3B

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Mth 434

Calculate the curvature and torsion of the following curves.

1. $\gamma(t) = (r\cos(t/r), r\sin(t/r), 0)$

$\gamma'(t) = (-\sin(t/r), \cos(t/r), 0)$

$\gamma''(t) = \frac{1}{r}(-\cos(t/r), -\sin(t/r), 0)$

$|v| = |\gamma'(t)| = 1 \implies a^{\perp} = a$ and so

$\boxed{K(t) = |a| = \frac{1}{r}}$ (as expected for a circle)

$T = \frac{v}{|v|} = (-\sin(t/r), \cos(t/r), 0)$

$n = \frac{a}{|a|} = (-\cos(t/r), -\sin(t/r), 0)$

$$b = T \times n = \begin{vmatrix} i & j & k \\ -\sin(t/r) & \cos(t/r) & 0 \\ -\cos(t/r) & -\sin(t/r) & 0 \end{vmatrix} = 1$$

$b' = 0 \implies \boxed{\tau = 0}$

as expected for a plane curve

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2.

$$\gamma(t) = \left(a\cos\left(\frac{t}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{t}{\sqrt{a^2+b^2}}\right), \frac{bt}{c}\right)$$

to make life easy let's define $c = \sqrt{a^2+b^2}$ so that

$$\gamma(t) = (a\cos(t/c), a\sin(t/c), bt/c)$$
$$\gamma'(t) = (-\tfrac{a}{c}\sin(t/c), \tfrac{a}{c}\cos(t/c), b/c)$$
$$\gamma''(t) = (-\tfrac{a}{c^2}\cos(t/c), -\tfrac{a}{c^2}\sin(t/c), 0)$$

$$|v| = |\gamma'(t)| = \sqrt{\frac{a^2}{c^2}+\frac{b^2}{c^2}} = \sqrt{\frac{c^2}{c^2}} = 1$$

$\Longrightarrow a^\perp = a$ and so

$$K(t) = |a| = \cancel{\sqrt{\tfrac{a^2}{c^2}\ldots}} = \boxed{\frac{a}{c^2}}$$

$$\mathbb{T} = \frac{v}{|v|} = (-\tfrac{a}{c}\sin(t/c), \tfrac{a}{c}\cos(t/c), b/c)$$

$$\cancel{\mathbb{N} = \frac{a^\perp}{|a^\perp|} = \frac{a}{|a|} \ldots}$$

$$\mathbb{N} = \frac{a^\perp}{|a^\perp|} = \frac{a}{|a|} = \frac{c^2}{a}\left(-\frac{a}{c^2}\cos(t/c), -\frac{a}{c^2}\sin(t/c), 0\right)$$

$$= (-\cos(t/c), -\sin(t/c), 0)$$

$$\mathbb{b} = \mathbb{T}\times\mathbb{N} = \begin{vmatrix} i & j & k \\ -\frac{a}{c}\sin(t/c) & \frac{a}{c}\cos(t/c) & b/c \\ -\cos(t/c) & -\sin(t/c) & 0 \end{vmatrix}$$

$$= \left(\frac{b}{c}\sin(t/c), -\frac{b}{c}\cos(t/c), \frac{a}{c}\right)$$

$$\mathbb{b}' = \left(\frac{b}{c^2}\cos(t/c), \frac{b}{c^2}\sin(t/c), 0\right)$$

$$\mathbb{B}' = \left(\frac{b}{c^2}\cos(t/c),\ \frac{b}{c^2}\sin(t/c),\ 0 \right)$$

$$\tau = \frac{-\langle \mathbb{B}', \mathbb{N} \rangle}{|v|}$$

$$= - \left(-\frac{b}{c^2}\cos^2(t/c) - \frac{b}{c^2}\sin^2(t/c) \right)$$

$$= \frac{b}{c^2}$$

Thus $\boxed{\kappa = \frac{a}{c^2} \qquad \tau = \frac{b}{c^2}}$

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$$\gamma(t) = (\cos(t), \sin(t), t)$$
$$\beta(t) = (\cos(t), \sin(t), -t)$$

$$\gamma(t) = (\cos(t), \sin(t), t)$$
$$\gamma'(t) = V_\gamma(t) = (-\sin(t), \cos(t), 1)$$
$$\gamma''(t) = a_\gamma(t) = (-\cos(t), -\sin(t), 0)$$

$$|\gamma'(t)| = |V_\gamma| = \sqrt{2} \implies a_\gamma^\perp = a_\gamma$$

$$\boxed{K_\gamma(t) = |a_\gamma(t)| = \sqrt{1} = 1}$$

$$t_\gamma = \frac{V_\gamma}{|V_\gamma|} = \frac{1}{\sqrt{2}}(-\sin(t), \cos(t), 1)$$

$$n_\gamma = \frac{a_\gamma^\perp}{|a_\gamma^\perp|} = \frac{a_\gamma}{|a_\gamma|} = (-\cos(t), -\sin(t), 0)$$

$$b_\gamma = t_\gamma \times n_\gamma = \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}(\sin(t), -\cos(t), 1)$$

$$b'(t) = \frac{1}{\sqrt{2}}(\cos(t), \sin(t), 0)$$

$$\tau_\gamma = -\frac{\langle b', n \rangle}{|v|} = -\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(-\cos^2(t) - \sin^2(t)\right)\right)$$

$$\boxed{\tau_\gamma(t) = \frac{1}{2}}$$

$\beta(t) = (\cos(t), \sin(t), -t)$
$\beta'(t) = (-\sin(t), \cos(t), -1) = V_\beta$
$\beta''(t) = (-\cos(t), -\sin(t), 0) = a_\beta$

$|\beta'| = |V| = \sqrt{2} \implies a^\perp = a$

$$\boxed{K_\beta(t) = |a_\beta| = \sqrt{1} = 1}$$

$t_\beta = \frac{V_\beta}{|V_\beta|} = \frac{1}{\sqrt{2}}(-\sin(t), \cos(t), -1)$

$n_\beta = \frac{a_\beta^\perp}{|a_\beta^\perp|} = \frac{a_\beta}{|a_\beta|} = (-\cos(t), -\sin(t), 0)$

$b_\beta = t_\beta \times n_\beta = \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ -\sin(t) & \cos(t) & -1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix}$

$= \frac{1}{\sqrt{2}}(-\sin(t), \cos(t), 1)$

$b'_\beta = \frac{1}{\sqrt{2}}(-\cos(t), -\sin(t), 0)$

$\tau_\beta = -\frac{\langle b'_\beta, n_\beta \rangle}{|V_\beta|}$

$= -\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\cos^2(t) + \sin^2(t))\right)$

$$\boxed{\tau_\beta = -\frac{1}{2}}$$

Thus the curvatures are the same as expected but the Torsions are opposite in sign. I think this is an indication of the reversal in the direction of the curve. $\gamma(t)$ turns to the left going up the z axis and β turns left going down the z axis.