Central Forces Homework 3

Due 5/16/18, 4 pm

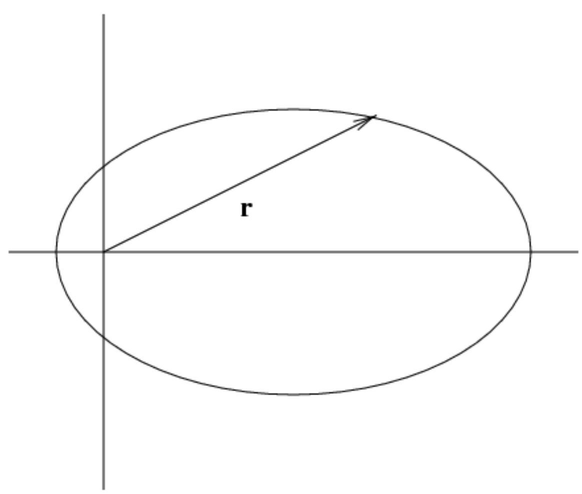
Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

PRACTICE:

1. Show that the plane polar coordinates we have chosen are equivalent to spherical coordinates if we make the choices:
   (a) The direction of z in spherical coordinates is the same as the direction of $\vec{L}$.
   (b) The θ of spherical coordinates is chosen to be $\pi/2$, so that the orbit is in the equatorial plane of spherical coordinates.
2. Show that the plane of the orbit is perpendicular to the angular momentum vector $\vec{L}$.

REQUIRED:

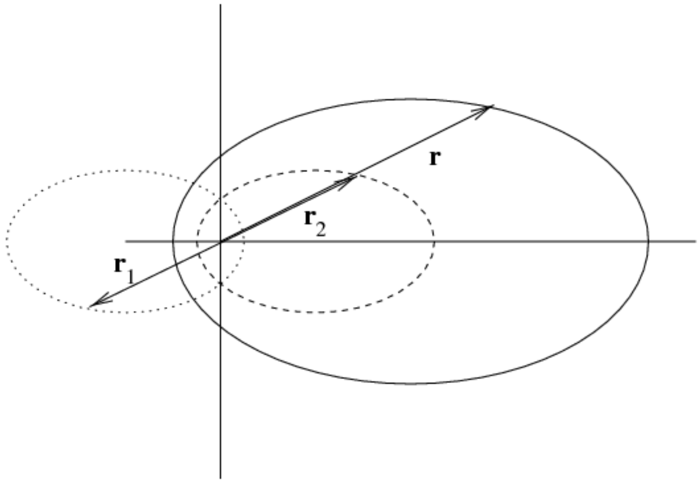
3. The figure below shows the position vector $\mathbf{r}$ and the orbit of a "fictitious" reduced mass.
   (a) Assuming that $m_2 = m_1$, draw on the figure the position vectors for m_1 and m_2 corresponding to $\mathbf{r}$. Also draw the orbits for m_1 and m_2. Describe a common physics example of central force motion for which $m_1 = m_2$.



Solution:

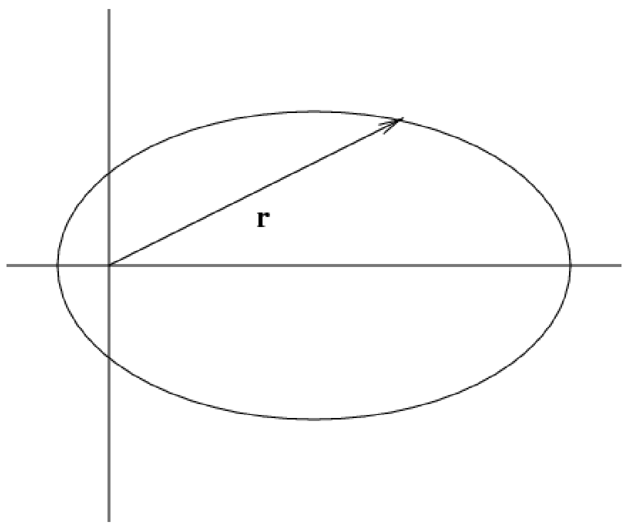
Our final answer should be a sketch of two orbits (one for each mass) with labeled position vectors.

