Final Exam **John Waczak**
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1. THE STANDARD MODELS

(a) Show that the Robertson-Walker cosmological models with $k = 1$ can be rewritten in the form

$$ds^2 = a(\eta)^2\Big(- d\eta^2 + d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2 d\theta d\phi^2\Big) \tag{1}$$

Solution:

The original Robertson-Walker metric is given by

$$ds^2 = -dt^2 + a(t)^2\Big(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big) \tag{2}$$

By setting $k = 1$, we obtain

$$ds^2 = -dt^2 + a(t)^2\Big(\frac{dr^2}{1 - r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big) \tag{3}$$

At this point, a closer inspection of our goal is warranted. Note that in (1), everything beyond the $d\eta^2$ term within the parentheses is in fact the regular metric for the 3-sphere. Therefore, we should make the substitution

$$r = \sin\psi \tag{4}$$

so that

$$r^2 = \sin^2\psi \qquad dr = \cos\psi d\psi \Rightarrow dr^2 = \cos^2\psi d\psi^2 \tag{5}$$

Finally, by the Pythagorean identity, we have

$$1 - r^2 = 1 - \sin^2\psi = \cos^2\psi \tag{6}$$

so that putting everything together, we find

$$ds^2 = -dt^2 + a(t)^2\Big(d\psi^2 + \sin\psi^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big) \tag{7}$$

Now the last step is to identify some kind of transformation $t \to \eta$. Let's define $\eta = \eta(t)$ such that

$$dt = a(\eta)d\eta \tag{8}$$

then we may replace $a(t) \to a(\eta)$ and $dt \to a(\eta)d\eta$. This results in

$$ds^2 = a(\eta)^2\Big(- d\eta^2 + d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)\Big) \tag{9}$$

(**b**) The dust-filled ($p = 0$), closed ($k = 1$) Robertson-Walker cosmology is given by

$$a = \frac{1}{2}C(1 - \cos\eta) \tag{10}$$

and the radiation-filled ($p = \rho/3$), closed ($k = 1$) Robertson-Walker cosmology is given by

$$a = B\sin\eta \tag{11}$$

where B and C are constant (and $\Lambda = 0$). *you do not need to verify these statements.* Show that a beam of light emitted radially ($\theta =$ const, $\phi =$ const, $\dot{\eta} > 0, \dot{\psi} > 0$) from $\psi = 0$ at $\eta = 0$ goes precisely once around the dust-filled universe during its lifetime, but only halfway around the radiation-filled universe during it's lifetime.

Solution:

For a radial beam of light, we have that that $ds^2 = 0$ and $d\theta = d\phi = 0$. Therefore, the metric reduces to

$$0 = -a^2 d\eta^2 + a^2 d\psi^2 \quad \Rightarrow \quad d\eta = d\psi \tag{12}$$

We can therefore say that if we start at $\psi = 0, \eta = 0$ traveling *around the universe* must mean that ψ goes from $0 \to 2\pi$ and therefore η also goes from $0 \to 2\pi$.

To unpack what is meant by the beam's *lifetime*, let's use the relationship we stated between η and t (equation 8).

For the dust filled-universe, we have

$$\begin{aligned} dt = a(\eta)d\eta &= \frac{1}{2}C(1 - \cos\eta)d\eta && (13)\\ \Rightarrow t &= \frac{1}{2}C\int_0^\eta (1 - \cos\eta')d\eta' && (14)\\ &= \frac{1}{2}C(\eta - sin\eta) && (15) \end{aligned}$$

Similarly, for the radiation-filled universe, we have

$$\begin{aligned} dt = a(\eta)d\eta &= B\int_0^\eta \sin\eta' d\eta' && (16)\\ &= B(-\cos\eta - (-\cos(0))) && (17)\\ &= B(1 - \cos\eta) && (18) \end{aligned}$$

therefore, we can write a system of *parametric* equation for the radius of the universe $a(\eta)$ and the far-away observer time $t(\eta)$

$$\text{dust-filled} \to \begin{cases} t = \frac{1}{2}C(\eta - \sin\eta) \\ a = \frac{1}{2}C(1 - \cos\eta) \end{cases} \tag{19}$$

$$\text{radiation-filled} \to \begin{cases} t = B(1 - \cos\eta) \\ a = B\sin\eta \end{cases} \tag{20}$$

The following figure illustrates the radius of the universe plotted against time as a function of η for the two scenarios.

