Geometric Anatomy of Theoretical Physics

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Lecture Playlist
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Contents

> Wherof one cannot speak, thereof one must be silent.
>
> — Wittgenstein

Overview

Theoretical physics is all about casting our ideas about the real world into *rigorous* mathematical form. We don't do this for mathematic's sake, but rather to fully explore what the implications of our ideas are.

Idea: If we have concepts that we can't cast into rigorous mathematical form, we have a strong indication that these concepts are not well understood.

From this perspective, then, mathematics is *just* the language. To draw physically meaningful conclusions, we must interpret the language.

Goals: Provide *proper* mathematical language for
- classical mechanics
- electromagnetism
- quantum mechanics
- statistical physics

So far, we have used a lot of mathematical tools such as
- Analysis
- Algebra
- Geometry

For example, we often appeal to geometric intuition for classical mechanics and electrodynamics. *Linear* algebra is leveraged throughout…

Differential Geometry can be thought of as the intersection of these three mathematical fields and is the primary subject for the course. The general structure will be to develop tools in the following sequence:

1. *Logic* (propositional and predicate logic)
2. *Axiomatic Set theory* (What are sets? How do we define them?)
3. *Topology* (What is continuity?)
4. *Topological Manifolds* (Toplogical spaces that “locally” look like $\mathbb{R}^d$)
5. *Differentiable Manifolds* (So that we can discuss derivatives)
6. *Bundles* (In order to introduce vector and tensor fields)
7. *Symplectic/Metric Geometry*
8. **PHYSICS**

1. Axiomatic Set Theory

1.1. Propositional Logic

The key notion of *propositional logic* is a proposition.

Definition 1.1.1: A *proposition* p is a variable that can take the values “true” or “false” and no others.

We can build new propositions from given ones using *logical operators*.

1. *unary* operators

p	$\neg p$	$\mathrm{id}\, p$	$\top p$	$\bot p$
T	F	T	T	F
F	T	F	T	F

2. *binary* operators ($2^4 = 16$ possible operators)

p	q	$p \wedge q$	$p \vee q$	…	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	…	T	T
T	F	F	T	…	F	F
F	T	F	T	…	T	F
F	F	F	F	…	T	T

The principal that anything can be implied from a false statement ($\Rightarrow$) goes by the latin *ex falso quod libet*.

Theorem 1.1.1:

$$(p \Rightarrow q) \Leftrightarrow ((\neg q) \Rightarrow (\neg p)) \tag{1.1.1}$$

Corollary 1.1.1.1: We can prove assertions by way of contradiction.

Proof: Consider the following truth table:

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$(\neg q) \Rightarrow (\neg p)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The result follows by taking (⟺) of final two columns. ■

NOTE: We will agree on the following order for binding strength: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

NOTE: All operators can be constructed from "nand" ($\neg \wedge$).

1.2. Predicate Logic

Definition 1.2.1: A *predicate* is a proposition-valued function of some variable(s)

Example 1.1.1.1: $P(x)$ is "True" or "False" dependent on x.

Strictly speaking, it is not the task of predicate logic to construct the predicates. We need some further "language" for how to describe x (what "set" do you take x from?). We can leave it open for now so that we can define the notion of set later. It will be the task of *Set Theory* to define some fundamental predicates, for example $Q(x, y) := x \in y$

Like with propositional logic, we can construct new predicates from given ones.

1. $Q(x, y, z) :\Leftrightarrow P(x) \wedge R(y, z)$
2. Convert predicate $P(x)$ into a proposition $\forall x : P(x)$ which reads as "for all x, $P(x)$ is true", i.e., $P(x) \Leftrightarrow \mathrm{True}$ independently of x.

   Example 1.1.1.2 *(feel-good)*: Given

   $$P(x) :\Leftrightarrow (x \text{ is a human being } \Rightarrow x \text{ has been created}), \tag{1.2.2}$$

   then $\forall x : P(x)$ is true.

3. Existence quantification: $\exists x : P(x)$ which reads as "there exists x such that $P(x)$ is true", i.e. $\neg(\forall x : (\neg P(x)))$

We can chain multiple quantifiers together for predicates of multiple variables, .e.g., $\forall x : \exists y : P(x, y)$

1.3. Axiomatic Systems and Theory of Proofs

What actually is a *proof*? It's a way of arguing that something is true, but we want to be more rigorous about it.

Definition 1.3.1: An *axiomatic system* is a finite sequence of propositions $a_1, a_2, ..., a_n$ which are called *axioms*.

