Homework 4 **John Waczak**
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1.) 543

2.) Let $V = \mathbb{R}^2$ as a vector space over $\mathbb{R}$ and $U = \text{span}\{e_1 + e_2\}$ where $\{e_1, e_2\}$ is the canonical basis.

a.) Give a geometric description of the left cosets of U in V.

Recall that for any $v \in V$ the coset $v + U = \{v + u : u \in U\}$. Geometrically, U in $\mathbb{R}^2$ is the line $y = x$ i.e. the line passing through the origin that makes an angle of $\pi/4$ to the x-axis. Therefore the set of cosets of U in V, ${}^{V}/_{U}$ is the set of all lines parallel to this $\pi/4$ line through the origin. This is roughly shown in the following diagram

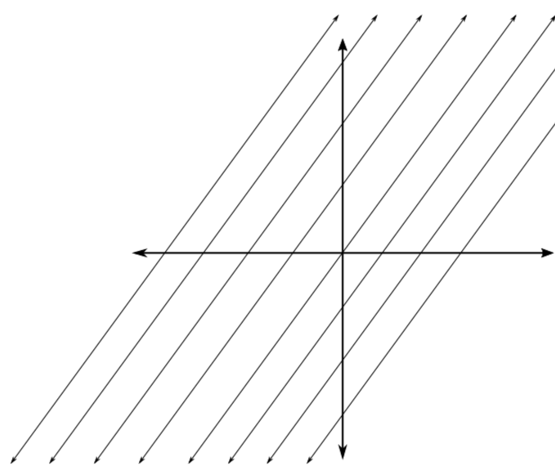

Figure 1: Some of the cosets of span$\{e_1 + e_2\}$

b.) Determine an explicit basis of ${}^{V}/_{U}$.

First recall that $\dim({}^{V}/_{U}) = \dim(V) - \dim(U)$ Therefore because $V = \mathbb{R}^2$ has dimension 2 and $U = \text{span}\{e_1 + e_2\}$ has $\dim(U) = 1$ we must have that $\dim({}^{V}/_{U})$ be 1. We already established that the cosets of U in V look like parallel lines to U. Therefore, let's take the simple choice, $\mathcal{B}_{{}^{V}/_{U}\frac{num}{den}} = \{e_1 - e_2 + U\}$ where we have chosen $v = e_1 - e_2$ to be the vector perpendicular to the basis vector for U. Certainly $e_1 - e_2$ is linearly independent from $e_1 + e_2$ and is therefore not in the span$\{e_1 + e_2\}$. Then any coset $v + U$ can be achieved by taking $\lambda \in \mathbb{R}$ such that $v = \lambda(e_1 - e_2)$. Thus we have a set

$$\mathcal{B}_{{}^{V}/_{U}} = \{e_1 - e_2 + U\}$$

is a basis for ${}^{V}/_{U}$.

3.) Let $V = \mathcal{P}_2(\mathbb{R})$. Define three linear functions, thus elements in V^*, by

$$l_1(p) = \int_0^1 p(x)dx \qquad l_2(p) = \int_0^2 p(x)dx \qquad l_3(p) = \int_0^{-1} p(x)dx$$

Show that $\{l_1, l_2, l_3\}$ is a basis of V^* by giving the basis of V to which it is a dual.

As the problem suggest we will construct a basis $\mathcal{B} = \{b_1, b_2, b_3\}$ for $\mathcal{P}_2(\mathbb{R})$ (it has dimension 3) to which $\{l_1, l_2, l_3\}$ forms the dual basis $\mathcal{B}^*$.

We begin this construction by considering the standard canonical basis for $\mathcal{P}_2(\mathbb{R})$ which is $\{1, x, x^2\}$. A general vector in $\mathcal{P}_2(\mathbb{R})$ may be represented as a linear combination of these basis vectors i.e. $p = \alpha + \beta x + \gamma x^2$ for $p \in \mathcal{P}_2(\mathbb{R})$, and some $\alpha, \beta, \gamma \in \mathbb{R}$. We can find the new basis vectors by allowing such a representation for each b_j and noting that we must have $l_i(b_j) = \delta_{ij}$ in order for the l_i to actually be the dual basis. As an example, for b_1 we have,

$$\begin{aligned} l_1(b_1) &= 1 \\ l_2(b_1) &= 0 \\ l_3(b_1) &= 0 \end{aligned}$$

