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AN APPLICATION OF THE STUDY DETERMINANT ON SKEW-CONINVOLUTORY QUATERNIONIC MATRICES

by

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Abstract

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John Aaron Q. Alcoseba
University of the Philippines Cebu, 2017

Adviser:
Dr. Lorna S. Almocera

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Chapter 1

Introduction

Ever since their discovery by William Rowan Hamilton in 1843, Quaternions have found extensive use in solving problems both in theoretical and applied mathematics - notably on the problem of 3D rotation.

Computations regarding 3D rotations use 4x4 matrices with real entries like the ones shown in Figure 1.1. We call any set of three angles that represent a rotation applied in some order around the principal axes as *Euler Angles* (in this case α, β, and γ) [4]. Computations with these matrices, however, are a bit tedious and require more elementary arithmetic operations [4]. It's also more difficult to determine the axis and angle of rotation using Euler angles [4]. Furthermore, this method is susceptible to a problem in mechanics known as the *Gimbal Lock* [2].

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a)Rotation by α in the x-axis

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)Rotation by β in the y-axis

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)Rotation by γ in the z-axis

Figure 1.1: 4x4 Rotation Matrices about the Principal Axes

The gimbal lock is a phenomenon that occurs when two of the moving axes x, y, and z (more commonly known as "pitch", "yaw", and "roll" respectively) coincide - resulting in a loss of one degree of freedom for the object being rotated [2].

Quaternions do not suffer from the gimbal lock. They are also found to be more compact - requiring less elementary arithmetic operations to perform rotation composition than rotation matrices [4]. The axis and angle of rotation can also be easily deduced. Let $\vec{q}$ be the purely imaginary parts of the quaternion $q = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$,i.e., $\vec{q} = b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$. It can be shown that

$$\frac{\vec{q}}{\sqrt{b^2+c^2+d^2}} \text{ is the axis of rotation and}$$

$$\theta \text{ satisfying } \sin\theta/2 = \sqrt{b^2+c^2+d^2} \text{ and } \cos\theta/2 = a \text{ is the angle of rotation. [4]}$$

Quaternions are used today in robotics, three-dimensional computer graphics, computer vision, crystallographic texture analysis, navigation, and molecular dynamics.

Mathematicians have made advancements in developing the theory of Quaternions. Notably, as one of the central points of this topic, we look into the concept of a *Quaternionic Matrix* and the implications it has on certain definitions that were already established in Linear Algebra.

One such implication is the concept of a determinant in the context of quaternionic matrices. In linear algebra, we saw that we can extend the definition of the determinant to matrices with complex entries [5]. This is possible because the complex numbers are commutative under complex multiplication [1].

Certain problems arise if we attempt to extend the classical definition to the quaternions because quaternions are not commutative under quaternion multiplication [1]. In [1], Aslaksen revisits the properties we associate with determinants and gives three conditions called *axioms* that should be satisfied in order for a definition of a determinant to be valid and useful:

1. $det(A) = 0$ if and only if A is singular.

2. $det(AB) = det(A)det(B)$ for all quaternionic matrices A and B.

3. If A' is obtained by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $det(A') = det(A)$.

Over the years, several mathematicians have come up with different ways to define a determinant for quaternionic matrices - the Cayley determinant (by Arthur Cayley in

1845), the Study determinant (by Eduard Study in 1920), the Dieudonne determinant, and Moore's determinant. Aslaksen showed whether or not these different definitions satisfy the above conditions [1].

We will look at a particular problem that will require the concept of a determinant - that is, to determine whether or not the set of all $n \times n$ *Skew-coninvolutory Quaternionic Matrices* (denoted by $\mathcal{D}_n(\mathbb{H})$) is empty when n is odd.

In [5], Sta. Maria provided a simple proof to the fact that the set of all $n \times n$ skew-coninvolutory *complex* matrices is empty when n is odd. The method of proof involved using the determinant defined for complex matrices (which is not different from the classical determinant for matrices with real entries).

In this paper, we will discuss the theory behind quaternionic determinants - particularly, the Study determinant. We will then use the Study determinant to extend the result by Sta. Maria to the set of all skew-coninvolutory quaternionic matrices, i.e., we will show that the set of all $n \times n$ skew-coninvolutory quaternionic matrices is empty when n is odd.

