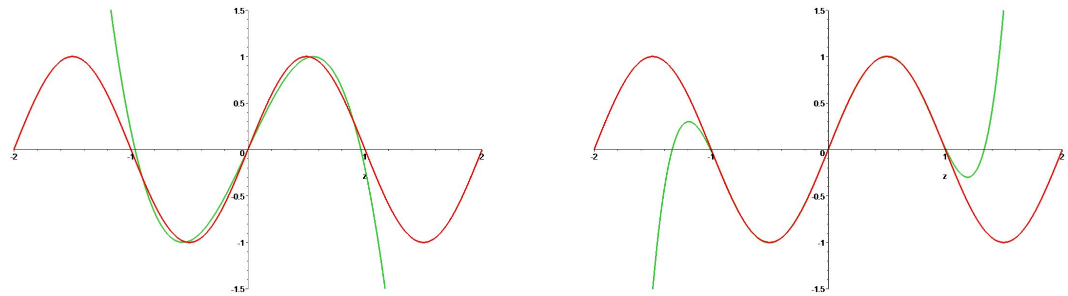

Figure 1: The Legendre Series approximation for the function $\sin(\pi z)$ terminating at $\ell = 3$ (left) and $\ell = 5$ (right).

Central Forces Homework 8

Due 6/1/18, 4 pm

Sensemaking: For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

REQUIRED:

1. Use your favorite tool (*e.g.* Maple, Mathematica, Matlab, pencil) to generate the Legendre polynomial expansion to the function $f(z) = \sin(\pi z)$. How many terms do you need to include in a partial sum to get a "good" approximation to $f(z)$ for $-1 < z < 1$? What do you mean by a "good" approximation? How about the interval $-2 < z < 2$? How good is your approximation? Discuss your answers. Answer the same set of questions for the function $g(z) = \sin(3\pi z)$

   Solution:

   The function $\sin(\pi\, z)$ is an odd function, therefore we only expect to get contributions from Legendre polynomials with odd values of ℓ. For $\ell = 1$, we will get a straight line approximation to the sine function, surely not good enough. For $\ell = 3$ we see that we are already getting the right number of peaks for the range $-1 < z < 1$, but the peaks are shifted and not quite the right shape. For $\ell = 5$, we are already getting a reasonably good fit, depending on our needs. As always, we need to say "a good fit compared to what?". Notice, however, that outside the range of $-1 < z < 1$, the fit gets worse and worse as we add more terms. See for example, this graph for $\ell = 19$. This behavior is expected. Series of orthogonal polynomials each have a specified range of convergence.

   For the function $\sin(3\pi\, z)$, terminating the series at $\ell = 3$ is a terrible fit. It doesn't even have enough terms to fit the correct number of peaks. Remember that the Legendre polynomials are **polynomials**, so terminating the series at $\ell = 3$ gives you only two peaks. For $\ell = 7$, we are at least getting the correct number of peaks. And for $\ell = 13$, we are finally getting the same kind of fit that we got in the previous example for $\ell = 5$.

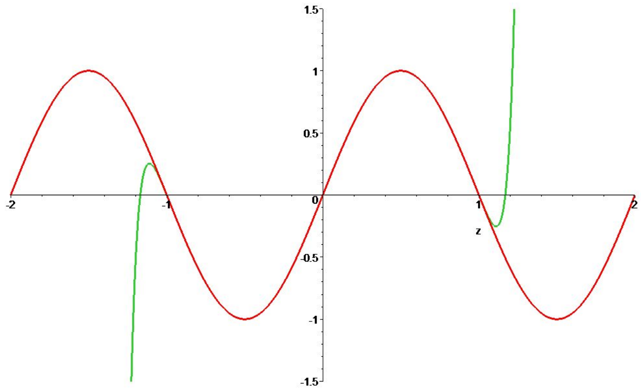

Figure 2: The Legendre Series approximation for the function $\sin(\pi z)$ terminating at $\ell = 19$.

