Complex Analysis - MTH 483 **John Waczak**
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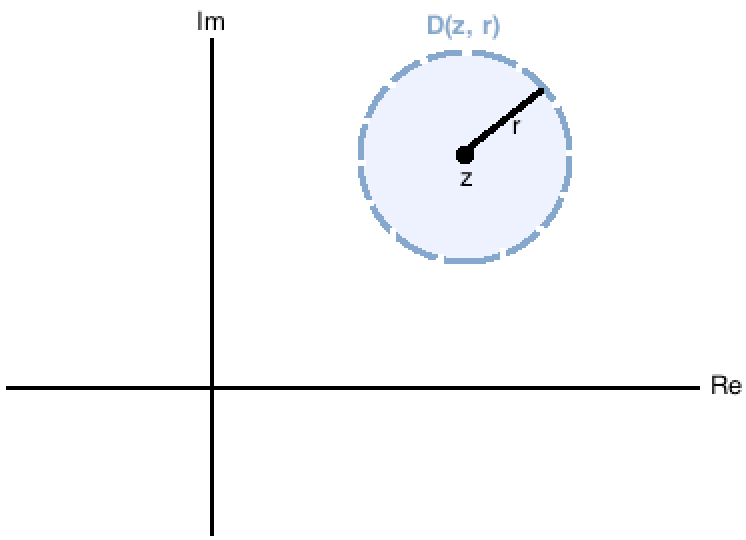
A little bit about topology

First a couple of standard notations...

Open Disk centered at $a \in \mathbb{C}$ with radius $r > 0$ is denoted by:

$$D_r(a) = \{z \in \mathbb{C} || z - a| < r\} \tag{1}$$



The **closed disk** is given by $\overline{D}_r(a) = \{z \in \mathbb{C} || z - a| \leq r\}$. I.e. includes the boundary.

A subset U of $\mathbb{C}$ is **open** if, $\forall\ z_0 \in U$ there exists $\epsilon > 0$ for which:

$$D_\epsilon(z_0) \subseteq U$$

In other words, every point of U is contained in an open disk entirely contained in U.

Examples: $\emptyset$, $\mathbb{C}$, $D_r(a)$, ... are all open sets.

Let U be an open subset of $\mathbb{C}$. We say that U is **connected** if $\nexists$ open subsets $A, B \subset U$ such that:

$$U = A \cup B \tag{2}$$
$$A \neq \emptyset, \quad B \neq \emptyset \tag{3}$$
$$A \cap B = \emptyset \tag{4}$$

Ex: $U = \mathbb{C} \setminus \{z = x + iy : y = 0\}$ is *not connected.*

Derivatives

Let $U \subset \mathbb{C}$ and let $z_0 \in U$ and let $G : U \setminus \{z_0\} \to \mathbb{C}$ be a function. We say that:

$$L = \lim_{z \to z_0} G(z) \tag{5}$$

exists if, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ (possibly depending on ϵ) such that, if:

$$|z - z_0| < \delta \to |G(z) - L| < \epsilon \tag{6}$$

Remark: The limit concerns values of $G(z)$ as z *approaches* z_0 from all directions... a much stronger condition than normal limit.

A **Region** is an *open, connected, nonempty* subset of $\mathbb{C}$.

Let $f : \Omega \to mathbbC$ be a function where Ω is a region in $\mathbb{C}$. Let $z_0 \in \Omega$. We say f is **Differentiable** at z_0 if:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists} \tag{7}$$

If it does exist, we call this limit $f'(z_0)$. If f is differentiable $\forall z_0 \in \Omega$ then we say f is **Holomorphic** on Ω.
If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, then we say f is **entire**.

Example: Show $f(z) = z^2$ is entire. Moreover, $f'(z) = 2z$.
Let $\epsilon > 0$, and let $z_0 \in \mathbb{C}$. Then,

$$\begin{aligned}
\left| \frac{f(z) - f(z_0)}{z - z_0} - 2z_0 \right| &= \left| \frac{z^2 - z_0^2}{z - z_0} - 2z_0 \right| \\
&= \left| \frac{(z - z_0)(z + z_0)}{(z - z_0)} - 2z_0 \right| \\
&= |z + z_0 - 2z_0| = |z - z_0|
\end{aligned}$$

Thus if we select $\delta = \epsilon$, then $|z - z_0| < \delta \to \left| \frac{f(z)-f(z_0)}{z-z_0} - 2z_0 \right| < \epsilon$.

It is *MUCH* easier to not be complex differentiable than real differentiable. Example: $f(z) = \overline{z}$ i.e. $f(x + iy) = x - iy$.

$$\begin{aligned}
\text{recall definition:} \quad A &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
\text{let } z_0 &= x_0 + iy_0 \\
\text{thus } A &= \lim_{h \to 0} \frac{x_0 - iy_0 + h - x_0 + iy_0}{h} \\
&= \frac{h}{h} = 1
\end{aligned}$$

Now let's take the limit vertically since we must be able to approach from *all* directions.

$$\begin{aligned}
B &= \lim_{t\to 0} \frac{f(z_0 + it) - f(z_0)}{it} \\
&= \lim_{t\to 0} \frac{x_0 - iy_0 - it - x_0 + iy_0}{it} \\
&= \lim_{t\to 0} \frac{-it}{it} = -1
\end{aligned}$$

Thus since $A \neq B$, f is not differentiable $\forall z_0$ and therefore f is *not* holomorphic.

Examples of holomorphic functions:

1. Polynomials are entire
2. Rational functions $\frac{p(z)}{q(z)}$ where $q(z) \neq 0$
3. Functions defined by convergent Power series e.g. $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$