Homework 5 **John Waczak**
MTH 443 Date: May 7, 2019
Dr. Schmidt Worked with: Garrett Jepson

1.) Let $\mathcal{B} = (b_1, ..., b_n)$ be an n-tuple of elements of $\mathbb{F}^n$. Let $M \in \mathcal{M}_n(\mathbb{F})$ be the matrix whose j-th column is b_j. Show that $\mathcal{B}$ is an ordered basis of $\mathbb{F}^n$ if and only if $\det(M) \neq 0$.

($\rightarrow$) Assume that $\mathcal{B}$ is an ordered basis. We must show that $\det(M) \neq 0$. As $\mathcal{B}$ is an ordered basis, its elements are linearly independent. That is no column of the matrix M whose columns are $b_j \in \mathcal{B}$ can be expressed as a linear combination of the other columns of M. This means that M reduced to the identity matrix. In order to use this information to calculate the determinant, we must recall the following theorems from the text:

Theorem 4.5 If $A \in \mathcal{M}_n(\mathbb{F})$ and B is a matrix obtained by switching any two rows of a, then

$$\det(B) = -\det(A)$$

Theorem 4.6 If $A \in \mathcal{M}_n(\mathbb{F})$ and B is a matrix obtained by adding a multiple of one row of A to another row of A. Then,

$$\det(B) = \det(A)$$

Therefore, because $\mathcal{B}$ is a basis, the columns of M are linearly independent and the matrix can therefore be row reduced in, say, k moves such that

$$\begin{aligned} det(M) &= (-1)^k \det(\mathrm{Id}_{\mathbb{F}}) \\ &= (-1)^k \cdot 1 \neq 0 \end{aligned}$$

From this we can see that if $\mathcal{B}$ is an ordered basis then $\det(M) \neq 0$.

($\leftarrow$) Assume that $\det(M) \neq 0$. Now, assume for contradiction that the $b_j \in \mathcal{B}$ do not form an ordered basis. It must be true that $\exists i \in \{1, ..., n\}$ such that b_i is a linear combination of the vectors of a subset of $\mathcal{B}$. Recall the following theorem from the text:
Theorem 4.8 For any $A \in \mathcal{M}_n(\mathbb{F})$, $\det(A^t) = \det(A)$.

Consider the matrix M^t in which the previous $b_j \in \mathcal{B}$ columns of M have become rows. For this new matrix M^t, we have that row b_i^t is a linear combination of some number of other rows. However, in class we proved that the determinant is zero if any two rows are linearly dependent. Thus, we have that

$$\det M^t = 0 = \det M$$

This contradicts the hypothesis and therefore we have that if $\det(M) \neq 0$, then $\mathcal{B}$ is an ordered basis. This completes the proof. $\square$

2.) Let V be an $\mathbb{R}$-vector space of dimension 2 and let T be a linear operator on V. Suppose $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$, for some basis $\mathcal{B}$. Determine all T-invariant subspaces of V.

Recall that a subspace W of V is called T-invariant if for all $w \in W$, $T(w) \in W$. From class, we saw that $\{0_V\}$ and V are certainly T-invariant subsapces for any linear operator as $T(0_V) = 0_V$ and $T(v) \in V$ by definition of a linear operator. Certainly if V has dimension 2 there can be no other 2-dimensional subspace than V itself as any such space must also contain it's span. Thus, the only other possible T-invariant subspaces must have dimension 1. To solve for such subspaces, consider a vector v such that span(v) is a T invariant subspace. In order for this to work, T must send v to span(v), that is,

$$T(v) = \lambda v$$

That is, the other T-invariant subspaces must be the eigenspaces of T. The matrix $[T]_{\mathcal{B}}$ has a characteristic polynomial

$$P_T(x) = (1 - x)(2 - x) + 2$$

Setting this to zero gives

$$\begin{aligned}(1 - x)(2 - x) + 2 &= 2 - x - 2x + x^2 + 2 \\ &= x^2 - 3x + 4 = 0\end{aligned}$$

Thus the eigenvalues of this operator are

$$\begin{aligned}\lambda_{1,2} &= \frac{3 \pm \sqrt{9 - 16}}{2} \\ &= \frac{3 \pm i\sqrt{7}}{2}\end{aligned}$$

Therefore, we can see that there can't be any such T-invariant subspaces of V because there are no real eigenvalues of T. Such complex eigenvalues would correspond to vectors in $\mathbb{C}^2$ to which our $\mathbb{R}$-vector space V is not isomorphic. Perhaps such subspaces could be allowed if we had V was a $\mathbb{C}$-vector space.