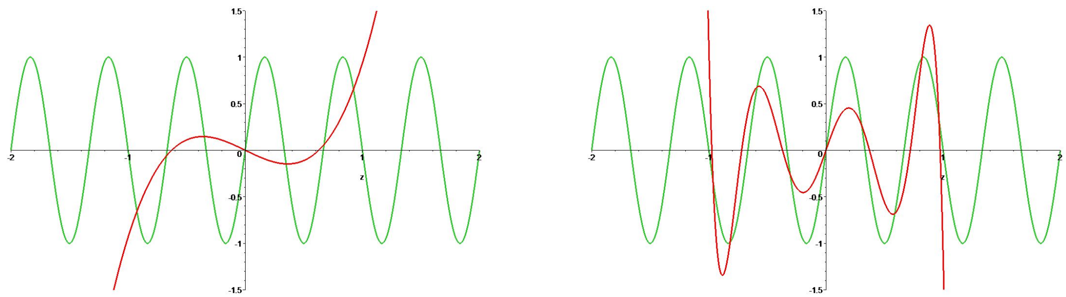

Figure 3: The Legendre Series approximation for the function $\sin(3\pi z)$ terminating at $\ell = 3$ (left) and $\ell = 7$ (right).

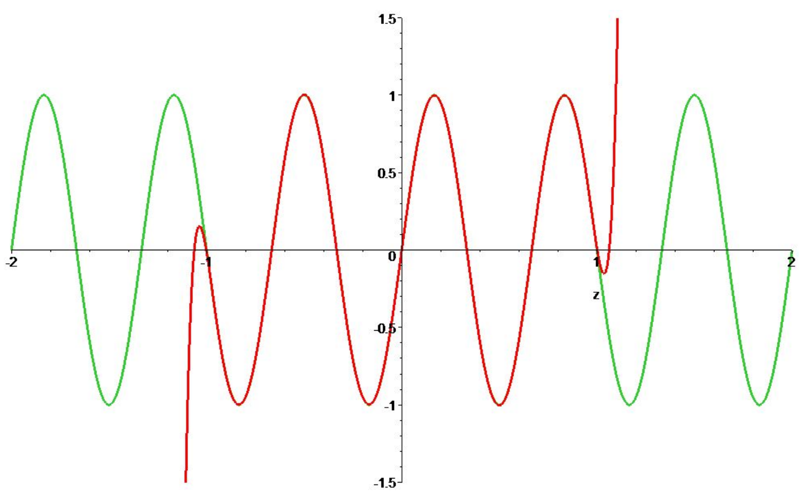

Figure 4: The Legendre Series approximation for the function $\sin(3\pi z)$ terminating at $\ell = 13$.

2. Show that if a linear combination of ring energy eigenstates is normalized, then the coefficients must satisfy

$$\sum_{m=-\infty}^{\infty} |c_m|^2 = 1$$

Solution:

The form of this answer should be a proof that the left hand side of the equation equals the right. We will start by noting that the coefficients may be written as inner products via Dirac notation:

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} |c_m|^2 &= \sum_{m=-\infty}^{\infty} |\langle m|\psi\rangle|^2 \\
&= \sum_{m=-\infty}^{\infty} \langle\psi|m\rangle\langle m|\psi\rangle \\
&= \langle\psi| \sum_{m=-\infty}^{\infty} [|m\rangle\langle m|]\, |\psi\rangle \\
&= \langle\psi|\hat{I}|\psi\rangle \\
&= \langle\psi|\psi\rangle \\
= 1
\end{aligned}$$

Here the identity operator has been substituted for the sum of the projection operators over all possible states $|m\rangle$.

3. Answer the following questions for a quantum mechanical particle confined to a ring. You may want to use the Mathematica activity on time dependence of a particle on the ring from the course website (cfqmring.nb) to help you figure out the answers.

   (a) Characterize the states for which the probability density does not depend on time.

