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Measurements

Given $A|\phi_n\rangle = a_n|\phi_n\rangle$; $\{|\phi_n\rangle\}$
If our system is non-degenerate then our possible measurements are the eigenvalues $\{a_n\}$. Mathematically, we say that the way this works is via projection, i.e.

$$P_n = |\phi_n\rangle\langle\phi_n| \tag{1}$$
$$P_n|\phi_n\rangle = \langle\phi_n|\psi\rangle|\phi_n\rangle \tag{2}$$

Example (L_x operator)**.**

$$L_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$L_z \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Our eigenvalues are given by the roots of the characteristic polynomial:

$$det[L_x - \lambda\mathbf{I}] = 0$$
$$-\lambda(\lambda^2 - \frac{1}{2}) - \frac{1}{\sqrt{2}}(-\frac{\lambda}{\sqrt{2}}) = 0$$
$$\Rightarrow \lambda \in \{0, \pm 1\}$$

Then we put these back into the matrix to solve for the Eigenvectors. Some linear algebra we can get something like:

$$|L_x = 0\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Example (L_z^2 operator)**.**

$$L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus our characteristic equation is:

$$-\lambda(1-\lambda)^2 = 0$$
$$\rightarrow \lambda = 0 \textit{ and } \lambda = 1 \textit{ with 2 fold degeneracy}$$

Our $\lambda = 0$ eigenvector comes from:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Unfortunately the second solution leads to degeneracy... how do we solve it?

Degeneracy

Given some a_n degenerate, we write:

$$A|\phi_n^i\rangle = a_n|\phi_n^i\rangle$$

And therefore we have:

$$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i |\phi_n^i\rangle ; c_n^i = \langle \phi_n^i | \phi_n^i \rangle \tag{3}$$

Then our probabilities change as well!

$$P(a_n) = \sum_{i=1}^{g_n} |c_n^i|^2 \tag{4}$$

$$P_n = \sum_{i=1}^{g_n} |\phi_n^i\rangle\langle\phi_n^i| \tag{5}$$

Previously, when we made a measurement, we knew what our final state would be with certainty. Now, our operator projects from a larger space to a smaller subspace. If $|\phi_i\rangle$ and $|\phi_j\rangle$ are degenerate, how do you know which you have projected onto? To figure this out, we need to look at our initial state. Say we have:

$$|\psi\rangle = \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2}}|\phi_n^1\rangle + \frac{\beta}{\sqrt{|\alpha|^2 + |\beta|^2}}|\phi_n^2\rangle$$

We can only determine the α, β given $|\psi\rangle$. i.e.

$$\alpha = \langle \phi_n^1 | \psi \rangle \;\; \beta = \langle \phi_n^1 | \psi \rangle$$

For the case of L_z^2 we had some degeneracy for the $\lambda = 1$ eigenvalue. Using $\lambda = 0$ gave

$$|L_z^2 = 0\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For the degenerate case we have:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{0}$$

We are constrained to have $c_2 = 0$ but we are free to make a choice for c_1, c_3. This allows us to pick nice choices for the subspace so long as we obey orhtonormality and completeness.

Phases

Given $|\psi\rangle$ and $|\psi'\rangle = e^{i\phi}|\psi\rangle$ are these physically equivalent? We say **yes** because $e^{i\phi}$ is an **overall** phase and is not measurable. Whenever we *measure* we will have $e^{i\phi}e^{-i\phi}$ which will disappear.

Alternatively if we have a state $|\psi\rangle = \lambda_1 e^{i\phi_1}|\psi_1\rangle + \lambda_2 e^{i\phi_2}$ has a **relative** phase $e^{i(\theta_2-\theta_1)}$ which will not disappear when we perform measurements on the state.