PH 424 review

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March 21, 2017

1 Oscillations

Oscillations are a repetitive variation of some quantity about a central equilibrium point. The easy way to think about it is that oscillations depend on space *or* time but not both (that's a wave). An oscillation is like a snapshot of a wave.

The following forms of oscillations are equivalent:

$$A : x(t) = Acos(\omega_0 t + \phi) \tag{1}$$
$$B : x(t) = B_p cos(\omega_0 t) + B_q sin(\omega t) \tag{2}$$
$$C : x(t) = Ce^{i\omega_0 t} + C^* e^{-i\omega t} \tag{3}$$
$$D : x(t) = Re[De^{i\omega t}] \tag{4}$$

Note that $C, D \in \mathbb{C}$ while $A, B_p, B_q \in \mathbb{R}$. There is another way to represent oscillations using differential equations.

Consider the differential equation for a spring mass system shown below.

$$m\ddot{x} = -kx \tag{5}$$

To solve differential equations, you make an educated guess, try it out, then form the general solution using a linear combination of the different possible solutions.

$$\begin{aligned} x(t) &= Ce^{pt} \\ p^2 &= -\frac{k}{m} \\ p &= \pm i\sqrt{\frac{k}{m}} \\ &= \pm i\omega_0 \end{aligned}$$

Thus the general solution (since there are two possible p values) is given by:

$$x(t) = Ce^{i\omega_0 t} + C^{'} e^{-i\omega_0 t} \tag{6}$$

In order to force our solution to be real, we apply the physical condition necessitating that $C^{'} = C^{*}$.

Similarly we can apply Kirchhoff's loop law to the LC circuit to derive the following differential equation:

$$\ddot{q} = -\frac{1}{LC}q \tag{7}$$

Adding a resistor and creating an LRC circuit makes things a little more complicated:

$$L\ddot{q} = -R\dot{q} - \frac{1}{C}q \tag{8}$$

The corresponding general differential equation for a damped oscillator is:

$$0 = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x \tag{9}$$

Solving this equation is as follow:

$$\begin{aligned} x(t) &= Ce^{pt} \\ 0 &= (p^2 + 2\beta p + \omega_0^2) \\ p_{\pm} &= -\beta \pm \sqrt{\beta^2 - \omega_0^2} \\ &= -\beta \pm i\omega_f, \quad \beta < \omega_0 \end{aligned}$$

This new system does not oscillate at ω_0 but instead at

$$\omega_f = (\omega_0^2 - \beta^2)^{\frac{1}{2}}$$

As mentioned in class there are three distinct cases of damped oscillation. The following table summarizes the relationships between ω_0 and β.

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$

Now we can define a metric for the 'damping-ness' of a system. The Q factor measures how many periods elapse during the damping time $\tau = \frac{1}{\beta}$.

$$Q = \frac{\omega_0}{2\beta} \tag{10}$$

Now let's reconsider the LCR circuit and see if we can derive equations for the voltage and current for a sinusoidal driving force.

$$V_0 e^{i\omega t} = L\ddot{q} + R\dot{q} + \frac{1}{C}q$$

This time our guess will be $|q|e^{i\phi}e^{i\omega t}$. Here we have a magnitude term, phase term, and oscillation term. Empirically this makes sense as we observe that driven systems always oscillate at the driving frequency.

$$\begin{aligned} q(t) &= |q_0|e^{i\phi+\omega t} \\ \dot{q}(t) &= i\omega|q_0|e^{i\phi+\omega t} \\ \ddot{q}(t) &= -\omega^2|q_0|e^{i\phi+\omega t} \\ |q_0|(-\omega^2 + 2i\omega\beta + \omega_0^2) &= \frac{V_0}{L}e^{i\omega t} \\ (\omega_0^2 - \omega^2)|q_0| + i2\beta\omega|q_0| &= \frac{V_0}{L}e^{-i\phi} \end{aligned}$$

Considering the separation of the left hand side into real and imaginary parts, we can construct equations to solve for $|q|$ and ϕ as functions of ω.

$$\phi_q = -arctan(\frac{-2\beta\omega}{w_0^2 - \omega^2}) \tag{11}$$

$$|q_0(\omega)| = \frac{V_0/L}{[(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2]^{\frac{1}{2}}} \tag{12}$$

