(2) Now we want to show that f restricts to an isometry between S and $f(S)$.

Recall that because $f = T_q \circ L_A$ and so by the chain rule $df = dT_q \circ dL_A$

Now by definition of the derivative,

$$d_pT_q(v) = \lim_{t\to 0} \frac{T_q(p+tv) - T_q(p)}{t}$$

$$= \lim_{t\to 0} \frac{p+tv+q - p+q}{t}$$

$$= \lim_{t\to 0} \frac{tv}{t} = v \quad \text{i.e. } dT_q = \text{identity}.$$

$$d_pL_A(v) = \lim_{t\to 0} \frac{L_A(p+tv) - L(p)}{t} \quad \left(L_A \text{ is linear operator}\right)$$

$$= \lim_{t\to 0} \frac{t\,L_A(v)}{t} = L_A(v)$$

and so we have $df = I \circ L_A = L_A$

Now to show df is an isometry let $x, y \in T_pS$. then

$$\langle df(x), df(y)\rangle = \langle L_A(x), L_A(y)\rangle$$

by proposition 1.55 (3) L_A preserves inner products, so $\langle L_A(x), L_A(y)\rangle = \langle x, y\rangle$

thus $\langle df(x), df(y)\rangle = \langle x, y\rangle$ which confirms that f is an isometry $\square$