Chapter 1

Introduction

Ever since their discovery by William Rowan Hamilton in 1843, Quaternions have found extensive use in solving problems both in theoretical and applied mathematics - notably on the problem of 3D rotation.

Computations regarding 3D rotations use 4x4 matrices with real entries like the ones shown in Figure 1.1. We call any set of three angles that represent a rotation applied in some order around the principal axes as *Euler Angles* (in this case α, β, and γ) [6]. Computations with these matrices, however, are a bit tedious and require more elementary arithmetic operations [6]. It's also more difficult to determine the axis and angle of rotation using Euler angles [6]. Furthermore, this method is susceptible to a problem in mechanics known as the *Gimbal Lock* [4].

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a)Rotation by α in the x-axis

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)Rotation by β in the y-axis

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)Rotation by γ in the z-axis

Figure 1.1: 4x4 Rotation Matrices about the Principal Axes

The gimbal lock is a phenomenon that occurs when two of the moving axes x, y, and z (more commonly known as "pitch", "yaw", and "roll" respectively) coincide - resulting in a loss of one degree of freedom for the object being rotated [4].

Quaternions do not suffer from the gimbal lock. They are also found to be more compact - requiring less elementary arithmetic operations to perform rotation composition than rotation matrices [6]. The axis and angle of rotation can also be easily deduced. Let $\vec{q}$ be the purely imaginary parts of the quaternion $q = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$,i.e., $\vec{q} = b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$. It can be shown that

$$\frac{\vec{q}}{\sqrt{b^2+c^2+d^2}} \text{ is the axis of rotation and}$$

$$\theta \text{ satisfying } \sin\theta/2 = \sqrt{b^2+c^2+d^2} \text{ and } \cos\theta/2 = a \text{ is the angle of rotation. [6]}$$

Quaternions are used today in robotics, three-dimensional computer graphics, computer vision, crystallographic texture analysis, navigation, and molecular dynamics.

Mathematicians have made advancements in developing the theory of Quaternions. Notably, as one of the central points of this topic, we look into the concept of a *Quaternionic Matrix* and the implications it has on certain definitions that were already established in Linear Algebra.

One such implication is the concept of a determinant in the context of quaternionic matrices. In linear algebra, we saw that we can extend the definition of the determinant to matrices with complex entries [7]. This is possible because the complex numbers are commutative under complex multiplication [1].

Certain problems arise if we attempt to extend the classical definition to the quaternions because quaternions are not commutative under quaternion multiplication [1]. In [1], Aslaksen revisits the properties we associate with determinants and gives three conditions called *axioms* that should be satisfied in order for a definition of a determinant to be valid and useful:

1. $det(A) = 0$ if and only if A is singular.

2. $det(AB) = det(A)det(B)$ for all quaternionic matrices A and B.

3. If A' is obtained by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $det(A') = det(A)$.

Over the years, several mathematicians have come up with different ways to define a determinant for quaternionic matrices - the Cayley determinant (by Arthur Cayley in

1845), the Study determinant (by Eduard Study in 1920), the Dieudonne determinant, and Moore's determinant. Aslaksen showed whether or not these different definitions satisfy the above conditions [1].

We will look at a particular problem that will require the concept of a determinant - that is, to determine whether or not the set of all $n \times n$ *Skew-coninvolutory Quaternionic Matrices* (denoted by $\mathcal{D}_n(\mathbb{H})$) is empty when n is odd.

In [7], Sta. Maria provided a simple proof to the fact that the set of all $n \times n$ skew-coninvolutory *complex* matrices is empty when n is odd. The method of proof involved using the determinant defined for complex matrices (which is not different from the classical determinant for matrices with real entries).

In this paper, we will discuss the theory behind quaternionic determinants - particularly, the Study determinant. We will then use the Study determinant to extend the result by Sta. Maria to the set of all skew-coninvolutory quaternionic matrices, i.e., we will show that the set of all $n \times n$ skew-coninvolutory quaternionic matrices is empty when n is odd.

1.1 List of Symbols

- $\mathrm{M}_n(\mathbb{R})$ - set of all $n \times n$ matrices with real entries.
- $\mathrm{M}_n(\mathbb{C})$ - set of all $n \times n$ matrices with complex entries.
- $\mathrm{M}_n(\mathbb{H})$ - set of all $n \times n$ matrices with quaternion entries.
- $\mathcal{D}_n(\mathbb{C})$ - set of all $n \times n$ skew-coninvolutory matrices with complex entries.
- $\mathcal{D}_n(\mathbb{H})$ - set of all $n \times n$ skew-coninvolutory matrices with quaternion entries.
- $det_{\mathbb{C}}()$ - determinant of a matrix in $\mathrm{M}_n(\mathbb{C})$.
- $det_{\mathbb{R}}()$ - determinant of a matrix in $\mathrm{M}_n(\mathbb{R})$.

Chapter 2

Preliminaries

In this chapter, we will be presenting terms and known results which are key to building and understanding the theory of quaternionic determinants.

2.1 A Review on Linear Maps

Definition 2.1 (Linear Map/ Linear Transformation)**.** *Let V, W be vector spaces over the field F. The function $L : V \to W$ is called a* ***linear map/ linear transformation*** *of V into W when the following properties are true $\forall u, v \in V$ and $\forall c \in F$:*

1. *$L(u+v) = L(u)+L(v)$*
2. *$L(cu) = cL(u)$*

Property 2 of definition 2.1 is interpreted as having the linear map commute with scalar multiplication. In later discussions, we will see that we can still define linear maps for non-commutative division rings like the quaternions provided that we only consider right scalar multiplication [8].

2.2 Complex Matrices

Our discussions regarding quaternionic matrices will mostly revolve around extending properties and definitions that already hold for complex matrices.

Complex Matrices are simply matrices that have complex entries. However, even with this little difference, complex matrices can give us deeper insight into concepts we've already seen in linear algebra - revisiting notions such as the definition of transposes and orthogonal matrices (see [7] for deeper discussion regarding this matter).

Definition 2.2 (Conjugate Matrix)**.** *A* conjugate matrix *is a matrix $\bar{E}$ obtained from E by taking the complex conjugate of every entry of E.*

Definition 2.3 (Coninvolutory Matrix)**.** *A matrix is said to be* coninvolutory *if $E\bar{E} = I_n$ for $E \in M_n(\mathbb{C})$.*

By manipulation, we obtain $E^{-1} = \bar{E}$. Hence, we may also say that a matrix whose inverse is its own conjugate matrix is a coninvolutory matrix.

