3. **Solution**: The equation of the orbit is

$$\frac{\alpha}{r} = 1 + \varepsilon \cos(\phi) \tag{1}$$

assuming without loss of generality that $\phi_0 = 0$. Here $\alpha = \dfrac{\ell^2}{\mu k}$ and $\varepsilon = \sqrt{1 + \dfrac{2E\ell^2}{mk^2}}$. Therefore, the radial distance r can vary from the maximum value $\dfrac{\alpha}{(1-\varepsilon)}$ to the minimum value $\dfrac{\alpha}{(1+\varepsilon)}$.

The angular velocity of the particle is given by

$$\omega = \dot{\phi} = \frac{\ell}{\mu r^2} \tag{2}$$

Thus, the maximum and minimum values of ω become,

$$\begin{cases} \omega_{\text{max}} = \dfrac{\ell}{\mu r_{\text{min}}^2} = \dfrac{\ell}{\mu \left[\frac{\alpha}{1+\varepsilon}\right]^2} \\[2ex] \omega_{\text{min}} = \dfrac{\ell}{\mu r_{\text{max}}^2} = \dfrac{\ell}{\mu \left[\frac{\alpha}{1-\varepsilon}\right]^2} \end{cases} \tag{3}$$

So that the ratio of the two is,

$$\frac{\omega_{\text{max}}}{\omega_{\text{min}}} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \equiv n \tag{4}$$

From which we have

$$\varepsilon = \frac{\sqrt{n}-1}{\sqrt{n}+1} \tag{5}$$

4. **Solution**: The force $f(r) = -k/r^3$ can be easily integrated to find the

corresponding central potential

$$V(r) = -\frac{k}{2r^2} \tag{6}$$

The corresponding effective potential is formed by adding in the kinetic energy due to rotations. That is,

$$V_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{2r^2} \tag{7}$$

The equation for the shape of an orbit is

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2 u^2}\left(-ku^3\right) \tag{8}$$

or,

$$\frac{d^2u}{d\phi^2} + \left[1 - \frac{\mu k}{\ell^2}\right] u = 0 \tag{9}$$

Let us consider the motion of various values of ℓ.

(a) $\ell^2 = \mu k$:
In this case, the effective potential vanishes and the orbit equation becomes

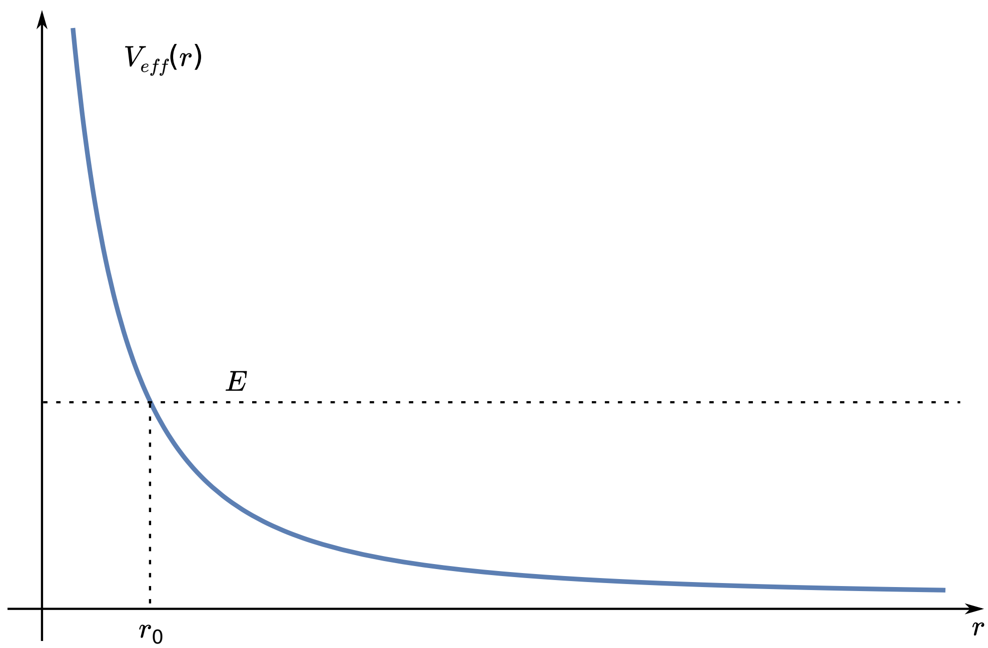
$$\frac{d^2u}{d\phi^2} = 0 \tag{10}$$

which leads to orbits of the form

$$u = \frac{1}{r} = A\phi + B \tag{11}$$

so that the particle spirals towards the force center.

(b) $\ell^2 > \mu k$: In this case the effective potential is positive and decreases monotonically with increasing r. For any value of the total energy E, the particle will approach the force center and will undergo a reversal of motion at $r = r_0$; the particle will then proceed again to an infinite distance. This is illustrated in the following sketch



Setting $1 - \mu k/\ell^2 = \beta^2 > 0$, then the differential equation becomes

$$\frac{d^2u}{d\phi^2} + \beta^2 u = 0 \tag{12}$$

with the solution

$$u(\phi) = \frac{1}{r} = A\cos\left(\beta\phi - \delta\right) \tag{13}$$

Since the minimum value of u is zero, this solution corresponds to unbounded motion, as expected from the form of the effective $V_{\text{eff}}(r)$.

(c) $\ell^2 < \mu k$: For this case we set $\mu k/\ell^2 - 1 = G^2 > 0$, and the orbit equation becomes

$$\frac{d^2u}{d\phi^2} - G^2 u = 0 \tag{14}$$

which leads to

$$u(\phi) = \frac{1}{r} = A\cosh(G\phi - \delta) \tag{15}$$

so that the particle spirals in towards the force center.