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Measurements

Given $A|\phi_n\rangle = a_n|\phi_n\rangle$; $\{|\phi_n\rangle\}$
If our system is non-degenerate then our possible measurements are the eigenvalues $\{a_n\}$. Mathematically, we say that the way this works is via projection, i.e.

$$P_n = |\phi_n\rangle\langle\phi_n| \tag{1}$$
$$P_n|\phi_n\rangle = \langle\phi_n|\psi\rangle|\phi_n\rangle \tag{2}$$

Example (L_x operator)**.**

$$L_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$
$$L_z \doteq \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Our eigenvalues are given by the roots of the characteristic polynomial:

$$det[L_x - \lambda\mathbf{I}] = 0$$
$$-\lambda(\lambda^2 - \frac{1}{2}) - \frac{1}{\sqrt{2}}(-\frac{\lambda}{\sqrt{2}}) = 0$$
$$\Rightarrow \lambda \in \{0, \pm 1\}$$

Then we put these back into the matrix to solve for the Eigenvectors. Some linear algebra we can get something like:

$$|L_x = 0\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$$

Example (L_z^2 operator)**.**

$$L_z^2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$
$$= \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Thus our characteristic equation is:

$$-\lambda(1-\lambda)^2 = 0$$
$$\rightarrow \lambda = 0 \text{ and } \lambda = 1 \text{ with 2 fold degeneracy}$$

Our $\lambda = 0$ eigenvector comes from:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Unfortunately the second solution leads to degeneracy... how do we solve it?

Degeneracy

Given some a_n degenerate, we write:

$$A|\phi_n^i\rangle = a_n|\phi_n^i\rangle$$

And therefore we have:

$$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i|\phi_n^i\rangle; c_n^i = \langle\phi_n^i|\phi_n^i\rangle \tag{3}$$

Then our probabilities change as well!

$$P(a_n) = \sum_{i=1}^{g_n} |c_n^i|^2 \tag{4}$$

$$P_n = \sum_{i=1}^{g_n} |\phi_n^i\rangle\langle\phi_n^i| \tag{5}$$

Previously, when we made a measurement, we knew what our final state would be with certainty. Now, our operator projects from a larger space to a smaller subspace. If $|\phi_i\rangle$ and $|\phi_j\rangle$ are degenerate, how do you know which you have projected onto? To figure this out, we need to look at our initial state. Say we have:

$$|\psi\rangle = \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2}}|\phi_n^1\rangle + \frac{\beta}{\sqrt{|\alpha|^2 + |\beta|^2}}|\phi_n^2\rangle$$

We can only determine the α, β given $|\psi\rangle$. i.e.

$$\alpha = \langle\phi_n^1|\psi\rangle \;\; \beta = \langle\phi_n^1|\psi\rangle$$

For the case of L_z^2 we had some degeneracy for the $\lambda = 1$ eigenvalue. Using $\lambda = 0$ gave

$$|L_z^2 = 0\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For the degenerate case we have:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{0}$$

We are constrained to have $c_2 = 0$ but we are free to make a choice for c_1, c_3. This allows us to pick nice choices for the subspace so long as we obey orhtonormality and completeness.

Phases

Given $|\psi\rangle$ and $|\psi'\rangle = e^{i\phi}|\psi\rangle$ are these physically equivalent? We say **yes** because $e^{i\phi}$ is an **overall** phase and is not measurable. Whenever we *measure* we will have $e^{i\phi}e^{-i\phi}$ which will disappear.

Alternatively if we have a state $|\psi\rangle = \lambda_1 e^{i\phi_1}|\psi_1\rangle + \lambda_2 e^{i\phi_2}$ has a **relative** phase $e^{i(\theta_2-\theta_1)}$ which will not disappear when we perform measurements on the state.

