Homework 2 **John Waczak**
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1 Consider a 1D harmonic oscillator of mass m, angular frequency ω_0 and charge q. Let $|n\rangle$ be eigenstates of the Hamiltonian H_0. For $t < 0$, the oscillator is in the ground state $|0\rangle$. At $t = 0$, it is subjected to an electric field pulse of amplitude $\mathcal{E}$ and duration τ. Let $\mathcal{P}_{0\to n}$ be the probability of finding the oscillator in the state $|n\rangle$ after the pulse.

(**a**) Write down the time-dependent perturbation to H_0.

Recall that the electric field is related to the electric potential by

$$\vec{\boldsymbol{E}} = -\nabla\Phi \tag{1}$$

From the above problem statement, we must have that after turning on, the electric field is of the form

$$\vec{\boldsymbol{E}} = \mathcal{E}\hat{\boldsymbol{x}} \tag{2}$$

where I have chosen x as the distance parameter. The potential energy V due to a charge q in an electric potential Φ is just $q\Phi$ and therefore, the complete perturbation potential may be written as

$$V(x,t) = \begin{cases} -qx\mathcal{E} & 0 \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(**b**) Calculate $\mathcal{P}_{0\to 1}$ by using first order time-dependent perturbation theory. How does $\mathcal{P}_{0\to 1}$ vary with τ for fixed ω_0?

The first order, coupled differential equations for the coefficients through the second order are

$$\lambda^0 : \qquad \imath\hbar\frac{d}{dt}c_n^0(t) = 0 \tag{4}$$

$$\lambda^1 : \qquad i\hbar\frac{d}{dt}c_n^1(t) = \sum_k V_{nk}(t)c_k^0(t)e^{i\omega_{nk}t} \tag{5}$$

$$\lambda^2 : \qquad i\hbar\frac{d}{dt}c_n^2(t) = \sum_k V_{nk}(t)c_k^1(t)e^{i\omega_{nk}t} \tag{6}$$

Each of these differential equations is separable in t and recursively requires the previous order correction. So long as $|i\rangle \neq |f\rangle$ (as is the case for parts b and c of this problem), then the first order correction is given by

$$c_f^1(t) = \frac{1}{i\hbar}\int_{t_0}^{t} V_{fi}(t')e^{i\omega_{fi}t'}dt' \tag{7}$$

which leads to a first order probability approximation given by

$$\mathcal{P}_{i\to f} \approx \frac{1}{\hbar^2}\left|\int_{t_0}^{t} V_{fi}(t')e^{i\omega_{fi}t'}dt'\right|^2 \tag{8}$$

To calculate this integral, we must first evaluate the matrix element in question. We have that $i = 0$ and $f = 1$ so that the matrix element is given by

$$\begin{aligned} V_{10}(t) &= \langle 1|V(t)|0\rangle && (9)\\ &= \langle 1|-qx\mathcal{E}|0\rangle && (10)\\ &= -q\mathcal{E}\,\langle 1|x|0\rangle && (11)\end{aligned}$$

Recall from PH651 that the ladder operator representation of the position operator is

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}\left(a + a^\dagger\right) \tag{12}$$

where

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle && (13)\\ a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle && (14)\end{aligned}$$

returning to the problem, we now have that

$$\begin{aligned} V_{10}(t) &= -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\left\langle 1\middle|\left(a+a^\dagger\right)\middle|0\right\rangle && (15)\\ &= -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\,\langle 1|1\rangle && (16)\\ &= -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}} && (17)\end{aligned}$$

In fact, the matrix element does not depend on t. It follows that the probability is given by

$$\begin{aligned} \mathcal{P}_{0\to 1} &= \frac{1}{\hbar^2}\left|\int_{t_0}^{t} -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}e^{i\omega_{10}t'}dt'\right|^2 && (18)\\ &= \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0}\left|\int_{t_0}^{t} e^{i\omega_{10}t'}dt'\right|^2 && (19)\\ \omega_{10} &= \left(\hbar\omega_0\frac{3}{2} - \hbar\omega_0\frac{1}{2}\right) = \omega_0 && (20)\\ \Rightarrow \mathcal{P}_{1\to 0} &= \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0}\left|\int_0^{\tau} e^{i\omega_0 t'}dt'\right|^2 && (21)\\ &= \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0}\left|\frac{1}{i\omega_0}(e^{i\omega_0\tau} - 1)\right|^2 && (22)\\ &= \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0}\left|\frac{e^{i\omega_0\tau/2}}{i\omega_0}(e^{i\omega_0\tau/2} - e^{-i\omega_0\tau/2})\right|^2 && (23)\\ &= \frac{q^2\mathcal{E}^2}{2m\hbar\omega_0}\left|\frac{e^{i\omega_0 t/2}}{i\omega_0}2\sin(\omega_0\tau/2)\right|^2 && (24)\\ &= \frac{2q^2\mathcal{E}^2}{m\hbar\omega_0^3}\sin^2(\omega_0\tau/2) && (25)\\ & && (26)\end{aligned}$$

