Homework 6 **John Waczak**
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1. INDEX GYMNASTICS

In a coordinate basis $\{dx^i\}$ of 1-forms, the components g_{ij} of the metric are defined by $ds^2 = g_{ij}dx^i dx^j$. The dual basis $\{\vec{\boldsymbol{e}}_i\}$ of vectors satisfies $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = g_{ij}$.
*These bases are **not** necessarily orthogonal. It is however still true that $d\vec{\boldsymbol{r}} = dx^i \vec{\boldsymbol{e}}_i$*

(a) Determine an expression for $\vec{\boldsymbol{e}}_i \cdot \vec{\nabla} f$ in terms of partial derivatives.

Expanding the total derivative of f, we have

$$df = \frac{\partial f}{\partial x^i} dx^i = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}} \tag{1}$$
$$= \vec{\nabla} f \cdot dx^i \vec{\boldsymbol{e}}_i \tag{2}$$
$$= \big(\vec{\boldsymbol{e}}_i \cdot \vec{\nabla} f\big) dx^i \tag{3}$$

$$\Rightarrow \vec{\boldsymbol{e}}_i \cdot \vec{\nabla} f = \frac{\partial f}{\partial x^i} \tag{4}$$

(b) Acting on 1-forms $F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, $G = \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{r}}$, the metric satisfies $g(F, G) = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}}$ for any vectors $\vec{\boldsymbol{F}}, \vec{\boldsymbol{G}}$. Express the components $g^{ij} = g(dx^i, dx^j)$ in terms of the components g_{ij}.
A derivation in 2 dimensions is acceptable if you don't see how to handle the general case.

In order to expand $g^{ij} = g(dx^i, dx^j)$, we need to find the fields related to dx^i so that we can take advantage of the above fact. To do this, consider the following inner products

$$\vec{\boldsymbol{e}}_i \cdot d\vec{\boldsymbol{r}} = \vec{\boldsymbol{e}}_i \cdot dx^k \vec{\boldsymbol{e}}_k \tag{5}$$
$$= dx^k g_{ik} \tag{6}$$
$$\vec{\boldsymbol{e}}_j \cdot d\vec{\boldsymbol{r}} = \vec{\boldsymbol{e}}_j \cdot dx^\ell \vec{\boldsymbol{e}}_\ell \tag{7}$$
$$= dx^\ell g_{j\ell} \tag{8}$$

This suggest that we should look at the inner product on these two fields.

$$g(dx^k g_{ik}, dx^\ell g_{j\ell}) = \vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = g_{ij} \tag{9}$$
$$\Rightarrow g_{ik}\ g_{j\ell}\ g(dx^k, dx^\ell) = g_{ij} \tag{10}$$
$$g_{ik}\ g_{j\ell}\ g^{k\ell} = g_{ij} \tag{11}$$

Now we note that this index equation may be reinterpreted as a matrix equation. If we take

$$\Big(g_{ij}\Big) \equiv G \qquad \Big(g^{ij}\Big) \equiv \tilde{G} \tag{12}$$

Then equation 11 reinterpreted in terms of these matrices says

$$G\left(G\tilde{G}\right) = G \tag{13}$$

Such an equation has a nontrivial solution only if

$$G\tilde{G} = \mathcal{I} \tag{14}$$

where $\mathcal{I}$ is the identity matrix. This further implies that we must have $\tilde{G} = G^{-1}$. Therefore each g^{ij} are the elements of the inverse matrix to G corresponding to all of the g_{ij}. There is no simple way to write this out as an inverse matrix is the adjugate matrix divided by the determinant. We would therefore expect each g^{ij} depends the determinant of many submatrices of G resulting in 16 linear equations that we can, in principle, solve.

2. DOUBLE-NULL COORDINATES

In 2-dimensional Minkowski space, let $u = t - x$, $v = t + x$.

(a) Express the line element $ds^2 = -dt^2 + dx^2$ in terms of the coordinate basis $\{du, dv\}$.

Taking the exterior derivative of the above, we find

$$du = dt - dx \qquad dv = dt + dx \tag{15}$$

so that

$$dt = \frac{1}{2}(dv + du) \tag{16}$$

$$dx = \frac{1}{2}(dv - du) \tag{17}$$

Therefore, the re-expressed line element is

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 && (18)\\ &= -\left[\frac{1}{2}(dv + du)\right]^2 + \left[\frac{1}{2}(dv - du)\right]^2 && (19)\\ &= -\frac{1}{2}dv\, du - \frac{1}{2}du\, dv && (20) \end{aligned}$$

I chose to leave equation (20) as it is to make identifying elements of the metric easier.