Compatible and incompatible observables

Recall that if we want to make measurements for operators A, B then,

Definition (compatible)**.** *Two operators A, B are compatible if* $[A, B] = 0$

Theorem. *If two observables are compatible, their operators possess a set of common eigenstate.*

Proof.

$$\begin{aligned}
A|\varphi_n\rangle &= a_n|\varphi_n\rangle \\
\langle\varphi_m|[A, B]|\varphi_n\rangle &= 0 \\
&= \langle\varphi_m|AB - BA|\varphi_n\rangle \\
&= a_m\langle\varphi_m|B - Ba_n|\varphi_n\rangle \\
&= (a_m - a_n)\langle\varphi_m|B|\varphi_n\rangle
\end{aligned}$$

From this we see that if $m \neq n$ then $\langle\varphi_m|B|\varphi_n\rangle = 0$ i.e. the off diagonal matrix element is zero. This indicates that B is represented in it's own eigen-basis. Since these were the eigen-basis for A, we conclude that A and B share eigenstates. $\square$

So if we have initial state $|\Psi\rangle$ and we measure B we get some state $|\varphi_n\rangle$. If we then measure A we get $|\varphi_n\rangle$ state again. We could flip the order of operators and the result would be the same, i.e.

$$A|\varphi_n\rangle = a_n|\varphi_n\rangle \tag{6}$$

$$B|\varphi_n\rangle = b_n|\varphi_n\rangle \tag{7}$$

Now what about incompatible operators?

Definition (incompatible)**.** *Two operators A, B are incompatible if* $[A, B] \neq 0$.

So if A, B are incompatible operators then if we have some $|\Psi\rangle$ and we measure A we will get some a_n and project to $|\varphi_n\rangle$. Then if we measure B we get some b_n and project again into some other state $|\chi_n\rangle$. Thus,

$$P(a_n, b_n) = |\langle\Psi||\varphi_n\rangle|^2|\langle\varphi_n||\chi_n\rangle|^2$$

But if we had changed the order of measurements, our total probability would be:

$$P(b_n, a_n) = |\langle\Psi||\chi_n\rangle|^2|\langle\chi_n||\varphi_n\rangle|^2$$

Example given $H|\varphi_n\rangle = n^2\varepsilon_0|\varphi_n\rangle$ and $A|\varphi_n\rangle = na_0|\varphi_{n+0}\rangle$. We expect that these are incompatible. First act with H then with A.

$$H|\varphi_3\rangle = 9\varepsilon_0|\varphi_3\rangle$$
$$A[9\varepsilon_0|\varphi_3\rangle] = 27a_0 \cdot \varepsilon_0$$

Now first act with A then with H

$$A|\varphi_3\rangle = 3a_0|\varphi_4\rangle$$
$$H[3a_0|\varphi_4\rangle] = 48a_0\epsilon_0|\varphi_4\rangle$$

Uncertainty Relations

Consider two arbitrary operators A, B and their expectation values $\langle A\rangle$, $\langle B\rangle$ with respect to some normalized state vector $|\psi\rangle$. i.e.

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$
$$\langle\hat{B}\rangle = \langle\psi|\hat{B}|\psi\rangle$$

The uncertainties ΔA and ΔB are defined by

$$\Delta A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2} \tag{8}$$
$$\Delta B = \sqrt{\langle\hat{B}^2\rangle - \langle\hat{B}^2\rangle} \tag{9}$$

Note that ΔA, ΔB are numbers, not operators according to the above definition. Sakurai uses,

$$\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle \tag{10}$$
$$\Delta\hat{B} = \hat{B} - \langle\hat{B}\rangle \tag{11}$$

What's the relationship between ΔA and $\Delta\hat{A}$?

$$\begin{aligned}(\Delta\hat{A})^2 &= (\hat{A} - \langle\hat{A}\rangle)^2 = \hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2 \\ \Rightarrow \langle(\Delta\hat{A})^2\rangle &= \langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle^2 + \langle\hat{A}\rangle^2 \\ &= \langle A^2\rangle - \langle\hat{A}\rangle^2 \\ &= (\Delta A)^2\end{aligned}$$

Now let's consider how the uncertainty operator acts on a state, i.e.

$$|\chi\rangle = \Delta\hat{A}|\psi\rangle$$
$$|\phi\rangle = \Delta\hat{B}|\psi\rangle$$

Then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle\chi|\phi\rangle|^2 &\leq \langle\chi|\chi\rangle\langle\phi|\phi\rangle \\ \langle\chi|\chi\rangle &= \langle\psi|\Delta\hat{A}^\dagger\Delta\hat{A}|\psi\rangle \\ &= \langle\psi|(\Delta\hat{A})^2|\psi\rangle \\ &= \langle(\Delta\hat{A})^2\rangle = (\Delta A)^2 \\ \Rightarrow \langle\phi|\phi\rangle &= \langle(\Delta\hat{B})^2\rangle\end{aligned}$$