From this we can see that the probability is oscillatory in τ. This (**c**) Show that, to obtain $\mathcal{P}_{0\to 2}$, the calculation must be done at least to the second order. Calculate $\mathcal{P}_{0\to 2}$ to this order.

In order to calculate the first order approximation, we need to determine the matrix element V_{20}. However, because x can be written in terms of raising and lowering operators, this matrix element must be zero as is shown below:

$$V_{20} = -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\left\langle 2\middle|\left(a+a^\dagger\right)\middle|0\right\rangle \tag{27}$$

$$= -q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\left\langle 2|1\right\rangle \tag{28}$$

$$= 0 \tag{29}$$

Therefore, we require the second order perturbation to determine the probability. Combining equation 7 with equation 6 yields

$$i\hbar\frac{d}{dt}c_n^2(t) = \sum_k V_{nk}e^{i\omega_{nk}t}\frac{1}{i\hbar}\int_0^\tau V_{ki}e^{i\omega_{ki}t}dt \tag{30}$$

Again, this equation is separable in t and can therefore be integrated to obtain the second order coefficients

$$c_f^2(t) = -\frac{1}{\hbar^2}\sum_k\int_0^\tau dt'\ V_{fk}(t')e^{i\omega_{fk}t'}\int_0^t dt''\ V_{ki}(t'')e^{i\omega_{ki}t''} \tag{31}$$

For this problem, $f = 2$ and $i = 0$. We therefore care about the matrix elements V_{2k} and V_{k0}. From both of our previous arguments, it is clear that the only surviving term in the series is for $k = 2$ as it is the only term for which V_{fk} and V_{ki} are not both zero. Thus, we have

$$c_2^2(t) = -\frac{1}{\hbar^2}\left(-q\mathcal{E}\sqrt{\frac{\hbar}{2m\omega_0}}\right)^2\int_0^\tau dt'\ \left\langle 2\middle|a+a^\dagger\middle|1\right\rangle e^{i\omega_0 t'}\int_0^{t'} dt''\ e^{i\omega_0 t''} \tag{32}$$

$$= -\frac{q^2\mathcal{E}^2}{2m\omega_0\hbar}\sqrt{2}\int_0^\tau dt'\ e^{i\omega_0 t'}\int_0^{t'} dt''\ e^{i\omega_0 t''} \tag{33}$$

$$= -\frac{q^2\mathcal{E}^2}{\sqrt{2}m\omega_0\hbar}\int_0^\tau dt'\ e^{i\omega_0 t'}\left(\frac{1}{i\omega_0}(e^{i\omega_0 t'}-1)\right) \tag{34}$$

$$= -\frac{q^2\mathcal{E}^2}{\sqrt{2}m\omega_0\hbar}\int_0^\tau dt'\ \left(\frac{1}{i\omega_0}(e^{2i\omega_0 t'}-e^{i\omega_0 t'})\right) \tag{35}$$

$$= -\frac{q^2\mathcal{E}^2}{\sqrt{2}m\omega_0\hbar}\left(-\frac{\left(e^{i\omega_o\tau}-1\right)^2}{2\omega_0}\right) \tag{36}$$

$$= \frac{q^2\mathcal{E}^2}{m\omega_0^3 2\hbar\sqrt{2}}\left(e^{i\omega_o\tau}-1\right)^2 \tag{37}$$

$$= \frac{2q^2\mathcal{E}^2e^{i\omega_0\tau}}{m\omega_0^3\hbar\sqrt{2}}\sin^2(\omega\tau/2) \tag{38}$$

and therefore, the probability is given by

$$\begin{aligned}
\mathcal{P}_{0\to 2} &= |c_2^1(t)_2^2(t)|^2 &(39)\\
&= |0 + c_2^2(t)|^2 &(40)\\
&= \frac{2q^4\mathcal{E}^4}{m^2\omega^6\hbar^2}\sin^4(\omega\tau/2) &(41)
\end{aligned}$$

2 A hydrogen atom is in the ground state at $t = -\infty$. An electric field $\vec{\boldsymbol{E}}(t) = \hat{\boldsymbol{k}}\mathcal{E}e^{-t^2/\tau^2}$ is applied until $t = +\infty$. What is the probability that the atom ends up in any of the $n = 2$ states to the first order? Does this answer depend on whether or not we incorporate spin in the picture?

