Central Forces Homework 2

Due 5/11/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

PRACTICE:

1. If a central force is the only force acting on a system of two masses (i.e. no external forces), what will the motion of the center of mass be?

2. Which of the following forces can be central forces? which cannot?

   (a) The force on a test mass m in a gravitational field $\vec{g}$, i.e. $m\vec{g}$

   (b) The force on a test charge q in an electric field $\vec{E}$, i.e. $q\vec{E}$

   (c) The force on a test charge q moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$, i.e. $q\vec{v}\times\vec{B}$

3. Using your favorite graphing package, make a plot of the reduced mass μ as a function of m_1 and m_2. What about the shape of this graph tells you something about the physical world that you would like to remember. You should be able to find at least three things.

   Solution:

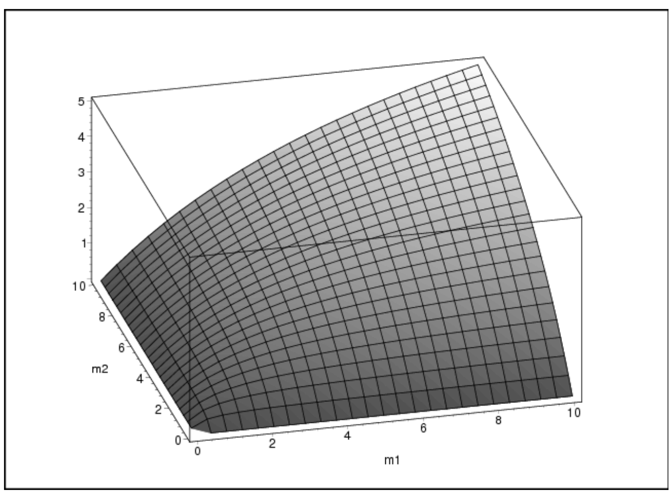

Figure 1: The reduced mass μ plotted as a function of the two masses that contribute to it, i.e. m_1 and m_2.

Some observations (not an exhaustive list):

(a) When m_1 or $m_2 = 0$: $\mu = 0$

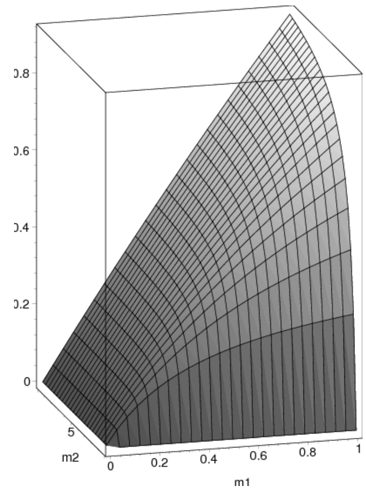

Figure 2: The reduced mass μ plotted as a function of the two masses that contribute to it for the special case $m_1 << m_2$.

(b) When $m_1 = m_2 = m$: $\mu = \frac{m}{2}$

(c) μ is always smaller than both m_1 or m_2

(d) The symmetry of the graph tells me that m_1 and m_2 are interchangeable. It doesn't matter which I choose to be the bigger one.

(e) Holding one of the masses constant (say, m_1, making sure that $m_1 \neq 0$), the slope of the graph gets shallower with increasing m_2. This means that the reduced mass is more sensitive to changes in the smaller mass.

(f) (Referring to Figure 2) If $m_2 > m_1$, the reduced mass changes linearly with m_1. (You can also see this algebraically by doing a power series expansion.)

REQUIRED:

4. Consider the differential equation $z^2y'' + zy' - (1-z)y = 0$. (This is known as Bessel's Equation.)

   (a) Using the change of variable $u = z^2$, rewrite Bessel's Equation in terms of u.

   Solution:

   Our answer at the end should be a differential equation with y as the dependent variable and u as the independent variable.
   Starting with $z = 2\sqrt{u}$, we have $dz = du/\sqrt{u}$. The differential operator then becomes:
   $\frac{d}{dz} \rightarrow \sqrt{u}\frac{d}{du}$

$$4u\sqrt{u}\frac{d}{du}(\sqrt{u}\frac{dy}{du}) + 2\sqrt{u}\sqrt{u}\frac{dy}{du} - (1-2\sqrt{u})y = 0$$
$$4u\sqrt{u}(\frac{1}{2\sqrt{u}}\frac{dy}{du} + \sqrt{u}\frac{d^2y}{du^2}) + 2u\frac{dy}{du} - (1-2\sqrt{u})y = 0$$
$$(2u\frac{dy}{du} + 4u^2\frac{d^2y}{du^2}) + 2u\frac{dy}{du} - (1-2\sqrt{u})y = 0$$
$$4u^2\frac{d^2y}{du^2} + 4u\frac{dy}{du} - (1-2\sqrt{u})y = 0$$

NOTE: This is actually the wrong change of variable. I meant to write $u = 2\sqrt{z}$ instead. Thus, it does not look as pretty as it could. Since the goal was to practice rewriting differential equations, it is okay we chose the wrong transformation. Feel free to try the "right" one and see what you get for more practice, since problems like this are likely to appear on the exam!

