John Waczak
Mth 434
17-C

3.75

write a general formula for the area of the surface of revolution

What is the area obtained by revolving around the z axis a circle of radius 1 centered at $(2,0,0)$ to get a Torus?

Recall $\|d\sigma\| = |\sigma_u \times \sigma_v|$

the surface patch for a general surface of revolution around the z axis is

$$\sigma(\theta, t) = (x(t)\cos(\theta), x(t)\sin\theta, z(t))$$

Therefore we have:

$$\sigma_\theta = (-x(t)\sin\theta, \ x(t)\cos\theta, 0)$$

$$\sigma_t = (x'(t)\cos\theta, x'(t)\sin\theta, z'(t))$$

$$\rightarrow \sigma_\theta \times \sigma_t = (x(t)z'(t)\cos(\theta), x(t)z'(t)\sin\theta, -x(t)x'(t)\sin^2\theta - x(t)x'(t)\cos^2\theta)$$

$$= (x(t)z'(t)\cos\theta, x(t)z'(t)\sin\theta, -x(t)x'(t))$$

Now we need to ~~normalize~~ find the norm

$$\sigma_\theta \times \sigma_t = (x(t)z'(t)\cos\theta,\ x(t)z'(t)\sin\theta,\ -x(t)x'(t))$$

$$|\sigma_\theta \times \sigma_t| = \left(x(t)^2 z'^2(t)\cos^2\theta + x(t)^2 z'(t)^2 \sin^2\theta + x(t)^2 x'(t)^2 \right)^{1/2}$$

$$= \left(x(t)^2 z'(t)^2 + x(t)^2 x'(t)^2 \right)^{1/2}$$

$$= x(t)\left(z'(t)^2 + x'(t)^2 \right)^{1/2}$$

Thus we have that in general

$$\| d\sigma \| = x(t)\sqrt{z'(t)^2 + x'(t)^2}$$

if $\gamma(t)$ is parametrized by arc length we can say $\sqrt{z'(t)^2 + x'(t)^2} = 1$ and so

$$\| d\sigma \| = x(t)$$

Now we want to use this to find the surface area of the torus given by

$$\sigma(\theta, t) = ((2+\cos t)\cos\theta,\ (2+\sin t)\sin\theta,\ \sin t)$$

$$t, \theta \in [0, 2\pi)$$

Now $\gamma(t) = (2+\cos t, 0, \sin t)$

$$\frac{d}{dt}\gamma(t) = (-\sin t, 0, \cos t)$$

which has $|\gamma(t)| = 1$ therefore from our previous conclusion we have

$$\|d\sigma\| = x(t) = 2+\cos t$$

and subsequently

$$A = \int_0^{2\pi}\int_0^{2\pi} 2+\cos t \, dt \, d\theta$$

$$= \int_0^{2\pi} 2t - \sin t \Big|_0^{2\pi} d\theta$$

$$= \int_0^{2\pi} 4\pi \, d\theta$$

$$= 8\pi^2$$

This makes sense as it agrees w/ general Torus S.A. $= (R^2 - r^2)\pi^2$ where here $r=1$, $R=3$ (inner + outer radii).