To get to current just start taking derivatives. Current is the time derivative of charge and back from our original charge equation we know that due to the phase, the current will have an extra $\pi/2$ phase shift.

$$|I(\omega)| = \omega|q_0(\omega)| \tag{13}$$

$$\phi_I = \frac{\pi}{2} + \phi_q = -arctan(\frac{\omega^2 - \omega_0^2}{2\beta\omega}) \tag{14}$$

You can consider an L circuit, R circuit, and C circuit in order to derive the relative phase shifts of each. The following table

component	$\Delta\phi_I$	Z
C	$\frac{\pi}{2}$	$\frac{1}{i\omega c}$
L	$-\frac{\pi}{2}$	$i\omega L$
R	0	R

2 Fourier Series

So far we have developed methods for evaluating ordinary differential equations including both damping and driving forces. However, we limited ourselves to sinusoidal driving forces as they appear frequently. The question remains: how do we deal with a non-sinusoidal driving term? The answer is that so long as the driving term is periodic, that is it repeats over some consistent interval, we can build a solution using sine and cosine terms of different frequencies or more generally by using complex exponentials (this should make sense due to Euler's identity).

The deep concept behind Fourier series is that if we consider the vector space of periodic functions (yes functions can be vectors since they obey the axioms of a vector space), every $e^{in\omega t}$ is mutually orthogonal. For finite dimensional spaces this is characterized using the dot product. As we learned in Quantum fundamentals, the dot product is a unique case of the more general inner product which is the equivalent operation that takes to vectors and returns a scalar. Essentially we need to show that all of the $e^{in\omega t}$ are mutually orthogonal. The inner product for this vector space is an integral and thus:

$$\int_0^T e^{in\omega t} \cdot e^{-im\omega t} dt = \frac{e^{2\pi(n-m)} - 1}{i\omega(n-m)} = 0, \quad m \neq n$$

Similarly for the case where n = m we have:

$$\int_0^T e^{in\omega t} \cdot e^{-in\omega t} dt = \int_0^T 1 dt = T, \quad n = m$$

Putting these together we can use the Kronecker delta function to summarize the whole condition:

$$\int_0^T e^{in\omega t} \cdot e^{-im\omega t} dt = T\delta_{n,m} \tag{15}$$

If we normalize using $\frac{1}{T}$ we now have *orthonormality*. Since we have shown that all of these exponentials (or sines and cosines by analogy) form a basis for our vector space, any 'vector' i.e. function can be represented as a linear combination of our basis vectors and so we have the important result that:

$$\psi(s) = \sum_{n=-\infty}^{n=\infty} c_n \cdot e^{in\omega s} \tag{16}$$

$$\psi(s) = \frac{1}{2}a_0 + \sum_{n=1}^{n=\infty} a_n \cdot cos(n\omega s) + \sum_{n=1}^{n=\infty} b_n \cdot sin(n\omega s) \tag{17}$$

$$\tag{18}$$

Here s is used as a dummy variable. This can replaced with x, t, or any other variable depending on the model in question. In both cases the bounds are such that we include all possible terms as well as the constant case for n=0. We can solve for the coefficients using our inner products for a T periodic system in the following way.

$$a_n = \frac{2}{T}\int_0^T \psi(s)cos(n\omega s)ds \tag{19}$$

$$b_n = \frac{2}{T}\int_0^T \psi(s)sin(n\omega s)ds \tag{20}$$

$$c_n = \frac{1}{T}\int_0^T \psi(s)e^{-in\omega s}ds \tag{21}$$

$$\tag{22}$$

Observe that $a_n, b_n \in \mathbb{R}$ while $c_n \in \mathbb{C}$. The pre-factors are determined from the orhtonormality conditions we previously derived. Note that they change if you use a 2T periodic function from -T to T.