Example 2.2.1. Consider the complex matrix, $E = \begin{pmatrix} -\frac{13}{17} + \frac{16}{17}\mathbf{i} & -\frac{8}{17} + \frac{2}{17}\mathbf{i} \\ \frac{16}{17} - \frac{4}{17}\mathbf{i} & \frac{19}{17} + \frac{8}{17}\mathbf{i} \end{pmatrix}$

We see that

$$\begin{aligned} E\bar{E} &= \begin{pmatrix} -\frac{13}{17} + \frac{16}{17}\mathbf{i} & -\frac{8}{17} + \frac{2}{17}\mathbf{i} \\ \frac{16}{17} - \frac{4}{17}\mathbf{i} & \frac{19}{17} + \frac{8}{17}\mathbf{i} \end{pmatrix} \begin{pmatrix} -\frac{13}{17} - \frac{16}{17}\mathbf{i} & -\frac{8}{17} - \frac{2}{17}\mathbf{i} \\ \frac{16}{17} + \frac{4}{17}\mathbf{i} & \frac{19}{17} - \frac{8}{17}\mathbf{i} \end{pmatrix} \\ &= \frac{1}{17}\frac{1}{17} \begin{pmatrix} -13 + 16\mathbf{i} & -8 + 2\mathbf{i} \\ 16 - 4\mathbf{i} & 19 + 8\mathbf{i} \end{pmatrix} \begin{pmatrix} -13 - 16\mathbf{i} & -8 - 2\mathbf{i} \\ 16 + 4\mathbf{i} & 19 - 8\mathbf{i} \end{pmatrix} \\ &= \frac{1}{289} \begin{pmatrix} 289 & 0 \\ 0 & 289 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence, we see that E is a coninvolutory matrix. **Side note:** We've obtained E using the factorization provided in Theorem 2.3 found in [7].

If we take the concept of coninvolutory matrices in the context of real matrices, we get $EE = E^2 = I_n$ since $\forall E \in \mathrm{M}_n(\mathbb{R})$, $E = \bar{E}$. This is what we call an *Involutory Matrix*, i.e., a matrix whose inverse is itself. We see that the coninvolutory matrix is, in some way, an extension of the concept of an involutory matrix in $\mathrm{M}_n(\mathbb{R})$. Also, notice that a coninvolutory matrix generalizes complex numbers with modulus 1 (complex numbers that lie on the unit circle of the complex plane), i.e., $z\bar{z} = |z|^2 = 1$ for $z \in \mathbb{C}$ [7].

Definition 2.4 (Skew-Coninvolutory Matrix)**.** *A matrix is said to be* skew-coninvolutory *if $E\bar{E} = -I_n$ for $E \in M_n(\mathbb{C})$.*

Again, we may say that a matrix whose inverse is the negative of its own conjugate matrix is a skew-coninvolutory matrix. If we take this in the context of real matrices, we get $EE = E^2 = -I_n$. Notice how this closely resembles a property of the complex number $\mathbf{i}$, i.e., $\mathbf{i}^2 = -1$. In fact, we have a special name for these linear maps: *complex structures* [9].

Definition 2.5 (Complex Structure)**.** *A* complex structure *of a vector space V is defined by the linear map (linear transformation) $J : V \to V$ such that $J^2 = -I$, where I is the identity map. [9]*

To put it simply, we can think of the complex structure in $\mathrm{M}_n(\mathbb{R})$ as the real matrix representation of the imaginary number $\mathbf{i}$ (or in the context of complex matrices, $\mathbf{i}I_n$ [1]). We can then write a matrix in $\mathrm{M}_n(\mathbb{R})$ as $A + BJ = A + JB$ where J is the complex structure in $\mathrm{M}_n(\mathbb{R})$, and $A, B \in \mathrm{M}_n(\mathbb{R})$ [3]. Notice that the matrices in $\mathrm{M}_n(\mathbb{R})$ will have to commute with the complex structure so that a complex vector space may manifest in $\mathrm{M}_n(\mathbb{R})$ [1] [3].

In later discussions, we will be looking into functions that make it possible for us to represent complex numbers and complex matrices (complex linear maps) as real matrices (real linear maps). In doing so, we will have to define a complex structure in the real matrices.

We can also take the determinants of complex matrices by using the classical definition. We see that computing for the determinant of a complex matrix will give us a complex number. We shall denote this determinant by $det_{\mathbb{C}}$. The following theorem shows one very useful result.

Theorem 2.1. *For a matrix $E \in M_n(\mathbb{C})$, $det_{\mathbb{C}}(\bar{E}) = \overline{det_{\mathbb{C}}(E)}$.*

Proof. We prove by mathematical induction.

Base Case: For $E = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$, $det_{\mathbb{C}}(\bar{E}) = \begin{vmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{vmatrix} = \overline{ad} - \overline{bc} = \overline{ad - bc} = \overline{det_{\mathbb{C}}(E)}$

Induction Hypothesis: Suppose $det_{\mathbb{C}}(\bar{E}) = \overline{det_{\mathbb{C}}(E)}$ holds for $E \in \mathrm{M}_n(\mathbb{C})$.

Let $X \in \mathrm{M}_{n+1}(\mathbb{C})$. Then,

$$det_{\mathbb{C}}(\overline{X}) = \overline{\sum_{j=1}^{n+1} a_{ij} c_{ij}} = \sum_{j=1}^{n+1} \overline{a_{ij}}\ \overline{c_{ij}}$$

is the i^{th} row expansion of an $(n+1) \times (n+1)$ matrix where $\overline{c_{ij}}$ is the cofactor of $\overline{a_{ij}}$. Note that $\overline{c_{ij}} = (-1)^{i+j}\overline{M_{ij}}$ where $\overline{M_{ij}}$ is the determinant of the $n \times n$ matrix obtained by deleting the i^{th} row and the j^{th} column of the original matrix.
By I.H., $\overline{M_{ij}}$ is the determinant of an $n \times n$ conjugate matrix. Thus, we see that we are computing for the determinant of an $(n+1) \times (n+1)$ conjugate matrix. □

Theorem 2.1 states that computing for the determinant commutes with conjugation, i.e., the determinant of the conjugate matrix is the conjugate of the determinant.

2.2.1 Skew-Coninvolutory Complex Matrices

We now show and prove a result concerning whether or not $\mathcal{D}_n(\mathbb{C})$ is empty when n is odd as seen in [7].

Theorem 2.2. *$\mathcal{D}_n(\mathbb{C})$ is empty when n is odd.*

Proof. If $E \in \mathcal{D}_n(\mathbb{C})$ then $E\bar{E} = -I_n$.
Taking the determinant of both sides,

$$\begin{aligned} det_{\mathbb{C}}(E\bar{E}) &= det_{\mathbb{C}}(-I_n) \\ det_{\mathbb{C}}(E)det_{\mathbb{C}}(\bar{E}) &= (-1)^n \\ det_{\mathbb{C}}(E)\overline{det_{\mathbb{C}}(E)} &= (-1)^n, \text{ by Theorem 2.1} \\ |det_{\mathbb{C}}(E)|^2 &= (-1)^n \end{aligned}$$

Since $|det_{\mathbb{C}}(E)|^2 > 0$, $(-1)^n > 0$. Hence, n must be even. □

The above theorem puts a restriction on the dimension of complex matrices that are skew-coninvolutory. In the context of real matrices, this means that $E^2 = -I_n$ only holds if E is a $2n \times 2n$ real matrix, i.e., a complex structure only exists for real matrices with even dimensions.

