Homework 3 **John Waczak**
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1. **INTEGRATION ON THE SPHERE** Consider $\mathbb{S}^2$ which can be viewed as the surface in $\mathbb{E}^3$ satisfying $x^2 + y^2 + z^2 =$ constant. Equivalently, it is the two-dimensional surface with line element $ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$

   (a) Let ω be the orientation on $\mathbb{S}^2$. Determine $\displaystyle\int_{\mathbb{S}^2} \omega$

   We can rewrite the line element in the more suggestive form

   $$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

   So that the orientation can be chosen as

   $$\omega = r d\theta \wedge r \sin\theta d\phi \tag{2}$$

   Then integration of ω is as simple as dropping the wedges and making sure that the final answer has the correct sign. That is,

   $$\int_{\mathbb{S}^2} \omega = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta d\theta d\phi \tag{3}$$
   $$= r^2 \int_0^{2\pi} (-\cos\theta)\Big|_0^{\pi} d\phi \tag{4}$$
   $$= 2r^2 \int_0^{2\pi} d\phi \tag{5}$$
   $$= 4\pi r^2 \tag{6}$$

   which is precisely what we should expect for the area of the 2-sphere.

   (b) Let $\alpha \in \bigwedge^1(\mathbb{S}^2)$. Use Stoke's theorem to compute $\displaystyle\int_{\mathbb{S}^2} d\alpha$.

   Recall that the sphere is a compact surface with no boundary (section 16.7). Then stokes theorem gives

   $$\int_{\mathbb{S}^2} d\alpha = \int_{\partial\mathbb{S}^2} \alpha \tag{7}$$
   $$= \int_{\varnothing} \alpha = 0 \tag{8}$$

   where in the last line we are integrating a 1-form over the empty set.

   (c) Find a 1-form on $\mathbb{S}^2$ such that $d\alpha = \omega$.

   We can write a general 1-form as

   $$\alpha = f\ r d\theta + h\ r \sin\theta d\phi \tag{9}$$

Zapping with d gives (**Note:** r is constant on $\mathbb{S}^2$)

$$d\alpha = r\left(\frac{\partial(h\sin\theta)}{\partial\theta} - \frac{\partial f}{\partial\phi}\right) d\theta \wedge d\phi \tag{10}$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial(h\sin\theta)}{\partial\theta} - \frac{\partial f}{\partial\phi}\right) rd\theta \wedge r\sin\theta d\phi \tag{11}$$

$$\omega = rd\theta \wedge r\sin\theta d\phi \tag{12}$$

$$\Rightarrow r\sin\theta = \left(\frac{\partial(h\sin\theta)}{\partial\theta} - \frac{\partial f}{\partial\phi}\right) \tag{13}$$

At this point we can make a choice. If we let $f = f(\theta)$ and $h = -r\cot(\theta)$, then we have

$$r\sin\theta = \frac{\partial(-r\cot\theta\sin\theta)}{\partial\theta} + 0 \tag{14}$$

$$= -r\frac{\partial\cos\theta}{\partial\theta} = r\sin\theta \tag{15}$$

Thus, if we let

$$\alpha = f(\theta)rd\theta - r^2\cos\theta d\phi \tag{16}$$

(d) How is this possible?

As we can see, this appears to be a contradiction. In part (a) we found that integrating ω gives a non-zero value. Part (b) illustrated that ω can be written as an exterior derivative for which the integral over the whole $\mathbb{S}^2$ must be zero! How can we resolve this? If we allow our $f(\theta) = 0$ then so far we have found that

$$\omega = d(-r^2\cos\theta d\phi) \tag{17}$$

As the text says in section 20.4, "the problem must be with $d\phi$" because $\cos\theta$ is certainly well defined over the whole circle. As the integral of ω is non-zero then ω must not be the derivative of a one form despite the fact that (17) is true. However, if (following the notation of section 20.4 of the text) we identify the orthonormal basis element as $r\sin\theta d\phi = \sigma^\phi$, then we may rewrite ω in terms of the orthonormal (well-behaved) basis element.

$$\omega = d(-r\cot\theta\sigma^\phi) \tag{18}$$

In this form it is easy to see that because of the cotangent function, ω is undefined at the poles and therefore, ω itself is not a well-defined 2-form. This discrepancy must be what allows the integral to be non-zero. Certainly Stoke's theorem is correct but we also know that integrating the orientation over a surface should give the surface area.

To aid in this consider the circle $\mathbb{S}^1$ for which $\omega = rd\theta$ If we integrate the orientation, we find that

$$\int_{\mathbb{S}^2} \omega = 2\pi r \tag{19}$$

If we then apply stokes theorem to a one form, say $f = r\phi$ then we have $df = \omega$. Therefore, stokes theorem should give that the same integral (19) is 0. The resolution to the dilemma is that in order to apply Stoke's theorem, our integrand must be a smooth function over the whole region. This is *not* the case for the coordinate function ϕ because there is a sharp discontinuity between the angle $\phi = 0$ and $\phi = 2\pi$ which both correspond to the same position on the circle.

On the 2-sphere, this issue persists so that, in fact, for half of the great circle between the north and south poles defined by $\phi = 0$ there is a very *not smooth* discontinuity in angle. This arc is shown in the following figure

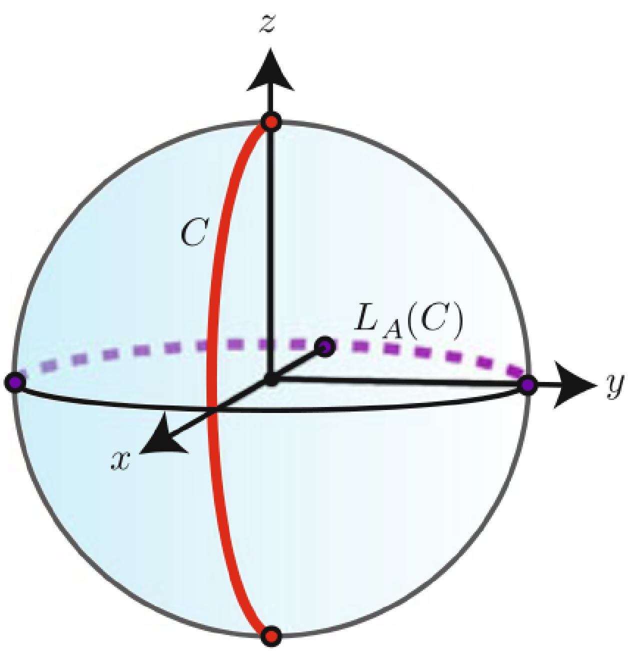

Figure 1: Figure showing the sphere $\mathbb{S}^2$ with the half-great circle C illustrating the discontinuity in the angle ϕ. Image taken from pg 131 of *Differential Geometry of Curves and Surfaces* by K. Tapp.

Thus, the problem we faced was not really a problem at all but rather our choice of coordinates for the sphere does not satisfy the requirements for the application of stokes theorem to ω. Our solution to part (b) is still correct because we never wrote the 1-form in terms of coordinates.