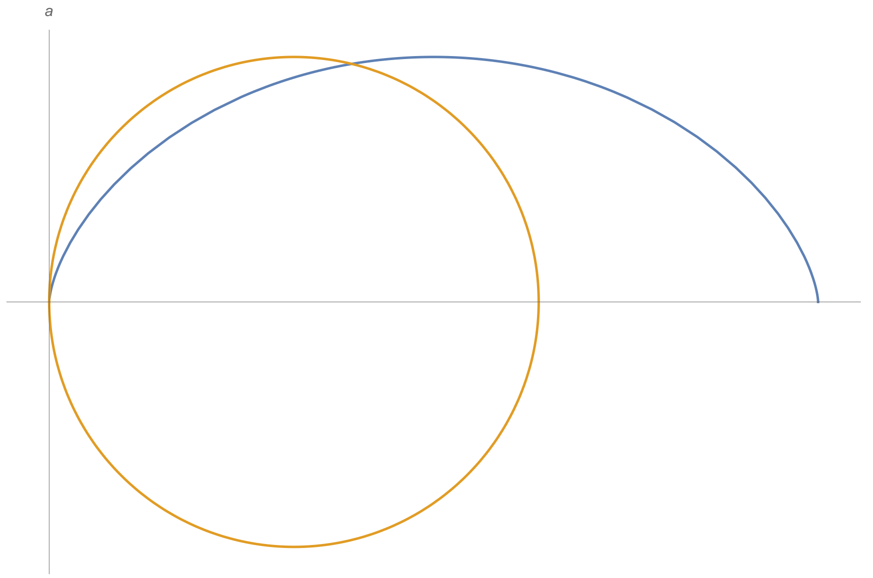

Figure 1: Parametric plots showing the radius of the dust-filled universe (orange) and the radiation-filled universe (blue) versus time

In the figure, I graphed for $\eta \in [0, 2\pi]$ so that when $\eta = 2\pi$ the dust-filled universe has $a = 0$ and when $\eta = \pi$ the radiation-filled universe has $a = 0$. This means that for a beam of light that begins to travel radially at $\psi = 0, \eta = 0$, the dust-filled universe expands and then contracts back to a point exactly when the light beam has traveled through all possible ψ values from 0 to 2π. Interestingly, for the radiation-filled universe, we have that the expansion and subsequent contraction happens faster so that the light beam has only traveled through π of the possible ψ values when the universe again shrinks back to a point.

(**c**) *Does this mean that one could see the back of one's head in either or both of these models?*

Solution:

No! For the entire lifetime, the particle travels radially. The light beam moves at the speed of light but the physical space the light moves through is shrinking. To illustrate the effect of this contraction of space we have chosen a coordinate system that maps infinity to $\psi = 2\pi$ by taking regular $\mathbb{E}^3$ and instead writing it as $\mathbb{S}^3$ multiplied by this weird $a(\eta)$ factor.

For the radiation-filled universe, we certainly don't see the back of our head because by the time the light beam makes it half-way through the universe, everything has shrunk to a point for which we no longer have a good concept of direction. For the dust-filled universe, we could maybe see the back of our head, but again, by the time the light makes it all the way around the universe to reach the back of our head, physical space has shrunk down to a point and I can no longer talk about direction. It seems as though all of the light in the universe must be hitting every side of my head at the same time.

2. NEW COORDINATES

Determine functions f and h so that the line element

$$ds^2 = -f(\rho)^2 dt^2 + h(\rho)^2 \Big(d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2) \Big) \tag{21}$$

is (locally) equivalent to the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 \Big(d\theta^2 + \sin^2\theta d\phi^2 \Big) \tag{22}$$

You may assume that ρ only depends on r, and that $r > 2m$. If you integrate by hand, you may find the substitution $r = m(1 + \cosh\alpha)$ to be helpful.

