Complex Variables: Assignment 2 **John Waczak**
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2.17

Where are the following functions differentiable? Where are they holomorphic? Determine their derivatives at points where they are differentiable

Recall that from Theorem 2.13 (b) if f is a complex function such that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous at z and satisfy the Cauchy-Riemann equations at z, then f is differentiable at z. Furthermore if f is differentiable for all points in some open disc around z then f is holomorphic at z. The derivative of f is given as:

$$f'(z) = u_x + iv_x$$

a) $f(z) = e^{-x}e^{-iy}$

Observe that we can rewrite $f(z) = e^{-x}(\cos(y) - i\sin(y))$ and thus identify $u(x, y) = e^{-x}\cos y$ and $v(x, y) = -e^{-x}\sin(y)$. Then the Cauchy Riemann equations yield:

$$\begin{aligned}
u_x &= -e^{-x}\cos y \\
u_y &= -e^{-x}\sin y \\
v_x &= e^{-x}\sin y \\
v_y &= -e^{-x}\cos y \\
\\
u_x &= v_y \Rightarrow -e^{-x}\cos y = -e^{-x}\cos y \\
u_y &= -v_x \Rightarrow -e^{-x}\sin y = -e^{-x}\sin y
\end{aligned}$$

All of the partials are continuous functions and the C-R relations are satisfied for all choices of x and y. Thus we have the $f(z)$ is differentiable for all $z \in \mathbb{C}$ and therefore is holomorphic on $\mathbb{C}$. At these points we have that $f'(z) = -e^{-x}\cos y + ie^{-x}\sin y = -f(z)$.

b) $f(z) = 2x + ixy^2$.

Identify $u(x,y) = 2x$ and $v(x,y) = xy^2$. Then the C-R relations give:

$$\begin{aligned}
u_x &= 2 \\
u_y &= 0 \\
v_x &= y^2 \\
v_y &= 2xy
\end{aligned}$$

$$\begin{aligned}
u_x = v_y &\Rightarrow 2 = 2xy \\
&\Rightarrow xy = 1 \\
u_y = -v_x &\Rightarrow 0 = y^2 \\
&\Rightarrow y = 0
\end{aligned}$$

As it is impossible to have both $y = 0$ and $xy = 1$ we have that f is nowhere differentiable and therefore not holomorphic.

c) $f(z) = x^2 + iy^2$

Identify $u(x,y) = x^2$ and $v(x,y) = y^2$. Then the C-R relations give:

$$\begin{aligned}
u_x &= 2x \\
u_y &= 0 \\
v_x &= 0 \\
v_y &= 2y
\end{aligned}$$

$$\begin{aligned}
u_x = v_y &\Rightarrow 2x = 2y \\
u_y = -v_x &\Rightarrow 0 = 0
\end{aligned}$$

The C-R relations are satisfied only on the line $x = y$. Thus, $f(z)$ is only differentiable when $x = y$ with derivative $f'(z) = 2x$. Because our differentiable points form a curve and not an area we cannot construct an open disc around any z such that all points in the disc are differentiable. This implies that $f(z)$ is not holomorphic for any $z \in \mathbb{C}$

d) $f(z) = e^x e^{-iy}$

Rewrite $f(z) = e^x(\cos y - i \sin y)$ and identify $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$. The C-R relations then give:

$$\begin{aligned}
u_x &= e^x \cos y \\
u_y &= -e^x \sin y \\
v_x &= -e^x \sin y \\
v_y &= -e^x \cos y
\end{aligned}$$

$$\begin{aligned}
u_x = v_y &\Rightarrow e^x \cos y = -e^x \cos y \\
&\Rightarrow \cos y = 0 \\
u_y = -v_x &\Rightarrow -e^x \sin y = e^x \sin y \\
&\Rightarrow \sin y = 0
\end{aligned}$$

Since $\nexists z \in \mathbb{R}$ such that $\sin y = 0$ and $\cos y = 0$ we have that $f(z)$ is nowhere differentiable and therefore is not holomorphic.

f) $f(z) = \mathrm{Im}(z)$.

Let $z = x + iy$ then $f(z) = y \in \mathbb{R}$. Thus we can identify $u(x, y) = y$ and $v(x, y) = 0$. The C-R relations give:

$$\begin{aligned}
u_x &= 0 \\
u_y &= 1 \\
v_x &= 0 \\
v_y &= 0
\end{aligned}$$

$$\begin{aligned}
u_x &= v_y \Rightarrow 0 = 0 \\
u_y &= -v_x \Rightarrow 1 = 0
\end{aligned}$$

This is impossible and therefore $f(z)$ is nowhere differentiable and thus nowhere holomorphic.

g) $f(z) = x^2 + y^2$

Identify $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Then the C-R relations give the following:

$$\begin{aligned}
u_x &= 2x \\
u_y &= 2y \\
v_x &= 0 \\
v_y &= 0
\end{aligned}$$

$$\begin{aligned}
u_x &= v_y \Rightarrow x = 0 \\
u_y &= -v_x \Rightarrow y = 0
\end{aligned}$$

Thus $f(z)$ is only differentiable when $x = y = 0$ i.e. at the origin. Here it has derivative $f'(z) = 2x \Rightarrow f'(0) = 0$.