Given the time-varying electric field from above, the corresponding perturbation potential is given by

$$V(z,t) = -zq\mathcal{E}e^{-t^2/\tau^2} \tag{42}$$

where I have made the assumption that $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$. Because the hydrogen atom is spherically symmetric, this choice should not affect our answer. There are four possible $n = 2$ states, namely one $2s$ state and three $2p$ states. We therefore must calculate the following matrix elements

$$\langle 200|V(t)|100\rangle \qquad \langle 210|V(t)|100\rangle \qquad \langle 21\pm 1|V(t)|100\rangle \tag{43}$$

As an operator, we may identify

$$z = \sqrt{\frac{4\pi}{3}}rY_1^0 = \sqrt{\frac{4\pi}{3}}rT_0^{(1)} \tag{44}$$

or, in other words, z can be thought of as a first rank spherical tensor operator. Then, the Wigner-Eckart theorem gives the selection rule that $m' = q + m$. Here $q = 0$ and $m = 0$ so that we can immediately say

$$\langle 211|V|100\rangle = 0 \qquad \langle 21-1|V|100\rangle = 0 \tag{45}$$

We can also see that $\langle 200|V|100\rangle = 0$ as the angular portion of the integral will look like

$$\int d\Omega\,\left(Y_0^0\right)^* Y_1^0 Y_0^0 \tag{46}$$

and must be zero due to parity. That leaves just 1 final element which we can directly integrate

$$\begin{aligned}
\langle 210|V|100\rangle &= -q\mathcal{E}e^{-t^2/\tau^2}\,\langle 210|z|100\rangle &(47)\\
&= -q\mathcal{E}e^{-t^2/\tau^2}\left\langle 210\middle|\sqrt{\frac{4\pi}{3}}rY_1^0\middle|100\right\rangle &(48)\\
&= -q\mathcal{E}e^{-t^2/\tau^2}\left(\frac{128\sqrt{2}a_0}{243}\right) &(49)
\end{aligned}$$

where the last calculation was performed using Mathematica. The probability is therefore given by

$$\mathcal{P}_{_{gs\to n=2}} = \frac{1}{\hbar^2}\left(\frac{128\sqrt{2}a_0}{243}\right)^2 \left|\int_{-\infty}^{\infty} -q\mathcal{E}e^{-t^2/\tau^2}e^{i\omega_{210,100}t}dt\right|^2 \tag{50}$$

$$= \frac{q^2\mathcal{E}^2}{\hbar^2}\left(\frac{128\sqrt{2}a_0}{243}\right)^2 \left|\int_{-\infty}^{\infty} e^{-t^2/\tau^2}e^{i\omega_{210,100}t}dt\right|^2 \tag{51}$$

$$= \frac{2^{15}}{3^{10}}\frac{a_0^2q^2\mathcal{E}^2}{\hbar^2}\tau^2\pi e^{-\omega^2\tau^2/2} \tag{52}$$

The calculations for which are shown on the following Mathematica worksheet.

In[14]:=
```
$Assumptions = a_0 > 0 and a_0 ∈ Reals and τ > 0
```
Out[14]= $a_0 > 0 \in$ and $\mathbb{R}$ $\tau > 0$

Calculate integral for matrix element

In[15]:=
$$\psi_{210}[r\_, \theta\_, \phi\_] := \frac{1}{2\sqrt{\pi}} \left(\frac{1}{2 * a_0}\right)\text{^}\left(3/2\right) * \frac{r}{a_0} * \text{Exp}\left[-r/\left(2 * a_0\right)\right] * \text{Cos}[\theta]$$

$$\psi_{100}[r\_, \theta\_, \phi\_] := \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_0}\right)\text{^}\left(3/2\right) * \text{Exp}[-r/(a_0)]$$