Example 2.2.2. We see that in the 1×1 case, $\bar{\mathbf{i}} = -\mathbf{i}$ but $\mathbf{i}\bar{\mathbf{i}} = -\mathbf{i}^2 = 1 \neq -1$. Whereas in the 2×2 case, consider $E = \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}$. Then $E\bar{E} = \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

We will see more manifestations of this fact in later discussions especially when we represent complex matrices as real matrices.

2.3 Quaternion Basics

In this section, we introduce properties and operations associated with quaternions including addition, multiplication, conjugation, inverse, and norm.

Definition 2.6 (Quaternion)**.** *The four-dimensional algebra of* Quaternions *is generated by the basis elements* $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ *such that* $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. $\mathbb{H} := \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} | a, b, c, d \in \mathbb{R}\}$. *[7]*

2.3.1 Multiplication and Addition

From definition 2.6 we can easily derive the following:

$$\mathbf{jk} = \mathbf{i} \qquad \mathbf{kj} = -\mathbf{i}$$
$$\mathbf{ki} = \mathbf{j} \qquad \mathbf{ik} = -\mathbf{j}$$
$$\mathbf{ij} = \mathbf{k} \qquad \mathbf{ji} = -\mathbf{k}$$

It is easy to see that the quaternions are not commutative under **multiplication**. However, they can commute with a scalar in $\mathbb{R}$.

Theorem 2.3 (Quaternion Multiplication)**.** *For quaternions* $q_1 = a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{j} + d_1\boldsymbol{k}$ *and* $q_2 = a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{j} + d_2\boldsymbol{k}$*,*

$$\begin{aligned} q_1q_2 = &(a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\boldsymbol{i} \\ &+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\boldsymbol{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\boldsymbol{k} \end{aligned}$$

Quaternions are, however, commutative under **addition** where $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$. We can clearly see (and it can be shown) that the quaternions form a **skew-field** [1] [5].

2.3.2 Other Operations and Properties

Definition 2.7 (ℍ-Conjugate). *The ℍ-Conjugate of a quaternion* $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ *is* $\bar{q} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$.

Definition 2.8 (ℍ-Norm). *The ℍ-Norm of a quaternion* $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ *is* $|q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$

Since $\mathbb{C} \subseteq \mathbb{H}$, we see that definition 2.7 reduces to the definition of a complex conjugate when $c, d = 0$. In a similar manner, the definition of an ℍ-norm 2.8 reduces to the definition of a modulus in $\mathbb{C}$ (Using Theorem 2.3, it can be shown that $q\bar{q} = a^2 + b^2 + c^2 + d^2$).

Definition 2.9 (Inverse). *The inverse of a quaternion* q *is* q^{-1} *such that* $q^{-1}q = qq^{-1} = 1$.

The following theorem shows many other quaternion properties including a way to compute for the inverse of a quaternion.

Theorem 2.4. *For* $q, p, r \in \mathbb{H}$,

1. *If* $q \neq 0$, *then* $q^{-1} = \bar{q}/|q|^2$.
2. $\overline{qp} = \bar{p}\bar{q}$.
3. $(qp)^{-1} = p^{-1}q^{-1}$ *provided that the inverses of* p *and* q *exist.*
4. $(qp)r = q(pr)$ *that is, quaternion multiplication is associative.*

It is important to note that the multiplicative inverse will not exist if $q = 0$ for $q \in \mathbb{H}$. Notice how these properties follow directly from the fact that the quaternions form a skew-field. For instance, property 4 holds because the quaternions form a group under multiplication. Properties 2 and 3 follow from results we've already obtained in group theory.

Theorem 2.5. *For* $z \in \mathbb{C}$, $z\boldsymbol{j} = \boldsymbol{j}\bar{z}$. *Alternatively,* $\boldsymbol{j}z = \bar{z}\boldsymbol{j}$. *[1]*

We can write $(c\mathbf{j} + d\mathbf{k})$ as $\mathbf{j}(c - d\mathbf{i}) = (c + d\mathbf{i})\mathbf{j}$ (notice how this follows from the definition or from theorem 2.5). Hence, a quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ can be written as $(a+b\mathbf{i})+\mathbf{j}(c-d\mathbf{i}) = (a+b\mathbf{i})+(c+d\mathbf{i})\mathbf{j}$. We can now, therefore, view ℍ as a 2-dimensional algebra over $\mathbb{C}$ [7] as much as we can view $\mathbb{C}$ as a 2-dimensional algebra over $\mathbb{R}$.

2.3.3 Quaternionic Matrices

Most of the definitions we've already mentioned for complex matrices can also be extended in the context of quaternionic matrices.

Definition 2.10 (Conjugate Quaternionic Matrix)**.** *A* conjugate quaternionic matrix *is a matrix $\bar{E}$ obtained from E by taking the $\mathbb{H}$-conjugate of every entry of E.*

Definition 2.11 (Skew-Coninvolutory Quaternionic Matrix)**.** *A quaternionic matrix E is said to be* Skew-Coninvolutory *if $E\bar{E} = -I_n$.*

We will continue our discussions involving definitions 2.10 and 2.11 in the main results.

Chapter 3

Results and Discussion

In this chapter, we discuss how the theory behind quaternionic determinants - specifically, the Study Determinant - is built. With this, we extend theorem 2.2 by showing that there are essentially no skew-coninvolutory quaternionic matrices of odd dimensions.

3.1 The Cayley Determinant and Aslaksen's Axioms

In 1845, 2 years after William Rowan Hamilton discovered quaternions, Arthur Cayley attempted to define the determinant of a quaternionic matrix using the usual formula (we denote the Cayley determinant by $Cdet$). Note that the following definition can be extended for $n \times n$ quaternionic matrices.

Definition 3.1 (2 × 2 Cayley Determinant)**.** *For a 2×2 quaternionic matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Cdet(A) = ad - cb$ *for* $a, b, c, d \in \mathbb{H}$ *[1].*

Notice that the order in which the entries are multiplied matters. We can see that for a 3×3 quaternionic matrix

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \; Cdet(B) = (aei + bfg + cdh) - (gec + hfa + idb).$$

The following examples are given in [1].

Example 3.1.1. Let $M = \begin{pmatrix} \mathbf{k} & \mathbf{j} \\ \mathbf{i} & 1 \end{pmatrix}$. Then, $Cdet(M) = \mathbf{k} - \mathbf{ij} = \mathbf{k} - \mathbf{k} = 0$. Hence, we can say that by the Cayley determinant, M is singular.

