QM III
Phys 653

Lecture #2

Time-dependent potentials: the interaction picture

Recall from Phys 651 ⇒ Schrödinger vs Heisenberg picture

Schrödinger

$$|\alpha, t_0; t\rangle_S = \underbrace{\hat{U}(t_0, t)}_{\text{propagator}} |\alpha, t_0\rangle$$

propagator $\Leftarrow e^{-\frac{i}{\hbar}H(t-t_0)}$ ← $+f(t)$

$$A_S(t) = A_S(0)$$

time-independent

Heisenberg

$$|\alpha, t_0; t\rangle_H = |\alpha, t_0\rangle \quad \text{(time-indep)}$$

$$A_H(t) = \hat{U}^+(t, t_0) A_S \hat{U}(t, t_0)$$

$$\frac{dA_H(t)}{dt} = \frac{1}{i\hbar}[A_H, H]$$

Intermediat (or *interaction*, or *Dirac*) picture

Both a state ket and an observable are time-dependent ⇒ useful for ⇒

$$H = \underbrace{H_0}_{\text{time-independent}} + V(t)$$

(2)

$\Rightarrow$ define $|\alpha, t_0; t\rangle_{\underset{\substack{\uparrow\\ \text{interaction}}}{I}} = e^{iH_0(t-t_0)/\hbar} |\alpha, t_0; t\rangle_S$

$$|\alpha, t_0; t_0\rangle_I = |\alpha, t_0; t_0\rangle_S$$

Consider $t_0 = 0$ for simplicity $\Rightarrow$

$$A_I = e^{\frac{iH_0 t}{\hbar}} A_S e^{-\frac{i}{\hbar}H_0 t}$$

$$\frac{dA_I}{dt} \underset{\substack{\uparrow\\ \text{show!}}}{=} \frac{1}{i\hbar}[A_I, H_0]$$

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = i\hbar\frac{\partial}{\partial t}\left(e^{\frac{i}{\hbar}H_0 t}|\alpha, t_0; t\rangle_S\right)$$

$$= -H_0 e^{\frac{i}{\hbar}H_0 t}|\alpha, t_0; t\rangle_S + \underbrace{e^{\frac{i}{\hbar}H_0 t}}\ \underbrace{i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_S}_{= (H_{0(S)} + V_{(S)}(t))|\alpha, t_0; t\rangle_S} =$$

$$= e^{\frac{i}{\hbar}H_0 t}\underbrace{V_S(t)}_{e^{-\frac{i}{\hbar}H_0 t} V_I e^{\frac{i}{\hbar}H_0 t}}|\alpha, t_0; t\rangle_S \Rightarrow$$

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I |\alpha, t_0; t\rangle_I \qquad (2.1)$$

Consider $H_0|n\rangle = E_n|n\rangle$

Let's say the system is in some initial state $|i\rangle$

Now apply some time-dependent potential $V(t)$, so the total Hamiltonian is

$$H = H_0 + V(t)$$

What is the probability that at some time t the system will be found in some state $|f\rangle$, where $f \neq i$? $\Rightarrow$ use interaction picture $\Rightarrow$

$$|\alpha, t_0; t\rangle_I = \sum_n \underbrace{c_n(t)}_{\uparrow \text{ due to } V(t)!} |n\rangle \Rightarrow \quad \text{Eq. (2.1)}$$

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle_I = V_I |\alpha, t_0; t\rangle_I \Rightarrow$$ multiply by $\langle n|$

$$i\hbar \frac{\partial}{\partial t} \underbrace{\langle n|\alpha, t_0; t\rangle_I}_{c_n(t)} = \sum_m \langle n|V_I|m\rangle \underbrace{\langle m|\alpha, t_0; t\rangle_I}_{c_m(t)}$$

$$\langle n|V_I|m\rangle = \langle n|e^{\frac{i}{\hbar}H_0 t} V(t) e^{-\frac{i}{\hbar}H_0 t}|m\rangle =$$

$$= e^{\frac{i}{\hbar}(E_n - E_m)t} \langle n|V(t)|m\rangle = V_{nm} e^{\frac{i}{\hbar}(E_n - E_m)t}$$

So, $i\hbar \frac{dC_n(t)}{dt} = \sum_m V_{nm} e^{i\omega_{nm} t} C_m(t) \Rightarrow$

$$\omega_{nm} = \frac{E_n - E_m}{\hbar}$$

$$i\hbar \begin{pmatrix} \dot{C}_1(t) \\ \dot{C}_2(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} e^{i\omega_{12} t} & \cdots \\ V_{21} e^{i\omega_{21} t} & V_{22} & \cdots \\ \cdots & & V_{33} \; \cdots \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \end{pmatrix} \quad (2.2)$$

So, to find a probability to end up in some state $|n\rangle$ after time t due to $V(t) \Rightarrow$ need to solve Eqs. (2.2) and then find $|C_n(t)|^2$.

Note: in most cases (2.2) is not solvable exactly! $\Rightarrow$ use time-dependent perturbation theory

$\Downarrow$

But there are exceptions! $\Rightarrow$

Two-level systems (NMR, Spin Magn Reson, Maser, ...)

$H_0 |n\rangle = E_n |n\rangle, \quad n = 1, 2$

E_2

E_1

$$H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|$$

Apply $V(t) = \gamma e^{i\omega t} |1\rangle\langle 2| + \gamma e^{-i\omega t} |2\rangle\langle 1|$; $\gamma, \omega > 0$ and real (2.3)

physical content: oscillating (with ω) electric or magnetic fields

Say, at $t=0 \Rightarrow \underbrace{c_1(0) = 1}_{\text{only level } E_1 \text{ is populated}}, \; c_2(0) = 0$

What happens at $t > 0$?

