Complex Variables: Assignment 3 **John Waczak**
MTH 483 Date: May 9, 2018

3.31

a.

$$\begin{aligned}
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\
&= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{-y+ix} - e^{-y-ix}}{2i} \\
&= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} \\
&= \frac{(e^{-y} - e^{y})\cos x + i(e^{-y} + e^{y})\sin x}{2i} \\
&= i\sinh y \cos x + \cosh y \sin x
\end{aligned}$$

□

b.

$$\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\
&= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \frac{e^{-y+ix} + e^{y-ix}}{2} \\
&= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2} \\
&= \frac{(e^{-y} + e^{y})\cos x + i(e^{-y} - e^{y})\sin x}{2} \\
&= \cosh y \cos x - i\sinh y \sin x
\end{aligned}$$

□

3.32

Proof. Notice that we can write $\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$. Thus since we are solving for the zeros and $2i \neq 0$, it follows that $e^{iz} = e^{-iz}$ and therefore $e^{2iz} = 1$. However since we have the periodicity of the exponential function we can write that $e^{2iz} = 1e^{0+i2\pi n}$ which gives the equation $2iz = 2i\pi n$ and therefore $z = \pi n$ where $n \in \mathbb{Z}$. Thus we have shown that all of the roots of $\sin z$ are real valued with precisely integer multiples of π. □

3.33

a. Let the set S be the line segment $z = iy$ with $0 \leq y \leq 2\pi$. Then the image of z under the exponential function is $\exp(S) = \exp(iy) = \cos(y) + i\sin(y)$. This image is the circle of radius one centered about the origin. □

b. Let the set S be the line segment $z = 1 + iy$ with $0 \le y \le 2\pi$. Then the image of this set under the exponential function is given by $\exp(S) = \exp(1 + iy) = e^1(\cos(y) + i\sin(y))$. This is just the circle of radius e around the origin. $\square$

c. Let $S = \{z = x + iy : 0 \le x \le 1, 0 \le y \le 2\pi\}$ be a rectangle. Then the image of of S under the exponential function is $\exp(S) = \exp(x + iy) = e^x(\cos(y) + i\sin(y))$. For every x we have the circle of radius e^x. Thus this image is the union of all such circles letting x go from 0 to 1 i.e. the closed disk of radius e centered at the origin. $\square$

3.40

Recall that the principal value of a^b with $a, b \in \mathbb{C}$ is defined as $a^b = \exp(bLog(a))$ where $Log(z)$ denotes the principal branch of the logarithm. Using this we have the following:

a.

$$\begin{aligned} Log(2i) &= \exp(1 \cdot Log(Log(2i))) \\ &= Log(2i) \\ &= \ln(2) + i\pi/2 \end{aligned}$$

$\square$

b.

$$\begin{aligned} (-1)^i &= \exp(iLog(-1)) \\ &= \exp(i(\ln(1) + i\pi)) \\ &= e^{-\pi} \approx 0.0432139 \end{aligned}$$

$\square$

c.

$$\begin{aligned} Log(-1 + i) &= \exp(Log(Log(-1 + i))) \\ &= Log(-1 + i) \\ &= \ln(\sqrt{2}) + i\frac{3\pi}{4} \end{aligned}$$

$\square$

3.41

a.

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 \end{aligned}$$

$\square$

b.

$$e^{\pi} = e^{\pi}$$

as it is a real number and so is already in the correct form

$\square$

c.

$$\begin{aligned} i^i &= \exp(iLog(i)) \\ &= \exp(i(\ln(1) + i\pi/2)) \\ &= \exp(-\pi/2) \end{aligned}$$

□

d.

$$\begin{aligned} e^{\sin i} &= e^{\frac{e^{ii} - e^{-ii}}{2i}} \\ &= e^{\sinh(1)i} \\ &= e^{\sinh(1)}e^{i} \\ &= e^{\sinh(1)}(\cos(1) + i\sin(1)) \\ &= e^{\sinh(1)}\cos(1) + ie^{\sinh(1)}\sin(1) \end{aligned}$$

□

e.

