Homework #6

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1. $x^2y'' + (x+1)y' - y = 0$

putting this in standard form yields:

$$y'' + \frac{(x+1)}{x^2}y' - \frac{1}{x^2}y = 0$$

and thus we can recognize $p(x) = \frac{x+1}{x^2}$ and $q(x) = \frac{1}{x^2}$

To use Frobenius's method we will use:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$
$$- \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+r)(n+r-1)+(n+r)-1]a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} = 0$$

now we reindex to get powers of x^{n+r-1}

$$a_0 r x^{r-1} + \sum_{n=1}^{\infty} [((n+r-1)(n+r-2)+(n+r-1)-1)a_{n-1} + (n+r)a_n]x^{n+r-1} = 0$$

which the first term implies $r=0$ if a_0 is nontrivial.

The recurrence relation comes from the summation and is:

$$((n+r-1)(n+r-2)+(n+r-1)-1)a_{n-1} + (n+r)a_n = 0$$
$$((n+r-1)^2 - 1)a_{n-1} + (n+r)a_n = 0$$

$$a_n = \frac{(1-(n+r-1)^2)}{(n+r)} a_{n-1}$$

substituting $r=0$ gives:

$$a_n = \left(\frac{1-(n-1)^2}{n}\right) a_{n-1}$$

and so $a_{n+1} = \left(\frac{1-n^2}{n+1}\right) a_n$

Letting $a_0 = a_0$ we have:

$$a_0 = a_0$$
$$a_1 = \frac{1}{1} a_0 = a_0$$
$$a_2 = \frac{0}{2} a_0 = 0$$

thus our first solution is $y_1(x) = a_0(1+x)$

from class we ~~know~~ we may obtain the ~~wronskian~~ second solution from the Wronskian i.e.

$$y_2 = y_1 W_0 \int_{x_1}^{x} \frac{\exp(-\int_{x_0}^{\xi} p(\eta)d\eta)}{y_1^2} d\xi$$

where W_0, x_1, x_0 are arbitrary. Thus we have

$$-\int_{x_0}^{\xi} p(\eta)d\eta = \int_{x_0=0}^{\xi} \frac{-\eta-1}{\eta^2} d\eta = \left[\frac{1}{\xi} - \ln(\xi)\right]$$

thus $\frac{y_2}{y_1} = \int_{x_1=0}^{x} \frac{e^{1/\xi - \ln(\xi)}}{(1+\xi)^2} d\xi = \frac{xe^{1/x}}{1+x}$

and so $y_2 = xe^{1/x}$ thus our final solution is

$$y = c_1(1+x) + c_2(xe^{1/x})$$

2. Use the Frobenius method to solve

$$x^2y'' - (2x+2x^2)y' + (x^2+2x+2)y = 0$$

To ~~as~~ rewrite this in Euler form allows us to identify

$$y(x) = \sum a_n x^{n+r} \qquad y'(x) = \sum (n+r) a_n x^{n+r-1} \qquad y''(x) = \sum (n+r)(n+r-1) a_n x^{n+r-2}$$

where $p(x) = \frac{-(2x+2x^2)}{x^2}$ $q(x) = \frac{(x^2+2x+2)}{x^2}$

plugging everything in yields

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty}a_n x^{n+r+2}$$
$$+ \sum_{n=0}^{\infty}2a_n x^{n+r+1} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

Simplifying yields

$$0 = \sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-2(n+r)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}\left[2-2(n+r-1)\right]a_{n-1}x^{n+r}$$
$$+ \sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

Which we can combine to give

$$0 = a_0[r(r-1)-2r+2]x^r + a_1[(1+r)(r)-2(1+r)+2]x^{1+r} + [2-2r]a_0 x^{1+r}$$
$$+ \sum_{n=2}^{\infty}\left[((n+r)(n+r-1)-2(n+r)+2)a_n + (2-2(n+r-1))a_{n-1} + a_{n-2}\right]x^{n+r}$$

From the x^r term we have the indicial equation:

$$r(r-1)-2r+2=0$$
$$r^2-3r+2=0$$
$$(r-1)(r-2)=0$$
$$\longrightarrow r=1,2$$

we will construct our first solution from $r=2$ and then use the Wronskian variation of parameters to find y_2.

