

1. The potential due to a ring of charge is given by:

$$V(s, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{s^2 + R^2 - 2sR \cos(\phi - \phi') + z^2}}$$

Expand this potential in a power series to fourth order, in the plane of the ring, for $s < R$. Warning: Make sure you keep **all** of the terms up to fourth order and none of the terms of higher order. This is tricky to do and is the most important lesson from this homework problem.

Solution:

To expand this potential in a power series, it would be nice to save some effort and use the series we have already memorized (Quiz 1). Recall,

$$(1 + u)^p = 1 + pu + \frac{p(p-1)}{2!}u^2 + \frac{p(p-1)(p-2)}{3!}u^3 + \frac{p(p-1)(p-2)(p-3)}{4!}u^4 + \dots \quad (1)$$

We are looking at the potential in the plane of the ring, so $z = 0$. We can also rewrite the square root as a power.

$$V(s, \phi, z = 0) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \left(s^2 + R^2 - 2sR \cos(\phi - \phi') \right)^{-1/2} d\phi' \quad (2)$$

The integrand is almost the same as equation (1) but we need it to match exactly for the power series to be valid. **Remember:** equation (1) is valid only for $|u| < 1$. We are interested in finding the potential where $s < R$, or in other words, $s/R < 1$ is a small quantity and we can *pull out* R^2 from the expression. That is,

$$\left(s^2 + R^2 - 2sR \cos(\phi - \phi') \right)^{-1/2} = \left[R^2 \left(1 + \frac{s^2}{R^2} - \frac{2s}{R} \cos(\phi - \phi') \right) \right]^{-1/2} \quad (3)$$

$$= \frac{1}{R} \left(1 + \frac{s^2}{R^2} - \frac{2s}{R} \cos(\phi - \phi') \right)^{-1/2} \quad (4)$$

$$u \equiv \frac{s^2}{R^2} - \frac{2s}{R} \cos(\phi - \phi'), \quad p = -1/2 \quad (5)$$

where in the final line I have identified our u and p for the series expansion.

Now, we need to expand the powers of u in order to find all of the fourth order terms in $\frac{s}{R}$. Yes, this is a lot of algebra.

$$p(u) = \left(\frac{s}{R} \right) \cos(\phi - \phi') - \frac{1}{2} \left(\frac{s}{R} \right)^2 \quad (6)$$

$$\frac{p(p-1)}{2!}u^2 = \frac{5}{2} \left(\frac{s}{R} \right)^2 \cos^2(\phi - \phi') - \frac{3}{2} \left(\frac{s}{R} \right)^3 \cos(\phi - \phi') + \frac{3}{8} \left(\frac{s}{R} \right)^4 \quad (7)$$

$$\frac{p(p-1)(p-2)}{3!}u^3 = \frac{5}{2} \left(\frac{s}{R} \right)^3 \cos^3(\phi - \phi') - \frac{15}{4} \left(\frac{s}{R} \right)^4 \cos^2(\phi - \phi') + \dots \quad (8)$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}u^4 = \frac{35}{8} \left(\frac{s}{R} \right)^4 \cos^4(\phi - \phi') + \dots \quad (9)$$

Using this to combine all terms with like powers in s/R , the integral reduces to

$$\begin{aligned}
V(s, \phi) \approx \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \frac{1}{R} \int_0^{2\pi} \left\{ 1 + \cos(\phi - \phi') \left(\frac{s}{R}\right) \right. \\
+ \left[\frac{3}{2} \cos^2(\phi - \phi') - \frac{1}{2} \right] \left(\frac{s}{R}\right)^2 \\
+ \left[\frac{5}{2} \cos^3(\phi - \phi') - \frac{3}{2} \cos(\phi - \phi') \right] \left(\frac{s}{R}\right)^3 \\
\left. + \left[\frac{3}{8} - \frac{15}{4} \cos^2(\phi - \phi') + \frac{35}{8} \cos^4(\phi - \phi') \right] \left(\frac{s}{R}\right)^4 \right\} d\phi'
\end{aligned} \tag{10}$$

The original equation for the potential can not be integrated analytically. Now that we have expanded the integrand, we have reduced the problem to a bunch of integrals of $\cos^n(\phi - \phi')$ which we can solve by brute force. If you wish to do the integrals by hand, take advantage of the exponential form of cosine. Otherwise, Mathematica is a great option for this sort of integral.

