Obtaining the Expressions for Converters without the use of Small-Ripple Approximation

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1 Goal

To derive the expressions for converters without the use of small-ripple approximation.

2 Approach

It is possible to solve converters exactly, without the use of the small-ripple approximation. We consider a boost converter to demonstrate this. First, we find the Laplace transform to write expressions for the waveforms of the circuits of Figs. 2(a) and 2(b). We then invert the transforms, match boundary conditions, and find the periodic steady-state solution of the circuit. Having done so, we then find the DC components of the waveforms and the peak values.

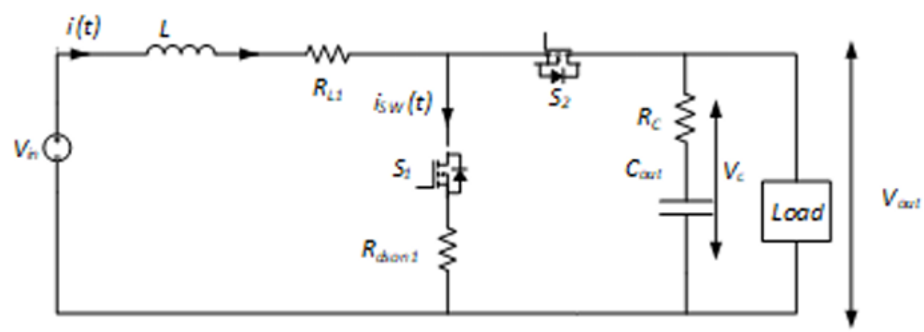
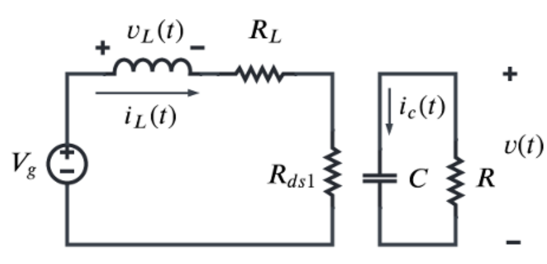
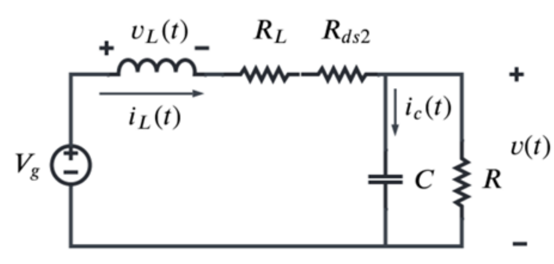


Figure 1: Boost Converter



(a) Switch 1 ON



(b) Switch 1 OFF

Figure 2: The two states of the boost converter

In the case when Switch 1 is ON, thus with Switch 2 being OFF,

$$v_g(t) = L\frac{di_L(t)}{dt} + (R_L + R_{ds1on})i_L(t) \tag{1}$$

$$C\frac{dv_C(t)}{dt} = -\frac{v_C(t)}{R} \tag{2}$$

In the case when Switch 2 is ON, thus with Switch 1 being OFF,

$$v_g(t) = L\frac{di_L(t)}{dt} + (R_L + R_{ds1on})i_L(t) + v_C(t) \tag{3}$$

$$C\frac{dv_C(t)}{dt} = i_L(t) - \frac{v_C(t)}{R} \tag{4}$$

Assumption 1: $v_g(t)$ is a constant DC input v_{in}

For simplifying the equations below, let α, ω and γ be:

$$\alpha_1 = \frac{\left(\frac{R_{net1}}{L} + \frac{1}{RC}\right)}{2}, \omega_1 = \sqrt{\frac{1}{LC}\left(\frac{R_{net1}}{R} + 1\right) - \alpha_1^2}, \gamma_1 = \frac{\left(\frac{1}{RC} - \frac{R_{net1}}{L}\right)}{2},$$

$$\alpha_2 = \frac{\left(\frac{R_{net2}}{L} + \frac{1}{RC}\right)}{2}, \omega_2 = \sqrt{\frac{1}{LC}\left(\frac{R_{net2}}{R} + 1\right) - \alpha_2^2}, \gamma_2 = \frac{\left(\frac{1}{RC} - \frac{R_{net2}}{L}\right)}{2},$$

where α_1, ω_1 and γ_1 are for state 1 and α_2, ω_2 and γ_2 are for state 2.

State 1: Switch ON($0 \leq t < DT_s$)

Given the differential equation for the inductor voltage when the switch is ON, we have:

$$L\frac{di_L(t)}{dt} = v_{in} - R_{net}i_L(t)$$

where $R_{net} = R_L + R_{ds1on}$.
The Laplace transform of the differential equation is:

$$L\left[sI_L(s) - I_L(0)\right] = \frac{v_{in}}{s} - R_{net}I_L(s)$$

$$LsI_L(s) - LI_L(0) = \frac{v_{in}}{s} - R_{net}I_L(s)$$

Combine terms involving $I_L(s)$:

$$LsI_L(s) + R_{net}I_L(s) = \frac{v_{in}}{s} + LI_L(0)$$

Factor out $I_L(s)$:

$$I_L(s)\left(Ls + R_{net}\right) = \frac{v_{in}}{s} + LI_L(0)$$

Solve for $I_L(s)$:

$$I_L(s) = \frac{\frac{v_{in}}{s} + LI_L(0)}{Ls + R_{net}}$$

Separate the terms in the numerator:

$$I_L(s) = \frac{v_{in}/s}{Ls + R_{net}} + \frac{LI_L(0)}{Ls + R_{net}}$$

$$I_L(s) = \frac{v_{in}}{s(Ls + R_{net})} + \frac{I_L(0)}{s + \frac{R_{net}}{L}}$$

We need to find the inverse Laplace transform of each term separately.
The inverse Laplace transform of the first term is:

$$\mathcal{L}^{-1}\left\{\frac{v_{in}}{Ls(s + \frac{R_{net}}{L})}\right\}$$

Using partial fraction decomposition:

$$\frac{v_{in}}{Ls(s + \frac{R_{net}}{L})} = \frac{A}{Ls} + \frac{B}{s + \frac{R_{net}}{L}}$$

Solving for A and B:

$$v_{in} = A(s + \frac{R_{net}}{L}) + BLs$$

For $s = 0$:

$$v_{in} = A\frac{R_{net}}{L} \quad \Rightarrow \quad A = \frac{v_{in}L}{R_{net}}$$

For $s = -\frac{R_{net}}{L}$:

$$v_{in} = BL(-\frac{R_{net}}{L}) \quad \Rightarrow \quad B = -\frac{v_{in}}{R_{net}}$$

So,

$$\frac{v_{in}}{Ls(s + \frac{R_{net}}{L})} = \frac{\frac{v_{in}L}{R_{net}}}{Ls} - \frac{\frac{v_{in}}{R_{net}}}{s + \frac{R_{net}}{L}}$$

The inverse Laplace transform is:

$$\mathcal{L}^{-1}\left\{\frac{v_{in}}{R_{net}}\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{v_{in}}{R_{net}}\frac{1}{s + \frac{R_{net}}{L}}\right\}$$
$$= \frac{v_{in}}{R_{net}} \cdot 1 - \frac{v_{in}}{R_{net}}e^{-\frac{R_{net}}{L}t}$$

Similarly, for the second term:

$$\mathcal{L}^{-1}\left\{\frac{I_L(0)}{s + \frac{R_{net}}{L}}\right\} = I_L(0)e^{-\frac{R_{net}}{L}t}$$

Combining both terms, we get the time-domain solution:

$$i_L(t) = \frac{v_{in}}{R_{net}}\left(1 - e^{-\frac{R_{net}}{L}t}\right) + I_L(0)e^{-\frac{R_{net}}{L}t}$$

Thus, the solution for $i_L(t)$ when the switch is ON is:

$$i_L(t) = \frac{v_{in}}{R_{net}}\left(1 - e^{-\frac{R_{net}}{L}t}\right) + I_L(0)e^{-\frac{R_{net}}{L}t}$$
$$i_L(t) = \frac{v_{in}}{R_{net}}\left(1 - e^{-(\alpha_1 - \gamma_1)t}\right) + I_L(0)e^{-(\alpha_1 - \gamma_1)t} \tag{5}$$

