Chapter 1

Introduction

1.1 Significance of Quaternions

Ever since their discovery by William Rowan Hamilton in 1843, Quaternions have found extensive use in solving problems both in theoretical and applied mathematics - notably on the problem of 3D rotation.

Definition 1.1.1 (Quaternion)**.** The four-dimensional algebra of *Quaternions* is generated by the basis elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. $\mathbb{H} := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} | a, b, c, d \in \mathbb{R}\}$.

Computations regarding 3D rotations use 4x4 matrices with real entries like the ones shown in Figure 1.1. We call any set of three angles that represent a rotation applied in some order around the principal axes as *Euler Angles* (in this case α, β, and γ) [1]. Computations with these matrices, however, are a bit tedious and require more elementary arithmetic operations [1]. It's also more difficult to determine the axis and angle of rotation using Euler angles [1]. Furthermore, this method is susceptible to a problem in mechanics known as the *Gimbal Lock* [2].

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a)Rotation by α in the x-axis

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)Rotation by β in the y-axis

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)Rotation by γ in the z-axis

Figure 1.1: 4x4 Rotation Matrices about the Principal Axes

The gimbal lock is a phenomenon that occurs when two of the moving axes x, y, and z (more commonly known as "pitch", "yaw", and "roll" respectively) coincide - resulting in a loss of one degree of freedom for the object being rotated [2].

Quaternions do not suffer from the gimbal lock. They are also found to be more compact - requiring less elementary arithmetic operations to perform rotation composition than rotation matrices [1]. The axis and angle of rotation can also be easily deduced. Let $\vec{q}$ be the purely imaginary parts of the quaternion $q = a + b\mathbf{i} +$

$c\mathbf{j} + d\mathbf{k}$,i.e., $\vec{q} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. It can be shown that

$$\frac{\vec{q}}{\sqrt{b^2 + c^2 + d^2}} \text{ is the axis of rotation and}$$

$$\theta \text{ satisfying } \sin\theta/2 = \sqrt{b^2 + c^2 + d^2} \text{ and } \cos\theta/2 = a \text{ is the angle of rotation. [1]}$$

Quaternions are used today in robotics, three-dimensional computer graphics, computer vision, crystallographic texture analysis, navigation, and molecular dynamics.

1.2 Determinants of Quaternionic Matrices

Mathematicians have made advancements in developing the theory of Quaternions. Notably, as one of the central points of this topic, we look into the concept of a *Quaternionic Matrix* and the implications it has on certain definitions that were already established in Linear Algebra.

One such implication is the concept of a determinant in the context of quaternionic matrices. In linear algebra, we saw that we can extend the definition of the determinant to matrices with complex entries [3]. This is possible because the complex numbers are commutative under complex multiplication [4].

Certain problems arise if we attempt to extend the classical definition to the quaternions because quaternions are not commutative under quaternion multiplication [4]. [4] revisits the properties we associate with determinants and gives three conditions called *axioms* that should be satisfied in order for a definition of a determinant to be valid and useful:

1. $det(A) = 0$ if and only if A is singular.
2. $det(AB) = det(A)det(B)$ for all quaternionic matrices A and B.
3. If A' is obtained by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $det(A') = det(A)$.

Over the years, several mathematicians have come up with different ways to define a determinant for quaternionic matrices - the Cayley determinant (by Arthur Cayley in 1845), the Study determinant (by Eduard Study in 1920), the Dieudonne determinant, and Moore's determinant. [4] showed whether or not these different definitions satisfy the above conditions. We will be discussing these determinants in greater detail in the following chapter.

1.3 Skew-Coninvolutory Quaternionic Matrices

[3] provided a simple proof to the fact that the set of all $n \times n$ Skew-Coninvolutory Matrices with complex entries (denoted by $\mathcal{D}_n(\mathbb{C})$) is empty when n is odd. The method of proof involved using the determinant defined for complex matrices (which is not different from the classical determinant for matrices with real entries).

In this paper, we attempt to extend this result for quaternionic matrices, i.e., we will investigate whether or not the set of all $n \times n$ Skew-Coninvolutory Matrices with quaternion entries (denoted by $\mathcal{D}_n(\mathbb{H})$) is, again, empty when n is odd. Furthermore, we will draw the same method of proof - using the concept of a determinant for quaternionic matrices to obtain the same results.