   Solution:

   The probability density is independent of time for all states that are associated with only a single energy eigenvalue. That is, states that are energy eigenstates. Due to the degeneracy in the energy levels of the Hamiltonian, some linear combinations of $|m\rangle$ states are also energy eigenstates. In particular, any superposition of the form $A|m\rangle + B|-m\rangle$, where A and B are arbitrary complex coefficients (subject to the normalization condition).

   (b) Characterize the states that are right-moving.

   Solution:

   The right-moving states are those that have only terms with positive m values, as these correspond to L_z positive, giving a rightward (counter-clockwise) sense of rotation. A similar statement can be made regarding left-moving and negative m values.

(c) Characterize the states that are standing waves.

Solution:

The "standing wave" solutions are those that have an equal contribution (in absolute value) from the $-m$ term for every m term that appears in the overall superposition. Note that you can have lots and lots of terms, just that each m should be offset by a $-m$ with a coefficient of equal magnitude. This can be thought of as representing an expectation value of $L_z = 0$. Note that standing wave may not be the best terminology to use here, as the probability density is never negative, but the general sense of oscillations up and down as compared to travelling waves is still present.

(d) Compare the time dependence of the three states:

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle + |-3\rangle\right) \\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle - |-3\rangle\right) \\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle + i|-3\rangle\right) \end{aligned}$$

Solution:

Usually, when a quantum state only contains two m states, the time-evolution causes the relative phase to cycle. In this case, changing the relative phase between the component eigenstates only represents (up to an overall phase) three different instants in time in the overall time evolution of the state.

In this example, however, the two states have different values of m but the same energy. As a result, this is an energy eigenstate and all three are constant (but different) in time.

4. Consider the following normalized state for the rigid rotor given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle$$

(a) What is the probability that a measurement of L_z will yield $2\hbar$? $-\hbar$? $0\hbar$?

Solution:

The state $|\psi\rangle$ is written in terms of the eigenstates of two operators (i.e. they have two labels). The first label tells us the eigenvalue of ℓ and the second of m. We are asked to find the probability that $m = 2$, -1, or 0.

$$\begin{aligned}
\mathcal{P}(m=2) &\doteq \sum_{\ell=0}^{\infty} |\langle \ell, 2|\psi\rangle|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \left\langle \ell, 2| \left(\frac{1}{\sqrt{2}} |1,-1\rangle + \frac{1}{\sqrt{3}} |1,0\rangle + \frac{i}{\sqrt{6}} |0,0\rangle \right) \right\rangle \right|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \left(\frac{1}{\sqrt{2}} \langle \ell, 2|1,-1\rangle + \frac{1}{\sqrt{3}} \langle \ell, 2|1,0\rangle + \frac{i}{\sqrt{6}} \langle \ell, 2|0,0\rangle \right) \right|^2 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(m=-1) &\doteq \sum_{\ell=0}^{\infty} |\langle \ell, -1|\psi\rangle|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \left\langle \ell, -1| \left(\frac{1}{\sqrt{2}} |1,-1\rangle + \frac{1}{\sqrt{3}} |1,0\rangle + \frac{i}{\sqrt{6}} |0,0\rangle \right) \right\rangle \right|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \left(\frac{1}{\sqrt{2}} \langle \ell, -1|1,-1\rangle + \frac{1}{\sqrt{3}} \langle \ell, -1|1,0\rangle + \frac{i}{\sqrt{6}} \langle \ell, -1|0,0\rangle \right) \right|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \frac{1}{\sqrt{2}} \delta_{\ell,1} \right|^2 \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(m=0) &\doteq \sum_{\ell=0}^{\infty} |\langle \ell, 0|\psi\rangle|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \left\langle \ell, 0| \left(\frac{1}{\sqrt{2}} |1,-1\rangle + \frac{1}{\sqrt{3}} |1,0\rangle + \frac{i}{\sqrt{6}} |0,0\rangle \right) \right\rangle \right|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \left(\frac{1}{\sqrt{2}} \langle \ell, 0|1,-1\rangle + \frac{1}{\sqrt{3}} \langle \ell, 0|1,0\rangle + \frac{i}{\sqrt{6}} \langle \ell, 0|0,0\rangle \right) \right|^2 \\
&= \sum_{\ell=0}^{\infty} \left| \frac{1}{\sqrt{3}} \delta_{\ell,1} + \frac{i}{\sqrt{6}} \delta_{\ell,0} \right|^2 \\
&= \left| \frac{1}{\sqrt{3}} \right|^2 + \left| \frac{i}{\sqrt{6}} \right|^2 \\
&= \frac{1}{2}
\end{aligned}$$

Be careful in the last last line of the last case. You need to take the square of the norm of each of the individual coefficients and *then* add, rather than adding the coefficients and then squaring. Why?