Given the differential equation for the capacitor voltage $v_C(t)$ when the switch is ON, we have:

$$C\frac{dv_C(t)}{dt} = -\frac{v_C(t)}{R}$$

The Laplace transform of the differential equation is:

$$C\left[sV_C(s) - V_C(0)\right] = -\frac{V_C(s)}{R}$$

$$CsV_C(s) - CV_C(0) = -\frac{V_C(s)}{R}$$

Combine terms involving $V_C(s)$:

$$CsV_C(s) + \frac{V_C(s)}{R} = CV_C(0)$$

Factor out $V_C(s)$:

$$V_C(s)\left(Cs + \frac{1}{R}\right) = CV_C(0)$$

Solve for $V_C(s)$:

$$V_C(s) = \frac{CV_C(0)}{Cs + \frac{1}{R}}$$

Simplify the fraction:

$$V_C(s) = \frac{V_C(0)}{s + \frac{1}{CR}}$$

We need to find the inverse Laplace transform of $V_C(s)$:

$$\mathcal{L}^{-1}\left\{\frac{V_C(0)}{s + \frac{1}{CR}}\right\}$$

The inverse Laplace transform is a standard form:

$$V_C(t) = V_C(0)e^{-\frac{t}{CR}}$$

Thus, the solution for $v_C(t)$ when the switch is ON is:

$$v_C(t) = V_C(0)e^{-\frac{t}{CR}}$$
$$v_C(t) = V_C(0)e^{-(\alpha_1+\gamma_1)t} \tag{6}$$

State 2: Switch OFF ($DT_s \leq t < T_s$)

- Given the differential equation for the capacitor voltage $v_C(t)$ when the switch is OFF, we have:

$$C\frac{dv_C(t)}{dt} = i_L(t) - \frac{v_C(t)}{R}$$

take laplace transform,

$$\begin{aligned}
C\left(sV_c(s) - V_c(DT_s)\right) &= I_L(s) - \frac{V_c(s)}{R} \\
CsV_c(s) - CV_c(DT_s) &= I_L(s) - \frac{V_C(s)}{R} \\
V_c(s)(s + \frac{1}{RC}) &= \frac{I_L(s)}{C} + V_C(DT_s) \\
V_C(s) &= \frac{1}{s + \frac{1}{RC}} \cdot \left(\frac{I_L(s)}{C} + V_C(DT_s)\right)
\end{aligned} \tag{7}$$

where $V_C(DT_s)$ is the capacitor voltage at DT_s

- Given the differential equation for the inductor current $i_L(t)$ when the switch is OFF, we have:

$$L\frac{di_L(t)}{dt} = v_{in} - R_{net}i_L(t) - v_C(t)$$

where $R_{net} = R_L + R_{ds2on}$.
The Laplace transform of the differential equation is:

$$L\left[sI_L(s) - I_L(DT_s)\right] = \frac{v_{in}}{s} - R_{net}I_L(s) - V_C(s)$$

$$LsI_L(s) - LI_L(DT_s) = \frac{v_{in}}{s} - R_{net}I_L(s) - V_C(s)$$

Combine terms involving $I_L(s)$:

$$LsI_L(s) + R_{net}I_L(s) = \frac{v_{in}}{s} + LI_L(DT_s) - V_C(s)$$

Factor out $I_L(s)$:

$$I_L(s)\left(Ls + R_{net}\right) = \frac{v_{in}}{s} + LI_L(DT_s) - V_C(s)$$

Solve for $I_L(s)$:

$$I_L(s) = \frac{\frac{v_{in}}{s} + LI_L(DT_s) - V_C(s)}{Ls + R_{net}}$$

Separate the terms in the numerator:

$$I_L(s) = \frac{\frac{v_{in}}{s}}{Ls + R_{net}} + \frac{LI_L(DT_s)}{Ls + R_{net}} - \frac{V_C(s)}{Ls + R_{net}}$$

$$I_L(s) = \frac{v_{in}}{s(Ls + R_{net})} + \frac{I_L(DT_s)}{s + \frac{R_{net}}{L}} - \frac{V_C(s)}{Ls + R_{net}}$$

substituting (7),

$$
\begin{aligned}
I_L(s) &= \frac{v_{in}}{Ls(s+\frac{R_{net}}{L})} + \frac{I_L(DT_s)}{s+\frac{R_{net}}{L}} - \frac{1}{(s+\frac{1}{RC})}\left(\frac{i_L(s)}{C} + V_C(DT_s)\right)\frac{1}{L(s+\frac{R_{net}}{L})} \\
&= \frac{v_{in}}{Ls(s+\frac{R_{net}}{L})} + \frac{I_L(DT_s)}{s+\frac{R_{net}}{L}} - \frac{I_L(s)}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})} - \frac{V_C(DT_s)}{L(s+\frac{1}{RC})(s+\frac{R_{net}}{L})} \\
I_L(s)\left(1+\frac{1}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})}\right) &= \frac{v_{in}}{Ls(s+\frac{R_{net}}{L})} + \frac{I_L(DT_s)}{s+\frac{R_{net}}{L}} - \frac{V_C(DT_s)}{L(s+\frac{1}{RC})(s+\frac{R_{net}}{L})} \\
I_L(s)\left(\frac{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})}\right) &= \frac{v_{in}}{Ls(s+\frac{R_{net}}{L})} + \frac{I_L(DT_s)}{s+\frac{R_{net}}{L}} - \frac{V_C(DT_s)}{L(s+\frac{1}{RC})(s+\frac{R_{net}}{L})}
\end{aligned}
$$

$$
\begin{aligned}
I_L(s) &= \left(\frac{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1}\right)\left(\frac{v_{in}}{Ls(s+\frac{R_{net}}{L})} + \frac{I_L(DT_s)}{s+\frac{R_{net}}{L}} - \frac{V_C(DT_s)}{L(s+\frac{1}{RC})(s+\frac{R_{net}}{L})}\right) \\
&= \frac{1}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1}\left(\frac{v_{in}C}{s}\left(s+\frac{1}{RC}\right) + \frac{I_L(DT_s)LC(s+\frac{1}{RC})}{1} - V_C(DT_s)C\right) \qquad (8)
\end{aligned}
$$

Lets perform partial fraction on each term separately,
simplifying term 1 of (8)

$$
\begin{aligned}
&\frac{V_{in}C\left(s+\frac{1}{RC}\right)}{s\left(LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1\right)} \\
&= V_{in}C\left(\frac{s+\frac{1}{RC}}{\left(LCs^2+\frac{L}{R}s\right)\left(s+\frac{R_{net2}}{L}\right)+s}\right) \\
&= V_{in}C\left(\frac{s+\frac{1}{RC}}{LCs^3+R_{net2}Cs^2+\frac{L}{R}s^2+\frac{Rnet}{R}s+s}\right) \\
&= V_{in}C\left(\frac{s+\frac{1}{RC}}{LCs^3+\left(R_{net2}C+\frac{L}{R}\right)s^2+\left(\frac{Rnet}{R}+1\right)s}\right) \\
&= \frac{V_{in}}{L}\left(\frac{s+\frac{1}{RC}}{s^3+\left(\frac{R_{net2}}{L}+\frac{1}{RC}\right)s^2+\left(\frac{Rnet}{RLC}+\frac{1}{LC}\right)s}\right) \\
&= \frac{V_{in}}{L}\left(\frac{s+\frac{1}{RC}}{s\left(s^2+\left(\frac{R_{net2}}{L}+\frac{1}{RC}\right)s+\left(\frac{Rnet}{RLC}+\frac{1}{LC}\right)\right)}\right) \\
&= \frac{V_{in}}{L}\left(\frac{s+\alpha_2+\gamma_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)}\right)
\end{aligned}
$$

since,

$$
\begin{aligned}
\alpha_2+\gamma_2 &= \frac{1}{RC} \\
\alpha_2^2+\omega_2^2 &= \frac{Rnet}{RLC}+\frac{1}{LC}
\end{aligned}
$$

simplifying further,

$$
\frac{V_{in}}{L}\left(\frac{s+\alpha_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} + \frac{\gamma_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)}\right) \qquad (9)
$$

