Fundamental theorem of Curves

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MTH 434

Tapp 1.76

let $\gamma(t) = (x(t), y(t), z(t))$ define $\hat{\gamma}(t)$ by $\hat{\gamma}(t) = (z(t), x(t), -y(t))$ describe how the curvature and torsion are related.

let K_γ, T_γ be curvature and torsion of γ and $K_{\hat{\gamma}}, T_{\hat{\gamma}}$ be for $\hat{\gamma}$. we can encode this transformation by:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ -y \end{pmatrix}$$

define $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ Note that

$$A^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$ Now we

compute

$$AA^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

thus $A \in O(3)$ and so is a rigid motion.

$$\det(A) = 1\left(\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = -1$$

thus by Proposition 1.64 $\boxed{K_\gamma = K_{\hat{\gamma}}}$ and $\boxed{T_\gamma = -T_{\hat{\gamma}}}$

Tapp 1.77.

$$\gamma = (\cos t, \sin t, t)$$
$$\beta = (\cos t, \sin t, -t)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ -t \end{pmatrix}$$

let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ clearly $A = A^T$

now $AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

thus $A \in O(3)$ is a rigid motion

$\det(A) = 1 \left(\det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = -1$ and so

A is improper thus by prop 1.64

$$\boxed{K_\gamma = K_\beta \text{ but } \tau_\gamma = -\tau_\beta}$$

Tapp 1.81

let $\gamma_1 = (\cos(t), \sin(t))$

$\gamma_2 (t, t^2)$

$f_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ "rotation by 90°"

$f_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ "reflection about y axis"

$f_1^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow f_1 f_1^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$f_2^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow f_2 f_2^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

thus $f_1, f_2 \in O(2)$

$\det(f_1) = 1 \Rightarrow$ proper rigid motion

$\det(f_2) = -1 \Rightarrow$ improper rigid motion

$f_1 \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$

this has same trace so $\exists$ reparametrization

$f_2 \gamma_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ t^2 \end{pmatrix} = \begin{pmatrix} -t \\ t^2 \end{pmatrix}$

$\phi : \mathbb{R} \to \mathbb{R}$ s.t. $t \mapsto -t$

is reparametrization.