Homework 4 **John Waczak**

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1. Consider a physical system whose Hamiltonian and initial state are given by $H = \varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $|\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, where ε_0 has the dimensions of energy.

(a) What values will we obtain when measuring the energy and with what probabilities?

The possible values of measurement are given by the eigenvalues of H. From its matrix representation, we have that the characteristic equation is

$$\begin{aligned} (-1-\lambda)[(1-\lambda)^2 - 1] &= 0 \\ \Rightarrow -1-\lambda &= 0 \\ \lambda &= -1 \\ \Rightarrow (1-\lambda)^2 - 1 &= 0 \\ \lambda &= 0, 2 \end{aligned}$$

Thus the possible measurements of H are $\{-1\varepsilon_0,\ 0\varepsilon_0,\ 2\varepsilon_0\}$. To figure out with what probabilities these are found we need to determine the eigenbasis for H and then evaluate $|\psi\rangle$ in this basis.

$$\begin{aligned} \begin{pmatrix} 1-2 & -1 & 0 \\ -1 & 1-2 & 0 \\ 0 & 0 & -1-2 \end{pmatrix} &= \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \\ &\cong \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \Rightarrow |E = 2\varepsilon_0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Continuing this procedure for the other eigenvalues leads to the eigenbasis

$$|E = 2\varepsilon_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad |E = 0\varepsilon_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad |E = -1\varepsilon_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now assuming that $|\psi\rangle$ is represented in this basis, we have the following probabilities

$$
\begin{aligned}
P(2\varepsilon_0) &= |\langle 2\varepsilon_0|\psi\rangle|^2 \\
&= 0 \\
P(0\varepsilon_0) &= |\langle 0\varepsilon_0|\psi\rangle|^2 \\
&= \left|\frac{1}{\sqrt{12}}(1+1)\right|^2 \\
&= \frac{1}{3} \\
P(-1\varepsilon_0) &= |\langle -1\varepsilon_0|\psi\rangle|^2 \\
&= \frac{4}{6} = \frac{2}{3}
\end{aligned}
$$

Note that $0 + 1/3 + 2/3 = 1$ and so we are confident that these are the correct probabilities.

(b) Calculate the expectation value of the Hamiltonian both ways: (i) using the eigenvalues and probabilities, and (ii) using the definition of expectation value with H and $|\psi\rangle$.

(i) $\langle H\rangle = \sum_n E_n P_n = 2\varepsilon_0 \cdot 0 + 0\varepsilon_0 \cdot \frac{1}{3} - \varepsilon_0 \cdot \frac{2}{3} = -\frac{2}{3}\varepsilon_0$

(ii)

$$
\begin{aligned}
\langle H\rangle &= \langle\psi|H|\psi\rangle \\
&= \frac{1}{6}\begin{pmatrix} 1 & 1 & 2\end{pmatrix}\varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix} \\
&= \frac{\varepsilon_0}{6}(-4) \\
&= -\frac{2}{3}\varepsilon_0
\end{aligned}
$$

So we see that the two definitions are equivalent.

2. Consider a system whose Hamiltonian and an operator A are given by the matrices

$$
H = \varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1\end{pmatrix}, \quad A = a_0 \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}
$$

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(a) If we measure energy, what values will we obtain?

This Hamiltonian is the same as in problem 1. Therefore, the possible energy values are $\{2\varepsilon_0, 0\varepsilon_0, -1\varepsilon_0\}$

(b) Suppose that when we measure the energy, we obtain a value of $-\varepsilon_0$. Immediately afterwards, we measure A. What values will we obtain for A and with what probabilities?

If we measure $E = -\varepsilon_0$, then the state has been projected into $|-\varepsilon_0\rangle$. To figure out the results of a subsequent A measurement we must determine the eigenvalues (possible measurements) of A as well as it's eigenvectors. Then we can find the coefficients of $|-\varepsilon_0\rangle$ in this basis for the probabilities.

