Tapp 3.29 John Waczak

$\gamma(t) = (1,0,0) + t(0,1,1)$ (Straight line)

Define $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ as $\sigma(\theta,t) = R_\theta(\gamma(t))$

where $R_\theta$ is rotation by $\theta$ about $z$ axis. i.e.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Demonstrate that $\sigma$ is a parametrized surface

To show this we must show that $d\sigma_q$ has rank 2 $\forall q \in \mathbb{R}^2$.

We can calculate $\sigma(\theta,t)$ as follows.

$$\sigma = R_\theta \gamma(t)$$

$$= \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta) - t\sin(\theta) \\ \sin(\theta) + t\cos(\theta) \\ t \end{pmatrix}$$

And so $\sigma(\theta, t) = (\cos\theta - t\sin\theta, \sin\theta + t\cos\theta, t)$

where
$$x(\theta,t) = \cos(\theta) - t\sin(\theta)$$
$$y(\theta,t) = \sin(\theta) + t\cos(\theta)$$
$$z(\theta,t) = t$$

Now we have the jacobian is

$$d\sigma_q = \begin{pmatrix} \frac{\partial x}{\partial \theta}(q) & \frac{\partial x}{\partial t}(q) \\ \frac{\partial y}{\partial \theta}(q) & \frac{\partial y}{\partial t}(q) \\ \frac{\partial z}{\partial \theta}(q) & \frac{\partial z}{\partial t}(q) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\theta - t\cos\theta & -\sin\theta \\ \cos\theta - t\sin\theta & \cos\theta \\ 0 & 1 \end{pmatrix}$$

Now we need to show this matrix has 2 rank $\forall t, \theta$.

$$\det\begin{pmatrix} -\sin\theta - t\cos\theta & -\sin\theta \\ \cos\theta - t\sin\theta & \cos\theta \end{pmatrix} =$$

$$= -\sin\theta\cos\theta - t\cos^2\theta - (-\sin\theta\cos\theta + t\sin^2\theta)$$

$$= -t\cos^2\theta - t\sin^2\theta = -t \neq 0$$

thus $\sigma$ is a parametrized surface.

Now let $G = \{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } x^2+y^2-z^2=1\}$

show that $G = \sigma(\mathbb{R}^2)$

to do this we simply need to calculate $x^2+y^2-z^2$.

$x^2 = (\cos\theta - t\sin\theta)^2 = \cos^2\theta + t^2\sin^2\theta - 2t\cos\theta\sin\theta$
$y^2 = (\sin\theta + t\cos\theta)^2 = \sin^2\theta + t^2\cos^2\theta + 2t\cos\theta\sin\theta$
$z^2 = t^2$

so $x^2+y^2-t^2 = \cos^2\theta + \sin^2\theta + t^2\sin^2\theta + t^2\cos^2\theta - t^2$

$= 1 + t^2 - t^2 = 1$

Thus $\sigma(\mathbb{R}^2) = G$. Therefore the hyperboloid is the union of a collection of non intersecting straight lines.

* perhaps we can construct this second collection by simply shifting our rotation as in the cylinder example but I'm not sure.

(b) Now define $\sigma(s,t) = (t,0,0) + s(0,1,t)$

we can simplify this to

$$\sigma(s,t) = (t, s, st)$$

where $x = t$ $y = s$, $z = st$

Now $d\sigma(s,t) = \begin{pmatrix} \frac{\partial t}{\partial s} & \frac{\partial t}{\partial t} \\ \frac{\partial s}{\partial s} & \frac{\partial s}{\partial t} \\ \frac{\partial st}{\partial s} & \frac{\partial st}{\partial t} \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ t & s \end{pmatrix}$$

which always has rank 2 as

$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0 \; \forall s,t.$

Now let $S = \{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } z = xy\}$

clearly $\sigma(\mathbb{R}^2) = S$ as

$$x = t, \; y = s, \; z = st = xy.$$

Tapp 3.30

$$f:\mathbb{R}^3 \to \mathbb{R} \text{ as } f(x,y,z) = x^2yz^3$$

for which values of $\lambda$ is $f^{-1}(\lambda)$ a regular surface?

let $\lambda = x^2yz^3$ then

~~$f^{-1}(\lambda)$~~

I'm not exactly sure how to set up the inverse function for this problem... Maybe let

$$\sigma^{-1}(\lambda) = \left( \frac{\lambda}{xyz^3}, \frac{\lambda}{x^2z^3}, \frac{\lambda}{x^2yz^2} \right)$$

~~because~~ I'm not sure how to proceed from here.