Bessel Beams

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1 Problem

Deduce the form of a cylindrically symmetric plane electromagnetic wave that propagates in vacuum.

A scalar, azimuthally symmetric wave of frequency ω that propagates in the positive z direction could be written as

$$\psi(\mathbf{r}, t) = f(\rho)\, e^{i(k_z z - \omega t)}, \tag{1}$$

where $\rho = \sqrt{x^2 + y^2}$. Then, the problem is to deduce the form of the radial function $f(\rho)$ and any relevant condition on the wave number k_z, and to relate that scalar wave function to a complete solution of Maxwell's equations.

The waveform (1) has both phase velocity and group velocity equal to ω/k_z. Comment on the apparent superluminal character of the wave in case that $k_z < k_f = \omega/c$, where c is the speed of light.

2 Solution

As the desired solution for the radial wave function proves to be a Bessel function, the cylindrical plane waves have come to be called Bessel beams, following their introduction by Durnin *et al.* [1, 2]. The question of superluminal behavior of Bessel beams has recently been raised by Mugnai *et al.* [3].

Bessel beams are a realization of super-gain antennas [4, 5, 6] in the optical domain. A simple experiment to generate Bessel beams is described in [7].

Sections 2.1 and 2.2 present two methods of solution for Bessel beams that satisfy the Helmholtz wave equation. The issue of group and signal velocity for these waves is discussed in sec. 2.3. Forms of Bessel beams that satisfy Maxwell's equations are given in sec. 2.4.

2.1 Solution via the Wave Equation

On substituting the form (1) into the wave equation,

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}, \tag{2}$$

we obtain

$$\frac{d^2 f}{d\rho^2} + \frac{1}{\rho} \frac{df}{d\rho} + (k_f^2 - k_z^2) f = 0. \tag{3}$$

This is the differential equation for Bessel functions of order 0, so that

$$f(\rho) = J_0(k'\rho), \tag{4}$$

where

$$k'^2 + k_z^2 = k_f^2. \tag{5}$$

The form of eq. (5) suggests that we introduce a (real) parameter α such that

$$k' = k_f \sin\alpha, \qquad \text{and} \qquad k_z = k_f \cos\alpha. \tag{6}$$

Then, the desired cylindrical plane wave has the form

$$\psi(\mathbf{r}, t) = J_0(k'\rho)\, e^{i(k_z z - \omega t)} = J_0(k_f \sin\alpha\, \rho)\, e^{i(k_f \cos\alpha\, z - \omega t)}, \tag{7}$$

which is commonly called a Bessel beam.

From the second of eq. (6) we have,

$$\omega = ck_f = \frac{ck_z}{\cos\alpha}\,. \tag{8}$$

The physical significance of parameter α, and that of the group velocity

$$v_g = \frac{d\omega}{dk_z} = \frac{\omega}{k_z} = v_p = \frac{c}{\cos\alpha} \tag{9}$$

for a family of Bessel beams of varying frequency but constant α, will be discussed in sec. 2.3.

While eq. (7) is a solution of the Helmholtz wave equation (2), assigning $\psi(\mathbf{r}, t)$ to be a single component of an electric field, say E_x, does not provide a full solution to Maxwell's equations. For example, if $\mathbf{E} = \psi\hat{\mathbf{x}}$, then $\boldsymbol{\nabla} \cdot \mathbf{E} = \partial\psi/\partial x \neq 0$. Bessel beams that satisfy Maxwell's equations are given in sec. 2.4.

2.2 Solution via Scalar Diffraction Theory

The Bessel beam (7) has large amplitude only for $|\rho| \lesssim 1/k_f \sin\alpha$, and maintains the same radial profile over arbitrarily large propagation distance z. This behavior appears to contradict the usual lore that a beam of minimum transverse extent a diffracts to fill a cone of angle $1/a$. Therefore, the Bessel beam (7) has been called "diffraction free" [2].

Here, we show that the Bessel beam does obey the formal laws of diffraction, and can be deduced from scalar diffraction theory.

