1. The potential due to a ring of charge is given by:

$$V(s,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{s^2+R^2-2sR\cos(\phi-\phi')+z^2}}$$

Expand this potential in a power series to fourth order, in the plane of the ring, for $s < R$. Warning: Make sure you keep **all** of the terms up to fourth order and none of the terms of higher order. This is tricky to do and is the most important lesson from this homework problem.

Solution:

To expand this potential in a power series, it would be nice to save some effort and use the series we have already memorized (Quiz 1). Recall,

$$(1+u)^p = 1 + pu + \frac{p(p-1)}{2!}u^2 + \frac{p(p-1)(p-2)}{3!}u^3 + \frac{p(p-1)(p-2)(p-3)}{4!}u^4 + \cdots \tag{1}$$

We are looking at the potential in the plane of the ring, so $z = 0$. We can also rewrite the square root as a power.

$$V(s,\phi,z=0) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\Big(s^2+R^2-2sR\cos(\phi-\phi')\Big)^{-1/2}d\phi' \tag{2}$$

The integrand is almost the same as equation (1) but we need it to match exactly for the power series to be valid. **Remember:** equation (1) is valid only for $|u| < 1$. We are interested in finding the potential where $s < R$, or in other words, $s/R < 1$ is a small quantity and we can *pull out* R^2 from the expression. That is,

$$\Big(s^2+R^2-2sR\cos(\phi-\phi')\Big)^{-1/2} = \Big[R^2\Big(1+\frac{s^2}{R^2}-\frac{2s}{R}\cos(\phi-\phi')\Big)\Big]^{-1/2} \tag{3}$$

$$= \frac{1}{R}\Big(1+\frac{s^2}{R^2}-\frac{2s}{R}\cos(\phi-\phi')\Big)^{-1/2} \tag{4}$$

$$u \equiv \frac{s^2}{R^2}-\frac{2s}{R}\cos(\phi-\phi'), \qquad p = -1/2 \tag{5}$$

where in the final line I have identified our u and p for the series expansion.

Now, we need to expand the powers of u in order to find all of the fourth order terms in $\frac{s}{R}$. Yes, this is a lot of algebra.

$$p(u) = \left(\frac{s}{R}\right)\cos(\phi-\phi') - \frac{1}{2}\left(\frac{s}{R}\right)^2 \tag{6}$$

$$\frac{p(p-1)}{2!}u^2 = \frac{5}{2}\left(\frac{s}{R}\right)^2\cos^2(\phi-\phi') - \frac{3}{2}\left(\frac{s}{R}\right)^3\cos(\phi-\phi') + \frac{3}{8}\left(\frac{s}{R}\right)^4 \tag{7}$$

$$\frac{p(p-1)(p-2)}{3!}u^3 = \frac{5}{2}\left(\frac{s}{R}\right)^3\cos^3(\phi-\phi') - \frac{15}{4}\left(\frac{s}{R}\right)^4\cos^2(\phi-\phi/) + ... \tag{8}$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}u^4 = \frac{35}{8}\left(\frac{s}{R}\right)^4\cos^4(\phi-\phi') + ... \tag{9}$$

Using this to combine all terms with like powers in s/R, the integral reduces to

$$
\begin{aligned}
V(s,\phi) \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{R}\int_0^{2\pi}\Bigg\{ & 1 + \cos(\phi-\phi')\left(\frac{s}{R}\right) \\
& + \left[\frac{5}{2}\cos^2(\phi-\phi') - \frac{1}{2}\right]\left(\frac{s}{R}\right)^2 \\
& + \left[\frac{5}{2}\cos^3(\phi-\phi') - \frac{3}{2}\cos(\phi-\phi')\right]\left(\frac{s}{R}\right)^3 \\
& + \left[\frac{3}{8} - \frac{15}{4}\cos^2(\phi-\phi') + \frac{35}{8}\cos^4(\phi-\phi')\right]\left(\frac{s}{R}\right)^4 \Bigg\} d\phi'
\end{aligned}
\tag{10}
$$

The original equation for the potential can not be integrated analytically. Now that we have expanded the integrand, we have reduced the problem to a bunch of integrals of $\cos^n(\phi-\phi')$ which we can solve by brute force. Before we proceed though, let's pause and think for a moment about the symmetries of the ring.

The ring has cylindrical symmetry. The charge Q is also evenly distributed along this ring. This means that the potential V can not depend on ϕ because if we were to close our eyes and rotate the ring, the new system would be indistinguishable from the original when we open our eyes again.

