Central Forces Homework 6

Due 5/25/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

PRACTICE:

1. Consider the normalized state $|\Phi\rangle$ for a quantum mechanical particle of mass μ constrained to move on a circle of radius r_0, given by:

$$|\Phi\rangle = \frac{\sqrt{3}}{2}|3\rangle + \frac{i}{2}|-2\rangle$$

   (a) What is the probability that a measurement of L_z will yield $2\hbar$? $3\hbar$?

   (b) What is the probability that a measurement of energy will yield $E = \frac{2\hbar^2}{I}$?

   (c) What is the expectation value of L_z in this state?

   (d) What is the expectation value of the energy in this state?

2. Consider the normalized state $|\Phi\rangle$ for a quantum mechanical particle of mass μ constrained to move on a circle of radius r_0, given by:

$$|\Phi\rangle = \frac{\sqrt{3}}{2}|3\rangle + \frac{i}{2}|-2\rangle$$

   (a) What is the probability that a measurement of L_z will yield $2\hbar$? $3\hbar$?

   (b) If you measured the z-component of angular momentum to be $3\hbar$, what would the state of the particle be immediately after the measurement is made?

   (c) What is the probability that a measurement of energy will yield $E = \frac{2\hbar^2}{I}$?

   (d) What is the expectation value of L_z in this state?

   (e) What is the expectation value of the energy in this state?

   (f) If you measured the z-component of angular momentum at some time $t \neq 0$, what is the probability that you would obtain $3\hbar$?

REQUIRED:

3. Consider the following normalized quantum state on a ring:

$$\Phi(\phi) = \sqrt{\frac{8}{3\pi}}\sin^2(3\,\phi)\cos(\phi)$$

(a) If you measured the z-component of angular momentum, what is the probability that you would obtain $\hbar$? $-3\hbar$? $-7\hbar$?

(b) If you measured the z-component of angular momentum, what other possible values could you obtain with non-zero probability?

(c) If you measured the energy, what is the probability that you would obtain $\frac{\hbar^2}{2I}$? $\frac{4\hbar^2}{2I}$? $\frac{25\hbar^2}{2I}$?

(d) If you measured the energy, what possible values could you obtain with non-zero probability?

(e) What is the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{4}$? In the region $\frac{\pi}{4} < \phi < \frac{3\pi}{4}$?

(f) Plot this wave function.

(g) What is the expectation value of L_z in this state?

4. In this problem, you will carry out calculations on the following normalized abstract quantum state on a ring:

$$|\Psi\rangle = \sqrt{\frac{1}{4}}\left(|1\rangle + \sqrt{2}\,|2\rangle + |3\rangle\right)$$

(a) You carry out a measurement to determine the energy of the particle at time $t = 0$. Calculate the probability that you measure the energy to be $\frac{4\hbar^2}{2I}$.

(b) You carry out a measurement to determine the z-component of the angular momentum of the particle at time $t = 0$. Calculate the probability that you measure the z-component of the angular momentum to be $3\hbar$.

(c) You carry out a measurement on the location of the particle at time, $t = 0$. Calculate the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{2}$.

(d) You carry out a measurement to determine the energy of the particle at time $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that you measure the energy to be $\frac{4\hbar^2}{2I}$.

(e) You carry out a measurement to determine the z-component of the angular momentum of the particle at time $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that you measure the z-component of the angular momentum to be $3\hbar$.

(f) You carry out a measurement on the location of the particle at time, $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{2}$.

(g) Write a short paragraph explaining what representation/basis you used for each of the calculations above and why you chose to use that representation/basis.

(h) In the above calculations, you should have found some of the quantities to be time dependent and others to be time independent. Briefly explain why this is so. That is, for a time dependent state like $|\Psi\rangle$ explain what makes some observables time dependent and others time independent.

5. Attached, you will find a table showing different representations of physical quantities associated with a particle-in-a-box. Make a similar table for a particle confined to a ring. Include all of the following information.

   - Hamiltonian
   - Eigenvalues of Hamiltonian
   - Normalized eigenstates of Hamiltonian
   - Coefficient of the nth eigenstate
   - Probability of measuring E_n
   - Expectation value of Hamiltonian
   - Z-component of angular momentum
   - Eigenvalues of z-component of angular momentum
   - Eigenstates of z-component of angular momentum
   - Coefficient of mth state of z-component of angular momentum
   - Probability of measuring $m\hbar$ for z-component of angular momentum
   - Expectation value of z-component of angular momentum

Particle in a Box

	Ket Representation	Wave Function Representation	Matrix Representation
Hamiltonian	$\hat{H}$	$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$	$\begin{pmatrix} E_1 & 0 & 0 & \cdots \\ 0 & E_2 & 0 & \cdots \\ 0 & 0 & E_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
Eigenvalues of Hamiltonian	$E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$	$E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$	$E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$
Normalized Eigenstates of Hamiltonian	$\lvert n\rangle$	$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \cdots$
Coefficient of n^{th} energy eigenstate	$c_n = \langle n \vert \psi\rangle$	$c_n = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)\psi(x)\,dx$	$\begin{pmatrix} 0 & \cdots & 1 & \cdots \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ \vdots \end{pmatrix}$
Probability of measuring E_n	$P(E_n) = \lvert c_n\rvert^2 = \lvert\langle n\vert\psi\rangle\rvert^2$	$P(E_n) = \lvert c_n\rvert^2 = \left\lvert\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)\psi(x)\,dx\right\rvert^2$	$P(E_n) = \lvert c_n\rvert^2 = \left\lvert\begin{pmatrix} 0 & \cdots & 1 & \cdots \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ \vdots \end{pmatrix}\right\rvert^2$
Expectation value of Hamiltonian	$\langle\psi\vert H\vert\psi\rangle = \sum_n \lvert c_n\rvert^2 E_n$	$\langle\psi\vert H\vert\psi\rangle = \int_0^L \psi^*(x)\hat{H}\psi(x)\,dx$	$\langle\psi\vert H\vert\psi\rangle = \begin{pmatrix} c_1^* & c_2^* & \cdots \end{pmatrix}\begin{pmatrix} E_1 & 0 & \cdots \\ 0 & E_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$

6. In spherical coordinates, the angular momentum operator is:

$$\hat{L} = -i\hbar \left(\hat{\phi} \frac{\partial}{\partial \theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right).$$

(a) Show that the orbital angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ are represented in spherical coordinates as

$$\hat{L}_x = i\hbar \left(\sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_y = -i\hbar \left(\cos\phi \frac{\partial}{\partial \theta} + \sin\phi \cot\theta \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}.$$

(b) Show that the operator $\hat{L}^2 = \hat{L} \cdot \hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ is:

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right].$$