(a)

Sensemaking: Our final solution must be a polynomial of the form

$$f(z) = a_0 + a_1(z-1) + a_2(z-1)^2 + a_3(z-1)^3 + ... \tag{1}$$

where we must specify the values for the coefficients $\{a_n\}$. If the recurrence relation allows for 1 solution, we will include the first four non-zero terms. If the recurrence relation indicates two possible solutions, we will write out the first five non-zero terms for each such solution $f_0(z)$ and f_1(z)

Solution: We wish to find the power series expansion centered around $z = 1$ for a differential equation whose recurrence relation is

$$a_{n+1} = \frac{1}{n+1}a_n \tag{2}$$

Because the recurrence relation does not skip indices ($n \to n+1$), knowing a single coefficient enables us to use (2) to solve for **all other coefficients**. This indicates that there is one possible solution. As per the instructions, we must find the first four non-zero terms. Take a_0 to be the first coefficient. Then, equation (2) gives:

$$n = 0: \qquad a_1 = a_0 = \frac{1}{1!}a_0 \tag{3}$$

$$n = 1: \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0 \tag{4}$$

$$n = 2: \qquad a_3 = \frac{1}{3}a_2 = \frac{1}{3!}a_0 \tag{5}$$

$$\begin{aligned} f(z) &\approx a_0 + \frac{a_0}{1!}(z-1) + \frac{a_0}{2!}(z-1)^2 + \frac{1}{3!}(z-1)^3 \\ &= a_0\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!}\right) \end{aligned} \tag{6}$$

In the final line, I have factored out the a_0 to make it clear that this value will be determined by our initial conditions. Take a good look inside the parenthesis in (6). Do you recognize which function this power series is for?

(b)

Sensemaking: Same as (a) except the expansion will be centered around zero. That is,

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... \tag{7}$$

For this problem, we wish to expand our solution around the point $z = 0$ for a differential equation that results in a recurrence relation given by

$$a_{n+2} = -\frac{(5-n)(6+n)}{(n+2)(n+1)} a_n \tag{8}$$

Because this recurrence relation *does* skip indices ($n \to n+2$), we expect two possible solutions. The first solution comes from starting with a_0. The second comes from starting with a_1. We must calculate 5 non-zero terms for each of these scenarios. Equation (8) gives

$$\begin{aligned}
& a_0 = a_0 && (9)\\
& a_1 = a_1 && (10)\\
n = 0: \quad & a_2 = -15a_0 && (11)\\
n = 1: \quad & a_3 = -\frac{14}{3}a_1 && (12)\\
n = 2: \quad & a_4 = -2a_2 = 30a_0 && (13)\\
n = 3: \quad & a_5 = -\frac{9}{10}a_3 = \frac{21}{5}a_1 && (14)\\
n = 4: \quad & a_6 = -\frac{1}{3}a_4 = -10a_0 && (15)\\
n = 5: \quad & a_7 = 0 && (16)\\
n = 6: \quad & a_8 = \frac{3}{14} = -\frac{15}{7}a_0 && (17)
\end{aligned}$$

Interestingly, the series involving a_1 terminates after $n = 3$. This comes from the $(5 - n)$ term in the numerator of (8). Putting these together, the general solution is

$$f(z) = a_0\left(1 - 15z^2 + 30z^4 - 10z^6 - \frac{15}{7}z^8 + ...\right) + a_1\left(z - \frac{14}{3}z^3 + \frac{21}{5}z^5\right) \tag{18}$$

Note that the solution involving a_1 is not a power series. We started by assuming a power series solution and have found a polynomial with a finite number of terms that exactly solves the differential equation.

(c)
Sensemaking: Same as (a) except the expansion will be centered around zero. That is,

$$f(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + ... \tag{19}$$

We are faced with a similar scenario to part(b). We wish to write the power series solution centered around $z = 0$ given the recurrence relation

$$a_{n+2} = -\frac{(3-n)}{(n+2)(n+1)} a_n \tag{20}$$

This recurrence relation skips indices ($n \to n+2$) so we expect two distinct solutions for a_0 and a_1. Using the relation, gives

$$\begin{aligned}
& a_0 = a_0 && (21)\\
& a_1 = a_1 && (22)\\
n = 0: \quad & a_2 = -\frac{3}{2}a_0 && (23)\\
n = 1: \quad & a_3 = -\frac{1}{3}a_1 && (24)\\
n = 2: \quad & a_4 = -\frac{1}{2}a_2 = \frac{3}{4}a_0 && (25)\\
n = 3: \quad & a_5 = 0 && (26)\\
n = 4: \quad & a_6 = \frac{1}{30}a_4 = \frac{1}{40}a_0 && (27)\\
n = 5: \quad & a_7 = 0 && (28)\\
n = 6: \quad & a_8 = \frac{3}{56}a_6 = \frac{3}{2240}a_0 && (29)
\end{aligned}$$

so that the general solution to the differential equation is

$$f(z) = a_0\left(1 - \frac{3}{2}z^2 + \frac{3}{4}z^4 + \frac{1}{40}z^6 + \frac{3}{2240}z^8 + ...\right) + a_1\left(z - \frac{1}{3}z^3\right) \tag{30}$$

Again, the series involving a_1 terminated!