Fundamental Theorem of Curves 5C

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Math 434

Tapp. 1.78 (ignore n=2)

For $c>0$ consider the dilation $d_c: \mathbb{R}^n \to \mathbb{R}^n$ $\vec{p} \mapsto c\vec{p}$
If γ is a regular curve and $\hat{\gamma} = \gamma \circ d_c$
how are the curvatures and torsions related?
Would the answers change if $c<0$?

Clearly d_c is not a rigid motion so we must calculate the individual curvatures/torsions and compare them.

let $\gamma(t) = (x(t), y(t), z(t))$ be a regular curve and $\hat{\gamma}(t) = c(x(t), y(t), z(t))$
Taking derivatives yields

$$\gamma = (x(t), y(t), z(t)) \qquad \hat{\gamma} = c\gamma$$
$$\gamma' = (x'(t), y'(t), z'(t)) \qquad \hat{\gamma}' = c\gamma'$$
$$\gamma'' = (x''(t), y''(t), z''(t)) \qquad \hat{\gamma}'' = c\gamma''$$

Recall curvature is defined as

$$K = \frac{|v(t) \times a(t)|}{|v(t)|^3}$$

Thus $K_\gamma = \dfrac{|\gamma' \times \gamma''|}{|\gamma'|^3}$

then $K_{\hat{\gamma}} = \dfrac{|c\gamma' \times c\gamma''|}{|c\gamma'|^3} = \dfrac{|c|^2 |\gamma' \times \gamma''|}{|c|^3 |\gamma'|}$

$$= \frac{1}{|c|} K_\gamma$$

Thus the curvatures are related by
$K_{\hat{\gamma}} = \frac{1}{|c|} K_\gamma$ and does not depend on the sign of c

Now for the torsion. Recall

$$T = \frac{V}{|V|}$$
$$N = \frac{T'}{|T'|}$$
$$B = T \times N$$

Calculating these for γ and $\hat{\gamma}$, we have that

$$T_{\hat{\gamma}} = \frac{c}{|c|} T_\gamma$$

$$T'_{\hat{\gamma}} = \frac{c}{|c|} T'_\gamma$$

Therefore $N_{\hat{\gamma}} = \dfrac{T'_{\hat{\gamma}}}{|T'_{\hat{\gamma}}|} = \dfrac{\frac{c}{|c|} T'_\gamma}{\left|\frac{c}{|c|} T'_\gamma\right|} = \dfrac{c}{|c|} N_\gamma$

since $\left|\frac{c}{|c|}\right| = 1$

Now $B_{\hat{\gamma}} = T_{\hat{\gamma}} \times N_{\hat{\gamma}} = \frac{c}{|c|} T_\gamma \times \frac{c}{|c|} N_\gamma$

$$= \left(\frac{c}{|c|}\right)^2 T_\gamma \times N_\gamma$$

$$= T_\gamma \times N_\gamma \quad \left(\left(\tfrac{c}{|c|}\right)^2 = 1 \right)$$

$$= B_\gamma$$

Thus the binormal vector doesn't change and so we have shown:

$B_{\hat{\gamma}} = B_\gamma$ which makes sense as T, N are unit vectors so B shouldn't scale. Now we have

$B'_{\hat{\gamma}} = B'_\gamma$ from the previous line and therefore

$$\tau_{\hat{\gamma}} = \frac{-\langle B'_{\hat{\gamma}}, N_{\hat{\gamma}} \rangle}{|V_{\hat{\gamma}}|} = \frac{-\langle B'_\gamma, \frac{c}{|c|} N_\gamma \rangle}{|c|\,|V_\gamma|}$$

$$\boxed{\tau_{\hat{\gamma}} = \frac{c}{|c|^2} \tau_\gamma}$$

thus torsion also scales like $\frac{1}{|c|}$ but here the sign will change if $c < 0$.