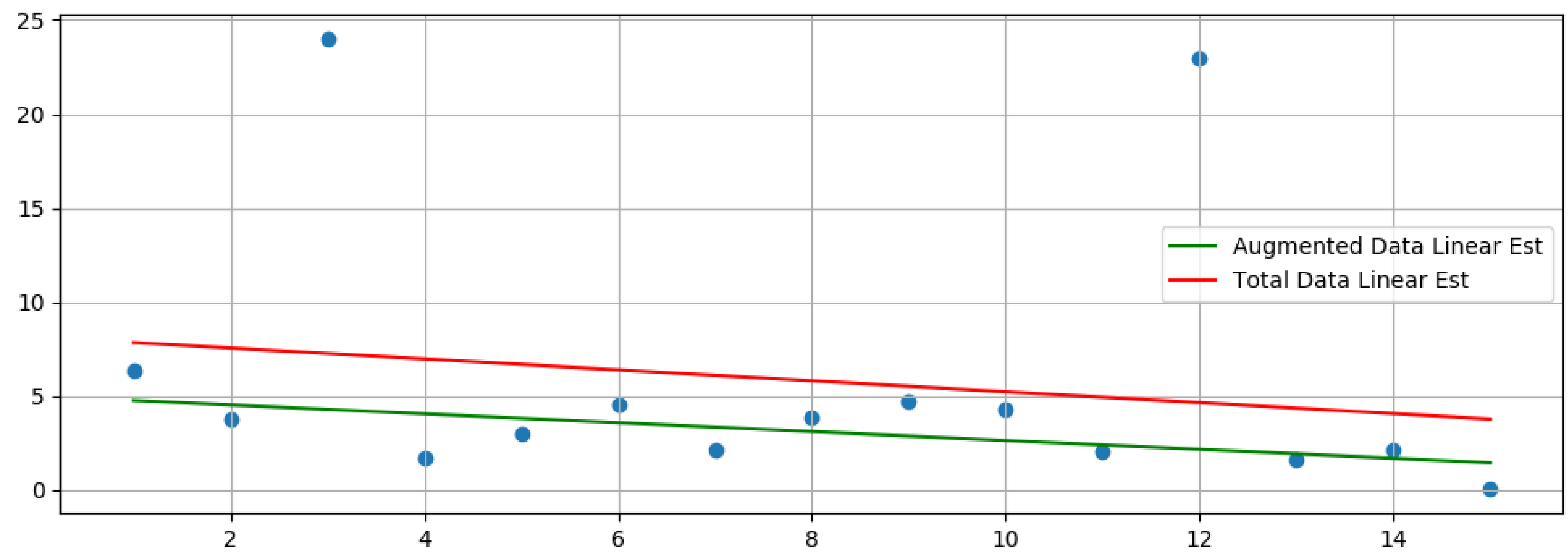
The Huber Loss

In [305]:
```
using JuMP, Gurobi

data = zeros(15)

data[1] = 6.31
data[2] = 3.78
data[3] = 24
data[4] = 1.71
data[5] = 2.99
data[6] = 4.53
data[7] = 2.11
data[8] = 3.88
data[9] = 4.67
data[10] = 4.25
data[11] = 2.06
data[12] = 23
data[13] = 1.58
data[14] = 2.17
data[15] = .02

A = zeros(15,2)
B = zeros(13,2)
j = 1
for i = 1:15
    A[i,1] = i
    A[i,2] = 1
    if i != 3 && i != 12
        B[j,1] = i
        B[j,2] = 1
        j += 1
    end
end

wopt = A\data;
west = A*wopt;

aug_data = zeros(13)

aug_data[1] = 6.31
aug_data[2] = 3.78
aug_data[3] = 1.71
aug_data[4] = 2.99
aug_data[5] = 4.53
aug_data[6] = 2.11
aug_data[7] = 3.88
aug_data[8] = 4.67
aug_data[9] = 4.25
aug_data[10] = 2.06
aug_data[11] = 1.58
aug_data[12] = 2.17
aug_data[13] = .02

aug_opt = B\aug_data
aug_est = B*aug_opt;
```

