Complex Analysis - MTH 483 **John Waczak**
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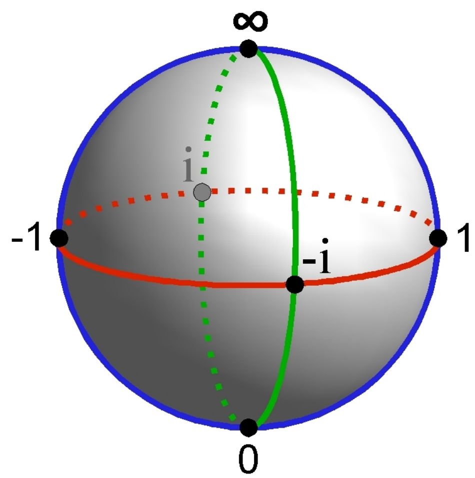
More Mobius examples

Recall: $f(z) = \frac{z-1}{i(z+1)}$ takes the unit circle $|z| = 1$ to the real line $y = 0$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to the set $L = \{y = 0\} \cup \{\infty\}$

Riemann Sphere



Example: Find the Mobius transformation that takes the circle $|z-1| = 2$ to the circle $|z+(1+i)| = 3$.

First the map $z \mapsto z - 1$ translates $|z-1| = 2$ to $|z| = 2$. And, $z \mapsto \frac{3}{2}z$ takes $|z| = 2$ to $|z| = 3$. Finally, $z \mapsto z - (1+i)$ takes $|z| = 3$ to $|z + (i+i)| = 3$. Now we just compose:

$$f(z) = \frac{3}{2}(z-1) - (1+i)$$
$$|z-1| = 2 \mapsto |z + (1+i)| = 3$$

Example: Find a Mobius transformation taking the circle $|z-1| = 2$ to the line $x = 1$.

First recall that $f(z) = \frac{z-1}{i(z+1)}$ takes unit circle to real line... Thus $z \mapsto z - 1$ takes $|z-1| = 2$ to $|z| = 2$. Now $z \mapsto z/2$ takes $|z| = 2$ to $|z| = 1$. Now $z \mapsto \frac{z-1}{i(z+1)}$ takes $|z| = 1$ to $y = 0$. Now we rotate by $\pi/2$ by multiplying by i since $i = e^{i\pi/2}$. Thus $z \mapsto iz$ takes $y = 0$ to $x = 0$. Now

$z \mapsto z+1$ takes $x=0$ to $x=1$. Therefore we have:

$$\begin{aligned}
f(z) &= i\frac{\frac{1}{2}(z-1)-1}{i(\frac{1}{2}(z-1)+1)}+1 \\
&= \frac{\frac{1}{2}z-\frac{3}{2}}{\frac{1}{2}z+\frac{1}{2}}+1 \\
&= \frac{z-3}{z+1}+1 \\
&= \frac{z-3+z+1}{z+1} \\
&= \frac{2z-2}{z+1}
\end{aligned}$$

Thus to go from circle to line just move circle to unit circle and then use $f(z)$. To go the other way just use the matrix inverse to find $f^{-1}(z)$.

Exponential sines and cosines

Define $z \mapsto e^z$ (sometimes written $\exp(z)$) by the power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \forall z \in \mathbb{C}$$

A few facts to report:

- This power series converges absolutely $\forall z \in \mathbb{C}$ and defines a holomorphic function $\mathbb{C} \to \mathbb{C}$.
- In terms of $z = x+iy$ we will see $e^{x+iy} = e^x(\cos(y)+i\sin(y))$. It would be *cheating* to use this as a definition (as in the book)...
- $e^{z+w} = e^z e^w$
- $|e^{x+iy}| = e^x$
- $e^0 = 1$
- $|e^{iy}| = 1$
- $\frac{1}{e^z} = e^{-z}$
- $e^z \neq 0$
- $e^{z+2\pi i} = e^z$

Recall the binomial theorem:

$$\begin{aligned}
(z+w)^n &= \sum_{j=0}^{n} (n,j) z^j w^{n-j} \\
(n,j) &= \frac{n!}{j!(n-j)!} \\
\text{ex:} \quad (z+w)^2 &= z^2+zw+w^2 \\
(z+w)^3 &= z^3+3z^2w+3zw^2+w^3
\end{aligned}$$

Now,

$$
\begin{aligned}
e^{z+w} &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} (n,j) z^j w^{n-j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{z^j w^{n-j}}{j!(n-j)!} \\
&= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{z^j w^{n-j}}{j!(n-j)!} \\
&= \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{n=j}^{\infty} \frac{w^{n-j}}{(n-j)!} \\
\text{let } m = n-j \quad &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{m=0}^{\infty} \frac{w^m}{m!} \\
&= \left(\sum_{j=0}^{\infty} \frac{z^j}{j!} \right) \left(\sum_{m=0}^{\infty} \frac{z^m}{m!} \right)
\end{aligned}
$$

Now we can define *cos* and *sin* in terms of power series:

1. $\cos z = \frac{e^{iz}+e^{-iz}}{2}$
2. $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$
3. $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$
4. $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} z^{2n+1}$

Test