Solution:

By direct comparison of the two line elements, we can identify the following relationships that must hold

$$f^2 = 1 - \frac{2m}{r} \qquad h^2 d\rho^2 = \frac{1}{1 - \frac{2m}{r}} dr^2 \qquad h^2\rho^2 = r \tag{23}$$

If we divide the middle equation by the rightmost equation, we find

$$\frac{1}{r^2 - 2mr} dr^2 = \frac{1}{\rho^2} d\rho^2 \tag{24}$$

which tells us

$$\frac{1}{\sqrt{r^2 - 2mr}} dr = \frac{1}{\rho} d\rho \tag{25}$$

This is a separable nonlinear ordinary differential equation that we can try and integrate to find $r(\rho)$. To aid in this integration, let

$$r = m(1 + \cosh\alpha) \tag{26}$$

so that

$$dr = m \sinh\alpha \, d\alpha \tag{27}$$

$$r^2 - 2mr = m^2(1 + \cosh^2\alpha + 2\cosh\alpha) - 2m(1 + \cosh\alpha) \tag{28}$$

$$= m^2 \left\{ \cosh^2\alpha - 1 \right\} = m^2 \sinh^2\alpha \tag{29}$$

Therefore, we have

$$\int \frac{dr}{\sqrt{r^2 - 2mr}} = \int \frac{m \sinh\alpha}{\sqrt{m^2 \sinh^2\alpha}} \, d\alpha \tag{30}$$

$$= \int d\alpha = \alpha \tag{31}$$

and the other integral is

$$\int \frac{d\rho}{\rho} = \ln(\rho) + K \tag{32}$$

where K is a dimensionful integration constant. In summary, we have found that

$$\alpha = \ln(\rho) + K \tag{33}$$

$$\Rightarrow r(\rho) = m(1 + \cosh(\ln(\rho) + K)) \tag{34}$$

$$= m\left(1 + \frac{e^{\ln\rho + K} + e^{-\ln\rho - K}}{2}\right) \tag{35}$$

$$= m\left(1 + \frac{e^{K}\rho + e^{-K}\frac{1}{\rho}}{2}\right) \tag{36}$$

we will now set the constant $K = 0$ for simplicity although this will make it impossible to think about dimensions. Thus

$$r(\rho) = m\left(1 + \frac{\rho^2 + 1}{2\rho}\right) = m\left(\frac{\rho^2 + 2\rho + 1}{2\rho}\right) = m\left(\frac{(\rho + 1)^2}{2\rho}\right) \tag{37}$$

From this expression, we can find the equations for $f(\rho)$ and $h(\rho)$. They are

$$h(\rho) = \frac{r}{\rho} = m\left(\frac{(\rho + 1)^2}{2\rho^2}\right) \tag{38}$$

$$f^2(\rho) = 1 - \frac{2m}{r} = 1 - \frac{2m}{m\left(\frac{(\rho+1)^2}{2\rho}\right)} \tag{39}$$

$$= 1 - \frac{4\rho}{(\rho + 1)^2} \tag{40}$$

$$\Rightarrow f(\rho) = 1 - \frac{4\rho}{(\rho + 1)^2} \tag{41}$$

3. GODEL GEOMETRY

The Godel geometry can be described in "rectangular" coordinates by the line element

$$ds^2 = \frac{1}{2\omega^2}\Big(-(dt + e^x dy)^2 + dx^2 + \frac{1}{2}e^{2x}dy^2 + dz^2\Big) \tag{42}$$

with ω constant and $t, x, y, z \in \mathbb{R}$.

(a) Show that this geometry satisfies Einstein's equation with a perfect fluid source.

Solution:

To do this, we will need to calculate the Einstein tensor in order to compare it to the stress-energy tensor for the perfect fluid. To do this, we can first simplify our calculation by defining an orthonormal basis of 1-forms from the metric i.e.

$$\sigma^T = \frac{1}{\sqrt{2}\omega}(dt + e^x dy) \tag{43}$$

$$\sigma^x = \frac{1}{\sqrt{2}\omega}dx \tag{44}$$

$$\sigma^y = \frac{1}{2\omega}e^x dy \tag{45}$$

$$\sigma^z = \frac{1}{\sqrt{2}\omega}dz \tag{46}$$

in this orthonormal basis of 1-forms, the line element becomes

$$ds^2 = -\left(\sigma^T\right)^2 + (\sigma^x)^2 + (\sigma^y)^2 + (\sigma^z)^2 \tag{47}$$

so that the (covariant) metric can be identified as

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

Using this metric and the above othonormal basis 1-forms, the Ricci curvature tensor R_{ij} can be calculated using Tevian's online front end to sage differential geometry package. The following shows how I used the Kerr geometry demo to calculate the Ricci tensor

Kerr Geometry

Initialization

Run the following code to initialize $\LaTeX$ output and load (Tevian's frontend to) the differential forms package, either by clicking on "Evaluate" or by typing Shift+Enter.

```
import urllib
url="http://oregonstate.edu/~drayt/MTH437/handouts/einstein.txt"
exec(eval(urllib.urlopen(url).read()))
Parallelism().set(nproc=8)
```

