1

Rigid Motions 4-C

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Mth 434
Homework 4C

Tapp. 1.73 prove that every proper rigid motion f in $\mathbb{R}^3$ that fixes the origin is a rotation about some axis.

If f is a proper rigid motion that fixes the origin by prop 1.57 $\exists A \in O(3)$ s.t. $f = L_A$. Now for A to be proper we require $\det(A) = 1$.

Let $f = L_A$ where $A \in O(3)$ w/ $\det(1)$. By the fundamental theorem of algebra we have that the characteristic polynomial for A can be written

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

where $\lambda_1\ \lambda_2\ \lambda_3$ are eigenvalues because $A \in O(3) \subset M_{3\times3}$ and so has a cubic characteristic polynomial.

In general λ_i could be $\in \mathbb{R}$ or $\mathbb{C}$. Assume W.L.O.G that $\lambda_3 \in \mathbb{C}$ then λ_2 or λ_1 must necessarily be equal to λ_3^* thus the third must be real so no matter what, we have at least 1 real eigenvalue. Otherwise all 3 must be real.

Let λ be real eigenvalue w/ normalized eigenvector. v. we have:

$$Av = \lambda v$$

recall that because $A \in O(3)$ it preserves distances and therefore norms. Thus by defintion 1.54

$$|Av| = |v| = |\lambda v|$$

$$\Longrightarrow |v| = |\lambda|\,|v|$$

and so because v is normalized $\lambda = \pm 1$.

Now consider $v = (x_1, x_2, x_3)$ we can always complet v to an orthonormal basis. let $u = (0, -x_3, x_2)$ then $\langle v, u \rangle = 0 - x_3 x_2 + x_3 x_2 = 0$ thus $v \perp u$. let u be normalized and define $w = v \times u$ then we have $w \perp v$, $w \perp u$ thus we have an orthonormal basis

$$\{v, u, w\}$$

Now we will find A in this new basis
Note that

$$Av = \lambda v$$

so the first column of a in the new basis must be $\begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$

Now if
$$Au = \alpha v + \beta u + \gamma w$$
$$Aw = \alpha' v + \beta' u + \gamma' w$$

by prop 1.55, A must preserve orthonormal basis. this means

$\langle Av, Au \rangle = 0$

$\langle Av, Aw \rangle = 0$

This is only possible if $\alpha = \alpha' = 0$ thus we can rewrite A as in the hint to have

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

By proposition 1.55 since the columns form an orthonormal basis, A in this new basis is still orthogonal. i.e.

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{pmatrix} =$$

$$\begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & a^2+b^2 & ac+bd \\ 0 & ac+bd & c^2+d^2 \end{pmatrix} = I$$

We know $\lambda^2 = 1$ and so we have $ac+bd=0$ and $a^2+b^2 = c^2+d^2 = 1$ thus the submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ multiplied by it's transpose $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ must also give the identity for $M_{2\times 2}$

This means $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$

Now recall that det is invariant under change of basis. Therefore

$$1 = \lambda \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

There are two cases. If $\lambda = 1$ then we know from Example 1.61 that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

because (a,c) and (b,d) must be unit vectors.
This is a rotation about θ.

If $\lambda = -1$ then 1.61 tells us that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

which is a reflection about line $\theta/2$ to. If this is the case then a vector

$$\vec{x} \in \text{span}\{(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)\}$$

will be unaffected by A and so is an eigenvector. Repeating the same proof using $\vec{x}$ as $\vec{v}$ would result in a rotation.

Thus any proper rigid motion f on $\mathbb{R}^3$ can w/ origin fixed is a rotation by some angel $\square$

T app 1.79

prove the following are equivalent

(1) A is orthogonal

(2) the rows of A form an orthonormal basis

(3) $AA^T = I$

$2 \to 3$ because the rows are orthonormal we have

$$\delta_{ij} = \langle (\text{row } i \text{ of } A), (\text{row } j \text{ of } A) \rangle$$

$$= \langle (\text{row } i \text{ of } A), (\text{column } j \text{ of } A^T) \rangle$$

$$= (A \cdot A^T)_{ij}$$

thus if rows are orthonormal then

$$(A \cdot A^T)_{ij} = \delta_{ij} \Rightarrow AA^T = I$$

$2 \leftarrow 3$ $AA^T = I$

means $\delta_{ij} = \langle (\text{row } i \text{ of } A), (\text{column } j \text{ of } A^T) \rangle$

$$= \langle (\text{row } i \text{ of } A), (\text{row } j \text{ of } A) \rangle$$

$\Rightarrow$ the rows of A form an orthonormal basis.

Thus we have show $2 \Leftrightarrow 3$

$(1) \to (3)$

let A be orthogonal. Then by 1.55

$$A^T A = I$$

if $A^T A = I$

we wts $AA^T = I$

Note that $A = AI = A(A^T A) = IA$

Now clearly

$$IA - AI = 0 \qquad (A - A = 0)$$

so $IA - A(A^T A) = 0$

$$IA - AA^T A = 0$$

$$(I - AA^T)A = 0$$

$$\implies I - AA^T = 0$$

$$\implies AA^T = I$$

thus $(1) \to (3)$

Now we will show

$(1) \leftarrow (3)$ $AA^T = I$

we w.t.s $\Rightarrow AA^T = I$

Note that $A = IA = (AA^T)A$

Now clearly we have that it must be true that

$$AI - IA = 0$$

so $AI - (AA^T)A = 0$

$$AI - AA^T A = 0$$

$$A(I - A^T A) = 0$$

thus $A^T A = I$ and by prop 1.55 A must be orthogonal.

thus we have shown $(1) \Leftrightarrow (3)$ and $(2) \Leftrightarrow (3)$ therefore the statements are equivalent.