MTH 434

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1 SPHERICAL COORDINATES II Consider spherical coordinates $\{r, \theta, \phi\}$ and the adapted orthonormal basis

$$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} = \left\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right\} \tag{1}$$

The "infinitesimal displacement vector" $d\vec{r}$ relates this basis to an orthonormal basis of 1-forms via

$$\vec{r} = dr \,\hat{r} + rd\theta \,\hat{\theta} + r\sin\theta d\phi \,\hat{\phi} \tag{2}$$

WARNING: these conventions imply $\tan \phi = \frac{y}{r}$

(a) Determine the exterior derivative of each basis vector (not 1-form) above, that is, compute $d\hat{\mathbf{r}}, d\hat{\boldsymbol{\theta}}$, and $d\hat{\boldsymbol{\phi}}$.

We begin by recalling the definition of *connection* 1-forms given by

$$d\hat{\mathbf{e}}_j = \omega^i{}_j \; \hat{\mathbf{e}}_i \tag{3}$$

i.e. the 1-form coefficients of the expansion of $d\hat{e}_j$ in the regular basis. In class, we also defined the metric compatibility and torsion free requirements in terms of connections as

$$\omega_{ij} + \omega_{ji} = 0 \tag{4}$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0 \tag{5}$$

where a connection with both indices downstairs is given by

$$\omega_{ij} = \hat{\boldsymbol{e}}_i \cdot d\hat{\boldsymbol{e}}_j \tag{6}$$

Equipped with these definitions we now can solve for the connection 1-forms. To evade a brute force calculation, recall the line element for \mathbb{E}^3 in spherical coordinates is given by

$$ds^{2} = dr^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2}$$
 (7)

so that

$$d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$
 (8)

however, we also know that

$$d\vec{r} = d(r\hat{r}) = dr\hat{r} + rd\hat{r} \tag{9}$$

so that direct comparison gives

$$\left| d\hat{\boldsymbol{r}} = d\theta \,\,\hat{\boldsymbol{\theta}} + \sin\theta d\phi \,\,\hat{\boldsymbol{\phi}} \right| \tag{10}$$

Knowing (10) significantly simplifies our job. For the other two derivatives we have

$$d\hat{\boldsymbol{\theta}} = \omega^r{}_{\theta} \,\,\hat{\boldsymbol{r}} + \omega^\theta{}_{\theta} \,\,\hat{\boldsymbol{\theta}} + \omega^\phi{}_{\theta} \,\,\hat{\boldsymbol{\phi}}$$

$$d\hat{\boldsymbol{\phi}} = \omega^r{}_{\phi} \,\,\hat{\boldsymbol{r}} + \omega^\theta{}_{\phi} \,\,\hat{\boldsymbol{\theta}} + \omega^\phi{}_{\phi} \,\,\hat{\boldsymbol{\phi}}$$

$$(11)$$

However, equation (4) further simplifies by allowing us to remove the diagonal terms.

$$d\hat{\boldsymbol{\theta}} = \omega^r{}_{\theta} \,\,\hat{\boldsymbol{r}} + \omega^{\phi}{}_{\theta} \,\,\hat{\boldsymbol{\phi}}$$

$$d\hat{\boldsymbol{\phi}} = \omega^r{}_{\phi} \,\,\hat{\boldsymbol{r}} + \omega^{\theta}{}_{\phi} \,\,\hat{\boldsymbol{\theta}}$$

$$(12)$$

The vector version of (4) states that

$$d(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j + d\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0$$
(13)

Thus, equation (13) yields the following:

$$\omega^r{}_{\theta} = -\hat{\boldsymbol{\theta}} \cdot d\hat{\boldsymbol{r}} = -d\theta \tag{14}$$

$$\omega^r{}_{\phi} = -\hat{\boldsymbol{\phi}} \cdot d\hat{\boldsymbol{r}} = -\sin\theta d\phi \tag{15}$$

