Homework 7 MTH 434

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1 SPHERICAL COORDINATES II Consider the sphere of radius r, in spherical coordinates  $(\theta, \phi)$ , with line element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

(a) Find the connection 1-forms  $\omega_{ij}$  in this basis.

Because we are confined to the surface of the sphere,  $d\vec{r}$  vector will not aid in our calculation. However, The metric compatibility requirement

$$\omega_{ij} + \omega_{ji} = 0 \tag{2}$$

Implies that every  $\omega_{ii}$  is identically zero. Furthermore, because r is constant on the sphere, there are only 4 connection 1-forms to calculate. The above asymmetry argument implies we need only explicitly calculate 1 of them. To do this we will consider the torsion free requirement which states

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0 \tag{3}$$

The leads to the following system of linear equations

$$\omega^{\theta}{}_{\phi} \wedge r \sin \theta d\phi = 0 \tag{4}$$

$$r\cos\theta d\theta \wedge d\phi + rd\theta \wedge \omega^{\theta}{}_{\phi} = 0 \tag{5}$$

where in equation (4) we note that dr = 0 for the sphere. Recalling that  $\omega^{i}_{j}$  are 1-forms and are expanded in the usual basis as

$$\omega^{i}{}_{j} = \Gamma^{i}{}_{jk} \,\sigma^{k} \tag{6}$$

equation (4) implies that  $\Gamma^{\theta}_{\phi\theta} = 0$  so that  $\omega^{\theta}_{\phi}$  has only a  $d\phi$  component. With this, equation (5) becomes

$$0 = r \cos \theta d\theta \wedge d\phi + r d\theta \wedge \Gamma^{\theta}{}_{\phi\phi} d\phi$$

$$= r \cos \theta d\theta \wedge d\phi + r \Gamma^{\theta}{}_{\phi\phi} d\theta \wedge d\phi$$

$$\Rightarrow \Gamma^{\theta}{}_{\phi\phi} = -\cos \theta$$
(7)

Where I have done the calculation in a coordinate basis for simplicity. Thus, we can conclude that the connection 1-forms for  $\mathbb{S}^2$  (in the orthonormal basis) are

$$\omega_{\theta\theta} = 0 \qquad \qquad \omega_{\theta\phi} = -\cos\theta \ d\phi = -\frac{\cot\theta}{r} r \sin\theta \ d\phi \qquad (8)$$

$$\omega_{\theta\theta} = 0 \qquad \omega_{\theta\phi} = -\cos\theta \ d\phi = -\frac{\cot\theta}{r} r \sin\theta \ d\phi \qquad (8)$$

$$\omega_{\phi\theta} = \cos\theta \ d\phi = \frac{\cot\theta}{r} r \sin\theta \ d\phi \qquad \omega_{\phi\phi} = 0 \qquad (9)$$

(b) Compute  $\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$  for i, j = 1, 2 (and where there is an implicit sum over k)

As in the last homework set, equation (2) combined with the fact that  $d\alpha = 0$  for any  $\alpha \in \bigwedge^1$  implies  $\Omega_{ii} = 0$  for all i. We also have that

$$\Omega_{ji} = -d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\Omega_{ij} \tag{10}$$

Therefore, we need only explicitly calculate one of the curvature 2-forms. For ease of calculation, I will begin in the coordinate basis and then convert back into the orthonormal basis to make the Gauss curvature obvious.

$$\Omega_{\theta\phi} = d\omega_{\theta\phi} + \omega_{\theta \ k} \wedge \omega_{k\phi} \tag{11}$$

$$= d\omega_{\theta\phi} + \omega_{\theta\theta} \wedge \omega_{\theta\phi} + \omega_{\theta\phi} \wedge \omega_{\phi\phi} \tag{12}$$

$$=d\omega_{\theta\phi} \tag{13}$$

$$= d(-\cos\theta \ d\phi) \tag{14}$$

$$= \sin\theta \ d\theta \wedge d\phi \tag{15}$$

$$= \frac{\sin \theta}{r^2 \sin \theta} \, r d\theta \wedge r \sin \theta \, d\phi \tag{16}$$

$$= \frac{1}{r^2} r d\theta \wedge r \sin \theta \ d\phi \tag{17}$$

$$=\frac{1}{r^2}\omega\tag{18}$$

where  $\omega$  is the orientation of  $\mathbb{S}^2$ .

In summary, we have found the following

$$\Omega_{\theta\theta} = 0 \qquad \Omega_{\theta\phi} = \frac{1}{r^2}\omega \qquad (19)$$

$$\Omega_{\theta\theta} = 0 \qquad \Omega_{\theta\phi} = \frac{1}{r^2}\omega 
\Omega_{\phi\theta} = -\frac{1}{r^2}\omega \qquad \Omega_{\phi\phi} = 0$$
(19)

From class, we know that the Gaussian curvature of a two dimensional surface is related to these curvature 2-forms by

$$\Omega^{1}{}_{2} = K\omega \tag{21}$$

and  $\frac{1}{r^2}$  is precisely the Gaussian curvature of  $\mathbb{S}^2$ 

(c) (Optional) Compare your answers (and your computations) with those from the previous homework assignment.

Looking at the previous homework, we see that the connection 1-forms are exactly the same for our computation in  $\mathbb{E}^3$  despite the fact that the computation on  $\mathbb{S}^2$  does not include any dr components. It is interesting to note that the dr components of the 1forms in the first structure equation from  $d\sigma^i$  exactly cancel out the dr components of the 1-forms from  $\omega^i{}_j \wedge \sigma^j$  leaving the  $\omega_{ij}$  unchanged as we go from  $\mathbb{E}^3$  to  $\mathbb{S}^2$ .

The exact opposite happens for the curvature 2-forms for which the dr components of the 2-forms in the second structure equation exactly cancel in 3-dimensions to give zero for each  $\Omega_{ij}$ . On the sphere, dr=0 which removes these canceling terms results in non-zero curvature 2-forms.

This whole process illustrates how we can derive the curvature for surfaces in  $\mathbb{E}^3$  by considering curvilinear coordinate systems and then setting a particular basis 1-form to 0.