1.1 List of Symbols

- $\mathrm{M}_n(\mathbb{R})$ - set of all $n \times n$ matrices with real entries.
- $\mathrm{M}_n(\mathbb{C})$ - set of all $n \times n$ matrices with complex entries.
- $\mathrm{M}_n(\mathbb{H})$ - set of all $n \times n$ matrices with quaternion entries.
- $\mathcal{D}_n(\mathbb{C})$ - set of all $n \times n$ skew-coninvolutory matrices with complex entries.
- $\mathcal{D}_n(\mathbb{H})$ - set of all $n \times n$ skew-coninvolutory matrices with quaternion entries.
- $det_{\mathbb{C}}()$ - determinant of a matrix in $\mathrm{M}_n(\mathbb{C})$.
- $det_{\mathbb{R}}()$ - determinant of a matrix in $\mathrm{M}_n(\mathbb{R})$.

Chapter 2

Preliminaries

In this chapter, we will be presenting terms and known results which are key to building and understanding the theory of quaternionic determinants.

2.1 Complex Matrices

Our discussions regarding quaternionic matrices will mostly revolve around extending properties and definitions that already hold for complex matrices.

To put it simply, *complex matrices* are simply matrices that have complex entries. However, even with this little difference, complex matrices can give us deeper insight into concepts we've already seen in linear algebra - revisiting notions such as the definition of transposes and orthogonal matrices (see [5] for deeper discussion regarding this matter).

Definition 2.1 (Conjugate Matrix)**.** *A* conjugate matrix *is a matrix $\bar{E}$ obtained from E by taking the complex conjugate of every entry of E.*

Definition 2.2 (Coninvolutory Matrix)**.** *A matrix is said to be* coninvolutory *if $E\bar{E} = I_n$ for $E \in M_n(\mathbb{C})$.*

By manipulation, we obtain $E^{-1} = \bar{E}$. Hence, we may also say that a matrix whose inverse is its own conjugate matrix is a coninvolutory matrix.

Example 2.1.1. Consider the complex matrix, $E = \begin{pmatrix} -\frac{13}{17} + \frac{16}{17}\mathbf{i} & -\frac{8}{17} + \frac{2}{17}\mathbf{i} \\ \frac{16}{17} - \frac{4}{17}\mathbf{i} & \frac{19}{17} + \frac{8}{17}\mathbf{i} \end{pmatrix}$

We see that

$$
\begin{aligned}
E\bar{E} &= \begin{pmatrix} -\frac{13}{17}+\frac{16}{17}\mathbf{i} & -\frac{8}{17}+\frac{2}{17}\mathbf{i} \\ \frac{16}{17}-\frac{4}{17}\mathbf{i} & \frac{19}{17}+\frac{8}{17}\mathbf{i} \end{pmatrix} \begin{pmatrix} -\frac{13}{17}-\frac{16}{17}\mathbf{i} & -\frac{8}{17}-\frac{2}{17}\mathbf{i} \\ \frac{16}{17}+\frac{4}{17}\mathbf{i} & \frac{19}{17}-\frac{8}{17}\mathbf{i} \end{pmatrix} \\
&= \frac{1}{17}\frac{1}{17} \begin{pmatrix} -13+16\mathbf{i} & -8+2\mathbf{i} \\ 16-4\mathbf{i} & 19+8\mathbf{i} \end{pmatrix} \begin{pmatrix} -13-16\mathbf{i} & -8-2\mathbf{i} \\ 16+4\mathbf{i} & 19-8\mathbf{i} \end{pmatrix} \\
&= \frac{1}{289} \begin{pmatrix} 289 & 0 \\ 0 & 289 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Hence, we see that E is a coninvolutory matrix. **Side note:** We've obtained E using the factorization provided in Theorem 2.3 found in [5].