And also,

$$\begin{aligned}\langle\chi|\phi\rangle &= \langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle \\ &= \langle(\Delta\hat{A}\Delta\hat{B})\rangle\end{aligned}$$

So our Cauchy-Schwarz inequality becomes:

$$|\langle(\Delta\hat{A}\Delta\hat{B})\rangle|^2 \leq (\Delta A)^2(\Delta B)^2 \tag{12}$$

Now let's look at $\Delta\hat{A}\Delta\hat{B}$.

$$\begin{aligned}\Delta\hat{A}\Delta\hat{B} &= \frac{1}{2}\Big(\Delta\hat{A}\Delta\hat{B}\Big) \\ &= 2\cdot\frac{1}{2}\Big(\Delta\hat{A}\Delta\hat{B}\Big) + \frac{1}{2}\Big(\Delta\hat{B}\Delta\hat{A}\Big) - \frac{1}{2}\Big(\Delta\hat{B}\Delta\hat{A}\Big) \\ &= \frac{1}{2}[\Delta\hat{A},\Delta\hat{B}] + \frac{1}{2}\{\Delta\hat{A},\Delta\hat{B}\}\end{aligned}$$

i.e. a combination of commutator and anti-commutator. Recall that $[,]$ was anti-Hermitian when the operators are Hermitian and that $\{,\}$ was Hermitian. Then we have that $\Delta\hat{A}\Delta\hat{B}$ has a real and an imaginary part. i.e.

$$\begin{aligned}\langle\Delta\hat{A}\Delta\hat{B}\rangle &= \frac{1}{2}\langle[\Delta\hat{A},\Delta\hat{B}]\rangle + \frac{1}{2}\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle \\ |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2 &= \frac{1}{4}|\langle[\Delta\hat{A},\Delta\hat{B}]\rangle|^2 + \frac{1}{4}|\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle|^2\end{aligned}$$

Now let's examine the term with $[\Delta\hat{A},\Delta\hat{B}]$:

$$\begin{aligned}[\Delta\hat{A},\Delta\hat{B}] &= \Delta\hat{A}\Delta\hat{B} - \Delta\hat{B}\Delta\hat{A} \\ &= (\hat{A}-\langle\hat{A}\rangle)(\hat{B}-\langle\hat{B}\rangle) - (\hat{B}-\langle\hat{B}\rangle)(\hat{A}-\langle\hat{A}\rangle) \\ &= \hat{A}\hat{B} - \hat{A}\langle\hat{B}\rangle - \langle\hat{A}\rangle\hat{B} + \langle\hat{A}\rangle\langle\hat{B}\rangle - \hat{B}\hat{A} + \hat{B}\langle\hat{A}\rangle + \langle\hat{B}\rangle\hat{A} - \langle\hat{B}\rangle\langle\hat{A}\rangle \\ &= [\hat{A},\hat{B}]\end{aligned}$$

Where in the last line we have equality because expectation values are scalars and can be moved to either side of the operator. Now returning to our previous calculation, we can simplify to find

$$|\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2 = \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 + \frac{1}{4}|\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle|^2$$

From this equality, we can confirm the inequality,

$$\begin{aligned}|\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2 &\geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 \\ \Rightarrow (\Delta A)^2(\Delta B)^2 &\geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 \quad \text{by eqn 12}\end{aligned}$$

Finally, by taking the square root of this equation, we arrive at the famous inequality:

$$\Delta A\Delta B \geq \frac{1}{2}\Big|\langle[\hat{A},\hat{B}]\rangle\Big| \tag{13}$$

which is the general form of the uncertainty principle.

Example: Heisenberg uncertainty relations

$$\hat{A} = X \quad ; \quad \hat{B} = P_x \tag{14}$$

$$[X, P_x] = i\hbar\mathbb{I} \tag{15}$$

Where $\mathbb{I}$ is the identity operator. Then,

$$\Delta x \Delta p_x \geq \frac{1}{2}|\langle i\hbar \rangle| = \frac{\hbar}{2}$$

The position and momentum of a microscopic system cannot be accurately measured both at once. If the position is measured with uncertainty Δx, the uncertainty in measuring the momentum cannot be smaller than $\hbar/2\Delta x$.