1.4 Symbols

- $\mathrm{M}_n(\mathbb{R})$ - set of all $n \times n$ matrices with real entries.
- $\mathrm{M}_n(\mathbb{C})$ - set of all $n \times n$ matrices with complex entries.
- $\mathrm{M}_n(\mathbb{H})$ - set of all $n \times n$ matrices with quaternion entries.
- $\mathcal{D}_n(\mathbb{C})$ - set of all $n \times n$ skew-coninvolutory matrices with complex entries.
- $\mathcal{D}_n(\mathbb{H})$ - set of all $n \times n$ skew-coninvolutory matrices with quaternion entries.

Chapter 2

Preliminaries

2.1 Complex Matrices

Definition 2.1.1 (Conjugate Matrix)**.** A *conjugate matrix* is a matrix $\bar{E}$ obtained from E by taking the complex conjugate of every entry of E.

Definition 2.1.2 (Coninvolutory Matrix)**.** A matrix is said to be *coninvolutory* if $E\bar{E} = I_n$ for $E \in \mathrm{M}_n(\mathbb{C})$.

Remark. By manipulation, we obtain $E^{-1} = \bar{E}$. Hence, we may also say that a matrix whose inverse is its own conjugate matrix is a coninvolutory matrix. Furthermore, we see that coninvolutory matrices are the extension of complex numbers with modulus 1 [3].

Definition 2.1.3 (Skew-Coninvolutory Matrix)**.** A matrix is said to be *skew-coninvolutory* if $E\bar{E} = -I_n$ for $E \in \mathrm{M}_n(\mathbb{C})$.

Remark. Again, we may say that a matrix whose inverse is the negative of its own conjugate matrix is a skew-coninvolutory matrix. This is analogous to the skewsymmetric matrices we've encountered in linear algebra.

Theorem 2.1.1. *For a matrix $E \in M_n(\mathbb{C})$, $det(\bar{E}) = \overline{det(E)}$.*

Proof. We prove by mathematical induction.

Base Case: For $E = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$, $det(\bar{E}) = \begin{vmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{vmatrix} = \overline{ad} - \overline{bc} = \overline{ad - bc} = \overline{det(E)}$

Induction Hypothesis: Suppose $det(\bar{E}) = \overline{det(E)}$ holds for $E \in \mathrm{M}_n(\mathbb{C})$.

Let $X \in \mathrm{M}_{n+1}(\mathbb{C})$. Then,

$$det(\overline{X}) = \overline{\sum_{j=1}^{n+1} a_{ij}c_{ij}} = \sum_{j=1}^{n+1} \overline{a_{ij}}\ \overline{c_{ij}}$$

is the i^{th} row expansion of an $(n+1) \times (n+1)$ matrix where $\overline{c_{ij}}$ is the cofactor of $\overline{a_{ij}}$.

Note that $\overline{c_{ij}} = (-1)^{i+j}\overline{M_{ij}}$ where $\overline{M_{ij}}$ is the determinant of the $n \times n$ matrix obtained by deleting the i^{th} row and the j^{th} column of the original matrix.

By I.H., $\overline{M_{ij}}$ is the determinant of an $n \times n$ conjugate matrix. Thus, we see that we are computing for the determinant of an $(n+1) \times (n+1)$ conjugate matrix. ■

2.1.1 Skew-Coninvolutory Complex Matrices

We now show and prove a result concerning whether or not $\mathcal{D}_n(\mathbb{C})$ is empty when n is odd as seen in [3].

Theorem 2.1.2. $\mathcal{D}_n(\mathbb{C})$ *is empty when n is odd.*

Proof. If $E \in \mathcal{D}_n(\mathbb{C})$ then $E\bar{E} = -I_n$.

Taking the determinant of both sides,

$$det(E\bar{E}) = det(-I_n)$$

$$det(E)det(\bar{E}) = (-1)^n$$

$$det(E)\overline{det(E)} = (-1)^n, \text{ by Theorem 2.1.1}$$

$$|det(E)|^2 = (-1)^n$$

Since $|det(E)|^2 > 0$, $(-1)^n > 0$. Hence, n must be even. ■

2.2 Quaternion Basics

2.2.1 Multiplication and Addition

Recall in Chapter 1 - the four-dimensional algebra of quaternions is generated by the basis elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \tag{2.1}$$

From the above equation, we can easily derive the following:

$$\mathbf{jk} = \mathbf{i} \qquad \mathbf{kj} = -\mathbf{i}$$

$$\mathbf{ki} = \mathbf{j} \qquad \mathbf{ik} = -\mathbf{j}$$

$$\mathbf{ij} = \mathbf{k} \qquad \mathbf{ji} = -\mathbf{k}$$

Notice that the quaternions are not commutative under *multiplication*. In general, for quaternions $q_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $q_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$,

$$
\begin{aligned}
q_1q_2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\
& + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}
\end{aligned}
$$

Quaternions are, however, commutative under *addition* where $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$.

