Complex Variables: Assignment 5 **John Waczak**

MTH 483 Date: June 5, 2018

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(a). Observe that we can rewrite the function accordingly

$$(z^2+1)^{-3}(z-1)^{-4} = ((z-i)(z+i))^{-3}(z-1)^{-4} = \frac{1}{(z-i)^3(z+i)^3(z-1)^4}$$

Then it is clear that the function has 3 poles $\{i, -i, 1\}$ with multiplicities $\{3, 3, 4\}$. $\square$

(b). Consider $f(z) = z\cot z$ which can be rewritten as $f(z) = \frac{z\cos z}{\sin z}$. Because $\sin z$ and $\cos z$ are $\pi/2$ out of phase, they never share a zero, thus we have poles of order 1 whenever $\sin(z) = 0$ i.e. when $z = n\pi$ such that $n \in \mathbb{Z} \setminus \{0\}$. Here 0 is a special case as we have the indeterminant form $\frac{0}{0}$. This singularity is removable as can be seen by the power series:

$$\begin{aligned} f(z) &= \frac{\cos z}{z^{-1}\sin(z)} \\ &= \frac{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}}{z^{-1}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}} \\ &= \frac{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}}{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}} \\ &= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots} \end{aligned}$$

And from the last line it is easy to see that as $z \to 0$, $f(z) \to 1$ indicating that $z = 0$ is in fact a removable singularity. $\square$

(c). The function $f(z) = \sin(z)z^{-5}$ has a pole of order 4 when $z = 0$ because (as we saw in the previous part) the power series for $\sin(z)$ is entire and has a power series looking like $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$. Thus dividing by z^5 cancels out the first z-term leaving z^{-4}. $\square$

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(a). We can find $\oint_\gamma \cot(z)dz$ by using the argument principle. Observe that $\cot z = \frac{\cos z}{\sin z} = \frac{\sin' z}{\sin z}$. The argument principal gives that $\oint_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i\big[Z(f,\gamma) - P(f,\gamma)\big]$. $\gamma = C[0,3]$ the circle of radius 3 centered about $z = 0$. Therefore we have that $f(z) = \sin(z)$ has no poles inside of γ and and one zero when $z = 0$. Thus

$$\oint_\gamma \cot(z)dz = 2\pi i$$

$\square$

(c). We want to find the value of $\oint_\gamma \frac{dz}{(z+4)(z^2+1)}$. We can do this using the Residue theorem. First observe that the integral can be rewritten as $\oint_\gamma \frac{dz}{(z+4)(z-i)(z+i)}$ which has 3 simple poles of order 1 at $\{-4, i, -i\}$. Only the poles at $i, -i$ lie within γ, thus

$$\begin{aligned}
\oint_\gamma \frac{dz}{(z+4)(z-i)(z+i)} &= 2\pi i \sum_i Res[f, z_i] \\
&= 2\pi i \Big[\frac{1}{(i+4)(i+i)} + \frac{1}{(-i+4)(-i-i)}\Big] \\
&= -\frac{2\pi i}{17}
\end{aligned}$$

□

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(c). We want to evaluate the following integral for $\gamma = C[0,2]$: $\oint_\gamma \frac{\exp(z)}{z^3+z} dz$. First rearrange the integral to become

$$\oint_\gamma \frac{\exp(z)}{z(z^2+1)} dz = \oint_\gamma \frac{\exp(z)}{z(z-i)(z+i)} dz$$

From this last equation we see that the integrand has 3 simple poles at $z = 0, i, -i$. All are included within γ. Therefore, our integral evaluates to

$$\begin{aligned}
\oint_\gamma f(z)dz &= 2\pi i \sum_i Res[f, z_i] \\
&= 2\pi i \Big[\frac{\exp(0)}{-i(i)} + \frac{\exp(i)}{i(2i)} + \frac{\exp(-i)}{-i(-2i)}\Big] \\
&= 2\pi i \Big[1 - \frac{\cos(1) + i\sin(1)}{2} - \frac{\cos(1) - i\sin(1)}{2}\Big] \\
&= 2\pi i(1 - \cos(1))
\end{aligned}$$

