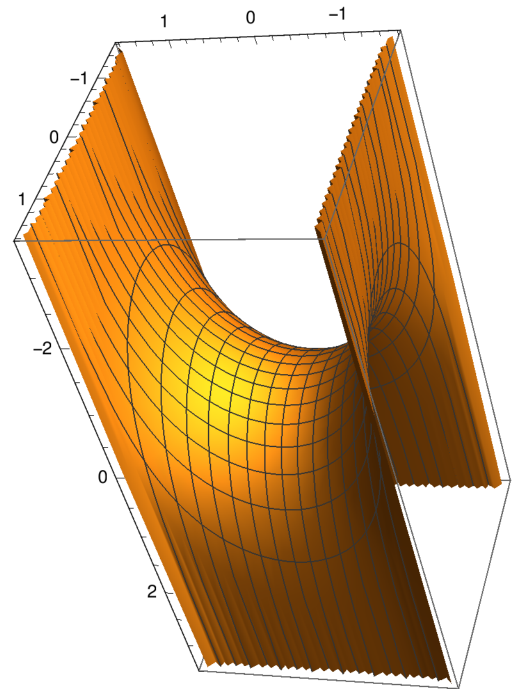
Graph of Scherk's Surface

In[1]:=
```
X[u_, v_] := u
Y[u_, v_] := v
Z[u_, v_] := Log[Cos[v] / Cos[u]]
```

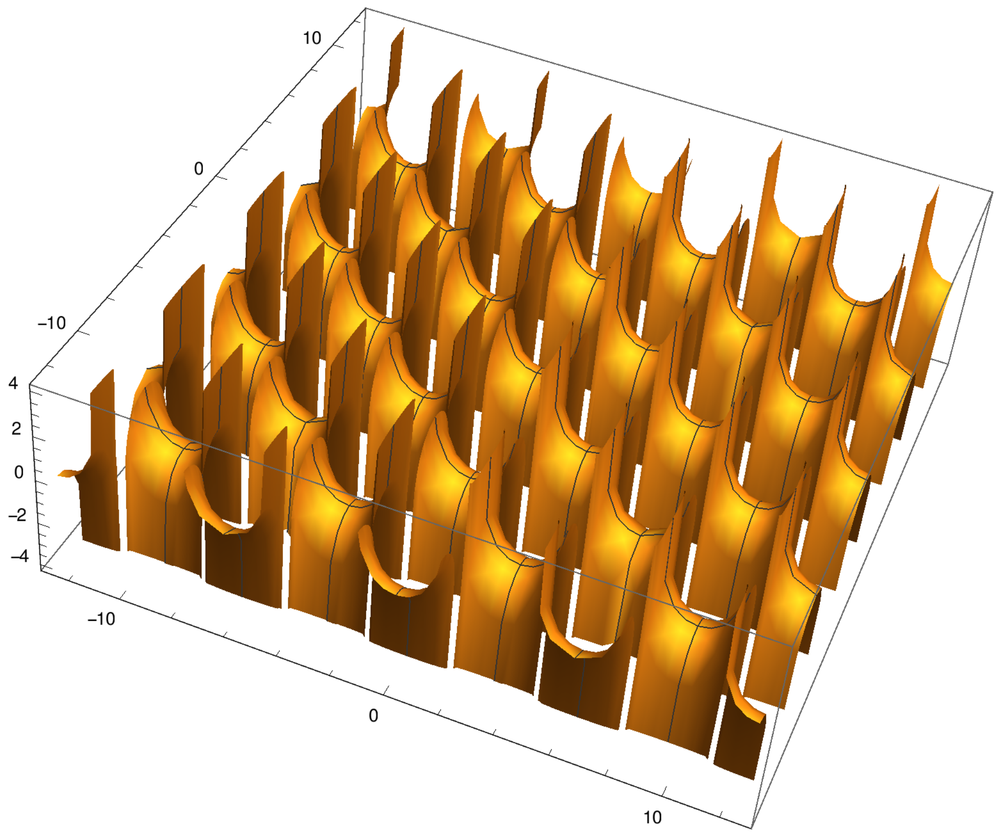
In[4]:=
```
ParametricPlot3D[{X[u, v], Y[u, v], Z[u, v]}, {u, -π / 2, π / 2}, {v, -π / 2, π / 2}]
```

Out[4]=



Here is the same surface on a larger domain

In[5]:=
```
ParametricPlot3D[{X[u, v], Y[u, v], Z[u, v]}, {u, -4 π, 4 π}, {v, -4 π, 4 π}]
```

