**3**. **Solution**: The equation of the orbit is

$$\frac{\alpha}{r} = 1 + \varepsilon \cos(\phi) \tag{1}$$

assuming without loss of generality that $\phi_0 = 0$. Here $\alpha = \dfrac{\ell^2}{\mu k}$ and $\varepsilon = \sqrt{1 + \dfrac{2E\ell^2}{mk^2}}$. Therefore, the radial distance $r$ can vary from the maximum value $\dfrac{\alpha}{(1-\varepsilon)}$ to the minimum value $\dfrac{\alpha}{(1+\varepsilon)}$.

The angular velocity of the particle is given by

$$\omega = \dot{\phi} = \frac{\ell}{\mu r^2} \tag{2}$$

Thus, the maximum and minimum values of $\omega$ become,

$$\begin{cases} \omega_{\max} = \dfrac{\ell}{\mu r_{\min}^2} = \dfrac{\ell}{\mu \left[\frac{\alpha}{1+\varepsilon}\right]^2} \\ \\ \omega_{\min} = \dfrac{\ell}{\mu r_{\max}^2} = \dfrac{\ell}{\mu \left[\frac{\alpha}{1-\varepsilon}\right]^2} \end{cases} \tag{3}$$

So that the ratio of the two is,

$$\frac{\omega_{\max}}{\omega_{\min}} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \equiv n \tag{4}$$

From which we have

$$\varepsilon = \frac{\sqrt{n}-1}{\sqrt{n}+1} \tag{5}$$

4. **Solution**: The force $f(r) = -k/r^3$ can be easily integrated to find the

corresponding central potential

$$V(r) = -\frac{k}{2r^2} \tag{6}$$

The corresponding effective potential is formed by adding in the kinetic energy due to rotations. That is,

$$V_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{2r^2} \tag{7}$$

The equation for the shape of an orbit is

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2 u^2}\left(-ku^3\right) \tag{8}$$

or,

$$\frac{d^2u}{d\phi^2} + \left[1 - \frac{\mu k}{\ell^2}\right] u = 0 \tag{9}$$

Let us consider the motion of various values of $\ell$.

(**a**) $\ell^2 = \mu k$:
In this case, the effective potential vanishes and the orbit equation becomes

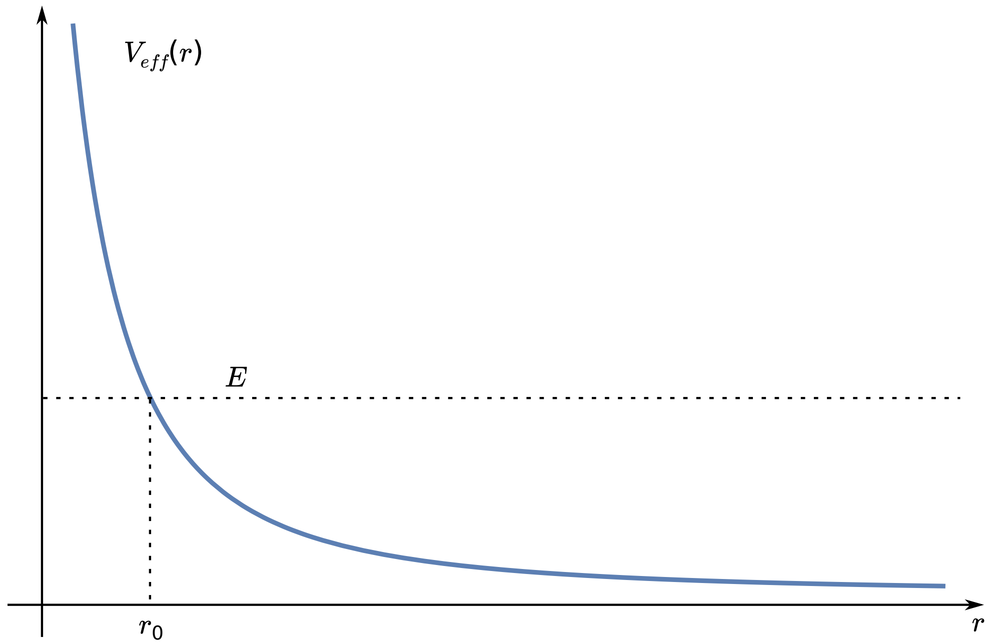
$$\frac{d^2u}{d\phi^2} = 0 \tag{10}$$

which leads to orbits of the form

$$u = \frac{1}{r} = A\phi + B \tag{11}$$

so that the particle spirals towards the force center.

(**b**) $\ell^2 > \mu k$: In this case the effective potential is positive and decreases monotonically with increasing $r$. For any value of the total energy $E$, the particle will approach the force center and will undergo a reversal of motion at $r = r_0$; the particle will then proceed again to an infinite distance. This is illustrated in the following sketch




Setting $1 - \mu k/\ell^2 = \beta^2 > 0$, then the differential equation becomes

$$\frac{d^2u}{d\phi^2} + \beta^2 u = 0 \tag{12}$$

with the solution

$$u(\phi) = \frac{1}{r} = A\cos\left(\beta\phi - \delta\right) \tag{13}$$

Since the minimum value of $u$ is zero, this solution corresponds to unbounded motion, as expected from the form of the effective $V_{\text{eff}}(r)$.

**(c)** $\ell^2 < \mu k$: For this case we set $\mu k/\ell^2 - 1 = G^2 > 0$, and the orbit equation becomes

$$\frac{d^2u}{d\phi^2} - G^2 u = 0 \tag{14}$$

which leads to

$$u(\phi) = \frac{1}{r} = A\cosh(G\phi - \delta) \tag{15}$$

so that the particle spirals in towards the force center.