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0. WARMUP

Determine the (nonzero) components of $R^i{}_{jkl}$ of the curvature 2-forms

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl} \sigma^k \wedge \sigma^l \tag{1}$$

for the Robertson-Walker geometry, with line element

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{2}$$

with $k = -1, 0, 1$, depending on whether the spatial cross-sections are hyperbolic, flat, or spherical respectively.

The curvature 2-forms for the Robertson-Walker geometry are given by

$$\left(\Omega^i{}_j\right) = \begin{pmatrix} 0 & \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^r & \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\theta & \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\phi \\ \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^r & 0 & \frac{\dot{a}^2+k}{a^2}\sigma^r \wedge \sigma^\theta & \frac{\dot{a}^2+k}{a^2}\sigma^r \wedge \sigma^\phi \\ \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\theta & -\frac{\dot{a}^2+k}{a^2}\sigma^r \wedge \sigma^\theta & 0 & \frac{\dot{a}^2+k}{a^2}\sigma^\theta \wedge \sigma^\phi \\ \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\phi & -\frac{\dot{a}^2+k}{a^2}\sigma^r \wedge \sigma^\phi & -\frac{\dot{a}^2+k}{a^2}\sigma^\theta \wedge \sigma^\phi & 0 \end{pmatrix} \tag{3}$$

Note that we have used the property that $\Omega_{ji} = -\Omega_{ij}$. Also note that the curvature two forms each only depend on one basis 2-form. We can now try and find the components of the curvature 2-forms using (1). These will clearly only be non-zero for non-zero $\Omega^i{}_j$.

Recall that by convention, $R^i{}_{jlk} = -R^i{}_{jkl}$ so that beginning with the first row of (3), we have

$$\Omega^t{}_r = \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^r = \frac{1}{2}\left(R^t{}_{rtr}\sigma^t \wedge \sigma^r + R^t{}_{rrt}\sigma^r \wedge \sigma^t\right) = R^t{}_{trt}\sigma^r \wedge \sigma^t \tag{4}$$

$$\Rightarrow R^t{}_{rtr} = \frac{\ddot{a}}{a}, \qquad R^t{}_{rrt} = -\frac{\ddot{a}}{a} \tag{5}$$

$$\Omega^t{}_\theta = \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\theta = \frac{1}{2}\left(R^t{}_{\theta t\theta}\sigma^t \wedge \sigma^\theta + R^t{}_{\theta\theta t}\sigma^\theta \wedge \sigma^t\right) = R^t{}_{\theta t\theta}\sigma^t \wedge \sigma^\theta \tag{6}$$

$$\Rightarrow R^t{}_{\theta t\theta} = \frac{\ddot{a}}{a}, \qquad R^t{}_{\theta\theta t} = -\frac{\ddot{a}}{a} \tag{7}$$

$$\Omega^t{}_\phi = \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\phi = \frac{1}{2}\left(R^t{}_{\phi t\phi}\sigma^t \wedge \sigma^\phi + R^t{}_{\phi\phi t}\sigma^\phi \wedge \sigma^t\right) = R^t{}_{\phi t\phi}\sigma^t \wedge \sigma^\phi \tag{8}$$

$$\Rightarrow R^t{}_{\phi t\phi} = \frac{\ddot{a}}{a}, \qquad R^t{}_{\phi\phi t} = -\frac{\ddot{a}}{a} \tag{9}$$

$$\Omega^r{}_\theta = \frac{\dot{a}^2+k}{a^2}\sigma^r \wedge \sigma^\theta = \frac{1}{2}\left(R^r{}_{\theta r\theta}\sigma^r \wedge \sigma^\theta + R^r{}_{\theta\theta r}\sigma^\theta \wedge \sigma^r\right) = R^r{}_{\theta r\theta}\sigma^r \wedge \sigma\theta \tag{10}$$

$$\Rightarrow R^r{}_{\theta r\theta} = \frac{\dot{a}^2+k}{a^2}, \qquad R^r{}_{\theta\theta r} = -\frac{\dot{a}^2+k}{a^2} \tag{11}$$

$$\Omega^r{}_\phi = \frac{\dot{a}^2+k}{a^2}\sigma^r\wedge\sigma^\phi = \frac{1}{2}\left(R^r{}_{\phi r\phi}\sigma^r\wedge\sigma^\phi + R^r{}_{\phi\phi r}\sigma^\phi\wedge\sigma^r\right) = R^r{}_{\phi r\phi}\sigma^r\wedge\sigma^\phi \tag{12}$$