Partial fraction of term 1 of (9),

$$\frac{s+\alpha_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} = \frac{A}{s}+\frac{Bs+C}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}$$
$$s+\alpha_2 = A((s+\alpha_2)^2+\omega_2^2)+Bs^2+Cs$$
$$\therefore A = \frac{\alpha_2}{\alpha_2^2+\omega_2^2}$$

substituting A,

$$s+\alpha_2 = \frac{\alpha_2}{\alpha_2^2+\omega_2^2}((s+\alpha_2)^2+\omega_2^2)+Bs^2+Cs$$

opening brackets and rearranging terms,

$$s+\alpha_2 = \left(\frac{\alpha_2}{\alpha_2^2+\omega_2^2}+B\right)s^2+\left(\frac{2\alpha_2^2}{\alpha_2^2+\omega_2^2}+C\right)s+\alpha_2$$

equating coefficients of both sides,

$$B = \frac{-\alpha_2}{\alpha_2^2+\omega_2^2}$$
$$C = \frac{\omega_2^2-\alpha_2^2}{\alpha_2^2+\omega_2^2}$$

Therefore,

$$\begin{aligned}\frac{s+\alpha_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} &= \frac{\alpha_2}{\alpha_2^2+\omega_2^2}\frac{1}{s}+\frac{\frac{-\alpha_2}{\alpha_2^2+\omega_2^2}s+\frac{\omega_2^2-\alpha_2^2}{\alpha_2^2+\omega_2^2}}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}\\ &= \frac{\alpha_2}{\alpha_2^2+\omega_2^2}\frac{1}{s}-\frac{\alpha_2}{\alpha_2^2+\omega_2^2}\frac{s+\alpha_2-\alpha_2}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}+\frac{\omega_2^2-\alpha_2^2}{\alpha_2^2+\omega_2^2}\frac{1}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}\frac{\omega_2}{\omega_2}\\ &= \frac{\alpha_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{\alpha_2}{\alpha_2^2+\omega_2^2}\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)+\frac{\omega_2^2-\alpha_2^2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\end{aligned} \tag{10}$$

Partial fraction of term 2 of (9),

$$\frac{\gamma_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} = \frac{A}{s}+\frac{Bs+C}{(s+\alpha_2)^2+\omega_2^2}$$
$$\gamma_2 = A((s+\alpha_2)^2+\omega_2^2)+Bs^2+Cs$$
$$\therefore A = \frac{\gamma_2}{\alpha_2^2+\omega_2^2}$$

Substituting A,

$$\gamma_2 = \frac{\gamma_2}{\alpha_2^2+\omega_2^2}((s+\alpha_2)^2+\omega_2^2)+Bs^2+Cs$$
$$\gamma_2 = \left(\frac{\gamma_2}{\alpha_2^2+\omega_2^2}+B\right)s^2+\left(\frac{2\alpha_2\gamma_2}{\alpha_2^2+\omega_2^2}+C\right)s+\gamma_2$$
$$\therefore B = \frac{-\gamma_2}{\alpha_2^2+\omega_2^2}$$
$$C = \frac{-2\alpha_2\gamma_2}{\alpha_2^2+\omega_2^2}$$

Therefore,

$$
\begin{aligned}
\frac{\gamma_2}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} &= \frac{\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}+\frac{-\gamma_2}{\alpha_2^2+\omega_2^2}\frac{s}{(s+\alpha_2)^2+\omega_2^2}+\frac{-2\alpha_2\gamma_2}{\alpha_2^2+\omega_2^2}\frac{1}{(s+\alpha_2)^2+\omega_2^2}\\
&= \frac{\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{s+\alpha_2-\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{2\alpha_2\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\\
&= \frac{\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{\gamma_2}{\alpha_2^2+\omega_2^2}\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)-\frac{2\alpha_2\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}
\end{aligned}
\tag{11}
$$

Substituting (10) and (11) in (9),

$$
\begin{aligned}
\frac{V_{in}}{L}\Bigg\{&\left[\frac{\alpha_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{\alpha_2}{\alpha_2^2+\omega_2^2}\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)+\frac{\omega_2^2-\alpha_2^2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right]\\
&+\left[\frac{\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{\gamma_2}{\alpha_2^2+\omega_2^2}\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)-\frac{2\alpha_2\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right]\Bigg\}
\end{aligned}
$$

Simplifying it, we get the first term of (8) to be,

$$
\frac{V_{in}}{L}\left\{\frac{\alpha_2+\gamma_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{\alpha_2+\gamma_2}{\alpha_2^2+\omega_2^2}\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)+\frac{\omega_2^2-\alpha_2^2-2\alpha_2\gamma_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right\}
$$

Taking inverse laplace, we get

$$
\frac{V_{in}}{L}\left\{\frac{\alpha_2+\gamma_2}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)+\frac{\omega_2^2-\alpha_2^2-2\alpha_2\gamma_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right\}
\tag{12}
$$

Simplifying term 2 of (8),

$$
\frac{I_L(DT_s)LC\left(s+\frac{1}{RC}\right)}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1}
$$

Taking LC out of the denominator and cancelling with numerator,

$$
\begin{aligned}
&= I_L(DT_s)\left(\frac{s+\frac{1}{RC}}{s^2+\left(\frac{R_{net2}}{L}+\frac{1}{RC}\right)s+\left(\frac{Rnet}{RLC}+\frac{1}{LC}\right)}\right)\\
&= I_L(DT_s)\left(\frac{s+\alpha_2+\gamma_2}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}\right)\\
&= I_L(DT_s)\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}+\frac{\gamma_2}{(s+\alpha_2)^2+\omega_2^2}\right)\\
&= I_L(DT_s)\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}+\frac{\gamma_2}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)
\end{aligned}
$$

Taking inverse laplace transform,

$$
I_L(DT_s)\left(e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\gamma_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)
\tag{13}
$$

Simplifying term 3 of (8),

$$V_C(DT_s)C\left(\frac{1}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1}\right)$$

Taking LC out and simplifying it like we did above, we get,

$$\frac{V_C(DT_s)}{L}\left(\frac{1}{(s+\alpha_2)^2+\omega_2^2}\right)$$
$$\frac{V_C(DT_s)}{L\omega_2}\left(\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)$$

Taking inverse laplace,

$$\frac{V_C(DT_s)}{L\omega_2}\left(e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right) \tag{14}$$

Thus, by combining (12), (13), and (14), the solution for $i_L(t)$ when the switch is OFF is:

$$\begin{aligned} i_L(t) =& \frac{V_i}{L}\left(A\left(1-e^{-\alpha_2(t-t_1)}\cos(\omega_2(t-t_1))+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(t-t_1)}\sin(\omega_2(t-t_1))\right)+Be^{-\alpha_2(t-t_1)}\sin(\omega_2(t-t_1))\right) \\ &+I_L(DT_s)\left(e^{-\alpha_2(t-t_1)}\cos(\omega_2(t-t_1))+\frac{\gamma_2}{\omega_2}e^{-\alpha_2(t-t_1)}\sin(\omega_2(t-t_1))\right) \\ &-\frac{V_C(DT_s)}{L\omega_2}e^{-\alpha_2(t-t_1)}\sin(\omega_2(t-t_1)) \end{aligned} \tag{15}$$

where

$$A=\frac{\alpha_2+\gamma_2}{\alpha_2^2+\gamma_2^2}$$
$$B=\frac{\omega_2^2-\alpha_2^2-2\alpha_2\gamma_2}{\alpha_2^2\omega_2+\omega_2^3}$$

Since $i_L(t)$ and $v_C(t)$ are continuous functions,

$$i_L(DT_s{}^-)=i_L(DT_s{}^+)$$
$$i_L(DT_s{}^-)=\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)+I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}$$
$$i_L(DT_s{}^+)=i_L(DT_s)$$
$$\therefore i_L(DT_s)=\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)+I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)} \tag{16}$$

Similarly,

$$v_C(DT_s)=V_C(0)e^{-(\alpha_1+\gamma_1)(DT)} \tag{17}$$

Similarly,

$$i_L(T_s{}^-)=i_L(T_s{}^+)$$
$$i_L(T_s{}^-)=i_L(T_s)$$
$$i_L(T_s{}^+)=i_L(0)$$
$$\therefore i_L(0)=i_L(T_s)$$