In [297]:
```
using PyPlot
figure(figsize=(12,4))
t = linspace(1,15,100)
y1 = aug_opt[1]*t + aug_opt[2]
y2 = wopt[1]*t + wopt[2]
plot(t,y1,"g-")
plot(t,y2,"r-")
grid()
scatter(1:15,data)
legend(["Augmented Data Linear Est", "Total Data Linear Est"]);
```



The green line has a better fit to the non-outlier data as we are not considering the outlier data. The red line lies above the green line for x = 1:15 because it considers the very large, outlier data.

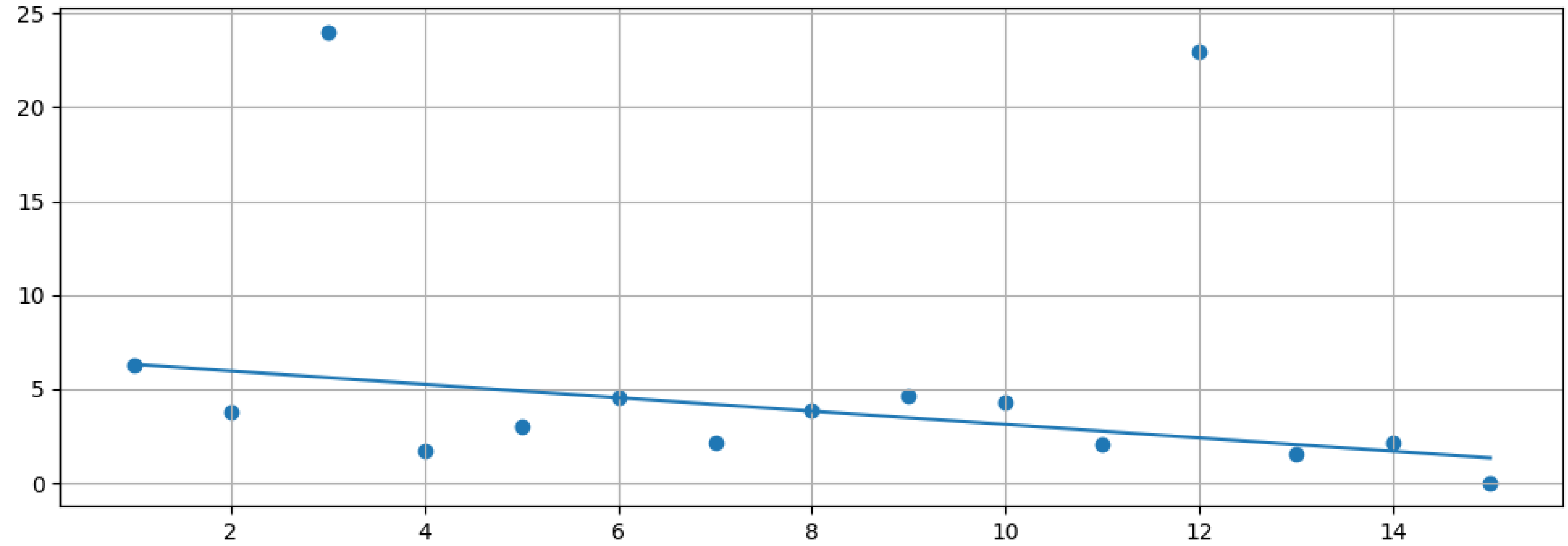
In [298]:

```
m = Model(solver=GurobiSolver())
@variable(m, z[1:15])
@variable(m, slope)
@variable(m, offset)

for i = 1:15
    @constraint(m, z[i] >= A[i,1]*slope + offset - data[i])
    @constraint(m, z[i] >= -(A[i,1]*slope + offset - data[i]))
end


@objective(m, Min, sum(z[i] for i=1:15))
solve(m)
slope = getvalue(slope)
offset = getvalue(offset);
```

```
Optimize a model with 30 rows, 17 columns and 90 nonzeros
Coefficient statistics:
  Matrix range     [1e+00, 2e+01]
  Objective range  [1e+00, 1e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [2e-02, 2e+01]
Presolve removed 15 rows and 0 columns
Presolve time: 0.00s
Presolved: 15 rows, 17 columns, 45 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0      handle free variables                          0s
      21    5.4030000e+01   0.000000e+00   0.000000e+00      0s

Solved in 21 iterations and 0.00 seconds
Optimal objective  5.403000000e+01
```