Now only two connections remain. However, we have that $\omega^{\phi}_{\theta} = -\omega^{\theta}_{\phi}$. It is therefore sufficient to solve for either one alone. To do this, consider the torsion free condition (eq 5). We have that

$$0 = d(rd\theta) + \omega^{\theta}{}_{r} \wedge dr + \omega^{\theta}{}_{\theta} \wedge rd\theta + \omega^{\theta}{}_{\phi} \wedge r\sin\theta d\phi$$
 (16)

$$= dr \wedge d\theta + \omega^{\theta}{}_{r} \wedge dr + \omega^{\theta}{}_{\phi} \wedge r \sin \theta d\phi \tag{17}$$

$$= dr \wedge d\theta + d\theta \wedge dr + \omega^{\theta}{}_{\phi} \tag{18}$$

$$\Rightarrow \omega^{\theta}_{\ \phi} \wedge r \sin \theta d\phi = 0 \tag{19}$$

Equation (19) tells us that ω^{θ}_{ϕ} must only include $d\phi$ and no other 1-forms in order for (19) to hold. Equation (5) also gives

$$0 = d(r\sin\theta d\phi) + \omega^{\phi}{}_{r} \wedge dr + \omega^{\phi}{}_{\theta} \wedge rd\theta + \omega^{\phi}{}_{\phi} \wedge r\sin\theta d\phi$$
 (20)

$$= \sin\theta dr \wedge d\phi + r\cos\theta d\phi \wedge d\phi + \omega^{\phi}{}_{r} \wedge dr + \omega^{\phi}{}_{\theta} \wedge rd\theta \tag{21}$$

$$= \sin\theta dr \wedge d\phi + \sin\theta d\phi \wedge dr + r\cos\theta d\theta \wedge d\phi + \omega^{\phi}{}_{\theta} \wedge rd\theta$$
 (22)

$$= r\cos\theta d\theta \wedge d\phi - rd\theta \wedge \omega^{\phi}{}_{\theta} \tag{23}$$

$$\Rightarrow \omega^{\phi}{}_{\theta} = \cos\theta d\phi \tag{24}$$

and
$$\omega^{\theta}_{\ \phi} = -\cos\theta d\phi$$
 (25)

...and that's all there is to it! In summary, we have

$$d\hat{\mathbf{r}} = d\theta \,\,\hat{\boldsymbol{\theta}} + \sin\theta d\phi \,\,\hat{\boldsymbol{\phi}}$$

$$d\hat{\boldsymbol{\theta}} = -d\theta \,\,\hat{\mathbf{r}} + \cos\theta d\phi \,\,\hat{\boldsymbol{\phi}}$$

$$d\hat{\boldsymbol{\phi}} = -\sin\theta d\phi \,\,\hat{\mathbf{r}} - \cos\theta d\phi \,\,\hat{\boldsymbol{\theta}}$$
(26)

(b) Compute $\omega_{ij} = \hat{e}_i \cdot d\hat{e}_j$ for i, j = 1, 2, 3. What sort of beast should you get?

This question asks us to identify the connection 1-forms. We can easily read these off from our solution to part (a) by comparing with equation (11). They are

$$\begin{aligned}
\omega_{rr} &= 0 & \omega_{\theta r} &= d\theta & \omega_{\phi r} &= \sin \theta d\phi \\
\omega_{r\theta} &= -d\theta & \omega_{\theta \theta} &= 0 & \omega_{\phi \theta} &= \cos \theta d\phi \\
\omega_{r\phi} &= -\sin \theta d\phi & \omega_{\theta \phi} &= -\cos \theta d\phi & \omega_{\phi \phi} &= 0
\end{aligned} \tag{27}$$

(c) Compute $\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ for i, j = 1, 2, 3 (and where there is an implicit sum over k). What sort of beast should you get?

