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2ϕ-C

Tapp 4.52 (Scherk's Surface)

$$f(x,y) = \ln\left(\frac{\cos(y)}{\cos(x)}\right)$$

we need $\phi = \dfrac{\cos(y)}{\cos(x)} > 0$ for $f(x,y)$ to be defined. $\cos(\theta) = 0 \Rightarrow \theta = \frac{\pi}{2} + n\pi$

thus we take:

$$\left\{ (x,y) \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \right\}$$

as domain as this ensures argument of Ln > 0 and that we don't divide by zero.

We will show graph is minimal surface by showing $H = 0 \;\forall\, x,y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$

from example 4.18 for a general graph of $f(x,y)$ we have

$$2H = \frac{f_{xx}(1+f_y^2) - 2f_{xy}f_xf_y + f_{yy}(1+f_x^2)}{(1+f_x^2+f_y^2)^{1/2}}$$

$f_x = \tan(x)$ $\qquad f_{xx} = \sec^2(x)$ $\qquad f_{xy} = 0$

$f_y = -\tan(x)$ $\qquad f_{yy} = -\sec^2(y)$

thus $$2H = \frac{\sec^2(x)(1+\tan^2(y)) - \sec^2(y)(1+\tan^2(x))}{\sqrt{1+\tan^2(x)+\tan^2(y)}}$$

$1 + \tan^2(x) = \sec^2(x)$

$1 + \tan^2(y) = \sec^2(y)$

thus we have

$$2H = \frac{\sec^2(x)\sec^2(y) - \sec^2(y)\sec^2(x)}{\sqrt{1+\tan^2(x)+\tan^2(y)}}$$

$\Longrightarrow H = 0 \quad \forall\ x, y$

and since $\tan(\theta)$ is defined for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and the denominator is $> 0 \ \forall\ x, y$ so H is defined. Therefore since $H = 0 \ \forall\ x, y \in$ Domain S is a minimum surface

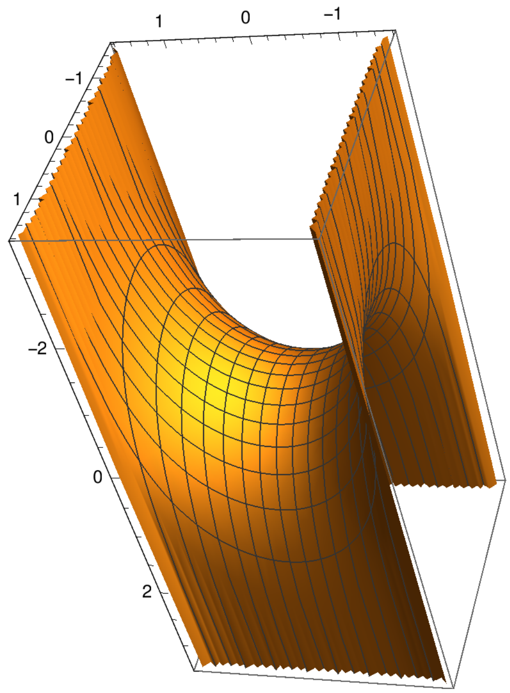
See attached pdf for graph of surface.