In [299]:

```
y3 = slope*t + offset
figure(figsize=(12,4))
plot(t,y3)
grid()
scatter(1:15,data);
```



The L_1 linear fit handles outlier data better than L_2 linear fits. This is because L_1 linear fitting takes a linear difference, whereas the L_2 linear fitting using the square of the difference. So large variances in L_2 makes a very large contribution to the norm, whereas large differences in L_1 is not as important.

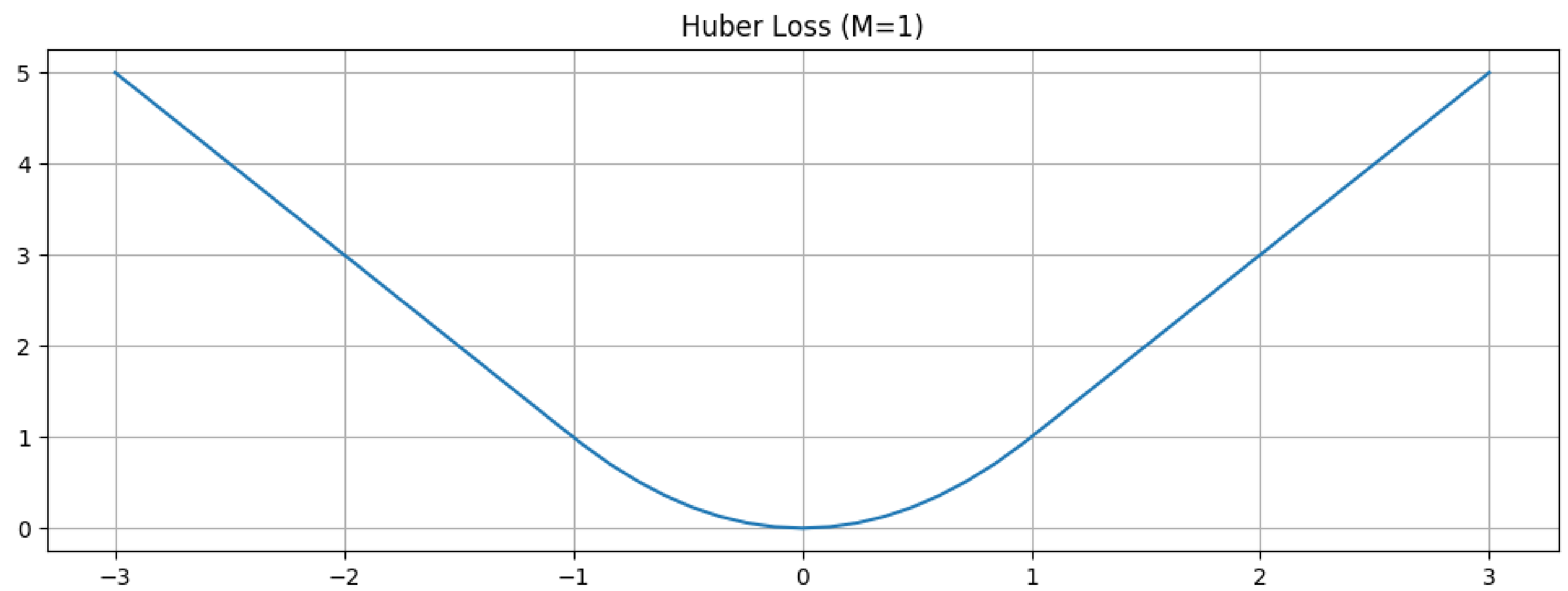
In [300]:

```
M = 1
I = 50
out = zeros(I + 1,2);
```