Eqs. (2.2) $\Rightarrow$

$$i\hbar \frac{dc_1}{dt} = V_{11} c_1 + V_{12} e^{i\omega_{12} t} c_2$$

$$i\hbar \frac{dc_2}{dt} = V_{21} e^{i\omega_{21} t} c_1 + V_{22} c_2$$

$\underbrace{V_{11} = V_{22} = 0}_{\text{from (2.3)}}$; $V_{12} = V_{21}^* = \gamma e^{i\omega t} \Rightarrow$

$$i\hbar \frac{dc_1(t)}{dt} = \gamma e^{i(\omega+\omega_{12})t} c_2(t) \quad \Rightarrow$$

$$i\hbar \frac{dc_2(t)}{dt} = \gamma e^{-i(\omega+\omega_{12})t} c_1(t)$$

$$i\hbar \frac{d^2 c_1}{dt^2} = i(\omega+\omega_{12}) \underbrace{\gamma e^{i(\omega+\omega_{12})t} c_2(t)}_{=\, i\hbar \frac{dc_1}{dt}} + \gamma e^{i(\omega+\omega_{12})t} \underbrace{\dot{c}_2(t)}_{\frac{1}{i\hbar}\gamma e^{-i(\omega+\omega_{12})t} c_1(t)} \quad \Rightarrow$$

$$i\hbar \ddot{c}_1 = -\hbar(\omega+\omega_{12})\dot{c}_1 - \frac{i}{\hbar}\gamma^2 c_1 \quad \Rightarrow$$

$$\ddot{c}_1 - i(\omega+\omega_{12})\dot{c}_1 + \frac{\gamma^2}{\hbar^2} c_1 = 0 \quad \underset{\text{solve!}}{\Rightarrow} \quad c_1 = ?$$

Or

$$i\hbar \frac{d^2 c_2}{dt^2} = -i(\omega+\omega_{12}) \cdot i\hbar \frac{dc_2}{dt} + \gamma \cancel{e^{-i(\omega+\omega_{12})t}} \cdot \frac{1}{i\hbar}\gamma \cancel{e^{i(\omega+\omega_{12})t}} c_2(t);$$

$$\ddot{c}_2(t) + i(\omega+\omega_{12})\dot{c}_2(t) + \frac{\gamma^2}{\hbar^2} c_2(t) = 0 \quad \underset{\text{solve!}}{\Rightarrow}$$

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\frac{\gamma^2}{\hbar^2} + (\omega+\omega_{12})^2/4} \sin^2\left[\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega+\omega_{12})^2}{4}}\; t\right]$$

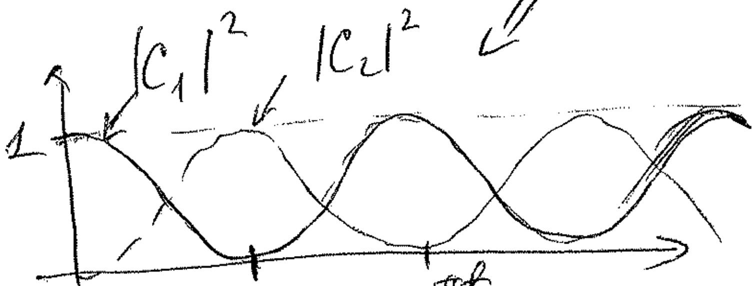
Denote $\Omega = \sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega+\omega_{12})^2}{4}}$ (Rabi frequency)

If $\omega \approx \omega_{21} = \frac{E_2 - E_1}{\hbar} \Rightarrow \omega + \omega_{12} = \omega - \omega_{21} \approx 0 \Rightarrow$

resonance condition

$\Omega = \frac{\gamma}{\hbar} \Rightarrow |C_2(t)|^2 \approx \sin^2 \Omega t$

$|C_1(t)|^2 + |C_2(t)|^2 = 1 \Leftarrow |C_1(t)|^2 \approx \cos^2 \Omega t$



$\frac{\gamma}{\hbar} t_1 = \frac{\pi}{2}$

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$t_1 = \frac{\pi\hbar}{2\gamma}$

$\Rightarrow$ $V(t)$ causes transitions from $|1\rangle$ to $|2\rangle$ (absorption) and then from $|2\rangle$ to $|1\rangle$ (emission)

(or spin flips if $|1\rangle = |+\rangle$, $|2\rangle = |-\rangle$)

Far away from resonance $\Rightarrow$ $\underbrace{|C_2(t)|^2_{max}}_{\frac{\gamma^2/\hbar^2}{\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{21})^2}{4}}} \neq 1 \Rightarrow$ modulation depth is reduced!

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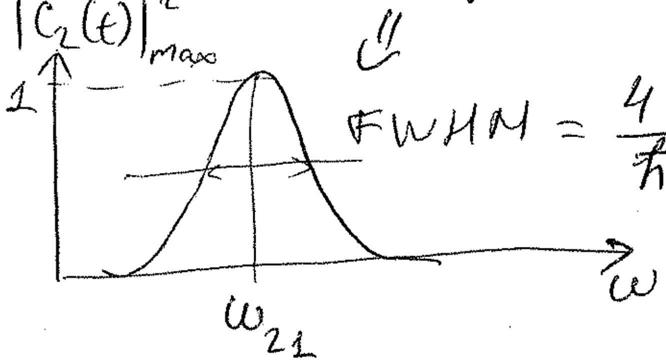

$\text{FWHM} = \frac{4\gamma}{\hbar}$

Reading assignment: Sakurai 5.5