Example 3.1.2. Consider the transpose of the matrix M, $M^T = \begin{pmatrix} \mathbf{k} & \mathbf{i} \\ \mathbf{j} & 1 \end{pmatrix}$. Then $Cdet(M^T) = \mathbf{k} - (\mathbf{ji}) = \mathbf{k} - (-\mathbf{k}) = 2\mathbf{k}$. Hence, we can say that by the Cayley determinant, M^T is invertible.

Taking into account the fact that the quaternions are non-commutative, one might ask whether or not this determinant behaves the way we expect - Will it really determine whether or not a quaternionic matrix is singular or not? Will the properties of the determinant still hold? Will the determinant still be a map from $M_n(G) \to G$ (in this case, $G = \mathbb{H}$)? The last question comes from the fact that the determinants of complex matrices is a map from $\mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$.

3.1.1 Brenner's Determinant Function and Aslaksen's Axioms

We take a step back and revisit what it means for a mapping to be a determinant. J.L. Brenner and Helmer Aslaksen offer different approaches to this, however, we will see how we can both arrive at the same conclusions. Brenner presented a general definition for a determinant function.

Definition 3.2 (Determinant Function)**.** *For a field F, a determinant over the matrices of $M_n(F)$ is a function det from $M_n(F)$ into F such that*

$$det(AB) = det(A)det(B) = det(B)det(A) \tag{3.1}$$

*holds either **(1)** $\forall A, B \in M_n(F)$ or **(2)** $\forall$ invertible $A, B \in M_n(F)$.*

Aslaksen, on the other hand, uses a more axiomatic approach in defining a determinant, presenting three determinant *axioms* which a determinant definition must satisfy in order for it to behave the way we expect, i.e., it has the properties we associate with determinants.

- **Axiom 1.** $det(A) = 0$ if and only if A is singular.
- **Axiom 2.** $det(AB) = det(A)det(B)$ for all quaternionic matrices A and B.
- **Axiom 3.** If A' is obtained by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $det(A') = det(A)$ (as we have already encountered in linear algebra, this operation can be described by an elementary matrix [1]).

Notice that a determinant is essentially a function that:

1. Maps to 0 if a matrix is singular

2. Preserves multiplication and,

3. Remains unchanged after applying the elementary operation of adding a left/right-multiple of one row/column to another row/column respectively.

Also notice that Alaksen's second axiom is the first condition in Brenner's determinant function.

We can trivially define the determinant merely as a function that constantly maps to 0 for all singular matrices and 1 for all non-singular matrices [5]. This will easily satisfy the above axioms [5] [1], however, we will mostly deal with determinants that are non-trivial (for instance the $det_{\mathbb{R}}()$, $det_{\mathbb{C}}$, and $Cdet$).

Theorem 3.1. *If det is not constantly equal to 1 or 0 (i.e., det is not a mapping* $det : M_n(F) \to F$ *where F is a field with two elements), then* $det(B) = 0$ *for all singular matrices.*

Theorem 3.1 not only shows how the determinant function in 3.2 holds for the non-trivial case, it also shows that conditions (1) and (2) of the determinant function are essentially equivalent [5].

Theorem 3.2. *If det satisfies all of Aslaksen's axioms, then* $det(M_n(\mathbb{H}))$ *is a commutative subset of* $\mathbb{H}$*. [1]*

Notice how theorem 3.2 is already implied in Brenner's determinant function in which it is already deemed necessary for the images to commute. By Theorem 3.2, we see that det cannot be a mapping onto $\mathbb{H}$. Since $Cdet$ is onto $\mathbb{H}$, by contrapositive of theorem 3.2, $Cdet$ does not satisfy one of the axioms - in fact, it doesn't satisfy any of them [1].

As an illustration (as seen in [1]), recall examples 3.1.1 and 3.1.2. Notice that,

$$
\begin{aligned}
M \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\implies \begin{pmatrix} \mathbf{k} & \mathbf{j} \\ \mathbf{i} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\implies & \mathbf{k}x + \mathbf{j}y = 0 \text{ and } \mathbf{i}x + y = 0 \\
\implies & x = 0 \text{ and } y = 0. \\
\implies & M \text{ is invertible. This contradicts with the fact that } Cdet(M) = 0.
\end{aligned}
$$

Also notice that,

$$
M^T \begin{pmatrix} -1 \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} \mathbf{k} & \mathbf{i} \\ \mathbf{j} & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

$$
\implies M^T \text{ is singular. This contradicts with the fact that } Cdet(M^T) = 2\mathbf{k}.
$$

Aslaksen provides counterexamples that show how the Cayley determinant fails to satisfy the other two axioms [1].

3.2 Matrix Homomorphisms

It is clear that the classical definition of the determinant cannot be extended to quaternionic matrices. It was not until 1920, that a new approach in defining a quaternionic determinant was presented in a paper by Eduard Study [1]. His idea involved transforming quaternionic matrices into complex matrices from which one could then just simply take the determinant [1]. The method involves homomorphisms between quaternionic, complex, and real matrices.

In this section, we take a closer look into these homomorphisms - first discussing the motivation behind them and then the theory.

3.2.1 Representing Complex Numbers as Real Matrices

In abstract algebra, we saw that we can define a bijection from the field of complex numbers to the 2D-plane ($\mathbb{R}^2$) - a mapping $\Theta : \mathbb{C} \to \mathbb{R}^2$ where a complex number $a + b\mathbf{i}$ is mapped to a vector/point (a, b) in the 2D-plane. Therefore, in order to represent complex numbers as real matrices, we have to find a way to view them as linear maps in $\mathbb{R}^2$.

Consider the complex function $f(z) = (a + b\mathbf{i})z$ (since $\mathbb{C}$ is a field, f is linear map). We see that the images of 1 and $\mathbf{i}$ are $a+b\mathbf{i}$ and $-b+a\mathbf{i}$ respectively. Under the function Θ (in which case 1 is mapped to $(1, 0)$ while $\mathbf{i}$ is mapped to $(0, 1)$), we seek a matrix in $\mathrm{M}_2(\mathbb{R})$ that maps $(1, 0)$ to $\Theta(a + b\mathbf{i}) = (a, b)$ and $(0, 1)$ to $\Theta(-b + a\mathbf{i}) = (-b, a)$. Let this matrix be $F = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}$ where $\alpha, \beta, \chi,$ and $\delta \in \mathbb{R}$. Then,

$$\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \chi \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \alpha = a; \chi = b \text{ and}$$

$$\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \implies \begin{pmatrix} \beta \\ \delta \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \implies \beta = -b; \delta = a$$

Therefore, $F = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The matrix F can be seen as the matrix representation of the linear map f which is defined by multiplying a complex number z by $a + b\mathbf{i}$.