One could object that we don't have numbers yet, but we could replace these symbols with a_i, a_{ii}, a_{iii}, etc. These are are considered to be *pre-mathematical numbers* by logicians. A professional logician would probably be more rigorous when publishing a new monograph…

Definition 1.3.2: A *proof* of a proposition p within an axiomatic system $a_1, ..., a_n$ is a finite sequence of propositions $q_1, ..., q_m$ (with $q_m \Leftrightarrow p$) such that for any $1 \leq j \leq m$ either

1. **(Axiomatic)** q_j is a proposition from the list of axioms
2. **(Tautological)** q_j is a tautology (statements that are true no matter what e.g., $p \vee \neg p$)
3. **(Modus ponens)** $\exists(1 \leq r, s < j) : q_r \wedge q_s \Rightarrow q_j$ is true

Most theorems are not proven in this fashion because it takes *many* lines to take this fully rigorous approach. This is like the machine code for proofs. We will use a more relaxed approach that is like the c++ of proofs, with the understanding being that a proof is only valid if other mathematicians agree that, *in principal*, what was written can be decomposed into proofs of this type.

Remark 1.3.2.1: If p can be proven from an axiomatic system $a_1, ..., a_n$, we often write $a_1, ..., a_n \vdash p$

Remark 1.3.2.2: This definition allows us to easily check a proof. An altogether different task is to *find* a proof.

Remark 1.3.2.3: Any tautology that occurs within the axioms of an axiomatic system can be removed without impairing the power of the axiomatic system. As an extreme example: the axiomatic system that describes propositional logic is the empty sequence because all we can prove are tautologies.

Definition 1.3.3: An axiomatic system is *consistent* if there exists a proposition q which *cannot* be proven from the axioms.

The idea behind Definition 1.3.3 is that a set of axioms for which anything can be proven is *useless*. Consider an axiomatic system containing contradictory propositions: $..., s, ..., \neg s, ...$ Then, by Definition 1.3.2, $s \wedge \neg s \Rightarrow q$ is a tautology. Any statement could be proven to be true.

Theorem 1.3.1: Propositional logic is consistent.

Proof: It suffices to show that there exists a proposition that can not be proven within propositional logic. Propositional logic has an empty sequence of axioms. The only rules for proofs are (T) and (M) from Definition 1.3.2. Therefore, only tautologies can be proven. $q \wedge \neg q$ can not be proven since it is not a tautology. ■

In general, it's very hard to prove a set of axioms are consistent.

Theorem 1.3.2 (Gödel): Any axiomatic system that is *powerful enough* to encode the elementary arithmetic of natural numbers is either inconsistent or contains a proposition that can neither be proven nor disproven.

This sent shockwaves through the mathematics world at the time (20th century) because the notion of truth had seemed to be clear: something was true *if it could be proven*. Gödel's theorem shows there are true statements that can't be proven!

Proof: Basic idea:

1. Assign to each (meta-)mathematical statement a number (called the Gödel nubmer)
2. Use a "The barber shaves all people in his village who do not shave themselves"-type argument to identify a proposition that is neither provable nor disprovable.

■

1.4. The ∈ - Relation

Set theory is build on the postulate that there is a fundamental relation, i.e., a predicate of two variables, called ∈. There will be **no** (strict) definition of what ∈ is, or of what a set is. Instead, we will write down 9 axioms that speak of ∈ and sets. These axioms will teach us how to use ∈ and what constitutes a set. Such an approach is necessary if we want to start *from scratch* without any other prior, assumed notions.

Using the ∈ - relation, we can immediately define:

$$\begin{aligned} x \notin y &:\Leftrightarrow \neg(x \in y) \\ x \subseteq y &:\Leftrightarrow \forall a : (a \in x \Rightarrow x \in y) \\ x = y &:\Leftrightarrow x \subseteq y \wedge y \subseteq x \end{aligned} \tag{1.4.3}$$

1.5. Zermelo-Frankel Axioms of Set Theory

Commonly, these are referred to as ZFC for **Z**ermelo-**F**rankel with the Axiom of **C**hoice.

Axiom 1.5.1 (Existance of sets): The statement $x \in y$ is a proposition if and only if x and y are both sets, i.e., $\forall x : \forall y : (x \in y) \,\overline{\vee}\, \neg(x \in y)$ (where $\overline{\vee}$ is exclusive or).

NOTE: there's some disagreement online about his treatment here. I suspect Schuller is just using an alternative formulation since we define "=" in terms of ∈

Axiom 1.5.2 (Existance of empty set): There exists a set that contains no elements, i.e., $\exists x : \forall y : y \notin x$

Theorem 1.5.1: The empty set is unique, and we give it the label $\varnothing$.

