Complex Analysis: Day 15 **John Waczak**
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Extension of Cauchy's integral formula (for derivative)

Recall: let $\Omega \subseteq \mathbb{C}$ be a region, γ a positively oriented, simple closed curve, and $f : \Omega \to \mathbb{C}$ a holomorphic function on Ω then,

$$f(w) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-w}dz$$

Often, γ is just a circle but we showed this for a general closed curve. We can use this in both directions. Sometimes you'd like to know the value of the integral and it's enough to just plug in w. On the other hand it is useful to go the other way so we can evaluate derivatives of functions easily!

$$\begin{aligned}
\text{Calculation:} \quad \frac{d}{dw}\Big(\frac{1}{z-w}\Big) &= \frac{1}{(z-w)^2} \\
\frac{d^2}{dw^2}\Big(\frac{1}{z-w}\Big) &= 2\frac{1}{(z-w)^3} \\
&. \\
&. \\
&. \\
\frac{d^n}{dw^n}\Big(\frac{1}{z-w}\Big) &= n!\frac{1}{(z-w)^(n+1)}
\end{aligned}$$

Theorem (Cauchy's integral formula for derivatives)**.** *With the hypotheses of Cauchy's integral formula for simple-closed curves we have that*

$$f'(w) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-w)^2}dz$$

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$$\textit{more generally} \quad f^{(n)}(w) = \frac{n!}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-w)^(n+1)}dz$$

Remark: the proof involves interchanging the order of a derivative with a path integral. The important *take-away* is that just assuming that f' is holomorphic (i.e. has one derivative) means that we get "for free" that all derivatives $f^{(n)}$ exist. Also notice that if we divide by the $n!$ we get something that looks very similar to the coefficient of a power series.

Examples

$$\text{Ex:} \qquad \oint_{|z|=1} \frac{\sin z}{z^2} dz$$

$$\text{take n=1 case in theorem } f(z) = \sin z$$
$$f'(z) = \cos z$$
$$\text{take w} = 0$$
$$f'(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\sin z}{z^2} dz = \cos 0 = 1$$
$$\Rightarrow \oint_{|z|=1} \frac{\sin z}{z^2} dz = 2\pi i$$

$$\oint_{|z|=2} \frac{1}{z^2(z-1)} dz =$$
$$\text{take straight vertical line between singularities and make two new paths}$$
$$= \oint_{\gamma_1} \frac{1}{z^(z-1)} dz + \oint_{\gamma_2} \frac{1}{z^2(z-2)} dz$$
$$\text{take } f_1(z) = \frac{1}{z-1}, \quad w = 0, \quad n = 1$$
$$f_1'(z) = \frac{-1}{(z-1)^2}$$
$$\Rightarrow \oint_{\gamma_1} \frac{f_1(z)}{z^2} dz = f_1'(0)2\pi i = -2\pi i$$
$$\text{take } f_2(z) = \frac{1}{z^2}, \quad w = 1, n = 0$$
$$\oint_{\gamma_2} \frac{f_2(z)}{z-1} dz = f_2'(1)2\pi i = 2\pi i$$
$$\text{thus } \oint_{|z|=2} \frac{1}{z^2(z-1)} dz = -2\pi i + 2\pi i = 0$$

Some fun applications of this theorem

Theorem (Fundamental theorem of Algebra)**.** *Let $p(z) = a_d z^d + a_{d-1} z^{d-1} + ...a_0 d^0$ be a non-constant polynomial with coefficients in $\mathbb{C}$. Then $p(z)$ has a root in $\mathbb{C}$ (this is false over $\mathbb{R}$).*

Proof. Assume without loss of generality that $a_d \neq 0$, $d \geq 1$. Note that $\exists R > 0$ for which $\frac{1}{2}|a_d||z|^d \leq |p(z)| \leq 2|a_d||z|^d$ whenever $|z| \geq R$.

$$p(z) = a_d z^d \Big(1 + \frac{a_{d-1}}{a_d z} + \frac{a_{d-2}}{a_d z^2} + ... + \frac{a_0}{a_d z^d}\Big)$$
$$\text{as } z \to \infty, \text{parentheses} \to 1$$

Now that we have this lemma let's prove the statement.

Assume that p has no roots in $\mathbb{C}$. Then $\frac{1}{p(z)}$ is entire. By Cauchy's integral formula with $f(z) = \frac{1}{p(z)}$ and R as in the lemma, we have

$$
\begin{aligned}
\frac{1}{p(0)} &= \frac{1}{2\pi i} \oint_{|z|=R} \frac{\frac{1}{p(z)}}{z} dz \\
|\frac{1}{p(0)}| &= \Big| \frac{1}{2\pi i} \oint_{|z|=R} \frac{\frac{1}{p(z)}}{z} dz \Big| \\
&\leq \frac{1}{2\pi} \text{length}(|z| = R) \cdot \max_{|z|=R} |\frac{1}{zp(z)} \\
&= \frac{d1}{2\pi} 2\pi R \cdot \frac{1}{R} \frac{2}{|a_d| R^d} \\
&= \frac{2}{|a_d| R^d} \to 0 \text{ as } R \to \infty
\end{aligned}
$$

But $p(0)$ is a constant that doesn't depend on R so the only choice is to make $R = 0$ thus we have $\frac{1}{p(0)} = 0$ which is a contradiction. $\square$