Complex Analysis - MTH 483 **John Waczak**
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More exponential stuff

Last time we defined the following:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ... \\ \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \end{aligned}$$

We defined the complex exponential function, e^z to be the power series. We haven't done a lot of power series so for now we just have to trust that this converges $\forall z$ and defines a holomorphic function. Using this we can define the cosine and sine functions.

We also mentioned that these are entire functions (holomorphic on $\mathbb{C}$). We also proved that $e^{z+w} = e^z e^w$. Here this isn't really algebra so we had to prove this using the power series definition. Furthermore we used this to show $e^0 = 1$.

From these definitions of cosine and sine we can derive the ordinary power series for those functions:

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ &= \frac{1}{2} \sum \frac{1}{n!}(iz)^n + \frac{1}{2} \sum \frac{1}{n!}(-iz)^n \\ &= \frac{1}{2} \sum \frac{1}{n!}(i^n + (-1)i^n)z^n \\ \text{let} \quad n &= 2m \quad \text{only even n contribute} \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{2m!} 2(-1)^m z^{2m} \\ \Rightarrow \cos z &= \sum \frac{(-1)^m}{2m!} z^{2m} \end{aligned}$$

We can do the same to derive the standard power series for the sine function. We can also check that $\frac{1}{e^z} = e^{-z} = (e^z)^\star$.

$$\begin{aligned} 1 = e^0 &= e^{z-z} = e^z e^{-z} \\ |e^{iy}| &= \sqrt{\cos^2 y + \sin^2 y} = 1 \\ \Rightarrow |e^{x+iy}| &= |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x \end{aligned}$$

Now let's consider periodicity of exponential representation for complex numbers:

$$e^{z+2\pi i} = e^{x+iy+2\pi i} = e^x e^{i(y+2\pi)} = e^{x+iy} = e^z$$

Finally, let's think about the derivative.

$$\begin{aligned}
f(z) = e^z &\Rightarrow f'(z) = e^z \\
\text{pf:} \quad \frac{d}{dt}\sum \frac{1}{n!}z^n &= \sum_{n=1}^{\infty} \frac{n}{n!} z^(n-1) \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\
\text{let } m = n - 1 \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} z^m = e^z
\end{aligned}$$

The complex logarithm

We want to try to define the inverse of the exponential function but this is tough because we don't have 1-to-1 since $e^{2\pi i} = e^z$. So we can't truly define an inverse for e^z (sort of). We can still *almsot* define an inverse. Recall that we are fine to define the inverse so long as we restrict the domain of the inverse to within the domain of periodicity (i.e. $(0, 2\pi)$).

Definition: Let $\Omega \subseteq \mathbb{C}$ be a region. A branch of the complex logarithm is a function $\log : \Omega \to \mathbb{C}$ satisfying the identity that $e^{\log z} = z$.

Remark: If log is a continuous branch of the logarithm, then so is the function $\log(z + 2\pi i)$. Because $e^{\log(z)+2\pi i} = e^{\log(z)} = z$. Therefore the logarithm is *not* unique.

Definition: Let $z \in \mathbb{C}$, $z = re^{i\phi}$ where $r = |z|$ and let $\phi \in (-\pi, \pi]$. Let $\arg z = \phi$. Assuming $z \neq 0$, define $Log z = \ln|z| + i\arg z$. This defines a function $Log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$. This function is called the **principal branch** of the logarithm.

$$e^{Log z} = e^{\ln|z| + i\arg z} = e^{\ln r} e^{i\phi} = re^{i\phi} = z$$

Warning: $Log z$ is not continuous along the non-positive real axis.

If $\log : \Omega \to \mathbb{C}$ is a branch of the logarithm, then log is holomorphic on Ω and $\log'(z) = \frac{1}{z}$.

$$\begin{aligned}
\text{proof sketch:} \\
e^{\log z} &= z \\
\Rightarrow (e^{\log z}) \log' z &= 1 \\
z \log'(z) &= 1 \\
\log'(z) &= \frac{1}{z}
\end{aligned}$$

Some examples

Examples:

$$\begin{aligned}
Log(2) &= \ln|2| \\
&= \ln(2) \\
Log(i) &= \ln|i| + i\arg(i) \\
&= \ln(1) + i\frac{\pi}{2} \\
&= 0 + i\frac{\pi}{2} \\
Log(-3) &= \ln(3) + i\arg(-3) \\
&= \ln(3) + i\pi \\
Log(-1+i) &= \ln|-1+i| + i\arg(-1+i) \\
&= \ln(\sqrt{2}) + i\frac{3\pi}{4}
\end{aligned}$$