Substituting (16), (17) and t=T in (15), we get,

$$\begin{aligned}
i_L(T_s) =& \frac{V_i}{L}\left(A\left(1-e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)+Be^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)\\
&+\left(\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)+I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}\right)\left(e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\gamma_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)\\
&-\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))
\end{aligned}$$

But $i_L(0) = i_L(T_s)$, so

$$\begin{aligned}
i_L(0) =& \frac{V_i}{L}\left(A\left(1-e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)+Be^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)\\
&+\left(\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)+I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}\right)\left(e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\gamma_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)\\
&-\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))
\end{aligned}$$

Bring the $I_L(0)$ to one side,

$$\begin{aligned}
i_L(0)\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot Expr\right) =& \frac{Vi}{L}\Big[A\left(1-e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)\\
&+Be^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\Big]+\frac{Vi}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)\cdot Expr\\
&-\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))
\end{aligned}$$

where

$$Expr = \left(e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\gamma_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)$$

Therefore,

$$\begin{aligned}
i_L(0) = \frac{1}{\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot Expr\right)}\Bigg(&\frac{Vi}{L}\Big[A\big(1-e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\big)\\
&+Be^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\Big]+\frac{Vi}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)\cdot Expr\\
&-\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\Bigg)
\end{aligned}$$

Let

$$\begin{aligned}
Expr1 = \frac{1}{\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot Expr\right)}\Bigg(&\frac{Vi}{L}\Big[A\big(1-e^{-\alpha_2(T-t_1)}\cos(\omega_2(T-t_1))+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\big)\\
&+Be^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\Big]+\frac{Vi}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)\cdot Expr\Bigg)
\end{aligned}$$

Therefore,

$$i_L(0) = Expr1 - \frac{1}{\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot Expr\right)}\left(\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right) \quad (19)$$

Let's derive the equation for $v_C(t)$ by substituting (15) in (20),

$$
\begin{aligned}
V_C(s) &= \frac{1}{s+\frac{1}{RC}} \cdot \left(\frac{I_L(s)}{C} + V_C(DT_s) \right) \\
&= \frac{I_L(s)}{C} \frac{1}{s+\frac{1}{RC}} + \frac{V_C(DT_s)}{s+\frac{1}{RC}}
\end{aligned} \tag{20}
$$

Let's simplify each term separately.
Substituting (8) in the first term of (20),

$$
\frac{I_L(s)}{C}\frac{1}{s+\frac{1}{RC}} = \frac{1}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1} \cdot \frac{1}{s+\frac{1}{RC}} \cdot \frac{1}{C}\left(\frac{v_{in}C}{s}\left(s+\frac{1}{RC}\right) + \frac{I_L(DT_s)LC(s+\frac{1}{RC})}{1} - V_C(DT_s)C \right)
$$

taking $\frac{1}{s+\frac{1}{RC}} \cdot \frac{1}{C}$ inside,

$$
= \frac{1}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1} \cdot \left(\frac{v_{in}}{s} + I_L(DT_s)L - \frac{V_C(DT_s)}{s+\frac{1}{RC}} \right) \tag{21}
$$

Lets perform partial fraction on each term of (21) separately,

$$
\begin{aligned}
&\frac{V_{in}}{s\left(LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1\right)} \\
&= \frac{V_{in}}{LC} \cdot \frac{1}{s\left(s^2+\left(\frac{R_{net2}}{L}+\frac{1}{RC}\right)s+\left(\frac{Rnet}{RLC}+\frac{1}{LC}\right)\right)} \\
&= \frac{V_{in}}{LC} \cdot \frac{1}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} \\
&= \frac{V_{in}}{LC} \cdot \frac{1}{s\left((s+\alpha_2)^2+\omega_2^2\right)} \\
\frac{1}{s\left((s+\alpha_2)^2+\omega_2^2\right)} &= \frac{A}{s} + \frac{Bs+C}{(s+\alpha_2)^2+\omega_2^2} \\
1 &= A((s+\alpha_2)^2+\omega_2^2) + Bs^2 + Cs \\
\therefore A &= \frac{1}{\alpha_2^2+\omega_2^2}
\end{aligned}
$$

Substituting A,

$$
\begin{aligned}
1 &= \frac{1}{\alpha_2^2+\omega_2^2}((s+\alpha_2)^2+\omega_2^2) + Bs^2 + Cs \\
1 &= \left(\frac{1}{\alpha_2^2+\omega_2^2}+B\right)s^2 + \left(\frac{2\alpha_2}{\alpha_2^2+\omega_2^2}+C\right)s + 1 \\
\therefore B &= \frac{-1}{\alpha_2^2+\omega_2^2} \\
C &= \frac{-2\alpha_2}{\alpha_2^2+\omega_2^2}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\frac{1}{s\left(s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)\right)} &= \frac{1}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}+\frac{-1}{\alpha_2^2+\omega_2^2}\frac{s}{(s+\alpha_2)^2+\omega_2^2}+\frac{-2\alpha_2}{\alpha_2^2+\omega_2^2}\frac{1}{(s+\alpha_2)^2+\omega_2^2} \\
&= \frac{1}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{1}{\alpha_2^2+\omega_2^2}\cdot\frac{s+\alpha_2-\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{2\alpha_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2} \\
&= \frac{1}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{s}-\frac{1}{\alpha_2^2+\omega_2^2}\left(\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right)-\frac{2\alpha_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}
\end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}
\frac{V_{in}}{s\left(LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1\right)} = \frac{V_{in}}{LC}\Bigg[&\frac{1}{\alpha_2^2+\omega_2^2}\left(\frac{1}{s}-\frac{s+\alpha_2}{(s+\alpha_2)^2+\omega_2^2}+\frac{\alpha_2}{\omega_2}\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\right) \\
&-\frac{2\alpha_2}{\alpha_2^2+\omega_2^2}\cdot\frac{1}{\omega_2}\cdot\frac{\omega_2}{(s+\alpha_2)^2+\omega_2^2}\Bigg]
\end{aligned}$$

Take inverse laplace, we get,

$$\frac{V_{in}}{LC}\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right] \tag{23}$$

Lets simplify the second term of (21),

$$\frac{I_L(DT_s)L}{LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1}$$

Take LC out of the denominator and simplify the denominator like before, we get

$$\frac{I_L(DT_s)}{C}\cdot\frac{1}{(s+\alpha_2)^2+\omega_2^2}$$

take inverse laplace,

$$\frac{I_L(DT_s)}{C\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1) \tag{24}$$

Lets simplify the third term of (21),

$$\frac{V_C(DT_s)}{(s+\frac{1}{RC})\left(LC(s+\frac{1}{RC})(s+\frac{R_{net}}{L})+1\right)}$$

Take LC out of the denominator and simplify the terms to α_2, γ_2, and ω_2, we get,

$$\frac{V_C(DT_s)}{LC}\left\{\frac{1}{(s+\alpha_2+\gamma_2)((s+\alpha_2)^2+\omega_2^2)}\right\}$$

$$\frac{1}{(s+\alpha_2+\gamma_2)((s+\alpha_2)^2+\omega_2^2)} = \frac{A}{s+\alpha_2+\gamma_2}+\frac{Bs+C}{(s+\alpha_2)^2+\omega_2^2}$$

$$1 = A((s+\alpha_2)^2+\omega_2^2)+Bs(s+\alpha_2+\gamma_2)+C(s+\alpha_2+\gamma_2)$$

$$\therefore A = \frac{1}{\gamma_2^2+\alpha_2^2}$$

substituting A,

$$1 = \frac{1}{\gamma_2^2+\alpha_2^2}((s+\alpha_2)^2+\omega_2^2)+Bs(s+\alpha_2+\gamma_2)+C(s+\alpha_2+\gamma_2)$$

rearranging terms,

$$1 = \left(\frac{1}{\gamma_2^2+\alpha_2^2}+B\right)s^2+\left(\frac{2\alpha_2}{\gamma_2^2+\alpha_2^2}+B(\gamma_2+\alpha_2)+C\right)s+\left(\frac{\alpha_2^2+\alpha_2^2}{\gamma_2^2+\alpha_2^2}+C(\alpha_2+\gamma_2)\right)$$