$$\Rightarrow R^r{}_{\phi r\phi} = \frac{\dot{a}^2+k}{a^2}, \qquad R^r{}_{\phi\phi r} = -\frac{\dot{a}^2+k}{a^2} \tag{13}$$

$$\Omega^\theta{}_\phi = \frac{\dot{a}^2+k}{a^2}\sigma^\theta\wedge\sigma^\phi = \frac{1}{2}\left(R^\theta{}_{\phi\theta\phi}\sigma^\theta\wedge\sigma^\phi + R^\theta{}_{\phi\phi\theta}\sigma^\phi\wedge\sigma^\theta\right) = R^\theta{}_{\phi\theta\phi}\sigma^\theta\wedge\sigma^\phi \tag{14}$$

$$\Rightarrow R^\theta{}_{\phi\theta\phi} = \frac{\dot{a}^2+k}{a^2}, \qquad R^\theta{}_{\phi\phi\theta} = -\frac{\dot{a}^2+k}{a^2} \tag{15}$$

For the items below the diagonal, we have

$$\Omega^r{}_t = \frac{\ddot{a}}{a}\sigma^t\wedge\sigma^r = \frac{1}{2}\left(R^r{}_{trt}\sigma^r\wedge\sigma^t + R^r{}_{ttr}\sigma^t\wedge\sigma^r\right) = R^r{}_{ttr}\sigma^t\wedge\sigma^r \tag{16}$$

$$\Rightarrow R^r{}_{ttr} = \frac{\ddot{a}}{a} \qquad R^r{}_{trt} = -\frac{\ddot{a}}{a} \tag{17}$$

$$\Omega^\theta{}_t = \frac{\ddot{a}}{a}\sigma^t\wedge\sigma^\theta = \frac{1}{2}\left(R^\theta{}_{t\theta t}\sigma^\theta\wedge\sigma^t + R^\theta{}_{tt\theta}\sigma^t\wedge\sigma^\theta\right) = R^\theta{}_{tt\theta}\sigma^t\wedge\sigma^\theta \tag{18}$$

$$\Rightarrow R^\theta{}_{tt\theta} = \frac{\ddot{a}}{a} \qquad R^\theta{}_{t\theta t} = -\frac{\ddot{a}}{a} \tag{19}$$

$$\Omega^\phi{}_t = \frac{\ddot{a}}{a}\sigma^t\wedge\sigma^\phi = \frac{1}{2}\left(R^\phi{}_{t\phi t}\sigma^\phi\wedge\sigma^t + R^\phi{}_{tt\phi}\sigma^t\wedge\sigma^\phi\right) = R^\phi{}_{tt\phi}\sigma^t\wedge\sigma^\phi \tag{20}$$

$$\Rightarrow R^\phi{}_{tt\phi} = \frac{\ddot{a}}{a} \qquad R^\phi{}_{t\phi t} = -\frac{\ddot{a}}{a} \tag{21}$$

$$\Omega^\theta{}_r = -\frac{\dot{a}^2+k}{a^2}\sigma^r\wedge\sigma^\theta = \frac{1}{2}\left(R^\theta{}_{r\theta r}\sigma^\theta\wedge\sigma^r + R^\theta{}_{rr\theta}\sigma^r\wedge\sigma^\theta\right) = R^\theta{}_{rr\theta}\sigma^r\wedge\sigma^\theta \tag{22}$$

$$\Rightarrow R^\theta{}_{rr\theta} = -\frac{\dot{a}^2+k}{a^2} \qquad R^\theta{}_{r\theta r} = \frac{\dot{a}^2+k}{a^2} \tag{23}$$

$$\Omega^\phi{}_r = -\frac{\dot{a}^2+k}{a^2}\sigma^r\wedge\sigma^\phi = \frac{1}{2}\left(R^\phi{}_{r\phi r}\sigma^\phi\wedge\sigma^r + R^\phi{}_{rr\phi}\sigma^r\wedge\sigma^\phi\right) = R^\phi{}_{rr\phi}\sigma^r\wedge\sigma^\phi \tag{24}$$

$$\Rightarrow R^\phi{}_{rr\phi} = -\frac{\dot{a}^2+k}{a^2} \qquad R^\phi{}_{r\phi r} = \frac{\dot{a}^2+k}{a^2} \tag{25}$$

$$\Omega^\phi{}_\theta = -\frac{\dot{a}^2+k}{a^2}\sigma^\theta\wedge\sigma^\phi = \frac{1}{2}\left(R^\phi{}_{\theta\phi\theta}\sigma^\phi\wedge\sigma^\theta + R^\phi{}_{\theta\theta\phi}\sigma^\theta\wedge\sigma^\phi\right) = R^\phi{}_{\theta\theta\phi}\sigma^\theta\wedge\sigma^\phi \tag{26}$$

$$\Rightarrow R^\phi{}_{\theta\theta\phi} = -\frac{\dot{a}^2+k}{a^2} \qquad R^\phi{}_{\theta\phi\theta} = \frac{\dot{a}^2+k}{a^2} \tag{27}$$

Having exhausted all of the non-zero curvature two-forms, I believe we have found all of the components.