(b) If you measured the z-component of angular momentum to be $-\hbar$, what would the state of the particle be immediately after the measurement is made? $0\hbar$?

Solution:

In order to find the state after taking a measurement, we have to use Postulate 5 (the projection postulate):

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

When $L_z = -\hbar$, the projection operator corresponds to states where $m = -1$, so for this situation the projection operator is

$$P_{-1} = \sum_{\ell=0}^{\infty} |\ell, -1\rangle\langle\ell, -1| = |1, -1\rangle\langle 1, -1|.$$

The new state is

$$\begin{aligned}
|\psi'\rangle &= \frac{|1,-1\rangle\langle 1,-1|\left(\frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle\right)}{\sqrt{\langle\psi|\left(|1,-1\rangle\langle 1,-1|\right)\left(\frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle\right)}} \\
&= \frac{\frac{1}{\sqrt{2}}|1,-1\rangle}{\sqrt{\langle\psi|\frac{1}{\sqrt{2}}|1,-1\rangle}} = \frac{\frac{1}{\sqrt{2}}|1,-1\rangle}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2}} \\
&= |1,-1\rangle,
\end{aligned}$$

which is what we would expect, since there is only one state that has $m = -1$.

When we make a measurement of $L_z = 0\hbar$, $m = 0$ and there are two states that are possible, $|1,0\rangle$ and $|0,0\rangle$, so the projection operator is

$$P_0 = \sum_{\ell=0}^{\infty} |\ell, 0\rangle\langle\ell, 0| = |1,0\rangle\langle 1,0| + |0,0\rangle\langle 0,0|$$

and the new state is

$$|\psi'\rangle = \frac{\left(|1,0\rangle\langle 1,0| + |0,0\rangle\langle 0,0|\right)\left(\frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle\right)}{\sqrt{\langle\psi|\left(|1,0\rangle\langle 1,0| + |0,0\rangle\langle 0,0|\right)\left(\frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle\right)}}$$

$$
\begin{aligned}
&= \frac{\frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle}{\sqrt{\langle\psi|\left(\frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{6}}|0,0\rangle\right)}} = \frac{\frac{1}{\sqrt{3}}|1,0\rangle + \frac{i}{\sqrt{2}}|0,0\rangle}{\sqrt{\frac{1}{3}+\frac{1}{6}}} \\
&= \sqrt{\frac{2}{3}}|1,0\rangle + i\frac{1}{\sqrt{3}}|0,0\rangle,
\end{aligned}
$$

which is just a re-normalized superposition of the the two $m = 0$ states.

(c) What is the expectation value of L_z in this state?

Solution:

The expectation value of L_z is given by:

