Homework 6 **John Waczak**
MTH 434 Date: April 1, 2019
Dr. Tevian Dray

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1 SPHERICAL COORDINATES II Consider spherical coordinates $\{r, \theta, \phi\}$ and the adapted orthonormal basis

$$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} = \left\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right\} \tag{1}$$

The "infinitesimal displacement vector" $d\vec{\boldsymbol{r}}$ relates this basis to an orthonormal basis of 1-forms via

$$\vec{\boldsymbol{r}} = dr\ \hat{\boldsymbol{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin\theta d\phi\ \hat{\boldsymbol{\phi}} \tag{2}$$

WARNING: *these conventions imply* $\tan\phi = \frac{y}{x}$

(**a**) Determine the exterior derivative of each basis vector (not 1-form) above, that is, compute $d\hat{\boldsymbol{r}}, d\hat{\boldsymbol{\theta}}$, and $d\hat{\boldsymbol{\phi}}$.

We begin by recalling the definition of *connection* 1-forms given by

$$d\hat{\boldsymbol{e}}_j = \omega^i{}_j\ \hat{\boldsymbol{e}}_i \tag{3}$$

i.e. the 1-form coefficients of the expansion of $d\hat{\boldsymbol{e}}_j$ in the regular basis. In class, we also defined the metric compatibility and torsion free requirements in terms of connections as

$$\omega_{ij} + \omega_{ji} = 0 \tag{4}$$
$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0 \tag{5}$$

where a connection with both indices downstairs is given by

$$\omega_{ij} = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j \tag{6}$$

Equipped with these definitions we now can solve for the connection 1-forms. To evade a brute force calculation, recall the line element for $\mathbb{E}^3$ in spherical coordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{7}$$

so that

$$d\vec{\boldsymbol{r}} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \tag{8}$$

however, we also know that

$$d\vec{\boldsymbol{r}} = d(r\hat{\boldsymbol{r}}) = dr\hat{\boldsymbol{r}} + rd\hat{\boldsymbol{r}} \tag{9}$$

so that direct comparison gives

$$\boxed{d\hat{\boldsymbol{r}} = d\theta\ \hat{\boldsymbol{\theta}} + \sin\theta d\phi\ \hat{\boldsymbol{\phi}}} \tag{10}$$

Knowing (10) significantly simplifies our job. For the other two derivatives we have

$$d\hat{\boldsymbol{\theta}} = {\omega^r}_\theta\, \hat{\boldsymbol{r}} + {\omega^\theta}_\theta\, \hat{\boldsymbol{\theta}} + {\omega^\phi}_\theta\, \hat{\boldsymbol{\phi}} \tag{11}$$
$$d\hat{\boldsymbol{\phi}} = {\omega^r}_\phi\, \hat{\boldsymbol{r}} + {\omega^\theta}_\phi\, \hat{\boldsymbol{\theta}} + {\omega^\phi}_\phi\, \hat{\boldsymbol{\phi}}$$

However, equation (4) further simplifies by allowing us to remove the diagonal terms.

$$d\hat{\boldsymbol{\theta}} = {\omega^r}_\theta\, \hat{\boldsymbol{r}} + {\omega^\phi}_\theta\, \hat{\boldsymbol{\phi}} \tag{12}$$
$$d\hat{\boldsymbol{\phi}} = {\omega^r}_\phi\, \hat{\boldsymbol{r}} + {\omega^\theta}_\phi\, \hat{\boldsymbol{\theta}}$$

The vector version of (4) states that

$$d(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j) = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j + d\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = 0 \tag{13}$$

Thus, equation (13) yields the following:

$${\omega^r}_\theta = -\hat{\boldsymbol{\theta}} \cdot d\hat{\boldsymbol{r}} = -d\theta \tag{14}$$
$${\omega^r}_\phi = -\hat{\boldsymbol{\phi}} \cdot d\hat{\boldsymbol{r}} = -\sin\theta d\phi \tag{15}$$