Proof (Standard textbook style): Assume x and x' are both empty sets. But then $\forall y : (y \in x) \Rightarrow (y \in x')$. This means that $x \subseteq x'$. Conversly, $\forall z : (z \in x') \Rightarrow (z \in x)$. Thus, $x' \subseteq x$. Therefore, $x = x'$. ■

Proof (formal version):

$$
\begin{aligned}
&a_1 \Leftrightarrow \forall y : y \notin x \\
&a_2 \Leftrightarrow \forall y : y \notin x' \\
&q_1 \Leftrightarrow (\forall y : y \notin x) \Rightarrow (\forall y : (y \in x \Rightarrow y \in x')) \\
&q_2 \Leftrightarrow a_1 \\
&q_3 \Leftrightarrow (\forall y : (y \in x \Rightarrow y \in x')) \Leftrightarrow x \subseteq x' \\
&q_4 \Leftrightarrow (\forall y : y \notin x') \Rightarrow (\forall y : (y \in x' \Rightarrow y \in x)) \\
&q_5 \Leftrightarrow a_2 \\
&q_6 \Leftrightarrow (\forall y : (y \in x' \Rightarrow y \in x)) \Leftrightarrow x' \subseteq x \\
&q_7 \Leftrightarrow x = x'
\end{aligned}
\tag{1.5.4}
$$

Therefore, the two empty sets are equivalent. ■

Going forward, we will not use this formal proof technique. In principal, we must be able to break down each proof to this level, if prodded.

Axiom 1.5.3 (Pair sets): Let x and y be sets. Then there exists a set that contains as its elements precisely the sets x and y, i.e., $\forall x : \forall y : \exists m : \forall u : (u \in m) \Leftrightarrow (u = x \vee u = y)$. The usual notation is to denote the set m by $\{x, y\}$

It appreas we have introduced an order here. Is $\{x, y\}$ the same as $\{y, x\}$? **Answer**: Yes, because $(a \in \{xy\}) \Leftrightarrow (a \in \{y, x\})$

Axiom 1.5.4 (Union sets): Let x be a set. Then there exists a set u whose elements are precisely the elements of the elements of x.

Notation: $u = \bigcup x$

Example 1.5.1.1: Let a, b, c be sets. Then $\{a\}$ and $\{b\}$ are sets and $x = \{\{a\}, \{b\}\} \Rightarrow \bigcup x = \{a, b\}$ is a set.

Example 1.5.1.2: Let $x = \{\{a\}, \{b, c\}\}$ a set. Then it follows that $\bigcup x$ is a set which is defined to be $\bigcup x \coloneqq \{a, b, c\}$.

Definition 1.5.1: Let $a_1, a_2, ..., a_n$ be sets. Define recursively for $n \geq 3$ the symbol $\{a_1, ..., a_n\} \coloneqq \bigcup \{\{a_1, ..., a_{n-1}\}, \{a_n\}\}$, i.e., we recursively apply the pair set axiom.

Axiom 1.5.5 (Replacemnt): Let R be a *functional* relation. Let m be a set. Then the *image* of m under R is again a set.

Here:
- A relation R is called *functional* if $\forall x : \exists! y : R(x, y)$, i.e. each x must be mapped to some y.
- The *image* of a set m under a functional relation R consists of those y for which there is an $x \in m$ such that $R(x, y)$

Axiom Axiom 1.5.5 implies an additional principle called *the principle of restricted comprehensions* which states that for a predicate P of one variable and a set m, the elements $y \in m$ for which $P(y)$ is true constitute a set, denoted $\{y \in m \mid P(y)\}$. This is our familiar "set builder" notation.

The principal of restricted comprehension is **not** to be confused with the *inconsistent* principal of universal comprehension which allowed us to write $\{y \mid P(y)\}$. This leads to problems like Russel's paradox. We must identify the set m from which y come!

Definition 1.5.2: Let $u \subseteq m$, then $m \setminus u := \{x \in m \mid x \notin u\}$

Now we move on to discuss the notion of *power sets*. Historically, in naive set theory, the principle of universal comprehension (PUC) was thought to be needed in order to define

$$\mathcal{P}(m) := \{y \mid y \subseteq m\} \tag{1.5.5}$$

Then, the rest of set theory would be created by repeated application of the power set. However, we cant write this definition without PUC. Instead, we postulate that the power set exists…

Axiom 1.5.6 (Existence of power sets): Let m be a set. Then there exists a set, denoted $\mathcal{P}(m)$, whose elements are precisely the subsets of m.

Example 1.5.1.3: Let $m = \{a, b\}$, then $\mathcal{P}(m) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Axiom 1.5.7 (Infinity): There exists a set that contains the empty set and, with each of its elements y, it also contains as an element $\{y\}$.