Inspection of the equation for each Ω_{ij} reveals some interesting structure given our solution to part (b) of the problem. Notice that the table in equation 27 in antisymmetric as a result of equation 4. Therefore, we have that.

$$\omega_{ik} \wedge \omega_{ki} = -\omega_{ik} \wedge \omega_{ik} \tag{28}$$

but each of our ω_{ik} are in \bigwedge^1 and therefore

$$\omega_{ik} \wedge \omega_{ik} = 0 \quad \forall i, k \tag{29}$$

Thus, because d(0) = 0 and because of (29) we have that $\Omega_{ii} = 0 \,\forall i$. If we zap equation (4) with d, we find that

$$d\omega_{ij} + d\omega_{ji} = 0 (30)$$

$$\Rightarrow d\omega_{ii} = -d\omega_{ij} \tag{31}$$

For the second half of the Ω_{ij} equation, we have that

$$\omega_{ik} \wedge \omega_{kj} = -\omega_{kj} \wedge \omega_{ik} \tag{32}$$

$$=\omega_{ik}\wedge\omega_{ik}\tag{33}$$

$$= -\omega_{ik} \wedge \omega_{ki} \tag{34}$$

Putting (31) together with (34) gives

$$\Omega_{ij} = -\Omega_{ji} \tag{35}$$

Therefore, it is sufficient to calculate elements with indices corresponding to the upper right half of the table in equation (27). Let's begin with $i = \theta$ and j = r.

$$d\omega_{\theta r} = d(d\theta) = 0 \tag{36}$$

$$\omega_{\theta k} \wedge \omega_{kr} = \omega_{\theta r} \wedge \omega_{rr} + \omega_{\theta \theta} \wedge \omega_{\theta r} + \omega_{\theta \phi} \wedge \omega_{\phi r} \tag{37}$$

$$= 0 + 0 + \omega_{\theta\phi} \wedge \omega_{\phi r} \tag{38}$$

$$= -\cos\theta d\phi \wedge \sin\theta d\phi \tag{39}$$

$$= -\cos\theta\sin\theta d\phi \wedge d\phi \tag{40}$$

$$=0 (41)$$

$$\Rightarrow \Omega_{\theta r} = 0 \tag{42}$$

For the next pair, we have

$$d\omega_{\phi r} = d(\sin\theta d\phi) = \cos\theta d\theta \wedge d\phi \tag{43}$$

$$\omega_{\phi k} \wedge \omega_{kr} = \omega_{\phi r} \wedge \omega_{rr} + \omega_{\phi \theta} \wedge \omega_{\theta r} + \omega_{\phi \phi} \wedge \omega_{\phi r} \tag{44}$$

$$=\omega_{\phi\theta}\wedge\omega_{\theta r}\tag{45}$$

$$=\cos\theta d\phi \wedge d\theta \tag{46}$$

$$= -\cos\theta d\theta \wedge d\phi \tag{47}$$

$$\Rightarrow \Omega_{\phi r} = 0 \tag{48}$$

finally,

$$d\omega_{\phi\theta} = d(\cos\theta d\phi) = -\sin\theta d\theta \wedge d\phi \tag{49}$$

$$\omega_{\phi k} \wedge \omega_{k\theta} = \omega_{\phi r} \wedge \omega_{r\theta} + \omega_{phi\theta} \wedge \omega_{\theta\theta} + \wedge_{\phi\phi} \wedge \omega_{\theta\phi}$$
 (50)

$$=\omega_{\phi r}\wedge\omega_{r\theta}\tag{51}$$

$$= \sin\theta d\phi \wedge -d\theta \tag{52}$$

$$= \sin\theta d\theta \wedge d\phi \tag{53}$$

$$\Rightarrow \Omega_{\phi\theta} = 0 \tag{54}$$

Thus we conclude that every $\Omega_{ij} = 0$ for \mathbb{E}^3 described in spherical coordinates. The Ω_{ij} are related to curvature, and therefore, this makes sense as we are considering regular Euclidean space which is supposed to be flat.