8c

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3.31 Define $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ s.t. $\sigma(u,v) = (u\cosh v, u\sinh v, u^2)$

Verify that the image of σ is contained in the graph $z = x^2 - y^2$

Let $G = \{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } x^2 - y^2 - z = 0\}$ we
W.T.S. $\sigma(\mathbb{R}^2) \subseteq G$. Observe that

$$x^2 - y^2 - z = u^2\cosh^2(v) - u^2\sinh^2(v) - u^2$$

$$= u^2(1) - u^2 = 0$$

thus the image of σ is contained in the graph G,

Now we want to check if the image is the entire graph which essentially means we want to find if $\sigma : \mathbb{R} \to G$ is onto. This means that $\forall q \in G\ \exists (u,v) \in \mathbb{R}^2$ s.t. $\sigma(u,v) = q$

After graphing the function it looks like the answer should be no, but let's test a point to check.

Note that the the point $(0,1,-1) \in G$

Now lets see if $\exists\ u,v \in \mathbb{R}^2$ s.t. $x=0$ and $y=1$, $z=-1$

this gives us the system

$$u\cosh v = 0$$
$$u\sinh v = 1$$
$$\rightarrow \quad u^2 = -1$$

as $u \in \mathbb{R}$ there is no solution to this problem and so we see that in fact $\sigma(\mathbb{R}^2)$ is not the entire graph as negative z values are not possible under this map.

Now we want to see if $\sigma(u,v)$ is a parametrized surface.

$$\sigma(u,v) = (u\cosh(v), u\sinh(v), u^2)$$

$$\rightarrow \quad d\sigma(u,v) = \begin{pmatrix} \cosh v & u\sinh(v) \\ \sinh v & u\cosh(v) \\ 2u & 0 \end{pmatrix}$$

Now lets look at $(0,0) \in \mathbb{R}^2$ this yields

$$d\sigma(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$ which

does <u>not</u> have rank 2 and therefore

$\sigma(u,v)$ is <u>not</u> a parametrized surface.

3.32 For an arbitrary integer $m \geq 2$ define

$$S_m = \{(x,y,z) \in \mathbb{R}^3 \mid x^m + y^m + z^m = 1\}$$

(1) prove that S_m is a regular value

define $f(x,y,z) = x^m + y^m + z^m$

then

$$df_p = (mx^{m-1}, my^{m-1}, mz^{m-1})$$

The only way for this to be the zero matrix is if $x = y = z = 0$ since $m \geq 2$. Now note that $0^m + 0^m + 0^m = 0 \neq 1$ and so $(0,0,0) \notin f^{-1}(1)$ and so 1 must be a regular value of $f(x,y,z)$. Because 1 is a regular value of f, S_m must be a regular surface $\forall m \geq 2$ by theorem 3.27 (page 133).

For (2) and Tapp 3.33 see attached mathematica code.