$$
\begin{aligned}
\langle\psi|L_z|\psi\rangle &= \langle\psi|\,L_z|\left(\frac{1}{\sqrt{2}}\,|1,-1\rangle + \frac{1}{\sqrt{3}}\,|1,0\rangle + \frac{i}{\sqrt{6}}\,|0,0\rangle\right) \\
&= \langle\psi|\left(\frac{1}{\sqrt{2}}(-1\hbar)\,|1,-1\rangle + \frac{1}{\sqrt{3}}(0\hbar)\,|1,0\rangle + \frac{i}{\sqrt{6}}(0\hbar)\,|0,0\rangle\right) \\
&= (-1\hbar)\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\,\langle 1,-1|1,-1\rangle + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\,\langle 1,0|1,-1\rangle + \frac{-i}{\sqrt{6}}\frac{1}{\sqrt{2}}\,\langle 0,0|1,-1\rangle\right) \\
&\quad +(0\hbar)\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\,\langle 1,-1|1,0\rangle + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\,\langle 1,0|1,0\rangle + \frac{-i}{\sqrt{6}}\frac{1}{\sqrt{3}}\,\langle 0,0|1,0\rangle\right) \\
&\quad +(0\hbar)\left(\frac{1}{\sqrt{2}}\frac{i}{\sqrt{6}}\,\langle 1,-1|0,0\rangle + \frac{1}{\sqrt{3}}\frac{i}{\sqrt{6}}\,\langle 1,0|0,0\rangle + \frac{-i}{\sqrt{6}}\frac{i}{\sqrt{6}}\,\langle 0,0|0,0\rangle\right) \\
&= -\frac{\hbar}{2}
\end{aligned}
$$

Notice that I've put a lot more steps in than you need so that you can see where all the terms go. Many of the terms are zero!!

Alternatively, since you know the probabilities and eigenvalues, you can use the weighted average notation.

$$
\langle\psi|L_z|\psi\rangle = \sum_{m=-\infty}^{\infty} m\hbar\mathcal{P}_{L_z=m\hbar} = 0\hbar\left(\frac{1}{2}\right) + (-1\hbar)\left(\frac{1}{2}\right) = -\frac{1}{2}\hbar
$$

(d) What is the expectation value of L^2 in this state?

Solution:

The expectation value of L^2 is given by:

$$
\begin{aligned}
\left\langle \psi | L^2 | \psi \right\rangle &= \langle\psi|\, L^2| \left(\frac{1}{\sqrt{2}} |1,-1\rangle + \frac{1}{\sqrt{3}} |1,0\rangle + \frac{i}{\sqrt{6}} |0,0\rangle \right) \\
&= \langle\psi| \left(\frac{1}{\sqrt{2}} (2\hbar^2) |1,-1\rangle + \frac{1}{\sqrt{3}} (2\hbar^2) |1,0\rangle + \frac{i}{\sqrt{6}} (0\hbar^2) |0,0\rangle \right) \\
&= (2\hbar^2) \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \langle 1,-1|1,-1\rangle + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \langle 1,0|1,-1\rangle + \frac{-i}{\sqrt{6}} \frac{1}{\sqrt{2}} \langle 0,0|1,-1\rangle \right) \\
&\quad + (2\hbar^2) \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \langle 1,-1|1,0\rangle + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \langle 1,0|1,0\rangle + \frac{-i}{\sqrt{6}} \frac{1}{\sqrt{3}} \langle 0,0|1,0\rangle \right) \\
&\quad + (0\hbar^2) \left(\frac{1}{\sqrt{2}} \frac{i}{\sqrt{6}} \langle 1,-1|0,0\rangle + \frac{1}{\sqrt{3}} \frac{i}{\sqrt{6}} \langle 1,0|0,0\rangle + \frac{-i}{\sqrt{6}} \frac{i}{\sqrt{6}} \langle 0,0|0,0\rangle \right) \\
&= \left(\frac{1}{2} + \frac{1}{3} \right) 2\hbar^2 \\
&= \frac{5}{3} \hbar^2
\end{aligned}
$$

Notice that I've put a lot more steps in than you need so that you can see where all the terms go. Many of the terms are zero, but fewer than in the previous case.

(e) What is the expectation value of the energy in this state?

