Complex Analysis - MTH 483 **John Waczak**
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Complex Curves (paths)

Definition: A curve (path) in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$. The real numbers have an orientation (from negative to positive). This gives curves a direction in $\mathbb{C}$, beginning at $\gamma(a)$ and ending at $\gamma(b)$.

A few examples: $\gamma : [0, 2] \to \mathbb{C}$, s.t. $\gamma(t) = (1 + 2i)t$. This is the parametrization of a straight line.

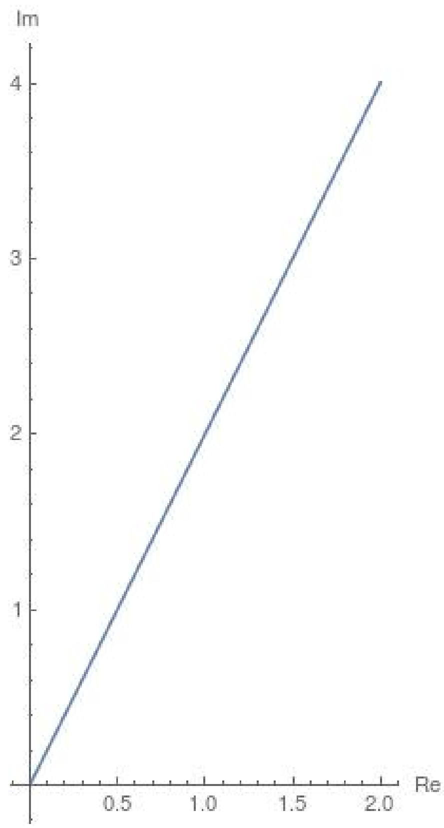

Figure 1: $\gamma(t) = (1 + 2i)t$

In general, to parametrize the **line segment** from z_1 to z_2 use: $\gamma(t) = (1 - t)z_1 + tz_2$ where $\gamma : [0, 1] \to \mathbb{C}$.

Now lets let $\gamma : [0, 1] \to \mathbb{C}$ s.t. $\gamma(t) = t + it^2$. This is a parametrization of the graph $y(t) = t^2$.

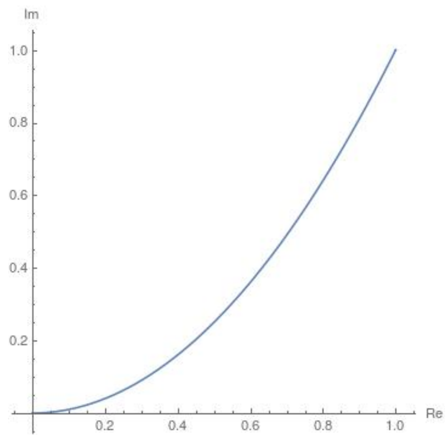

Figure 2: $\gamma(t) = t + it^2$

Here's a circle: $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = 3 + 2e^{it}$.

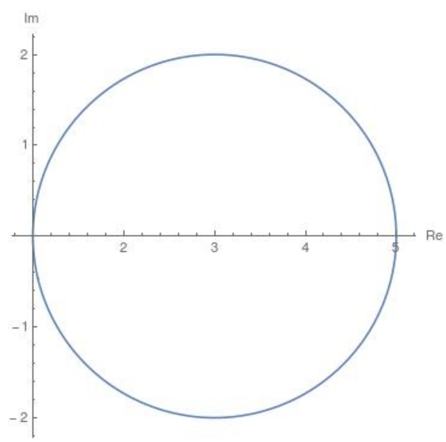

Figure 3: $\gamma(t) = 3 + 2e^{it}$

Thus in general we have that a **Circle** of radius r centered at z_0 is given by: $\gamma(t) = z_0 + re^{it}$ This parametrization is **positively oriented** i.e. counterclockwise. For negative orientation, let $t \mapsto -t$.

Properties of curves in $\mathbb{C}$

Remark: All curves can be assumed to be piecewise differentiable for finitely many jumps... i.e. only finitely many sharp corners.

Definition The derivative of a complex curve $f(t)$ is accomplished componentwise i.e. if $f(t) = x(t) + iy(t)$ then $f'(t) = x'(t) + iy'(t)$.

Definition A curve is *Simple* if $\gamma(t) \neq \gamma(tz)$ whenever $a \leq t_1 < t_2 \leq b$, except possible when $t_1 = a$ and $t_2 = b$. (Curves shouldn't intersect itself except at the end points).

Definition A curve is *Closed* if $\gamma(a) = \gamma(b)$. Think circle.

Complex integration

If $g : [a,b] \to \mathbb{C}$, then we can write $g(t) = x(t) + iy(t)$ for real valued functions $x, y : \mathbb{R} \to \mathbb{R}$. Then the integral

$$\int_a^b g(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt$$

Definition let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Let $f(z)$ be a complex valued function which is continuous on the path γ. Then we define the integral of $f(z)$ over the path γ as the following:

$$\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Example: $\gamma : [0,1] \to \mathbb{C}$ s.t. $\gamma(t) = (1-t) + (2+i)t$. Let $f(z) = z^2$.

$$\begin{aligned}
\gamma(t) &= (1-t) + (2+i)t \\
&= 1 + t + it \\
\gamma'(t) &= 1 + i \\
\Rightarrow \int_\gamma f(z)dz &= \int_0^1 f(1+t+it)(1+i)dt \\
&= \int_0^1 (1+t+it)^2(1+i)dt \\
&= (1+i)\int_0^1 (1+(1+i)t)^2 dt \\
&= (1+i)\int_0^1 (1+2(1+i)t+(1+i)^2t^2)dt \\
&= (1+i)\Big[t + (1+i)t^2 + \frac{(1+i)^2}{3}t^3\Big]_0^1 \\
&= (1+i)\Big[1 + (1+i_+ \frac{(1+i)^2}{3})\Big]
\end{aligned}$$

Example Integrate the function $f(z) = z^2$ over the positively oriented unit circle $\gamma(t) = e^{it}$.

$$\begin{aligned}
\gamma(t) &= e^{it} \\
\gamma'(t) &= ie^{it} \\
\Rightarrow \int_\gamma f(z)dz &= \int_0^{2\pi} (e^{it})^2 ie^{it}dt \\
&= i\int_0^{2\pi} e^{3it}dt \\
&= i\Big[\frac{1}{3i}e^{3it}\Big]_0^{2\pi} \qquad = 0
\end{aligned}$$