Example 3.2.1. Take the complex number $z = -3 + 2\mathbf{i}$. Then its real matrix representation is $\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}$. We can use this real matrix representation to multiply z with another complex number, say $1 + 2\mathbf{i}$. $\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$ which corresponds to the complex number obtained by multiplying $z(1 + 2\mathbf{i}) = (1 + 2\mathbf{i})z = -7 - 4\mathbf{i}$.

It is important to note that F in general, doesn't represent the complex number $a+b\mathbf{i}$ itself but the function associated with multiplying a complex number by $a + b\mathbf{i}$. In this case, F can represent both left and right multiplication because complex numbers are commutative under multiplication.

3.2.2 Homomorphisms from $\mathbf{M}_n(\mathbb{C})$ to $\mathbf{M}_{2n}(\mathbb{R})$

In order to represent complex matrices as real matrices, notice that every complex matrix can be represented as the sum of a real matrix and a purely imaginary matrix, i.e., for an $n \times n$ complex matrix Z, $Z = A + B\mathbf{i}$ where $A, B \in \mathrm{M}_n(\mathbb{R})$ [1]. We define a mapping

$$\phi(A + B\mathbf{i}) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} [1]$$

Notice how this mapping is simply a generalization of the matrix F in 3.2.1. If we can show that ϕ is an injective homomorphism (i.e., it preserves multiplication and is one-to-one), we can essentially represent any complex matrix as a $2n \times 2n$ real matrix. Also notice how the dimension of the real matrix is necessarily even. This is a direct consequence of theorem 2.2.

Theorems 3.3 and 3.4 will come in handy in proving that ϕ is an injective homomorphism.

Theorem 3.3. *For matrices $A, B, C \in M_n(\mathbb{H})$, $A(B + C) = AB + AC$.*

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}] \in \mathrm{M}_n(\mathbb{C})$. Then $B + C = [b_{ij} + c_{ij}]$ and

$$\begin{aligned} A(B + C) &= [\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})] = [\sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj})] \\ &= [\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}] = [\sum_{k=1}^{n} a_{ik}b_{kj}] + [\sum_{k=1}^{n} a_{ik}c_{kj}] = AB + AC \end{aligned} \tag{3.2}$$

□

The same method of proof can be used for the right distributive law. Furthermore, since $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$, theorem 3.3 holds for matrices in $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{C})$ (we can show this by seeing that the matrices in $\mathrm{M}_n(\mathbb{H})$ form a ring).

Theorem 3.4. *For matrices $A_{ij}, B_{ij} \in M_n(\mathbb{H})$ where $i, j = 1, 2, ..., m$,*

$$\begin{pmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mm} \end{pmatrix} \begin{pmatrix} B_{11} & \dots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \dots & B_{mm} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{m} A_{1k}B_{k1} & \cdots & \sum_{k=1}^{m} A_{1k}B_{km} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{m} A_{mk}B_{k1} & \cdots & \sum_{k=1}^{m} A_{mk}B_{km} \end{pmatrix}$$

Proof.

$$\begin{pmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mm} \end{pmatrix} \begin{pmatrix} B_{11} & \dots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \dots & B_{mm} \end{pmatrix}$$

$$= \begin{pmatrix} a_{1111} & \dots & a_{111n} & & a_{1m11} & \dots & a_{1m1n} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ a_{11n1} & \dots & a_{11nn} & & a_{1mn1} & \dots & a_{1mnn} \\ & \vdots & & \ddots & & \vdots & \\ a_{m111} & \dots & a_{m11n} & & a_{mm11} & \dots & a_{mm1n} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ a_{m1n1} & \dots & a_{m1nn} & & a_{mmn1} & \dots & a_{mmnn} \end{pmatrix} \begin{pmatrix} b_{1111} & \dots & b_{111n} & & b_{1m11} & \dots & b_{1m1n} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ b_{11n1} & \dots & b_{11nn} & & b_{1mn1} & \dots & b_{1mnn} \\ & \vdots & & \ddots & & \vdots & \\ b_{m111} & \dots & b_{m11n} & & b_{mm11} & \dots & b_{mm1n} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ b_{m1n1} & \dots & b_{m1nn} & & b_{mmn1} & \dots & b_{mmnn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{m}[\sum_{l=1}^{n} a_{1kil}b_{k1lj}] & \dots & \sum_{k=1}^{m}[\sum_{l=1}^{n} a_{1kil}b_{kmlj}] \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{m}[\sum_{l=1}^{n} a_{mkil}b_{k1lj}] & \dots & \sum_{k=1}^{m}[\sum_{l=1}^{n} a_{mkil}b_{kmlj}] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{m} A_{1k}B_{k1} & \cdots & \sum_{k=1}^{m} A_{1k}B_{km} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{m} A_{mk}B_{k1} & \cdots & \sum_{k=1}^{m} A_{mk}B_{km} \end{pmatrix}$$

□

Theorem 3.4 will make it convenient for us to multiply square matrices with square matrices as entries. Again, this theorem also holds for matrices in $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{C})$ because $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$.

Theorem 3.5. *Let* $\phi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ *such that* $C+D\boldsymbol{i} \mapsto \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$ *where* $C+D\boldsymbol{i} \in M_n(\mathbb{C})$. *Then* ϕ *is an injective homomorphism.*

Proof.

1-1:

$$\begin{aligned}
&\phi(A + B\mathbf{i}) = \phi(C + D\mathbf{i}) \\
&\implies \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} C & -D \\ D & C \end{pmatrix} \\
&\implies A = C \text{ and } B = D \text{ by Matrix Equality} \\
&\implies A + B\mathbf{i} = C + D\mathbf{i} \\
&\implies \phi \text{ is injective.}
\end{aligned}$$

Homomorphism:

Let $A + B\mathbf{i}$, $C + D\mathbf{i} \in \mathrm{M}_n(\mathbb{C})$. Then

$$\begin{aligned}
\phi[(A + B\mathbf{i})(C + D\mathbf{i})] &= \phi[(A + B\mathbf{i})C + (A + B\mathbf{i})D\mathbf{i}] \text{ by Theorem 3.3} \\
&= \phi[AC + BC\mathbf{i} + AD\mathbf{i} - BD] \\
&= \phi[(AC - BD) + (BC + AD)\mathbf{i}] \\
&= \begin{pmatrix} AC - BD & -(BC + AD) \\ BC + AD & AC - BD \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\phi[(A + B\mathbf{i})]\phi[(C + D\mathbf{i})] &= \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} C & -D \\ D & C \end{pmatrix} \\
&= \begin{pmatrix} AC - BD & -(BC + AD) \\ BC + AD & ac - bd \end{pmatrix} \text{ by Theorem 3.4.}
\end{aligned}$$

□

Alternatively, we can define a matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Notice that the matrix J is the image of $iI \in \mathrm{M}_n(\mathbb{C})$ under ϕ [1]. Intuitively, this means that J represents the imaginary number $\mathbf{i}$ in $\mathrm{M}_{2n}(\mathbb{R})$. It can be easily shown that $J^2 = -I$. Therefore, J gives a *complex structure* in $\mathbb{R}^{2n}$. Recall in Chapter 2 that in order for a subset of matrices in $\mathrm{M}_{2n}(\mathbb{R})$ to "mimic" a complex vector space, every linear map in the said subset must commute with the complex structure. We see that the subset of images of complex matrices under ϕ satisfy this. We can then write $\phi(\mathrm{M}_n(\mathbb{C})) = \{P \in \mathrm{M}_{2n}(\mathbb{R}) | JP = PJ\}$[1].

