4. let $f: [a,b] \to \mathbb{R}$ be strictly increasing. Show that $\int_a^b f(x)\,dx$ exists

John Waczak

4

Recall $f: [a,b] \to \mathbb{R}$ is intlable on $[a,b]$ iff $\forall \varepsilon > 0$ $\exists$ step functions f_1, f_2 s.t.

$$f_1(x) \le f(x) \le f_2(x) \quad \forall x \in [a,b]$$

and

$$\int_a^b f_2(x) - f_1(x)\,dx < \varepsilon$$

Lets define f_2 to be a step function defined as

$$f_2(x) = \sum_{i=1}^{N} f(x_i)\, \mathbb{1}_{(x_{i-1}, x_i)}(x) \quad \text{(right end p}$$

and

$$f_1(x) = \sum_{i=1}^{N} f(x_{i-1})\, \mathbb{1}_{(x_{i-1}, x_i)}(x) \quad \text{(left end pt}$$

then $\forall x \in [a,b]$ it is true that

$$f_1(x) \le f(x) \le f_2(x)$$

Now WTS $\int_a^b f_2(x) - f_1(x)\,dx < \varepsilon$.

by linearity of int we have

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

(next page)