According to that theory [8], a cylindrically symmetric wave $f(\rho)$ of frequency ω at the plane $z = 0$ propagates to point $\mathbf{r}$ with amplitude

$$\psi(\mathbf{r}, t) = \frac{k_f}{2\pi i} \int\!\!\int \rho'\; d\rho' d\phi\; f(\rho') \frac{e^{i(k_f R - \omega t)}}{R}, \tag{10}$$

where R is the distance between the source and observation point. Defining the observation point to be $(\rho, 0, z)$, we have

$$R^2 = z^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\phi, \tag{11}$$

so that for large z,

$$R \approx z + \frac{\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi}{2z}. \tag{12}$$

In the present case, we desire the amplitude to have form (1). As usual, we approximate R by z in the denominator of eq. (10), while using approximation (12) in the exponential factor. This leads to the integral equation

$$\begin{aligned} f(\rho)e^{ik_z z} &= \frac{k_f}{2\pi i}\frac{e^{ik_f z}e^{ik_f\rho^2/2z}}{z}\int_0^\infty \rho'\ d\rho'\ f(\rho')e^{ik_f\rho'^2/2z}\int_0^{2\pi} d\phi\ e^{-ik_f\rho\rho'\cos\phi/z} \\ &= \frac{k_f}{i}\frac{e^{ik_f z}e^{ik_f\rho^2/2z}}{z}\int_0^\infty \rho'\ d\rho'\ f(\rho')J_0(k_f\rho\rho'/z)\, e^{ik_f\rho'^2/2z}, \end{aligned} \tag{13}$$

using a well-known integral representation of the Bessel function J_0.

It is now plausible that the desired eigenfunction $f(\rho)$ is a Bessel function, say $J_0(k'\rho)$, and on consulting a table of integrals of Bessel functions we find an appropriate relation [9],

$$\int_0^\infty \rho'\ d\rho'\ J_0(k'\rho')J_0(k_f\rho\rho'/z)e^{ik_f\rho'^2/2z} = \frac{iz}{k_f}\, e^{-ik_f\rho^2/2z}e^{-ik'^2 z/2k_f}\, J_0(k'\rho). \tag{14}$$

Comparing this with eq. (13), we see that $f(\rho) = J_0(k'\rho)$ is indeed an eigenfunction provided that

$$k_z = k_f - \frac{k'^2}{2k_f}. \tag{15}$$

Thus, if we write $k' = k_f \sin\alpha$, then for small α,

$$k_z \approx k_f(1 - \alpha^2/2) \approx k_f \cos\alpha, \tag{16}$$

and the wave (1) again has form (7).

Strictly speaking, the scalar diffraction theory reproduces the “exact” result (7) only for small α. But the scalar diffraction theory is only an approximation, and we predict with confidence that an “exact” diffraction theory would lead to the form (7) for all values of parameter α. That is, “diffraction-free” beams are predicted within diffraction theory.

It remains that the theory of diffraction predicts that an infinite aperture is needed to produce a beam whose transverse profile is invariant with longitudinal distance. That a Bessel beam is no exception to this rule is reviewed in sec. 2.3.

The results of this section were inspired by [10]. One of the first solutions for Gaussian laser beams was based on scalar diffraction theory cast as an eigenfunction problem [11].

2.3 Superluminal Behavior

In general, the group velocity (9) of a Bessel beam exceeds the speed of light. However, this apparently superluminal behavior cannot be used to transmit signals faster than lightspeed.

An important step towards understanding this comes from the interpretation of parameter α as the angle with respect to the z axis of the wave vectors of an infinite set of ordinary plane waves whose superposition yields the Bessel beam [12]. To see this, we invoke an integral representation of the Bessel function to write eq. (7) as[1]

[1]Using the well-known representation of the Bessel function $J_n(\rho)$,

$$J_n(\rho) = \frac{1}{2\pi}\int_\beta^{2\pi+\beta} e^{\pm i(\rho\cos\theta - n\theta)}d\theta, \tag{17}$$

$$
\begin{aligned}
\psi(\mathbf{r},t) &= J_0(k_f \sin\alpha\,\rho)\,e^{i(k_f\cos\alpha\, z-\omega t)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{i(k_f\sin\alpha\, x\cos\phi + k_f\sin\alpha\, y\sin\phi + k_f\cos\alpha\, z-\omega t)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{i(\mathbf{k}_f\cdot\mathbf{r}-\omega t)},
\end{aligned}
\tag{19}
$$

where the wave vector $\mathbf{k}_f$, given by

$$
\mathbf{k}_f = k_f(\sin\alpha\cos\phi, \sin\alpha\sin\phi, \cos\alpha), \tag{20}
$$

makes angle α to the z axis as claimed.