2.2.2 Other Operations and Properties

Definition 2.2.1 ($\mathbb{H}$-Conjugate)**.** The $\mathbb{H}$-Conjugate of a quaternion $q = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ is $\bar{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.

Remark. Notice that $q\bar{q} = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}) = a^2 + b^2 + c^2 + d^2$.

Definition 2.2.2 ($\mathbb{H}$-Norm)**.** The $\mathbb{H}$-Norm of a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is $|q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$

Definition 2.2.3 (Inverse)**.** The inverse of a quaternion q is q^{-1} such that $q^{-1}q = qq^{-1} = 1$.

Theorem 2.2.1. *For $q, p, r \in \mathbb{H}$,*

1. $|q|^2 = q\bar{q}$.

2. *If $q \neq 0$, then $q^{-1} = \bar{q}/|q|^2$.*

3. $\overline{qp} = \bar{p}\bar{q}$.

4. $(qp)^{-1} = p^{-1}q^{-1}$ *provided that the inverses of* p *and* q *exist.*

5. $(qp)r = q(pr)$ *that is, quaternion multiplication is associative.*

Remark. Notice that most of the properties we see in quaternions are merely extensions of the properties we see in complex numbers.

2.2.3 Quaternionic Matrices

Most of the definitions we've already mentioned for complex matrices can also be extended in the context of quaternionic matrices.

Definition 2.2.4 (Conjugate Quaternionic Matrix)**.** A *conjugate quaternionic matrix* is a matrix $\bar{E}$ obtained from E by taking the $\mathbb{H}$-conjugate of every entry of E.

Definition 2.2.5 (Skew-Coninvolutory Quaternionic Matrix)**.** A quaternionic matrix E is said to be *Skew-Coninvolutory* if $E\bar{E} = -I_n$.

2.3 Matrix Homomorphisms

We look into functions that make it possible for us to represent complex numbers and quaternions as matrices. These functions are of extreme importance as they are used to define some of the quaternionic determinants we will encounter.

2.3.1 Representing Complex Numbers as Real Matrices

In abstract algebra, we saw that we can define a bijection from the field of complex numbers to the 2D-plane ($\mathbb{R}^2$) - a mapping $\Theta : \mathbb{C} \to \mathbb{R}^2$ where a complex number $a + b\mathbf{i}$ is mapped to a vector/point (a, b) in the 2D-plane. Therefore, in order to represent complex numbers as real matrices, we have to find a way to view them as linear transformations over $\mathbb{R}^2$.

Consider the complex function $f(z) = (a + b\mathbf{i})z$. We see that the images of 1 and $\mathbf{i}$ are $a + b\mathbf{i}$ and $-b + a\mathbf{i}$ respectively. Under the function Θ (in which case 1 is mapped to $(1, 0)$ while $\mathbf{i}$ is mapped to $(0, 1)$), we seek a matrix in $\mathrm{M}_2(\mathbb{R})$ that maps $(1, 0)$ to $\Theta(a + b\mathbf{i}) = (a, b)$ and $(0, 1)$ to $\Theta(-b + a\mathbf{i}) = (-b, a)$.
Let this matrix be $F = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}$ where $\alpha, \beta, \chi,$ and $\delta \in \mathbb{R}$. Then,

$$\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \chi \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \alpha = a; \chi = b \text{ and}$$

$$\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \implies \begin{pmatrix} \beta \\ \delta \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \implies \beta = -b; \delta = a$$

Therefore, $F = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The matrix F can be seen as the matrix representation of the function f which is defined by multiplying a complex number z by $a + b\mathbf{i}$. We can therefore see the matrix F as the real matrix representation of the complex number $a + b\mathbf{i}$.

Remark. Notice that the column vectors of the matrix F are where the vectors $(1,0)$ and $(0,1)$ are mapped to. This pattern shows up in most of the matrices that we will be dealing with - the column vectors of a matrix are the images of the basis vectors (standard) under the linear transformation.

2.3.2 Homomorphisms from $\mathbf{M}_n(\mathbb{C})$ to $\mathbf{M}_{2n}(\mathbb{R})$

In the previous subsection, we saw that we can represent complex numbers as 2×2 real matrices. We can then define a mapping from $\mathbb{C}$ to $\mathrm{M}_2(\mathbb{R})$. We can also show that this mapping is a homomorphism.