Graph of Scherk's Surface

In[1]:=
```
X[u_, v_] := u
Y[u_, v_] := v
Z[u_, v_] := Log[Cos[v] / Cos[u]]
```

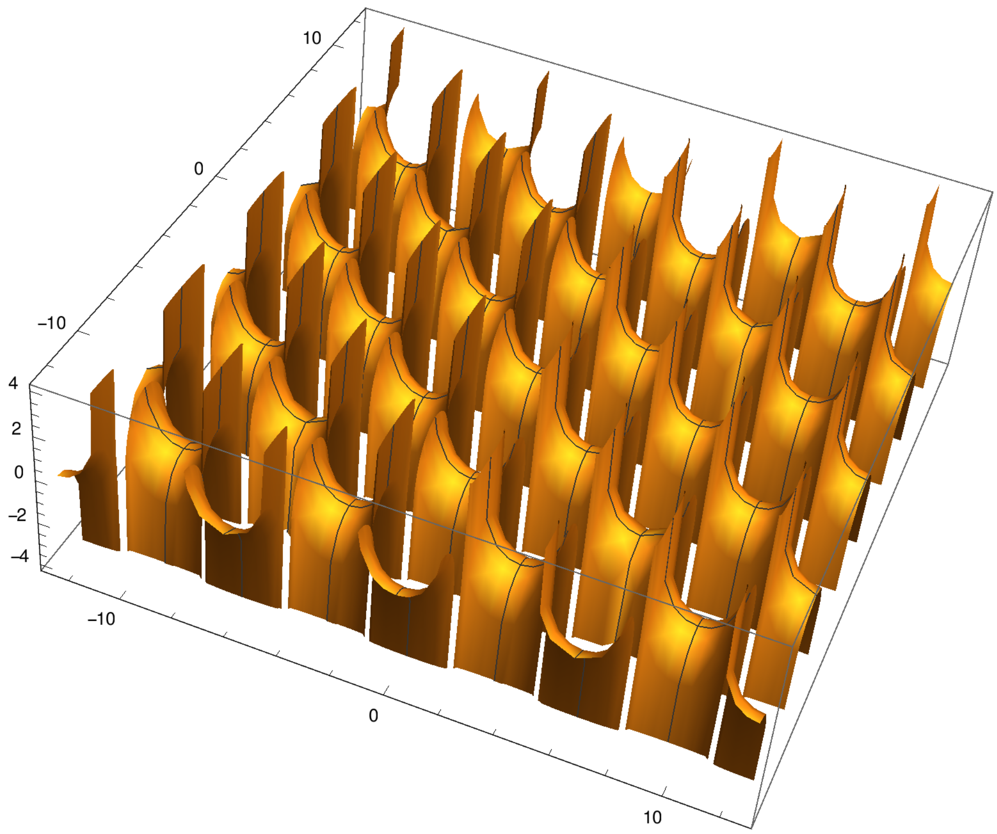
In[4]:=
```
ParametricPlot3D[{X[u, v], Y[u, v], Z[u, v]}, {u, -π / 2, π / 2}, {v, -π / 2, π / 2}]
```

Out[4]=



Here is the same surface on a larger domain

In[5]:=
```
ParametricPlot3D[{X[u, v], Y[u, v], Z[u, v]}, {u, -4 π, 4 π}, {v, -4 π, 4 π}]
```

Out[5]=


Tapp 4.53) verify that the following parametrized surface is minimal

σ(u,v) = [u-sin(u)cosh(v), 1-cos(u)cosh(v), -4sin(u/2)sinh(v/2)]

I do not want to prove that this map is conformal (I'm not really sure how to do that) so I will crank out H by brute force and show that it is 0 for all u,v in the domain

In[6]:=
```
σ[u_, v_] := {u - Sin[u] * Cosh[v], 1 - Cos[u] * Cosh[v], -4 * Sin[u / 2] * Sinh[v / 2]}
```

In[7]:=
```
σ1[u_, v_] := D[σ[u, v], u]
```

In[8]:=
```
σ1[u, v]
```

Out[8]= $\left\{1 - \text{Cos}[u]\,\text{Cosh}[v],\ \text{Cosh}[v]\,\text{Sin}[u],\ -2\,\text{Cos}\left[\frac{u}{2}\right]\text{Sinh}\left[\frac{v}{2}\right]\right\}$

^ is my calculation of σ_u

In[9]:=
```
σ2[u_, v_] := D[σ[u, v], v]
```

In[10]:= $\sigma_2[u, v]$

Out[10]= $\left\{-\text{Sin}[u]\ \text{Sinh}[v],\ -\text{Cos}[u]\ \text{Sinh}[v],\ -2\ \text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right\}$

Now we need to calculate the unit normal field N. First we will find the cross product and then we will normalize it.

In[11]:= $\text{GaussMap}[u\_, v\_] := \sigma_1[u, v] \times \sigma_2[u, v] / (\text{Norm}[\sigma_1[u, v] \times \sigma_2[u, v]])$

In[12]:= $\text{FullSimplify}\left[\text{GaussMap}[u, v]\right]$

Out[12]=
$$\left\{\left(2\ \text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]\ (\text{Cos}[u]-\text{Cosh}[v])\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{3v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\left(4\ \text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right),\right.$$
$$\left(2\ \text{Cosh}\left[\frac{v}{2}\right]\ (-\text{Cos}[u]+\text{Cosh}[v])\ \text{Sin}\left[\frac{u}{2}\right]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{3v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\left(4\ \text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right),$$
$$\left.\left((-\text{Cos}[u]+\text{Cosh}[v])\ \text{Sinh}[v]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{3v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\left(4\ \text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\ \text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right)\right\}$$

Now we just need to find the 2nd partial derivatives and then we will have everything we need to calculate E,F,G,e,f,g

In[13]:= $\sigma_{11}[u\_, v\_] := D[\sigma_1[u, v], u]$
$\sigma_{11}[u, v]$

Out[14]= $\left\{\text{Cosh}[v]\ \text{Sin}[u],\ \text{Cos}[u]\ \text{Cosh}[v],\ \text{Sin}\left[\frac{u}{2}\right]\ \text{Sinh}\left[\frac{v}{2}\right]\right\}$

