4. continued...

John Waczak

5

Recall that for any step function $f:[a,b]\to\mathbb{R}$

$$\int_a^b f\,dx = \sum_{i=1}^{N} c_i (x_i - x_{i-1})$$

So we can write

$$\int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx = \sum_{i=1}^{N} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(x_{i-1})(x_i - x_{i-1})$$

$$= \sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$

because $f$ is strictly increasing $f(x_i) - f(x_{i-1}) > 0 \ \forall i$ so since we can make $(x_i - x_{i-1})$ arbitrarily small, we can make $\sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$ arbitrarily small and thus

$$\int_a^b f_2(x) - f_1(x)\,dx < \varepsilon.$$

Therefore if $f$ is strictly increasing on $[a,b]$ it is integrable.