If we take the concept of coninvolutory matrices in the context of real matrices, we get $EE = E^2 = I_n$ since $\forall E \in \mathrm{M}_n(\mathbb{R})$, $E = \bar{E}$. This is what we call an *Involutory Matrix*, i.e., a matrix whose inverse is itself. We see that the coninvolutory matrix is, in some way, an extension of the concept of an involutory matrix in $\mathrm{M}_n(\mathbb{R})$. Also, notice that a coninvolutory matrix generalizes complex numbers with modulus 1 (complex numbers that lie on the unit circle of the complex plane), i.e., $z\bar{z} = |z|^2 = 1$ for $z \in \mathbb{C}$ [5].

Definition 2.3 (Skew-Coninvolutory Matrix)**.** *A matrix is said to be* skew-coninvolutory *if $E\bar{E} = -I_n$ for $E \in M_n(\mathbb{C})$.*

Again, we may say that a matrix whose inverse is the negative of its own conjugate matrix is a skew-coninvolutory matrix. If we take this in the context of real matrices, we get $EE = E^2 = -I_n$. Notice how this closely resembles a property of the complex number $\mathbf{i}$, i.e., $\mathbf{i}^2 = -1$. In fact, we have a special name for these linear maps: *complex structures* [7].

Definition 2.4 (Complex Structure)**.** *A* complex structure *of a vector space V is defined by the linear map (linear transformation) $J : V \to V$ such that $J^2 = -I$, where I is the identity map. [7]*

In later discussions, we will be looking into functions that make it possible for us to represent complex numbers and complex matrices (complex linear maps) as real matrices (real linear maps). In doing so, we will have to define a complex structure in the real matrices.

Recall in linear algebra that a linear map *must* commute with scalar multiplication, i.e. $L(cv) = cL(v)$ for a linear map L, $v \in V$ (where V is a vector space over the field F) and $c \in F$. Hence, representing complex linear maps as real linear maps requires the latter to commute with a complex structure of its vector space (this applies to any associated linear maps between different vector spaces) [1] [6].

We can also take the determinants of complex matrices by using the classical definition. We see that computing for the determinant of a complex matrix will give us a complex number. We shall denote this determinant by $det_{\mathbb{C}}$. The following theorem shows one very useful result.

Theorem 2.1. *For a matrix $E \in M_n(\mathbb{C})$, $det_{\mathbb{C}}(\bar{E}) = \overline{det_{\mathbb{C}}(E)}$.*

Proof. We prove by mathematical induction.

Base Case: For $E = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$, $det_{\mathbb{C}}(\bar{E}) = \begin{vmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{vmatrix} = \overline{ad} - \overline{bc} = \overline{ad - bc} = \overline{det_{\mathbb{C}}(E)}$

Induction Hypothesis: Suppose $det_{\mathbb{C}}(\bar{E}) = \overline{det_{\mathbb{C}}(E)}$ holds for $E \in \mathrm{M}_n(\mathbb{C})$.
Let $X \in \mathrm{M}_{n+1}(\mathbb{C})$. Then,

$$det_{\mathbb{C}}(\overline{X}) = \overline{\sum_{j=1}^{n+1} a_{ij}c_{ij}} = \sum_{j=1}^{n+1} \overline{a_{ij}}\ \overline{c_{ij}}$$

is the i^{th} row expansion of an $(n+1) \times (n+1)$ matrix where $\overline{c_{ij}}$ is the cofactor of $\overline{a_{ij}}$.
Note that $\overline{c_{ij}} = (-1)^{i+j}\overline{M_{ij}}$ where $\overline{M_{ij}}$ is the determinant of the $n \times n$ matrix obtained by deleting the i^{th} row and the j^{th} column of the original matrix.
By I.H., $\overline{M_{ij}}$ is the determinant of an $n \times n$ conjugate matrix. Thus, we see that we are computing for the determinant of an $(n+1) \times (n+1)$ conjugate matrix. □

Theorem 2.1 states that computing for the determinant commutes with conjugation, i.e., the determinant of the conjugate matrix is the conjugate of the determinant.

2.1.1 Skew-Coninvolutory Complex Matrices

We now show and prove a result concerning whether or not $\mathcal{D}_n(\mathbb{C})$ is empty when n is odd as seen in [5].

