Complex Analysis: Day 18 **John Waczak**
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More power series stuff

If $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ has radius of convergence R, then it defines a holomorphic function on the disc $|z-c| < R$. (If $R = +\infty$ then the function is entire). Moreover:

$$f'(z) = \sum_{n=1}^{\infty} a_n n(z-c)^{n-1}$$

which has the same radius of convergence $\sqrt[n]{n} \to 1$ as $n \to \infty$. Thus by reapplication of the theorem to subsequent derivatives we have that $f(z)$ has infinitely many derivatives.

A useful formula is:

$$\begin{aligned}
a_n &= \frac{f^{(n)}(c)}{n!} \\
f(z) &= \sum_{k=0}^{\infty} a_k(z-c)^k \\
f'(z) &= \sum_{k=1}^{\infty} a_k(k)(z-c)^{k-1} \\
&\vdots \\
f^{(n)}(z) &= \sum_{k=n}^{\infty} a_k k(k-1)(k-2)...(k-(n-1))(z-c)^{k-n}
\end{aligned}$$

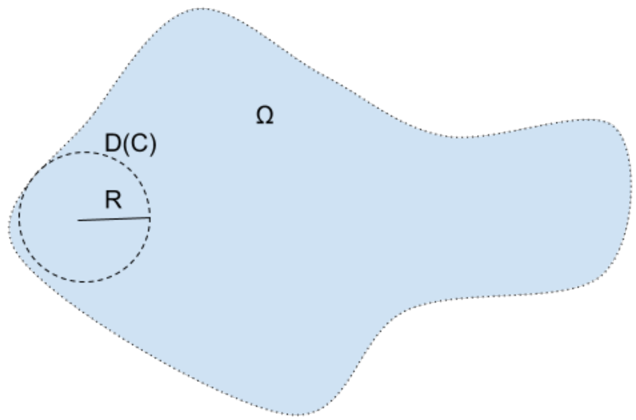
So it seems like functions that can be written as power-series are infinitely better than just our regular holomorphic functions.

Theorem. *Let $f : \Omega \to \mathbb{C}$ be holomorphic and let $c \in \Omega$, $R > 0$ for which $D_R(c) \subseteq \Omega$. Then f can be represented as a power series:*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$$

for all $z \in D_R(c)$. Moreover, the radius of convergence of the power series is $\geq R$. Finally, the coefficient a_n is given by $a_n = \frac{f^{(n)}(c)}{n!}$.

You might ask what is the difference between a Taylor and power series? A power series is a function defined by the infinite sum. A Taylor series is when we start with a function and then define the a_n's in the sum. For us they are essentially the same thing.



Proof. Main ingredient: Cauchy's integral formula. Take $c = 0$ for simplicity. Given $z \in D_R(0)$, define r so that $r = \frac{|z|+R}{2}$ and $D_r(0) \subseteq D_R(0)$. By CIF we have that:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w - z} dw \\
&= \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w(1 - \frac{z}{w})} dw \\
\text{now } |z/w| &= |z|/r < 1 \\
&= \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k dw \\
&= \sum_{k=0}^{\infty} \left\{ \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(w)}{w^{k+1}} dw \right\} z^k \\
\text{let } a_k &= \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w^{k+1}} dw
\end{aligned}$$

□

Theorem (Corollary)**.** *Let $f : \Omega \to \mathbb{C}$ be holomorphic and let $c \in \Omega$, then the power series for $f(z)$ given by*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (z - c)^n$$

has radius of convergence at least the distance from c to the nearest point on the boundary of Ω.