In [301]:
```
for i = 0:I
    x = -3 + i*6/I
    m = Model(solver=GurobiSolver())
    @variable(m, v >= 0)
    @variable(m, w <= M)
    @constraint(m, w + v >=  x)
    @constraint(m, w + v >= -x)
    @objective(m, Min, w^2 + 2*M*v)
    solve(m);
    out[i+1,1] = x
    out[i+1,2] = getobjectivevalue(m)
end
```

```
Optimize a model with 2 rows, 2 columns and 4 nonzeros
Model has 1 quadratic objective term
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [2e+00, 2e+00]
  QObjective range [2e+00, 2e+00]
  Bounds range     [1e+00, 1e+00]
  RHS range        [3e+00, 3e+00]
Presolve removed 2 rows and 2 columns
Presolve time: 0.00s
Presolve: All rows and columns removed

Barrier solved model in 0 iterations and 0.00 seconds
Optimal objective 5.00000000e+00
Optimize a model with 2 rows, 2 columns and 4 nonzeros
Model has 1 quadratic objective term
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [2e+00, 2e+00]
```

In [302]:
```
figure(figsize=(12,4))
plot(out[:,1],out[:,2])
title("Huber Loss (M=1)")
grid()
```

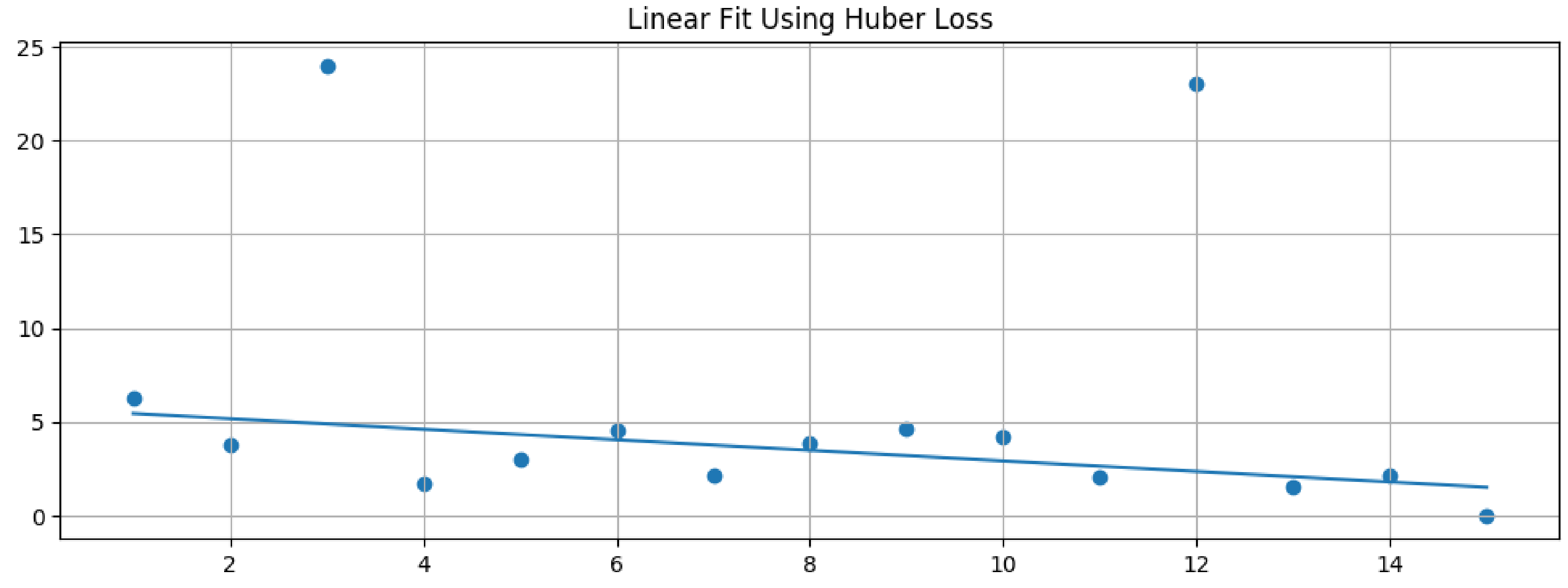


In [303]:
```
m = Model(solver=GurobiSolver())
@variable(m, x)
@variable(m, b)
@variable(m, w[1:15] <= M)
@variable(m, v[1:15] >= 0)
for i = 1:15
    @constraint(m, w[i] + v[i] >= i*x + b - data[i])
    @constraint(m, w[i] + v[i] >= -(i*x + b - data[i]))
end
@objective(m, Min, sum((w[i])^2 + 2*M*v[i] for i = 1:15))
solve(m)
```

```
Optimize a model with 30 rows, 32 columns and 120 nonzeros
Model has 15 quadratic objective terms
Coefficient statistics:
  Matrix range     [1e+00, 2e+01]
  Objective range  [2e+00, 2e+00]
  QObjective range [2e+00, 2e+00]
  Bounds range     [1e+00, 1e+00]
  RHS range        [2e-02, 2e+01]
Presolve time: 0.00s
Presolved: 30 rows, 32 columns, 120 nonzeros
Presolved model has 15 quadratic objective terms
Ordering time: 0.00s

Barrier statistics:
 Dense cols : 2
 Free vars  : 2
 AA' NZ     : 7.500e+01
 Factor NZ  : 1.080e+02
 Factor Ops : 3.800e+02 (less than 1 second per iteration)
 Threads    : 1

