- 1. INTEGRATION ON THE SPHERE Consider \mathbb{S}^2 which can be viewed as the surface in \mathbb{E}^3 satisfying $x^2 + y^2 + z^2 = \text{constant}$. Equivalently, it is the two-dimensional surface with line element $ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$
 - (a) Let ω be the orientation on \mathbb{S}^2 . Determine $\int_{\mathbb{S}^2} \omega$

We can rewrite the line element in the more suggestive form

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{1}$$

So that the orientation can be chosen as

$$\omega = rd\theta \wedge r\sin\theta d\phi \tag{2}$$

Then integration of ω is as simple as dropping the wedges and making sure that the final answer has the correct sign. That is,

$$\int_{\mathbb{S}^2} \omega = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\phi \tag{3}$$

$$=r^2 \int_0^{2\pi} \left(-\cos\theta\right) \Big|_0^{\pi} d\phi \tag{4}$$

$$=2r^2 \int_0^{2\pi} d\phi \tag{5}$$

$$=4\pi r^2\tag{6}$$

which is precisely what we should expect for the area of the 2-sphere.

(b) Let $\alpha \in \bigwedge^1(\mathbb{S}^2)$. Use Stoke's theorem to compute $\int_{\mathbb{S}^2} d\alpha$.

Recall that the sphere is a compact surface with no boundary (section 16.7). Then stokes theorem gives

$$\int_{\mathbb{S}^2} d\alpha = \int_{\partial \mathbb{S}^2} \alpha \tag{7}$$

$$= \int_{\varnothing} \alpha = 0 \tag{8}$$

where in the last line we are integrating a 1-form over the empty set.

(c) Find a 1-form on \mathbb{S}^2 such that $d\alpha = \omega$.

We can write a general 1-form as

$$\alpha = f \, r d\theta + h \, r \sin \theta d\phi \tag{9}$$

Zapping with d gives (**Note:** r is constant on \mathbb{S}^2)

$$d\alpha = r \left(\frac{\partial (h \sin \theta)}{\partial \theta} - \frac{\partial f}{\partial \phi} \right) d\theta \wedge d\phi \tag{10}$$

$$= \frac{1}{r \sin \theta} \left(\frac{\partial (h \sin \theta)}{\partial \theta} - \frac{\partial f}{\partial \phi} \right) r d\theta \wedge r \sin \theta d\phi \tag{11}$$

$$\omega = rd\theta \wedge r\sin\theta d\phi \tag{12}$$

$$\Rightarrow r\sin\theta = \left(\frac{\partial(h\sin\theta)}{\partial\theta} - \frac{\partial f}{\partial\phi}\right) \tag{13}$$

At this point we can make a choice. If we let $f = f(\theta)$ and $h = -r \cot(\theta)$, then we have

$$r\sin\theta = \frac{\partial(-r\cot\theta\sin\theta)}{\partial\theta} + 0\tag{14}$$

$$= -r \frac{\partial \cos \theta}{\partial \theta} = r \sin \theta \tag{15}$$

Thus, if we let

$$\alpha = f(\theta)rd\theta - r^2\cos\theta d\phi \tag{16}$$

(d) How is this possible?

As we can see, this appears to be a contradiction. In part (a) we found that integrating ω gives a non-zero value. Part (b) illustrated that ω can be written as an exterior derivative for which the integral over the whole \mathbb{S}^2 must be zero! How can we resolve this? If we allow our $f(\theta) = 0$ then so far we have found that

$$\omega = d(-r^2 \cos \theta d\phi) \tag{17}$$

As the text says in section 20.4, "the problem must be with $d\phi$ " because $\cos\theta$ is certainly well defined over the whole circle. As the integral of ω is non-zero then ω must not be the derivative of a one form despite the fact that (17) is true. However, if (following the notation of section 20.4 of the text) we identify the orthonormal basis element as $r \sin\theta d\phi = \sigma^{\phi}$, then we may rewrite ω in terms of the orthonormal (well-behaved) basis element.

$$\omega = d(-r\cot\theta\sigma^{\phi}) \tag{18}$$

In this form it is easy to see that because of the cotangent function, ω is undefined at the poles and therefore, ω itself is not a well-defined 2-form. This discrepancy must be what allows the integral to be non-zero. Certainly Stoke's theorem is correct but we also know that integrating the orientation over a surface should give the surface area.

To aid in this consider the circle \mathbb{S}^1 for which $\omega = rd\theta$ If we integrate the orientation, we find that

$$\int_{\mathbb{S}^2} \omega = 2\pi r \tag{19}$$

If we then apply stokes theorem to a one form, say $f = r\phi$ then we have $df = \omega$. Therefore, stokes theorem should give that the same integral (19) is 0. The resolution to the dilemma is that in order to apply Stoke's theorem, our integrand must be a smooth function over the whole region. This is *not* the case for the coordinate function ϕ because there is a sharp discontinuity between the angle $\phi=0$ and $\phi=2\pi$ which both correspond to the same position on the circle.

On the 2-sphere, this issue persists so that, in fact, for half of the great circle between the north and south poles defined by $\phi = 0$ there is a very *not smooth* discontinuity in angle. This arc is shown in the following figure

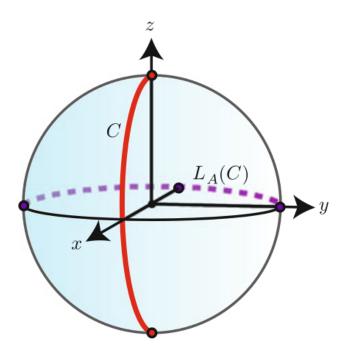


Figure 1: Figure showing the sphere \mathbb{S}^2 with the half-great circle C illustrating the discontinuity in the angle ϕ . Image taken from pg 131 of *Differential Geometry of Curves and Surfaces* by K. Tapp.

Thus, the problem we faced was not really a problem at all but rather our choice of coordinates for the sphere does not satisfy the requirements for the application of stokes theorem to ω . Our solution to part (b) is still correct because we never wrote the 1-form in terms of coordinates.