Out[5]=


Tapp 4.53) verify that the following parametrized surface is minimal

σ(u,v) = [u-sin(u)cosh(v), 1-cos(u)cosh(v), -4sin(u/2)sinh(v/2)]

I do not want to prove that this map is conformal (I'm not really sure how to do that) so I will crank out H by brute force and show that it is 0 for all u,v in the domain

In[6]:=

```
σ[u_, v_] := {u - Sin[u] * Cosh[v], 1 - Cos[u] * Cosh[v], -4 * Sin[u / 2] * Sinh[v / 2]}
```

In[7]:=
```
σ1[u_, v_] := D[σ[u, v], u]
```

In[8]:=
```
σ1[u, v]
```

Out[8]= $\left\{1 - \mathrm{Cos}[u]\,\mathrm{Cosh}[v],\ \mathrm{Cosh}[v]\,\mathrm{Sin}[u],\ -2\,\mathrm{Cos}\left[\frac{u}{2}\right]\mathrm{Sinh}\left[\frac{v}{2}\right]\right\}$

^ is my calculation of σ_u

In[9]:=
```
σ2[u_, v_] := D[σ[u, v], v]
```

In[10]:= $\sigma_2[u, v]$

Out[10]= $\left\{-\text{Sin}[u]\ \text{Sinh}[v],\ -\text{Cos}[u]\ \text{Sinh}[v],\ -2\ \text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right\}$

Now we need to calculate the unit normal field N. First we will find the cross product and then we will normalize it.

In[11]:= **GaussMap[u_, v_] := σ_1[u, v] × σ_2[u, v] / (Norm[σ_1[u, v] × σ_2[u, v]])**

In[12]:= **FullSimplify[GaussMap[u, v]]**

Out[12]=
$$\left\{\left(2\ \text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]\ (\text{Cos}[u]-\text{Cosh}[v])\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{3v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\left(4\ \text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right),\right.$$
$$\left(2\ \text{Cosh}\left[\frac{v}{2}\right]\ (-\text{Cos}[u]+\text{Cosh}[v])\ \text{Sin}\left[\frac{u}{2}\right]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{3v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\left(4\ \text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right),$$
$$\left.\left((-\text{Cos}[u]+\text{Cosh}[v])\ \text{Sinh}[v]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{3v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\left(4\ \text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right)\right\}$$

Now we just need to find the 2nd partial derivatives and then we will have everything we need to calculate E,F,G,e,f,g

In[13]:= **σ_{11}[u_, v_] := D[σ_1[u, v], u]**
σ_{11}[u, v]

Out[14]= $\left\{\text{Cosh}[v]\ \text{Sin}[u],\ \text{Cos}[u]\ \text{Cosh}[v],\ \text{Sin}\left[\frac{u}{2}\right]\ \text{Sinh}\left[\frac{v}{2}\right]\right\}$

In[15]:= **σ_{12}[u_, v_] := D[σ_1[u, v], v]**
σ_{12}[u, v]

Out[16]= $\left\{-\text{Cos}[u]\ \text{Sinh}[v],\ \text{Sin}[u]\ \text{Sinh}[v],\ -\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]\right\}$

In[17]:= **σ_{22}[u_, v_] := D[σ_2[u, v], v]**
σ_{22}[u, v]

Out[18]= $\left\{-\text{Cosh}[v]\ \text{Sin}[u],\ -\text{Cos}[u]\ \text{Cosh}[v],\ -\text{Sin}\left[\frac{u}{2}\right]\ \text{Sinh}\left[\frac{v}{2}\right]\right\}$

Now we will calculate E,F,G

In[19]:= **E_e[u_, v_] := Dot[σ_1[u, v], σ_1[u, v]]**

In[20]:= `FullSimplify[Ee[u, v]]`

Out[20]= $2\,\mathrm{Cosh}\left[\frac{v}{2}\right]^2 \left(-\mathrm{Cos}[u] + \mathrm{Cosh}[v]\right)$

I used the subscript because Mathematica uses E for Euler's constant

In[21]:= `F[u_, v_] := Dot[σ1[u, v], σ2[u, v]]`

In[22]:= `FullSimplify[F[u, v]]`

Out[22]= 0

In[27]:= `G[u_, v_] := Dot[σ2[u, v], σ2[u, v] ]`
`FullSimplify[G[u, v]]`

Out[28]= $2\,\mathrm{Cosh}\left[\frac{v}{2}\right]^2 \left(-\mathrm{Cos}[u] + \mathrm{Cosh}[v]\right)$

