Complex Analysis - MTH 483 **John Waczak**
Day 11 Date: April 25, 2018

Moving towards Cauhcy's Theorem... Primitives

Definition let $f : \Omega \to \mathbb{C}$ be holomorphic. A *primitive* a.k.a. anti-derivative for f is a holomorphic function say $F : \Omega \to \mathbb{C}$ s.t. $F'(z) = f(z)$.

Example $f(z) = 2z$ is holomorphic on $\mathbb{C}$ and $F = z^2$ is a primitive.

Exmaple let $\Omega = \mathbb{C} \setminus \{0\}$ and let $f(z) = \frac{1}{z^n}$ for $n \geq 1$.

- Case 1: $n \geq 2$ in this case $F(z)$ is $\frac{1}{-n+1} z^{-n+1} = \frac{1}{(1-n)z^{n-1}}$ is a primitive.
- Case 2: if $n = 1$ i.e. $f(z) = \frac{1}{z}$ does *not* have a primitive on $\Omega = \mathbb{C} \setminus \{0\}$. However if we consider $\Omega = \mathbb{C} \setminus (-\infty, 0]$. On this region, $f(z)$ has $F(z) = Log(z)$.

Evaluating integrals with primitives

Theorem Let $f : \Omega \to \mathbb{C}$ be holomorphic with primitive F. Suppose we have a path γ in Ω. Then

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

pf. $F'(z) = f(z)$. Use the F.T.C. with inverse chain rule in integrand.

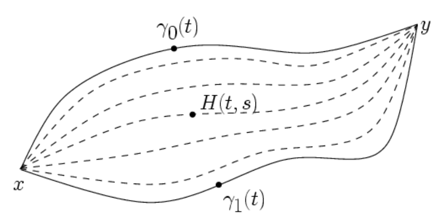
Theorem let $f : \Omega \to \mathbb{C}$ be holomorphic with primitive $F : \Omega \to \mathbb{C}$ and let γ be closed curve in Ω. Then $\int_\gamma f(z)dz = 0$. pf use previous theorem.

Now we see why $f(z) = 1/z$ does not have a primitive on $\mathbb{C} \setminus \{0\}$. If it did, then we would have $\int_{|z|=1} \frac{1}{z} dz = 0$. This is wrong. In fact, let $\gamma(t) = e^{it}$ then $\int_0^{2\pi} = \frac{1}{e^{it}} ie^{it} dt = 2\pi i \neq 0$.

Homotopy - continuously deforming one curve into another

Definition let Ω be a region in $\mathbb{C}$ and let $\gamma_0, \gamma_1 : [0, 1] \to \mathbb{C}$ be two closed curves in Ω. We say that γ_0 is Ω-homotopic to γ_1 if $\exists$ continuous function $h : [0, 1] \times [0, 1] \to \Omega$ such that $h(t, 0) = \gamma_0(t)$ and $h(t, 1) = \gamma_1(t)$. Finally, we want $h(0, s) = h(1, s)$ for closure.

Notaiton: $\gamma_0 \sim_\Omega \gamma_1$ signifies homotopy.



Theorem Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $\gamma_0 \sim_\Omega \gamma_1$ be curves in Ω. Then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$