Taylor - Chapter 5 Notes - Oscillations

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1 Hooke's Law

Hooke's spring law for a general spring mass system or other oscillatory motion:

$$F_x(x) = -kx \tag{1}$$

Here x is the displacement from the equilibrium position of the spring. And k is a positive number called the "force" or "spring" constant. The Hooke's law force is a *restoring force* meaning that the direction of the force always points towards the equilibrium postion for positive values of k.

Recall the force is simply the minus gradient of the Potential energy. Therefore an equivalent formulation of Hooke's Law:

$$U(x) = \frac{1}{2}kx^2 \tag{2}$$

2 Simple Harmonic Motion

Consider the differential equation for Newton's second law of motion for a mass m that is displaced from stable equilibrium.

$$m\ddot{x} = F_x = -kx \tag{3}$$

$$\ddot{x} = -\frac{k}{m}x \tag{4}$$

For convenience we denote $\omega = \frac{k}{m}$ which turns out to be the angular frequency of oscillation.

2.1 Exponential solutions and other forms

Equation (4) is a 2nd order, linear, homogeneous differential equation. The general method for solving these is to make some kind of informed guess (ansatz according to Matt) which in this case is either sin or cos in exponential form (as the behave under the rules of equation 4).

Taking any linear combination of these two solutions clearly also satisfies the differential equation so we have:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{5}$$

This linear combination is more commonly denoted as a *superposition solution* in physics and is an idea that appears all over the place. Any particular solution can be created by tweaking C_1 and C_2 appropriately. The following forms for the solution to (4) are equivalent:

$$\begin{aligned}
x(t) &= B_1 \cos \omega t + B_2 \sin \omega t \\
x(t) &= A \cos (\omega t + \phi) \\
x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\
x(t) &= Re[De^{i\omega t}]
\end{aligned}$$

2.2 Energy consideration

Consider now the energy of the oscillator...

$$\begin{aligned}
U &= \frac{1}{2} k x^2 \\
U &= \frac{1}{2} A^2 \cos^2(\omega t + \phi) \\
T &= \frac{1}{2} m v^2 \\
T &= \frac{1}{2} m \omega^2 A^2 sin^2(\omega t + \phi)
\end{aligned}$$

Interestingly enough, when you consider the total energy, due to the pythagorean identity you simply get:

$$E = T + U = \frac{1}{2} A^2 \tag{6}$$

Which is always constant as it should be for a conservative force.

3 Damped Oscillations

Now we consider the idea that there may be resistive forces damping the oscillations so that they "die out". Let the resistive force be defined as $F_r = -b\mathbf{v}$. Then Newton's second law (4) transforms into:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{7}$$

Kirchoff's loop law for an LRC circuit gives a similar differential equation.

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0 \tag{8}$$

In order to solve (7) it is convenient to divide through by m and rename some variables. i.e. let $\omega_0 = \sqrt{\frac{k}{m}}$ be the natural frequency and $\frac{b}{l}m = 2\beta$ be the damping constant. Large β correspond to large damping forces. Applying these definitions reframes the differential equation as:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = 0 \tag{9}$$

As previously, solving this equation begins with a reasonable guess.

$$\begin{aligned} x(t) &= e^{rt} \\ \dot{x}(t) &= re^{rt} \\ \ddot{x}(t) &= r^2 e^{rt} \\ &\leftrightarrow \\ r^2 + 2\beta r + \omega_0^2 &= 0 \end{aligned}$$

The last equation is called the characteristic polynomial... it's roots are the different possible values of omega in the final linear combination of solutions.

$$\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}$$

This gives the general dampened solution:

$$x(t) = e^{-\beta t}(C_1 e^{\sqrt{\beta^2 - \omega_0^2}} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}}) \tag{10}$$

3.1 Undamped oscillations

In the case of undamped oscillation, $\beta = 0$ and so the solution is purely imaginary and oscillates at frequency ω_0

3.2 Weak Damping a.k.a. underdamping

This occurs in the case where $\beta < \omega_0$. The square root is imaginary, so we can simplify and write:

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_f$$

ω_f Is a frequency that is less than the natural frequency ω_0. For small β this difference is essentially zero, giving us the undamped solution. Using this new parameter, the solution becomes:

$$x(t) = Ae^{-\beta t}\cos(\omega_f t + \phi) \tag{11}$$

In this form it is easy to see that $\frac{1}{\beta}$ is the time it takes for the decay term $Ae^{-\beta t}$ to reach $\frac{1}{e}$ of the initial value. So, despite having the same units as frequency, we can view this as the "decay" parameter.