Example 3.2.2. Take the complex matrix

$$Z = \begin{pmatrix} 1+2\mathbf{i} & 3\mathbf{i} \\ 1 & -2+\mathbf{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \mathbf{i}. \text{ Then, } \phi(Z) = \begin{pmatrix} 1 & 0 & -2 & -3 \\ 1 & -2 & 1 & -1 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

3.2.3 Representing Quaternions as Complex Matrices

Recall in Chapter 2 that we can view the quaternions as a 2-dimensional algebra over $\mathbb{C}$ by having $a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k} = (a+b\mathbf{i})+\mathbf{j}(c-d\mathbf{i})$. In general, we can write any quaternion as $x + \mathbf{j}y$ where $x, y \in \mathbb{C}$. Because of this, we see that we can define a bijection $\Omega : \mathbb{H} \to \mathbb{C}^2$ where a quaternion $q = x+\mathbf{j}y$ is mapped to $(x, y) \in \mathbb{C}^2$. In order to represent quaternions as complex matrices, we have to find a way to view them as linear maps in $\mathbb{C}^2$.

Let us consider the quaternionic function $s(q) = (x + \mathbf{j}y)q$. We see that,

$$\begin{aligned}
&s(1) = x + \mathbf{j}y \\
&s(\mathbf{i}) = (x + \mathbf{j}y)\mathbf{i} = x\mathbf{i} + \mathbf{j}(y\mathbf{i}) \\
&s(\mathbf{j}) = (x + \mathbf{j}y)\mathbf{j} = x\mathbf{j} + \mathbf{j}y\mathbf{j} = \mathbf{j}\bar{x} + \mathbf{j}^2\bar{y} = -\bar{y} + \mathbf{j}\bar{x} \\
&s(\mathbf{k}) = (x + \mathbf{j}y)\mathbf{k} = x\mathbf{k} + \mathbf{j}y\mathbf{k} = -x\mathbf{j}\mathbf{i} + \bar{y}\mathbf{j}\mathbf{k} = \bar{y}\mathbf{i} - \mathbf{j}(\bar{x}\mathbf{i})
\end{aligned}$$

Under the function Ω, we see that the images of 1, $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are $(1, 0), (\mathbf{i}, 0), (0, 1)$ and $(0, -\mathbf{i})$ respectively and the images of $s(1), s(\mathbf{i}), s(\mathbf{j})$ and $s(\mathbf{k})$ are $(x, y), (x\mathbf{i}, y\mathbf{i}), (-\bar{y}, \bar{x})$ and $(\bar{y}\mathbf{i}, -\bar{x}\mathbf{i})$. We seek a matrix in $\mathrm{M}_2(\mathbb{C})$ such that $(1, 0) \mapsto (x, y)$, $(\mathbf{i}, 0) \mapsto (x\mathbf{i}, y\mathbf{i})$, $(0, 1) \mapsto (-\bar{y}, \bar{x})$, and $(0, -\mathbf{i}) \mapsto (\bar{y}\mathbf{i}, -\bar{x}\mathbf{i})$.

Let this matrix be $S = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$, where κ, λ, μ, and $\nu \in \mathbb{C}$. Then,

$$\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} \kappa \\ \mu \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \kappa = x \text{ and } \mu = y$$

$$\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ 0 \end{pmatrix} = \begin{pmatrix} x\mathbf{i} \\ y\mathbf{i} \end{pmatrix} \implies \begin{pmatrix} \kappa\mathbf{i} \\ \mu\mathbf{i} \end{pmatrix} = \begin{pmatrix} x\mathbf{i} \\ y\mathbf{i} \end{pmatrix} \implies \kappa = x \text{ and } \mu = y$$

$$\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} \implies \begin{pmatrix} \lambda \\ \nu \end{pmatrix} = \begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} \implies \lambda = -\bar{y} \text{ and } \nu = \bar{x}$$

$$\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} \begin{pmatrix} 0 \\ -\mathbf{i} \end{pmatrix} = \begin{pmatrix} \bar{y}\mathbf{i} \\ -\bar{x}\mathbf{i} \end{pmatrix} \implies \begin{pmatrix} -\lambda\mathbf{i} \\ -\nu\mathbf{i} \end{pmatrix} = \begin{pmatrix} \bar{y}\mathbf{i} \\ -\bar{x}\mathbf{i} \end{pmatrix} \implies \lambda = -\bar{y} \text{ and } \nu = \bar{x}$$

Hence, $S = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}$. The matrix S can be seen as the complex matrix representation of the quaternionic linear map s which is defined by performing a left-multiplication by a quaternion $x + \mathbf{j}y$. Notice that we only need to look into the images of 1 and $\mathbf{j}$ (or $\mathbf{i}$ and $\mathbf{k}$) to obtain S. This is because their images under the function Ω form a basis for $\mathbb{C}^2$. Also notice that s is a linear transform. This is why we can obtain a matrix representation of s in $\mathbb{C}$.

Example 3.2.3. Take the quaternion $q = -2 + \mathbf{i} - 5\mathbf{j} + 2\mathbf{k} = -2 + \mathbf{i} + \mathbf{j}(-5 - 2\mathbf{i})$. Then its complex matrix representation is $\begin{pmatrix} -2+\mathbf{i} & 5-2\mathbf{i} \\ -5-2\mathbf{i} & -2-\mathbf{i} \end{pmatrix}$. We can use this matrix to multiply q with another quaternion, say $1 + \mathbf{i} + 3\mathbf{j} - 4\mathbf{k} = 1 + \mathbf{i} + \mathbf{j}(3 + 4\mathbf{i})$. We have, $\begin{pmatrix} -2+\mathbf{i} & 5-2\mathbf{i} \\ -5-2\mathbf{i} & -2-\mathbf{i} \end{pmatrix} \begin{pmatrix} 1+\mathbf{i} \\ 3+4\mathbf{i} \end{pmatrix} = \begin{pmatrix} 20+13\mathbf{i} \\ -5-18\mathbf{i} \end{pmatrix}$ which corresponds to the quaternion obtained by multiplying $q(1 + \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 20 + 13\mathbf{i} - 5\mathbf{j} + 18\mathbf{k} = 20 + 13\mathbf{i} + \mathbf{j}(-5 - 18\mathbf{i})$ on the left.