The only other T-invariant subspaces we have encountered before are the range $T(V)$ and the kernel $\ker(V)$. One can easily verify that $\ker(T) = \{0_V\}$ because the columns of $[T]_{\mathcal{B}}$ are linearly independent. By rank nullity, we have that the dimension of the image is 2 and therefore must also span all of V. Thus we can conclude that the only two T-invariant subspaces of V are $\{0_V\}$ and V.

3.) (543)

4.) Give an example of a continuous function $v : \mathbb{R} \to \mathbb{R}^3$ such that $v(t_1), v(t_2), v(t_3)$ form an $\mathbb{R}$-basis for $\mathbb{R}^3$ whenever t_1, t_2, t_3 are distinct points of $\mathbb{R}$.

First, let's consider possible solutions for a function $f : \mathbb{R} \to \mathbb{R}^2$ with similar properties to aid in our construction. Certainly we can not have a constant function as $f(t_1) = f(t_2)$ $\forall t_1 \neq t_2 \in \mathbb{R}$. Thus we can check the naive next step:

$$f(t) = (1, t)^t$$

In order for pairs of t_1, t_2 to form a basis for $\mathbb{R}^2$ we need that $det[f(t_1)\ f(t_2)] \neq 0$ Fortunately this determinant is

$$\det \begin{pmatrix} 1 & 1 \\ t_1 & t_2 \end{pmatrix} = t_1 - t_2$$

which certainly isn't zero if t_1, t_2 are distinct. Thus we have that for any two points of $\mathbb{R}$ f applied to these points forms a basis for $\mathbb{R}^2$.

Based on this information we can consider how we should add a third component try and extend the property of the function f to $\mathbb{R}^3$. One simple way could be to increase the final component for successive values of t. Thus we shall consider the function

$$v(t) = (1, t, e^t)^t$$

Certainly this function $v(t)$ is continuous as each of its components consists of a continuous function. If we take $t_1, t_2, t_3 \in \mathbb{R}$ to be three distinct points, we must have that

$$\det[v(t_1)\ v(t_2)\ v(t_3)] \neq 0$$

expanding this out gives

$$t_2 e^{t_3} - e^{t_2} t_3 - t_1 e^{t_3} + e^{t_1} t_3 + t_1 e^{t_2} - e^{t_1} t_2 \neq 0$$

The functions t, e^t are both strictly increasing and $e^t > t$ $\forall t \in \mathbb{R}$. It is not clear to me how to show that this determinant function is always nonzero but I think it is sufficient to show how the function behaves for an example from each of $t_1 < t_2 < t_3 < 0$, $\quad t_1 < 0$, $t_3 > 0$ and $t_1 < t_2 < t_3$, and lastly $0 < t_1 < t_2 < t_3$ as the behavior of the component functions is well understood.

For the case of $t_1 = -10$, $t_2 = -5$, $t_3 = -1$ we have that the determinant is

$$det[v(t_1)\ \ v(t_2)\ \ v(t_3)] = \frac{4 - 9e^5 + 5e^9}{e^{10}} \neq 0$$

For the case of $t_1 = -1$, $t_2 = 0$, $t_3 = 1$ we have

$$det[v(t_1)\ \ v(t_2)\ \ v(t_3)] = -2 + \frac{1}{e} + e \neq 0$$

and finally for the case of $t_1 = 1$, $t_2 = 10$, $t_3 = 20$ we have that

$$det[v(t_1)\ \ v(t_2)\ \ v(t_3)] = e(10 - 19e^9 + 9e^{19}) \neq 0$$

Therefore, I believe I have found an example of such a function.