3.3 Strong Damping a.k.a. overdamping

Suppose now that beta, our damping constant becomes much larger, specifically that $\beta > \omega_0$. This forces the square root of our exponents in (10) to be real and so we have:

$$x(t) = C_1 e^{-(\beta-\sqrt{\beta^2-\omega_0^2})} + C_2 e^{-(\beta+\sqrt{\beta^2-\omega_0^2})} \tag{12}$$

3.4 Critical damping

The bound that defines the difference between under and over damping is called critical damping. This occurs specifically when $\beta = \omega_0$. This actually kills one of our guesses from eqn (10) and so we are forced to find another in order to write the general solution as a linear combination. Fortunately we may observe that $te^{-\beta t}$ satisfies the differential equation giving our general solution to be:

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} \tag{13}$$

3.5 Summary

The table below summarizes the values of the decay parameter that correspond to each situation previously described:

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$

4 Driven Damped Oscillations

Damping forces drain the energy of oscillators and eventual bring them to a halt. Thus, for oscillations to continue, an external driving force is arranged to keep the system oscillating. The differential equation becomes:

$$m\ddot{x} + b\dot{x} + kx = F(x) \tag{14}$$

Its counterpart for the LRC circuit is the differential equation:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = \varepsilon(t) \tag{15}$$

In general, a damped driven oscillator has differential equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \tag{16}$$

Solutions to these kinds of problems are not too terrible hard to form. Recall that the rule we previously developed by which a second order ODE must have at least two different solutions in its general solution basis. Thus if we can guess one solution x_p where p stands for particular and then find the homogeneous solution for $f(t) = 0$, we can combine the two to form our general solution. If D was the linear "differential operator" we would have:

$$D(x_p + x_h) = f(t) + 0 = f(t)$$

So clearly this solution holds.

For simplicity let's make the forcing function sinusoidal. i.e.

$$f(t) = f_0 e^{i\omega t}$$

Since our forcing function is complex, whatever solution we end up with we will just take the real part to be *physical* solution.

A reasonable guess could be $z(t) = Ce^{i\omega t}$ and so we will plug this into our differential equation and solve for C:

$$\begin{aligned} \ddot{z} + 2\beta\dot{z} + \omega_0^2 z &= f_0 e^{i\omega t} \\ (-\omega^2 + 2i\beta\omega + \omega_0^2)Ce^{i\omega t} &= f_0 e^{i\omega t} \end{aligned}$$

This has solutions iff:

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} \tag{17}$$

For convenience we will put this into exponential form:

$$C = Ae^{-i\phi}$$

with

$$\begin{aligned} A^2 &= \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \\ \phi &= \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}$$

This results in the particular solution:

$$z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \phi)} \tag{18}$$

with real part:

$$x(t) = A\cos(\omega t - \phi) \tag{19}$$

Combining this with the homogeneous solution as previously mentioned gives our general solution to be:

$$x(t) = A\cos(\omega t - \phi) + C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{20}$$

The exponential terms from our homogeneous solution die out over time and so are called *transients.*

5 Resonance

After the transients die out, we showed the driven damped oscillator follows the equation of motion:

$$x(t) = A\cos(\omega t - \phi)$$

with an amplitude given by,

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

and phase,

$$\phi = \arctan(\frac{2\beta\omega}{\omega_0^2 - \omega^2})$$

Observe that the amplitude A is proportional to the driving force f_0. A also clearly depends on the two important frequencies: the driving frequency ω and the natural frequency ω_0. By itself, the oscillator wants to vibrate at the natural frequency. If instead we want it to vibrate at the driving frequency, we find that the system only responds well for $\omega \approx \omega_0$. We call this phenomenon - the dramatically greater response of the oscillator when driven at the correct frequency - *Resonance.*

The amplitude is maximized ("at resonance") when the denominator is minimized, that is, $(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$. There are a lot of frequencies to talk about. Here is a table summarizing them:

symbol	value	description
ω_0	$\sqrt{\frac{k}{m}}$	natural frequency of undamped oscillator
$\omega_f = \omega_1$	$\sqrt{\omega_0^2 - \beta^2}$	frequency of damped oscillator
ω	arbitrary	driving frequency
ω_2	$\sqrt{\omega_0^2 - 2\beta^2}$	value of ω at which response is max

The maximum possible amplitude can then be found by setting $\omega \approx \omega_0$ to give:

$$A_{max} \approx \frac{f_0}{2\beta\omega_0}$$

Other important equations for characterizing resonance include:

$$Q = \frac{\omega_0}{2\beta}$$

$$(decay \quad time) = \frac{1}{\beta}$$

$$T = \frac{2\pi}{\omega_0}$$

$$Q = \pi\frac{decay \quad time}{period}$$

$$\phi_{lag} = \arctan(\frac{-2\beta\omega}{\omega_0^2 - \omega^2})$$

And here's one more table for the phase difference between passive circuit components and their complex impedences:

component	$\Delta\phi_I$	Z
C	$\frac{\pi}{2}$	$\frac{1}{i\omega c}$
L	$-\frac{\pi}{2}$	$i\omega L$
R	0	R