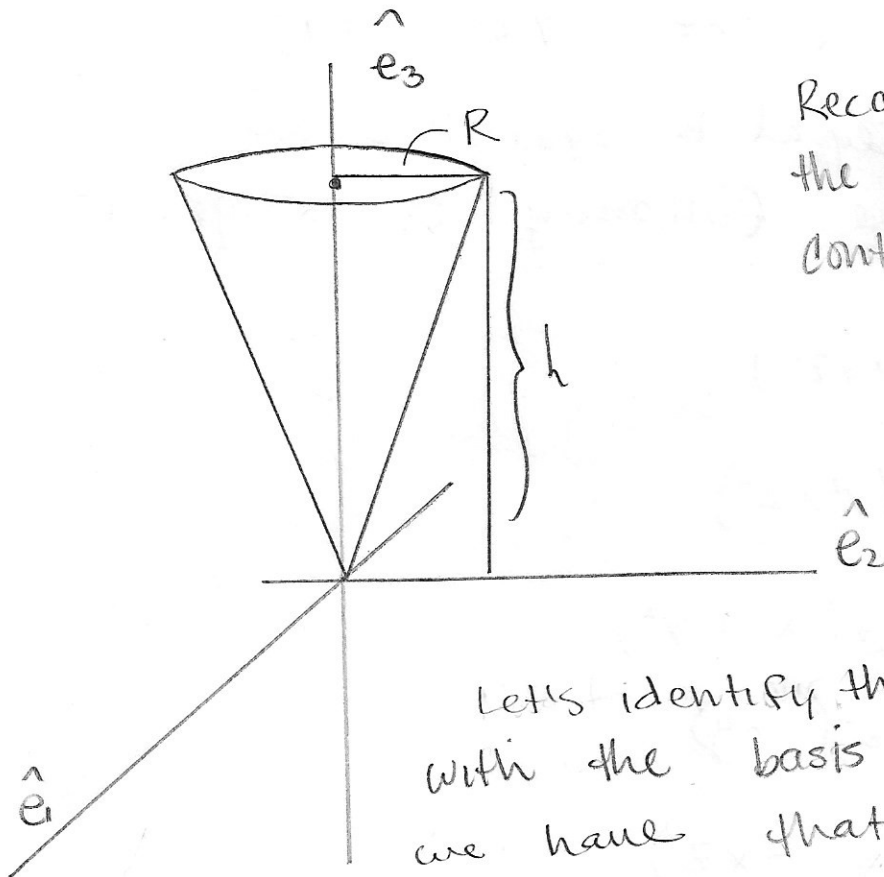


- (1) Homogenous circular cone of height h and base radius R . Calculate inertial tensor in the following basis.

①



Recall the equation for the inertial tensor in continuum

$$I_{ij} = \int d^3\vec{x} \rho(\vec{x}) (\delta_{ij} x^2 - x_i x_j)$$

$$\text{where } x^2 = x_1^2 + x_2^2 + x_3^2$$

Let's identify the coordinates x_1, x_2, x_3 with the basis vectors $\{\hat{e}_i\}$ then we have that the matrix formed

$$\text{by } (\delta_{ij} x^2 - x_i x_j) = \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 - x_1^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_2^2 + x_3^2 - x_2^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 + x_3^2 - x_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{pmatrix}$$

to make things nicer let's let

$$x = x_1 \quad y = x_2 \quad z = x_3 \quad \text{then}$$

the I.T. is

$$I = \rho \int_V dx dy dz \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix}$$

and so because the integral is symmetric we need to perform the following 6 integrals

- 1) $I_{11} = \rho \int_V dx dy dz (y^2 + z^2)$
- 2) $I_{22} = \rho \int_V dx dy dz (x^2 + z^2)$
- 3) $I_{33} = \rho \int_V dx dy dz (x^2 + y^2)$
- 4) $I_{12} = \rho \int_V dx dy dz (-xy)$
- 5) $I_{13} = \rho \int_V dx dy dz (-xz)$
- 6) $I_{23} = \rho \int_V dx dy dz (-yz)$

we also have that since the cone is homogeneous
that $\rho = \frac{M}{V_{\text{cone}}} = \boxed{\frac{M}{\frac{1}{3}\pi R^2 h} = \rho}$

