John Walczak

Tapp: 3.80

(1) prove that if f is a rigid motion and $S \subset \mathbb{R}^3$ is a regular surface then $f(S)$ is a regular surface.

Since S is a regular surface we have

$$\sigma : U \subset \mathbb{R}^2 \to V \subset \mathbb{R}^3 \text{ is a smooth bijection}$$

and U is an open set in $\mathbb{R}^2$.

Now we want to show $f(S)$ is a regular surface. Because f is a rigid motion we can say $f = T_q \circ L_A$ where T_q is a translation by q and L_A is an orthogonal matrix. Now we already have that U is open in $\mathbb{R}^2$. Observe the following diagram for the map $f \circ \sigma$

$$f \circ \sigma : U \subset \mathbb{R}^2 \longrightarrow f(V) \subset \mathbb{R}^3$$

this is equivalent to

$$U \subset \mathbb{R}^2 \xrightarrow{\sigma} V \subset \mathbb{R}^3 \xrightarrow{f} f(V) \subset \mathbb{R}^3.$$

Thus all we need to show is that because σ is a diffeomorphism, $f \circ \sigma$ must also be a diffeomorphism

σ is a smooth bijection and so σ^{-1} exists and is smooth. Now

$$f \circ \sigma = T_q \circ L_A \circ \sigma.$$

Now the inverse of f, is simply

$$f^{-1} = L_A^{-1} \circ T_{-q} \quad \text{and so}$$

$$(f \circ \sigma)^{-1} = \sigma^{-1} \circ f^{-1} = \sigma^{-1} \circ L_A^{-1} \circ T_{-q}$$

since ~~L_A~~ $A \in O(3)$ we know it is invertible and so L_A^{-1} is defined. Thus we have a map and its inverse and so all we must show is that they are smooth.

The derivatives of T_q, T_{-q} are the identity and so those functions are smooth.

Similarly because L_A is a rotation or reflection (we showed this in exercise 1.17.3) then at most ~~[illegible]~~ L_A multiplies the components of the vector in $\mathbb{R}^3$ by some combination of sine, cosine and ~~or~~ constant. These three functions are smooth so $T_q \circ L_A \circ \sigma$ and $\sigma^{-1} \circ L^{-1} \circ T_{-q}$ are smooth. Thus ~~F~~ we have an open set $U \subset \mathbb{R}^2$, a neighborhood $f(V)$ of $p \in f(S)$ and a diffeomorphism ~~F~~ $T_q \circ L_A \circ \sigma$.

Therefore, $f(S)$ is a regular surface.

we also know $dL_A = L_A$ so it is infinitely differentiable

(2) Now we want to show that f restricts to an isometry between S and $f(S)$.

Recall that because $f = T_q \circ L_A$ and so by the chain rule $df = dT_q \circ dL_A$

Now by definition of the derivative,

$$d_pT_q(v) = \lim_{t\to 0} \frac{T_q(p+tv) - T_q(p)}{t}$$

$$= \lim_{t\to 0} \frac{p+tv+q - p+q}{t}$$

$$= \lim_{t\to 0} \frac{tv}{t} = v \quad \text{i.e. } dT_q = \text{identity.}$$

$$d_pL_A(v) = \lim_{t\to 0} \frac{L_A(p+tv) - L(p)}{t} \quad \left(L_A \text{ is Linear operator}\right)$$

$$= \lim_{t\to 0} \frac{t\,L_A(v)}{t} = L_A(v)$$

and so we have $df = I \circ L_A = L_A$

Now to show df is an isometry let $x, y \in T_pS$. then

$\langle df(x), df(y)\rangle = \langle L_A(x), L_A(y)\rangle$

by proposition 1.55 (3) L_A preserves inner products, so $\langle L_A(x), L_A(y)\rangle = \langle x, y\rangle$

thus $\langle df(x), df(y)\rangle = \langle x, y\rangle$ which confirms that f is an isometry $\square$

Tapp 3.85

Let S be the graph of the equation $z = xy$
classify the linear rigid motions of $\mathbb{R}^3$
that induce isometries of S.

we have that $\sigma(u,v) = (u, v, uv)$
defines the surface patch for the surface
created. We know that the surface is a
regular surface because $\cancel{x}$ u, v, uv are smooth.

Now we want to "classify" the linear
rigid motions that induce isometries.

From the first problem we already showed that
if f is a rigid motion then $f(S)$ is an
isometry for a regular surface S.

The linear rigid motions are L_A w/ $A \subset O(3)$
since T_q translations arent linear.

The question also mentions ~~in~~ in parentheses that
it wants only the isometries that map
S to itself.

we know L_A is defined by $A \subset O(3)$ thus we have $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ s.t.

$A^T = A^{-1}$ Now we want to take

$\sigma(u,v) = (u, v, uv)$ and so

Now $A\,\sigma(u,v) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$

$A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$ so

$A^T A^T = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

$=$

For this to work all we should do is apply an orthogonal ~~map~~ $L_A \in O(3)$ to $\sigma(u,v) = (u, v, uv)$ (I think)

then if $A\,\sigma(u,v) = (x, y, z)$

we just need to show that to show we've sent $x \cdot y = z$ to itself.

S to itself.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} u \\ v \\ uv \end{pmatrix} =$$

$$\rightarrow \quad au + bv + cuv = x$$
$$du + ev + fuv = y$$
$$gu + hv + iuv = Y = xy$$

$$\rightarrow \quad gu + hv + iuv = (au + bv + cuv)(du + ev + fuv)$$

$$\rightarrow \quad gu + hv + iuv = adu^2 \ldots$$

I'm getting really lost doing this....
I'm not sure how to continue from here and unfortunately I've run out of time. It's 11:48 and I have to stop to scan everything.