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4. Let $f:[a,b]\to\mathbb{R}$ be strictly increasing function. Show $\int_a^b f(x)dx$ exists

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Recall $f:[a,b]\to\mathbb{R}$ is intlable iff $\forall\, \varepsilon>0$ $\exists$ step functions f_1, f_2 on $[a,b]$ s.t.

$$f_1(x)\le f(x)\le f_2(x) \quad \forall x\in[a,b]$$

and

$$\int_a^b f_2(x)-f_1(x)\,dx < \varepsilon$$

Let's define f_2 to be a step function defined as

$$f_2(x)\equiv \sum_{i=1}^{N} f(x_i)\,\mathbb{1}_{(x_{i-1},x_i)}(x)$$

(right ~~hand~~ end pt.)

and f_1 to be

$$f_1(x)\equiv \sum_{i=1}^{N} f(x_{i-1})\,\mathbb{1}_{(x_{i-1},x_i)}(x)$$

(left end pt)

then $\forall x\in[a,b]$ it is true that

$$f_1(x)\le f(x)\le f_2(x)$$

Now WTS

$$\int_a^b f_2(x)-f_1(x)\,dx < \varepsilon$$

by linearity we have then that

$$\int_a^b f_2(x)dx - \int_a^b f_1(x)dx < \varepsilon$$