5. **Laguerre Polynomials**

The differential equation for Laguerre polynomials $L_m(z)$ is given by

$$zL'' + (1-z)L' + nL = 0$$

Find a polynomial solution of this differential equation for the case $n = 4$. For what values of z is your solution valid?

Solution:

Our answer at the end should be a polynomial in z: $L_4(z) = c_0 + c_1 z + c_2 z^2 +$ a finite number of terms. Our task is to find the c coefficients.
We start by assuming

$$y(z) = \sum_{m=0}^{\infty} c_m z^m,$$

giving

$$y'(z) = \sum_{m=1}^{\infty} m c_m z^{m-1}$$

and

$$y''(z) = \sum_{m=2}^{\infty} m(m-1) c_m z^{m-2}$$

Rearranging the terms in the sums gives us:

$$z\sum_{m=2}^{\infty} m(m-1)c_m z^{m-2} + (1+z)\sum_{m=1}^{\infty} m c_m z^{m-1} + n\sum_{m=0}^{\infty} c_m z^m = 0$$

$$\sum_{m=0}^{\infty}\left[(m+1)^2 c_{m+1}-(m-n)c_m\right] z^m = 0$$

The parentheses must be equal to 0 for all m, so:
$c_{m+1} = \frac{m-n}{(m+1)^2} c_m$
Note that this recursion relation does not skip anything, so we only get a single solution. This is a consequence of choosing a "bad" point to expand around. The solution we do get is still perfectly valid because it is a polynomial, which has no convergence issues.

$$\begin{gathered} z^0 : c_1 = -4c_0 \\ z^1 : c_2 = -3c_1/4 = 3c_0 \\ z^2 : c_3 = -2c_2/9 = -2c_0/3 \\ z^3 : c_4 = -c_3/16 = c_0/24 \\ z^4 : c_5 = 0 \end{gathered}$$

$L_4(z) = c_0(1 - 4z + 3z^2 - 2z^3/3 + z^4/24)$
As discussed above, this solution is valid for all z because it is a terminating polynomial.

6. Consider two particles of equal mass m. The forces on the particles are $\vec{F}_1 = 0$ and $\vec{F}_2 = F_0\hat{x}$. If the particles are initially at rest at the origin, find the position, velocity, and acceleration of the center of mass as functions of time. Solve this problem in two ways, with or without theorems about the center of mass motion. Write a short description comparing the two solutions.

Solution:

Our answer at the end should be functions $a_{cm}(t) =$, $v_{cm}(t) =$, and $x_{cm}(t) =$. **Method 1** - solving for the motion of one of the particles, then see what happens to the center of mass.

Find the acceleration of particle 2 using Newton's 2nd Law:

$$a_{2,x} = \frac{F_0}{m}$$

Since the acceleration is only in the x-direction, only the x-coordinate will change with time. Integrate to get the velocity and position as a function of time:

$$\begin{aligned}
\int_{t'=0}^{t} a_{2,x} dt' &= \int_{t'=0}^{t} \frac{F_0}{m}\, dt' \\
\int_{t'=0}^{t} \frac{d^2x}{dt^2}\, dt' &= \frac{F_0}{m} t \\
v_{2,x}(t) - v(t=0) &= \frac{F_0}{m} t \\
v_{2,x}(t) &= \frac{F_0}{m} t \\
\int_{t'=0}^{t} v_{2,x}(t) dt' &= \int_{t'=0}^{t} \frac{F_0}{m} t dt' \\
x(t) - x(t=0) &= \frac{F_0}{m} t^2 \\
x(t) &= \frac{F_0}{2m} t^2
\end{aligned}$$

We have used the fact that the initial velocity and position of both particles is zero. Now, evaluate the center of mass:

$$\begin{aligned}
x_{CM} &= \frac{m_1}{M} x_1(t) + \frac{m_2}{M} x_2(t) \\
&= \frac{1}{2} \left(\frac{F_0}{2m} t^2 \right) \\
&= \frac{F_0}{4m} t^2
\end{aligned}$$

Method 2 - using center of mass theorems.

