Central Forces Homework 5

Due 5/23/18, 4 pm

1. Express the cartesian coordinates (x, y, z) in terms of the spherical coordinates (r, θ, ϕ)

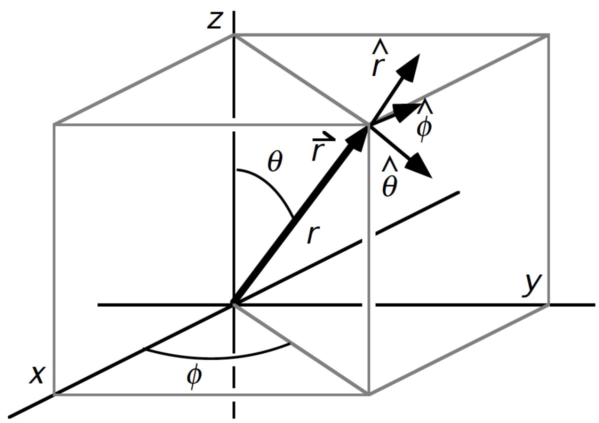

Figure 1: The Spherical Coordinate System

$$\begin{cases} x & = \\ y & = \\ z & = \end{cases}$$

2. The unit vectors in the spherical coordinate system ($\hat{r}, \hat{\theta}$, and $\hat{\phi}$) are functions of position. It is convenient to express them in terms of the spherical coordinates and the unit vectors of the cartesian coordinate system ($\hat{\imath}, \hat{\jmath}$, and $\hat{k}$), which themselves are <u>NOT</u> functions of position.

$$\begin{cases} \hat{r} & = \\ \hat{\phi} & = \\ \hat{\theta} & = \end{cases}$$

3. Sometimes, it is also useful to express the unit vectors of the cartesian coordinate system ($\hat{i}, \hat{j}$, and $\hat{k}$) in terms of the unit vectors in the spherical coordinate system ($\hat{r}, \hat{\theta}$, and $\hat{\phi}$):

$$\begin{cases} \hat{i} &= \\ \\ \hat{j} &= \\ \\ \hat{k} &= \end{cases}$$

4. Now derive the variations of unit vectors in the spherical coordinate system:

$$\frac{\partial \hat{r}}{\partial r} =$$
$$\frac{\partial \hat{r}}{\partial \theta} =$$
$$\frac{\partial \hat{r}}{\partial \phi} =$$
$$\frac{\partial \hat{\phi}}{\partial r} =$$
$$\frac{\partial \hat{\phi}}{\partial \theta} =$$
$$\frac{\partial \hat{\phi}}{\partial \phi} =$$
$$\frac{\partial \hat{\theta}}{\partial r} =$$
$$\frac{\partial \hat{\theta}}{\partial \theta} =$$
$$\frac{\partial \hat{\theta}}{\partial \phi} =$$

5. The path increment $d\vec{r}$ for an infinitesimal displacement from (r, θ, ϕ) to $(r + dr, \theta + d\theta, \phi + d\phi)$ is:

$$d\vec{r} =$$

6. The differential volume $dV = dxdydz$ expressed in spherical coordinates is:

$$dV =$$

7. Show that in the spherical coordinate system

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

8. The divergence of a vector $\vec{A}$ in the spherical coordinate system is:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}.$$

Now show that the Laplacian operator ∇^2 in the spherical coordinate system can be written as:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$