□

(d). We want to evaluate the integral $\oint_\gamma \frac{dz}{z^2 \sin z}$ where $\gamma = C[0,1]$. This function has only one pole inside of γ when $z = 0$ with order 3. Thus we have that

$$\begin{aligned}
\oint_\gamma f(z)dz &= 2\pi i Res(f, z=0) \\
&= 2\pi i \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2} z^3 f(z) \\
&= \pi i \lim_{z\to 0} \frac{d^2}{dz^2} \frac{z}{\sin z} \\
&= \pi i \lim_{z\to 0} \csc(z)[z\cot^2(z) - 2\cot(z) + z\csc^2(z)] \\
&= \frac{\pi i}{3}
\end{aligned}$$

□

(e). We want to evaluate the integral $\oint_\gamma f(z)dz$ for $f(z) = \frac{\exp(z)}{(z+2)^2\sin(z)}$ $\gamma = C[0,3]$. This function has a simple pole when $z = 0$ and a pole of multiplicity 2 when $z = -2$. Both of these poles lie within γ and therefore,

$$\begin{aligned}\oint_\gamma f(z)dz &= 2\pi i[Res(f, z = 0) + Res(f, z = -2)] \\ &= 2\pi i\Big[\lim_{z\to 0} zf(z) + \lim_{z\to -2}\frac{d}{dz}(z+2)^2 f(z)\Big] \\ &= 2\pi i\Big[\frac{1}{4} + \frac{((-1-\cot(2))\csc(2))}{e^2}\Big]\end{aligned}$$

$\square$

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(a). We want to find the number of zeros of $3\exp(z) - z$ in $\overline{D}[0,1]$. By Rouche's theorem we can see that $|-z| = 1$ is less than $|3\exp(z)| = 3$ $\forall z \in \gamma$ (the boundary). So define $f(z) = 3\exp(z)$ and $g(z) = -z$. Then,

$$Z(f + g, \gamma) = Z(f, \gamma)$$

However here we see that the function $f(z) = 3\exp(z)$ has no zeros and therefore as we shrink the boundary, we do not find any zeros. Thus we conclude that the function $3\exp(z) - z$ has no zeros inside of $\overline{D}[0,1]$. $\square$

(b). We want to find the number of zeros of $\frac{1}{3}\exp(z) - z$ in $\overline{D}[0,1]$. In analogy to part (a), observe that $|f(z)| = |\frac{1}{3}\exp(z)| = \frac{1}{3}e^{\Re(z)} \le \frac{1}{3}e^1 = \frac{1}{3}e$ and $|g(z)| = |-z| = 1$ on the boundary. Thus by applying Rouche's theorem, we have that the number of zeros $Z(f + g, \gamma) = Z(g, \gamma)$ since $|g(z)| \ge |f(z)|$. Therefore, there are no zeros until we shrink γ down to radius 0 at which point $g(z) = -z$ has a zero. Therefore we conclude that $f(z) + g(z) = \frac{1}{3}\exp(z) - z$ has one zero inside of $\overline{D}[0,1]$. $\square$

(c). We want to find the zeros of $z^4 - 5z + 1$ inside of $\{z \in \mathbb{C} : 1 \le |z| \le 2\}$. Define $f(z) = z^4$ and $g(z) = -5z + 1$. The total number of zeros in our region should be the zeros inside of $C[0,2]$ minus those in $C[0,1]$. Thus for the first case observe that $|f(z)| = |z^4| \le 2^4 = 16$ on the outer boundary. Then $|g(z)| = |-5z + 1| \le 5 \cdot 2 + 1 = 11$ Therefore Rouche's theorem gives us that the number of zeros for $f(z) + g(z)$ inside of $C[0,2]$ is given by the number of zeros of $f(z)$ which is 4 by the fundamental theorem of algebra.

For the second one we have that $|g(z)| = |-5z + 1| \le 5 + 1 = 6$ on the boundary of $C[0,1]$ and $|f(z)| = |z^4| \le 1^4 = 1$ on the same boundary. Thus the number of zeros for $f(z) + g(z)$ in $C[0,1]$ is given by the number of zeros of $g(z)$ in $C[0,1]$. This number is 1 (when $z = 1/5$). Therefore we conclude that that the total number of zeros in the annulus is $4 - 1 = \mathbf{3}$.

$\square$