Central Forces Homework 9

Due 6/6/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

REQUIRED:

1. Show that the angular momentum operators L^2 and L_z commute with the central force Hamiltonian H, where

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$L_z = -i\hbar \frac{\partial}{\partial\phi}$$

$$H = -\frac{\hbar^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] + V(r)$$

Show that $[H, L^2] = 0$

$$H = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + V(r)$$

$$= \left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2}\right] + V(r)$$

$$[H, L^2] = \left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2} + V(r)\right]L^2$$

$$- L^2\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2} + V(r)\right]$$

$$= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L^2 - \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L^2 + \underbrace{\frac{L^4}{2\mu r^2}}_{③} + V(r)L^2$$

$$+ \frac{\hbar^2}{2\mu}L^2\frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{2\mu}L^2\frac{2}{r}\frac{\partial}{\partial r} \underbrace{- L^2\frac{L^2}{2\mu r^2}}_{⑦} - L^2V(r)$$

Note that ① the third and seventh terms sum to zero.

② L^2 has angular dependence only

③ $\frac{\partial^2}{\partial r^2}$, $\frac{2}{r}\frac{\partial}{\partial r}$ and $V(r)$ only depend on r

Thus The orders of L^2 with $\frac{\partial^2}{\partial r^2}$, $\frac{2}{r}\frac{\partial}{\partial r}$ and $V(r)$ are interchangeable.

$$\Rightarrow [H, L^2] = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L^2 - \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L^2 + V(r)L^2$$

$$+ \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L^2 + \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L^2 + V(r)L^2 = 0 \;\square$$

show that $[H, L_z] = 0$

To show that $[H, L_z]=0$, Let us show

$$[L^2, L_z] = 0$$

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z]$$

$$= [L_x^2, L_z] + [L_y^2, L_z] + \cancel{[L_z^2, L_z]}^{0}$$

$$= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$

$$= L_x(-i\hbar)L_y + (-i\hbar)L_yL_x + L_y(i\hbar)L_x + i\hbar L_xL_y$$

$$= -i\hbar L_xL_y - i\hbar L_yL_x + i\hbar L_yL_x + i\hbar L_xL_y$$

$$= \phi$$

$$H = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2} + V(r)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$[H, L_z] = \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}, L_z\right] + \left[-\frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}, L_z\right] + \left[\frac{L^2}{2\mu r}, L_z\right] + [V(r), L_z]$$

Because L^2 only depends θ and ϕ, and L_z only depends on ϕ, the Last two terms are both equal to Zero

Thus

$$[H, L_z] = \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}, L_z\right] + \left[-\frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}, L_z\right]$$

$$= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L_z + \frac{\hbar^2}{2\mu}L_z\frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L_z + \frac{\hbar^2}{2\mu}L_z\frac{2}{r}\frac{\partial}{\partial r}$$

$$= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}\left(-i\hbar\frac{\partial}{\partial\phi}\right) + \frac{\hbar^2}{2\mu}\left(-i\hbar\frac{\partial}{\partial\phi}\frac{\partial^2}{\partial r^2}\right)$$

$$- \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}\left(-i\hbar\frac{\partial}{\partial\phi}\right) + \frac{\hbar^2}{2\mu}\frac{2}{r}(-i\hbar)\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}$$

$$= \frac{i\hbar^3}{2\mu}\frac{\partial^2}{\partial r^2}\frac{\partial}{\partial\phi} - \frac{i\hbar^3}{2\mu}\frac{\partial}{\partial\phi}\frac{\partial^2}{\partial r^2}$$

$$+ \frac{i\hbar^3}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi} - \frac{i\hbar^3}{2\mu}\frac{2}{r}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}$$

Because the orders of these partial derivatives are interchangeable, $[H, L_z] = 0$ $\square$

2. Write out the first 9 terms in the sum:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell,m} Y_{\ell,m}$$

Describe the energy degeneracy of the rigid rotor system, i.e. give the number of eigenstates that all have the same energy.