We now see that a Bessel beam is rather simple to produce in principle [2]. Just superpose all possible plane waves with equal amplitude and a common phase that make angle α to the z axis,

According to this prescription, we expect the z axis to be uniformly illuminated by the Bessel beam. If that beam is created at the plane $z = 0$, then any annulus of equal radial extent in that plane must project equal power into the beam. For large ρ this is readily confirmed by noting that $J_0^2(k_f \sin\alpha\,\rho) \approx \cos^2(k_f \sin\alpha\,\rho + \delta)/(k_f \sin\alpha\,\rho)$, so the integral of the power over an annulus of one radial period, $\Delta\rho = \pi/(k_f \sin\alpha)$, is independent of radius.

Thus, from an energy perspective a Bessel beam is not confined to a finite region about the z axis. If the beam is to propagate a distance z from the plane $z = 0$, it must have radial extent of at least $\rho = z\tan\alpha$ at $z = 0$. An arbitrarily large initial aperture, and arbitrarily large power, is required to generate a Bessel beam that retains its "diffraction-free" character over an arbitrarily large distance.

Each of the plane waves that makes up the Bessel beam propagates with velocity c along a ray that makes angle α to the z axis. The intersection of the z axis and a plane of constant phase of any of these wave moves forward with superluminal speed $c/\cos\alpha$, which is equal to the phase and group velocities (9).

This superluminal behavior does not represent any violation of special relativity, but is an example of the "scissors paradox" that the point of contact of a pair of scissors could move faster than the speed of light while the tips of the blades are moving together at sublightspeed. A ray of sunlight that makes angle α to the surface of the Earth similarly leads to a superluminal velocity $c/\cos\alpha$ of the point of contact of a wave front with the Earth.

However, we immediately see that a Bessel beam could not be used to send a signal from, say, the origin, $(0, 0, 0)$, to a point $(0, 0, z)$ at a speed faster than light. A Bessel beam at $(0, 0, z)$ is made of rays of plane waves that intersect the plane $z = 0$ at radius $\rho = z\tan\alpha$. Hence, to deliver a message from $(0, 0, 0)$ to $(0, 0, z)$ via a Bessel beam, the information must

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with $\theta = \phi - \varphi$, we have

$$
J_n(\rho) = \frac{1}{2\pi}\int_{\beta+\varphi}^{2\pi+\beta+\varphi} e^{\pm i(\rho\cos\varphi\cos\phi+\rho\sin\varphi\sin\phi-n\phi+n\varphi)}d\phi = \frac{e^{\pm in\varphi}}{2\pi}\int_0^{2\pi} e^{\pm i(x\cos\phi+y\sin\phi-n\phi)}d\phi, \tag{18}
$$

where for $\mathbf{r} = (\rho, \varphi, z)$ we define $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, and we set $\beta = -\varphi$. Then, eq. (19) follows $n = 0$.

first propagate from the origin out to at least radius $\rho = z \tan\alpha$ at $z = 0$ to set up the beam. Then, the rays must propagate distance $z/\cos\alpha$ to reach point z with the message. The total distance traveled by the information is thus $z(1+\sin\alpha)/\cos\alpha$, and the signal velocity v_s is given by

$$v_s \approx c\frac{\cos\alpha}{1+\sin\alpha}, \tag{21}$$

which is always less than c. The group velocity and signal velocity for a Bessel beam are very different. Rather than being a superluminal carrier of information at its group velocity $c/\cos\alpha$, a modulated Bessel beam could be used to deliver messages only at speeds well below that of light.