Now all we need to do is get the limits of integration and we can have mathematica crank it out. (2)

our equation for the cone is given by

$$R^2 z^2 = h^2 x^2 + h^2 y^2$$

i.e. when $z=0$ we get $r=0$ and when $z=h$ we get $r=R$. We can solve this for z to find

$$z = \frac{h}{R} \sqrt{x^2 + y^2}$$

Since we have a maximum height of h it must be true that

$$\frac{h}{R} \sqrt{x^2 + y^2} \leq z \leq h$$

Now looking down from above the cone, the x - y projection is just a circle of radius R at height $z=h$ so that we have

$$x^2 + y^2 \leq R^2$$

if we solve this for y we have that
 $-\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}$
and finally by inspection
 $-R \leq x \leq R$. Thus our integration order is
 $dz dy dx$ giving the general
integral

$$I_{ij} = \int_{-R}^R \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \int_{\frac{h}{R} \sqrt{x^2 + y^2}}^h \rho(\delta_{ij} r^2 - x_i x_j) dz dy dx$$

~~See attached mathematica code for the evaluation of the integrals~~

Thus we have that the off diagonal elements are zero and the full tensor is

$$\mathbf{I} = \begin{pmatrix} \frac{3}{20} M (4h^2 + R^2) & 0 & 0 \\ 0 & \frac{3}{20} M (4h^2 + R^2) & 0 \\ 0 & 0 & \frac{3}{10} M R^2 \end{pmatrix}$$

this makes sense as we know rotations around x and y axis should have same \mathbf{I} by symmetry. This verifies that $\{\hat{e}_i\}$ given in the problem form the principal axes for the cone as well as an orthonormal basis

In[40]:= \$Assumptions = Element[R, Reals] && Element[M, Reals] && Element[h, Reals] && R > 0

$$\rho = \frac{M}{\frac{1}{3} \pi R^2 h}$$

Out[40]= $R \in \text{Reals} \&\& M \in \text{Reals} \&\& h \in \text{Reals} \&\& R > 0$

$$\text{Out[41]} = \frac{3 M}{h \pi R^2}$$

$$\text{In[45]} := I_{11} = \rho * \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{\frac{h*\sqrt{x^2+y^2}}{R}}^h (y^2 + z^2) \, dz \, dy \, dx$$

$$\text{Out[45]} = \frac{3}{20} M (4 h^2 + R^2)$$

$$\text{In[46]} := I_{22} = \rho * \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{\frac{h*\sqrt{x^2+y^2}}{R}}^h (x^2 + z^2) \, dz \, dy \, dx$$

$$\text{Out[46]} = \frac{3}{20} M (4 h^2 + R^2)$$

$$\text{In[47]} := I_{33} = \rho * \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{\frac{h*\sqrt{x^2+y^2}}{R}}^h (x^2 + y^2) \, dz \, dy \, dx$$

$$\text{Out[47]} = \frac{3 M R^2}{10}$$

$$\text{In[48]} := I_{12} = \rho * \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{\frac{h*\sqrt{x^2+y^2}}{R}}^h (-x * y) \, dz \, dy \, dx$$

$$\text{Out[48]} = 0$$

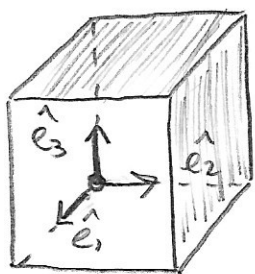
$$\text{In[49]} := I_{13} = \rho * \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{\frac{h*\sqrt{x^2+y^2}}{R}}^h (-x * z) \, dz \, dy \, dx$$

