Complex Analysis: Day 13 **John Waczak**
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More on Cauchy's Integral Formula

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a region containing a closed disc $\overline{D_R(c)} \subseteq \Omega$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let $w \in D_r(c)$. Then*

$$f(w) = \frac{1}{2\pi i} \int\limits_{|z-c|=R} \frac{f(z)}{z-w} dz$$

Where we have integrated around the circle in the clockwise direction.

Proof. Without loss of generality we may assume that w is the center of a circle. Use a straight line homotopy from $|z-c| = R$ to a smaller circle $|z-w| = r$. Then

$$\frac{1}{2\pi i} \int\limits_{|z-c|=R} \frac{f(z)}{z-w} dz = \frac{1}{2\pi i} \int\limits_{|z-w|=r} \frac{f(z)}{z-w} dz$$

Now let's calculate something that will be usefull:

$$\frac{1}{2\pi i} \int\limits_{|z-w|=r} \frac{1}{z-w} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{w + re^{it} - w} rie^{it} dt = 1$$

Let $A = f(w) - \frac{1}{2\pi i} \int\limits_{|z-w|=r} \frac{f(z)}{z-w} dz$. We must show that $A = 0$.

$$|A| = \left| f(w) - \frac{1}{2\pi i} \int\limits_{|z-w|=r} \frac{f(z)}{z-w} dz \right|$$

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$|A| = 0$ some details left out

□

Definition A **Jordan curve** is a simple, closed curve which is positively oriented (counter-clockwise, does not cross itself, ends where it begins).

Theorem (Jordan Curve Theorem)**.** *If γ is a Jordan Curve then $\mathbb{C} \setminus \gamma$ is the union of two regions, one bounded and one unbounded (ie. the "inside" and the "outside".)*

Now we may interpret **positively-oriented** to mean that the bounded region is on the left as we traverse the curve.

Theorem (Cauchy's theorem for Jordan Curves)**.** *Let $\Omega \subseteq \mathbb{C}$ be a region, let γ be a Jordan curve in Ω such that Ω contains the entire region bounded by γ. Let w be a point in the region bounded by γ. Then if $f : \Omega \to \mathbb{C}$ is holomorphic,*

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz$$

The proof of this theorem follows from the theorem for circles as we can show that γ is homotopic to a circle centered at w in $\Omega \setminus \{w\}$.

Examples

Ex: $\int\limits_{|z-i|=1} \frac{1}{z^2+1} dz$ We could use partial fraction decomposition and just grind it out but we can do it with less effort using Cauchy's theorem. Let $f(z) = \frac{1}{z+i}$. This is holomorphic on $\mathbb{C} \setminus \{-i\}$. Furthermore, it is holomorphic on our region of integration which means we can use Cauchy's theorem.

$$\begin{aligned}
&\text{let } w = i \\
&f(i) = \frac{1}{2\pi i} \int\limits_{|z-i|=1} \frac{f(z)}{z-i} dz \\
\text{we want } \int\limits_{|z-i|=1} \frac{1}{z^2+1} dz &= \int\limits_{|z-i|=1} \frac{\frac{1}{z+i}}{z-i} dz \\
&= 2\pi i f(i) \\
&= \pi
\end{aligned}$$

Ex: $\int\limits_{|z|=3} \frac{e^z}{z^2-2z} dz$.

$$\begin{aligned}
\int\limits_{|z|=3} \frac{e^z}{z^2-2z} dz &= \int\limits_{|z|=3} \frac{-\frac{1}{2}e^z}{z} + \frac{\frac{1}{2}e^z}{z-2} dz \\
&= -\frac{1}{2} \int\limits_{|z|=3} \frac{e^z}{z} dz + \frac{1}{2} \int\limits_{|z|=3} \frac{e^z}{z-2} dz \\
&= -\frac{1}{2} 2\pi i e^0 + \frac{1}{2} 2\pi i e^2 \\
&= \pi i (e^2 - 1)
\end{aligned}$$