Completeness of commuting observables

Consider an observable $A \to a_n$, $\{|\varphi_n\rangle\}$. Is $\{A\}$ a complete set of commuting observables (C.S.C.O)?

If a_n is not degenerate so that $a_n \iff |\varphi_n\rangle$ and a_n are not degenerate, then $\{A\}$ is a complete set of commuting observables. Recall

$$L_z \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This has nondegenerate eigenvalues and therefor is a C.S.C.O. On the other hand

$$L_z^2 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Has a degeneracy of 2 for eigenvalue $\lambda = 1$. So we say that L_z^2 is not a C.S.C.O.

So how do we specify the vectors with no ambiguity? Given A has degenerate a_n's. Find B such that $[A, B] = 0$. Then we hope to find a pair of numbers $\{a_n, b_n\}$ which will specify your vector completely without ambiguity. Then the set $\{A, B\}$ is a C.S.C.O.

Ex: Given $A \Rightarrow a_1 = 0$, $a_{2,3} = 1$. Is $\{A\}$ a C.S.C.O?

NO, but imagine we are given that

$$|a_1 = 0\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|a_2 = 1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|a_3 = 1\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then when we find a B such that $[A, B] = 0$ where $b_1 = 0, b_2 = 1, b_3 = -1$. Then if we have

$$|b_1 = 0\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|b_2 = 1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|b_3 = -1\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then we can be specific and say that

$$|a_1 = 0, b = 0\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|a_2 = 1, b = 1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|a_3 = 1, b = -1\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Which eliminates the ambiguity. Furthermore because $[A, B] = 0$ we know these are valid eigenstates for A. Then the set $\{A, B\}$ is a C.S.C.O. So we can see that we are okay so long as degeneracy in A and degeneracy in B do not overlap. If we have this problem, we keep looking observables until have a complete description. **NOTE** observables in nature are

EX 2: Is the set $\{X, Y, Z\}$ of position operators a CSCO?
Is the set $\{X\}$ in 3 dimensional position space a CSCO?
Is the set $\{X\}$ in 1 dimensional position space a CSCO?

For the first case, yes because $|x, y, z\rangle$ is enough to pinpoint where you are in space. For that same reason the second set is not CSCO. However, $\{X\}$ in 1-dimensional space is a CSCO.

Unitary transformations

A unitary transformation U is given by $\hat{U}^\dagger \hat{U} = \mathbb{I}$

$$|\psi'\rangle = \hat{U}|\psi\rangle$$
$$\langle\psi'| = \langle\psi|\hat{U}^\dagger$$

Given some operator $\hat{A}$ such that $\hat{A}|\psi\rangle = |\phi\rangle$. What happens when we act a unitary operator on the whole thing to get $\hat{A}'|\psi'\rangle = |\phi'\rangle$?

$$\hat{U}^\dagger \hat{A}'|\psi\rangle = \hat{U}^\dagger \hat{U}|\phi\rangle = |\phi\rangle$$

This leads us to conclude the change of basis formula

$$\hat{A} = \hat{U}^\dagger \hat{A}' \hat{U}$$
$$\hat{A}' = \hat{U} \hat{A} \hat{U}^\dagger$$

properties of unitary operators

If $\hat{A}$ is Hermitian, then $\hat{A}'$ is Hermitian.

Proof.

$$\hat{A}^\dagger = (\hat{U}^\dagger \hat{A}' \hat{U})^\dagger = \hat{U}^\dagger \hat{A}'^\dagger \hat{U} = \hat{U}^\dagger \hat{A}' \hat{U}$$

$\square$

Eigenvalues:

$$\begin{aligned}
\hat{A}' |\varphi_n'\rangle &= (\hat{U} \hat{A} \hat{U}^\dagger)(\hat{U} |\varphi_n\rangle) \\
&= \hat{U} \hat{A} |\varphi_n\rangle \\
&= a_n \hat{U} |\varphi_n\rangle
\end{aligned}$$

so the eigenvalues don't change by applying unitary transformations.

