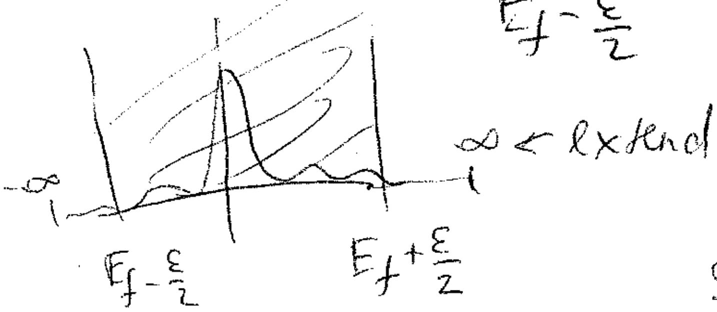
(b) now let $E_f - E_\beta \approx \mathcal{E} \gg \frac{2\pi\hbar}{T}$






In this case

$$\int_{E_f - \frac{\mathcal{E}}{2}}^{E_f + \frac{\mathcal{E}}{2}} \frac{\sin^2 \frac{\omega_{\alpha\beta} T}{2}}{\left(\frac{\omega_{\alpha\beta}}{2}\right)^2} dE \approx \int_{-\infty}^{+\infty} \frac{\sin^2 \frac{\omega_{\alpha\beta} T}{2}}{\left(\frac{\omega_{\alpha\beta}}{2}\right)^2} dE \approx$$




$$= \hbar \int_{-\infty}^{+\infty} \frac{\sin^2 \frac{\omega T}{2}}{\left(\frac{\omega}{2}\right)^2} d\omega = \underline{2\pi\hbar T}$$

$\omega_{\alpha\beta} \approx \omega$

HW!

Then, $\quad \mathcal{P}_{\beta \to B(E)} = \frac{1}{\hbar^2} |V_{\alpha\beta}|^2 \rho_\alpha(E_f) \cdot 2\pi\hbar T =$

$$= \underline{\frac{2\pi}{\hbar} |V_{\alpha\beta}|^2 \rho_\alpha(E)\, T}$$

note $E \approx E_f$ since $\omega_{\alpha\beta} \approx 0$

$\Downarrow$

<u>energy-conserving</u>

<u>transitions</u>

Introduce transition probability per unit time $\Rightarrow$

$$p_{\beta \to B(E)} = \frac{d\mathcal{P}_{\beta \to B(E)}}{dt} = \frac{2\pi}{\hbar} |V_{\alpha\beta}|^2 \rho_\alpha(E) \quad (7.2)$$

Fermi's Golden rule

for energy-conserving transitions $\Rightarrow$ time-indep probability per unit time

For energy non-conserving

$\Downarrow$

$\mathcal{P}_{\beta \to B(E)}$ is time-independent $\Rightarrow$ $p_{\beta \to B(E)} = 0$

Validity of Eq. (7.2) $\Rightarrow$ $T$ is long enough to guarantee that $\varepsilon \gg \frac{2\pi\hbar}{T}$. From another side $T$ is short enough to justify the 1st-order pert. theory, i.e. $\omega_{\alpha\beta} t \Big|_{t=T} \ll 1$.

What if we have another perturbation? $\Rightarrow$

$V(t) = V e^{-i\omega t}$ $\Rightarrow$ Lecture # 4 $\Rightarrow$

$$\mathcal{P}_{i \to f} = \frac{1}{\hbar^2} |V_{if}|^2 \left| \int_0^t e^{i(\omega_{fi} - \omega)t'} dt' \right|^2 =$$

$$= \frac{|V_{if}|^2}{\hbar^2} \frac{\sin^2 \frac{\omega_{fi}-\omega}{2} t}{\left(\frac{\omega_{fi}-\omega}{2}\right)^2}$$

What if the perturbation is applied from $-\frac{T}{2}$ to $\frac{T}{2}$ and $T \to \infty$ ? $\Rightarrow$

$$\mathcal{P}_{i\to f} = \frac{1}{\hbar^2} |V_{if}|^2 \Big| \underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(\omega_{fi}-\omega)t'} dt'}_{\Downarrow T\to\infty \atop 2\pi\delta(\omega_{fi}-\omega)} \Big|^2 =$$

$$= \frac{4\pi^2}{\hbar^2} |V_{if}|^2 \underbrace{\delta(\omega_{fi}-\omega)\,\delta(\omega_{fi}-\omega)}_{\parallel} = \frac{2\pi}{\hbar^2} |V_{if}|^2 \delta(\omega_{fi}-\omega) \cdot \lim_{T\to\infty} T$$

$$\lim_{T\to\infty} \underbrace{\delta(\omega_{fi}-\omega)}_{\downarrow \atop 0 \text{ unless } \omega_{fi}=\omega} \frac{1}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} \underbrace{e^{i(\omega_{fi}-\omega)t}}_{\parallel \atop 1} dt =$$

$$= \delta(\omega_{fi}-\omega) \lim_{T\to\infty} \frac{T}{2\pi}$$

Average transition rate $\Rightarrow$

$$P_{i\to f} = \frac{d\mathcal{P}}{dt} = \frac{2\pi}{\hbar} |V_{if}|^2 \delta\left(E_f^{(0)} - E_i^{(0)} - \hbar\omega\right)$$