Comparing coefficients,

$$B = \frac{-1}{\gamma_2^2+\alpha_2^2}$$

Substituting B,

$$\frac{2\alpha_2}{\gamma_2^2+\alpha_2^2}+\frac{-1}{\gamma_2^2+\alpha_2^2}(\gamma_2+\alpha_2)+C=0$$

$$C = \frac{-(\alpha_2-\gamma_2)}{\gamma_2^2+\alpha_2^2}$$

Therefore,

$$\frac{V_C(DT_s)}{LC}\left\{\frac{1}{(s+\alpha_2+\gamma_2)((s+\alpha_2)^2+\omega_2^2)}\right\} = \frac{V_C(DT_s)}{LC}\left(\frac{\frac{1}{\gamma_2^2+\alpha_2^2}}{s+\alpha_2+\gamma_2}+\frac{\frac{-1}{\gamma_2^2+\alpha_2^2}s}{(s+\alpha_2)^2+\omega_2^2}+\frac{\frac{-(\alpha_2-\gamma_2)}{\gamma_2^2+\alpha_2^2}}{(s+\alpha_2)^2+\omega_2^2}\right)$$

Taking inverse laplace,

$$\frac{V_C(DT_s)}{LC}\left\{\frac{1}{\gamma_2^2+\alpha_2^2}\cdot e^{-(\alpha_2+\gamma_2)(t-t_1)}-\frac{1}{\gamma_2^2+\alpha_2^2}\cdot\left(e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)-\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)\right.$$
$$\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right\} \tag{25}$$

Combining (23),(24), and (25), we get,

$$\frac{I_L(s)}{C}\frac{1}{s+\frac{1}{RC}} = \frac{V_{in}}{LC}\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)\right.$$
$$\left.-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right]+\frac{I_L(DT_s)}{C\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)$$
$$-\frac{V_C(DT_s)}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(t-t_1)}-e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)\right.$$
$$\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right] \tag{26}$$

Take laplace inverse of the second term of (20).

$$\frac{V_C(DT_s)}{s+\frac{1}{RC}} = V_C(DT_s)e^{-(\alpha_2+\gamma_2)(t-t_1)} \tag{27}$$

Combining (26) and (27), we get,

$$v_C(t) = \frac{V_{in}}{LC}\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)\right.$$
$$\left.-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right]+\frac{I_L(DT_s)}{C\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)$$
$$-\frac{V_C(DT_s)}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(t-t_1)}-e^{-\alpha_2(t-t_1)}\cos\omega_2(t-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right)\right.$$
$$\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}\cdot e^{-\alpha_2(t-t_1)}\sin\omega_2(t-t_1)\right]$$
$$+V_C(DT_s)e^{-(\alpha_2+\gamma_2)(t-t_1)} \tag{28}$$

Now that we have both $i_L(t)$ and $v_C(t)$ in terms of $i_L(0)$ and $v_C(0)$, we can solve them to find out the expressions for $i_L(0)$ and $v_C(0)$.
Since $v_C(t)$ is a continuous function,

$$v_C(T_s^-) = v_C(T_s^+)$$
$$v_C(T_s^-) = v_C(T_s)$$
$$v_C(T_s^+) = v_C(0)$$
$$\therefore v_C(0) = v_C(T_s)$$

$$V_C(0) = \frac{V_{in}}{LC}\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.$$
$$\left.-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+\frac{I_L(DT_s)}{C\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)$$
$$-\frac{V_C(DT_s)}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(T-t_1)}-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.$$
$$\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+V_C(DT_s)e^{-(\alpha_2+\gamma_2)(T-t_1)}$$

Lets substitute $I_L(DT_s)$ and $V_C(DT_s)$,

$$
\begin{aligned}
V_C(0) = \frac{V_{in}}{LC}&\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.\\
&\left.-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+\frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)+I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}}{C\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\\
&-\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(T-t_1)}-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.\\
&\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}\cdot e^{-(\alpha_2+\gamma_2)(T-t_1)}
\end{aligned}
$$

Take $I_L(0)$ to LHS and $V_C(0)$ to RHS,

$$
\begin{aligned}
\frac{-I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}}{C\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1) &= -V_C(0)\\
&+\frac{V_{in}}{LC}\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.\\
&\left.-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+\frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)}{C\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\\
&-\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(T-t_1)}-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.\\
&\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}\cdot e^{-(\alpha_2+\gamma_2)(T-t_1)}
\end{aligned}
$$

Take $V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}$ common,

$$
\begin{aligned}
\frac{-I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}}{C\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1) &= -V_C(0)\\
&+\frac{V_{in}}{LC}\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.\\
&\left.-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]+\frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)}{C\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\\
&+V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}\left\{e^{-(\alpha_2+\gamma_2)(T-t_1)}-\frac{1}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(T-t_1)}-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)\right.\right.\right.\\
&\left.\left.\left.+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]\right\}
\end{aligned}
\tag{29}
$$

Let us shorten these equations by representing some parts with acronyms. These acronyms have no particular meaning:

$$
expr10=\left[\frac{1}{\alpha_2^2+\omega_2^2}\left(1-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)-\frac{2\alpha_2}{\alpha_2^2\omega_2+\omega_2^3}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]
$$

$$
\begin{aligned}
expr11=&\left\{e^{-(\alpha_2+\gamma_2)(T-t_1)}-\frac{1}{LC}\left[\frac{1}{\gamma_2^2+\alpha_2^2}\left(e^{-(\alpha_2+\gamma_2)(T-t_1)}-e^{-\alpha_2(T-t_1)}\cos\omega_2(T-t_1)+\frac{\alpha_2}{\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)\right.\right.\\
&\left.\left.-\frac{\alpha_2-\gamma_2}{\gamma_2^2\omega_2+\alpha_2^2\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right]\right\}
\end{aligned}
$$

Therefore, (29) becomes,

$$\frac{-I_L(0)e^{-(\alpha_1-\gamma_1)(DT_s)}}{C\omega_2}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1) = -V_C(0)+\frac{V_{in}}{LC}\cdot expr10 + \frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)}{C\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)+V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}\cdot expr11$$

$$i_L(0) = \frac{-C\omega_2}{e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)}\left(-V_C(0)+\frac{V_{in}}{LC}\cdot expr10 + \frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)}{C\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)+V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}\cdot expr11\right)$$

separate $V_C(0)$ terms,

$$i_L(0) = V_C(0)\cdot\frac{-C\omega_2}{e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)}\cdot\left(e^{-(\alpha_1+\gamma_1)(DT)}\cdot expr11-1\right) + \frac{-C\omega_2}{e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)}\left(\frac{V_{in}}{LC}\cdot expr10+\frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)}{C\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)$$

Let

$$expr13 = \frac{-C\omega_2}{e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)}\cdot\left(\frac{V_{in}}{LC}\cdot expr10+\frac{\frac{v_{in}}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\right)}{C\omega_2}e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)\right)$$

$$expr14 = \frac{-C\omega_2}{e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot e^{-\alpha_2(T-t_1)}\sin\omega_2(T-t_1)}\cdot\left(e^{-(\alpha_1+\gamma_1)(DT)}\cdot expr11-1\right)$$

Therefore $I_L(0)$ becomes,

$$i_L(0) = V_C(0)\cdot expr14+expr13 \tag{30}$$

Equate (19) and (30),

$$Expr1-\frac{1}{\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot Expr\right)}\left(\frac{V_C(0)e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right) = V_C(0)\cdot expr14+expr13$$

Let

$$expr12 = \frac{1}{\left(1-e^{-(\alpha_1-\gamma_1)(DT_s)}\cdot Expr\right)}\left(\frac{e^{-(\alpha_1+\gamma_1)(DT)}}{L\omega_2}e^{-\alpha_2(T-t_1)}\sin(\omega_2(T-t_1))\right)$$

Therefore,

$$Expr1 - V_C(0)\cdot expr12 = V_C(0)\cdot expr14+expr13$$

Isolate $V_C(0)$,

$$V_C(0) = \frac{Expr1-expr13}{expr14+expr12}$$

We can then substitute $V_C(0)$ in (19) to find $I_L(0)$.