The recurrence relation (general) is:

$$[(n+r)(n+r-1)-2(n+r)+2]a_n + [2-2(n+r-1)]a_{n-1} + a_{n-2} = 0$$
$$[(n+r)(n+r-1)-2(n+r)+2]a_n = [2(n+r-1)-2]a_{n-1} - a_{n-2}$$
$$[(n+r)(n+r-3)+2]a_n = [2(n+r-1)-2]a_{n-1} - a_{n-2}$$
$$a_n = \frac{[2(n+r-1)-2]a_{n-1} - a_{n-2}}{(n+r)(n+r-3)+2}$$

And the x^{1+r} terms give a condition on a_1 and a_0

i.e. $$a_1 = \frac{[2r-2]a_0}{(1+r)(r-2)+2}$$

if we substitute $r=2$ these equations become

$$a_n = \frac{2n\,a_{n-1} - a_{n-2}}{(n+2)(n-1)+2} \qquad a_1 = \frac{2a_0}{2} = a_0$$

with this we can write out the first few terms to find a pattern

$$a_0 = a_0 \qquad a_2 = \frac{4a_0 - a_0}{4+2} = \frac{a_0}{2} \qquad a_4 = \frac{2\cdot 4\frac{a_0}{6} - \frac{a_0}{2}}{6\cdot 3+2} = \frac{a_0}{24}$$

$$a_1 = a_0 \qquad a_3 = \frac{3a_0 - a_0}{12} = \frac{a_0}{6}$$

which follows $a_n = a_0/n!$ and so our solution is

$$y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n = a_0 x^2 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = a_0 x^2 e^x$$

and so our y_1 is

$$y_1(x) = x^2 e^x$$

to find y_2 we have $y_2 = y_1 w_0 \int_{x_0}^{x} \frac{\exp(-\int_{x_1}^{\xi} p(\eta)d\eta)}{y_1^2} d\xi$

$$p(x) = \frac{-(2x+2x^2)}{x^2} \longrightarrow \int_{x_1}^{\xi} \frac{2\eta + 2\eta^2}{\eta^2} d\eta = 2\xi + 2\ln(\xi)$$

thus $$y_2 = y_1 \int_{x_0}^{x} \frac{e^{2\xi + 2\ln(\xi)}}{(\xi^2 e^{\xi})^2} d\xi = y_1 \int_{x_0}^{x} \frac{1}{\xi^2} d\xi$$

$$= \frac{\cancel{x^2}e^x}{x} = xe^x$$

thus the general solution is a linear combination of the two:

$$\boxed{y(x) = C_1 x^2 e^x + C_2 x e^x}$$

and the ξ^{m+2} gives

$$a_2 = \frac{-(\alpha a_1 + \frac{1}{2}Ea_0)}{((2+r)^2 - \frac{m^2}{4})} = \frac{-\left(\frac{-\alpha^2}{(1+r)-\frac{m^2}{4}}a_0\right)}{((2+r)^2 - \frac{m^2}{4})}$$

which give us our first 3 terms. Plugging in $r = \frac{m}{2}$ yields

$$a_0 = a_0 \qquad a_1 = \frac{\alpha}{\frac{m^2}{4} - (1+\frac{m}{2})^2}a_0 = \frac{\alpha a_0}{\frac{m^2}{4} - 1 - m - \frac{m^2}{4}} = \frac{-\alpha a_0}{1+m}$$

$$a_2 = \frac{-\left(\left(\frac{-\alpha^2}{1+m}\right)a_0 + \frac{1}{2}Ea_0\right)}{(2+\frac{m}{2})^2 - \frac{m^2}{4}} = \frac{\frac{\alpha^2 a_0}{(1+m)} - \frac{1}{2}Ea_0}{4+2m}$$

$$= \frac{\alpha^2 - \frac{1}{2}E(m+1)}{(4+2m)(m+1)}a_0$$

So we have $a_0 = a_0$, $a_1 = -\frac{\alpha a_0}{1+m}$, $a_2 = \frac{(\alpha^2 - \frac{1}{2}E(1+m))}{(m+1)(2m+4)}a_0$

thus the solution for the larger of the two roots is

$$\boxed{u(\xi) = a_0\xi^{m/2} - \frac{\alpha}{(1+m)}a_0\xi^{3m/2} + \frac{(\alpha^2 - \frac{1}{2}E(1+m))}{(m+1)(2m+4)}a_0\xi^{5m/2} + \sum_{n=3}^{\infty}\left[((n+\tfrac{m}{2})^2 - \tfrac{m^2}{4}]a_n + \alpha a_{n-1} + \tfrac{1}{2}Ea_{n-2} - \tfrac{1}{4}Fa_{n-3}\right]\xi^{n+m/2}}$$

b. from the boxed solution we can see that we will need to include at least 4 terms to observe the Stark Effect.