In[17]:=
$$\sqrt{\frac{4\pi}{3}} * \text{Integrate}[\psi_{210}[r, \theta, \phi] * r * \text{SphericalHarmonicY}[1, 0, \theta, \phi] * \psi_{100}[r, \theta, \phi] * r\text{^}2 * \text{Sin}[\theta], \{r, 0, \infty\}, \{\theta, 0, \pi\}, \{\phi, 0, 2\pi\}]$$

Out[17]=
$$\text{ConditionalExpression}\left[\frac{128\sqrt{2}\ a_0}{243}, \text{Re}[a_0] > 0\right]$$

Integrate for the probability

In[18]:=
$$\text{Nasty\_Int} = \text{Integrate}\left[\text{Exp}\left[-t\text{^}2/\tau\text{^}2\right] * \text{Exp}[i * \omega * t], \{t, -\infty, \infty\}\right]$$

Out[18]=
$$\text{ConditionalExpression}\left[\frac{e^{-\frac{1}{4}\tau^2\omega^2}\sqrt{\pi}}{\sqrt{\frac{1}{\tau^2}}}, \text{Re}\left[\tau^2\right] > 0\right]$$

In[23]:=
$$\frac{q^2 * \epsilon^2}{h^2} * \left(\frac{128\sqrt{2} * a_0}{243}\right)^2 * \left(\frac{e^{-\frac{1}{4}\tau^2\omega^2}\sqrt{\pi}}{\sqrt{\frac{1}{\tau^2}}}\right)^2$$

Out[23]=
$$\frac{32\,768\, e^{-\frac{1}{2}\tau^2\omega^2}\pi q^2 \epsilon^2 \tau^2 a_0^2}{59\,049\, h^2}$$

In[24]:=
$$\frac{2^{15}\, e^{-\frac{1}{2}\tau^2\omega^2}\pi q^2 \epsilon^2 \tau^2 a_0^2}{3^{10}\, h^2}$$

Out[24]=
$$\frac{32\,768\, e^{-\frac{1}{2}\tau^2\omega^2}\pi q^2 \epsilon^2 \tau^2 a_0^2}{59\,049\, h^2}$$

Does this change if we consider the spin interaction? Well, we know that if we consider the spin of the electron as forming a small current loop, then we can define a magnetic moment $\vec{\boldsymbol{\mu}}$. Although this $\vec{\boldsymbol{\mu}}$ does not interact directly with $\vec{\boldsymbol{E}}$, it does interact with the magnetic field.

$$V = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}} \tag{53}$$

we can expand the above equation further by identifying the relationship between electron spin and the magnetic moment, that is

$$V = g\frac{e}{2m}\vec{\boldsymbol{S}} \cdot \vec{\boldsymbol{B}} \tag{54}$$

In order to tell if this interaction takes place, we must try and figure out what $\vec{\boldsymbol{B}}$ is. Recall that Faraday's law in differential form gives

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial}{\partial t}\vec{\boldsymbol{B}} \tag{55}$$

However, we have that

$$\vec{\boldsymbol{E}}(t) = \hat{\boldsymbol{k}}\mathcal{E}e^{-t^2/\tau^2} \tag{56}$$

which is a constant vector field in space for a given time t. Based on this fact,

$$\nabla \times \vec{\boldsymbol{E}} = (\nabla \times \hat{\boldsymbol{k}})\mathcal{E}e^{-t^2/\tau^2} = 0 \tag{57}$$

and, therefore, $\vec{\boldsymbol{B}}$ must be constant in time. This result suggests that the interaction would be time independent and looks just like the Zeeman effect.

3 A 1D harmonic oscillator is in the state $n = 0$ at $t = -\infty$. The perturbation is applied from $t = -\infty$ to $t = +\infty$. Show that if

$$V(t) = -e\mathcal{E}x/\left[1 + (t/\tau)^2\right], \tag{58}$$

then to the first order, $\mathcal{P}_{0\to 1} = \frac{e^2\mathcal{E}^2\pi^2\tau^2}{2m\omega\hbar}e^{-2\omega\tau}$

