

0. WARMUP

Determine the (nonzero) components of R^i_{jkl} of the curvature 2-forms

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} \sigma^k \wedge \sigma^l \quad (1)$$

for the Robertson-Walker geometry, with line element

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \quad (2)$$

with $k = -1, 0, 1$, depending on whether the spatial cross-sections are hyperbolic, flat, or spherical respectively.

The curvature 2-forms for the Robertson-Walker geometry are given by

$$\left(\Omega^i_j \right) = \begin{pmatrix} 0 & \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^r & \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\theta & \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\phi \\ \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^r & 0 & \frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\theta & \frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\phi \\ \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\theta & -\frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\theta & 0 & \frac{\dot{a}^2 + k}{a^2} \sigma^\theta \wedge \sigma^\phi \\ \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\phi & -\frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\phi & -\frac{\dot{a}^2 + k}{a^2} \sigma^\theta \wedge \sigma^\phi & 0 \end{pmatrix} \quad (3)$$

Note that we have used the property that $\Omega_{ji} = -\Omega_{ij}$. Also note that the curvature two forms each only depend on one basis 2-form. We can now try and find the components of the curvature 2-forms using (1). These will clearly only be non-zero for non-zero Ω^i_j .

Recall that by convention, $R^i_{jlk} = -R^i_{jkl}$ so that beginning with the first row of (3), we have

$$\Omega^t_r = \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^r = \frac{1}{2} (R^t_{rtr} \sigma^t \wedge \sigma^r + R^t_{rrt} \sigma^r \wedge \sigma^t) = R^t_{trt} \sigma^r \wedge \sigma^t \quad (4)$$

$$\Rightarrow R^t_{rtr} = \frac{\ddot{a}}{a}, \quad R^t_{rrt} = -\frac{\ddot{a}}{a} \quad (5)$$

$$\Omega^t_\theta = \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\theta = \frac{1}{2} (R^t_{\theta t\theta} \sigma^t \wedge \sigma^\theta + R^t_{\theta\theta t} \sigma^\theta \wedge \sigma^t) = R^t_{\theta t\theta} \sigma^\theta \wedge \sigma^t \quad (6)$$

$$\Rightarrow R^t_{\theta t\theta} = \frac{\ddot{a}}{a}, \quad R^t_{\theta\theta t} = -\frac{\ddot{a}}{a} \quad (7)$$

$$\Omega^t_\phi = \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\phi = \frac{1}{2} (R^t_{\phi t\phi} \sigma^t \wedge \sigma^\phi + R^t_{\phi\phi t} \sigma^\phi \wedge \sigma^t) = R^t_{\phi t\phi} \sigma^\phi \wedge \sigma^t \quad (8)$$

$$\Rightarrow R^t_{\phi t\phi} = \frac{\ddot{a}}{a}, \quad R^t_{\phi\phi t} = -\frac{\ddot{a}}{a} \quad (9)$$

$$\Omega^r_\theta = \frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\theta = \frac{1}{2} (R^r_{\theta r\theta} \sigma^r \wedge \sigma^\theta + R^r_{\theta\theta r} \sigma^\theta \wedge \sigma^r) = R^r_{\theta r\theta} \sigma^\theta \wedge \sigma^r \quad (10)$$

$$\Rightarrow R^r_{\theta r\theta} = \frac{\dot{a}^2 + k}{a^2}, \quad R^r_{\theta\theta r} = -\frac{\dot{a}^2 + k}{a^2} \quad (11)$$

$$\Omega^r_\phi = \frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\phi = \frac{1}{2} \left(R^r_{\phi r \phi} \sigma^r \wedge \sigma^\phi + R^r_{\phi \phi r} \sigma^\phi \wedge \sigma^r \right) = R^r_{\phi r \phi} \sigma^r \wedge \sigma^\phi \quad (12)$$

$$\Rightarrow R^r_{\phi r \phi} = \frac{\dot{a}^2 + k}{a^2}, \quad R^r_{\phi \phi r} = -\frac{\dot{a}^2 + k}{a^2} \quad (13)$$