4. Use the Frobenius method to show that if the potential is harmonic & the system E must be quantized.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x)$$

$$\psi'' + \left[\frac{2mE}{\hbar^2} - \frac{m^2\omega^2x^2}{\hbar^2}\right]\psi = 0$$

now we let $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x \longrightarrow \xi^2 = \frac{m\omega}{\hbar}x^2$ and so

$$\psi'' + \left[\frac{2mE}{\hbar^2} - \frac{m\omega}{\hbar}\xi^2\right]\psi = 0$$

note that $d\xi = \sqrt{\frac{m\omega}{\hbar}}\,dx \rightarrow dx^2 = \frac{\hbar}{m\omega}d\xi^2$

$$\frac{d^2}{\frac{\hbar}{m\omega}d\xi^2}\psi + \left[\frac{2mE}{\hbar^2} - \frac{m\omega}{\hbar}\xi^2\right]\psi = 0$$

$$\psi''(\xi) + \left[\frac{2E}{\hbar\omega} - \xi^2\right]\psi = 0$$

Now if we define $K = \frac{2E}{\hbar\omega}$

$$\boxed{\psi'' + [K - \xi^2]\psi = 0}$$

Now if we take the limit suggested in the hint that $\xi^2 >> K$ we have that this reduces to

$$\psi'' = \xi^2\psi$$

this is satisfied by the equation $\psi = \varphi(\xi)e^{-\xi^2/2}$ which we make (–) in exponent to insure normalizability. This substitution will let us get rid of the ξ dependence in the diff eq.

taking derivates gives:

$$\psi' = \varphi' e^{-\xi^2/2} - \xi \varphi e^{-\xi^2/2}$$
$$\psi'' = \varphi'' e^{-\xi^2/2} - \xi \varphi' e^{-\xi^2/2} - \varphi e^{-\xi^2/2} - \xi \varphi' e^{-\xi^2/2} + \xi^2 \varphi e^{-\xi^2/2}$$

and so $\psi'' = e^{-\xi^2/2}[\varphi'' - \xi\varphi' - \varphi - \xi\varphi' + \xi^2 \varphi]$

and so the diff eq becomes:

$$e^{-\xi^2/2}[\varphi'' - \xi\varphi' - \varphi - \xi\varphi' + \xi^2\varphi] - \xi^2 \varphi e^{-\xi^2/2} = 0$$

and so $\varphi'' - 2\xi\varphi' - \varphi = 0$ if we extend this to the original eqn. the substitution becomes:

$$\varphi'' - 2\xi\varphi' + (K-1)\varphi = 0$$

we can solve this problem directly using series solutions.

$$\varphi(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \quad \varphi'(\xi) = \sum_{n=1}^{\infty} n a_n \xi^{n-1} \quad \varphi''(\xi) = \sum_{n=2}^{\infty} n(n-1) a_n \xi^{n-2}$$

$$\sum_{n=2}^{\infty} (n)(n-1) a_n \xi^{n-2} - \sum_{n=1}^{\infty} 2 n a_n \xi^n + \sum_{n=0}^{\infty} (K-1) a_n \xi^n = 0$$

Now we reindex to get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \xi^n - \sum_{n=1} 2 n a_n \xi^n + \sum_{n=0} (K-1) a_n \xi^n$$

so $2a_2 + (K-1)a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n + (K-1)a_n]\xi^n$

$2a_2 + (K-1)a_0 = 0$

$a_2 = \frac{(1-K)}{2} a_0$ $\qquad$ $a_{n+2} = \frac{2n-(K-1)}{(n+2)(n+1)} a_0$

Now there are two possibilities, the series of a_n's terminates or it doesnt.
Let's assume it doesn't, then we have

$$a_{n+2} = \frac{2n-(K-1)}{(n+2)(n+1)} a_n$$

so for large n i.e. $n \gg 1$ we have that this ratio becomes $\frac{a_{n+2}}{a_n} \propto \frac{2}{n}$.

Now note that this is similar to the behavior of e^{ξ^2} for large n.

$$e^{\xi^2} = 1 + \xi^2 + \frac{\xi^4}{2!} + \frac{\xi^6}{3!} + \dots$$

$$= \sum_{n=0,2,4,6,\dots}^{\infty} \frac{1}{(\frac{n}{2})!} \xi^n$$

~~For~~ and so the ratio is $\frac{b_{n+2}}{b_n} = \frac{(\frac{n}{2})!}{((n+2)/2)!}$

using wolfram this is equivalent to

$$= \frac{1}{\frac{n}{2}+1} = \frac{2}{n+2}$$ thus for $n \gg 1$

we have the ratio $\frac{b_{n+2}}{b_n} \propto \frac{2}{n}$ ~~so~~

Therefore if the series does not terminate

then $\psi \propto e^{\xi^2} e^{-\xi^2/2} = e^{\xi^2/2}$

which is <u>not</u> normalizable and is therefore <u>not</u> physical thus the series must terminate which means there must exist some N where the series stops i.e.

$$\frac{(2N-(K-1))}{(N+2)(N+1)} = 0 \longrightarrow 2N - K + 1 = 0$$

thus we have

$$K = 2N + 1$$

and so we have shown K is quantized. It can be verified that this holds so long as $N \in \{0, 1, 2, \dots\}$

and so because $K = \frac{2E}{\hbar\omega}$ we conclude

$$E = \hbar\omega\left(N + \tfrac{1}{2}\right) \quad N \in \{0, 1, 2, 3, \dots\}$$