Evaluate

```
Initialization loaded
```

Help | Powered by SageMath

Code

Now enter a line element in the box below, adapting the given code as needed. First declare any parameters or functions, then provide a list of coordinates using the `MakeC` command as shown below (note the double parentheses). Finally, set the (covariant) metric to the (inverse of the) matrix relating the coordinate and orthonormal 1-form bases. The result should be the line element in tensor notation.

```
m,w=var('m,omega')
MakeC(('t','x','y','z'))
Q=M.automorphism_field()
#Qinv=matrix([[sqrt(Delta)/rho,0,0,sqrt(Delta)/rho*a*sin(theta)^2],[0,rho/sqrt(Delta),0,0],[0,0,rho,0],[a*sin(theta)/rho,0,0,sin(theta
Qinv = matrix([[(1/(sqrt(2)*w)),0,(1/(sqrt(2)*w))*exp(x),0],[0, (1/(sqrt(2)*w)),0,0],[0,0,(1/(2*w))*exp(x),0],[0,0,0,(1/(sqrt(2)*w))]]
Q[:]=Qinv.inverse()
e=XX.frame().new_frame(Q,'e')
M.set_default_frame(e)
g=M.metric('g',M._dim-2)
g[1,1],g[2,2],g[3,3],g[4,4]=-1,1,1,1
g.display(XX.frame())
```

Evaluate

$$g = -\frac{1}{2\,\omega^2}\mathrm{d}t\otimes\mathrm{d}t - \frac{e^{x}}{2\,\omega^2}\mathrm{d}t\otimes\mathrm{d}y + \frac{1}{2\,\omega^2}\mathrm{d}x\otimes\mathrm{d}x - \frac{e^{x}}{2\,\omega^2}\mathrm{d}y\otimes\mathrm{d}t - \frac{e^{(2\,x)}}{4\,\omega^2}\mathrm{d}y\otimes\mathrm{d}y + \frac{1}{2\,\omega^2}\mathrm{d}z\otimes\mathrm{d}z$$

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List the nonzero components of the (covariant) metric tensor (in a coordinate basis!).

```
g.display_comp(XX.frame())
```

Evaluate

$$
\begin{aligned}
g_{tt} &= -\frac{1}{2\,\omega^2} \\
g_{ty} &= -\frac{e^x}{2\,\omega^2} \\
g_{xx} &= \frac{1}{2\,\omega^2} \\
g_{yt} &= -\frac{e^x}{2\,\omega^2} \\
g_{yy} &= -\frac{e^{(2\,x)}}{4\,\omega^2} \\
g_{zz} &= \frac{1}{2\,\omega^2}
\end{aligned}
$$

List the nonzero Christoffel symbols (in an orthonormal frame).

```
nab=g.connection()
nab.display()
```

Evaluate

$$
\begin{aligned}
\Gamma^{1}{}_{2\,3} &= \omega \\
\Gamma^{1}{}_{3\,2} &= -\omega \\
\Gamma^{2}{}_{1\,3} &= \omega \\
\Gamma^{2}{}_{3\,1} &= \omega \\
\Gamma^{2}{}_{3\,3} &= -\sqrt{2}\omega \\
\Gamma^{3}{}_{1\,2} &= -\omega \\
\Gamma^{3}{}_{2\,1} &= -\omega \\
\Gamma^{3}{}_{2\,3} &= \sqrt{2}\omega
\end{aligned}
$$

Compute and display the components of the Ricci tensor R_{ij}. The Kerr geometry is a vacuum solution of Einstein's equation!

```
ric=g.ricci()
ric[:]
```

Evaluate

$$
\begin{pmatrix}
2\,\omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

Enter any further code you wish below.

```

```

Evaluate

As we can see from the calculation, the Ricci curvature tensor is

$$(R_{ij}) = \begin{pmatrix} 2\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{49}$$

The Einstein tensor is defined as

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \tag{50}$$

where R is the Ricci scalar which for this case is just $R = 2\omega^2$. Putting this all together, we have

$$(G_{ij}) = \begin{pmatrix} 2\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \omega^2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3\omega^2 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{pmatrix} \tag{51}$$

The stress-tensor for a perfect fluid in its rest frame is given by

$$(T_{ij}) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{52}$$

And therefore, we see that Einstein's equation

$$G_{ij} + \Lambda g_{ij} = 8\pi T_{ij} \tag{53}$$

is satisfied by the above as each of these matrices is diagonal with the three spatial components being the same.