$$\text{In[12]:= } u = \left(\frac{s}{R}\right)^2 - 2 * \left(\frac{s}{R}\right) * \text{Cos}[\phi - \phi_0]$$

$$\text{Out[12]= } \frac{s^2}{R^2} - \frac{2 s \text{Cos}[\phi - \phi_0]}{R}$$

Find terms in expansion up to u^4

$$\text{In[13]:= } p = -(1/2);$$

$$\text{In[14]:= } \text{Expand}\left[\frac{p}{1} * u\right]$$

$$\text{Expand}\left[\frac{p (p - 1)}{2!} * u^2\right]$$

$$\text{Expand}\left[\frac{p (p - 1) (p - 2)}{3!} * u^3\right]$$

$$\text{Expand}\left[\frac{p (p - 1) (p - 2) (p - 3)}{4!} * u^4\right]$$

$$\text{Out[14]= } -\frac{s^2}{2 R^2} + \frac{s \text{Cos}[\phi - \phi_0]}{R}$$

$$\text{Out[15]= } \frac{3 s^4}{8 R^4} - \frac{3 s^3 \text{Cos}[\phi - \phi_0]}{2 R^3} + \frac{3 s^2 \text{Cos}[\phi - \phi_0]^2}{2 R^2}$$

$$\text{Out[16]= } -\frac{5 s^6}{16 R^6} + \frac{15 s^5 \text{Cos}[\phi - \phi_0]}{8 R^5} - \frac{15 s^4 \text{Cos}[\phi - \phi_0]^2}{4 R^4} + \frac{5 s^3 \text{Cos}[\phi - \phi_0]^3}{2 R^3}$$

$$\text{Out[17]= } \frac{35 s^8}{128 R^8} - \frac{35 s^7 \text{Cos}[\phi - \phi_0]}{16 R^7} + \frac{105 s^6 \text{Cos}[\phi - \phi_0]^2}{16 R^6} - \frac{35 s^5 \text{Cos}[\phi - \phi_0]^3}{4 R^5} + \frac{35 s^4 \text{Cos}[\phi - \phi_0]^4}{8 R^4}$$

Collect like terms of $\left(\frac{s}{R}\right)^k$

$$\begin{aligned}
 \text{In[18]:= Integrand1} &= \frac{s \cos[\phi - \phi_0]}{R} \\
 \text{Integrand2} &= -\frac{s^2}{2 R^2} + \frac{3 s^2 \cos[\phi - \phi_0]^2}{2 R^2} \\
 \text{Integrand3} &= -\frac{3 s^3 \cos[\phi - \phi_0]}{2 R^3} + \frac{5 s^3 \cos[\phi - \phi_0]^3}{2 R^3} \\
 \text{Integrand4} &= \frac{3 s^4}{8 R^4} - \frac{15 s^4 \cos[\phi - \phi_0]^2}{4 R^4} + \frac{35 s^4 \cos[\phi - \phi_0]^4}{8 R^4}
 \end{aligned}$$

$$\text{Out[18]= } \frac{s \cos[\phi - \phi_0]}{R}$$

$$\text{Out[19]= } -\frac{s^2}{2 R^2} + \frac{3 s^2 \cos[\phi - \phi_0]^2}{2 R^2}$$

$$\text{Out[20]= } -\frac{3 s^3 \cos[\phi - \phi_0]}{2 R^3} + \frac{5 s^3 \cos[\phi - \phi_0]^3}{2 R^3}$$

$$\text{Out[21]= } \frac{3 s^4}{8 R^4} - \frac{15 s^4 \cos[\phi - \phi_0]^2}{4 R^4} + \frac{35 s^4 \cos[\phi - \phi_0]^4}{8 R^4}$$

Calculate the nasty integral

$$\begin{aligned}
 \text{In[22]:= Int} &= \text{Expand} \left[\int_0^{2\pi} \left(1 + \right. \right. \\
 &\quad \cos[\phi - \phi_0] * \left(\frac{s}{R} \right) + \\
 &\quad \left(\frac{3}{2} \cos[\phi - \phi_0]^2 - \frac{1}{2} \right) * \left(\frac{s}{R} \right)^2 + \\
 &\quad \left(\frac{5}{2} \cos[\phi - \phi_0]^3 - \frac{3}{2} \cos[\phi - \phi_0] \right) * \left(\frac{s}{R} \right)^3 + \\
 &\quad \left. \left(\frac{3}{8} - \frac{15}{4} \cos[\phi - \phi_0]^2 + \frac{35}{8} \cos[\phi - \phi_0]^4 \right) * \left(\frac{s}{R} \right)^4 \right) d\phi_0 \Big]
 \end{aligned}$$

$$\text{Out[22]= } 2 \pi + \frac{\pi s^2}{2 R^2} + \frac{9 \pi s^4}{32 R^4}$$

Multiply by all of the constants to get the potential --> $V = \frac{Q}{4 \pi \epsilon_0} * \frac{1}{2 \pi} * \text{Integral}$

In[23]:=

$$\text{In[24]:= } V = \frac{Q}{4 \pi \epsilon_0} * \text{Expand} \left[\frac{1}{2 \pi} * \text{Int} \right]$$

$$\text{Out[24]= } \frac{Q \left(1 + \frac{s^2}{4 R^2} + \frac{9 s^4}{64 R^4} \right)}{4 \pi \epsilon_0}$$

Therefore, our solution for the electric potential in the plane of the ring to fourth order in s is

$$V(s, \phi, z = 0) \approx \frac{Q}{4\pi\epsilon_0} \left\{ \frac{1}{R} + \frac{1}{4} \frac{s^2}{R^3} + \frac{9}{64} \frac{s^4}{R^5} \right\} \quad (11)$$

NOTE: Our solution does not depend on ϕ *and* is an even function in s . Why is that?

CHECK: Does our solution agree with the original integral equation at the origin?