Theorem 2.2. *$\mathcal{D}_n(\mathbb{C})$ is empty when n is odd.*

Proof. If $E \in \mathcal{D}_n(\mathbb{C})$ then $E\bar{E} = -I_n$.
Taking the determinant of both sides,

$$\begin{aligned} det_{\mathbb{C}}(E\bar{E}) &= det_{\mathbb{C}}(-I_n) \\ det_{\mathbb{C}}(E)det_{\mathbb{C}}(\bar{E}) &= (-1)^n \\ det_{\mathbb{C}}(E)\overline{det_{\mathbb{C}}(E)} &= (-1)^n, \text{ by Theorem 2.1} \\ |det_{\mathbb{C}}(E)|^2 &= (-1)^n \end{aligned}$$

Since $|det_{\mathbb{C}}(E)|^2 > 0$, $(-1)^n > 0$. Hence, n must be even. $\square$

The above theorem puts a restriction on the dimension of complex matrices that are skew-coninvolutory. In the context of real matrices, this means that $E^2 = -I_n$ only holds if E is a $2n \times 2n$ real matrix, i.e., a complex structure only exists for real matrices with even dimensions. We will see manifestations of this fact in later discussions especially when we represent complex matrices as real matrices.

2.2 Quaternion Basics

In this section, we introduce properties and operations associated with quaternions including addition, multiplication, conjugation, inverse, and norm.

Definition 2.5 (Quaternion)**.** *The four-dimensional algebra of* Quaternions *is generated by the basis elements $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ such that $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. $\mathbb{H} := \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} | a, b, c, d \in \mathbb{R}\}$. [5]*

2.2.1 Multiplication and Addition

From definition 2.5 we can easily derive the following:

$$\mathbf{jk} = \mathbf{i} \qquad \mathbf{kj} = -\mathbf{i}$$
$$\mathbf{ki} = \mathbf{j} \qquad \mathbf{ik} = -\mathbf{j}$$
$$\mathbf{ij} = \mathbf{k} \qquad \mathbf{ji} = -\mathbf{k}$$

Notice that the quaternions are not commutative under **multiplication**.

Theorem 2.3 (Quaternion Multiplication)**.** *For quaternions* $q_1 = a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{j} + d_1\boldsymbol{k}$ *and* $q_2 = a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{j} + d_2\boldsymbol{k}$*,*

$$\begin{aligned} q_1q_2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\boldsymbol{i} \\ & + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\boldsymbol{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\boldsymbol{k} \end{aligned}$$

Quaternions are, however, commutative under **addition** where $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$. We can clearly see (and it can be shown) that the quaternions form a *skew-field* [1] [3].

2.2.2 Other Operations and Properties

Definition 2.6 ($\mathbb{H}$-Conjugate)**.** *The* $\mathbb{H}$*-Conjugate of a quaternion* $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ *is* $\bar{q} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$*.*

Since $\mathbb{C} \subseteq \mathbb{H}$, we see that definition 2.6 reduces to the definition of a complex conjugate when $c, d = 0$. Also notice that $q\bar{q} = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}) = a^2 + b^2 + c^2 + d^2$.

Definition 2.7 ($\mathbb{H}$-Norm)**.** *The* $\mathbb{H}$*-Norm of a quaternion* $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ *is* $|q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$

Definition 2.8 (Inverse)**.** *The inverse of a quaternion* q *is* q^{-1} *such that* $q^{-1}q = qq^{-1} = 1$*.*

Theorem 2.4. *For* $q, p, r \in \mathbb{H}$*,*

1. $|q|^2 = q\bar{q}$.

2. *If* $q \neq 0$, *then* $q^{-1} = \bar{q}/|q|^2$.

3. $\overline{qp} = \bar{p}\bar{q}$.

4. $(qp)^{-1} = p^{-1}q^{-1}$ *provided that the inverses of* p *and* q *exist.*

5. $(qp)r = q(pr)$ *that is, quaternion multiplication is associative.*

Notice that most of the properties we see in quaternions are merely extensions of the properties we see in complex numbers.

2.2.3 Quaternionic Matrices

Most of the definitions we've already mentioned for complex matrices can also be extended in the context of quaternionic matrices.

