Complex Analysis Summary **John Waczak**
Date: October 6, 2017

Analytic Functions A function f of the complex variable z is analytic at a point z_0 if $f'(z)$ exists at z_0 and in some neighborhood of z_0 i.e. $\{z : |z_0 - z| < \epsilon\}$

Singular Point (singularity) If a function is analytic at every point in a neighborhood of z_0 except at $z = z_0$, we call z_0 a singular point.

Cauchy-Riemann Conditions Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic at z_0. Then at z_0 we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Complex Integrals Suppose $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = u(t) + iv(t)$. Then given two numbers $a, b \in \mathbb{R}$ we define the definite integral of f as:

$$\int_a^b f(z)dz = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Cauchy Goursat Theorem let $f(z)$ be analytic over the simply connected in a region U. Let C be a closed contour through this region U. Then we have that:

$$\oint_C f(z)dz = 0$$

Proof: let $f(z) = u(x, y) + iv(x, y)$ then by the definition of the integral we have:

$$\oint f(z)dz = \oint (u + iv)(dx + idy) = \oint udx - vdy + i\oint vdx + udy$$

Now we apply Green's theorem for surface integrals to achieve:

$$\oint vdx - udy = \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$
$$\oint vdx + udy = \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

Now, because f is analytic on the closed contour and in the interior, we have that the Cauchy-Riemann conditions must hold making both of the integrands 0. Thus we have:

$$\oint_C f(z)dz = 0 \quad \square$$

Cauchy Integral Formula Let f be analytic everywhere within and on a closed contour C. If z_0 is any point on the interior to C, then:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz,$$

where the integral is taken in the positive direction around C.

Derivative of Analytic Functions If a function f is analytic at a point then its derivatives of all orders, f', f", ..., are also analytic functions at that point. The n^{th} derivative of f at z_0 is given by the formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}} \qquad (n = 1, 2, 3...)$$

Taylor Series Let f be analytic at all points within a circle C_0 with a center at z_0 and radius r_0. Then at each point z inside C_0:

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$