3 Testing

This inductor current equation was coded in matlab and compared with the inductor current equation of the small-ripple approximation method using the following parameters. All the tests used the same parameters:

$$
\begin{aligned}
Vi &= 10; \\
R_L &= 0.5; \\
Rds1on &= 0.4; \\
Rds2on &= 0; \\
Rnet1 &= R_L + Rds1on; \\
Rnet2 &= R_L + Rds2on; \\
R &= 10; \\
L &= 0.0012; \\
C &= 0.033; \\
T &= 1/10000; \\
D &= 0.5; \\
D1 &= 1 - D;
\end{aligned}
$$

3.1 Tests using $I_L(0)$ and $V_C(0)$ equations from small ripple approximation CRADA report

```matlab
V=D1*R*Vi / (R_L+D*Rds1on+D1*Rds2on + D1^2*R);
I = V/(D1*R);

delta_iL = (-D1*Rds1on + D1*Rds2on + D1^2*R)*Vi*D*T / (2*(R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L);

delta_v = D1*Vi*D*T/(2*(R_L+D*Rds1on+D1*Rds2on + D1^2*R)*C);


delta_iL2 = (-D*Rds1on + D*Rds2on + D1*D*R)*Vi*D1*T / (2*(R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L);

delta_v2 = D1*Vi*D*T/(2*(R_L+D*Rds1on+D1*Rds2on + D1^2*R)*C);

% t=linspace(0,D*T);
%
% i_L=((-D1*Rds1on + D1*Rds2on + D1^2*R)*Vi/((R_L+D*Rds1on+D1*Rds2on + D1^2*R)*L))*t + I - delta_iL;


val1 = @(x) ((-D1*Rds1on + D1*Rds2on + D1^2*R)*Vi / ((R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L))*x + I - delta_iL;

val2 = @(x) ((D*Rds1on - D*Rds2on + D1*D*R)*Vi / ((R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L))*(-x+D*T) + I + delta_iL2;


% Define the piecewise function as a regular MATLAB function
piecewise_func = @(x) (x <= D * T) .* val1(x) + (x > D * T) .* val2(x);

% Define time vector for one period
t_single_period = linspace(0, T, 1000); % 1000 points in one period
y_single_period = piecewise_func(t_single_period);

% Repeat the single period values
num_periods = 1;
y6 = repmat(y_single_period, 1, num_periods);
t6 = linspace(0, T * num_periods, length(y6));

% Plot the repeated signal
figure;
plot(t6, y6);
title('Repeated Piecewise Function Plot');
xlabel('Time (s)');
ylabel('Inductor Current (A)');
grid on;
%set(gca,'ytick', 0:1:10);
hold on;
```

Figure 3: The MATLAB code for the small-ripple approximation method.

```matlab
i_L0 = I-delta_iL;
vo_0 = V+delta_v;
t1 = D * T;
alpha1 = ((Rnet1 / L) + (1 / (R * C))) / 2;
gamma1 = ((1 / (R * C)) - (Rnet1 / L)) / 2;
alpha2 = ((Rnet2 / L) + (1 / (R * C))) / 2;
gamma2 = ((1 / (R * C)) - (Rnet2 / L)) / 2;
omega2 = sqrt((1 / (L * C)) * ((Rnet2 / R)+1) - alpha2^2);

t2_end = T - t1;

% Define val1 and val2 expressions
val1 = @(x) i_L0 * exp(-(alpha1 - gamma1) .* x) + (Vi / Rnet1) .* (1 - exp(-(alpha1 - gamma1) .* x));

i_L_DT = (Vi / Rnet1) + exp(-(Rnet1 / L) * D * T) * (i_L0 - (Vi / Rnet1));
v_C_DT = vo_0*exp(-D*T/(C*R));

A = (alpha2 + gamma2) / (alpha2^2 + gamma2^2);
B = (omega2^2 - alpha2^2 - 2 * alpha2 * gamma2) / (alpha2^2 * omega2 + omega2^3);
C = (v_C_DT / (L * omega2));

val2 = @(x) (Vi / L) * (A * (1 - exp(-alpha2 .* (x-t1)) .* cos(omega2 .* (x-t1)) +...
                        (alpha2 / omega2) * exp(-alpha2 .* (x-t1)) .* sin(omega2 .* (x-t1))) - B * exp(-alpha2 .* (x-t1)) .* sin(omega2 .* (x-t1))) ...
           + i_L_DT * (exp(-alpha2 .* (x-t1)) .* cos(omega2 .* (x-t1)) + (gamma2 / omega2) * exp(-alpha2 .* (x-t1)) .* sin(omega2 .* (x-t1))) ...
           - C * exp(-alpha2 .* (x-t1)) .* sin(omega2 .* (x-t1));

% Define the piecewise function as a regular MATLAB function
piecewise_func2 = @(x) (x < D * T) .* val1(x) + (x >= D * T) .* val2(x);

% Define time vector for one period
t_single_period = linspace(0, T, 1000); % 1000 points in one period
y_single_period = piecewise_func2(t_single_period);

% Repeat the single period values
num_periods = 1;
y6 = repmat(y_single_period, 1, num_periods);
t6 = linspace(0, T * num_periods, length(y6));

% Plot the repeated signal
plot(t6, y6);
grid on;
```


Figure 4: The MATLAB code for the Laplace method discussed above.

The plot for one time period using the code was obtained. The blue line is the small-ripple equation and the orange line is the laplace method.

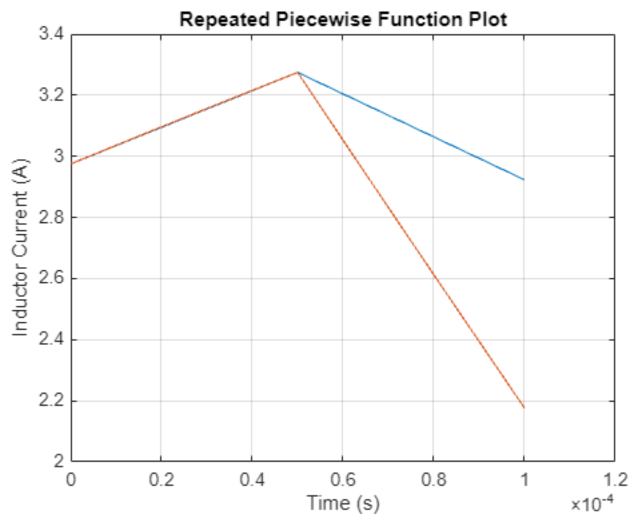


Figure 5: Plot where the initial values (i_L0 and vo_0 in the code) of the steady state intervals are the same for both methods.

When we changed the initial value of the capacitor to 0 (i.e. made vo_0=0 in the code), the slope of the second interval became closer to the slope of the small-ripple equation.

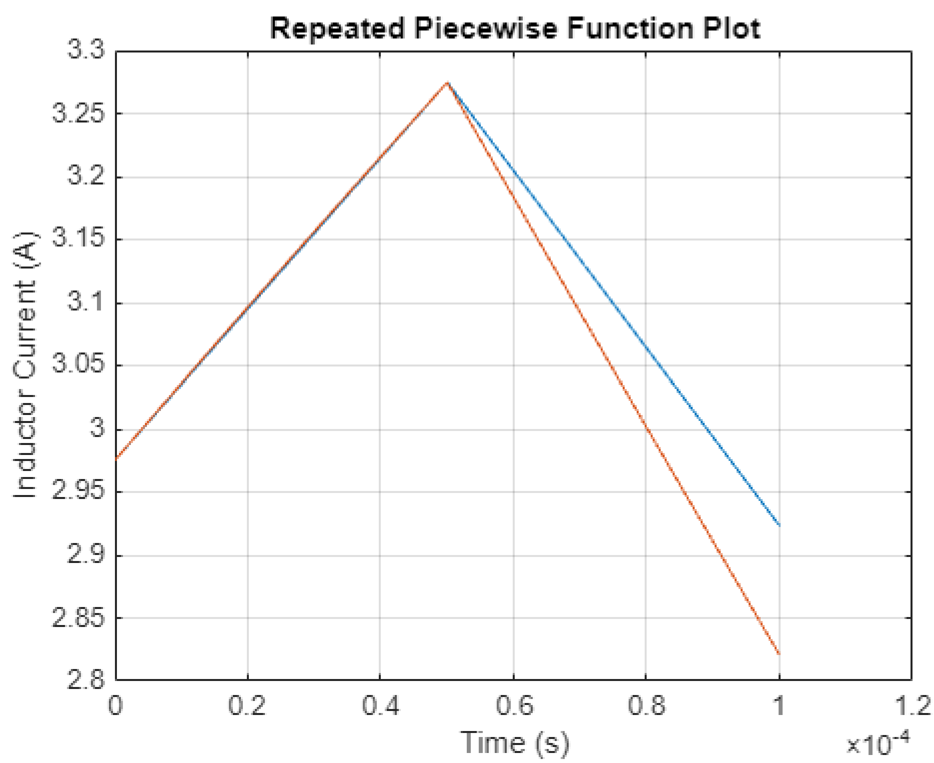


Figure 6: Plot with different initial values

Comment:
From Fig. 5 and 6, we see that the second interval's slope changes when the capacitor voltage's initial value (vo_0) changes. This might be because the initial values that the laplace equations used to plot in matlab were obtained by using the small-ripple approximation. Therefore, the inconsistency arises.
We need to find the initial values of inductor current and capacitor voltage in each interval of the steady state, without using the small-ripple approximation. This could be done by using Laplace transform's initial value theorem and then match boundary conditions between Switch On state and Switch Off state.