3 Wave mechanics - non-dispersive

Just as ordinary differential equations can be used to characterize oscillatory motion, partial differential equations that consider both space and time are used to characterize waves. A *wave* is a disturbance in some medium that travels from one location to another. We shall consider two ubiquitous types of waves: the standing wave and the traveling wave. Both rely on the ever-present physics concept of superposition which states that the amplitudes of two waves traveling through the same space at the same time add. The following equations illustrates one form of a standing wave:

$$\psi(x,t) = Asin(kx)cos(\omega t) \tag{23}$$

Here it is clear to see that nothing is really 'traveling' because the spatial and temporal dependencies are disjoint. The cosine term changes the amplitude of the total wave as time progresses but there is no apparent lateral movement. The next equation is one form for a traveling wave:

$$\psi(x,t) = Asin(kx - \omega t) \tag{24}$$

Here the the extra argument of $-\omega t$ makes it so that as time progresses the horizontal shift of the entire function increases and so the wave travels (in this case to the right). Both of these are solutions to the **wave equation**.

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \tag{25}$$

As we did before, we will solve this equation by trying a guess $\psi x, t = X(x)T(t)$ and then forming our general solution as a linear combination of acceptable particular solutions. For this derivation, we consider as string with fixed ends. So that we may write our solution in full generality we will suspend application of initial conditions.

$$\psi(x,t) = X(x)T(t)$$
$$X^{''}(x)T(t) = \frac{1}{v^2}X(x)\ddot{T}(t)$$

The trick to these equations is that at this step you can divide through by your guess to make the equation separable into two sides that each only depend on one variable.

$$\frac{X^{''}(x)}{X(x)} = \frac{\ddot{T}(t)}{v^2T(t)}$$

Now observe that since both sides depend on separate variables we can imagine that a small change in x does not affect the right hand side. Similarly the a small change in time won't affect the left hand side. Thus we can equate both sides to some constant A. Next we will look at the left hand side and solve for X(x).

$$\frac{X^{''}(x)}{X(x)} = A$$
$$X^{''}(x) = AX(x)$$
$$X(x) = Pe^{\sqrt{A}x}$$

For this equation to make physical sense A must be negative because otherwise our solution will be purely real exponential and will not meet the boundary conditions for being zero at both ends- definitely not a wave. Thus:

$$X(x) = Ce^{i\sqrt{|A|}x} = Acos(\sqrt{|A|}x) + Bsin(\sqrt{|A|}x)$$
$$X(L) = 0 \rightarrow \sqrt{|A|} = \frac{2\pi n}{L}$$
$$|A| = \frac{4\pi^2n^2}{L^2}$$
$$X(x) = Acos(\frac{2\pi nx}{L}) + Bsin(\frac{2\pi nx}{L})$$

Now that we know A, we can solve the right hand side for the T equation and craft our solution.

$$\ddot{T}(t) = -\frac{4v^2n^2\pi^2}{L^2}T(t)$$
$$T(t) = Pe^{-\frac{2vn\pi}{L}t} = Ccos(\frac{2\pi nvt}{L}) + Dsin(\frac{2\pi nvt}{L})$$

Now we have everything we need to craft our general solution. To make things cleaner we use $k = \frac{2\pi}{L}$ and $\omega = k * v$.

$$\psi(x,t) = X(t)T(t) \tag{26}$$
$$= (Acos(nkx) + Bsin(nkx)) \cdot (Ccos(n\omega t) + Dsin(n\omega t)) \tag{27}$$

This is the general solution to the wave equation. If we start applying boundary conditions for the initial shape and velocity, things will simplify and we can get back to our standing and/or travelling wave equations with a little bit of trig. Solving for the coefficients will typically involve Fourier series expansions of your initial shape $\psi(x,0)$ but be smart, if the question is some strange combination of sines and cosines there is probably a trig identity that will turn it back into sines and cosines of different integer frequencies that will tell you easily what the Fourier expansion is.

4 Reflection and Transmission

Now we want to consider what happens for a traveling wave that encounters a boundary. One example of such a boundary would be for a rope where the mass density μ suddenly changes. We observe that there is both a reflected and transmitted wave. We can write this as two equations using:

$$f_1(x,t) = Ae^{i(k_1x-\omega t)} + Be^{i(-k_1x-\omega t)} \tag{28}$$
$$f_2(x,t) = Ce^{i(k_2x-\omega t)} \tag{29}$$

We have two physical boundary conditions for this system to make sense. At the boundary, the wave itself must be continuous. Similarly the velocity must also be continuous. To make things nice we'll put the boundary at x=0. Applying these gives:

$$A + B = C, \quad f_1(0,t) = f_2(0,t)$$
$$k_1(A - B) = k_2C, \quad \dot{f}_1(0,t) = \dot{f}_2(0,t)$$

