Math 443/543, HW #5; due Friday, Nov. 9

1.) Let $\mathcal{B} = (b_1, \ldots, b_n)$ be an n-tuple of elements of $\mathbb{F}^n$. Let $M \in \mathcal{M}_n(\mathbb{F})$ be the matrix whose j-th column is b_j. Show that $\mathcal{B}$ is an ordered basis of $\mathbb{F}^n$ if and only if $\det(M) \neq 0$.

2.) Let V be an $\mathbb{R}$-vector space of dimension 2 and let T be a linear operator on V. Suppose $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$, for some basis $\mathcal{B}$. Determine all T-invariant subspaces of V.

3.) (543) Let x_i be indeterminants (or: variables). Consider the $n \times n$ matrix, V whose ith row is $(1, x_i, x_i^2, \ldots, x_i^{n-1})$ for $i = 1, \ldots, n$. (The matrix V is called a Vandermonde matrix.) Show that the determinant of V equals $\prod_{i>j} (x_i - x_j)$.

Hint. Use induction. After proving the result for a base case, consider the case of an $n \times n$ Vandermonde matrix. Reduce by using neighboring columns; factor appropriately; use properties of the determinant to allow the induction hypothesis to finish the argument.

4.) Give an example of a continuous function $v : \mathbb{R} \to \mathbb{R}^3$ such that $v(t_1)$, $v(t_2)$ and $v(t_3)$ form an $\mathbb{R}$-basis for $\mathbb{R}^3$ whenever t_1, t_2 and t_3 are distinct points of $\mathbb{R}$.

Optional Problems —
these are mainly for review, they will not be graded

O 1.) FIS p. 21 # 20. (Unions of subspaces)

O 2.) FIS p. 35 # 17. (Finite number of generating sets)

O 3.) FIS p. 41 # 9. (Linear independence of two vectors)