3.2 Tests using $I_L(0)$ and $V_C(0)$ derived above

Here we tested the equations we derived using the property of continuous functions for both $i_L(t)$ and $v_C(t)$ (i.e. equation (19) and (30). We did not use the small ripple approximation to derive these equations.

```
t1 = D * T;
alpha1 = ((Rnet1 / L) + (1 / (R * C))) / 2;
gamma1 = ((1 / (R * C)) - (Rnet1 / L)) / 2;
alpha2 = ((Rnet2 / L) + (1 / (R * C))) / 2;
gamma2 = ((1 / (R * C)) - (Rnet2 / L)) / 2;
omega2 = sqrt((1 / (L * C)) * ((Rnet2 / R)+1) - alpha2^2);


A = (alpha2 + gamma2) / (alpha2^2 + gamma2^2);
B = (omega2^2 - alpha2^2 - 2 * alpha2 * gamma2) / (alpha2^2 * omega2 + omega2^3);

Expr = exp(-alpha2 .* (T-t1)) .* cos(omega2 .* (T-t1)) + (gamma2 / omega2) * exp(-alpha2 .* (T-t1)) .* sin(omega2 .* (T-t1));

expr2 = (Vi / L) .* (A .* (1 - exp(-alpha2 .* (T-t1)) .* cos(omega2 .* (T-t1)) +...
        (alpha2 / omega2) * exp(-alpha2 .* (T-t1)) .* sin(omega2 .* (T-t1))) + B * exp(-alpha2 .* (T-t1)) .* sin(omega2 .* (T-t1)));

expr3 = (Vi/Rnet1)*(1-exp(-(alpha1-gamma1)*D*T))*Expr ;

expr10 = (1/(alpha2^2+omega2^2)) * (1 - exp(-alpha2*(T-t1)) * cos(omega2*(T-t1)) +(alpha2/omega2)* exp(-alpha2*(T-t1)) * sin(omega2*(T-t1)) ) ...
    - (2*alpha2/(alpha2^2*omega2 - omega2^3)) * exp(-alpha2*(T-t1)) * sin(omega2*(T-t1));

expr11 = exp(-(alpha2+gamma2)*(T-t1)) - (1/L*C) * ((1/(alpha2^2+omega2^2)) * (exp(-(alpha2+gamma2)*(T-t1)) - exp(-alpha2*(T-t1)) * cos(omega2*(T-t1)) ...
    + (alpha2/omega2)*exp(-alpha2*(T-t1)) * sin(omega2*(T-t1)))...
    - ((alpha2-gamma2)/(alpha2^2*omega2+omega2^3))*exp(-alpha2*(T-t1)) * sin(omega2*(T-t1)));

expr12 = (1/(1-exp(-(alpha1-gamma1)*D*T) .* Expr)) * ((1/(L*omega2)) * exp(-(alpha2+gamma2)*D*T) *  exp(-alpha2*(T-t1)) * sin(omega2*(T-t1)));

expr13 = (-C*omega2/exp(-(alpha1-gamma1)*D*T)) * ((Vi*expr10/(L*C)) ...
    + (1/(L*omega2)) * ((Vi/Rnet1)*(1-exp(-(alpha1-gamma1)*D*T))) * exp(-alpha2*(T-t1)) * sin(omega2*(T-t1)));

expr14 = (-C*omega2/exp(-(alpha1-gamma1)*D*T)) * (exp(-(alpha2+gamma2)*D*T) * expr11 - 1);

Expr1 = (1/(1-exp(-(alpha1-gamma1)*D*T) .* Expr)) .* (expr2 + expr3) ;

vo_0 = (Expr1 - expr13)/(expr14 + expr12);



v_C_DT = vo_0*exp(-D*T/(C*R));

C1 = (v_C_DT / (L * omega2));

expr4 = C1 * exp(-alpha2 .* (T-t1)) .* sin(omega2 .* (T-t1));

i_L0 = (1/(1-exp(-(alpha1-gamma1)*D*T) .* Expr)) .* (expr2 + expr3 - expr4) ;


i_L_DT = (Vi / Rnet1) + exp(-(Rnet1 / L) * D * T) * (i_L0 - (Vi / Rnet1));
```

Figure 7: The MATLAB code for $I_L(0)$ and $V_C(0)$

The plot for one time period using the code was obtained. The blue line is the small-ripple equation and the orange line is the laplace method.

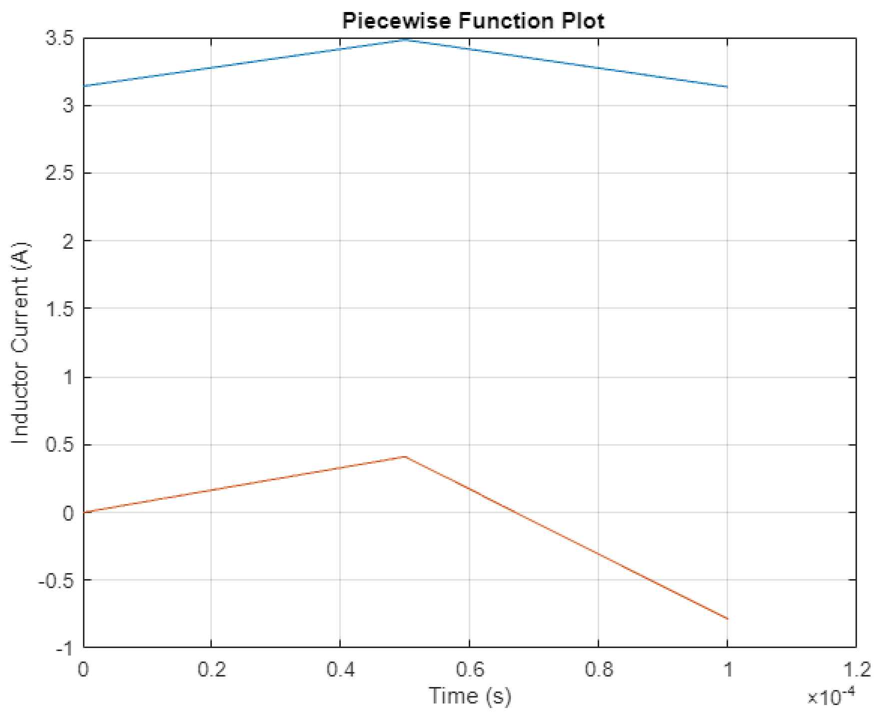


Figure 8: Small-ripple plot and Laplace plot with different equations for initial values

Comment:
As you can see, the two plots are different from each other. This could be due to many reasons:
1) Our assumption of continuous functions is incorrect or is lacking something.
2) There could be an error in the derivations that was not found or an error in the Matlab code.
3) The approach to use Laplace transform on two separate states (i.e. when switch 1 is on and when switch 2 is on) is incorrect.

4 Next steps

Look for a new approach to solve converters without using small ripple approximation.

Some papers that I found that used Laplace Transform are:

1) **“”A Switch Mode Power Supply Simulation Technique Based in Laplace Transform””** Acácio M. R. Amaral, A. J. Marques Cardoso.