2.19

Prove that if f is holomorphic the the region $G \subseteq \mathbb{C}$ and always real valued, then f is constant in G.

Define $f(x, y) = u(x, y) + iv(x, y)$. Since f is always real valued we have that $v(x, y) = 0 \quad \forall z \in G$. Then it follows from the C-R relations that $u_x = v_y \Rightarrow u_x = 0$ and $v_x = 0$ since $v = 0$. Then $f'(z) = u_x + iv_x = 0 + i0 = 0$. Since the derivative of $f(z)$ is zero $\forall z \in G$ then $f(z)$ must be a constant in G. $\square$

2.20

Prove that if $f(z)$ and $\overline{f(z)}$ are holomorphic in the region $G \subseteq \mathbb{C}$ then $f(z)$ is constant in G.

let $f(z) = u(x, y) + iv(x, y)$. Then $\overline{f(z)} = u(x, y) - iv(x, y)$. Applying the C-R relations to both functions gives the following:

$$f(z) \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\overline{f(z)} \begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$$

If we add the first equation from each case we get:

$$u_x = 0$$

If we subtract the second equations from each case we get:

$$v_x = 0$$

Thus because $u_x = v_x = 0$ we have that $f'(z) = 0 + i0 = 0$ for all $z \in G$. This implies that $f(z)$ is constant in G. $\square$

2.24

For each of the following functions u, find a function v such that $u + iv$ is holomorphic in some region. Maximize that region.

a) $u(x, y) = x^2 - y^2$

The C-R relations give that $u_x = v_y$ and $u_y = -v_x$. Thus differentiating u gives the following:

$$v_y = 2x$$
$$v_x = -(-2y) = 2y$$

Thus if we integrate either equation with respect to the type of derivative it represents we should be able to determine a functional form for v.

$$\int v_y dy = \int 2x dy = 2xy + C$$
$$\int v_x dx = \int 2y dx = 2xy + c$$

Thus it appears that $v(x, y) = 2xy$ as both integrals agree. The domain of this function is maximized as it is all of $\mathbb{C}$.

d) $u(x, y) = \frac{x}{x^2+y^2}$

Following the same pattern as for the previous part we have that:

$$v_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$
$$v_x = -\Big(- \frac{2xy}{(x^2 + y^2)^2}\Big) = \frac{2xy}{(x^2 + y^2)^2}$$

Now we can integrate both equations to determine v.

$$\int v_y dy = \int \frac{x^2 - y^2}{(x^2 + y^2)^2} dy = \frac{-y}{x^2 + y^2} + C$$
$$\int v_x dx = \int \frac{2xy}{(x^2 + y^2)^2} dx = \frac{-y}{x^2 + y^2} + c$$

Since both equations agree we have the $v(x, y) = \frac{-y}{x^2+y^2}$. This function is defined for all $z \in \mathbb{C} \setminus \{0\}$ and is therefore maximized.

3.7

Show that the Mobius transformation $f(z) = \frac{1+z}{1-z}$ *maps the unit circle onto the imaginary axis*

Recall that in polar form all z s.t. $|z| = 1$ can be written as $z = e^{i\phi}$. Then using this in $f(z)$ yields the following.

$$\begin{aligned} f(z) &= \frac{1 + e^{i\phi}}{1 - e^{i\phi}} \\ &= \frac{1 + e^{i\phi}}{1 - e^{i\phi}} \Big(\frac{\overline{1 - e^{i\phi}}}{\overline{1 - e^{i\phi}}} \Big) \\ &= \frac{(1 + e^{i\phi})(1 - e^{-i\phi})}{|1 - e^{i\phi}|^2} \\ &= \frac{e^{i\phi} - e^{-i\phi}}{|1 - e^{i\phi}|^2} \\ &= \frac{2i \sin\phi}{|1 - e^{i\phi}|^2} \\ &= i \frac{2 \sin\phi}{|1 - e^{i\phi}|^2} \end{aligned}$$

Thus we see that $f(z)$ is just i times the transformation that maps the unit circle to the real line. Since $i = e^{i\pi/2}$, this is just a rotation by $\pi/2$ of the aforementioned transformation. When you rotate the real line by $\pi/2$ you get the imaginary axis and therefore this transformation must map the unit circle to the imaginary axis.