Notice that S only represents left multiplication in the quaternions. One would probably ask whether or not we can have a complex matrix representation for right multiplication.

Consider the quaternionic function defined by multiplying a quaternion q by some quaternion $x + \mathbf{j}y$ on the right, $s_R(q) = q(x + \mathbf{j}y)$ such that $y \neq 0$. We see that,

$$\begin{aligned} s_R(1) &= x + \mathbf{j}y \\ s_R(\mathbf{i}) &= \mathbf{i}x - \mathbf{j}\mathbf{i}y \\ s_R(\mathbf{j}) &= -y + \mathbf{j}x \\ s_R(\mathbf{k}) &= -\mathbf{i}y - \mathbf{j}\mathbf{i}x \end{aligned}$$

We use the same method in obtaining the complex matrix representation. We let $S_R = \begin{pmatrix} \kappa_R & \lambda_R \\ \mu_R & \nu_R \end{pmatrix}$ be the complex matrix representation of this function.

$$\begin{pmatrix} \kappa_R & \lambda_R \\ \mu_R & \nu_R \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} \kappa_R \\ \mu_R \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \kappa_R = x \text{ and } \mu_R = y$$

$$\begin{pmatrix} \kappa_R & \lambda_R \\ \mu_R & \nu_R \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{i}x \\ -\mathbf{i}y \end{pmatrix} \implies \begin{pmatrix} \kappa_R\mathbf{i} \\ \mu_R\mathbf{i} \end{pmatrix} = \begin{pmatrix} \mathbf{i}x \\ -\mathbf{i}y \end{pmatrix} \implies \kappa_R = x \text{ and } \mu_R = -y$$

$$\begin{pmatrix} \kappa_R & \lambda_R \\ \mu_R & \nu_R \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \implies \begin{pmatrix} \lambda_R \\ \nu_R \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \implies \lambda_R = -y \text{ and } \nu_R = x$$

$$\begin{pmatrix} \kappa_R & \lambda_R \\ \mu_R & \nu_R \end{pmatrix} \begin{pmatrix} 0 \\ -\mathbf{i} \end{pmatrix} = \begin{pmatrix} -\mathbf{i}y \\ -\mathbf{i}x \end{pmatrix} \implies \begin{pmatrix} -\lambda_R\mathbf{i} \\ -\nu_R\mathbf{i} \end{pmatrix} = \begin{pmatrix} -\mathbf{i}y \\ -\mathbf{i}x \end{pmatrix} \implies \lambda_R = y \text{ and } \nu_R = x$$

This implies that $\mu_R = \lambda_R = 0$. Hence, the complex matrix representation of right multiplication by a quaternion is $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. However, we see that, $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ which doesn't correspond to $s_R(1) = x + \mathbf{j}y$. We will later see that there is no matrix representation for right multiplication by a quaternion because it is not a linear transform in $\mathbb{H}$. However, notice that every linear transform in $\mathbb{H}$ commutes with right multiplication, i.e., $\begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}$. We will discuss this matter further in the next subsection.

3.2.4 Homomorphisms from $\mathbf{M}_n(\mathbb{H})$ to $\mathbf{M}_{2n}(\mathbb{C})$

Before proceeding, let's first generalize the matrix S and define a mapping from $\mathrm{M}_n(\mathbb{H})$ to $\mathrm{M}_{2n}(\mathbb{C})$. In order to represent quaternionic matrices as complex matrices, notice that every quaternionic matrix can be represented as the sum $Q = X + \mathbf{j}Y$ where $X, Y \in \mathrm{M}_n(\mathbb{C})$ [1]. We define a mapping

$$\psi(X + \mathbf{j}Y) = \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} [1]$$

Again, if we can show that ψ is an injective homomorphism, we can essentially represent any quaternionic matrix as a $2n \times 2n$ complex matrix.

Theorem 3.6. *Let $\psi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ such that $X + \boldsymbol{j}Y \mapsto \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix}$ where $X + \boldsymbol{j}Y \in M_n(\mathbb{H})$. Then ψ is an injective homomorphism.*

Proof.

1-1:

$$\begin{aligned}
&\psi(X + \mathbf{j}Y) = \psi(V + \mathbf{j}W) \\
&\implies \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} = \begin{pmatrix} V & -\overline{W} \\ W & \overline{V} \end{pmatrix} \\
&\implies X = V \text{ and } Y = W \text{ by Matrix Equality} \\
&\implies X + \mathbf{j}Y = V + \mathbf{j}W \\
&\implies \psi \text{ is injective.}
\end{aligned}$$

Homomorphism:

Let $X + \mathbf{j}Y,\ V + \mathbf{j}W \in \mathrm{M}_n(\mathbb{H})$. Then

$$\begin{aligned}
\psi[(X+\mathbf{j}Y)(V+\mathbf{j}W)] &= \psi[X(V+\mathbf{j}W) + \mathbf{j}Y(V+\mathbf{j}W)] \text{ by Theorem 3.3} \\
&= \psi[XV + X\mathbf{j}W + \mathbf{j}YV + \mathbf{j}Y\mathbf{j}W] \\
&= \psi[XV + \mathbf{j}\overline{X}W + \mathbf{j}YV + \mathbf{j}^2\overline{Y}W] \\
&= \psi[(XV - \overline{Y}W) + \mathbf{j}(\overline{X}W + YV)] \\
&= \begin{pmatrix} XV - \overline{Y}W & -\overline{(\overline{X}W + YV)} \\ \overline{X}W + YV & \overline{XV - \overline{Y}W} \end{pmatrix} \\
&= \begin{pmatrix} XV - \overline{Y}W & -X\overline{W} - \overline{YV} \\ \overline{X}W + YV & \overline{XV} - Y\overline{W} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\psi[(X+\mathbf{j}Y)]\psi[(V+\mathbf{j}W)] &= \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} \begin{pmatrix} V & -\overline{W} \\ W & \overline{V} \end{pmatrix} \\
&= \begin{pmatrix} XV - \overline{Y}W & -X\overline{W} - \overline{YV} \\ \overline{X}W + YV & \overline{XV} - Y\overline{W} \end{pmatrix} \text{ by Theorem 3.4.}
\end{aligned}$$

□

The non-commutativity of quaternions presents some problems in representing *quaternionic linear maps* as complex linear maps. If we consider a quaternionic linear map say $L(v) = Av$ for $A \in \mathrm{M}_n(\mathbb{H})$ where we take in quaternions as scalars, then, $cAv = cL(v) = L(cv) = Acv$ which is false [8]. However, $Avc = L(v)c = L(vc) = Avc$. Hence, we now see that any quaternionic linear map commutes with right scalar multiplication by a quaternion (as was seen in subsection 3.2.3). However, right scalar multiplication itself is not a linear map in $\mathbb{H}$ (in order for it to be a linear map, it in turn, has to commute with other quaternions)[8] [1]. This poses a problem because it implies that there is no matrix representation for right scalar multiplication [1].