Solution:

The form of our answer here should be a list of terms in the sum, each with a coefficient and a spherical harmonic. Specifically, we want the terms with the lowest values of ℓ:

$$\approx c_{0,0}Y_0^0 + c_{1,-1}Y_1^{-1} + c_{1,0}Y_1^0 + c_{1,1}Y_1^1 + c_{2,-2}Y_2^{-2} + c_{2,-1}Y_2^{-1} + c_{2,0}Y_2^0 + c_{2,1}Y_2^1 + c_{2,2}Y_2^2$$

Since m runs in integer steps from $-\ell$ to ℓ, the total number of states with the same value of ℓ is ℓ positive values, ℓ negative values, and one value corresponding to $m = 0$, which overall is $2\ell + 1$.

3. Consider the normalized function:

$$f(\theta, \phi) = \begin{cases} N\left(\frac{\pi^2}{4} - \theta^2\right) & 0 < \theta < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \theta < \pi \end{cases}$$

where

$$N = \frac{1}{\sqrt{\frac{\pi^5}{8} + 2\pi^3 - 24\pi^2 + 48\pi}}$$

(a) Find the $|\ell, m\rangle = |0,0\rangle$, $|1,-1\rangle$, $|1,0\rangle$, and $|1,1\rangle$ terms in a spherical harmonics expansion of $f(\theta, \phi)$.

(b) If a quantum particle, confined to the surface of a sphere, is in the state above, what is the probability that a measurement of the square of the total angular momentum will yield $2\hbar^2$? $4\hbar^2$?

(c) If a quantum particle, confined to the surface of a sphere, is in the state above, what is the probability that the particle can be found in the region $0 < \theta < \frac{\pi}{6}$ and $0 < \phi < \frac{\pi}{6}$? Repeat the question for the region $\frac{5\pi}{6} < \theta < \pi$ and $0 < \phi < \frac{\pi}{6}$. Plot your approximation from part (a) above and check to see if your answers seem reasonable.

Solution:

See attached Mathematica worksheet Sphere.nb.

4. Make a table, similar to the one you made for a particle confined to a ring, showing the different representations of the physical quantities associated with the rigid rotor. Include information about the operators $\hat{H}$, $\hat{L}_z$, and $\hat{L}^2$.

Particle on a Ring

	Ket Representation	Wave Function Representation	Matrix Representation
Hamiltonian	$\hat{H}$	$-\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}$	$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & E_1 & 0 & 0 & \cdots \\ \cdots & 0 & E_0 & 0 & \cdots \\ \cdots & 0 & 0 & E_{-1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \hbar^2/2I & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \hbar^2/2I & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
Eigenvalues of Hamiltonian	$E_m = \frac{\hbar^2}{2I}m^2$	$E_m = \frac{\hbar^2}{2I}m^2$	$E_m = \frac{\hbar^2}{2I}m^2$
Normalized Eigenstates of Hamiltonian	$\vert m\rangle$	$\Phi_m(\phi) = \sqrt{\frac{1}{2\pi r_0}}e^{im\phi}$	$\ldots, \begin{pmatrix} \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \ \ldots$
Coefficient of m^{th} energy eigenstate	$c_m = \langle m \vert \Phi\rangle$	$c_m = \int_0^{2\pi} \sqrt{\frac{1}{2\pi r_0}}e^{-im\phi}\,\Phi(\phi)\,r_0 d\phi$	$\begin{pmatrix} \cdots & 1 & \cdots & 0 & \cdots \end{pmatrix}\begin{pmatrix} \vdots \\ c_m \\ \vdots \\ c_0 \\ \vdots \end{pmatrix}$
Probability of measuring E_m	$P(E_m) = \vert c_{+m}\vert^2 + \vert c_{-m}\vert^2$ $= \vert\langle +m \vert \Phi\rangle\vert^2 + \vert\langle -m \vert \Phi\rangle\vert^2$	$P(E_m) = \left\vert \int_0^{2\pi} \sqrt{\frac{1}{2\pi r_0}}e^{-im\phi}\,\Phi(\phi)\,r_0 d\phi \right\vert^2$ $+ \left\vert \int_0^{2\pi} \sqrt{\frac{1}{2\pi r_0}}e^{im\phi}\,\Phi(\phi)\,r_0 d\phi \right\vert^2$	$P(E_m) = \left\vert \begin{pmatrix} \cdots & 1 & \cdots & 0 & \cdots \end{pmatrix}\begin{pmatrix} \vdots \\ c_m \\ \vdots \\ c_0 \\ \vdots \end{pmatrix} \right\vert^2 + \left\vert \begin{pmatrix} \cdots & 0 & \cdots & 1 & \cdots \end{pmatrix}\begin{pmatrix} \vdots \\ c_0 \\ \vdots \\ c_{-m} \\ \vdots \end{pmatrix} \right\vert^2$
Expectation value of Hamiltonian	$\langle \Phi \vert H \vert \Phi \rangle = \sum_m \vert c_m\vert^2 E_m$	$\langle \Phi \vert H \vert \Phi \rangle = \int_0^{2\pi} \Phi^*(\phi)\ \hat{H}\ \Phi(\phi)\,r_0 d\phi$	$\langle \Phi \vert H \vert \Phi \rangle = \begin{pmatrix} \cdots & c_1^* & c_0^* & c_{-1}^* & \cdots \end{pmatrix}\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & E_1 & 0 & 0 & \cdots \\ \cdots & 0 & E_0 & 0 & \cdots \\ \cdots & 0 & 0 & E_{-1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ c_1 \\ c_0 \\ c_{-1} \\ \vdots \end{pmatrix}$