                  Objective                Residual
Iter       Primal          Dual         Primal    Dual     Compl     Time
   0   3.09994877e+05 -2.99850223e+04  0.00e+00 1.00e+03  1.00e+06     0s
   1   1.41170553e+06 -1.10717831e+06  0.00e+00 3.69e+00  4.55e+04     0s
   2   2.11112505e+05 -1.60845142e+05  0.00e+00 3.69e-06  6.20e+03     0s
   3   2.58533189e+04 -2.29553984e+04  0.00e+00 3.71e-12  8.13e+02     0s
   4   4.89577697e+03 -3.34776285e+03  0.00e+00 1.24e-14  1.37e+02     0s
   5   7.46863396e+02 -4.54750634e+02  0.00e+00 7.11e-15  2.00e+01     0s
   6   1.97368833e+02  4.98426352e-01  0.00e+00 1.13e-14  3.28e+00     0s
   7   1.12980111e+02  7.85726950e+01  0.00e+00 6.44e-15  5.73e-01     0s
   8   9.72702625e+01  9.27694671e+01  0.00e+00 7.11e-15  7.50e-02     0s
   9   9.59533887e+01  9.50239427e+01  0.00e+00 7.11e-15  1.55e-02     0s
  10   9.55271477e+01  9.54662891e+01  0.00e+00 3.55e-15  1.01e-03     0s
  11   9.55036500e+01  9.54956672e+01  0.00e+00 8.44e-15  1.33e-04     0s
  12   9.55002892e+01  9.54991709e+01  0.00e+00 3.55e-15  1.86e-05     0s
  13   9.54998155e+01  9.54996573e+01  0.00e+00 1.09e-14  2.64e-06     0s
  14   9.54997485e+01  9.54997261e+01  0.00e+00 8.66e-15  3.73e-07     0s
  15   9.54997390e+01  9.54997358e+01  0.00e+00 3.55e-15  5.27e-08     0s
  16   9.54997377e+01  9.54997372e+01  0.00e+00 4.44e-16  7.45e-09     0s

Barrier solved model in 16 iterations and 0.00 seconds
Optimal objective 9.54997377e+01
```

Out[303]: :Optimal

In [304]:
```
y3 = getvalue(x)*t + getvalue(b)
figure(figsize=(12,4))
plot(t,y3)
grid()
title("Linear Fit Using Huber Loss")
scatter(1:15,data);
```



Hyperbolic Program

Part a

We start with the expression: $t(a^T x + b) \geq 1$, where $t \geq 0$ and $x \in \mathbb{R}^n$. We use the equivalence from the homwork, which is: $w^T w \leq xy$, where $x, y \geq 0 \iff \left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \leq x + y$We do this by letting $w = 1$, $x = (a^t x + b)$, and $y = t$.