I can treat the system as if all the mass is located at the center of mass and the force is being exerted directly onto the center of mass.

$$a_{CM,x} = \frac{F_0}{2m}$$

$$\int_{t'=0}^{t} a_{CM} dt' = \int_{t'=0}^{t} \frac{F_0}{2m} dt'$$

$$\int_{t'=0}^{t} \frac{d^2x}{dt^2} dt' = \frac{F_0}{2m} t$$

$$v_{CM,x}(t) = \frac{F_0}{2m} t$$

$$\int_{t'=0}^{t} v_{CM,x}(t) dt' = \int_{t'=0}^{t} \frac{F_0}{2m} t dt'$$

$$x_{CM}(t) - x(t=0) = \frac{F_0}{m} t^2$$

$$x_{CM}(t) = \frac{F_0}{4m} t^2$$

In method 1, all of the force is acting on the second particle and the first one doesn't move at all. In method 2, the force is acting on the fictitious center of mass which, at all times, is half way (half because the particles have equal masses) between the two particles. The two methods give the same answer.

7. (a) Find $\mathbf{r}_{\text{sun}} - \mathbf{r}_{\text{cm}}$ and μ for the Sun–Earth system. Compare $\mathbf{r}_{\text{sun}} - \mathbf{r}_{\text{cm}}$ to the radius of the Sun and to the distance from the Sun to the Earth. Compare μ to the mass of the Sun and the mass of the Earth.

Solution:

Our answer at the end should be a direct comparison (for example, a ratio) between radii and a different comparison between masses. **Sun↔Earth**

$$\begin{aligned} |\mathbf{r}_{\text{s}} - \mathbf{r}_{\text{e}}| &= 148 \times 10^9 m \\ R_{\text{s}} &= \text{radius of Sun} \\ &= 7 \times 10^8 m \\ m_{\text{s}} &= \text{mass of Sun} \\ &= 2 \times 10^{30} kg \\ m_{\text{e}} &= \text{mass of Earth} \\ &= 5.98 \times 10^{24} kg \end{aligned}$$

Therefore:

$$\mu = \frac{m_{\text{e}} m_{\text{s}}}{m_{\text{e}} + m_{\text{s}}} = 5.98 \times 10^{24} kg \approx m_{\text{e}} = 3 \times 10^{-6} m_{\text{s}}$$

and:

$$\begin{aligned}
|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{cm}| &= \left|\mathbf{r}_\mathrm{s} - \frac{m_\mathrm{e}\mathbf{r}_\mathrm{e} + m_\mathrm{s}\mathbf{r}_\mathrm{s}}{m_\mathrm{e} + m_\mathrm{s}}\right| \\
&= \frac{m_\mathrm{e}}{m_\mathrm{e} + m_\mathrm{s}}|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{e}| \\
&= 4.425 \times 10^5 m
\end{aligned}$$

Comparisons:

$$\begin{aligned}
\frac{|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{cm}|}{R_\mathrm{s}} &= 6.3 \times 10^{-4} \\
\frac{|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{cm}|}{|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{e}|} &= 3.0 \times 10^{-6}
\end{aligned}$$

i.e. The center-of-mass, about which the sun and earth both rotate, is well inside the sun itself.

(b) Repeat the calculation for the Sun–Jupiter system.

Solution:

Sun↔Jupiter

$$\begin{aligned}
|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{j}| &= 778 \times 10^9 m \\
R_\mathrm{s} &= \text{radius of Sun} \\
&= 7 \times 10^8 m \\
m_\mathrm{s} &= \text{mass of Sun} \\
&= 2 \times 10^{30} kg \\
m_\mathrm{j} &= \text{mass of Jupiter} \\
&= 1.90 \times 10^{27} kg
\end{aligned}$$

Therefore:

$$\mu = \frac{m_\mathrm{j} m_\mathrm{s}}{m_\mathrm{j} + m_\mathrm{s}} = 1.898 \times 10^{27} kg \approx m_\mathrm{j} = .95 \times 10^{-3} m_\mathrm{s}$$

and:

$$\begin{aligned}
|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{cm}| &= \left|\mathbf{r}_\mathrm{s} - \frac{m_\mathrm{j}\mathbf{r}_\mathrm{j} + m_\mathrm{s}\mathbf{r}_\mathrm{s}}{m_\mathrm{j} + m_\mathrm{s}}\right| \\
&= \frac{m_\mathrm{j}}{m_\mathrm{j} + m_\mathrm{s}}|\mathbf{r}_\mathrm{s} - \mathbf{r}_\mathrm{j}| \\
&= 7.38 \times 10^8 m
\end{aligned}$$

Comparisons:

$$
\begin{aligned}
\frac{|\mathbf{r}_{\rm s} - \mathbf{r}_{\rm cm}|}{R_{\rm s}} &= 1.05 \\
\frac{|\mathbf{r}_{\rm s} - \mathbf{r}_{\rm cm}|}{|\mathbf{r}_{\rm s} - \mathbf{r}_{\rm j}|} &= 9.5 \times 10^{-4}
\end{aligned}
$$

i.e. The center-of-mass, about which the Sun and Jupiter both rotate, is actually (just) outside the Sun itself!