$$\begin{aligned} \exp(Log(3+4i)) &= \exp(\ln(5) + i\arctan(4/3)) \\ &= 5e^{i\arctan(4/3)} \\ &= 5(\cos\arctan(4/3) + i\sin\arctan(4/3)) \\ &= 5(3/5 + i4/5) \\ &= 3 + 5i \text{ as expected since } \exp(Log(z)) = z \end{aligned}$$

□

f.

$$\begin{aligned} (1+i)^{1/2} &= \exp(1/2Log(1+i)) \\ &= \exp(1/2(\ln(\sqrt{2}) + i\pi/4)) \\ &= 2^{1/4}e^{i\pi/8} \\ &= 2^{1/4}(\cos\pi/8 + i\sin\pi/8) \end{aligned}$$

□

4.1

Recall that the length of a curve $\gamma(t)$ is given as $\int |\gamma'(t)|dt$. Thus we have the following:

b.

$$\begin{aligned}\gamma(t) &= (-1-i) + (2i - (-1-i))t \\ &= -1 + t - i + 3it \\ \gamma'(t) &= 1 + 3i \\ \Rightarrow \text{length}(\gamma(t)) &= \int_0^1 \sqrt{(1+3i)(1-3i)}dt \\ &= \sqrt{10} \cdot 1 \\ &= \sqrt{10}\end{aligned}$$

□

c. Top half of circle $C[0,34]$.

$$\begin{aligned}\gamma(t) &= 34e^{it} \\ \gamma'(t) &= 34ie^{it} \\ \Rightarrow \text{length}(\gamma(t)) &= \int_0^\pi \sqrt{\big(34ie^{it}\big)\big(-34ie^{-it}\big)}dt \\ &= \int_0^\pi 34dt \\ &= 34\pi \text{ which is exactly half of the circumference of the circle}\end{aligned}$$

□

4.4

Proof. Recall Cauchy's integral forumla which states that

$$f(w) = \frac{1}{2\pi i} \oint\limits_{C[w,R]} \frac{f(z)}{z-w} dz$$

. whenever $f(z)$ is holomorphic on $\overline{D}[w,R]$. Thus choosing $f(z) = 1$ and $w = 0$ gives us that:

$$\begin{aligned}1 &= \frac{1}{2\pi i} \oint\limits_{C[0,1]} \frac{dz}{z} \\ \Rightarrow \oint\limits_{C[0,1]} \frac{dz}{z} &= 2\pi i\end{aligned}$$

Because we have chosen $f(z)$ to be a constant function for all $z \in \mathbb{C}$ then $f(w) = 1 \ \forall w$. This means that reapplying the theorem gives

$$\begin{aligned}1 &= \frac{1}{2\pi i} \oint\limits_{C[w,R]} \frac{dz}{z-w} \\ \Rightarrow \oint\limits_{C[w,R]} \frac{dz}{z-w} &= 2\pi i\end{aligned}$$

□

4.5

b. Observe that $f(z) = z^2 - 2z + 3$ is a holomorphic function as it is the addition of monomials which are holomorphic. Furthermore the curve of integration $\gamma = C[0,2]$ is contractible. Thus by *Corollary 4.20* we have that $\int_\gamma f(z)dz = 0$. $\square$

d. Since we have that $f(z) = xy$ we can verify with the C-R equations that this function is not everywhere differentiable. Thus we can not safely apply *Corollary 4.20*. Instead we can perform the integration by recognizing that in general $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$. Using this we have that:

$$\begin{aligned}
\gamma(t) &= \sqrt{2}\cos t + i\sqrt{2}\sin t \\
\gamma'(t) &= -\sqrt{2}\sin t + i\sqrt{2}\cos t \\
\Rightarrow \int_\gamma f(z)dz &= \int_0^{2\pi} \left(\sqrt{2}\cos t\sqrt{2}\sin t\right)\Big[-\sqrt{2}\sin t + i\sqrt{2}\cos t\Big]dt \\
&= 2\sqrt{2}\Big[\int_0^{2\pi} \cos t\sin^2 dt + i\int_0^{2\pi} \cos^2 t\sin t dt\Big] \\
&= 2\sqrt{2}(0+0) \\
&= 0
\end{aligned}$$

