Geometric Anatomy of Theoretical Physics

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Lecture Playlist
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Contents

> Wherof one cannot speak, thereof one must be silent.
>
> — Wittgenstein

Overview

Theoretical physics is all about casting our ideas about the real world into *rigorous* mathematical form. We don't do this for mathematic's sake, but rather to fully explore what the implications of our ideas are.

Idea: If we have concepts that we can't cast into rigorous mathematical form, we have a strong indication that these concepts are not well understood.

From this perspective, then, mathematics is *just* the language. To draw physically meaningful conclusions, we must interpret the language.

Goals: Provide *proper* mathematical language for
- classical mechanics
- electromagnetism
- quantum mechanics
- statistical physics

So far, we have used a lot of mathematical tools such as
- Analysis
- Algebra
- Geometry

For example, we often appeal to geometric intuition for classical mechanics and electrodynamics. *Linear* algebra is leveraged throughout…

Differential Geometry can be thought of as the intersection of these three mathematical fields and is the primary subject for the course. The general structure will be to develop tools in the following sequence:
1. *Logic* (propositional and predicate logic)
2. *Axiomatic Set theory* (What are sets? How do we define them?)
3. *Topology* (What is continuity?)
4. *Topological Manifolds* (Toplogical spaces that “locally” look like $\mathbb{R}^d$)
5. *Differentiable Manifolds* (So that we can discuss derivatives)
6. *Bundles* (In order to introduce vector and tensor fields)
7. *Symplectic/Metric Geometry*
8. **PHYSICS**

1. Axiomatic Set Theory

1.1. Propositional Logic

The key notion of *propositional logic* is a proposition.

Definition 1.1.1: A *proposition* p is a variable that can take the values “true” or “false” and no others.

We can build new propositions from given ones using *logical operators*.

1. *unary* operators

p	$\neg p$	$\mathrm{id}\, p$	$\top p$	$\bot p$
T	F	T	T	F
F	T	F	T	F

2. *binary* operators ($2^4 = 16$ possible operators)

p	q	$p \wedge q$	$p \vee q$	$\dots$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	…	T	T
T	F	F	T	…	F	F
F	T	F	T	…	T	F
F	F	F	F	…	T	T

The principal that anything can be implied from a false statement ($\Rightarrow$) goes by the latin *ex falso quod libet*.

Theorem 1.1.1:

$$(p \Rightarrow q) \Leftrightarrow ((\neg q) \Rightarrow (\neg p))$$

Corollary 1.1.1.1: We can prove assertions by way of contradiction.

Proof: Consider the following truth table:

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$(\neg q) \Rightarrow (\neg p)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The result follows by taking ($\Longleftrightarrow$) of final two columns. ■

NOTE: We will agree on the following order for binding strength: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

NOTE: All operators can be constructed from "nand" ($\neg\wedge$).

1.2. Predicate Logic

Definition 1.2.1: A *predicate* is a proposition-valued function of some variable(s)

Example 1.1.1.1: $P(x)$ is "True" or "False" dependent on x.

Strictly speaking, it is not the task of predicate logic to construct the predicates. We need some further "language" for how to describe x (what "set" do you take x from?). We can leave it open for now so that we can define the notion of set later. It will be the task of *Set Theory* to define some fundamental predicates, for example $Q(x, y) := x \in y$

Like with propositional logic, we can construct new predicates from given ones.

1. $Q(x, y, z) :\Leftrightarrow P(x) \wedge R(y, z)$
2. Convert predicate $P(x)$ into a proposition $\forall x : P(x)$ which reads as "for all x, $P(x)$ is true", i.e., $P(x) \Leftrightarrow \mathrm{True}$ independently of x.

   Example 1.1.1.2 *(feel-good)*: Given

   $$P(x) :\Leftrightarrow (x \text{ is a human being } \Rightarrow x \text{ has been created}),$$

   then $\forall x : P(x)$ is true.

3. Existence quantification: $\exists x : P(x)$ which reads as "there exists x such that $P(x)$ is true", i.e. $\neg(\forall x : (\neg P(x)))$

We can chain multiple quantifiers together for predicates of multiple variables, .e.g., $\forall x : \exists y : P(x, y)$

1.3. Axiomatic Systems and Theory of Proofs

What actually is a *proof*? It's a way of arguing that something is true, but we want to be more rigorous about it.

Definition 1.3.1: An *axiomatic system* is a finite sequence of propositions $a_1, a_2, ..., a_n$ which are called *axioms*.

One could object that we don't have numbers yet, but we could replace these symbols with a_i, a_{ii}, a_{iii}, etc. These are are considered to be *pre-mathematical numbers* by logicians. A professional logician would probably be more rigorous when publishing a new monograph…

Definition 1.3.2: A *proof* of a proposition p within an axiomatic system $a_1, ..., a_n$ is a finite sequence of propositions $q_1, ..., q_m$ (with $q_m \Leftrightarrow p$) such that for any $1 \leq j \leq m$ either

1. **(Axiomatic)** q_j is a proposition from the list of axioms
2. **(Tautological)** q_j is a tautology (statements that are true no matter what e.g., $p \vee \neg p$)
3. **(Modus ponens)** $\exists (1 \leq r, s < j) : q_r \wedge q_s \Rightarrow q_j$ is true

Most theorems are not proven in this fashion because it takes *many* lines to take this fully rigorous approach. This is like the machine code for proofs. We will use a more relaxed approach that is like the c++ of proofs, with the understanding being that a proof is only valid if other mathematicians agree that, *in principal*, what was written can be decomposed into proofs of this type.

Remark 1.3.2.1: If p can be proven from an axiomatic system $a_1, ..., a_n$, we often write $a_1, ..., a_n \vdash p$

Remark 1.3.2.2: This definition allows us to easily check a proof. An altogether different task is to *find* a proof.

Remark 1.3.2.3: Any tautology that occurs within the axioms of an axiomatic system can be removed without impairing the power of the axiomatic system. As an extreme example: the axiomatic system that describes propositional logic is the empty sequence because all we can prove are tautologies.

Definition 1.3.3: An axiomatic system is *consistent* if there exists a proposition q which *cannot* be proven from the axioms.

The idea behind Definition 1.3.3 is that a set of axioms for which anything can be proven is *useless*. Consider an axiomatic system containing contradictory propositions: $..., s, ..., \neg s, ...$ Then, by Definition 1.3.2, $s \wedge \neg s \Rightarrow q$ is a tautology. Any statement could be proven to be true.

Theorem 1.3.1: Propositional logic is consistent.

Proof: It suffices to show that there exists a proposition that can not be proven within propositional logic. Propositional logic has an empty sequence of axioms. The only rules for proofs are (T) and (M) from Definition 1.3.2. Therefore, only tautologies can be proven. $q \wedge \neg q$ can not be proven since it is not a tautology. ■

In general, it's very hard to prove a set of axioms are consistent.

Theorem 1.3.2 (Gödel): Any axiomatic system that is *powerful enough* to encode the elementary arithmetic of natural numbers is either inconsistent or contains a proposition that can neither be proven nor disproven.

This sent shockwaves through the mathematics world at the time (20th century) because the notion of truth had seemed to be clear: something was true *if it could be proven*. Gödel's theorem shows there are true statements that can't be proven!

Proof: Basic idea:

1. Assign to each (meta-)mathematical statement a number (called the Gödel nubmer)
2. Use a "The barber shaves all people in his village who do not shave themselves"-type argument to identify a proposition that is neither provable nor disprovable.

■

1.4. The Axioms of Set Theory