Now only two connections remain. However, we have that ${\omega^\phi}_\theta = -{\omega^\theta}_\phi$. It is therefore sufficient to solve for either one alone. To do this, consider the torsion free condition (eq 5). We have that

$$0 = d(rd\theta) + {\omega^\theta}_r \wedge dr + {\omega^\theta}_\theta \wedge rd\theta + {\omega^\theta}_\phi \wedge r\sin\theta d\phi \tag{16}$$
$$= dr \wedge d\theta + {\omega^\theta}_r \wedge dr + {\omega^\theta}_\phi \wedge r\sin\theta d\phi \tag{17}$$
$$= dr \wedge d\theta + d\theta \wedge dr + {\omega^\theta}_\phi \tag{18}$$
$$\Rightarrow {\omega^\theta}_\phi \wedge r\sin\theta d\phi = 0 \tag{19}$$

Equation (19) tells us that ${\omega^\theta}_\phi$ must only include $d\phi$ and no other 1-forms in order for (19) to hold. Equation (5) also gives

$$0 = d(r\sin\theta d\phi) + {\omega^\phi}_r \wedge dr + {\omega^\phi}_\theta \wedge rd\theta + {\omega^\phi}_\phi \wedge r\sin\theta d\phi \tag{20}$$
$$= \sin\theta dr \wedge d\phi + r\cos\theta d\phi \wedge d\phi + {\omega^\phi}_r \wedge dr + {\omega^\phi}_\theta \wedge rd\theta \tag{21}$$
$$= \sin\theta dr \wedge d\phi + \sin\theta d\phi \wedge dr + r\cos\theta d\theta \wedge d\phi + {\omega^\phi}_\theta \wedge rd\theta \tag{22}$$
$$= r\cos\theta d\theta \wedge d\phi - rd\theta \wedge {\omega^\phi}_\theta \tag{23}$$
$$\Rightarrow {\omega^\phi}_\theta = \cos\theta d\phi \tag{24}$$
$$\text{and } {\omega^\theta}_\phi = -\cos\theta d\phi \tag{25}$$

...and that's all there is to it! In summary, we have

$$\boxed{\begin{aligned} d\hat{\boldsymbol{r}} &= d\theta\, \hat{\boldsymbol{\theta}} + \sin\theta d\phi\, \hat{\boldsymbol{\phi}} \\ d\hat{\boldsymbol{\theta}} &= -d\theta\, \hat{\boldsymbol{r}} + \cos\theta d\phi\, \hat{\boldsymbol{\phi}} \\ d\hat{\boldsymbol{\phi}} &= -\sin\theta d\phi\, \hat{\boldsymbol{r}} - \cos\theta d\phi\, \hat{\boldsymbol{\theta}} \end{aligned}} \tag{26}$$

(b) Compute $\omega_{ij} = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j$ for $i, j = 1, 2, 3$. *What sort of beast should you get?*

This question asks us to identify the connection 1-forms. We can easily read these off from our solution to part (a) by comparing with equation (11). They are

$$\boxed{\begin{array}{lll} \omega_{rr} = 0 & \omega_{\theta r} = d\theta & \omega_{\phi r} = \sin\theta d\phi \\ \omega_{r\theta} = -d\theta & \omega_{\theta\theta} = 0 & \omega_{\phi\theta} = \cos\theta d\phi \\ \omega_{r\phi} = -sin\theta d\phi & \omega_{\theta\phi} = -\cos\theta d\phi & \omega_{\phi\phi} = 0 \end{array}} \tag{27}$$

(**c**) Compute $\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ for $i, j = 1, 2, 3$ (and where there is an implicit sum over k). *What sort of beast should you get?*

Inspection of the equation for each Ω_{ij} reveals some interesting structure given our solution to part (b) of the problem. Notice that the table in equation 27 in antisymmetric as a result of equation 4. Therefore, we have that.