$$\text{Out[49]} = 0$$

$$\text{In[51]} := I_{23} = \rho * \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{\frac{h*\sqrt{x^2+y^2}}{R}}^h (-y * z) \, dz \, dy \, dx$$

$$\text{Out[51]} = 0$$

- (3)
- (2) calculate the inertial tensor for
 of a homogenous cube of
 density ρ , mass M , and side
 of length b w.r.t. the 2 following
 body fixed coordinate systems.
- * I will do this for an arbitrary rectangular prism then simplify *
- (a) $\{\hat{e}_i\}$ w/ origin at corner of cube



let the box have side lengths
 l_1, l_2, l_3 corresponding to
 $\hat{e}_1, \hat{e}_2, \hat{e}_3$

As we had in the last problem, our
 inertial tensor components are

$$I_{ij} = \int_V \rho(\vec{x}) (\delta_{ij} x^2 - x_i x_j) d\vec{x}$$

where x is the distance to the origin and

$$\rho = \frac{M}{V} = \frac{M}{l_1 l_2 l_3} \text{ (rectangular prism)} = \frac{M}{b^3} \text{ (cube)}$$

I is always symmetric and so just as before
 we need only calculate 6 integrals.

$$I_{11} = \rho \int_V dx dy dz (y^2 + z^2)$$

$$I_{22} = \rho \int_V dx dy dz (x^2 + z^2)$$

$$I_{33} = \rho \int_V dx dy dz (x^2 + y^2)$$

$$I_{12} = \rho \int_V dx dy dz (-xy)$$

$$I_{13} = \rho \int_V dx dy dz (-xz)$$

$$I_{23} = \rho \int_V dx dy dz (-yz)$$

for these integrals the bounds are simply

$$0 \leq x \leq l_1 \quad 0 \leq y \leq l_2 \quad 0 \leq z \leq l_3$$

so

$$\begin{aligned}
 I_{11} &= \rho \int_0^{l_1} \int_0^{l_2} \int_0^{l_3} (y^2 + z^2) dz dy dx \\
 &= \rho \int_0^{l_1} \int_0^{l_2} y^2 l_3 + \frac{1}{3} l_3^3 dy dx \\
 &= \rho \int_0^{l_1} \left(\frac{1}{3} l_3 l_2^3 + \frac{1}{3} l_3^3 l_2 \right) dx \\
 &= \rho \frac{1}{3} (l_1 l_2^3 l_3 + l_1 l_2 l_3^3) \\
 &= \frac{M}{l_1 l_2 l_3} \cdot \frac{1}{3} (l_1 l_2 l_3) (l_2^2 + l_3^2)
 \end{aligned}$$

thus $I_{11} = \frac{1}{3} M (l_2^2 + l_3^2)$

and by symmetry we should find that

$$I_{22} = \frac{1}{3} M (l_1^2 + l_3^2)$$

$$I_{33} = \frac{1}{3} M (l_1^2 + l_2^2)$$

Now for I_{12}, I_{13}, I_{23} we have

$$I_{12} = \rho \int_0^{l_3} \int_0^{l_2} \int_0^{l_1} -xy \, dx \, dy \, dz$$

$$= \rho \int_0^{l_3} \int_0^{l_2} -\frac{1}{2} l_1^2 y \, dy \, dz$$

$$= \rho \int_0^{l_3} -\frac{1}{4} l_1^2 l_2^2 \, dz$$

$$= -\frac{1}{4} \rho l_1^2 l_2^2 l_3$$

$$= -\frac{M}{4} l_1 l_2$$

Thus by symmetry we have

$$I_{13} = -\frac{M}{4} l_1 l_3$$

$$\text{and } I_{23} = -\frac{M}{4} l_2 l_3$$

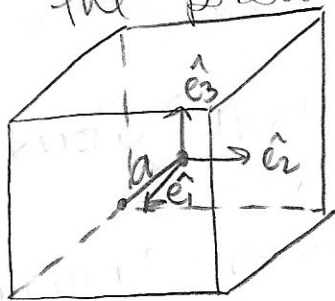
therefore our moment of inertia for the general rectangular prism is just