Theorem 2.3.1. *Let $\phi : \mathbb{C} \to M_2(\mathbb{R})$ such that $a + b\boldsymbol{i} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then ϕ is an injective homomorphism from $\mathbb{C}$ to $M_2(\mathbb{R})$.*

Remark. We will not include the proof for this theorem as this is merely a special case of Theorem 2.3.3 (when $n = 1$).

We will extend the definition of ϕ to hold for complex matrices in general. To do this, notice that every complex matrix can be represented as the sum of a real matrix and a purely imaginary matrix, i.e., for an $n \times n$ matrix N, $N = C + D\mathbf{i}$ where $C, D \in \mathrm{M}_n(\mathbb{R})$. We see that we can intuitively extend the definition of ϕ by defining

$$\phi(C + D\mathbf{i}) = \begin{pmatrix} C & -D \\ D & C \end{pmatrix} [4]$$

Alternatively, we can define a matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

and define ϕ more generally as $\phi(\mathrm{M}_n(\mathbb{C})) = \{P \in \mathrm{M}_{2n}(\mathbb{R}) | JP = PJ\}$.

Before we prove that the mapping ϕ is a homomorphism, we first prove that the left distributive laws hold for matrices in $\mathrm{M}_n(\mathbb{C})$.

Theorem 2.3.2. *For matrices $A, B, C \in M_n(\mathbb{C})$, $A(B+C) = AB + AC$.*

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}] \in \mathrm{M}_n(\mathbb{C})$. Then $B + C = [b_{ij} + c_{ij}]$ and

$$\begin{aligned} A(B+C) &= [\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})] = [\sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj})] \\ &= [\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}] = [\sum_{k=1}^{n} a_{ik}b_{kj}] + [\sum_{k=1}^{n} a_{ik}c_{kj}] = AB + AC \end{aligned} \tag{2.2}$$

■

Remark. The same method of proof can be used for the right distributive law. Furthermore, this also holds for matrices in $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{H})$.

Theorem 2.3.3. *Let $\phi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that $C + D\boldsymbol{i} \mapsto \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$ where $C + D\boldsymbol{i} \in M_n(\mathbb{C})$. Then ϕ is an injective homomorphism.*

Proof.

1-1:

$$\phi(A + B\mathbf{i}) = \phi(C + D\mathbf{i}) \implies \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$$

$\implies A = C$ and $B = D$ by Matrix Equality $\implies A + B\mathbf{i} = C + D\mathbf{i} \implies \phi$ is injective.

Homomorphism:

Let $A + B\mathbf{i}$, $C + D\mathbf{i} \in \mathrm{M}_n(\mathbb{C})$. Then

$$\phi[(A+B\mathbf{i})(C+D\mathbf{i})] = \phi[(A+B\mathbf{i})C + (A+B\mathbf{i})D\mathbf{i}] \text{ by Theorem 2.3.2}$$

$$= \phi[AC + BC\mathbf{i} + AD\mathbf{i} - BD] = \phi[(AC-BD) + (BC+AD)\mathbf{i}]$$

$$= \begin{pmatrix} (AC-BD) & -(BC+AD) \\ (BC+AD) & (AC-BD) \end{pmatrix}$$

$$\phi[(A+B\mathbf{i})]\phi[(C+D\mathbf{i})] = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \dots & a_{1n} & -b_{11} & \dots & -b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} & -b_{n1} & \dots & -b_{nn} \\ b_{11} & \dots & b_{1n} & a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} & a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} c_{11} & \dots & c_{1n} & -d_{11} & \dots & -d_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} & -d_{n1} & \dots & -d_{nn} \\ d_{11} & \dots & d_{1n} & c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & \dots & d_{nn} & c_{n1} & \dots & c_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^n a_{1k}c_{k1} - \sum_{k=1}^n b_{1k}d_{k1} & \dots & -\sum_{k=1}^n a_{1k}d_{kn} - \sum_{k=1}^n b_{1k}c_{kn} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n b_{nk}c_{k1} + \sum_{k=1}^n a_{nk}d_{k1} & \dots & -\sum_{k=1}^n b_{nk}d_{kn} + \sum_{k=1}^n a_{nk}c_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} (AC-BD) & -(BC+AD) \\ (BC+AD) & (AC-BD) \end{pmatrix} \qquad \blacksquare$$

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