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I took roughly **1.5 hours** to finish and type up this homework

1

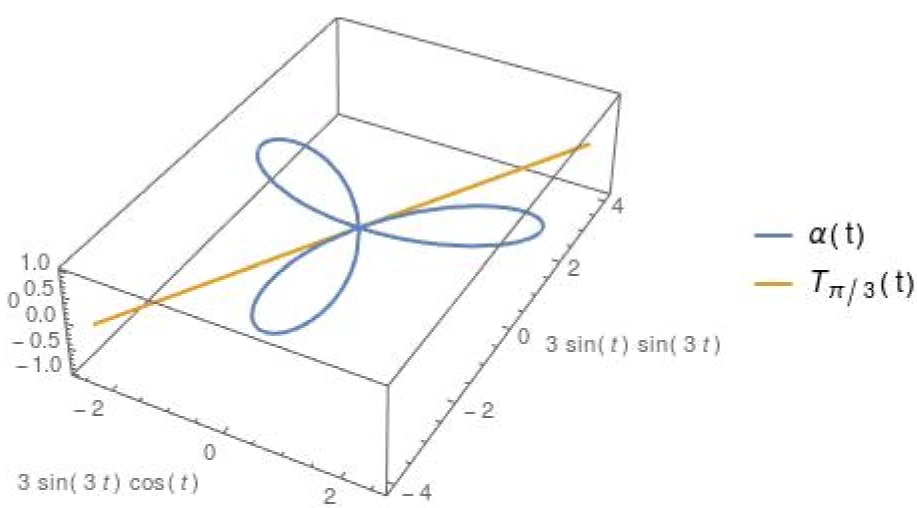
a) Show that $\alpha(t) = (\sin(3t)\cos(t), \sin(3t)\sin(t), 0)$ is regular curve.

$$\begin{aligned}
\alpha'(t) &= (3\cos(3t)\cos(t) - \sin(3t)\sin(t), 3\cos(3t)\sin(t) + \sin(3t)\cos(t), 0) \\
&= (\cos(2t) + 2\cos(4t), 2\sin(4t) - \sin(2t), 0) \\
|\alpha'(t)| &= \sqrt{(\cos(2t) + 2\cos(4t))^2 + (2\sin(4t) - \sin(2t))^2} \\
&= \sqrt{4\cos(6t) + 5} \\
&|\sqrt{4\cos(6t)+5}| \geq 1 \quad \forall t \\
\Rightarrow |\alpha'(t)| &\geq 0 \quad \forall t \\
&\text{therefore} \quad \alpha(t) \quad \text{is a regular curve} \quad \square
\end{aligned}$$

b) Find the equation of the tangent line to alpha at $t = \pi/3$.

$$\begin{aligned}
T_{\pi/3}(t) &= \alpha(\pi/3) + \alpha'(\pi/3)t \\
\alpha(\pi/3) &= (0,0,0) \\
\alpha'(\pi/3) &= (-\frac{3}{2}, -\frac{3\sqrt{3}}{2}, 0) \\
\Rightarrow T_{\pi/3}(t) &= (0,0,0) + (-\frac{3}{2}, -\frac{3\sqrt{3}}{2}, 0)t \\
&= (-\frac{3}{2}t, -\frac{3\sqrt{3}}{2}t, 0)
\end{aligned}$$

c) plot $\alpha(t)$.



I have included the tangent line from part (b) as proof that it is in fact a tangent line.

1.3

Use an improper integral to show that such a restriction has finite arc length even though it makes infinitely many loops around the origin.

$$
\begin{aligned}
\gamma(t) &= c(e^{\lambda t}\cos(t), e^{\lambda t}\sin(t)) \\
\gamma'(t) &= c(e^{\lambda t}(\lambda\cos(t) - \sin(t)), e^{\lambda t}(\lambda\sin(t) + \cos(t))) \\
|\gamma'(t)| &= c\sqrt{e^{2\lambda t}((\lambda\cos(t) - \sin(t))^2 + (\lambda\sin(t) + \cos(t))^2)} \\
&= ce^{\lambda t}\sqrt{(\lambda\cos(t) - \sin(t)^2 + (\lambda\sin(t) + \cos(t))^2} \\
&= ce^{\lambda t}\sqrt{\lambda^2 + 1} \\
\int_0^\infty ce^{\lambda t}\sqrt{\lambda^2 + 1}dt &= c\sqrt{\lambda^2 + 1}\Big[\frac{e^{\lambda t}}{\lambda}\Big]_0^\infty \\
&\text{Recall that } \lambda < 0, \quad \text{thus} \\
&= c\sqrt{\lambda^2 + 1}[-\frac{1}{\lambda}] \\
&= \frac{-c\sqrt{\lambda^2 + 1}}{\lambda} > 0
\end{aligned}
$$

Thus we have shown that the logarithmic spiral has finite arc length when restricted to $[0, \infty)$ (which is nuts!)

1.16

Let $\gamma(t)$ be a logarithmic spiral. Prove that the angle between $\gamma(t), \gamma'(t)$ is a constant function.

$$
\begin{aligned}
\gamma(t) &= c(e^{\lambda t}\cos(t), e^{\lambda t}\sin(t)) \\
\gamma'(t) &= c(e^{\lambda t}(\lambda\cos(t) - \sin(t)), e^{\lambda t}(\lambda\sin(t) + \cos(t))) \\
\cos(\theta) &\equiv \frac{\langle\gamma(t), \gamma'(t)\rangle}{|\gamma(t)||\gamma'(t)|} \\
|\gamma(t)| &= c\sqrt{e^{2\lambda t}\cos^2(t) + e^{2\lambda t}\sin^2(t)} \\
&= ce^{\lambda t} \\
|\gamma'(t)| &= ce^{\lambda t}\sqrt{\lambda^2 + 1} \\
\langle\gamma(t), \gamma'(t)\rangle &= c^2e^{2\lambda t}(\lambda\cos^2(t) - \cos(t)\sin(t) + \lambda\sin^2(t) + \cos(t)\sin(t)) \\
&= \lambda c^2e^{2\lambda t} \\
\Rightarrow \cos(\theta) &= \frac{\lambda c^2e^{2\lambda 2}}{ce^{\lambda t}ce^{\lambda t}\sqrt{\lambda^2 + 1}} \\
&= \frac{\lambda}{\sqrt{\lambda^2 + 1}} \\
\Rightarrow \theta &= \cos^{-1}(\frac{\lambda}{\sqrt{\lambda^2 + 1}}) = \text{constant}
\end{aligned}
$$

Not that because $\sqrt{\lambda^2 + 1} > |\lambda|$ the argument of the inverse cosine is always between (-1, 1) and so is well defined.