In order to find the first order correction, we first need to find the matrix element V_{10} which is given by

$$V_{10}(t) = -\frac{e\mathcal{E}}{1 + (t/\tau)^2}\langle 1|x|0\rangle \tag{59}$$

$$= -\frac{e\mathcal{E}}{1 + (t/\tau)^2}\sqrt{\frac{\hbar}{2m\omega}}\left\langle 1\middle|a + a^\dagger\middle|0\right\rangle \tag{60}$$

$$= -\frac{e\mathcal{E}}{1 + (t/\tau)^2}\sqrt{\frac{\hbar}{2m\omega}} \tag{61}$$

Using this, the probability of transitioning is given by

$$\mathcal{P}_{0\to 1} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} V_{10}(t)e^{i\omega t}dt\right|^2 \tag{62}$$

$$= \frac{e^2\mathcal{E}^2}{2\hbar m\omega}\left|\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1 + (t/\tau)^2}dt\right|^2 \tag{63}$$

$$\tag{64}$$

This integral is not solvable by ordinary methods, however, as math gurus we can recognize that the denominator may be factored over the complex numbers yielding

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(1+i\frac{t}{\tau})(1-i\frac{t}{\tau})} dt \tag{65}$$

so that over the complex numbers, this integral has two simple poles at $t = \pm i\tau$. To find the value of the integral, we may consider alternatively a contour in the complex plane as shown in the following image

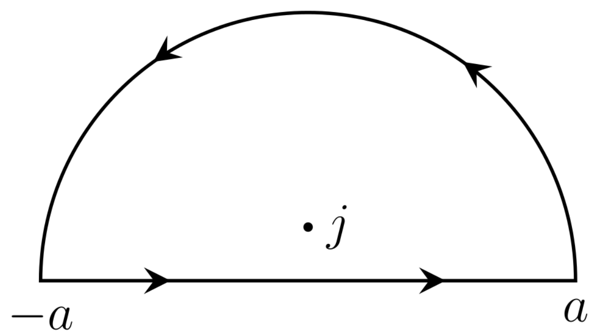

Figure 1: Contour integral in the complex plane showing one of the simple poles at the point $j = i\tau$

If we denote the path in the above figure as C then,

$$\oint_C \frac{e^{i\omega z}}{(1+i\frac{z}{\tau})(1-i\frac{z}{\tau})} dz = 2\pi i \mathrm{Res}(z = i\tau) \tag{66}$$

$$= 2\pi i \frac{-i\tau e^{-\omega\tau}}{2} \tag{67}$$

$$= \pi\tau e^{-\omega\tau} \tag{68}$$

We can also break up the path into two parts: the path along the real line, and the half circle. Our solution does not depend on a and in the limit that $a \to \infty$ the integral along the half circle path vanishes so that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1+(t/\tau)^2} dt = \pi\tau e^{-\omega\tau} \tag{69}$$

Plugging this into the equation for our probability yields

$$\mathcal{P}_{0\to 1} = \frac{e^2\mathcal{E}^2}{2\hbar m\omega} \left|\pi\tau e^{-\omega\tau}\right|^2 \tag{70}$$

$$= \frac{e^2\mathcal{E}^2\pi^2\tau^2}{2\hbar m\omega} e^{-2\omega\tau} \tag{71}$$

which confirms the result!

4 Consider a system containing two spin-1/2 particles. At $t = 0$, the system is in the state $|m_{s1}, m_{s2}\rangle = |+,-\rangle$. The unperturbed Hamiltonian H_0 is spin-independent and can be taken as zero. At $t = 0$, a time-independent perturbation is applied $V = \frac{4\Delta}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2$, where S_1 and S_2 are spin operators, and Δ is a constant.

(**a**) Find the state of the system at an arbitrary time $t > 0$.

We can re-write this perturbation to take advantage of the coupled basis.

$$V = \frac{2\Delta}{\hbar^2}\left(\vec{\boldsymbol{S}}^2 - \vec{\boldsymbol{S}}_1^2 - \vec{\boldsymbol{S}}_2^2\right) \tag{72}$$

At time $t = 0$ we are in the state, $|+,-\rangle$ which in the coupled basis corresponds to

$$|+,-\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,0\rangle) \tag{73}$$

Each component has its energy eigenvalue equation

$$V\,|1,0\rangle = \Delta\,|1,0\rangle \qquad V\,|0,0\rangle = -3\Delta\,|0,0\rangle \tag{74}$$

which we can use to immediately write the time dependence, e.g.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\Delta t/\hbar}\,|1,0\rangle + e^{3i\Delta t/\hbar}\,|0,0\rangle\right) \tag{75}$$

(**b**) Using the result of part (a), find the probabilities that after a time t the system will end up in $|+,+\rangle$, $|-,-\rangle$, $|+,-\rangle$, and $|-,+\rangle$ states.

