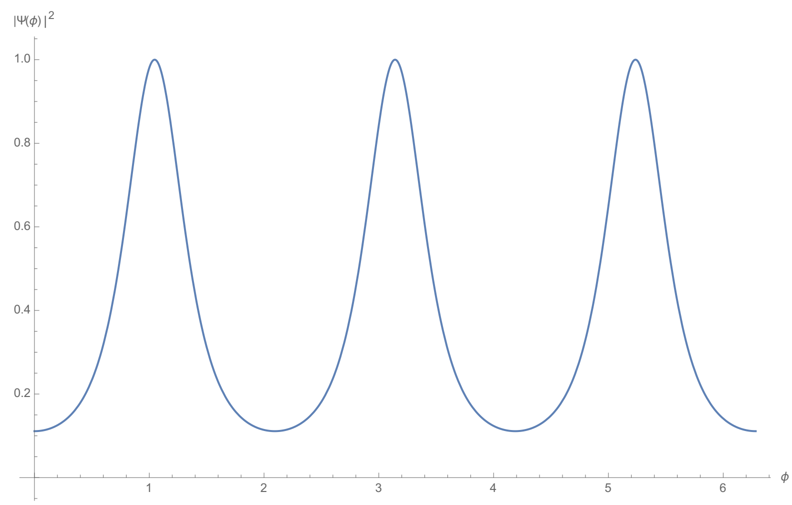
2. Solution:

(**a**) Plot $\dfrac{1}{[2+\cos(3\phi)]^2}$ from $\phi = 0$ to $\phi = 2\pi$

Look closely at what you are asked to graph. In fact, this is the probability density $\Psi(\phi)^*\Psi(\phi)$ which tells us the probability of finding the quantum particle in a region of space $r_0 d\phi$ on the ring. The following graph of this probability density is for N=1 made using Mathematica. We will find the true value of N in the next part of the problem.



The fact that this plot is periodic at 0 and 2π should convince you that $|\Psi\rangle$ is a valid state for a particle on the ring.

(**b**) Determine the normalization constant N.

We can not easily rewrite the wave function $\Psi(\phi)$ in terms of the eigenstates $\Phi_m(\phi)$. The best approach is to stay in function land and use the integral representation of the inner product. That is,

$$1 = \langle\Psi|\Psi\rangle = \int_0^{2\pi} \left| \frac{N}{[2+3\cos\phi]^2} \right|^2 r_0 d\phi \tag{1}$$

$$= |N|^2 r_0 \int_0^{2\pi} \frac{1}{4+4\cos 3\phi+\cos^2 3\phi} d\phi \tag{2}$$

$$= |N|^2 r_0 \left[\frac{4}{9\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{3\phi}{2}\right) - \frac{\sin 3\phi}{9(2+\cos(3\phi))} \right]_0^{2\pi} \tag{3}$$

Directly plugging in 2π and 0 in order to evaluate (3) yields a value of 0. That is **wrong**. In part (a) we graphed the integrand. It is a perfectly well behaved function; it's smooth and doesn't blow up. In fact, the problem is that the antiderivative involves a term of the form $\tan^{-1}(\tan(x))$. This is **not** well defined over the entire interval $[0, 2\pi]$ because the inverse tangent function has a restricted range from $(-\pi/2, \pi/2)$. If you are not convinced, try graphing $\tan^{-1}(\tan(x))$ and the indefinite integral from equation (3).

To get around this problem, we can recognize that there is symmetry in our graph from part (a). Instead, let's integrate from 0 to $2\pi/6$ and then multiply by 6.

$$1 = 6|N|^2 r_0 \left[\frac{4}{9\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{3\phi}{2}\right) - \frac{\sin 3\phi}{9(2+\cos(3\phi))} \right]_0^{2\pi/6} \tag{4}$$

$$= 6|N|^2 r_0 \left[\frac{4}{9\sqrt{3}} \frac{\pi}{2} \right] = |N|^2 r_0 \left[\frac{4\pi}{3\sqrt{3}} \right] \tag{5}$$

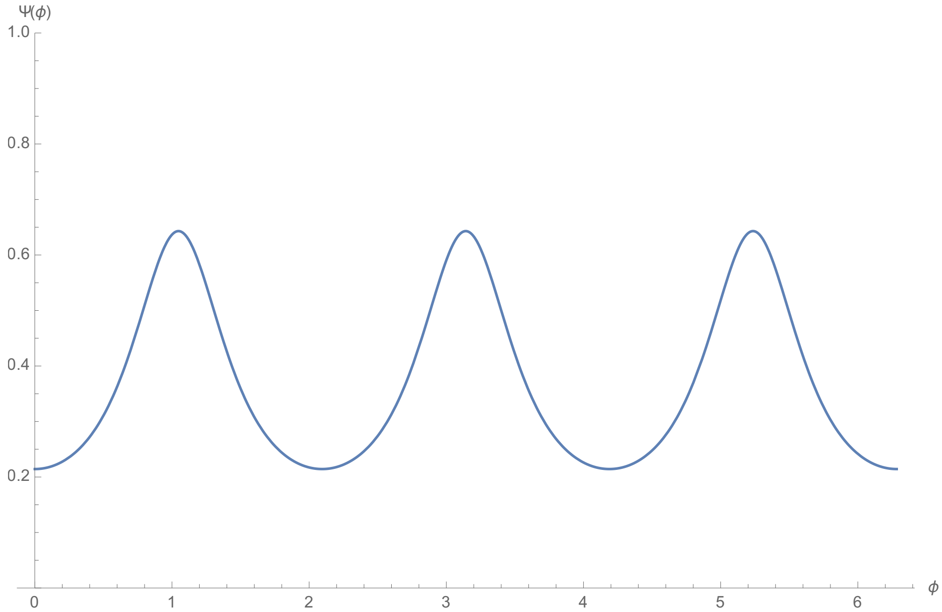
$$\Rightarrow N = \sqrt{\frac{3\sqrt{3}}{4\pi r_0}} \tag{6}$$

The normalized wave function is therefore

$$\Psi(\phi) = \sqrt{\frac{3\sqrt{3}}{4\pi r_0}} \frac{1}{2+\cos 3\phi} \tag{7}$$

(**c**) Plot the wave function

A plot of our result from (b) is shown below. Note that we have set $r_0 = 1$ for convenience.



(**d**) What is the expectation value of L_z in this state?

The expecation value is

$$\begin{aligned}
\langle L_z \rangle &= \langle \Psi | L_z | \Psi \rangle && (8) \\
&= \int_0^{2\pi} \Psi^*(\phi) \left(-i\hbar \frac{\partial}{\partial \phi} \right) \Psi(\phi) r_0 d\phi && (9) \\
&= \frac{3\sqrt{3}}{4\pi r_0} \int_0^{2\pi} \frac{1}{2 + \cos 3\phi} \left(-i\hbar \frac{\partial}{\partial \phi} \right) \frac{1}{2 + \cos 3\phi} r_0 d\phi && (10) \\
&= -i\hbar \frac{3\sqrt{3}}{4\pi} \int_0^{2\pi} \frac{3 \sin 3\phi}{2 + \cos 3\phi} d\phi && (11) \\
&= 0 && (12)
\end{aligned}$$