3. Compute $\int_0^1 x\,dx$ directly from definition assuming only that the integral exists

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③

Since the integral exists we know $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $\{x_i\}$ is a partition of width $< \delta$ then

$|S - A| < \varepsilon$ where S is Reimann sum and A is $\int_a^b f(x)\,dx$

let $\{x_i\}$ be a regular partition of width $\frac{b-a}{N} = \frac{1}{N}$ i.e. $x_i = 0 + \frac{i}{N} = \frac{i}{N}$

we can make the width arbitrarily small by controlling N so by the above def:

$$\int_0^1 x\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} x_i\left(\frac{1}{N}\right)$$

choosing to use right end points gives

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N}\left(\frac{1}{N}\right) = \lim_{N\to\infty} \frac{1}{N^2}\sum_{i=1}^{N} i$$

$$= \lim_{N\to\infty} \frac{1}{N^2}\frac{N(N+1)}{2} = \lim_{N\to\infty} \frac{1}{2} + \frac{1}{2N} = \frac{1}{2}$$

thus $\int_0^1 x\,dx = \frac{1}{2}$ which is the same answer we get if we use the FTC to calculate the integral instead!

$\blacksquare$