Because $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \left(1 + \frac{m_2}{m_1}\right) \mathbf{r}_2 = 2\,\mathbf{r}_2$, we have



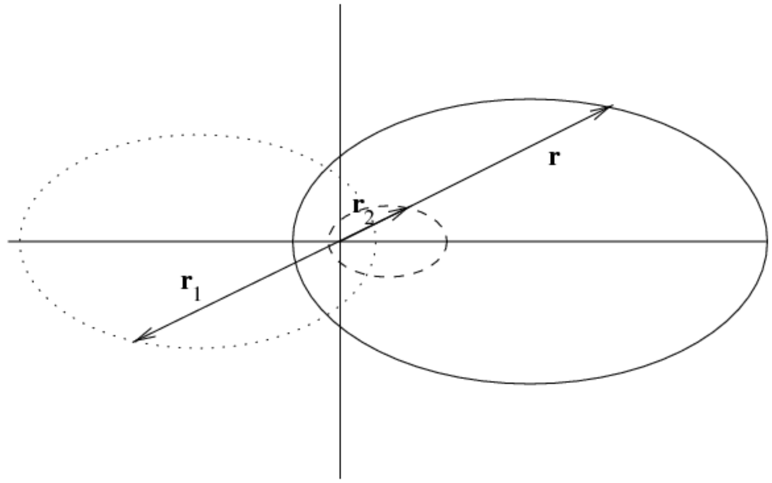
Notice that $|\mathbf{r}| = |\mathbf{r}_1| + |\mathbf{r}_2|$ and that the orbits are reflections of each other through the origin. A common physics example would be motion of equal mass binary stars. A quantum example would be positronium.

(b) Repeat the previous problem for $m_2 = 3m_1$.



Solution:

In this case we have $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \left(1 + \frac{m_2}{m_1}\right) \mathbf{r}_2 = \alpha\, \mathbf{r}_2$, where $\alpha > 2$. The example I have plotted below is for $\alpha = 4$, i.e. $m_2 = 3m_1$. We have



Notice that even for this relatively small value of α that the orbit of m_2 is closer to lying inside the orbit of m_1. An interesting challenge question is: What is the smallest value of α for which one orbit lies entirely inside the other?

A common physics example of $m_2 > m_1$ would be a neutron star rotating around a black hole or a planet rotating around a star.

4. Consider a system of two particles.

(a) Show that the total kinetic energy of the system is the same as that of two "fictitious" particles: one of mass $M = m_1 + m_2$ moving with the speed of the CM (center of mass) and one of mass μ (the reduced mass) moving with the speed of the relative position $\vec{r} = \vec{r}_2 - \vec{r}_1$.

Solution:

In this problem, you are trying to show that the total kinetic energy of the system of two particles m_1 and m_2 is the same as the total kinetic energy of the two "fictitious" particles:

$$T_{total} = \frac{1}{2}m_1|\dot{\vec{r}_1}|^2 + \frac{1}{2}m_2|\dot{\vec{r}_2}|^2 = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2$$

You can start from either side of this equation and show that it is equal to the other side - here I will start from the left. The center of mass position is $\vec{R} = \frac{m_1}{M}\vec{r_1} + \frac{m_2}{M}\vec{r_2}$ and the relative position is $\vec{r} = \vec{r_2} - \vec{r_1}$.

$$\vec{r_2} = \vec{r} - \vec{r_1}$$
$$\implies \vec{R} = \frac{m_1}{M}\vec{r_1} + \frac{m_2}{M}(\vec{r} - \vec{r_1}) = \frac{m_2}{M}\vec{r} + \frac{m_1 + m_2}{M}\vec{r_1}$$
$$\implies \vec{r_1} = \vec{R} - \frac{m_2}{M}\vec{r}$$

$$\text{Likewise: } \vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}$$

Using these relationships:

$$\begin{aligned}
T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\
&= \frac{1}{2}m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2 |\dot{\vec{r}}_2|^2 \\
&= \frac{1}{2}m_1 \left|\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right|^2 + \frac{1}{2}m_2 \left|\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right|^2 \\
&= \frac{1}{2}m_1 \left(\dot{\vec{R}}^2 - \frac{2m_2}{M}\dot{\vec{R}}\cdot\dot{\vec{r}} + \frac{m_2^2}{M^2}\dot{\vec{r}}^2\right) + \frac{1}{2}m_2 \left(\dot{\vec{R}}^2 + \frac{2m_1}{M}\dot{\vec{R}}\cdot\dot{\vec{r}} + \frac{m_1^2}{M^2}\dot{\vec{r}}^2\right) \\
&= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2 (m_1+m_2)}{M^2}\dot{\vec{r}}^2 \quad \text{(cross terms cancel)} \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2
\end{aligned}$$

where M is the total mass, $\dot{\vec{R}}$ is the center of mass velocity, μ is the reduced mass, and $\dot{\vec{r}}$ is the relative velocity.