Definition 2.9 (Conjugate Quaternionic Matrix). *A* conjugate quaternionic matrix *is a matrix* $\bar{E}$ *obtained from* E *by taking the* $\mathbb{H}$*-conjugate of every entry of* E*.*

Definition 2.10 (Skew-Coninvolutory Quaternionic Matrix). *A quaternionic matrix* E *is said to be* Skew-Coninvolutory *if* $E\bar{E} = -I_n$.

Chapter 3

Results and Discussion

3.1 The Cayley Determinant and Aslaksen's Axioms

In 1845, 2 years after William Rowan Hamilton discovered quaternions, Arthur Cayley attempted to define the determinant of a quaternionic matrix using the usual formula (we denote the Cayley determinant by $Cdet$), i.e., for a 2×2 quaternionic matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Cdet(A) = ad - cb$ for $a, b, c, d \in \mathbb{H}$ [1]. The same goes for 3×3 matrices and so on. Taking into account the fact that the quaternions are non-commutative (and the implications it has on linear mappings as will be discussed later), we might ask whether or not this determinant behaves the way we expect - Will it really determine whether or not a quaternionic matrix is singular or not? Will the properties of the determinant still hold? Will the determinant still be a map from $M_n(G) \to G$ (in this case, $G = \mathbb{H}$)? The last question comes from the fact that the determinants of complex matrices is a map from $\mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$.

3.1.1 The Determinant Function

We take a step back and revisit what it means for a mapping to be a determinant. We present the determinant function as defined by J.L. Brenner.

Definition 3.1 (Determinant Function)**.** *For a field F, a determinant over the matrices of $M_n(F)$ is a function det from $M_n(F)$ into F such that*

$$det(AB) = det(A)det(B) = det(B)det(A) \tag{3.1}$$

*holds either **(1)** $\forall A, B \in M_n(F)$ or **(2)** $\forall$ invertible $A, B \in M_n(F)$.*

Notice that the definition requires the images of A and B to commute (this is not possible for skew-fields like the quaternions). We can see this matter viewed in a more rigorous manner (also in a manner more specific to the quaternions) while discussing Aslaksen's axioms.

We see that if *det* is a constant function that only maps to either 0 or 1 (with 0 for singular matrices and 1 for invertible matrices), then *det* satisfies the above definition [3]. The following theorem by Brenner shows that this holds for non-trivial determinants as well and that conditions (1) and (2) are essentially equivalent [3].

Theorem 3.1. *If det is not constantly equal to 1 or 0 (i.e., det is not a mapping* $det : M_n(F) \to F$ *where F is a field with two elements), then* $det(B) = 0$ *for all singular matrices.*

3.1.2 Aslaksen's Axioms

In the *Mathematical Intelligencer*, Helmer Aslaksen presented 3 determinant *axioms* which a determinant definition must satisfy in order for it to behave the way we expect, i.e., it has the properties we associate with determinants. These axioms were already introduced in Chapter 1 and we will be discussing them in greater detail here.

- **Axiom 1.** $det(A) = 0$ if and only if A is singular.
- **Axiom 2.** $det(AB) = det(A)det(B)$ for all quaternionic matrices A and B.
- **Axiom 3.** If A' is obtained by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $det(A') = det(A)$.

Lemma 3.2.

Lemma 3.3.

Theorem 3.4.

3.2 The Study Determinant

It was not until 1920, that a new approach in defining a quaternionic determinant was presented in a paper by Eduard Study [1]. His idea involved transforming quaternionic matrices into complex matrices from which one could then just simply take the determinant [1]. The method involves homomorphisms between quaternionic, complex, and real matrices.

3.2.1 Matrix Homomorphisms

We look into functions that make it possible for us to represent complex numbers and quaternions as matrices.

Homomorphisms from $\mathbf{M}_n(\mathbb{C})$ to $\mathbf{M}_{2n}(\mathbb{R})$

In order to represent complex matrices as real matrices, notice that every complex matrix can be represented as the sum of a real matrix and a purely imaginary matrix, i.e., for an $n \times n$ matrix Z, $Z = A + B\mathbf{i}$ where $A, B \in \mathrm{M}_n(\mathbb{R})$ [1]. We define a mapping

$$\phi(A + B\mathbf{i}) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} [1]$$

Before we show that this mapping is an injective homomorphism, we first show that the left distributive laws hold for matrices in $\mathrm{M}_n(\mathbb{C})$.