Inner product

$$\begin{aligned}
\langle \psi' | \hat{A}' | \chi' \rangle &= (\langle \psi | \hat{U}^\dagger)(\hat{U} \hat{A} \hat{U}^\dagger)(\hat{U} | \chi \rangle) && (16) \\
&= \langle \psi | \hat{A} | \chi \rangle && (17)
\end{aligned}$$

This means that expectation values, norms, inner products, etc... are invariant under unitary transformation. This makes sense as unitary operators have eigenvalues with norm 1. This means that a unitary operator can't re-scale the vectors only translate, reflect, etc...

If $\hat{A} = \beta \hat{B} + \gamma \hat{C}$ then $\hat{A}' = \beta \hat{B}' + \gamma \hat{C}'$.

Given $[\hat{A}, \hat{B}] = a \in \mathbb{C}$. Then $[\hat{A}', \hat{B}'] = \hat{U}[\hat{A}, \hat{B}]\hat{U}^\dagger = a\hat{U}\hat{U}^\dagger = a$. So most things we care about will remain the same.

1 Infinitesimal Unitary Transformations

Recall $\hat{U}^\dagger = \hat{U}^{-1}$. We had some nice properties that really all of the properties we care about do not change by application of unitary operators. Now we want to illustrate how this is useful for physics. Consider a unitary operator

$$\hat{U}_\varepsilon(\hat{G}) = \mathbb{I} + i\varepsilon \hat{G}$$

Where $\hat{G}$ is the *generator of transformation* and ε is a real infinitesimal. Under what circumstances is something like this unitary.

$$\begin{aligned}
\hat{U}_\varepsilon \hat{U}_\varepsilon^\dagger &= (\mathbb{I} + i\varepsilon \hat{G})(\mathbb{I} - i\varepsilon \hat{G}^\dagger) \\
&= \mathbb{I} + i\varepsilon(\hat{G} - \hat{G}^\dagger) + \varepsilon^2 \hat{G}\hat{G}^\dagger \\
&= \mathbb{I} \quad \text{if } \hat{G} \text{ is Hermitian}
\end{aligned}$$

And so for this to be unitary, we need that $\hat{G}$ is Hermitian. Now let's act this on a ket

$$\begin{aligned}|\psi'\rangle &= (\mathbb{I} + i\varepsilon\hat{G})|\psi\rangle \\ &= |\psi\rangle + i\varepsilon\hat{G}|\psi\rangle\end{aligned}$$

So all of your conservation laws will work in this form where our quantity is some generator. Now we want to see how operators transform.

$$\begin{aligned}\hat{A}' &= \hat{U}\hat{A}\hat{U}^\dagger \\ &= (\mathbb{I} + i\varepsilon\hat{G})\hat{A}(\mathbb{I} - i\varepsilon\hat{G}^\dagger) \\ &= (\mathbb{I} + i\varepsilon\hat{G})(\hat{A} - i\varepsilon\hat{A}\hat{G}^\dagger) \\ &= \hat{A} + i\varepsilon\hat{G}\hat{A} - i\varepsilon\hat{A}\hat{G}^\dagger + \varepsilon^2\hat{G}\hat{A}\hat{G}^\dagger \\ &= \hat{A} - ie[\hat{A}, \hat{G}]\end{aligned}$$

Where in the last line we take ε^2 to be small enough to neglect. This informs us that $\hat{A}$ represents some conserved quantity.

Finite unitary transformations

Now consider the following case

$$\begin{aligned}\hat{U}_\alpha(\hat{G}) &= \hat{U}_\varepsilon(\hat{G})\hat{U}_\varepsilon(\hat{G})...\hat{U}_\varepsilon(\hat{G}) \\ &= \lim_{N\to\infty} \Pi_{k=1}^N(\mathbb{I} + i\frac{\alpha}{N}\hat{G}) \\ &= \lim_{N\to\infty}(1 + i\frac{\alpha}{N}\hat{G})^N \\ &= e^{i\alpha\hat{G}}\end{aligned}$$

Where we have recalled that

Now to show this is unitary, we have

$$(e^{i\alpha\hat{G}})^\dagger = e^{-i\alpha\hat{G}^\dagger} = (e^{i\alpha\hat{G}})^{-1}$$

