Complex Analysis: Day 19 **John Waczak**
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Poles

Recall: If $f : \Omega \to \mathbb{C}$ is holomorphic and $a \in \Omega$ then there exists some $R > 0$ for which f can be expressed in a power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for all $z \in D_R(a)$, and $D_R(a) \subseteq \Omega$. Where

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition: If $f(z)$ is holomorphic and $f(a) = 0$, we call a a *zero* of f.

Theorem. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and $a \in \Omega$ be a zero of f. Then one of the following two cases occurs:*

- *a There exists a disc D centered at a for which $f(z) = 0$ $\forall z \in D$.*
- *b There exists an integer $m \geq 1$ and a holomorphic function $g : \Omega \to \mathbb{C}$ such that $f(z) = (z-a)^m g(z)$ for all $z \in \Omega$ and $g(a) \neq 0$.*

Example: $f(z) = z^3 - 2z^2 + z$, $a = 1$.

$$\begin{aligned} f(1) &= 1^3 - 2 + 1 = 0 \\ f(z) &= z(z^2 - 2z + 1) \\ &= z(z-1)^2 \\ \textit{thus } m = 2 &\quad g(z) = z \end{aligned}$$

Definition In case (b) we say that m is the *multiplicity* of the zero a.

Proof. Expand f into its power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where $|z - a| < R$ with $R > 0$. Let $D = D_R(a)$. If $c_n = 0$ $\forall n$ then clearly case (a) holds. Otherwise we have $c_m \neq 0$ for some $m \geq 1$ and we make take m minimal with this property. We already know that $c_0 = 0$ since $f(a) = c_0$ and we are assuming that a is a zero. Thus the power series starts at m and looks like

$$f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n$$

and so we can observe that

$$\begin{aligned} &= (z-a)^m \Big\{ c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + ... \Big\} \\ &= (z-a)^m g(z) \\ \text{where } g(z) &= \sum_{k=0}^{\infty} c_{m+k}(z-a)^k \end{aligned}$$

If instead $z \neq a$, also define

$$g(z) = \frac{f(z)}{(z-a)^m} \quad (z \in \Omega \setminus \{a\})$$

This defines $g : \Omega \to \mathbb{C}$. **Note:** points $z \in D \setminus \{a\}$ have two definitions of g(z), but the two definitions are equal. □

Example $f(z) = \sin^2(z)$, $a = 0$

$$\begin{aligned} \sin^2(z) &= \sum_{n=0}^{\infty} c_n z^n \\ f'(z) &= 2\sin(z)\cos(z) \\ f''(z) &= 2\cos^2(z) - 2\sin^2(z) \\ \Rightarrow c_0 &= f(0) = 0 \\ c_1 &= f'(0) = 0 \\ c_2 &= f''(0)/2 = 1 \neq 0 \\ \Rightarrow m &= 1 \end{aligned}$$

the function

$$g(z) = \begin{cases} \frac{\sin^2(z)}{z^2} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is holomorphic on $\mathbb{C}$ and $\sin^2(z) = z^2 g(z)$.