

Homework 5

MTH 443

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Date: April 1, 2019

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1.) Let $\mathcal{B} = (b_1, \dots, b_n)$ be an n -tuple of elements of \mathbb{F}^n . Let $M \in \mathcal{M}_n(\mathbb{F})$ be the matrix whose j -th column is b_j . Show that \mathcal{B} is an ordered basis of \mathbb{F}^n if and only if $\det(M) \neq 0$.

(\rightarrow) Assume that \mathcal{B} is an ordered basis. We must show that $\det(M) \neq 0$. As \mathcal{B} is an ordered basis, its elements are linearly independent. That is no column of the matrix M whose columns are $b_j \in \mathcal{B}$ can be expressed as a linear combination of the other columns of M . This means that M reduced to the identity matrix. In order to use this information to calculate the determinant, we must recall the following theorems from the text:

Theorem 4.5 If $A \in \mathcal{M}_n(\mathbb{F})$ and B is a matrix obtained by switching any two rows of A , then

$$\det(B) = -\det(A)$$

Theorem 4.6 If $A \in \mathcal{M}_n(\mathbb{F})$ and B is a matrix obtained by adding a multiple of one row of A to another row of A . Then,

$$\det(B) = \det(A)$$

Therefore, because \mathcal{B} is a basis, the columns of M are linearly independent and the matrix can therefore be row reduced in, say, k moves such that

$$\begin{aligned} \det(M) &= (-1)^k \det(\text{Id}_{\mathbb{F}}) \\ &= (-1)^k \cdot 1 \neq 0 \end{aligned}$$

From this we can see that if \mathcal{B} is an ordered basis then $\det(M) \neq 0$.

(\leftarrow) Assume that $\det(M) \neq 0$. Now, assume for contradiction that the $b_j \in \mathcal{B}$ do not form an ordered basis. It must be true that $\exists i \in \{1, \dots, n\}$ such that b_i is a linear combination of the vectors of a subset of \mathcal{B} . Recall the following theorem from the text:

Theorem 4.8 For any $A \in \mathcal{M}_n(\mathbb{F})$, $\det(A^t) = \det(A)$.

Consider the matrix M^t in which the previous $b_j \in \mathcal{B}$ columns of M have become rows. For this new matrix M^t , we have that row b_i^t is a linear combination of some number of other rows. However, in class we proved that the determinant is zero if any two rows are linearly dependent. Thus, we have that

$$\det M^t = 0 = \det M$$

This contradicts the hypothesis and therefore we have that if $\det(M) \neq 0$, then \mathcal{B} is an ordered basis. This completes the proof. \square

- 2.) Let V be an \mathbb{R} -vector space of dimension 2 and let T be a linear operator on V . Suppose $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$, for some basis \mathcal{B} . Determine all T -invariant subspaces of V .

Recall that a subspace W of V is called T -invariant if for all $w \in W$, $T(w) \in W$. From class, we saw that $\{0_V\}$ and V are certainly T -invariant subspaces for any linear operator as $T(0_V) = 0_V$ and $T(v) \in V$ by definition of a linear operator. Certainly if V has dimension 2 there can be no other 2-dimensional subspace than V itself as any such space must also contain its span. Thus, the only other possible T -invariant subspaces must have dimension 1. To solve for such subspaces, consider a vector v such that $\text{span}(v)$ is a T invariant subspace. In order for this to work, T must send v to $\text{span}(v)$, that is,

$$T(v) = \lambda v$$

That is, the other T -invariant subspaces must be the eigenspaces of T . The matrix $[T]_{\mathcal{B}}$ has a characteristic polynomial

$$P_T(x) = (1 - x)(2 - x) + 2$$

Setting this to zero gives

$$\begin{aligned} (1 - x)(2 - x) + 2 &= 2 - x - 2x + x^2 + 2 \\ &= x^2 - 3x + 4 = 0 \end{aligned}$$

Thus the eigenvalues of this operator are

$$\begin{aligned} \lambda_{1,2} &= \frac{3 \pm \sqrt{9 - 16}}{2} \\ &= \frac{3 \pm i\sqrt{7}}{2} \end{aligned}$$

Therefore, we can see that there can't be any such T -invariant subspaces of V because there are no real eigenvalues of T . Such complex eigenvalues would correspond to vectors in \mathbb{C}^2 to which our \mathbb{R} -vector space V is not isomorphic. Perhaps such subspaces could be allowed if we had V was a \mathbb{C} -vector space.

