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MTH 443 Last Updated: October 7, 2018
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Notation Comments

Remark. *The notation* $L_A : \mathbb{F}^n \to F^m$ *when* $A \in M_n(\mathbb{F})$ *has the letter* L *to indicate left multiplication by* A *on column vectors.*

Remark. *Given any linear operator* $T : V \to V$*, and finite ordered bases* B, C *for* V*. The matrix of* T *with respect to* B *and* C *is denoted* $[T]_B^C$*. In particular,*

$$[Id_v]_C^B = \left([Id_v]_b^C\right)^{-1} \tag{1}$$

From this,

$$[T]_C^C = [Id_v]_B^C \, [T]_B^B \, [Id_v]_C^B \tag{2}$$
$$= Q^{-1} \, [T]_B^C \, Q \tag{3}$$

where $Q = [Id_v]_C^B$ *is the change of basis matrix.*

Cosets

If U is a subspace of V and $v \in V$ then the left coset of U in V represented by v is

$$v + U = \{v + u | u \in U\} \tag{4}$$

The **set of left cosets** of U in V is

$$V/U = \{v + U | v \in V\} \tag{5}$$

Note that if $v \in V$ and $u \in U$ then $v + U = (v + u) + U$. Naively, we could hope that

$$V/U \times V/U \to V/U$$
$$(v_1 + U, v_2 + U) \mapsto (v_1 + v_2) + U$$

actually defines a function. We have to be sure that when you choose some v_1', v_2' that the resulting coset is the same... i.e. that we need to check that this really is a function for which inputs have exactly one output. That is,

$$(v_1 + U, v_2 + U) \mapsto v_1 + v_2 + U$$

is well-defined, in the sense that the right hand side value is independent of choice of coset representatives of the initial cosets. Here, if $v_1' = v_1 + u_1$, $v_2' = v_2 + u_2$ with $u_i \in U$. Now

$$v_1' + v_2' + U = \Big[(v_1 + u_1) + (v_2 + u_2)\Big] + U$$

Thus,

$$\begin{aligned} v_1' + v_2' + U &= (v_1 + u_1 + v_2) + (u_2 + U) \\ &= (v_1 + v_2 + u_1) + U \\ &= v_1 + v_2 + U \end{aligned}$$

That is, since addition on V is Abelian, every subgroup U is normal and thus the naive formula does give a well-defined function. We now check if

$$\begin{aligned} \mathbb{F} \times V/U &\to V/U \\ (\lambda, v + U) &\mapsto \lambda v + U \end{aligned}$$

is a well-defined function (**IT IS**). So the family of cosets of subspace U in vector space V is itself a vector space over $\mathbb{F}$.

Lemma. *Suppose U is a vector subspace of V, and B is a basis of U. Let $C = B \cup B'$ be any basis of V extending B. Then, $\{v + U | v \in B'\}$ is a basis of our quotient vector space V/U.*

Proof. Suppose $\sum_i \lambda_i (v_i + U) = 0_{V/U}$ for some $\lambda_1, ..., \lambda_n \in \mathbb{F}$. and $v_1, ..., v_n \in B'$. Since $0_{V/U} = 0_v + U = U$ is our zero vector, thus

$$\Big(\sum_i^n \lambda_i v_i \Big) + U = U$$

This holds if and only if

$$\sum_i^n \lambda_i v_i \in U$$

However, the $v_i \in B'$ and hence are linearly independent of the $sp(B)$. Therefore, this linear combination can only be $0_V \in U$. But C is a basis and thus all of the $\lambda_i = 0$. Note if $U = V$ then V/U is only $\{0_v + U\}$ and one uses logical statements. □