The characteristic equation for A leads to eigenvalues $\{0a_0, a_0\sqrt{17}, -a_0\sqrt{17}\}$. The eigenvectors corresponding to these values are

$$|-\sqrt{17}a_0\rangle = \begin{pmatrix} 2\sqrt{2/17} \\ -1/\sqrt{2} \\ 1/\sqrt{34} \end{pmatrix} \quad |\sqrt{17}a_0\rangle = \begin{pmatrix} 2\sqrt{2/17} \\ 1/\sqrt{2} \\ 1/\sqrt{34} \end{pmatrix} \quad |0a_0\rangle = \begin{pmatrix} -\sqrt{1/17} \\ 0 \\ 4/\sqrt{17} \end{pmatrix}$$

Thus the possible measurements are their probabilities are

$$\begin{aligned} P\left(a = -\sqrt{17}a_0\right) &= \left|\langle -\sqrt{17}a_0 | -\varepsilon_0\rangle\right|^2 \\ &= \left|\frac{1}{\sqrt{34}}\right|^2 \\ &= 1/34 \\ P\left(a = \sqrt{17}a_0\right) &= \left|\langle \sqrt{17}a_0 | -\varepsilon_0\rangle\right|^2 \\ &= 1/34 \\ P\left(a = 0a_0\right) &= \left|\langle 0a_0 | -\varepsilon_0\rangle\right|^2 \\ &= 32/34 \end{aligned}$$

(c) What is the expectation value of A?

The expectation value of A is given by

$$\begin{aligned} \langle A\rangle &= \langle -\varepsilon_0|A|\varepsilon_0\rangle \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} a_0 \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 0a_0 \end{aligned}$$

3. Consider a physical system whose state and two observables A and B are represented by

$$|\psi\rangle = \frac{1}{6}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \quad A = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(a) We first measure A and then B. Find the probability of obtaining a value of 0 for A and a value of 1 for B.

First, note that $|\psi\rangle$ is not normalized. For the values in the column, the normalization factor should instead be $1/sqrt17$. I will use this re-normalized ket in the following calculations. In analogy to rolling two dice, the probabilities simply multiply. Note though that there is degeneracy in the B=1 measurement so we must add together the probability due to each state corresponding to this eigenvalue. Thus

$$P(a=0,\ \text{then } b=1) = |\langle a=0|\psi\rangle|^2 \left(\left|\langle b^1=1|a=0\rangle\right|^2 + \left|\langle b^2=1|a=0\rangle\right|^2 \right)$$

Assuming 0 and 1 are in fact eigenvalues of A and B, we need to find the corresponding eigenvectors in order to perform the above calculations. This gives us

$$|a=0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix}$$
$$|b^1=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix}$$
$$|b^2=1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

And so the probabilities are

$$|\langle a=0|\psi\rangle|^2 = \left|\frac{4i}{\sqrt{17}\sqrt{2}}\right|^2 = \frac{16}{34} = \frac{8}{17}$$
$$\left|\langle b^1=1|a=0\rangle\right|^2 = \left|\frac{2}{2}\right|^2 = 1$$
$$\left|\langle b^2=1|a=0\rangle\right|^2 = \left|\frac{0}{2}\right|^2 = 0$$

Thus, we conclude that

$$P\left(a=0 \text{ then } b=1\right) = \frac{8}{17}\cdot\Big(1+0\Big) = \frac{8}{17}$$

(b) If we measure B then A what is the probability of obtaining 1 for B and 0 for A?