$\square$

4.6

a. γ is the line segment from 0 to $1-i$. Thus we have that:

$$\begin{aligned}
\gamma(t) &= 0 + (1 - i - 0)t \\
&= (1-i)t \\
\gamma'(t) &= (1-i) \\
\int_\gamma x dz &= \int_0^1 t(1-i)dt = \frac{1}{2}(1-i) \\
\int_\gamma y dz &= \int_0^1 -it(1-i)dt = \frac{-i}{2}(1-i) \\
\Rightarrow \int_\gamma z dz &= \int_\gamma x dz + i\int_\gamma y dz \\
&= \frac{1}{2}(1-i) + i\frac{-i}{2}(1-i) = (i-1) \\
\int_\gamma \overline{z} dz &= \int_\gamma x dz - i\int_\gamma y dz \\
&= \frac{1}{2}(1-i) - i\frac{-i}{2}(1-i) \\
&= \frac{1}{2}(1-i) - \frac{1}{2}(1-i) = 0
\end{aligned}$$

$\square$

c. γ is $C[a,r]$. Let $a = a_x + ia_y$. Then,

$$\begin{aligned}
\gamma(t) &= \sqrt{r}\cos t + a_x + i(\sqrt{r}\sin t + a_y) \\
\gamma'(t) &= -\sqrt{r}\sin t + i\sqrt{r}\cos t \\
\int_\gamma x dz &= \int_0^{2\pi} \Big(\sqrt{r}\cos t + a_x\Big)\Big(-\sqrt{r}\sin t + i\sqrt{r}\cos t\Big) \\
&= r\int_0^{2\pi} \cos^2 t dt = \pi r \\
\int_\gamma y dz &= \int_0^{2\pi} \Big(i\sqrt{r}\sin t + ia_y\Big)\Big(-\sqrt{r}\sin t + i\sqrt{r}\cos t\Big) \\
&= -ir\int_0^{2\pi} \sin^2 t dt = -i\pi r \\
\Rightarrow \int_\gamma z dz &= \int_\gamma x dz + i\int_\gamma y dz \\
&= 2\pi r \\
\int_\gamma \overline{z} dz &= \int_\gamma x dz - i\int_\gamma y dz \\
&= \pi r - \pi r = 0
\end{aligned}$$

where I have used the fact that $\sin t, \cos t, \sin t\cos t$ all integrate to zero over a full period of $[0, 2\pi]$. $\square$

4.7

a. γ is a line segment from 1 to i. Then we have that:

$$\begin{aligned}
\gamma(t) &= 1 + (i-1)t \\
&= 1 - t + it \\
\gamma'(t) &= (i-1) \\
\int_\gamma \exp(3z) dz &= \int_0^1 e^{3(1+(i-1)t)}(i-1)dt \\
&= (i-1)e^3\int_0^3 e^{3(i-1)t}dt \\
&= \frac{1}{3}e^3\Big[e^{3(i-1)t}\Big]_0^1 \\
&= \frac{1}{3}e^3\Big[e^{3(i-1)} - 1\Big]
\end{aligned}$$

$\square$

b. γ is $C[0,3]$. This means that $\gamma(t) = \sqrt{3}\cos t + i\sqrt{3}\sin t$. Notice that $f(z) = \exp(3z)$ is holomorphic because it is the composition of two holomorphic functions $\exp(z)$ and $3z$. γ is contractible and thus we have that by *Corollary 4.20*

$$\int_\gamma \exp(3z) dz = 0$$

$\square$

c. $y = x^2$ is our curve which we are integrating over from $x = 0$ to $x = 1$.

$$\begin{aligned} \gamma(t) &= t + it^2 \\ \gamma'(t) &= 1 + 2it \\ \Rightarrow \int_\gamma \exp(3z)dz &= \int_0^1 e^{3(t+it^2)}(1+2it)dt \\ &= \frac{1}{3}e^{3(t+it^2)}\Big|_0^1 \\ &= \frac{1}{3}e^{3+3i} - \frac{1}{3} \end{aligned}$$

$\square$