Rigid Rotor/Particle on a Sphere

	Ket Representation	**Wave Function Representation**	**Matrix Representation**
Hamiltonian	$\hat{H}$	$\hat{H} = \frac{1}{2I}L^2 \doteq -\frac{\hbar^2}{2I}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}\right)$	$H \doteq \begin{pmatrix} E_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & E_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & E_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & E_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & E_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2\frac{\hbar^2}{2I} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2\frac{\hbar^2}{2I} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2\frac{\hbar^2}{2I} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 6\frac{\hbar^2}{2I} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
Eigenvalues of Hamiltonian	$E_\ell = \frac{\hbar^2}{2I}\ell(\ell+1)$	$E_\ell = \frac{\hbar^2}{2I}\ell(\ell+1)$	$E_\ell = \frac{\hbar^2}{2I}\ell(\ell+1)$
Normalized Eigenstates of Hamiltonian	$\lvert \ell m\rangle$	$Y_\ell^m(\theta,\phi) = (-1)^{(m+\lvert m\rvert)/2}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-\lvert m\rvert)!}{(\ell+\lvert m\rvert)!}}P_\ell^m(\cos\theta)e^{im\phi}$	$\lvert 00\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \lvert 11\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \lvert 10\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \lvert 1,-1\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \ \ldots$
Coefficient of the energy eigenstate with quantum numbers ℓ, m	$c_{\ell m} = \langle \ell m \vert \psi\rangle$	$c_{\ell m} = \int_0^{2\pi}\int_0^{\pi} Y_\ell^{m*}(\theta,\phi)\psi(\theta,\phi)\sin\theta d\theta d\phi$	$c_{\ell m} = \begin{pmatrix} \cdots & 1 & \cdots \end{pmatrix}\begin{pmatrix} \vdots \\ c_{\ell m} \\ \vdots \end{pmatrix}$
Probability of measuring $E_{\ell,m}$	$\mathcal{P}_{E_\ell} = \sum_{m=-\ell}^{\ell}\left\lvert\langle \ell m \vert \psi\rangle\right\rvert^2 = \sum_{m=-\ell}^{\ell}\left\lvert c_{\ell m}\right\rvert^2$	$\mathcal{P}_{E_\ell} = \sum_{m=-\ell}^{\ell}\left\lvert\int_0^{2\pi}\int_0^{\pi} Y_\ell^{m*}(\theta,\phi)\psi(\theta,\phi)\sin\theta d\theta d\phi\right\rvert^2$	$\mathcal{P}_{E_\ell} = \sum_{m=-\ell}^{\ell}\left\lvert\begin{pmatrix} \cdots & 1 & \cdots \end{pmatrix}\begin{pmatrix} \vdots \\ c_{\ell m} \\ \vdots \end{pmatrix}\right\rvert^2$

5 (Challenge Problem) Let $\mathbf{J}$ be an angular momentum with a set of three observables J_x, J_y, and J_z that satisfy:

$$\begin{aligned}
[J_x, J_y] &= i\hbar J_z \\
[J_y, J_z] &= i\hbar J_x \\
[J_z, J_x] &= i\hbar J_y
\end{aligned}$$

$\mathbf{J}^2$, J_+, and J_- are three operators that are defined as following:

$$\begin{aligned}
\mathbf{J}^2 &= J_x^2 + J_y^2 + J_z^2 \\
J_+ &= J_x + iJ_y \\
J_- &= J_x - iJ_y
\end{aligned}$$