In[15]:= $\sigma_{12}[u\_, v\_] := D[\sigma_1[u, v], v]$
$\sigma_{12}[u, v]$

Out[16]= $\left\{-\text{Cos}[u]\ \text{Sinh}[v],\ \text{Sin}[u]\ \text{Sinh}[v],\ -\text{Cos}\left[\frac{u}{2}\right]\ \text{Cosh}\left[\frac{v}{2}\right]\right\}$

In[17]:= $\sigma_{22}[u\_, v\_] := D[\sigma_2[u, v], v]$
$\sigma_{22}[u, v]$

Out[18]= $\left\{-\text{Cosh}[v]\ \text{Sin}[u],\ -\text{Cos}[u]\ \text{Cosh}[v],\ -\text{Sin}\left[\frac{u}{2}\right]\ \text{Sinh}\left[\frac{v}{2}\right]\right\}$

Now we will calculate E,F,G

In[19]:= $E_e[u\_, v\_] := \text{Dot}[\sigma_1[u, v], \sigma_1[u, v]]$

In[20]:=
```
FullSimplify[Eₑ[u, v]]
```

Out[20]= $2\,\text{Cosh}\left[\frac{v}{2}\right]^2\,\left(-\text{Cos}[u]+\text{Cosh}[v]\right)$

I used the subscript because Mathematica uses E for Euler's constant

In[21]:=
```
F[u_, v_] := Dot[σ1[u, v], σ2[u, v]]
```

In[22]:=
```
FullSimplify[F[u, v]]
```

Out[22]= 0

In[27]:=
```
G[u_, v_] := Dot[σ2[u, v], σ2[u, v] ]
FullSimplify[G[u, v]]
```

Out[28]= $2\,\text{Cosh}\left[\frac{v}{2}\right]^2\,\left(-\text{Cos}[u]+\text{Cosh}[v]\right)$

Now I will Calculate e,f,g

In[29]:=
```
eₑ[u_, v_] := Dot[σ11[u, v], GaussMap[u, v]]
```

In[30]:=
```
FullSimplify[eₑ[u, v]]
```

Out[30]= $$\left(2\,\text{Cosh}\left[\frac{v}{2}\right]^3\,\left(\text{Cos}[u]-\text{Cosh}[v]\right)\,\text{Sin}\left[\frac{u}{2}\right]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3\,u}{2}\right]\,\text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\,\text{Cosh}\left[\frac{3\,v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\,\left(4\,\text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\,\text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right)$$

In[31]:=
```
f[u_, v_] := Dot[σ12[u, v], GaussMap[u, v]]
```

In[33]:=
```
FullSimplify[f[u, v]]
```

Out[33]= $$\left(2\,\text{Cos}\left[\frac{u}{2}\right]\,\text{Cosh}\left[\frac{v}{2}\right]^2\,\left(-\text{Cos}[u]+\text{Cosh}[v]\right)\,\text{Sinh}\left[\frac{v}{2}\right]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3\,u}{2}\right]\,\text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\,\text{Cosh}\left[\frac{3\,v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\,\left(4\,\text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\,\text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right)$$

In[34]:=
```
g[u_, v_] := Dot[σ22[u, v], GaussMap[u, v]]
```

In[35]:=
```
FullSimplify[g[u, v]]
```

Out[35]= $$\left(2\,\text{Cosh}\left[\frac{v}{2}\right]^3\,\left(-\text{Cos}[u]+\text{Cosh}[v]\right)\,\text{Sin}\left[\frac{u}{2}\right]\right)\Big/\left(\sqrt{\left(\text{Abs}\left[\text{Cos}\left[\frac{3\,u}{2}\right]\,\text{Cosh}\left[\frac{v}{2}\right]-\text{Cos}\left[\frac{u}{2}\right]\,\text{Cosh}\left[\frac{3\,v}{2}\right]\right]^2+\text{Abs}[\text{Cos}[u]-\text{Cosh}[v]]^2\,\left(4\,\text{Abs}\left[\text{Cosh}\left[\frac{v}{2}\right]\,\text{Sin}\left[\frac{u}{2}\right]\right]^2+\text{Abs}[\text{Sinh}[v]]^2\right)\right)}\right)$$

Now we can calculate H directly from the definition

In[36]:=
$$H[u\_, v\_] := (1/2) * \frac{e_e[u, v] * G[u, v] - 2 * f[u, v] * F[u, v] + g[u, v] * E_e[u, v]}{E_e[u, v] * G[u, v] - F[u, v]\,\hat{}\,2}$$

In[38]:=
```
FullSimplify[H[u, v]]
```
Out[38]=
```
0
```

^ Thus we have shown that for all u,v in the domain that the mean curvature is zero. Therefore we conclude that the surface is minimal. Below is a graph of the surface for fun.

In[43]:=
```
ParametricPlot3D[{σ[u, v][[1]], σ[u, v][[2]], σ[u, v][[3]]}, {u, -1, 1}, {v, -1, 1}]
```

Out[43]=