2.4 Solution via the Vector Potential

To deduce all components of the electric and magnetic fields of a Bessel beam that satisfies Maxwell's equation starting from a single scalar wave function, we follow the suggestion of Davis [13] and seek solutions for a vector potential $\mathbf{A}$ that has only a single component. Of course, we must recall that only in rectangular coordinates do the components of a vector potential that obeys the (vector) wave equation also satisfy the scalar wave equation [14].

We work in the Lorentz gauge (and Gaussian units), so that the scalar potential Φ is related by

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0. \tag{22}$$

The vector potential can therefore have a nonzero divergence, which permits solutions having only a single component. The electric and magnetic fields can be deduced from the potentials via

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{23}$$

and

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{24}$$

For this, the scalar potential must first be deduced from the vector potential using the Lorentz condition (22). We consider waves of frequency ω and time dependence of the form $e^{-i\omega t}$, so that $\partial\Phi/\partial ct = -ik_f\Phi$. Then, the Lorentz condition yields

$$\Phi = -\frac{i}{k_0}\boldsymbol{\nabla}\cdot\mathbf{A}, \tag{25}$$

and the electric field is given by

$$\mathbf{E} = ik_f\left[\mathbf{A} + \frac{1}{k_f^2}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})\right]. \tag{26}$$

This electric field obeys $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, since $\nabla^2(\boldsymbol{\nabla}\cdot\mathbf{A}) + k_f^2(\boldsymbol{\nabla}\cdot\mathbf{A}) = 0$ for a vector potential $\mathbf{A}$ of frequency ω that satisfies the wave equation (2), *etc.*

We already have a scalar solution (7) to the wave equation, which we now interpret as the only nonzero component, A_j, of the vector potential for a Bessel beam that propagates in the $+z$ direction,

$$A_j(\mathbf{r},t) = \psi(\mathbf{r},t) \propto J_0(k_f \sin\alpha\,\rho)\,e^{i(k_f\cos\alpha\, z-\omega t)}. \tag{27}$$

We consider the three choices for the meaning of index j, namely x, y, and z, which lead to three types of Bessel beams. Of these, only the case of $j = z$ corresponds to azimuthally symmetric fields, and so perhaps should be called the Bessel beam.

2.4.1 $j = x$

In this case, recalling that $J_0' = -J_1$,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial \psi}{\partial x} = -\frac{k_f \sin\alpha\, x}{\rho} J_1(k_f \sin\alpha\, \rho)\, e^{i(k_f \cos\alpha\, z - \omega t)}. \tag{28}$$

In calculating $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})$ we use the identity

$$J_1'(\varrho) = J_0(\varrho) - \frac{J_1(\varrho)}{\varrho}\,. \tag{29}$$

Also, we divide $\mathbf{E}$ and $\mathbf{B}$ as obtained from eqs. (24) and (26) by the factor ik_f to present the results in a simpler form. We find,

$$\begin{aligned}
E_x &= \left[J_0(\varrho)\left(1 - \sin^2\alpha \frac{x^2}{\rho^2}\right) + \sin\alpha \frac{x^2 - y^2}{k_f \rho^3} J_1(\varrho)\right] e^{i(k_f \cos\alpha\, z - \omega t)}, \\
E_y &= \frac{\sin\alpha\, xy}{\rho^2}\left[\frac{2J_1(\varrho)}{k_f \varrho} - \sin\alpha J_0(\varrho)\right] e^{i(k_f \cos\alpha\, z - \omega t)}, \\
E_z &= -i \sin 2\alpha \frac{x}{2\rho} J_1(\varrho)\, e^{i(k_f \cos\alpha\, z - \omega t)},
\end{aligned} \tag{30}$$

where

$$\varrho \equiv k_f \sin\alpha\, \rho, \tag{31}$$

and

$$\begin{aligned}
B_x &= 0, \\
B_y &= \cos\alpha\, J_0(\varrho)\, e^{i(k_f \cos\alpha\, z - \omega t)}, \\
B_z &= -i \sin\alpha \frac{x}{\rho} J_1(\varrho)\, e^{i(k_f \cos\alpha\, z - \omega t)}.
\end{aligned} \tag{32}$$

This Bessel beam that obeys Maxwell's equations and has purely x polarization of its electric field on the z axis, but includes nonzero y and z polarization at points off that axis, and does not exhibit the azimuthal symmetry of the underlying vector potential. In the limit that $\alpha = 0$ this Bessel beam becomes an ordinary plane wave with x polarization of its electric field.