If we assume b_j to be a vector in the general form discussed above, then integrating over the different ranges defined by each l_i leads to a system of equations easily encoded as

$$Mb_i = e_i$$

Where the matrix M is given by

$$M = \begin{pmatrix} \int_0^1 1dx & \int_0^1 xdx & \int_0^1 x^2dx \\ \int_0^2 1dx & \int_0^2 xdx & \int_0^2 x^2dx \\ \int_0^{-1} 1dx & \int_0^{-1} xdx & \int_0^{-1} x^2dx \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$$

and represents the coefficients from the integration. Solving this system for each b_j (see attached Mathematica code) gives the following vectors

$$\begin{aligned} b_1 &= 1 + x - \frac{3}{2}x^2 \\ b_2 &= -\frac{1}{6} + \frac{1}{2}x^2 \\ b_3 &= -\frac{1}{3} + x - \frac{1}{2}x^2 \end{aligned}$$

These vectors are linearly independent as their coordinate representations in $\mathbb{R}^3$ have a determinant of 1 when written as rows in a matrix (and the determinant is zero if there are any two linearly dependent rows). Therefore we see that there is a basis $\mathcal{B} = \{b_1, b_2, b_3\}$ of $\mathcal{P}_2(\mathbb{R})$ for which $\{l_1, l_2, l_3\}$ is the dual. Thus, we conclude that $\{l_1, l_2, l_3\}$ is a basis for the dual space.

4.) Consider the real vector space of real 2×2 matrices with its usual scalar multiplication and addition. Let $\mathcal{V}$ be the subspace spanned by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

and let $T : \mathcal{V} \to \mathbb{R}$ be the function that takes each $M \in \mathcal{V}$ to its trace. That is, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $\mathrm{Tr}(M) = a + d$.

a.) Show that T is a linear transformation

Let $M_1, M_2 \in \mathcal{V}$ with

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

and let $\lambda \in \mathbb{R}$. Then we have that

$$\begin{aligned} \mathrm{Tr}\left(\lambda M_1 + M_2\right) &= \mathrm{Tr}\left(\begin{pmatrix} \lambda a_1 + a_2 & \lambda b_1 + b_2 \\ \lambda c_1 + c_2 & \lambda d_1 + d_2 \end{pmatrix}\right) \\ &= \lambda a_1 + a_2 + \lambda d_1 + d_2 \\ &= \lambda(a_1 + d_1) + a_2 + d_2 \\ &= \lambda\,\mathrm{Tr}(M_1) + \mathrm{Tr}(M_2) \end{aligned}$$

therefore we see that the trace is a linear transformation.

b.) Choose explicit ordered bases $\mathcal{B}$ of $\mathcal{V}$ and $\mathcal{C}$ of $\mathbb{R}$ and give the matrix representation of T with respect ot these bases.

First note that the dimension of $\mathcal{M}_2$ is 4 and therefore it should have 4 elements in any basis. Since $\mathcal{V} \subseteq \mathcal{M}_2$ it would be wise to check that this is a proper subset. We can identify each matrix through isomorphism with a column vector in $\mathbb{R}^4$. Putting these together in a matrix K gives

$$K = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

This row reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This tells us that A, B, and C are linearly independent but $D = 2A - B + C$. Therefore a basis for $\mathcal{V}$ can be sufficiently described as

$$\mathcal{B} = \{A, B, C\}$$

We also identify the basis $\mathcal{C}$ for $\mathbb{R}$ to be the obvious $\mathcal{C} = \{1\}$. Then the matrix representation of T between these two bases is given by

$$\begin{aligned}[T]_{\mathcal{B}}^{\mathcal{C}} &= \begin{pmatrix} \mathrm{Tr}(A) & \mathrm{Tr}(B) & \mathrm{Tr}(C) \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 & 0 \end{pmatrix}\end{aligned}$$

This form makes sense as the dimension of our subspace $\mathcal{V}$ is 3 and therefore it is isomorphic to $\mathbb{R}^3$. Thus we need a row vector of size 3 in order to have a properly defined functional Tr via the usual matrix multiplication.

c.) Let $\mathcal{W}$ be the kernel of T. Give an explicit basis for $\mathcal{W}$.

Recall that the kernel is the set of vectors in the domain mapped to the zero vector in the codomain. That is

$$(2 \quad 2 \quad 0) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0_{\mathbb{R}}$$

where x, y, and z are the coefficients for a linear combination of the ordered basis elements A, B and C. This equation gives that $x = -y$ and therefore, the kernel is spanned by the basis

$$\mathcal{D} = \{A - B, C\} \tag{1}$$