(b) Show that the total angular momentum of the system can be similarly decomposed into the angular momenta of these two fictitious particles.

Solution:

For this part, you are trying to show that the total angular momentum of the system of two particles m_1 and m_2 is the same as the total angular momentum of the system of two "fictitious" particles. Using the same relationships found in part (a) between $\vec{r}_1$, $\vec{r}_2$, $\vec{R}$, $\vec{r}$:

$$\begin{aligned}
\vec{L} &= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \\
&= \vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_2 \times m_2\dot{\vec{r}}_2 \\
&= \left(\vec{R} - \frac{m_2}{M}\vec{r}\right) \times m_1 \left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right) + \left(\vec{R} + \frac{m_1}{M}\vec{r}\right) \times m_2 \left(\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right) \\
&= m_1\vec{R}\times\dot{\vec{R}} - \frac{m_1 m_2}{M}\vec{r}\times\dot{\vec{R}} - \frac{m_1 m_2}{M}\vec{R}\times\dot{\vec{r}} + \frac{m_1 m_2^2}{M^2}\vec{r}\times\dot{\vec{r}} \\
&\quad m_2\vec{R}\times\dot{\vec{R}} + \frac{m_1 m_2}{M}\vec{r}\times\dot{\vec{R}} + \frac{m_1 m_2}{M}\vec{R}\times\dot{\vec{r}} + \frac{m_1^2 m_2}{M^2}\vec{r}\times\dot{\vec{r}} \\
&= (m_1+m_2)\vec{R}\times\dot{\vec{R}} + \frac{m_1 m_2 (m_1+m_2)}{M^2}\vec{r}\times\dot{\vec{r}} \\
&= \vec{R}\times M\dot{\vec{R}} + \vec{r}\times\mu\dot{\vec{r}} \\
&= \vec{R}\times\vec{P}_{CM} + \vec{r}\times\vec{p}_{rel} \quad \left(\text{where } \vec{P}_{CM} = M\dot{\vec{R}} \text{ and } \vec{p}_{rel} = \mu\dot{\vec{r}}\right)
\end{aligned}$$

5. The general equation for a straight line in polar coordinates is given by:

$$r(\phi) = \frac{r_0}{\cos(\phi - \delta)}$$

Find the polar equation for the following straight lines:

(a) $y = 3$

Solution:

Our answer in each case should be a specific polar equation $r(\phi)$ that does not have x or y in it.

In general, the geometric interpretation of this equation is that r_0 represents the closest distance of the line from the origin (you can see this because largest the denominator can be is 1, making r_0 the smallest value of r(ϕ)), and δ is the angle that the line makes with the y-axis measured counterclockwise (you can see this by plotting several lines with different values of δ).

For this particular line, $y = 3$, the point of the line closest to the origin is (0,3), so $r_0 = 3$. The line is a horizontal straight line, so $\delta = \frac{\pi}{2}$.

$$r(\phi) = \frac{3}{\cos(\phi - \frac{\pi}{2})}$$

(b) $x = 3$

Solution:

This is a vertical straight line (the angle relative to the y-axis is zero), and the closet point to the origin is $(3, 0)$.

$$r(\phi) = \frac{3}{\cos\phi}$$

(c) $y = -3x + 2$

Solution:

This is a line with negative slope that crosses the y-axis at $(0, 2)$ and crosses the x-axis at $\left(0, \frac{2}{3}\right)$. The angle between the line and the y-axis is:

$$\tan(\delta) = \frac{\left(\frac{2}{3}\right)}{2}$$

$$\delta = \arctan\left(\frac{1}{3}\right)$$

The point of closest approach is where there is an intersection between this line and the line that is perpendicular to it and also passes through the origin. Lines that are perpendicular to $y = -3x + 2$ have slope $m = +\frac{1}{3}$. The line with slope $m = +\frac{1}{3}$ and passes through the origin is:

$$y = \frac{1}{3}x$$

The point where these two lines intersect is:

$$\frac{1}{3}x = -3x + 2$$
$$\frac{10}{3}x = 2$$
$$x = \frac{3}{5}, y = \frac{1}{5}$$

The value of r_0 is then:

$$\begin{aligned} r_0 &= \sqrt{x^2 + y^2} \\ &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)^2} \\ &= \frac{\sqrt{10}}{5} \end{aligned}$$

$$r(\phi) = \frac{\sqrt{10}}{5\cos\left(\phi - \arctan\ \left(\frac{1}{3}\right)\right)}$$