2.4.2 $j = y$

This case is very similar to that of $j = x$.

2.4.3 $j = z$

In this case the electric and magnet fields retain azimuthal symmetry, so that it is convenient to display the ρ, ϕ and z components of the fields. First,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial\psi}{\partial z} = ik_f\cos\alpha\, J_0(k_f\sin\alpha\,\rho)\, e^{i(k_f\cos\alpha\, z-\omega t)}. \tag{33}$$

Then, we divide the electric and magnetic fields obtained from eqs. (24) and (26) by $k_f\sin\alpha$ to find the relatively simple forms:

$$\begin{aligned} E_\rho &= \cos\alpha\, J_1(\varrho)\, e^{i(k_f\cos\alpha\, z-\omega t)}, \\ E_\phi &= 0, \\ E_z &= i\sin\alpha\, J_0(\varrho)\, e^{i(k_f\cos\alpha\, z-\omega t)}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} B_\rho &= 0, \\ B_\phi &= J_1(\varrho)\, e^{i(k_f\cos\alpha\, z-\omega t)}, \\ B_z &= 0. \end{aligned} \tag{35}$$

This Bessel beam is a transverse magnetic (TM) wave. The radial electric field E_ρ vanishes on the z axis (as it must if that axis is charge free), while the longitudinal electric field E_z is maximal there.

The time-average electromagnetic energy density $\langle u\rangle$ in the Bessel beam (34)-(35) is

$$\langle u\rangle = \frac{|E|^2+|B|^2}{8\pi} = \frac{(1+\cos^2\alpha)J_1^2(\varrho)+\sin^2\alpha J_0^2(\varrho)}{8\pi}, \tag{36}$$

and the time-average energy-flow (Poynting) vector $\langle\mathbf{S}\rangle$ is

$$\langle\mathbf{S}\rangle = \frac{c}{8\pi}Re(\mathbf{E}\times\mathbf{B}^\star) = \frac{c}{8\pi}\cos\alpha J_1^2(\varrho)\,\hat{\mathbf{z}} \tag{37}$$

The energy flow $\langle\mathbf{S}\rangle$ is not equal the energy density $\langle u\rangle$ times the group velocity $c\,\hat{\mathbf{z}}/\cos\alpha$, which is another indication that the Bessel beam should not be thought of as an ordinary plane wave, but rather as the sum of a family of waves that make angle α to the z axis, as discussed in sec. 2.3.

As is well known, corresponding to each TM wave solution to Maxwell's equations in free space, there is a TE (transverse electric) mode obtained by the duality transformation $\mathbf{E}_{\rm TE} = \mathbf{B}_{\rm TM}$, $\mathbf{B}_{\rm TE} = -\mathbf{E}_{\rm TM}$. Hence, there exists a Bessel beam in which the electric field has only the component $E_\phi = J_1(\varrho)\, e^{i(k_f\cos\alpha\, z-\omega t)}$, *etc.*

A Appendix: Bessel Beams with Orbital Angular Momentum

This Appendix was written in January 2008 and updated April 2009.

