Abstract Surfaces 2 **John Waczak**
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Example: Torus

Let $T^2 \subset \mathbb{R}^3$ be the torus of revolution with center $(0,0,0)$. Now let's define the antipodal map $A : T^2 \to T^2$ such that

$$(x,y,z) \mapsto (-x,-y,-z)$$

Define $K = T^2/N$ be the set of equivalence relations defined by

$$p \equiv q \in T^2 \iff q = A(p)$$

This gives us the projection map $\pi : T^2 \to K$ such that $p \mapsto [p] = \{p, A(p)\}$. Now we cover T^2 with coordinate charts $x_\alpha : U_\alpha \to T^2$ such that $x_\alpha(U_\alpha) \cap A \circ x_\alpha(U_\alpha) = \emptyset$. As before K with $\{U_\alpha, \pi \circ x_\alpha\}$ is the abstract surface called the **Klein Bottle**

Tangent Plane

So far we had defined T_pS to be $\{v \in \mathbb{R}^3 : v = \alpha'(0)\text{for some curve in S with } \alpha(0) = p\}$. We want to free ourselves from $\mathbb{R}^3$.

Define $\alpha(-\varepsilon, \varepsilon) \to \mathbb{R}^2$ be differentiable with $\alpha(0) = p$ write $\alpha'(0) = (u'(0), v'(0) = w$. Let f be differentiable, real valued function, defined in a neighborhood of p. ($f : U9(p) \to \mathbb{R}$). Restrict f to α then the directional derivative of f relative to w is given by

$$\begin{aligned}
\frac{d(f \circ \alpha)}{dt}\Big|_{t=0} &= \Big(\frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}\Big)\Big|_{t=0} \\
&= \Big\{u'(0)\frac{\partial}{\partial u}_0 + v'(0)\frac{\partial}{\partial v}_0\Big\} f
\end{aligned}$$

The directional derivative in direction w is an operator on differentiable functions which depends only on w.

Definition. *A differentiable map $\alpha : (-\varepsilon, \varepsilon) \to S$ is called a Curve on S. Assume that $\alpha(0) = p$ and let D be the set of real-valued functions on S which are differentiable at p. The* ***Tangent vector*** *to the curve α at $t = 0$ is the function $\alpha'(0) : D \to \mathbb{R}$ given by $\alpha'(0)[f] = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0}$*