$$\Omega^\theta_\phi = \frac{\dot{a}^2 + k}{a^2} \sigma^\theta \wedge \sigma^\phi = \frac{1}{2} \left(R^\theta_{\phi \theta \phi} \sigma^\theta \wedge \sigma^\phi + R^\theta_{\phi \phi \theta} \sigma^\phi \wedge \sigma^\theta \right) = R^\theta_{\phi \theta \phi} \sigma^\theta \wedge \sigma^\phi \quad (14)$$

$$\Rightarrow R^\theta_{\phi \theta \phi} = \frac{\dot{a}^2 + k}{a^2}, \quad R^\theta_{\phi \phi \theta} = -\frac{\dot{a}^2 + k}{a^2} \quad (15)$$

For the items below the diagonal, we have

$$\Omega^r_t = \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^r = \frac{1}{2} \left(R^r_{trt} \sigma^r \wedge \sigma^t + R^r_{ttr} \sigma^t \wedge \sigma^r \right) = R^r_{ttr} \sigma^t \wedge \sigma^r \quad (16)$$

$$\Rightarrow R^r_{ttr} = \frac{\ddot{a}}{a} \quad R^r_{trt} = -\frac{\ddot{a}}{a} \quad (17)$$

$$\Omega^\theta_t = \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\theta = \frac{1}{2} \left(R^\theta_{t\theta t} \sigma^\theta \wedge \sigma^t + R^\theta_{tt\theta} \sigma^t \wedge \sigma^\theta \right) = R^\theta_{tt\theta} \sigma^t \wedge \sigma^\theta \quad (18)$$

$$\Rightarrow R^\theta_{tt\theta} = \frac{\ddot{a}}{a} \quad R^\theta_{t\theta t} = -\frac{\ddot{a}}{a} \quad (19)$$

$$\Omega^\phi_t = \frac{\ddot{a}}{a} \sigma^t \wedge \sigma^\phi = \frac{1}{2} \left(R^\phi_{t\phi t} \sigma^\phi \wedge \sigma^t + R^\phi_{tt\phi} \sigma^t \wedge \sigma^\phi \right) = R^\phi_{tt\phi} \sigma^t \wedge \sigma^\phi \quad (20)$$

$$\Rightarrow R^\phi_{tt\phi} = \frac{\ddot{a}}{a} \quad R^\phi_{t\phi t} = -\frac{\ddot{a}}{a} \quad (21)$$

$$\Omega^\theta_r = -\frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\theta = \frac{1}{2} \left(R^\theta_{r\theta r} \sigma^\theta \wedge \sigma^r + R^\theta_{rr\theta} \sigma^r \wedge \sigma^\theta \right) = R^\theta_{rr\theta} \sigma^r \wedge \sigma^\theta \quad (22)$$

$$\Rightarrow R^\theta_{rr\theta} = -\frac{\dot{a}^2 + k}{a^2} \quad R^\theta_{r\theta r} = \frac{\dot{a}^2 + k}{a^2} \quad (23)$$

$$\Omega^\phi_r = -\frac{\dot{a}^2 + k}{a^2} \sigma^r \wedge \sigma^\phi = \frac{1}{2} \left(R^\phi_{r\phi r} \sigma^\phi \wedge \sigma^r + R^\phi_{rr\phi} \sigma^r \wedge \sigma^\phi \right) = R^\phi_{rr\phi} \sigma^r \wedge \sigma^\phi \quad (24)$$

$$\Rightarrow R^\phi_{rr\phi} = -\frac{\dot{a}^2 + k}{a^2} \quad R^\phi_{r\phi r} = \frac{\dot{a}^2 + k}{a^2} \quad (25)$$

$$\Omega^\phi_\theta = -\frac{\dot{a}^2 + k}{a^2} \sigma^\theta \wedge \sigma^\phi = \frac{1}{2} \left(R^\phi_{\theta\phi\theta} \sigma^\phi \wedge \sigma^\theta + R^\phi_{\theta\theta\phi} \sigma^\theta \wedge \sigma^\phi \right) = R^\phi_{\theta\theta\phi} \sigma^\theta \wedge \sigma^\phi \quad (26)$$

$$\Rightarrow R^\phi_{\theta\theta\phi} = -\frac{\dot{a}^2 + k}{a^2} \quad R^\phi_{\theta\phi\theta} = \frac{\dot{a}^2 + k}{a^2} \quad (27)$$

Having exhausted all of the non-zero curvature two-forms, I believe we have found all of the components.