3.9

Fix $a \in \mathbb{C}$ with $|a| < 1$ and consider $f_a(z) = \frac{z-a}{1-\overline{a}z}$

a) Show that $f_a(z)$ is a Mobius transformation

Recall that a Mobius transformation is a function of the form $f(z) = \frac{az+b}{cz+d}$ such that $ad - bc \neq 0$. Here we have $a = 1$, $b = -a$, $c = -\overline{a}$, $d = 1$. Thus $ad - bc = 1 - a\overline{a} = 1 - |a|^2$. Since $|a| < 1$ this can never evaluate to zero and therefore we have that $f_a(z)$ is a Mobius transformation.

b) Show that $f_a^{-1}(z) = f_{-a}(z)$

Recall from class that if $f(z) = \frac{az+b}{cz+d}$ is a Mobius transformation, it is invertible with inverse: $f^{-1}(z) = \frac{dz-b}{-cz+a}$. Using this and the results from a we have that:

$$\begin{aligned} f_a^{-1}(z) &= \frac{z+a}{\overline{a}z+1} \\ &= \frac{z-(-a)}{1-(-\overline{a})z} \\ &= f_{-a}(z) \end{aligned}$$

b) Prove that $f(z)$ maps the unit disc $D_1(0)$ to itself

We must show that if $|z| < 1$ then $|f(z)| < 1$. Observe the following:

$$\begin{aligned} |f(z)| &= \frac{|z-a|}{|1-\overline{a}z|} < 1 \\ \Rightarrow |z-a|^2 &< |1-\overline{a}z|^2 \\ (z-a)(\overline{z}-\overline{a}) &< (1-\overline{a}z)(1-a\overline{z}) \\ |z|^2 - a\overline{z} - \overline{a}z + |a|^2 &< 1 - a\overline{z} - \overline{a}z + |az| \\ |z|^2 + |a|^2 &< 1 + |az| = 1 + |a||z| \\ \Rightarrow 0 &< 1 + |a||z| - |z|^2 - |a|^2 \\ &= 1 + (|z| - |a|)^2 - |a||z| \end{aligned}$$

I am not sure how exactly to continue from here. We know that $|a| < 1$ by hypothesis.

3.14

Find Mobius transformations satisfying each of the following...

First recall from class that the Mobius transformation that sends $\alpha_1 \mapsto 0$, $\alpha_2 \mapsto 1$ and $\alpha_3 \mapsto \infty$ is given by $f(z) = \frac{(z-\alpha_1)(\alpha_2-\alpha_3)}{(z-\alpha_3)(\alpha_2-\alpha_1)}$.

a) $1 \mapsto 0,\ 2 \mapsto 1,\ 3 \mapsto \infty.$

$$f(z) = \frac{(z-1)(2-3)}{(z-3)(2-1)} = \frac{1-z}{z-3}$$

b) $1 \mapsto 0,\ 1+i \mapsto 1,\ 2 \mapsto \infty.$

$$f(z) = \frac{(z-1)(1+i-2)}{(z-2)(1+i-1)} = \frac{(i-1)z+(1-i)}{iz-2i}$$

c) $0 \mapsto i,\ 1 \mapsto 1,\ \infty \mapsto -i$

In oder to solve this final transformation we will first find the reverse mapping and then take the inverse.

$$f^{-1}(z) = \frac{(z-i)(1+i)}{(z+i)(1-i)} = \frac{(1+i)z-(i-1)}{(1-i)z+(i+1)}$$
$$a = 1+i, \quad b = 1-i, \quad c = 1-i, \quad d = 1+i$$
$$\text{thus } f(z) = \frac{(1+i)z-(1-i)}{(-1-i)z+(1+i)}$$

3.16

Let γ be the unit circle. Find a Mobius transformation that transforms γ to γ and transforms 0 to $\frac{1}{2}$.

First, we know that $f(z) = \frac{az+1}{cz+2}$ in order to send 0 to $\frac{1}{2}$. Now in order to map the unit circle to itself let's try $f(1) = 1$ and $f(-1) = -1$.

$$\begin{aligned}
f(1) = 1 &= \frac{a+1}{c+2} \\
f(-1) = -1 &= \frac{-a+1}{-c+2} \\
\text{adding eqns: } 0 &= \frac{a+1}{c+2} + \frac{-a+1}{-c+2} \\
\frac{a-1}{-c+2} &= \frac{a+1}{c+2} \\
(a+1)(-c+2) &= (a-1)(c+2) \\
2a - ac - c + 2 &= ac + 2a - c - 2 \\
-ac + 2 &= ac - 2 \\
ac &= 2 \\
\Rightarrow a &= \frac{2}{c} \\
f(1) = 1 &= \frac{\frac{2}{c}+1}{c+2} \\
c + 2 &= \frac{2}{c} + 1 \\
c^2 + 2c &= 2 + c \\
c^2 + c - 2 &= 0 \Rightarrow c = -2, \quad c = 1 \\
c &\neq -2 \text{ since } f(1) \text{ would equal } 0 \\
\Rightarrow c &= 1 \Rightarrow a = \frac{2}{c} = 2 \\
\text{thus } f(z) &= \frac{2z+1}{z+2}
\end{aligned}$$