Show that the operators J_+, J_-, J_z, and $\mathbf{J}^2$ satisfy the following commutation relations:

$$\begin{aligned}
[\mathbf{J}^2, J_z] &= [\mathbf{J}^2, J_+] = [\mathbf{J}^2, J_-] = 0 \\
[J_z, J_+] &= +\hbar J_+ \\
[J_z, J_-] &= -\hbar J_- \\
[J_+, J_-] &= 2\hbar J_z
\end{aligned}$$

Show that $[H, L^2] = 0$

$$H = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + V(r)$$

$$= \left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2}\right] + V(r)$$

$$[H, L^2] = \left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2} + V(r)\right]L^2$$

$$- L^2\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2} + V(r)\right]$$

$$= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L^2 - \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L^2 + \underbrace{\frac{L^4}{2\mu r^2}}_{3} + V(r)L^2$$

$$+ \frac{\hbar^2}{2\mu}L^2\frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{2\mu}L^2\frac{2}{r}\frac{\partial}{\partial r} \underbrace{- L^2\frac{L^2}{2\mu r^2}}_{7} - L^2 V(r)$$

Note that ① the third and seventh terms sum to zero.

② L^2 has angular dependence only

③ $\frac{\partial^2}{\partial r^2}$, $\frac{2}{r}\frac{\partial}{\partial r}$ and $V(r)$ only depend on r

Thus The orders of L^2 with $\frac{\partial^2}{\partial r^2}$, $\frac{2}{r}\frac{\partial}{\partial r}$ and $V(r)$ are interchangeable.

$$\Rightarrow [H, L^2] = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L^2 - \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L^2 + V(r)L^2$$

$$+ \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L^2 + \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L^2 + V(r)L^2 = 0 \; \square$$

show that $[H, L_z] = 0$

To show that $[H, L_z] = 0$, Let us show

$$[L^2, L_z] = 0$$

$$
\begin{aligned}
[L^2, L_z] &= [L_x^2 + L_y^2 + L_z^2, L_z] \\
&= [L_x^2, L_z] + [L_y^2, L_z] + \underbrace{[L_z^2, L_z]}_{0} \\
&= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y \\
&= L_x(-i\hbar)L_y + (-i\hbar)L_yL_x + L_y(i\hbar)L_x + i\hbar L_xL_y \\
&= -i\hbar L_xL_y - i\hbar L_yL_x + i\hbar L_yL_x + i\hbar L_xL_y \\
&= \phi
\end{aligned}
$$

$$H = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{2\mu r^2} + V(r)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$[H, L_z] = \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}, L_z\right] + \left[-\frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}, L_z\right] + \left[\frac{L^2}{2\mu r}, L_z\right] + [V(r), L_z]$$

Because L^2 only depends θ and ϕ, and L_z only depends on ϕ, the Last two terms are both equal to Zero

Thus

$$[H, L_z] = \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}, L_z\right] + \left[-\frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}, L_z\right]$$

$$= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}L_z + \frac{\hbar^2}{2\mu}L_z\frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}L_z + \frac{\hbar^2}{2\mu}L_z\frac{2}{r}\frac{\partial}{\partial r}$$

$$= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}\left(-i\hbar\frac{\partial}{\partial\phi}\right) + \frac{\hbar^2}{2\mu}\left(-i\hbar\frac{\partial}{\partial\phi}\frac{\partial^2}{\partial r^2}\right)$$

$$- \frac{\hbar^2}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}\left(-i\hbar\frac{\partial}{\partial\phi}\right) + \frac{\hbar^2}{2\mu}\frac{2}{r}(-i\hbar)\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}$$

$$= \frac{i\hbar^3}{2\mu}\frac{\partial^2}{\partial r^2}\frac{\partial}{\partial\phi} - \frac{i\hbar^3}{2\mu}\frac{\partial}{\partial\phi}\frac{\partial^2}{\partial r^2}$$

$$+ \frac{i\hbar^3}{2\mu}\frac{2}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi} - \frac{i\hbar^3}{2\mu}\frac{2}{r}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}$$

Because the orders of these partial derivatives are interchangeable, $[H, L_z] = 0$ $\square$