1. Using the relationships

$$R_{ij} = R^m{}_{imj} \quad (28)$$

$$G^i{}_j = R^i{}_j - \frac{1}{2}\delta^i{}_j R \quad (29)$$

Compute the (nonzero) components $G^i{}_j$ of the *Einstein tensor* for the Robertson-Walker geometry.

Given the curvature two-form components $R^i{}_{jkl}$ from problem 0, we can now find the components of the Ricci curvature tensor. Let's start with the diagonal terms

$$R_{tt} = R^m{}_{tmt} = R^t{}_{ttt} + R^r{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t} \quad (30)$$

$$= 0 - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} = -3\frac{\ddot{a}}{a} \quad (31)$$

$$R_{rr} = R^m{}_{rmr} = R^t{}_{rtr} + R^r{}_{rrr} + R^\theta{}_{r\theta r} + R^\phi{}_{r\phi r} \quad (32)$$

$$= \frac{\ddot{a}}{a} + 0 + \frac{\dot{a}^2 + k}{a^2} + \frac{\dot{a}^2 + k}{a^2} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \quad (33)$$

$$R_{\theta\theta} = R^m{}_{\theta m\theta} = R^t{}_{\theta t\theta} + R^r{}_{\theta r\theta} + R^\theta{}_{\theta\theta\theta} + R^\phi{}_{\theta\phi\theta} \quad (34)$$

$$= \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + 0 + \frac{\dot{a}^2 + k}{a^2} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \quad (35)$$

$$R_{\phi\phi} = R^m{}_{\phi m\phi} = R^t{}_{\phi t\phi} + R^r{}_{\phi r\phi} + R^\theta{}_{\phi\theta\phi} + R^\phi{}_{\phi\phi\phi} \quad (36)$$

$$= \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + \frac{\dot{a}^2 + k}{a^2} + 0 = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \quad (37)$$

Putting all of these together, we can calculate the trace of the Ricci curvature $R^i{}_i$. Remember that $R^i{}_j = g^{ik}R_{kj}$.

$$R = g^{tk}R_{kt} + g^{rk}R_{kr} + g^{\theta k}R_{k\theta} + g^{\phi k}R_{k\phi} = 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2 + k}{a^2} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \quad (38)$$

Note that all of the non-zero curvature components are of the form $R^m{}_{nnm}$ or $R^m{}_{nmn}$. Therefore, it follows that all off diagonal Ricci curvature components $R_{ij} = 0$ where $i \neq j$.

Using this result, we conclude that the *Einstein tensor* is diagonal and therefore we only need to calculate four elements. Better yet, $R_{rr} = R_{\theta\theta} = R_{\phi\phi}$ and so we really only have to

calculate two. That is,

$$G^t_t = R^t_t - \frac{1}{2}\delta^t_t R \quad (39)$$

$$= g_{tk}R^k_t - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \quad (40)$$

$$= 3\frac{\ddot{a}}{a} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \quad (41)$$

$$= -3\frac{\dot{a}^2 + k}{a^2} \quad (42)$$

$$G^r_r = R^r_r - \frac{1}{2}\delta^r_r R \quad (43)$$

$$= g_{rk}R^k_r - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \quad (44)$$

$$= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \quad (45)$$

$$= -2\frac{\ddot{a}a}{a^2} - \frac{\dot{a}^2 + k}{a^2} \quad (46)$$

$$= -\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} \quad (47)$$

$$G^\theta_\theta = G^\phi_\phi = G^r_r = -\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} \quad (48)$$

Thus, we have specified all four non-zero Einstein tensor components for the Robertson-Walker geometry. These results agree with equations 9.15 and 9.16 in the textbook.