Homework 6 Mth 311

John Waczak

2) Define $f(x) \equiv \begin{cases} x & \text{if } x \in [0,1] \text{ and } x \in \mathbb{Q} \\ -x & \text{if } x \in [0,1] \text{ and } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

prove $f: [0,1] \to \mathbb{R}$ is not integrable.

Because both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in the ~~rationals~~ reals $\forall$ interval $[\alpha, \beta]$ w/ $\alpha, \beta \in \mathbb{R}$ contains points $\in \mathbb{Q}$ and points in $\mathbb{R}\setminus\mathbb{Q}$. Hence $\forall$ partition $\{x_i\}$ of $[0,1]$ $\forall$ width $\delta > 0$, we can choose x_i' to be such that $x_i' \in \mathbb{Q}$ $\forall i$ or $x_i' \in \mathbb{R}\setminus\mathbb{Q}$ $\forall i$.

Let $\{x_i\}$ be a partition for $[0,1]$. Then we define

$$S_1 = \sum_{i=1}^{N} f(x_i^{1\prime})(x_i - x_{i-1}) \quad \text{and}$$

$$S_2 = \sum_{i=1}^{N} f(x_i^{2\prime})(x_i - x_{i-1})$$

such that

$$x_i^{1\prime} \in \mathbb{Q} \ \forall i$$

$$x_i^{2\prime} \in \mathbb{R}\setminus\mathbb{Q} \ \forall i$$

Then $\forall \delta > 0$ (width of partition), we have

$\cancel{|S_1 - S_2|}$

$|S_1 - S_2| > 0$ since S_2 must be negative

since $f(x_i^{2\prime}) \leq 0$ $\forall i$ by def

so $\exists \varepsilon$ s.t. $|S_1 - S_2| \geq \varepsilon$ thus f is not integrable

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9) suppose $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ continuous
prove $\int_a^b |f+g| \le \int_a^b |f| + \int_a^b |g|$

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observe that because f,g continuous on $[a,b]$ $\exists \int_a^b f$, $\int_a^b g$. By linearity of integration we have that $\exists \int_a^b f+g$.

Now observe that $\forall\, \alpha,\beta \in \mathbb{R}$ $\alpha\neq\beta$ by triangle inequality we know

$$|\alpha+\beta| \le |\alpha| + |\beta|$$

Since f and g are real valued we can say $|f+g| \le |f| + |g|$ $\forall x\in[a,b]$

all the absolute value does is make the functions strictly positive so $|f+g|, |f|, |g|$ are continuous and therefore intable on $[a,b]$.

$\therefore$ by corollary 1 (page 117) we have that

$|f+g|$ and $|f|+|g|$ are intable on $[a,b]$ ~~so~~
and $|f+g| \le |f|+|g|$ so

$$\int_a^b |f+g| \le \int_a^b |f|+|g| = \int_a^b |f| + \int_a^b |g|$$

∎

3. Compute $\int_0^1 x\,dx$ directly from definition assuming only that the integral exists

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③

Since the integral exists we know

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $\{X_i\}$ is a partition of width $< \delta$ then

$$|S - A| < \varepsilon \text{ where } S \text{ is Reimann sum and } A \text{ is } \int_a^b f(x)\,dx$$

let $\{X_i\}$ be a regular partition of width

$$\frac{b-a}{N} = \frac{1}{N} \quad \text{i.e. } X_i = 0 + \frac{i}{N} = \frac{i}{N}$$

we can make the width arbitrarily small by controlling N so by the above def:

$$\int_0^1 x\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} x_i^*\left(\frac{1}{N}\right)$$

choosing to use right end points gives

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N}\left(\frac{1}{N}\right) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} i$$

$$= \lim_{N\to\infty} \frac{1}{N^2} \frac{N(N+1)}{2} = \lim_{N\to\infty} \frac{1}{2} + \frac{1}{2N} = \frac{1}{2}$$

thus $\int_0^1 x\,dx = \frac{1}{2}$ which is the same answer we get if we use the FTC to calculate the integral instead!

∎

4. Let $f:[a,b]\to\mathbb{R}$ be strictly increasing. Show that $\int_a^b f(x)\,dx$ exists

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Recall $f:[a,b]\to\mathbb{R}$ is int'lable on $[a,b]$ iff $\forall \varepsilon>0$ $\exists$ step functions f_1, f_2 s.t.

$$f_1(x) \le f(x) \le f_2(x) \quad \forall x\in[a,b]$$

and

$$\int_a^b f_2(x)-f_1(x)\,dx < \varepsilon$$

Lets define f_2 to be a step function defined as

$$f_2(x)=\sum_{i=1}^{N} f(x_i)\,\mathbb{1}_{(x_{i-1},x_i)}(x) \quad \text{(right end p}$$

and

$$f_1(x)=\sum_{i=1}^{N} f(x_{i-1})\,\mathbb{1}_{(x_{i-1},x_i)}(x) \quad \text{(left end pt}$$

then $\forall x\in[a,b]$ it is true that

$$f_1(x)\le f(x)\le f_2(x)$$

Now WTS $\int_a^b f_2(x)-f_1(x)\,dx<\varepsilon$.

by linearity of int we have

$$=\int_a^b f_2(x)\,dx-\int_a^b f_1(x)\,dx$$

(next page)

4. continued

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Recall that $\forall$ step function $f: [a,b] \to \mathbb{R}$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} c_i (x_i - x_{i-1})$$ where c_i's are heights of each ste

Thus we can say

$$\int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx = \sum_{i=1}^{N} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(x_{i-1})(x_i - x_{i-1})$$

$$= \sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$

Now if we ~~also~~ know that $x_i - x_{i-1} \leq \delta$ the width so

$$\leq \delta \sum_{i=1}^{N} f(x_i) - f(x_{i-1}) = \delta (f(x_a) - f(x_b))$$

strictly because δ is can be bigger than $x_i - x_{i-1}$

Thus if $\delta = \dfrac{\varepsilon}{f(x_a) - f(x_b)}$ then

$$\int_a^b f_2(x) - f_1(x)\,dx < \varepsilon$$

$\blacksquare$