However, in [1], we see that we can consider a linear map $\widetilde{R_j}$ in $\mathbb{C}^{2n}$ as the image of right scalar multiplication by $\mathbf{j}$ under the homomorphism. $\widetilde{R_j}$ corresponds to multiplying $v \in \mathbb{C}^{2n}$ by the matrix J and then conjugating [1]. This gives a quaternionic structure in $\mathbb{C}^{2n}$ and thus, a quaternionic linear map corresponds to a complex linear map Q that commutes with $\widetilde{R_j}$, i.e., $Q\overline{Jv} = \overline{JQv}$ for $v \in \mathbb{C}^{2n}$. It can be easily shown that the latter holds if and only if $\overline{Q}J = JQ$ using the fact that $Q\overline{Jv} = \overline{QJv}$. Thus, $\psi(\mathrm{M}_n(\mathbb{H})) = \{Q \in \mathrm{M}_{2n}(\mathbb{C}) | \overline{Q}J = JQ\}$.

Example 3.2.4. Take the quaternionic matrix

$$Q = \begin{pmatrix} 1+2\mathbf{i}-3\mathbf{j}+\mathbf{k} & 2\mathbf{i}+5\mathbf{k} \\ 1-\mathbf{i} & 3+\mathbf{j}+\mathbf{k} \end{pmatrix} = \begin{pmatrix} 1+2\mathbf{i} & 2\mathbf{i} \\ 1-\mathbf{i} & 3 \end{pmatrix} + \mathbf{j} \begin{pmatrix} -3-\mathbf{i} & -5\mathbf{i} \\ 0 & 1-\mathbf{i} \end{pmatrix}.$$

Hence, $\psi(Q) = \begin{pmatrix} 1+2\mathbf{i} & 2\mathbf{i} & 3-\mathbf{i} & -5\mathbf{i} \\ 1-\mathbf{i} & 3 & 0 & -1-\mathbf{i} \\ -3-\mathbf{i} & -5\mathbf{i} & 1-2\mathbf{i} & -2\mathbf{i} \\ 0 & 1-\mathbf{i} & 1+\mathbf{i} & 3 \end{pmatrix}$.

3.3 The Study Determinant

Definition 3.3 (Study Determinant)**.** *For $M \in M_n(\mathbb{H})$, the* Study Determinant *is defined by $SdetM = det_{\mathbb{C}}(\psi(M)) = \sqrt{det_{\mathbb{R}}(\phi(\psi(M)))}$.*

Notice that we can compute for the Study determinant in two different ways:

1. Simply getting the complex matrix representation of the quaternionic matrix and proceeding to take its complex determinant or;

2. Getting the real matrix representation of the quaternionic matrix by composing ϕ and ψ, and then taking the square root of the real determinant.

In the following example, we take the Study determinant of the quaternionic matrix in Example 3.1.1 for which the Cayley determinant failed to identify as non-singular.

Example 3.3.1. Let $M = \begin{pmatrix} \mathbf{k} & \mathbf{j} \\ \mathbf{i} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathbf{i} & 1 \end{pmatrix} + \mathbf{j} \begin{pmatrix} -\mathbf{i} & 1 \\ 0 & 0 \end{pmatrix}$. Then,

1. $Sdet(M) = det_{\mathbb{C}}(\psi(M)) = det_{\mathbb{C}} \begin{pmatrix} 0 & 0 & -\mathbf{i} & -1 \\ \mathbf{i} & 1 & 0 & 0 \\ -\mathbf{i} & 1 & 0 & 0 \\ 0 & 0 & -\mathbf{i} & 1 \end{pmatrix} = 4.$

2. We see that $\psi(M) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \mathbf{i}.$

$$\sqrt{det_{\mathbb{R}}(\phi(\psi(M)))} = \left(det_{\mathbb{R}} \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right)^{1/2} = \sqrt{16} = 4$$

Hence, we can see that by the Study determinant, the matrix M is invertible which it really is as shown in section 3.1.

Aslaksen has shown that the Study Determinant satisfies all of his axioms [1]. Furthermore, he also showed that $Sdet(M) \geq 0$ [1].

3.4 Skew-Coninvolutory Quaternionic Matrices of Odd Dimensions

Proposition 1 (Main Result)**.** *$\mathcal{D}_n(\mathbb{H})$ is empty when n is odd.*

Proof. If $F \in \mathcal{D}_n(\mathbb{H})$ then $F\bar{F} = -I_n$.
Taking the Study determinant of both sides,

$$\begin{aligned} Sdet(E\bar{E}) &= Sdet(-I_n) \\ Sdet(E)Sdet(\bar{E}) &= (-1)^n \\ Sdet(E)\overline{Sdet(E)} &= (-1)^n, \text{ by Theorem 2.1} \\ |Sdet(E)|^2 &= (-1)^n \end{aligned}$$

Since $|Sdet(E)|^2 > 0$, $(-1)^n > 0$. Hence, n must be even. □

Theorem 2.1 applies because the Study determinant can be viewed as a complex determinant.

Example 3.4.1. Again consider the 1×1 case where, $\mathbf{j}\bar{\mathbf{j}} = -\mathbf{j}^2 = 1 \neq -1$. Whereas in the 2×2 case, consider $E = \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix}$. Then, $E\overline{E} = \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

List of References

[1] H. Aslaksen, *Quaternionic determinants*, The Mathematical Intelligencer, (1996).

[2] G. Balmes, *Rotating a unit vector in 3d using quaternions.* Wolfram Demonstrations Project, February 2016.

[3] J. Bell, *Complexification, complex structures, and linear ordinary differential equations*, April 2014.

[4] Y.-B. Jia, *Rotation in the space*, September 2016.

[5] B. J.L., *Applications of the dieudonne determinant*, Linear Algebra and Its Applications, (1968).

[6] A. Lerios, *Rotations and quaternions*, December 1995.

[7] J. P. B. S. Maria, *The sum of coninvolutory matrices*, Master's thesis, University of the Philippines, Diliman, Quezon City, 2012.

[8] M. van Leeuwen (https://math.stackexchange.com/users/18880/marc-van leeuwen), *On complexifying vector spaces.* Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/85198 (version: 2011-11-24).

[9] E. W. Weisstein, *Complex structure, from mathworld - a wolfram web resource.*