Theorem 3.5. *For matrices $A, B, C \in M_n(\mathbb{C})$, $A(B + C) = AB + AC$.*

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}] \in \mathrm{M}_n(\mathbb{C})$. Then $B + C = [b_{ij} + c_{ij}]$ and

$$\begin{aligned} A(B + C) &= [\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})] = [\sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj})] \\ &= [\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}] = [\sum_{k=1}^{n} a_{ik}b_{kj}] + [\sum_{k=1}^{n} a_{ik}c_{kj}] = AB + AC \end{aligned} \tag{3.2}$$

□

The same method of proof can be used for the right distributive law. Furthermore, this also holds for matrices in $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{H})$.

Theorem 3.6. *Let $\phi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that $C + D\boldsymbol{i} \mapsto \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$ where $C + D\boldsymbol{i} \in M_n(\mathbb{C})$. Then ϕ is an injective homomorphism.*

Proof.

1-1:

$$\phi(A + B\mathbf{i}) = \phi(C + D\mathbf{i}) \implies \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$$

$\implies A = C$ and $B = D$ by Matrix Equality $\implies A + B\mathbf{i} = C + D\mathbf{i} \implies \phi$ is injective.

Homomorphism:

Let $A + B\mathbf{i}$, $C + D\mathbf{i} \in \mathrm{M}_n(\mathbb{C})$. Then

$$\begin{aligned}\phi[(A+B\mathbf{i})(C+D\mathbf{i})] &= \phi[(A+B\mathbf{i})C + (A+B\mathbf{i})D\mathbf{i}] \text{ by Theorem 3.5}\\ &= \phi[AC + BC\mathbf{i} + AD\mathbf{i} - BD] = \phi[(AC-BD) + (BC+AD)\mathbf{i}]\\ &= \begin{pmatrix}(AC-BD) & -(BC+AD)\\ (BC+AD) & (AC-BD)\end{pmatrix}\end{aligned}$$

$$\phi[(A+B\mathbf{i})]\phi[(C+D\mathbf{i})] = \begin{pmatrix}A & -B\\ B & A\end{pmatrix}\begin{pmatrix}C & -D\\ D & C\end{pmatrix}$$

$$= \begin{pmatrix}a_{11} & \dots & a_{1n} & -b_{11} & \dots & -b_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & \dots & a_{nn} & -b_{n1} & \dots & -b_{nn}\\ b_{11} & \dots & b_{1n} & a_{11} & \dots & a_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ b_{n1} & \dots & b_{nn} & a_{n1} & \dots & a_{nn}\end{pmatrix}\begin{pmatrix}c_{11} & \dots & c_{1n} & -d_{11} & \dots & -d_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ c_{n1} & \dots & c_{nn} & -d_{n1} & \dots & -d_{nn}\\ d_{11} & \dots & d_{1n} & c_{11} & \dots & c_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ d_{n1} & \dots & d_{nn} & c_{n1} & \dots & c_{nn}\end{pmatrix}$$

$$= \begin{pmatrix}\sum_{k=1}^{n} a_{1k}c_{k1} - \sum_{k=1}^{n} b_{1k}d_{k1} & \dots & -\sum_{k=1}^{n} a_{1k}d_{kn} - \sum_{k=1}^{n} b_{1k}c_{kn}\\ \vdots & \ddots & \vdots\\ \sum_{k=1}^{n} b_{nk}c_{k1} + \sum_{k=1}^{n} a_{nk}d_{k1} & \dots & -\sum_{k=1}^{n} b_{nk}d_{kn} + \sum_{k=1}^{n} a_{nk}c_{kn}\end{pmatrix}$$

$$= \begin{pmatrix}(AC-BD) & -(BC+AD)\\ (BC+AD) & (AC-BD)\end{pmatrix} \qquad \square$$