(b) Determine the components g^{ij} in this basis.

Recall from problem 1 that $ds^2 = g_{ij}dx^i dx^j$. Therefore, by inspection, we have that

$$\begin{cases} g_{uu} = g_{vv} = 0 \\ g_{uv} = g_{vu} = -\frac{1}{2} \end{cases} \tag{21}$$

so that as a matrix, the metric is

$$\left(g_{ij}\right) = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \tag{22}$$

Now we wish to find all of the g^{ij}. Recall from problem 1 that $g^{ij} \equiv g(dx^i, dx^j)$. Therefore,

$$\begin{aligned}
g^{uu} &= g(du, du) = g(dt - dx, dt - dx) && (23)\\
&= g(dt, dt) + g(dx, dx) && (24)\\
&= -1 + 1 = 0 && (25)\\
g^{uv} &= g(du, dv) = g(dt - dx, dt + dx) && (26)\\
&= g(dt, dt) - g(dx, dx) && (27)\\
&= -1 - 1 = -2 && (28)\\
g^{vu} &= g(dv, du) = g(dt + dx, dt - dx) && (29)\\
&= g(dt, dt) - g(dx, dx) && (30)\\
&= -1 - 1 = -2 && (31)\\
g^{vv} &= g(dv, dv) = g(dt + dx, dt + dx) && (32)\\
&= g(dt, dt) + g(dx, dx) && (33)\\
&= -1 + 1 = 0 && (34)
\end{aligned}$$

To summarize, the matrix corresponding to g^{ij} is

$$\left(g^{ij}\right) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \tag{35}$$

which is in fact the inverse matrix to $\left(g_{ij}\right)$ as we predicted in problem 1.

(**c**) Compute $g_{ij}g^{jk}$. What sort of beast is it the components of?

There are four terms we must compute. Using our results from (b), they are

$$\begin{aligned}
g_{uu}g^{uu} + g_{uv}g^{vu} &= 1 = \delta_u{}^u && (36)\\
g_{uu}g^{vu} + g_{uv}g^{vv} &= 0 = \delta_u{}^v && (37)\\
g_{vu}g^{uu} + g_{vv}g^{uv} &= 0 = \delta_v{}^u && (38)\\
g_{vu}g^{uv} + g_{vv}g^{vv} &= 1 = \delta_v{}^v && (39)
\end{aligned}$$

Apparently, $g_{ij}g^{jk} = \delta_i{}^k$ but we know this to be the case from problem 1 where we concluded that this kind of "beast" corresponds to an identity matrix. In other words,

$$\left(g_{ij}g^{jk}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{40}$$

3. TRACES

Suppose that two vector-valued 1-forms $\vec{\boldsymbol{G}} = G^i{}_j\sigma^j\vec{\boldsymbol{e}}_i$ and $\vec{\boldsymbol{R}} = R^i{}_j\sigma^j\vec{\boldsymbol{e}}_i$ have components that are related by

$$G^i{}_j = R^i{}_j - \frac{1}{2}\delta^i{}_jR, \tag{41}$$

where $R = R^i{}_i$. Find an expression for the trace $G = G^i{}_i$ of $\vec{\boldsymbol{G}}$ in terms of R.
*R is called the **trace** of $\vec{\boldsymbol{R}}$; more precisely, it is the trace of the matrix of components $\left(R^i{}_j\right)$. You may assume if desired that the underlying geometry is 4-dimensional, with signature 1.*

To find the trace, we simply let $j = i$. Doing this yields

$$G^i{}_i = R^i{}_i - \frac{1}{2}\delta^i{}_i R \tag{42}$$

$$= R^i{}_i - \frac{1}{2}\left(\sum_i 1\right) R \tag{43}$$

$$= R^i{}_i - \frac{N}{2} R = R\left(1 - \frac{N}{2}\right) \tag{44}$$

where N is the dimensionality of the space. For the particular case of 4-dimensional spacetime with signature 1, we have

$$G^i{}_i = R(1-2) = -R \tag{45}$$

which a quick internet search confirms. Apparently, equation (45) is the reason why $\vec{\boldsymbol{G}}$, the Einstein tensor, is also known as the trace-reversed Ricci tensor.