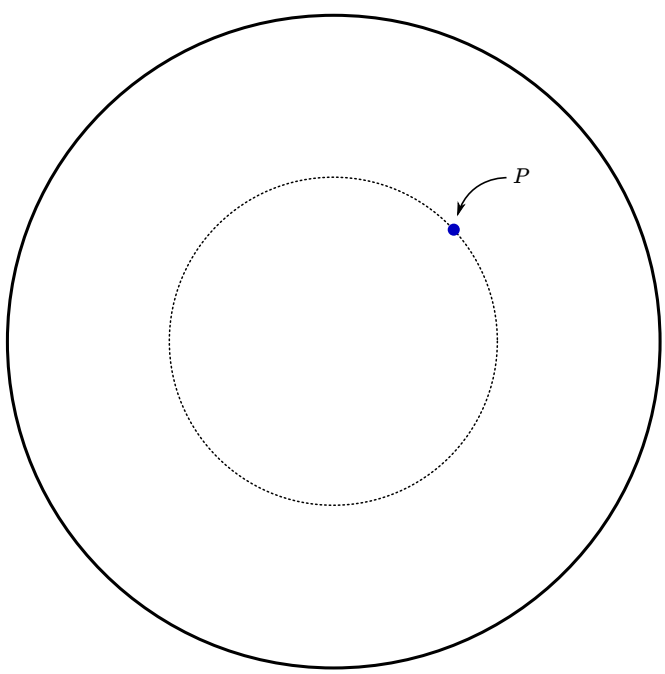

Figure 1: A point p within a ring of evenly distributed charge. All points along the dashed circle must have the same value of potential due to the symmetry of the charge distribution.

From the above argument we know that the final solution can have no ϕ dependence. We can therefore choose a particular value of ϕ to simplify the integral. For convenience, let's say $\phi = 0$. Then our integral is reduced to the following

$$
\begin{aligned}
V(s) \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{R}\int_0^{2\pi}\Bigg\{ & 1+\cos(\phi')\left(\frac{s}{R}\right) \\
& + \left[\frac{5}{2}\cos^2(\phi') - \frac{1}{2}\right]\left(\frac{s}{R}\right)^2 \\
& + \left[\frac{5}{2}\cos^3(\phi') - \frac{3}{2}\cos(\phi')\right]\left(\frac{s}{R}\right)^3 \\
& + \left[\frac{3}{8} - \frac{15}{4}\cos^2(\phi') + \frac{35}{8}\cos^4(\phi')\right]\left(\frac{s}{R}\right)^4 \Bigg\} d\phi'
\end{aligned} \tag{11}
$$

In solving the above, the following integrals will be useful (look them up or try using the exponential form of cosine to solve).

$$\int \cos(x) = \sin(x) + C \tag{12}$$

$$\int \cos^2(x) = \int \frac{1}{2}(\cos(2x)+1)dx = \frac{1}{2}(x + \sin(x)\cos(x)) + C \tag{13}$$

$$\int \cos^3(x) = \frac{1}{12}(9\sin(x) + \sin(3x)) + C \tag{14}$$

$$\int \cos^4(x) = \frac{1}{32}(12x + 8\sin(2x) + \sin(4x)) + C \tag{15}$$

$$
\begin{aligned}
V(s) \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{R}\Bigg\{ & 2\pi + \left(\frac{s}{R}\right)\int_0^{2\pi}\cos(\phi')d\phi' \\
& + \left(\frac{s}{R}\right)^2\int_0^{2\pi}\left[\frac{5}{2}\cos^2(\phi') - \frac{1}{2}\right]d\phi' \\
& + \left(\frac{s}{R}\right)^3\int_0^{2\pi}\left[\frac{5}{2}\cos^3(\phi') - \frac{3}{2}\cos(\phi')\right]d\phi' \\
& + \left(\frac{s}{R}\right)^4\int_0^{2\pi}\left[\frac{3}{8} - \frac{15}{4}\cos^2(\phi') + \frac{35}{8}\cos^4(\phi')\right]d\phi' \Bigg\}
\end{aligned} \tag{16}
$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{R}\left\{2\pi + \frac{3\pi}{2}\left(\frac{s}{R}\right)^2 - 3\pi\left(\frac{s}{R}\right)^4\right\} \tag{17}$$

$$= \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{R} + \frac{3}{4}\frac{s^2}{R^3} - \frac{3}{2}\frac{s^4}{R^5}\right\} \tag{18}$$

$$\tag{19}$$

NOTE: our solution is an even function in s. Why is that?

CHECK: Does our solution agree with the original integral equation at the origin?