So now assuming $\hat{G}$ is Hermitian,

$$\begin{aligned}\hat{A}' &= \hat{U}\hat{A}\hat{U}^\dagger \\ &= e^{i\alpha\hat{G}}\hat{A}e^{-i\alpha\hat{G}} \\ &= \hat{A} + i\alpha[\hat{G}, \hat{A}], \frac{(i\alpha)^2}{2!}[\hat{G}, [\hat{G}, \hat{A}]] + ...\end{aligned}$$

Applications of unitary transformations

Time Translations $\hat{G} = -\frac{1}{\hbar}\hat{H}$ where $\hat{H}$ is the Hamiltonian. For simplicity let's work with infinitesimal translations in time $\varepsilon \to \delta t$.

$$\begin{aligned}
\hat{U}_{\delta t}(\hat{H}) &= \mathbb{I} + \frac{i}{\hbar}(-\hat{H})\delta t \\
\hat{U}_{\delta t}|\psi(T)\rangle &= (\mathbb{I} + \frac{i}{\hbar}\delta t \hat{H})|\psi(t)\rangle \\
&= |\psi(t)\rangle + \delta t \frac{\partial}{\partial t}|\psi(t)\rangle \\
&\approx |\psi(t+\delta t)\rangle
\end{aligned}$$

Where we have noticed that our equation is a Taylor expansion. Now let's try spatial translators

$$\begin{aligned}
\hat{G} = \frac{1}{\hbar}\hat{P}_x &= \mathbb{I} + \frac{i}{\hbar}\varepsilon \hat{P}_x \\
\hat{U}_\varepsilon(\hat{P}_x)\psi(x) = \psi(x) &+ \varepsilon\frac{\partial}{\partial x}\psi(x) = \psi(x+\varepsilon)
\end{aligned}$$

Where again we are using the Taylor expansion. Now how about observable operators?

$$\begin{aligned}
\hat{A} &= X \\
\hat{X}' &= (\mathbb{I} + \frac{i}{\hbar}\varepsilon\hat{P}_x)X(\mathbb{I} - \frac{i}{\hbar}\varepsilon\hat{P}_x) \\
&= X + \frac{i}{\hbar}\varepsilon[\hat{P}_x, X] = X + \varepsilon
\end{aligned}$$

Since P and X don't commute, momentum does not conserve position and the effect is a translation as expected.

Rotations $\hat{G} = \frac{1}{\hbar}\hat{J}_z$ where $\hat{J}_z$ is the z component of the orbital angular momentum. Then

$$\hat{U}_{d\phi} = \mathbb{I} + \frac{i}{\hbar}d\phi\hat{J}_z$$

THIS IS SO FREAKING BEAUTIFUL! THIS MATHEMATICAL FORMALISM IS GIVING US ALL OF THE CONSERVATION LAWS!!!

2 Position, momentum, translation

For this we need to know how to deal with continuum variables. First, let's list what we know about discrete vs continuum variables

$$\begin{array}{rl}
\text{Discrete} & \text{Continuum} \\
A|\varphi_n\rangle = a_n|\varphi_n\rangle & B|\xi'\rangle = b|\xi'\rangle \\
\langle\varphi_n|\varphi_m\rangle = \delta_{nm} & \delta(x-x_0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{ik(x-x_0)}dk \\
\displaystyle\sum_n |\varphi_n\rangle\langle\varphi_n| = \mathbb{I} & \displaystyle\int d\xi'|\xi'\rangle\langle\xi'| = \mathbb{I} \\
|\psi\rangle = \displaystyle\sum_n |\varphi_n\rangle\langle\varphi_n||\psi\rangle & |\psi\rangle = \displaystyle\int d\xi'|\xi'\rangle\langle\xi'||\psi\rangle \\
\langle\chi|\psi\rangle = \displaystyle\sum_n \langle\chi|\phi_n\rangle\langle\phi_n|\psi\rangle & \langle\chi|\psi\rangle = \displaystyle\int d\xi'\langle\chi|\xi'\rangle\langle\xi'|\psi\rangle \\
\langle\phi_n|A|\phi_m\rangle = a_m\delta_{mn} & \langle\chi''|B|\xi'\rangle = b\delta(\xi''-\xi')
\end{array}$$