The probabilities are given by the squared norm of the coefficients of $|\psi(t)\rangle$ in the uncoupled basis, e.g.

$$\mathcal{P}_{++} = |\langle +,+|\psi(t)\rangle|^2 \tag{76}$$

$$= |\langle 1,1|\psi(t)\rangle|^2 = 0 \tag{77}$$

$$\mathcal{P}_{--} = |\langle -,-|\psi(t)\rangle|^2 \tag{78}$$

$$= |\langle 1,-1|\psi(t)\rangle|^2 = 0 \tag{79}$$

$$\mathcal{P}_{+,-} = |(\langle 1,0| + \langle 0,0|)\,|\psi(t)\rangle|^2 \tag{80}$$

$$= \frac{1}{4}\left|e^{-i\Delta t/\hbar} + e^{3i\Delta t/\hbar}\right|^2 \tag{81}$$

$$= \frac{1 + \cos(4\Delta t/\hbar)}{2} \tag{82}$$

$$\mathcal{P}_{+,-} = |(\langle 1,0| - \langle 0,0|)\,|\psi(t)\rangle|^2 \tag{83}$$

$$= \frac{1}{4}\left|e^{-i\Delta t/\hbar} - e^{3i\Delta t/\hbar}\right|^2 \tag{84}$$

$$= \frac{1 - \cos(4\Delta t/\hbar)}{2} \tag{85}$$

(**c**) Now treat V as a perturbation (applied at $t = 0$) and calculate the probabilities of the transitions described in part (b) using first-order perturbation theory.

To first order, the probability is given by

$$\mathcal{P}_{i\to f} = \left|c_f^0(t) + c_f^1(t)\right|^2 = \left|\delta_{f,i} - \frac{i}{\hbar}\int_0^t V_{fi}(t')e^{i\omega_{fi}t'}dt'\right|^2 \tag{86}$$

V is clearly diagonal for the coupled states. Therefore, if our final state is either $|+,+\rangle = |1,1\rangle$ or $|-,-\rangle = |1,-1\rangle$ the probability must be zero by orthogonality. For the other two possibilities, we therefore have

$$\mathcal{P}_{i\to f} = \left|1 - \frac{i}{\hbar}\int_0^t V_{f,i}(t')e^{i\omega_{fi}t'}dt'\right|^2 \tag{87}$$

so that the probabilities become

$$\mathcal{P}_{+,-} = \left|1 - \frac{i}{\hbar}\int_0^t \frac{1}{2}(\Delta\langle 1,0|1,0\rangle - 3\Delta\langle 0,0|0,0\rangle)e^{i0t'}dt'\right|^2 \tag{88}$$

$$= \left|1 + \frac{i\Delta t}{\hbar}\right|^2 = 1 + \frac{\Delta^2 t^2}{\hbar^2} \tag{89}$$

$$\mathcal{P}_{-,+} = \left|0 - \frac{i}{\hbar}\int_0^t \frac{1}{2}(\Delta\langle 1,0|1,0\rangle + 3\Delta\langle 0,0|0,0\rangle)e^{i0t'}dt'\right|^2 \tag{90}$$

$$= \left|-\frac{i2\Delta t}{\hbar}\right|^2 = \frac{4\Delta^2 t^2}{\hbar^2} \tag{91}$$

(d) Compare the exact solution found in part (b) with the approximate one found in part (c) and comment.

The perturbation theory agrees exactly with the analytic solution for the two final states $|+,+\rangle$ and $|-,-\rangle$. This makes sense because if a state can never be reach it is unlikely that we can add successively smaller terms using a perturbation approach and reach a zero value. For the other two approximations, let's consider the Power series for our exact solutions

$$\frac{1+\cos(4\Delta t/\hbar)}{2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\left(\frac{16\Delta^2 t^2}{2!\hbar^2}\right) \tag{92}$$

$$= 1 - \frac{4\Delta^2 t^2}{\hbar^2} \tag{93}$$

$$\frac{1-\cos(4\Delta t/\hbar)}{2} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\left(\frac{16\Delta^2 t^2}{2!\hbar^2}\right) \tag{94}$$

$$= \frac{4\Delta^2 t^2}{\hbar^2} \tag{95}$$

Therefore, $mathcalP_{+,-}$ is almost exact up to the coefficent of the t^2 term and $\mathcal{P}_{-,+}$ is exact.