(**b**) What are the allowed values of the cosmological constant Λ, the energy density ρ, and the pressure p? *Make reasonable physical assumptions.*

Solution:

To decide what the allowed values should be, let's write out the two equations that (53) leads to for the perfect fluid. They are

$$3\omega^2 - \Lambda = 8\pi\rho \tag{54}$$

$$-\omega^2 + \Lambda = 8\pi p \tag{55}$$

It is physically reasonable to assume a positive energy density $\rho > 0$ because stuff exists and it's moving around. It is also reasonable to expect pressure density to be positive in the sense that gravity is attractive, not repulsive... (to be honest, I'm not entirely sure here). However, if we do require that $\rho, p > 0$ then it must follow that $\Lambda > 0$ because $\omega^2 > 0$. After searching around the internet, I think that taking $\Lambda = 2\omega^2$ would lead to an interesting result, namely

$$\omega^2 = 8\pi\rho \tag{56}$$

$$\omega^2 = 8\pi p \tag{57}$$

in which case we have that ρ and p are of the same magnitude and therefore, the resulting model is neither matter dominated nor radiation dominated. I think this is the assumption made on the wikipedia entry found at `https://en.wikipedia.org/wiki/G%C3%B6del_metric` under "Einstein Tensor".

(**c**) The Godel line element can be "rewritten" in cylindrical coordinates as:

$$ds^2 = \frac{2}{\omega^2}\Big[- (dT + \sqrt{2}\sinh^2(r)d\phi)^2 + dr^2 + (\sinh^2(r) + \sinh^4(r))d\phi^2 + dz^2\Big] \tag{58}$$

with $T, r, z \in \mathbb{R}$ and $\phi \in \mathbb{S}$ (the unit circle). *You do not need to verify this equivalence!*. Consider circles with T, r, and z constant. Are these curves timelike, spacelike, or null?

Solution:

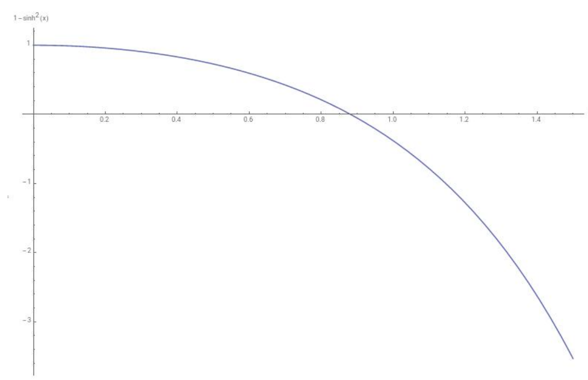
For this case, the line element simplifies to

$$\begin{aligned} ds^2 &= \frac{2}{\omega^2}\Big(- 2\sinh^4(r) + \sinh^2(r) + \sinh^4(r)\Big)d\phi^2 &(59)\\ &= \frac{2}{\omega^2}\Big(\sinh^2(r) - \sinh^4(r)\Big)d\phi^2 &(60)\end{aligned}$$

In order to classify these curves, we need to decide if $ds^2 > 0$, $ds^2 < 0$ or $ds^2 = 0$. For ds^2 to be non-trivially zero, we require

$$\sinh^2(r) = \sinh^4(r) \quad \Rightarrow \quad 1 = \sinh^2(r) \tag{61}$$

So solutions to the above equation will tell us the regions where such curves are spacelike, null, and timelike. The following graph shows what the plot of $1 - \sinh^2(r)$ looks like for $r \geq 0$. There is clearly an inner region for small r where such curves are spacelike



($ds^2 > 0$) one intersection point where the curve is null, and then (surprisingly!) curves that are completely timelike! That's really weird!

(**d**) Comment *briefly* on the implications of your result in part (c).

Solution:

We just found an infinite number of curves for which traveling in a circle i.e. ϕ going from $0 \to 2\pi$, is completely timelike. That means I just found a trajectory through spacetime that allows for travel (without going faster than the speed of light) to the past! Apparently such curves are commonly called Closed-Timelike-Curves (CTC) and this geometry which contains them was an attempt by Godel to show that GR can be inconsistent with reality and needs fixing.