We look for solutions to the wave equation,

$$\begin{aligned}
\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= \left[\frac{\partial^2 A_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 A_\rho}{\partial \phi^2} - A_\rho\right) + \frac{\partial^2 A_\rho}{\partial z^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi} - \frac{1}{c^2}\frac{\partial^2 A_\rho}{\partial t^2}\right]\hat{\boldsymbol{\rho}} \\
&= \left[\frac{\partial^2 A_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 A_\phi}{\partial \phi^2} - A_\phi\right) + \frac{\partial^2 A_\phi}{\partial z^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi} - \frac{1}{c^2}\frac{\partial^2 A_\phi}{\partial t^2}\right]\hat{\boldsymbol{\phi}} \\
&= \left[\frac{\partial^2 A_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2}\right]\hat{\mathbf{z}} = 0 \qquad (38)
\end{aligned}$$

for the vector potential $\mathbf{A}$ in cylindrical coordinates, where the scalar wave functions A_ρ, A_ϕ, and A_z have the form

$$\psi(\mathbf{r}, t) = f(\rho)\, e^{i(k_z z - \omega t \pm m\phi)}, \qquad (39)$$

where m is a non-negative integer (so that $\psi(\phi = 2\pi) = \psi(\phi = 0)$). For the case that $A_\rho = A_\phi = 0$ and $A_z = \psi$, we obtain

$$\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} + \left(k_f^2 - k_z^2 - \frac{m^2}{\rho^2}\right) f = 0. \qquad (40)$$

This is the differential equation for Bessel functions of order m, so that one set of Bessel beams are generated by the vector potential

$$A_\rho = A_\phi = 0, \qquad A_z(\mathbf{r}, t) = J_m(k'\rho)\, e^{i(k_z z - \omega t \pm m\phi)}, \qquad (41)$$

where

$$k_z = k_f \cos\alpha, \qquad k' = k_f \sin\alpha, \qquad \varrho = k'\rho. \qquad (42)$$

We can also consider the solution $A_\rho = \psi$, $A_\phi = \pm i\psi$, $A_z = 0$, for which both the $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ terms of eq. (38) lead to

$$\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} + \left(k_f^2 - k_z^2 - \frac{(m-1)^2}{\rho^2}\right) f = 0. \qquad (43)$$

Thus, a second set of Bessel beams are generated by the vector potential

$$A_\rho(\mathbf{r}, t) = J_{m-1}(k'\rho)\, e^{i(k_z z - \omega t \pm m\phi)}, \qquad A_\phi = \pm i A_\rho, \qquad A_z = 0. \qquad (44)$$

Since the Bessel function J_m is real, surfaces of constant phase (at fixed time) in the wave functions (41) and (44) are not planes of constant z, but rather they are "screws" of pitch $2\pi m/k_f \cos\alpha = m\lambda_0/\cos\alpha$, where $\lambda_0 = 2\pi/k_f = 2\pi c/\omega$. The wave vector is

$$\mathbf{k} = \boldsymbol{\nabla}(k_z z \pm m\phi) = k_z\,\hat{\mathbf{z}} \pm \frac{m}{\rho}\,\hat{\boldsymbol{\phi}}, \qquad (45)$$

which is perpendicular to the surfaces of constant phase has lines that form helices of constant radius ρ. The wave vector $\mathbf{k}$ makes angle β to the $\hat{\mathbf{z}}$-axis given by

$$\tan\beta = \pm\frac{m}{k_z\rho} = \pm\frac{m\tan\alpha}{\varrho}\,. \tag{46}$$

As for Bessel beams with zero orbital angular momentum, the dispersion relation is given by eq. (8), *i.e.*, $\omega(\mathbf{k}) = k_f c = k_z c/\cos\alpha$, so the group velocity vector is again

$$\mathbf{v}_g = \boldsymbol{\nabla}_{\mathbf{k}}\omega = \frac{c}{\cos\alpha}\,\hat{\mathbf{z}}. \tag{47}$$

A.1 Bessel Beams Based on A_z

We first consider the case that the only nonzero component of the vector potential is A_z with the form (41), for which

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial\psi}{\partial z} = ik_z\,J_m(\varrho)\,e^{i(k_z z-\omega t\pm m\phi)}. \tag{48}$$