This is similar to the previous problem except the order has been switched. This means we have to deal with the degeneracy in B twice. It follows that the probability is given by

$$P(b=1,\ \text{then } a=0) = \left(\left|\langle b^1=1|\psi\rangle\right|^2 \left|\langle a=0|b^1=1\rangle\right|^2 \right) + \left(\left|\langle b^2=1|\psi\rangle\right|^2 \left|\langle a=0|b^2=1\rangle\right|^2 \right)$$

The probabilities are

$$
\begin{aligned}
\left|\langle b^1 = 1|\psi\rangle\right|^2 &= \left|\frac{1}{\sqrt{2}\sqrt{17}}(-4i)\right|^2 \\
&= \frac{16}{34} = \frac{8}{17} \\
\left|\langle b^2 = 1|\psi\rangle\right|^2 &= \left|\frac{1}{\sqrt{17}}(1)\right|^2 \\
&= \frac{1}{17} \\
\left|\langle a = 0|b^1 = 1\rangle\right|^2 &= \left|\frac{1}{2}(1+1)\right|^2 \\
&= 1 \\
\left|\langle a = 0|b^2 = 1\rangle\right|^2 &= \left|\frac{1}{\sqrt{2}}(0)\right|^2 \\
&= 0
\end{aligned}
$$

Which gives a final probability of

$$
P(b = 1 \text{ then } a = 0) = \left(\frac{8}{17}\cdot 1\right) + \left(\frac{1}{17}\cdot 0\right) = \frac{8}{17}
$$

Note that these probabilities aren't the same.

(c) Compare the results of (a) and (b) and explain.

In the first example A projects $|\psi\rangle$ onto $|a = 0\rangle$ whcih can then be seen as some linear combination of $|b_i\rangle$, the B basis. In the second case, we let B project $|\psi\rangle$ to $|b = 1\rangle$ which can be thought of as a superposition of A's $|a_i\rangle$ basis states. A and B share common eigenstates though so it should not be a surprise that the values are the same as they form a C.S.C.O.

4 (a) Is the state $\psi(\theta, \varphi) = e^{-3i\varphi}\cos\theta$ an eigenfunction of the operators $A_\varphi = \partial/\partial\varphi$ and $B_\theta = \partial/\partial\theta$.

To check if ψ is an eigenfunction, we can let A_φ and B_θ act upon $\psi(\theta, \varphi)$.

$$
\begin{aligned}
A_\varphi\psi &= \frac{\partial}{\partial\varphi}e^{-3i\varphi}\cos\theta \\
&= -3ie^{-3i\varphi}\cos\theta \\
&= -3i\psi(\theta, \varphi) \\
B_\theta\psi &= \frac{\partial}{\partial\theta}e^{-3i\varphi}\cos\theta \\
&= -e^{-3i\varphi}\sin\theta
\end{aligned}
$$

So we see that ψ is an eigenfunction of A_φ with eigenvalue $-3i$ and but is not an eigenfunction of B_θ.

(b) Are A_φ and B_θ Hermitian?

To check if these operators are Hermitian, we will use the inner-product definition.

$$\langle\psi|A_\varphi|\chi\rangle = \int_0^\pi \int_0^{2\pi} r^2 \sin\theta d\varphi d\theta \; \psi^*(\theta,\varphi)\frac{\partial}{\partial\varphi}\chi(\theta,\varphi)$$
$$= \int_0^\pi d\theta \left(r^2 \sin\theta \; \psi^*(\theta,\varphi)\chi(\theta,\varphi)\Big|_0^{2\pi} - \int_0^{2\pi} r^2 \sin\theta d\varphi \; \chi(\theta,\varphi)\frac{\partial}{\partial\varphi}\psi^*(\theta,\varphi)\right)$$

Because we are using spherical coordinates, we may impose a continuity periodicity condition on all wavefunctions so that $\psi(\varphi = 0) = \psi(\varphi = 2\pi)$ and $\chi(\varphi = 0) = \chi(\varphi = 2\pi)$. From this the evaluation term cancels, leaving us with

$$= -\int_0^\pi \int_0^{2\pi} r^2 \sin\theta d\varphi d\theta \; \chi(\theta,\varphi)\frac{\partial}{\partial\varphi}\psi^*(\theta,\varphi) = -\langle\chi|A_\varphi|\psi\rangle^*$$

Thus A_φ is anti-Hermitian.