Now I will Calculate e,f,g

In[29]:= `ee[u_, v_] := Dot[σ11[u, v], GaussMap[u, v]]`

In[30]:= `FullSimplify[ee[u, v]]`

Out[30]= $\left(2\,\mathrm{Cosh}\left[\frac{v}{2}\right]^3 \left(\mathrm{Cos}[u] - \mathrm{Cosh}[v]\right) \mathrm{Sin}\left[\frac{u}{2}\right]\right) \Big/ \left(\sqrt{\left(\mathrm{Abs}\left[\mathrm{Cos}\left[\frac{3u}{2}\right] \mathrm{Cosh}\left[\frac{v}{2}\right] - \mathrm{Cos}\left[\frac{u}{2}\right] \mathrm{Cosh}\left[\frac{3v}{2}\right]\right]^2 + \mathrm{Abs}[\mathrm{Cos}[u] - \mathrm{Cosh}[v]]^2 \left(4\,\mathrm{Abs}\left[\mathrm{Cosh}\left[\frac{v}{2}\right] \mathrm{Sin}\left[\frac{u}{2}\right]\right]^2 + \mathrm{Abs}[\mathrm{Sinh}[v]]^2\right)\right)}\right)$

In[31]:= `f[u_, v_] := Dot[σ12[u, v], GaussMap[u, v]]`

In[33]:= `FullSimplify[f[u, v]]`

Out[33]= $\left(2\,\mathrm{Cos}\left[\frac{u}{2}\right] \mathrm{Cosh}\left[\frac{v}{2}\right]^2 \left(-\mathrm{Cos}[u] + \mathrm{Cosh}[v]\right) \mathrm{Sinh}\left[\frac{v}{2}\right]\right) \Big/ \left(\sqrt{\left(\mathrm{Abs}\left[\mathrm{Cos}\left[\frac{3u}{2}\right] \mathrm{Cosh}\left[\frac{v}{2}\right] - \mathrm{Cos}\left[\frac{u}{2}\right] \mathrm{Cosh}\left[\frac{3v}{2}\right]\right]^2 + \mathrm{Abs}[\mathrm{Cos}[u] - \mathrm{Cosh}[v]]^2 \left(4\,\mathrm{Abs}\left[\mathrm{Cosh}\left[\frac{v}{2}\right] \mathrm{Sin}\left[\frac{u}{2}\right]\right]^2 + \mathrm{Abs}[\mathrm{Sinh}[v]]^2\right)\right)}\right)$

In[34]:= `g[u_, v_] := Dot[σ22[u, v], GaussMap[u, v]]`

In[35]:= `FullSimplify[g[u, v]]`

Out[35]= $\left(2\,\mathrm{Cosh}\left[\frac{v}{2}\right]^3 \left(-\mathrm{Cos}[u] + \mathrm{Cosh}[v]\right) \mathrm{Sin}\left[\frac{u}{2}\right]\right) \Big/ \left(\sqrt{\left(\mathrm{Abs}\left[\mathrm{Cos}\left[\frac{3u}{2}\right] \mathrm{Cosh}\left[\frac{v}{2}\right] - \mathrm{Cos}\left[\frac{u}{2}\right] \mathrm{Cosh}\left[\frac{3v}{2}\right]\right]^2 + \mathrm{Abs}[\mathrm{Cos}[u] - \mathrm{Cosh}[v]]^2 \left(4\,\mathrm{Abs}\left[\mathrm{Cosh}\left[\frac{v}{2}\right] \mathrm{Sin}\left[\frac{u}{2}\right]\right]^2 + \mathrm{Abs}[\mathrm{Sinh}[v]]^2\right)\right)}\right)$

Now we can calculate H directly from the definition

In[36]:= $H[u\_, v\_] := (1/2) * \frac{e_e[u, v] * G[u, v] - 2 * f[u, v] * F[u, v] + g[u, v] * E_e[u, v]}{E_e[u, v] * G[u, v] - F[u, v]\,\verb|^|2}$

In[38]:=
```
FullSimplify[H[u, v]]
```
Out[38]=
```
0
```

^ Thus we have shown that for all u,v in the domain that the mean curvature is zero. Therefore we conclude that the surface is minimal. Below is a graph of the surface for fun.

In[43]:=
```
ParametricPlot3D[{σ[u, v][[1]], σ[u, v][[2]], σ[u, v][[3]]}, {u, -1, 1}, {v, -1, 1}]
```

Out[43]=