Now to check that this does indeed send the unit circle to itself we need to show that $|f(z)| = 1$ when $|z| = 1$. Let $z = e^{i\phi}$. Then

$$\begin{aligned}
|f(z)| &= \frac{|2e^{i\phi}+1|}{|e^{i\phi}+2|} \\
&= \frac{|2\cos\phi + 1 + i2\sin\phi|}{|\cos\phi + 2 + i\sin\phi|} \\
&= \frac{\sqrt{(2\cos\phi+1)^2 + 4\sin^2\phi}}{\sqrt{(\cos\phi+2)^2+\sin^2\phi}} \\
&= \frac{\sqrt{1 + 4\cos\phi + 4\cos^2\phi + 4\sin^2\phi}}{\sqrt{4 + 4\cos\phi + \cos^2\phi + \sin^2\phi}} \\
&= \frac{\sqrt{5+4\cos\phi}}{\sqrt{5+4\cos\phi}} = 1
\end{aligned}$$

□

3.19

Show that if $f = u + iv$ is holomorphic then the Jacobian equals $|f'(z)|^2$.

Because we have that f is holomorphic, the C-R relations apply and give us that $u_x = v_y$ and $u_y = -v)x$. Thus the Jacobian becomes:

$$\begin{aligned} J &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} \\ &= u_x^2 + v_x^2 = |f'(z)|^2 \text{ since } f'(z) = u_x + iv_x \end{aligned}$$

□

3.21

Find each Mobius transformation f such that

a) f maps $0 \mapsto 1,\ 1 \mapsto \infty,\ \infty \mapsto 0$

First, recall Corollary 3.8 which extends the Mobius transformations to include infinities in a way such that:

$$\begin{cases} \frac{az+b}{cz+d} & if \quad z \in \mathbb{C} \setminus \{-\frac{d}{c}\} \\ \infty & if \quad z = -\frac{d}{c} \\ \frac{a}{c} & if \quad z = \infty \end{cases}$$

Using this corollary we see that

$$\begin{aligned} f(\infty) = 0 = a/c \Rightarrow a = 0 \\ f(1) = \infty \rightarrow 1 = -d/c \rightarrow -c = d \\ f(0) = 1 = -b/c \Rightarrow c = -b \\ \text{thus } f(z) = \frac{b}{-bz+b} \text{ is the desired trans.} \end{aligned}$$

b) f maps $1 \mapsto 1,\ -1 \mapsto i,\ -i \mapsto -1$

We will solve this transformation by creating a composition of two transformations of the form used in problem 3.14.

$$\begin{aligned}
\text{let } f(\alpha_1) &= \beta_1, \quad f(\alpha_2) = \beta_2, \quad f(\alpha_3) = \beta_3 \\
\text{define } g_1 &= \frac{(z-\alpha_1)(\alpha_2-\alpha_3)}{(z-\alpha_3)(\alpha_2-\alpha_1)} \\
g_2 &= \frac{(z-\beta_1)(\beta_2-\beta_3)}{(z-\beta_3)(\beta_2-\beta_1)} \\
\text{then } g_1 &= \frac{(z-1)(-1+i)}{(z+i)(-1-1)} = \frac{(-1+i)z+(1-i)}{-2z-2i} \\
g_2 &= \frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{(i+1)z+(-1-i)}{(i-1)z+(i-1)} \\
g_2^{-1} &= \frac{(i-1)z+(1+i)}{(1-i)z+(1+i)} \\
\Rightarrow g_2^{-1} \circ g_1 &= \frac{(i-1)\frac{(z-1)(-1+i)}{(z+i)(-1-1)}+(1+i)}{(1-i)\frac{(z-1)(-1+i)}{(z+i)(-1-1)}+(1+i)}
\end{aligned}$$

And some further simplification with Mathematica leads to:

$$f(z) = \frac{4}{(-1+2i)+z} + (1+2i)$$

This works as we can verify $f(1) = -2i + 1 + 2i = 1$ and so on.

c) f maps x-axis to $y = x$ and y-axis to $y = -x$, and the unit circle to itself

Recall from linear algebra that the map that obeys the first two transformations above is a simple rotation counter-clockwise by $\pi/4$. Thus if we define the Mobius transformation with $a = e^{i\pi/4}$, $b = c = d = 0$ then we have the simple rotation desired. To make sure that the unit circle maps to itself observe that $|e^{i\pi/4}| = 1$ and thus $|f(z)| = 1|z| = 1$ whenever $|z| = 1$.