Then, we divide the electric and magnetic fields obtained from eqs. (24) and (26) by k' to find the relatively simple TM forms:

$$\begin{aligned}
E_\rho &= \cos\alpha\,\frac{J_{m+1}(\varrho)-J_{m-1}(\varrho)}{2}\,e^{i(k_z z-\omega t\pm m\phi)},\\
E_\phi &= \frac{\mp im\cos\alpha}{\varrho}\,J_m(\varrho)\,e^{i(k_z z-\omega t\pm m\phi)},\\
E_z &= i\sin\alpha\,J_m(\varrho)\,e^{i(k_z z-\omega t\pm m\phi)},
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
B_\rho &= \frac{\pm im}{\varrho}\,J_m(\varrho)\,e^{i(k_z z-\omega t\pm m\phi)},\\
B_\phi &= \frac{J_{m+1}(\varrho)-J_{m-1}(\varrho)}{2}\,e^{i(k_z z-\omega t\pm m\phi)},\\
B_z &= 0.
\end{aligned} \tag{50}$$

We can also obtain TE waves by the duality transform, $\mathbf{E}_{\mathrm{TE}} = \mathbf{B}_{\mathrm{TM}}$, $\mathbf{B}_{\mathrm{TE}} = -\mathbf{E}_{\mathrm{TM}}$.

The time-average electromagnetic energy density $\langle u\rangle$ in the Bessel beam (49)-(50) is

$$\begin{aligned}
\langle u\rangle &= \frac{|E|^2+|B|^2}{8\pi} = \frac{\sin^2\alpha}{8\pi}J_m^2(\varrho) + \frac{1+\cos^2\alpha}{8\pi}\left(\frac{m^2J_m^2(\varrho)}{\varrho^2} + \frac{[J_{m+1}(\varrho)-J_{m-1}(\varrho)]^2}{4}\right)\\
&= \frac{\sin^2\alpha}{8\pi}J_m^2(\varrho) + \frac{1+\cos^2\alpha}{16\pi}[J_{m+1}^2(\varrho)+J_{m-1}^2(\varrho)],
\end{aligned} \tag{51}$$

using the fact that $2mJ_m/\varrho = J_{m+1}+J_{m-1}$.

The time-average energy-flow (Poynting) vector $\langle \mathbf{S} \rangle$ is

$$\begin{aligned}
\langle \mathbf{S} \rangle &= \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{B}^\star) \\
&= \frac{c}{8\pi} \left(\frac{\mp m \sin\alpha}{\varrho} J_m^2(\varrho)\, \hat{\boldsymbol{\phi}} + \cos\alpha \left[\frac{m^2 J_m^2(\varrho)}{\varrho^2} + \frac{[J_{m+1}(\varrho) - J_{m-1}(\varrho)]^2}{4} \right] \hat{\mathbf{z}} \right) \\
&= \frac{c}{8\pi} \left(\frac{\mp m \sin\alpha}{\varrho} J_m^2(\varrho)\, \hat{\boldsymbol{\phi}} + \cos\alpha \frac{J_{m+1}^2(\varrho) + J_{m-1}^2(\varrho)}{2}\, \hat{\mathbf{z}} \right). \qquad (52)
\end{aligned}$$

Lines of the energy flow vector $\langle \mathbf{S} \rangle$, like those of the wave vector $\mathbf{k}$, are helices of constant radius ρ. From eq. (52) we see that the energy-flow vector makes angle γ to the $\hat{\mathbf{z}}$-axis where $\tan\gamma = \mp(m \tan\alpha/\varrho)\, 2J_m^2(\varrho)/[J_{m+1}^2(\varrho) + J_{m-1}^2(\varrho)]$. Thus, the energy-flow vector $\langle \mathbf{S} \rangle$ and the wave vector $\mathbf{k}$ are not in the same directions when index m is nonzero. For large $|m|$ the wave vector $\mathbf{k}$ (parallel to the phase velocity) and the Poynting vector $\langle \mathbf{S} \rangle$ point in opposite directions; lines of these vector fields form very tight, opposing helices. The group velocity vector (47) corresponds to the energy-flow velocity averaged over the helical lines of the Poynting vector.