Solution:

For the rigid rotor, the Hamiltonian $\hat{H}$ is proportional to L^2, by construction. (This will not be the case for the hydrogen atom.) We can see this by looking at the differential operators:

$$
\begin{aligned}
\hat{H} &= -\frac{\hbar^2}{2\mu r_0^2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \\
&= \frac{1}{2\mu r_0^2} L^2
\end{aligned}
$$

Therefore, the expectation value of $\hat{H}$ is proportional to the expectation value of L^2 (see the previous part) with this same proportionality constant.

$$
\left\langle \psi | \hat{H} | \psi \right\rangle = \frac{1}{2\mu r_0^2} \left\langle \psi | L^2 | \psi \right\rangle = \frac{5}{3} \frac{\hbar^2}{2\mu r_0^2}
$$

5. Let $P_l(z)$ be a solution of the Legendre's equation

$$
(1 - z^2) \frac{d^2 y(z)}{dz^2} - 2z \frac{dy(z)}{dz} + l(l+1) y(z) = 0.
$$

Show that $(1-z^2)^{m/2}\dfrac{d^m P_l(z)}{dz^m}$ is a solution of the associated Legendre equation

$$(1-z^2)\frac{d^2y(z)}{dz^2} - 2z\frac{dy(z)}{dz} + \left[l(l+1) - \frac{m^2}{1-z^2}\right] y(z) = 0.$$

Hint: You may want to start by differentiating the Legendre's equation m times with respect to z and apply Leibniz's theorem for the mth derivative of a product

$$\frac{d^m}{dz^m}\left[f(z)g(z)\right] = \sum_{r=0}^{m} \binom{m}{r} \frac{d^r f(z)}{dz^r}\frac{d^{m-r}g(z)}{dz^{m-r}}$$

(Strictly, this equation is true for $k \geqslant 1$, since it is for this range of values of k that all quantities involved are well defined; but we may also make it true for $k = 0$ provided we *define* $\mathrm{f}_{-1} = 0$, for then

$$\begin{aligned}\mathrm{f}_0 &= P_1(x)P_0(y) - P_0(x)P_1(y)\\ &= x - y \qquad \text{(by equations (3.21))}\end{aligned}$$

and $x - y$ is also equal to

$$(2k+1)(x-y)P_k(x)P_k(y) + \mathrm{f}_{k-1} \text{ with } k = 0.)$$

If we now sum the set of equations (3.31) from $k=0$ to $k=l$, we obtain

$$\begin{aligned}\sum_{k=0}^{l} \mathrm{f}_k &= \sum_{k=0}^{l} (2k+1)(x-y)P_k(x)P_k(y) + \sum_{k=0}^{l} \mathrm{f}_{k-1}\\ &= \sum_{k=0}^{l} (2k+1)(x-y)P_k(x)P_k(y) + \sum_{k=1}^{l} \mathrm{f}_{k-1}\\ &= \sum_{k=0}^{l} (2k+1)(x-y)P_k(x)P_k(y) + \sum_{k=0}^{l-1} \mathrm{f}_k.\end{aligned}$$

Hence

$$\sum_{k=0}^{l} \mathrm{f}_k - \sum_{k=0}^{l-1} \mathrm{f}_k = \sum_{k=0}^{l} (2k+1)(x-y)P_k(x)P_k(y)$$

so that

$$\mathrm{f}_l = (x-y)\sum_{k=0}^{l}(2k+1)P_k(x)P_k(y)$$

and if we remember the definition of f_l we obtain

$$\frac{(l+1)}{(x-y)}\{P_{l+1}(x)P_l(y) - P_l(x)P_{l+1}(y)\} = \sum_{k=0}^{l}(2k+1)P_k(x)P_k(y).$$

3.8 ASSOCIATED LEGENDRE FUNCTIONS

Theorem 3.9

If z is a solution of Legendre's equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0$$

then $(1-x^2)^{m/2}(\mathrm{d}^mz/\mathrm{d}x^m)$ is a solution of the equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left\{l(l+1) - \frac{m^2}{1-x^2}\right\}y = 0$$

(known as the associated Legendre equation).