$$I = \begin{pmatrix} \frac{1}{3}M(l_1^2 + l_3^2) & -\frac{1}{4}Ml_1l_2 & -\frac{1}{4}Ml_1l_3 \\ -\frac{1}{4}Ml_1l_2 & \frac{1}{3}M(l_1^2 + l_3^2) & -\frac{1}{4}Ml_2l_3 \\ -\frac{1}{4}Ml_1l_3 & -\frac{1}{4}Ml_2l_3 & \frac{1}{3}M(l_1^2 + l_2^2) \end{pmatrix}$$

For the cube where $l_1 = l_2 = l_3 = b$ this becomes

$$I = \begin{pmatrix} \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 \end{pmatrix}$$

Now for (2) we want to compute I from the CM coordinates a distance \vec{a} to the previous origin



This actually makes the integrals a bit nicer as our new bounds are simply:

$$-\frac{l_1}{2} \leq x \leq \frac{l_1}{2}$$

$$-\frac{l_2}{2} \leq y \leq \frac{l_2}{2}$$

$$-\frac{l_3}{2} \leq z \leq \frac{l_3}{2}$$

I will again only compute I_{11} and I_{12} arguing the rest by symmetry

(5)

$$I_{11} = \rho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \int_{-\frac{l_3}{2}}^{\frac{l_3}{2}} (y^2 + z^2) dz dy dx$$

$$= \rho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \left[y^2 z + \frac{1}{3} z^3 \right]_{-\frac{l_3}{2}}^{\frac{l_3}{2}} dy dx$$

$$= \rho \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \left[y^2 l_3 + \frac{1}{12} l_3^3 \right] dy dx$$

$$= \rho \left[\frac{1}{12} l_1 l_2^3 l_3 + \frac{1}{12} l_1 l_2 l_3^3 \right]$$

$$I_{11} = \frac{1}{12} M (l_2^2 + l_3^2)$$

and thus by symmetry

$$I_{22} = \frac{1}{12} M (l_1^2 + l_3^2)$$

$$I_{33} = \frac{1}{12} M (l_1^2 + l_2^2)$$

Now we will find I_{12}

$$I_{12} = \rho \int_{-l_3/2}^{l_3/2} \int_{-l_2/2}^{l_2/2} \int_{-l_1/2}^{l_1/2} (-xy) dx dy dz$$

$$= \rho \int_{-l_3/2}^{l_3/2} \int_{-l_2/2}^{l_2/2} -\frac{1}{2} x^2 y \Big|_{-l_1/2}^{l_1/2} dy dz = 0$$

therefore by symmetry $I_{13}, I_{23} = 0$ which makes sense as our axes fall along principal axes.

Therefore for the prism we have

$$I = \begin{pmatrix} \frac{1}{12}(l_2^2 + l_3^2) & 0 & 0 \\ 0 & \frac{1}{12}(l_1^2 + l_3^2) & 0 \\ 0 & 0 & \frac{1}{12}(l_1^2 + l_2^2) \end{pmatrix}$$

for the cube this simplifies to

$$I = \begin{pmatrix} \frac{1}{6} b^2 & 0 & 0 \\ 0 & \frac{1}{6} b^2 & 0 \\ 0 & 0 & \frac{1}{6} b^2 \end{pmatrix}$$

This makes sense as by the parallel axis theorem we can relate this to our first inertial tensor " \tilde{I} " by

$$\tilde{I}_{ij} = I_{ij} + M(a^2 \delta_{ij} - a_i a_j)$$

for $i \neq j$ this difference is $-\frac{M}{4} b^2$ and when

$i = j$ we have $+M(\frac{3}{4} b^2 - \frac{1}{4} b^2) = \frac{M b^2}{2}$ which

gets us back the \tilde{I} from part (1)!