Complex Analysis: Day 17 **John Waczak**
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Limsup

Def let a_n be a sequence of nonnegative real numbers.

- If $\{a_n\}$ has a subsequence that is unbounded, we say
$$\limsup_{n\to+\infty} a_n = +\infty$$
- If $\{a_n\}$ is bounded, we define:
$$\limsup_{n\to+\infty} a_n$$
to be the largest limit of any convergent subsequence of $\{a_n\}$

Examples

- $\limsup_{n\to+\infty}(1+\frac{1}{n}) = 1$
- $a_n = 4 + (-1)^n = \{3, 5, 3, 5, ...\}$ thus $\limsup_{n\to+\infty} a_n = \lim_{n\to\infty}\{5, 5, 5, 5, ...\} = 5$.

The **big-picture** goal is

- to show that functions defined by power series are holomorphic where they converge absolutely.
- Conversely, if $f : \Omega \to \mathbb{C}$ is holomorphic and $c \in \Omega$, then there exists a disc $D_r(c) \subseteq \Omega$ on wich f is given by a power series.

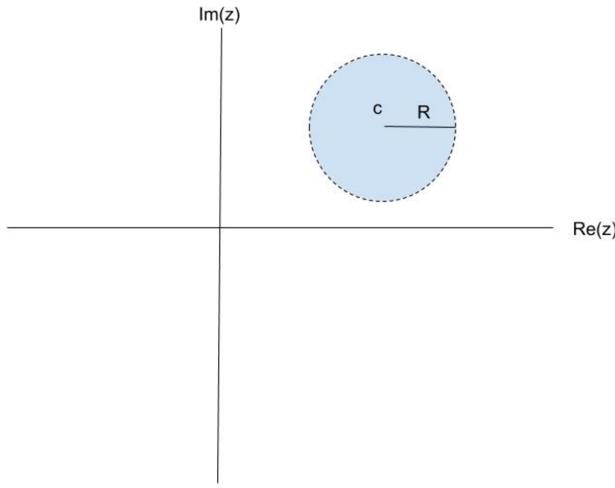
More on power series

Theorem. *Let* $\sum\limits_{n=0}^{\infty} a_n(z-c)^n$ *be a power series where* $a_n, c \in \mathbb{C}$. *Then* $\exists 0 \leq R \leq +\infty$ *such that:*

a If $|z-c| < R$ *then the series converges absolutely*

b If $|z-c| > R$ *the series diverges.*

Moreover, we have that $R = \frac{1}{L}$ *where* $L = \limsup_{n\to+\infty} |a_n|^{1/n}$ *Interpret* $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$.



Some helpful facts...

- $n^{1/n} \to 1$ as $n \to +\infty$
- $(1/n!)^{1/n} \to 0$ as $n \to +\infty$

Example define $f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ then,

$$L = \limsup (1/n!)^(1/n) = 0 \Rightarrow R = +\infty$$

Example define $\sum z^n$ then,

$$L = \limsup 1^{1/n} = 1 \to R = 1/1 = 1$$

Thus the series converges absolutely for $|z| < 1$. On this disc we have $\sum z^n = \frac{1}{1-z}$

Proof of theorem:

Proof. Assume $c = 0$ and assume that $0 < L < +\infty$ (other cases are dealt with similarly). (a) Assume $|z| < R$. We need to show that the series converges absolutely. From the assumption we have that $|z| < \frac{1}{L}$ which means that $L|z| < 1$. Then let $\epsilon > 0$ such that $r := (L + \epsilon)|z| < 1$. By the definition of L we have that $|a_n|^{1/n} \leq L + \epsilon$ for large enough n. For the nth term of the series we have

$$\begin{aligned} |a_n z^n| &= (|a_n|^{1/n}|z|)^n \\ &\leq ((L + \epsilon)|z|)^n \\ &= r^n \end{aligned}$$

and $r < 1$ so $\sum |a_n z^n|$ converges by comparison test ot the convergent geometric series $\sum r^n$ with $r < 1$.

If $|z| > R$ a similar argument show that the sequence of terms $a_n z^n$ do not even converge to zero and hence the series diverges. $\square$

Theorem. *Let* $f(z) = \sum a_n (z-c)^n$ *be a function defined by a power series with radius of convergence* $0 < R \leq \infty$ *Then* $f(z)$ *is holomorphic in the disc* $D_R(c)$*, and has derivative*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-c)^{n-1}$$

Moreover, the radius of convergence for $f'(z)$ *is also* R*.*

Once we show that all holomorphic functions are given by a power series than if we have $f(z)$ is once differentiable, it is infinitely differentiable.

Example $e^z = \sum \frac{1}{n!} z^n$

$$\begin{aligned} (e^z)' &= \sum_{n=1}^{\infty} n \frac{1}{n!} z^{n-1} \\ \text{let } m &= n-1 \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} z^m = e^z \end{aligned}$$