The only other T -invariant subspaces we have encountered before are the range $T(V)$ and the kernel $\ker(T)$. One can easily verify that $\ker(T) = \{0_V\}$ because the columns of $[T]_{\mathcal{B}}$ are linearly independent. By rank nullity, we have that the dimension of the image is 2 and therefore must also span all of V . Thus we can conclude that the only two T -invariant subspaces of V are $\{0_V\}$ and V .

- 3.) (543)

- 4.) Give an example of a continuous function $v : \mathbb{R} \rightarrow \mathbb{R}^3$ such that $v(t_1), v(t_2), v(t_3)$ form an \mathbb{R} -basis for \mathbb{R}^3 whenever t_1, t_2, t_3 are distinct points of \mathbb{R} .

First, let's consider possible solutions for a function $f : \mathbb{R} \rightarrow \mathbb{R}^2$ with similar properties to aid in our construction. Certainly we can not have a constant function as $f(t_1) = f(t_2) \forall t_1 \neq t_2 \in \mathbb{R}$. Thus we can check the naive next step:

$$f(t) = (1, t)^t$$

In order for pairs of t_1, t_2 to form a basis for \mathbb{R}^2 we need that $\det[f(t_1) \ f(t_2)] \neq 0$. Fortunately this determinant is

$$\det \begin{pmatrix} 1 & 1 \\ t_1 & t_2 \end{pmatrix} = t_1 - t_2$$

which certainly isn't zero if t_1, t_2 are distinct. Thus we have that for any two points of \mathbb{R} f applied to these points forms a basis for \mathbb{R}^2 .

Based on this information we can consider how we should add a third component try and extend the property of the function f to \mathbb{R}^3 . One simple way could be to increase the final component for successive values of t . Thus we shall consider the function

$$v(t) = (1, t, e^t)^t$$

Certainly this function $v(t)$ is continuous as each of its components consists of a continuous function. If we take $t_1, t_2, t_3 \in \mathbb{R}$ to be three distinct points, we must have that

$$\det[v(t_1) \ v(t_2) \ v(t_3)] \neq 0$$

expanding this out gives

$$t_2 e^{t_3} - e^{t_2} t_3 - t_1 e^{t_3} + e^{t_1} t_3 + t_1 e^{t_2} - e^{t_1} t_2 \neq 0$$

The functions t, e^t are both strictly increasing and $e^t > t \forall t \in \mathbb{R}$. It is not clear to me how to show that this determinant function is always nonzero but I think it is sufficient to show how the function behaves for an example from each of $t_1 < t_2 < t_3 < 0$, $t_1 < 0, t_3 > 0$ and $t_1 < t_2 < t_3$, and lastly $0 < t_1 < t_2 < t_3$ as the behavior of the component functions is well understood.

For the case of $t_1 = -10, t_2 = -5, t_3 = -1$ we have that the determinant is

$$\det[v(t_1) \ v(t_2) \ v(t_3)] = \frac{4 - 9e^5 + 5e^9}{e^{10}} \neq 0$$

For the case of $t_1 = -1, t_2 = 0, t_3 = 1$ we have

$$\det[v(t_1) \ v(t_2) \ v(t_3)] = -2 + \frac{1}{e} + e \neq 0$$

and finally for the case of $t_1 = 1, t_2 = 10, t_3 = 20$ we have that

$$\det[v(t_1) \ v(t_2) \ v(t_3)] = e(10 - 19e^9 + 9e^{19}) \neq 0$$

Therefore, I believe I have found an example of such a function.