PROOF

Since z is a solution of Legendre's equation, we must have

$$(1-x^2)\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}z}{\mathrm{d}x} + l(l+1)z = 0. \tag{3.32}$$

Now let us differentiate equation (3.32) m times with respect to x:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{(1-x^2)\frac{\mathrm{d}^2z}{\mathrm{d}x^2}\right\} - 2\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{x\frac{\mathrm{d}z}{\mathrm{d}x}\right\} + l(l+1)\frac{\mathrm{d}^mz}{\mathrm{d}x^m} = 0$$

which, when we use Leibniz's theorem for the mth derivative of a product,† becomes

$$\begin{aligned}(1-x^2)\frac{\mathrm{d}^{m+2}z}{\mathrm{d}x^{m+2}} + m\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2).\frac{\mathrm{d}^{m+1}z}{\mathrm{d}x^{m+1}} + \frac{m(m-1)}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1-x^2).\frac{\mathrm{d}^mz}{\mathrm{d}x^m}\\ - 2\left\{x\frac{\mathrm{d}^{m+1}z}{\mathrm{d}x^{m+1}} + m\frac{\mathrm{d}}{\mathrm{d}x}x.\frac{\mathrm{d}^mz}{\mathrm{d}x^m}\right\} + l(l+1)\frac{\mathrm{d}^mz}{\mathrm{d}x^m} = 0\end{aligned}$$

(since higher derivatives of $1 - x^2$ and x vanish).

Collecting terms in $\mathrm{d}^{m+2}z/\mathrm{d}x^{m+2}$, $\mathrm{d}^{m+1}z/\mathrm{d}x^{m+1}$ and $\mathrm{d}^mz/\mathrm{d}x^m$, we obtain

$$(1-x^2)\frac{\mathrm{d}^{m+2}z}{\mathrm{d}x^{m+2}} - 2x(m+1)\frac{\mathrm{d}^{m+1}z}{\mathrm{d}x^{m+1}} + \{l(l+1) - m(m-1) - 2m\}\frac{\mathrm{d}^mz}{\mathrm{d}x^m} = 0,$$

which, on denoting $\mathrm{d}^mz/\mathrm{d}x^m$ by z_1, becomes

$$(1-x^2)\frac{\mathrm{d}^2z_1}{\mathrm{d}x^2} - 2(m+1)x\frac{\mathrm{d}z_1}{\mathrm{d}x} + \{l(l+1) - m(m+1)\}z_1 = 0. \tag{3.33}$$

If we now write

$$z_2 = (1-x^2)^{m/2}z_1 = (1-x^2)^{m/2}\frac{\mathrm{d}^mz}{\mathrm{d}x^m}$$

equation (3.33) becomes

$$\begin{aligned}(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\{z_2(1-x^2)^{-m/2}\} - 2(m+1)x\frac{\mathrm{d}}{\mathrm{d}x}\{z_2(1-x^2)^{-m/2}\}\\ + \{l(l+1) - m(m+1)\}z_2(1-x^2)^{-m/2} = 0.\end{aligned} \tag{3.34}$$

But

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{z_2(1-x^2)^{-m/2}\} &= \frac{\mathrm{d}z_2}{\mathrm{d}x}(1-x^2)^{-m/2} + z_2.-\frac{m}{2}(1-x^2)^{-(m/2)-1}.-2x\\ &= \frac{\mathrm{d}z_2}{\mathrm{d}x}(1-x^2)^{-m/2} + mz_2x(1-x^2)^{-(m/2)-1}\end{aligned}$$

† $\dfrac{\mathrm{d}^m}{\mathrm{d}x^m}(uv) = \displaystyle\sum_{r=0}^{m} {}^mC_r\frac{\mathrm{d}^ru}{\mathrm{d}x^r}\frac{\mathrm{d}^{m-r}v}{\mathrm{d}x^{m-r}}$.

so that

$$\begin{aligned}
\frac{d^2}{dx^2}\{z_2(1-x^2)^{-m/2}\} \\
&= \frac{d^2z_2}{dx^2}(1-x^2)^{-m/2} + \frac{dz_2}{dx}\cdot -\frac{m}{2}(1-x^2)^{-(m/2)-1}.(-2x) \\
&\quad + m\left\{\frac{dz_2}{dx}x(1-x^2)^{-(m/2)-1} + z_2(1-x^2)^{-(m/2)-1}\right. \\
&\quad \left. + z_2x\left(-\frac{m}{2}-1\right)(1-x^2)^{-(m/2)-2}.-2x\right\} \\
&= \frac{d^2z_2}{dx^2}(1-x^2)^{-m/2} + \frac{dz_2}{dx}mx(1-x^2)^{-(m/2)-1} + m\frac{dz_2}{dx}x(1-x^2)^{-(m/2)-1} \\
&\quad + mz_2(1-x^2)^{-(m/2)-1} + mz_2x^2(m+2)(1-x^2)^{-(m/2)-2}.
\end{aligned}$$