1. Using the relationships

$$R_{ij} = R^m{}_{imj} \tag{28}$$

$$G^i{}_j = R^i{}_j - \frac{1}{2}\delta^i{}_j R \tag{29}$$

Compute the (nonzero) components $G^i{}_j$ of the *Einstein tensor* for the Robertson-Walker geometry.

Given the curvature two-form components $R^i{}_{jkl}$ from problem 0, we can now find the components of the Ricci curvature tensor. Let's start with the diagonal terms

$$R_{tt} = R^m{}_{tmt} = R^t{}_{ttt} + R^r{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t} \tag{30}$$

$$= 0 - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} = -3\frac{\ddot{a}}{a} \tag{31}$$

$$R_{rr} = R^m{}_{rmr} = R^t{}_{rtr} + R^r{}_{rrr} + R^\theta{}_{r\theta r} + R^\phi{}_{r\phi r} \tag{32}$$

$$= \frac{\ddot{a}}{a} + 0 + \frac{\dot{a}^2 + k}{a^2} + \frac{\dot{a}^2 + k}{a^2} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \tag{33}$$

$$R_{\theta\theta} = R^m{}_{\theta m\theta} = R^t{}_{\theta t\theta} + R^r{}_{\theta r\theta} + R^\theta{}_{\theta\theta\theta} + R^\phi{}_{\theta\phi\theta} \tag{34}$$

$$= \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + 0 + \frac{\dot{a}^2 + k}{a^2} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \tag{35}$$

$$R_{\phi\phi} = R^m{}_{\phi m\phi} = R^t{}_{\phi t\phi} + R^r{}_{\phi r\phi} + R^\theta{}_{\phi\theta\phi} + R^\phi{}_{\phi\phi\phi} \tag{36}$$

$$= \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + \frac{\dot{a}^2 + k}{a^2} + 0 = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \tag{37}$$

Putting all of these together, we can calculate the trace of the Ricci curvature $R^i{}_i$. Remember that $R^i{}_j = g^{ik}R_{kj}$.

$$R = g^{tk}R_{kt} + g^{rk}R_{kr} + g^{\theta k}R_{k\theta} + g^{\phi k}R_{k\phi} = 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2 + k}{a^2} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \tag{38}$$

Note that all of the non-zero curvature components are of the form $R^m{}_{nnm}$ or $R^m{}_{nmn}$. Therefore, it follows that all off diagonal Ricci curvature components $R_{ij} = 0$ where $i \neq j$.

Using this result, we conclude that the *Einstein tensor* is diagonal and therefore we only need to calculate four elements. Better yet, $R_{rr} = R_{\theta\theta} = R_{\phi\phi}$ and so we really only have to calculate two. That is,

$$
\begin{aligned}
G^t{}_t &= R^t{}_t - \frac{1}{2}\delta^t{}_t R && (39)\\
&= g_{tk}R^k{}_t - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) && (40)\\
&= 3\frac{\ddot{a}}{a} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) && (41)\\
&= -3\frac{\dot{a}^2+k}{a^2} && (42)\\
G^r{}_r &= R^r{}_r - \frac{1}{2}\delta^r{}_r R && (43)\\
&= g_{rk}R^k{}_r - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) && (44)\\
&= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2+k}{a^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) && (45)\\
&= -2\frac{\ddot{a}a}{a^2} - \frac{\dot{a}^2+k}{a^2} && (46)\\
&= -\frac{2a\ddot{a}+\dot{a}^2+k}{a^2} && (47)\\
G^\theta{}_\theta &= G^\phi{}_\phi = G^r{}_r = -\frac{2a\ddot{a}+\dot{a}^2+k}{a^2} && (48)
\end{aligned}
$$

Thus, we have specified all four non-zero Einstein tensor components for the Robertson-Walker geometry. These results agree with equations 9.15 and 9.16 in the textbook.