$$\langle\psi|B_\theta|\chi\rangle = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi \; \psi^*(\theta,\varphi)\frac{\partial}{\partial\theta}\chi(\theta,\varphi)$$
$$= \int_0^{2\pi} d\varphi \left(r^2 \sin\theta \; \psi^*(\theta,\varphi)\chi(\theta,\varphi)\Big|_0^{\pi} - \int_0^{\pi} d\theta r^2 \sin\theta \; \chi(\theta,\varphi)\frac{\partial}{\partial\theta}\psi^*(\theta,\varphi)\right)$$

Since $\sin(\pi) = \sin(0) = 0$, we have that this becomes

$$= \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi \; \chi(\theta,\varphi)\frac{\partial}{\partial\theta}\psi^*(\theta,\varphi) = -\langle\chi|B_\theta|\psi\rangle^*$$

And thus B_θ is anti-Hermitian and not Hermitian.

(c) Calculate the expectation values $\langle A_\varphi\rangle$ and $\langle B_\theta\rangle$

$$\langle A_\varphi\rangle = \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \; e^{3i\varphi}\cos\theta\frac{\partial}{\partial\theta}\left(e^{-3i\varphi}\cos\theta\right)$$
$$= -4i\pi r^2$$
$$\langle B_\theta\rangle = \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \; e^{3i\varphi}\cos\theta\frac{\partial}{\partial\theta}e^{-3i\varphi}\cos\theta d\varphi d\theta$$
$$= \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\cos\theta(-\sin\theta) d\varphi d\theta$$
$$= 0$$

(d) Find the commutator $[A_\varphi, B_\theta]$.

Consider some general wavefunction $\Phi(\theta, \varphi)$. Then

$$A_\varphi B_\theta \Phi = A_\varphi \frac{\partial}{\partial \theta}\Phi = \frac{\partial^2 \Phi}{\partial \varphi \partial \theta}$$
$$B_\theta B_\varphi \Phi = B_\theta \frac{\partial}{\partial \varphi}\Phi = \frac{\partial^2 \Phi}{\partial \theta \partial \varphi}$$

Because mixed partial derivatives are equal, these terms are the same and therefore we have that $[A_\varphi, B_\theta] = 0$ i.e. A_φ and B_θ commute.

Consider a physical system which has a number of observables that are represented by the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & -i & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

(a) Which among these observables are compatible? Find the Results of the measurements of the compatible observables.

See the attached Mathematica notebook for direct calculations

$$[A, B] = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -2i & 4 \\ 1 & -4 & 2i \end{pmatrix}$$
$$[B, C] = \begin{pmatrix} 0 & -3 & 1 \\ 3 & 6i & -12 \\ -1 & 12 & -6i \end{pmatrix}$$
$$[A, C] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From these commutators we see that operators A and C are the only compatible observables. The results of measurements of the compatible observables were also calculated using Mathematica (to save a bit of time). The eigenvalues are

$$\{a_n\} = \{-1, 1, 1\}$$
$$\{c_n\} = \{4, -2, 2\}$$

(b) Give a basis of eigenvectors common to these observables.

From the Mathematica calculations, we have the following eigenbasis for the compatible observables

$$|a = -1, c = -2\rangle = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$|a = 1, c = 4\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$|a = 1, c = 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) Do the following constitute a C.S.C.O: $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, $\{B, C\}$, $\{A, C\}$?

Based off of our calculations, we can see that $\{A\}$ is not a C.S.C.O. because it has a degenerate eigenvalue of 1. The sets $\{B\}$ and $\{C\}$ do constitute a C.S.C.O. as they do not have degenerate eigenvalues.

The sets $\{A, B\}$ and $\{B, C\}$ do not constitute C.S.C.O. as those operators do not commute. $\{A, C\}$ does form a C.S.C.O. though as we showed in part (b) that you can form a set of eigenvectors for $\{A, C\}$ which has non-degenerate eigenvalues.