So now we can write our original expression as $t(a^T + b) \geq 1 \iff \left\| \begin{bmatrix} 2 \\ a^T x + b - t \end{bmatrix} \right\| \leq a^T x + b + .t$

Then we can note that $\begin{bmatrix} 2 \\ a^T x + b - t \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 2 \\ b - t \end{bmatrix}$.

So by letting $A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_n \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ b - t \end{bmatrix}$, $c = a$, and $d = b + t$, we see the final desired form of:

$$\|Ax + b\| \leq c^T x + d$$

Part b

First let $s_i = (a_i^T x + b_i)^{-1}$. For each s_i, we introduce a slack variable t_i such that $t_i \geq 1/(a_i^T x + b_i)$ This necessary condition can be written as: $t_i(a_i^T x + b_i) \geq 1$, which we recognize from part a.

So our objective function can be written as: $\sum_{i=1}^{p} t_i$ subject to the given constraints and the additional constraints: $t_i(a_i^T x + b_i) \geq 1$ $\forall i = 1, .., p$.

The original contraints are transformed with little effort: $\|(-1)\begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T \vec{x} + (-d_i)\| \leq 0 \;\; \forall i = 1, .., q$

The additional constraints are added by following the logic behind part a, as follows:

$$\left\| \begin{bmatrix} 0 & \dots & 0 \\ a_{i,1} & \dots & a_{i,n} \end{bmatrix} + \begin{bmatrix} 2 \\ b_i - t_i \end{bmatrix} \right\| \leq a_i^T x + b_i + t_i \, \forall i = 1, .., p$$

Note that by imposing the additional constraint of $t_i \geq 0 \, \forall i = 1, .., p$, we have, along with the additional constraints added above, that $a_i^T x + b_i > 0 \forall i = 1, .., p$. This is because t_i's are positive and $t_i(a_i^T x + b_i) \geq 1$, which implies that $(a_i^T x + b_i)$ must also be positive.

So finally we can write out the SOCP as:

$$min: \sum_{i=1}^{p} t_i \quad subject\ to$$
$$\|(-1)\begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T \vec{x} + (-d_i)\| \leq 0 \;\; \forall i = 1, .., q$$
$$\left\| \begin{bmatrix} 0 & \dots & 0 \\ a_{i,1} & \dots & a_{i,n} \end{bmatrix} + \begin{bmatrix} 2 \\ b_i - t_i \end{bmatrix} \right\| \leq a_i^T x + b_i + t_i$$
$$-t_i \leq 0 \;\; \forall i = 1, .., p$$

Heat Pipe Design

Using the information from the problem, we construct the following geometric program:

$$max: \;\; \alpha_4 T r^2 \; subject\ to$$

$$T \leq T_{max}, \;\; -T \leq T_{min}$$
$$r \leq r_{max}, \;\; -r \leq r_{min}$$
$$w \leq w_{max}, \;\; -w \leq w_{min}$$
$$w \leq 0.1r$$

Then we use the following variable substitutions:

$$x := logT$$
$$y := logr$$
$$z := logw$$

and rewrite our program as:

$$max: x + 2y + \log \alpha_4 \; subject\ to$$

$$y + \log(e^{\log \alpha_1 - z + x} + \alpha_2 + e^{\log \alpha_3 + z}) \leq \log C_{max}$$

$$\log T_{min} \leq x \leq \log T_{max}$$
$$\log r_{min} \leq y \leq \log r_{max}$$
$$\log w_{min} \leq z \leq \log w_{max}$$

$$z - y \leq \log 0.1$$

This is a convex optimization problem.