Hence equation (3.34) becomes

$$\begin{aligned}
\frac{d^2z_2}{dx^2}(1-x^2)^{-(m/2)+1} &+ 2mx(1-x^2)^{-m/2}\frac{dz_2}{dx} + mz_2(1-x^2)^{-m/2} \\
&+ m(m+2)(1-x^2)^{-(m/2)-1}x^2z_2 \\
&- 2(m+1)x\left\{(1-x^2)^{-m/2}\frac{dz_2}{dx} + mx(1-x^2)^{-(m/2)-1}z_2\right\} \\
&+ \{l(l+1)-m(m+1)\}z_2(1-x^2)^{-m/2} = 0
\end{aligned}$$

which, on cancelling a common factor of $(1-x^2)^{-m/2}$ and collecting like terms, becomes

$$\begin{aligned}
(1-x^2)\frac{d^2z_2}{dx^2} &+ \{2mx - 2(m+1)x\}\frac{dz_2}{dx} \\
&+ \left\{m + \frac{m(m+2)}{1-x^2}x^2 - \frac{2(m+1)mx^2}{1-x^2} + l(l+1) - m(m+1)\right\}z_2 = 0. \quad (3.35)
\end{aligned}$$

The coefficient of dz_2/dx is just $-2x$, while the coefficient of z_2 is

$$\begin{aligned}
l(l+1) + \frac{(m^2+2m-2m^2-2m)x^2}{1-x^2} + m - m^2 - m \\
&= l(l+1) - \frac{m^2x^2}{1-x^2} - m^2 \\
&= l(l+1) - \frac{m^2}{1-x^2}.
\end{aligned}$$

Thus equation (3.35) reduces to

$$(1-x^2)\frac{d^2z_2}{dx^2} - 2x\frac{dz_2}{dx} + \left\{l(l+1) - \frac{m^2}{1-x^2}\right\}z_2 = 0$$

so that z_2 satisfies the associated Legendre equation which, by the definition of z_2, proves the theorem.

COROLLARY

The associated Legendre functions $P_l^m(x)$ defined by

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \quad (3.36)$$

satisfy the associated Legendre equation.

PROOF

This result follows immediately from the theorem, since $P_l(x)$ satisfies Legendre's equation.

Using Rodrigues' formula (theorem 3.2), it is possible to rewrite definition (3.36) in the form

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

The right-hand side of this expression is well defined for negative values of m such that $l + m \geqslant 0$, i.e., $m \geqslant -l$, whereas the original definition (3.36) of $P_l^m(x)$ was only valid for $m \geqslant 0$. Thus we may use this new form to define $P_l^m(x)$ for values of m such that $m \geqslant -l$.

It is easy to verify that if we consider m positive, the function $P_l^{-m}(x)$ defined in such a way is a solution of Legendre's associated equation as well as $P_l^m(x)$. In fact, it is not an independent solution; it may be proved that

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \quad (3.37)$$

(see problem 3 at the end of this chapter).

3.9 PROPERTIES OF THE ASSOCIATED LEGENDRE FUNCTIONS

Theorem 3.10

(i) $P_l^0(x) = P_l(x)$.
(ii) $P_l^m(x) = 0$ if $m > l$.

PROOF

(i) This result is immediately obvious from definition (3.36).

(ii) Since $P_l(x)$ is a polynomial of degree l, it will reduce to zero when differentiated more than l times. Thus $\dfrac{d^m}{dx^m}P_l(x) = 0$ for $m > l$, and the required result then follows from definition (3.36).