$$\omega_{ik} \wedge \omega_{ki} = -\omega_{ik} \wedge \omega_{ik} \tag{28}$$

but each of our ω_{ik} are in $\bigwedge^1$ and therefore

$$\omega_{ik} \wedge \omega_{ik} = 0 \quad \forall\, i, k \tag{29}$$

Thus, because $d(0) = 0$ and because of (29) we have that $\Omega_{ii} = 0 \;\forall\; i$. If we zap equation (4) with d, we find that

$$d\omega_{ij} + d\omega_{ji} = 0 \tag{30}$$

$$\Rightarrow d\omega_{ji} = -d\omega_{ij} \tag{31}$$

For the second half of the Ω_{ij} equation, we have that

$$\begin{aligned} \omega_{ik} \wedge \omega_{kj} &= -\omega_{kj} \wedge \omega_{ik} && (32) \\ &= \omega_{jk} \wedge \omega_{ik} && (33) \\ &= -\omega_{jk} \wedge \omega_{ki} && (34) \end{aligned}$$

Putting (31) together with (34) gives

$$\Omega_{ij} = -\Omega_{ji} \tag{35}$$

Therefore, it is sufficient to calculate elements with indices corresponding to the upper right half of the table in equation (27). Let's begin with $i = \theta$ and $j = r$.

$$\begin{aligned} d\omega_{\theta r} &= d(d\theta) = 0 && (36) \\ \omega_{\theta k} \wedge \omega_{kr} &= \omega_{\theta r} \wedge \omega_{rr} + \omega_{\theta\theta} \wedge \omega_{\theta r} + \omega_{\theta\phi} \wedge \omega_{\phi r} && (37) \\ &= 0 + 0 + \omega_{\theta\phi} \wedge \omega_{\phi r} && (38) \\ &= -\cos\theta d\phi \wedge \sin\theta d\phi && (39) \\ &= -\cos\theta \sin\theta d\phi \wedge d\phi && (40) \\ &= 0 && (41) \\ \Rightarrow \Omega_{\theta r} &= 0 && (42) \end{aligned}$$

For the next pair, we have

$$\begin{aligned}
d\omega_{\phi r} &= d(\sin\theta d\phi) = \cos\theta d\theta \wedge d\phi && (43)\\
\omega_{\phi k} \wedge \omega_{kr} &= \omega_{\phi r} \wedge \omega_{rr} + \omega_{\phi\theta} \wedge \omega_{\theta r} + \omega_{\phi\phi} \wedge \omega_{\phi r} && (44)\\
&= \omega_{\phi\theta} \wedge \omega_{\theta r} && (45)\\
&= \cos\theta d\phi \wedge d\theta && (46)\\
&= -\cos\theta d\theta \wedge d\phi && (47)\\
\Rightarrow \Omega_{\phi r} &= 0 && (48)
\end{aligned}$$

finally,

$$\begin{aligned}
d\omega_{\phi\theta} &= d(cos\theta d\phi) = -\sin\theta d\theta \wedge d\phi && (49)\\
\omega_{\phi k} \wedge \omega_{k\theta} &= \omega_{\phi r} \wedge \omega_{r\theta} + \omega_{phi\theta} \wedge \omega_{\theta\theta} + \wedge_{\phi\phi} \wedge \omega_{\theta\phi} && (50)\\
&= \omega_{\phi r} \wedge \omega_{r\theta} && (51)\\
&= \sin\theta d\phi \wedge -d\theta && (52)\\
&= \sin\theta d\theta \wedge d\phi && (53)\\
\Rightarrow \Omega_{\phi\theta} &= 0 && (54)
\end{aligned}$$

Thus we conclude that every $\Omega_{ij} = 0$ for $\mathbb{E}^3$ described in spherical coordinates. The Ω_{ij} are related to curvature, and therefore, this makes sense as we are considering regular Euclidean space which is supposed to be flat.