Abstract Surfaces **John Waczak**
MTH 435 Date: May 31, 2018
Dr. Christine Escher

Problem 1

Introduce a metric on the projective plane $\mathbb{R}P^2$ so that the natural projection $\pi : S^2 \to \mathbb{R}P^2$ is a local isometry. What is the Gaussian curvature of such a metric?

Proof. Recall that the natural projection $\pi : S^2 \to \mathbb{R}P^2$ is defined such that $\forall p \in S^2$, $\pi(p) = [p] = \{p, A(p)\}$ where $A : S^2 \to S^2$ is the antipodal map. We want to choose a metric $\langle , \rangle$ for $\mathbb{R}P^2$ such that π is an isometry. This mean that for $p \in S^2$ and $\forall x, y \in T_pS^2$ we need

$$\left\langle x, y \right\rangle_p = \left\langle d\pi(x), d\pi(y) \right\rangle_{\pi(p)}$$

Recall that $d\pi : T_pS^2 \to T_{\pi(p)}\mathbb{R}P^2$ is a linear transformation. We also showed in class that specific charts $X_i : U \subset \mathbb{R}^2 \to S^2$ induce an associated basis on the tangent space. Thus, let $p \in S^2$ such that $p = X_i(u, v)$ for some $u, v \in U$ and let $q \in \mathbb{R}P^2$ such that $q = \pi(p)$. Then $\forall a, b \in T_{\pi(p)}\mathbb{R}P^2$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, we can write

$$\begin{aligned} a &= \alpha\Big(\pi \circ X_i(u, v)\Big)_u + \beta\Big(\pi \circ X_i(u, v)\Big)_v \\ b &= \gamma\Big(\pi \circ X_i(u, v)\Big)_u + \delta\Big(\pi \circ X_i(u, v)\Big)_v \end{aligned}$$

Where I have written $\pi(p)_u, \pi(p)_v$ to denote the associated basis for the tangent space. Now define $a', b' \in T_{p=X_i(u,v)}S^2$ such that

$$\begin{aligned} a' &= \alpha\Big(X_i(u, v)\Big)_u + \beta\Big(X_i(u, v)\Big)_v \\ b' &= \gamma\Big(X_i(u, v)\Big)_u + \delta\Big(X_i(u, v)\Big)_v \end{aligned}$$

Where a', b' are the points in the tangent space associated with the scaling factors $\{\alpha, \beta, \gamma\delta\}$. From the linearity of $d\pi$ we can write

$$\begin{aligned} d\pi(a') &= d\pi\Big[\alpha\Big(X_i(u, v)\Big)_u + \beta\Big(X_i(u, v)\Big)_v\Big] \\ &= \alpha d\pi\Big(X_i(u, v)\Big)_u + \beta d\pi\Big(X_i(u, v)\Big)_v \\ &= \alpha\Big(\pi \circ X_i(u, v)\Big)_u + \beta\Big(\pi \circ X_i(u, v)\Big)_v = a \end{aligned}$$

So $d\pi(a') = a$ and repeating this process for b' gives $d\pi(b') = b$. Therefore, to induce an isometry, choose a metric $\langle , \rangle$ such that

$$\langle a, b \rangle_{\pi(p)} \equiv \langle a', b' \rangle_p$$

Then $\forall x, y \in T_pS^2$ we have that

$$\left\langle x, y \right\rangle_p = \left\langle d\pi(x), d\pi(y) \right\rangle_{\pi(p)}$$

Because our choice of metric on $\mathbb{R}P^2$ induces an isometry with S^2 then the Gaussian curvature of S^2 is preserved by the isometry π. Thus the curvature of $\mathbb{R}P^2$ under our metric must be 1. $\square$

Problem 2 (The Infinite Mobius Strip)

Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be a cylinder and $A : C \to C$ be the map (antipodal map) such that $A(x, y, z) = (-x, -y, -z)$. Let M be the quotient of C by the equivalence relation $p \sim A(p)$ and let $\pi : C \to M$ be the map $\pi(p) = [p] = \{p, A(p)\}$, $p \in C$.

(a) *Show that M can be given a differentiable structure so that π is a local diffeomorphism (M is then called the infinite Mobius Strip)*

(b) *Prove that M is non-orientable*

(c) *Introduce a Riemannian metric on M so that π is a local isometry. What is the curvature of such a metric.*

(a). Recall that a **differentiable structure** is the family of open subsets of $\mathbb{R}^2$ denoted U_α together with coordinate charts on those open subsets that take $U_\alpha \to S$. These charts are denoted X_α and are referred to collectively as $\{U_\alpha, X_\alpha\}$.

We need to show that M can be given a differentiable structure $(U_\alpha, \pi \circ X_\alpha)$ so that π is a local diffeomorphism. To be a local diffeomorphism we need the image $\pi \circ X_\alpha(U_\alpha)$ to be open in M for all α and that $\pi \circ X_\alpha$ is a smooth bijection with smooth inverse. First, it is not hard to establish that the antipodal map $A(p)$ is a bijection. The only operation is multiplying each coordinate by the scalar (-1) which is smooth. It's inverse also looks pretty much exactly the same.

I think I'm on to something but I'm really confused on how to proceed.

$\square$

(b). I have no idea what to do for this part... $\square$

(c). repeat the construction from problem 1. If we make π an isometry, then it must preserve the curvature of the cylinder C which we know to be 0. $\square$

Problem 3

(a) *Show that the projection $\pi : S^2 \to \mathbb{R}P^2$ from the sphere onto the projective plane has the following properties. (1) π is continuous and $\pi(S^2) = \mathbb{R}P^2$. (2) each point $p \in \mathbb{R}P^2$ has a neighborhood U such that $\pi^{-1}(U) = V_1 \cup V_2$ where V_1 and V_2 are disjoint upen subsets of S^2 and the restriction of π to each V_i, $\quad i = 1, 2$ is a homeomorphism onto U. Thus π satisfies formally the conditions for a covering map with two sheets. Beacaus of this, we say that S^2 is an orientable double covering of $\mathbb{R}P^2$.*

(b) *Show that in this sense, the torus T is an orientable double covering of the Klein bottle K and that the cylinder is an orientable double covering of the infinite Mobius strip.*

Problem 4

Extend the Gauss-Bonnet theorem to orientable Riemannian 2-manifolds and apply it to prove the following fact: There is no Riemannian metric on an abstract surface T