One such set is, informally thinking, the set with the elements $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$

In "elementary school", we introduce a "trivial" notation:
- $\emptyset := 0$
- $\{\emptyset\} := 1$
- $\{\{\emptyset\}\} := 2$
- $\{\{\{\emptyset\}\}\} := 3$
- and so on…

Corollary 1.5.1.1: The set of non-negative integers, $\mathbb{N}$, is a set.

As a *set*, $\mathbb{R}$ can be understood/defined as $\mathbb{R} := \mathcal{P}(\mathbb{N})$. All of the sets that we are familiar with will be boostrapped from the empty set in this fashion.

Axiom 1.5.8 (Axiom of Choice): Let x be a set whose elements are
- non-empty
- mutually disjoint

Then, there exists a set y which contains exactly one element of each element of x.

Sometimes people call y a *dark set* since it's not clear *how* to pick the element from each element of x. The intuition is:

$$x = \{\{\text{left shoe } 1, \text{right shoe } 1\}, \{\text{left shoe } 2, \text{right shoe } 2\}, \ldots\}$$
$$y = \{\text{left shoe } 1, \text{left shoe } 2, \ldots\}$$

does not need the axiom of choice. But, if instead, we were to consider sets of socks:

$$x = \{\{\text{sock } 1 \text{ left}, \text{sock } 1 \text{ right}\}, \{\text{sock } 2 \text{ left}, \text{sock } 2 \text{ right}\}\ldots\}$$

but left/right socks look the same, so we need to invoke Axiom 1.5.8 to construct y.

Another observation is that Axiom 1.5.8 is *independent* of the other 8 axioms so that we could have a set theory with it omitted.

Finally, we note that
- the proof that every vector space has a basis requires the axiom of choice.
- the proof that there exists a complete system of representatives of an equivalence relation requires the axiom of choice.

Axiom 1.5.9 (Axiom of Foundation): Every non-empty set x contains an element y such that x and y are disjoint sets.

An immediate implication is that there is *no* set that contains itself as an element.

1.6. Classification of sets

A recurrent theme in mathematics is the *classification* of *spaces* by means of structure-preserving *maps* between those spaces. The *space* is usually meant to be some set equipped with some *structure*.

Definition 1.6.1: A *map* $\phi : A \to B$ is a *relation* such that $\forall a \in A$ there exists exactly one $b \in B$ such that $\phi(a, b)$.

Standard notation:

$$\begin{aligned} \phi : A &\to B \\ a &\mapsto \phi(a) \end{aligned} \tag{1.6.6}$$

Terminology:
- A is the domain of ϕ.
- B is the target of ϕ.
- $\phi(A) \equiv \mathrm{im}_{\phi(A)} := \{\phi(a) \in B \mid a \in A\}$ is the image.

Definition 1.6.2: Let $\phi : A \to B$ be a map. Then, ϕ is
- *injective* (one-to-one) if $\phi(a_1) = \phi(a_2) \Rightarrow a_1 = a_2$
- *surjectivie* (onto) if $\phi(A) = B$
- *bijective* if *injective* and *surjective*.

Definition 1.6.3: Two sets A and B are called (set-theoretically) *isomorphic*, written $A \underset{\text{set}}{\cong} B$, if there exists some bijection $\phi : A \to B$.

Remark 1.6.3.1: If there *is* any bijection $\phi : A \to B$, then generically, there are many.

Definition 1.6.4: A set A is
- *infinite* if there exists a proper subset $B \subset A$ such that $B \underset{\text{set}}{\cong} A$.
  - *countably infinite* if $A \underset{\text{set}}{\cong} \mathbb{N}$
  - *uncountably infinite* otherwise.
- *finite* otherwise

Definition 1.6.5: Given two maps $A \xrightarrow{\phi} B$ and $B \xrightarrow{\psi} C$, one can construct a new map

$$\begin{aligned} \psi \circ \phi : A &\to C \\ a &\mapsto \psi(\phi(a)) \end{aligned} \tag{1.6.7}$$

called the *composition* of maps ϕ and ψ. When reading $\psi \circ \phi$, we say "ϕ after ψ".

Definition 1.6.6: Let $\phi : A \to B$ be a bijection. Then the *inverse* map of ϕ is the map

$$\begin{aligned} &\phi^{-1} : B \to A \\ &\text{Defined uniquely by} \\ &\phi^{-1} \circ \phi = \mathrm{id}_A \\ &\phi \circ \phi^{-1} = \mathrm{id}_B \end{aligned} \tag{1.6.8}$$