Homework 6 Mth 311

John Waczak

2) Define $f(x) \equiv \begin{cases} x & \text{if } x \in [0,1] \text{ and } x \in \mathbb{Q} \\ -x & \text{if } x \in [0,1] \text{ and } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

prove $f : [0,1] \to \mathbb{R}$ is not integrable.

Because both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in the ~~rationals~~ reals $\forall$ interval $[\alpha, \beta]$ w/ $\alpha, \beta \in \mathbb{R}$ contains points $\in \mathbb{Q}$ and points in $\mathbb{R}\setminus\mathbb{Q}$.
Hence $\forall$ partition $\{x_i\}$ of $[0,1]$ $\forall$ width $\delta > 0$, we can choose x_i' to be such that $x_i' \in \mathbb{Q}$ $\forall i$ or $x_i' \in \mathbb{R}\setminus\mathbb{Q}$ $\forall i$.

Let $\{x_i\}$ be a partition for $[0,1]$. Then we define

$$S_1 = \sum_{i=1}^{N} f(x_i^{1\prime})(x_i - x_{i-1}) \quad \text{and}$$

$$S_2 = \sum_{i=1}^{N} f(x_i^{2\prime})(x_i - x_{i-1})$$

such that

$$x_i^{1\prime} \in \mathbb{Q} \ \forall i$$

$$x_i^{2\prime} \in \mathbb{R}\setminus\mathbb{Q} \ \forall i$$

Then $\forall \delta > 0$ (width of partition), we have

$$|S_1 - S_2| > 0 \quad \text{since } S_2 \text{ must be negative}$$

since $f(x_i^{2\prime}) \leq 0 \ \forall i$ by def

so $\exists \varepsilon$ s.t. $|S_1 - S_2| \geq \varepsilon$ thus f is not integrable