Complex Analysis: Day 22 **John Waczak**
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Heading towards Residues

Recall that last time we defined 3 types of singularities

1. Removable
2. Pole
3. Essential

Removable singularities mean it's possible to find an alternative function that agrees with f and is defined at the singularity. A *pole* had $\lim_{z\to z_0} = +\infty$. Essential was neither removable nor a pole.

Theorem. *If $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and there exists $r > 0$ for which $f(z)$ is bounded on $D_r(z_0) \setminus \{z_0\}$, then the singularity is removable. Using this we can show the following*

Proposition. *If $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, then f has a pole at z_0 if and only f $\frac{1}{f(z_0)}$ has a zero when $f(z)$ has a singularity.*

Example $f(z) = \frac{z^2}{(z-1)(z+2)}$ $g(z) = \frac{(z-1)(z-2)}{z^2}$
f(z) has zero at $z = 0$ and poles at $z = 1, z = -2$.

Proposition. *Let $f : \Omega \setminus \{a\} \to \mathbb{C}$ be holomorphic w/ a pole at a of multiplicity $m \geq 1$. Then there exists $r > 0$ such that for all $z \in D_r(a) \setminus \{a\}$ we have that:*

$$f(z)\frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + ... + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + ... = \sum_{k=-m}^{\infty} a_k(z-a)^k \tag{1}$$