The time-average electromagnetic angular momentum density $\langle \mathbf{l} \rangle$ is

$$\begin{aligned}
\langle \mathbf{l} \rangle &= \mathbf{r} \times \langle \mathbf{p} \rangle = (\rho\, \hat{\boldsymbol{\rho}} + z\, \hat{\mathbf{z}}) \times \frac{\langle \mathbf{S} \rangle}{c^2} \\
&= \frac{1}{8\pi c} \left(\frac{\pm m z \sin\alpha}{\varrho} J_m^2(\varrho)\, \hat{\boldsymbol{\rho}} - \rho \cos\alpha \frac{J_{m+1}^2(\varrho) + J_{m-1}^2(\varrho)}{2}\, \hat{\boldsymbol{\phi}} + \frac{\mp m \rho \sin\alpha}{\varrho} J_m^2(\varrho)\, \hat{\mathbf{z}} \right) \\
&= \frac{1}{8\pi} \left(\frac{\pm m z \sin\alpha}{c\varrho} J_m^2(\varrho)\, \hat{\boldsymbol{\rho}} - \rho \cos\alpha \frac{J_{m+1}^2(\varrho) + J_{m-1}^2(\varrho)}{2c}\, \hat{\boldsymbol{\phi}} + \frac{\mp m}{\omega} J_m^2(\varrho)\, \hat{\mathbf{z}} \right), \qquad (53)
\end{aligned}$$

recalling that the electromagnetic momentum density is given by $\mathbf{p} = \mathbf{S}/c^2$. The transverse components of the angular momentum density $\langle \mathbf{l} \rangle$ integrate to zero over any transverse plane at fixed z. The longitudinal component can be interpreted as orbital angular momentum about the z axis. But because the Bessel beam is the result of a superposition of converging plane waves, the orbital angular momentum does not obey the simple relation $\langle \mathbf{l} \rangle = m \langle u \rangle / \omega$, as would hold if the beam were comprised only of plane waves propagating in the z direction.[2]

A.2 Bessel Beams Based on A_ρ and A_ϕ

We now consider the case that vector potential has the form (44), for which

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_\rho}{\partial \rho} + \frac{A_\rho}{\rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} = -k'\, J_m(\varrho)\, e^{i(k_z z - \omega t \pm m\phi)}, \qquad (54)$$

[2]For comparison, so-called Gaussian-Laguerre laser beams also can carry orbital angular momentum, but since these beams are well approximated by plane waves that are essentially radial when far from the beam "waist", the orbital angular momentum is given by $\langle \mathbf{l} \rangle = m \langle u \rangle / \omega$ [15]. For a discussion of how Bessel beams (and Gaussian-Laguerre beams) with high orbital angular momentum can be described in terms of skewed bundles of straight rays in the approximation of geometrical optics, see [17].

noting that $dJ_{m-1}(k'\rho)/d\rho = -k' J_m(k'\rho) + [(m-1)/\rho] J_{m-1}(k'\rho)$. Then, we divide the electric and magnetic fields obtained from eqs. (24) and (26) by $\pm k_z$ to find the forms:

$$\begin{aligned}
E_\rho &= \pm i \left(\frac{m \sin^2\alpha}{\varrho \cos\alpha} J_m(\varrho) + \cos\alpha J_{m-1}(\varrho) \right) e^{i(k_z z - \omega t \pm m\phi)}, \\
E_\phi &= \left(\frac{m \cos\alpha}{\varrho} J_m(\varrho) - \frac{J_{m-1}(\varrho)}{\cos\alpha} \right) e^{i(k_z z - \omega t \pm m\phi)}, \\
E_z &= \mp \sin\alpha\, J_m(\varrho)\, e^{i(k_z z - \omega t \pm m\phi)},
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
B_\rho &= J_{m-1}(\varrho) e^{i(k_z z - \omega t \pm m\phi)}, \\
B_\phi &= \pm i J_{m-1}(\varrho)\, e^{i(k_z z - \omega t \pm m\phi)}, \\
B_z &= -i \tan\alpha J_m(\varrho)\, e^{i(k_z z - \omega t \pm m\phi)},
\end{aligned} \tag{56}$$

The dual solutions, $\mathbf{E}' = \mathbf{B}$, $\mathbf{B}' = -\mathbf{E}$, correspond to circularly polarized Bessel beams, which may be the most physically interesting of the four sets of solutions exhibited here.

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