Now we can solve for the reflection and transmission coefficients.

$$R = \frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$
$$T = \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

Alternatively we do this in terms of the impedance. We have two equations for impedance. For a circuit $Z = \frac{V_{ext}}{I}$. For displacements, we can write $Z = \frac{F_{ext}}{(\frac{\partial f}{\partial t})_x}$. For a string this comes out to $Z = sqrt\mu T$. Thus we can redefine our wave speed in terms of Z. After doing this it is clear that so long as ω is constant,

$Z \propto k$. So our Transmission and Reflection terms become:

$$R_q = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{30}$$

$$T_q = \frac{2Z_1}{Z_1 + Z_2} \tag{31}$$

For a coaxial cable, finding these coefficients for the voltage forms uses the fact that $V_x = \frac{1}{C_0}\frac{\partial q}{\partial x}$. This gives:

$$R_v = -R = \frac{Z_2 - Z_1}{Z_1 + Z_2} \tag{32}$$

$$T_v = \frac{Z_2 T}{Z_1} = \frac{2Z_2}{Z_1 + Z_2} \tag{33}$$

5 Wave mechanics - damping considerations

ω is not always proportional to k. For example, if we consider the partial differential equation describing the voltage in a coaxial cable we have:

$$\frac{1}{L_0 C_0} V_{xx} = V_{tt} + \Gamma V_t, \quad \Gamma = \frac{R_0}{L_0} \tag{34}$$

In general this is written as:

$$v^2 V_{xx} = V_{tt} + \Gamma V_t \tag{35}$$

Let's make our guess a traveling wave given by $f(x,t) = Ae^{i(kx-\omega t)}$. Plugging this into the differential equation gives the following:

$$\begin{aligned} v^2(ik)^2 V(x,t) &= [(-i\omega)^2 - i\omega\Gamma]V(x,t) \\ k^2 &= \frac{1}{v^2}(\omega^2 + i\omega\Gamma) \\ k &= \pm\frac{\omega}{v}(1 + \frac{i\Gamma}{\omega})^{0.5} \\ k &\approx \pm\frac{\omega}{v} + \frac{i\Gamma}{2v} \end{aligned}$$

The final line uses Taylor expands for the limit where $\frac{\Gamma}{\omega} << 1$. This is the weak damping case. Otherwise we would not be able to separate the above equations in to real and imaginary parts. When we do, we get:

$$V(x,t) = Ae^{-\frac{\Gamma}{2v}x} e^{i(\frac{\omega}{v}x - \omega t)} \tag{36}$$

For simplicity we will define $\beta = \frac{\Gamma}{2v}$. Observe that:

$$\beta = \frac{\Gamma}{2v} = \frac{R_0 C_0 v}{2}$$

Using this we can reevaluate the transmission and reflection coefficients to be:

$$R_v = -Re^{-2\beta L} \tag{37}$$

$$T_v = \frac{Z_2}{Z_1} T e^{-\beta L} \tag{38}$$

$1/\beta$ is the attenuation length - an important metric for characterizing waves.

6 Energy

By the conservation of energy we know that the total must be given by the sum of the kinetic and potential energies.

$$E_{tot} = W(x) + U(x) \tag{39}$$

I'm losing steam here so I'm just going to list a bunch of important energy equations...

$$E_0 = \frac{Z}{2v}(\frac{\partial f}{\partial t})^2 + \frac{Zv}{2}(\frac{\partial f}{\partial x})^2 \tag{40}$$

$$P = F_{applied}\frac{\partial f}{\partial t} = -Zvf_x f_t \tag{41}$$

7 Fourier Transforms

For the final topic of this class we ask the question: What if my forcing function isn't periodic? Well we can model this by using our Fourier series in the limit of an infinite period. E.g. $T \to \infty, \quad \omega \to d\omega$. This leads us to the concept of the Fourier transform which takes an input in one space (time) and converts it to the reciprocal space (frequency). The transforms are given by:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{42}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega \tag{43}$$

For the equivalent versions using spatial variables replace t with x and ω with k.