In [308]:
```
using Mosek

m = Model(solver=MosekSolver())

C_max = 500

@variable(m, X[1:3])

@constraint(m, X[3] - X[2] <= log(0.1))
@NLconstraint(m, X[2] + log(exp(X[1]-X[3]) + 1 + exp(X[3])) <= log(C_max))
@objective(m, Max, X[1] + 2X[2])
solve(m)
```

```
Problem
  Name                   :
  Objective sense        : max
  Type                   : GECO (general convex optimization problem)
  Constraints            : 2
  Cones                  : 0
  Scalar variables       : 3
  Matrix variables       : 0
  Integer variables      : 0

Optimizer started.
Interior-point optimizer started.
Presolve started.
Linear dependency checker started.
Linear dependency checker terminated.
Eliminator started.
Freed constraints in eliminator : 0
Eliminator terminated.
Eliminator - tries                  : 1                 time                   : 0.00
Lin. dep.  - tries                  : 1                 time                   : 0.00
Lin. dep.  - number                 : 0
Presolve terminated. Time: 0.00
Matrix reordering started.
Local matrix reordering started.
Local matrix reordering terminated.
Matrix reordering terminated.
Optimizer  - threads                : 2
Optimizer  - solved problem         : the primal
Optimizer  - Constraints            : 2
Optimizer  - Cones                  : 0
Optimizer  - Scalar variables       : 5                 conic                  : 0
Optimizer  - Semi-definite variables: 0                 scalarized             : 0
Factor     - setup time             : 0.00              dense det. time        : 0.00
Factor     - ML order time          : 0.00              GP order time          : 0.00
Factor     - nonzeros before factor : 15                after factor           : 17
Factor     - dense dim.             : 0                 flops                  : 2.70e+01
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   5.3e+00  2.5e+00  3.8e+00  0.00e+00   0.000000000e+00   2.813410717e+00   1.0e+00  0.00
1   1.2e+00  6.0e-01  9.5e-01  -4.85e-01  3.771113669e+00   4.496832620e+00   2.5e-01  0.00
2   2.4e-01  3.0e-01  2.7e-01  2.22e-01   9.266063993e+00   9.639624658e+00   6.7e-02  0.00
3   1.2e-02  1.4e-02  4.0e-02  7.97e-01   1.074281033e+01   1.080359853e+01   7.3e-03  0.00
4   2.9e-03  1.2e-03  5.0e-03  1.00e+00   1.082709581e+01   1.082734469e+01   1.4e-03  0.00
5   2.5e-04  1.7e-04  1.4e-04  9.85e-01   1.084432319e+01   1.084350063e+01   1.0e-04  0.00
6   3.3e-06  2.5e-06  7.3e-07  9.96e-01   1.084554724e+01   1.084553303e+01   1.3e-06  0.00
7   1.7e-08  1.3e-08  3.6e-09  1.00e+00   1.084556109e+01   1.084556102e+01   6.5e-09  0.00
Interior-point optimizer terminated. Time: 0.00.

Optimizer terminated. Time: 0.01

Interior-point solution summary
  Problem status  : PRIMAL_AND_DUAL_FEASIBLE
  Solution status : OPTIMAL
  Primal.  obj: 1.0845561091e+01    nrm: 4e+00    Viol.  con: 6e-08    var: 0e+00
  Dual.    obj: 1.0845561018e+01    nrm: 2e+00    Viol.  con: 0e+00    var: 4e-08

Basic solution summary
  Problem status  : UNKNOWN
  Solution status : UNKNOWN
  Primal.  obj: 0.0000000000e+00    nrm: 0e+00    Viol.  con: 5e+00    var: 0e+00
  Dual.    obj: 0.0000000000e+00    nrm: 0e+00    Viol.  con: 0e+00    var: 2e+00
```

Out[308]: :Optimal

In [309]:
```
println("The optimal heat transfer is: ", exp(getobjectivevalue(m)))
```

```
The optimal heat transfer is: 51305.90289873848
```