We define a matrix

$$J = \begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix}$$

. Notice that the matrix J is the image of $iI \in \mathrm{M}_n(\mathbb{C})$ under ϕ [1]. It can be easily shown that $J^2 = -I$. It is obvious that J gives a *complex structure* in $\mathbb{R}^{2n}$. Hence, $\phi(\mathrm{M}_n(\mathbb{C})) = \{P \in \mathrm{M}_{2n}(\mathbb{R}) | JP = PJ\}$, i.e., the real matrix representations of complex matrices are the linear maps in $\mathrm{M}_{2n}(\mathbb{R})$ that commute with the complex structure [1].

Homomorphisms from $\mathbf{M}_n(\mathbb{H})$ to $\mathbf{M}_{2n}(\mathbb{C})$

To represent quaternionic matrices as complex matrices, notice that every quaternionic matrix can be represented as the sum $Y = C + \mathbf{j}D$ where $C, D \in \mathrm{M}_n(\mathbb{C})$ [1]. We define a mapping

$$\psi(C + \mathbf{j}D) = \begin{pmatrix} C & -\overline{D} \\ D & \overline{C} \end{pmatrix} [1]$$

We can show that ψ is an injective homomorphism using the same proof outline in the previous subsection [1].

The non-commutativity of quaternions presents some problems in representing *quaternionic linear maps* as complex linear maps. If we consider a quaternionic linear map say $L(v) = Av$ for $A \in \mathrm{M}_n(\mathbb{H})$ where we take in quaternions as scalars, then, $cAv = cL(v) = L(cv) = Acv$ which is false (considering the base case for 1×1 matrices) [6]. However, $Avc = L(v)c = L(vc) = Avc$. Hence, we now see that any quaternionic linear map commutes with right scalar multiplication by a quaternion which itself is not a linear map in $\mathbb{H}$ (in order for it to be a linear map, it in turn, has to commute with other quaternions)[6] [1]. This poses a problem because it implies that there is no matrix representation for right scalar multiplication [1]. However, in [1], we see that we can consider a linear map $\widetilde{R_j}$ in $\mathbb{C}^{2n}$ as the image of right scalar multiplication by $\mathbf{j}$ under the homomorphism. $\widetilde{R_j}$ corresponds to multiplying $v \in \mathbb{C}^{2n}$ by the matrix J and then conjugating [1]. This gives a quaternionic structure in $\mathbb{C}^{2n}$ and thus, a quaternionic linear map corresponds to a complex linear map Q that commutes with $\widetilde{R_j}$, i.e., $Q\overline{Jv} = \overline{JQv}$ for $v \in \mathbb{C}^{2n}$. It can be easily shown that the latter holds if and only if $\overline{Q}J = JQ$ using the fact that $Q\overline{Jv} = \overline{QJv}$. Thus, $\psi(\mathrm{M}_n(\mathbb{H})) = \{Q \in \mathrm{M}_{2n}(\mathbb{C}) | \overline{Q}J = JQ\}$.

3.2.2 Study Determinant

Definition 3.2. *The Study Determinant is defined by* $SdetM = det_{\mathbb{C}}(\psi M) = \sqrt{det_{\mathbb{R}}(\phi(\psi(M)))}$.

It can be shown that the Study Determinant satisfies all of Aslaksen's axioms [1].

3.3 Main Result

Proposition 1. *$\mathcal{D}_n(\mathbb{H})$ is empty when n is odd.*

Proof. If $F \in \mathcal{D}_n(\mathbb{H})$ then $F\bar{F} = -I_n$.
Taking the Study determinant of both sides,

$$\begin{aligned} Sdet(E\bar{E}) &= Sdet(-I_n) \\ Sdet(E)Sdet(\bar{E}) &= (-1)^n \\ Sdet(E)\overline{Sdet(E)} &= (-1)^n, \text{ by Theorem 2.1} \\ |Sdet(E)|^2 &= (-1)^n \end{aligned}$$

Since $|Sdet(E)|^2 > 0$, $(-1)^n > 0$. Hence, n must be even. $\square$

Theorem 2.1 holds because the Study determinant is a complex determinant.

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