2) **“A Laplace transform approach to the simulation of DC-DC converters”** Grasso, Francesco ; Manetti, Stefano ; Piccirilli, Maria Cristina ; Reatti, Alberto International journal of numerical modelling, 2019-09, Vol.32 (5), p.n/a

It is based on the use of the laplace transform and of dedicated models for controlled switches and diodes. Time-domain waveforms are obtained from network functions representing the circuit under analysis without assuming small-signal approximation.

Using the method of Laplace transform applied to DC-DC converters as presented in the previous section, the authors have developed a new symbolic program named SapWin for power electronics (SapWinPE). It is part of the integrated package Symbolic Analysis Program for Windows (SapWin), one of the existing symbolic programs for circuit analysis, also developed by the authors.

The theoretical waveforms of the inductor voltage, v_L, inductor current, i_L, diode current, i_D, and capacitor voltage, v, are sketched in Figure 2 in the paper. Equations (1) to (5) in the paper, obtained starting from some properties in time domain, show the time-variant nature of the circuit because of the presence of the switching cell. By modeling the switching cell so as to account for the state of the switches via parameter settings, it is possible to use the Laplace transform and to obtain the network functions characterizing the circuit. They depend on the capacitor and inductor initial conditions and on the parameters of the switching cell model. The circuit working conditions determine the values of the cell parameters,which, consequently, are variable. This means that the time-domain responses cannot be derived by using a "once-for-all" inverse Laplace transform of the corresponding network functions but it is necessary to take into account the different switching intervals. The switching cell model developed by the authors is reported in the paper in Figure 3.

This paper is very detailed and has our same objective using a boost converter. From what I read and understood, the paper seems credible. In their results, the plots obtained using laplace transform were almost similar to the plot obtained by small ripple approximation, and thereby, validating the approximation to be accurate for real-world applications.

NOTE: The method below was a trial using published references as a guide. However, this method hasn't been fully understood. There were errors in the published papers that were discovered later on in the derivation process, which damages the published papers' credibility as well as the method's credibility. A key error that was discovered is when $I_L(s)$ and $V_C(s)$ was converted to the constants $i_{Ll,p}$ and $v_{ol,p}$ in (22). An explanation could not be found for this change. Therefore, we discontinued this method. The errors are discussed in detail at the end of this method.

Deriving Accurate Expressions for Inductor Current and Capacitor Voltage

John Abraham

June 2024

A Goal

To derive the expressions for a boost converter using Laplace Transform.

B Approach

It is possible to solve converters exactly, without the use of the small-ripple approximation. We consider a boost converter to demonstrate this. First, we find the Laplace transform to write expressions for the waveforms of the circuit. We then invert the transforms, match boundary conditions, and find the periodic steady-state solution of the circuit.

C A MATHEMATICAL MODELING METHOD[1]

The circuit of a boost dc-dc converter is shown in Fig. 1. In this investigation, the switch S and the diode D are not ideal and have R_{ds1on} and R_{ds2on} to enhance accuracy respectively.

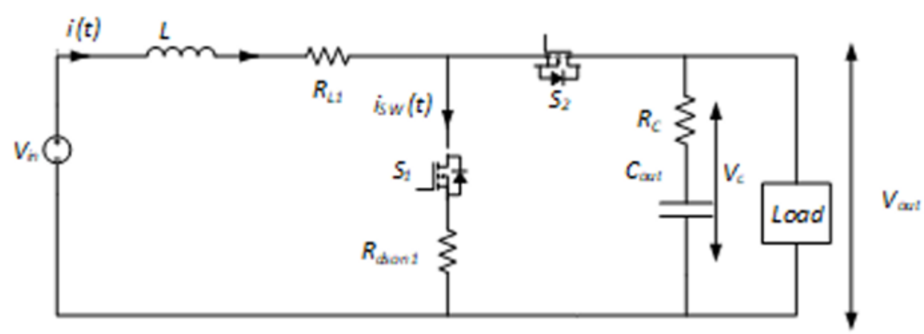
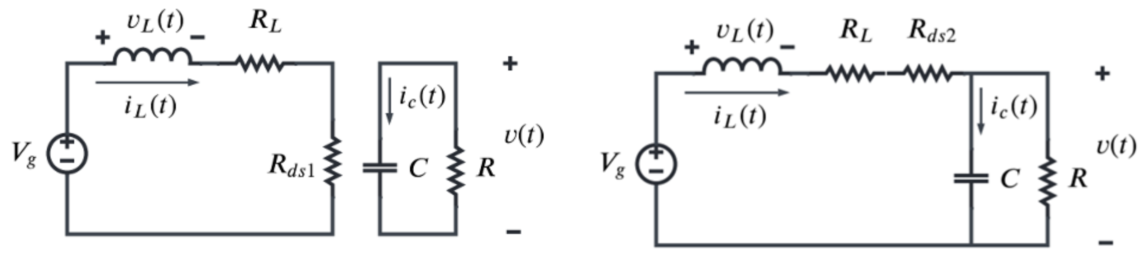


Figure 1: Boost Converter



(a) Switch 1 ON (b) Switch 1 OFF

Figure 2: The two states of the boost converter

A. Analysis of Converter in CCM

In the case when Switch 1 is ON, thus with Switch 2 being OFF,

$$v_g(t) = L\frac{di_L(t)}{dt} + (R_L + R_{ds1on})i_L(t)$$
$$C\frac{dv_C(t)}{dt} = -\frac{v_C(t)}{R}$$

In the case when Switch 2 is ON, thus with Switch 1 being OFF,

$$v_g(t) = L\frac{di_L(t)}{dt} + (R_L + R_{ds1on})i_L(t) + v_C(t)$$
$$C\frac{dv_C(t)}{dt} = i_L(t) - \frac{v_C(t)}{R}$$

There are two fundamentally different operating modes for the converter [3]. The first, continuous-conduction mode (CCM), is where energy in the inductor flows continuously during the operation of the converter. The increase of stored energy in the inductor during the ON time of the switch is equal to the energy discharged into the output during the OFF time of the switch, ensuring steady-state operation. At the end of the discharge interval, residual energy remains in the inductor. During the next ON interval of the switch, energy builds from that residual level to that required by the load for the next switching cycle.

For investigating the stability of the boost dc-dc converter in the continuous conduction mode (CCM), we need one equation that represents the equations of both state 1 and state 2 of the boost converter. Below is the mathematical modeling method that was proposed in [1] and [2]. Let $v_g(t)$ be V_i and $v_c(t)$ be v_o.

$$L\frac{di_L}{dt} + R_{net}i_L = V_i - f(t)v_o \tag{1}$$
$$C\frac{dv_o}{dt} + \frac{v_o}{R} = f(t)i_L \tag{2}$$

where $R_{net} = R_L + R_{dson}$.

(1) and (2) are the general equations of the voltage and the current of the boost dc-dc converter. In these equations, $f(t)$ is the switching function which is implemented for modeling the switch S. By specifying the value of the function $f(t)$ and by applying the value of this function in (1) and (2), the related equations to each of the switching intervals is obtained. The function $f(t)$ is defined as follows for determining the converter equations in time intervals t_1 (the switch S on) and t_2 (the switch S off).

$$f(t) = \sum_{p=0}^{\infty} \left[u\left(t - t_1 - pT\right) - u(t - T - pT)\right] \tag{3}$$

In (3) T is the switching period and p stands for the p^{th} switching interval. Fig. 3 shows the f(t) waveform and as can be seen in (0, t1) time interval, it is equal to zero, so in this condition the power switch will be deactivated and is open circuit in boost converter structure and for (t1,T) the switch is active and short-circuit. So Eq. (1) and (2) can be written in two different modes according to switch conditions. So f(t) is the total of difference between the two-step functions for p switching intervals.

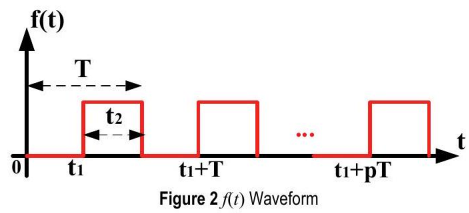


Figure 3: f(t) waveform (Ghaderi et al.)

In order to be able to make a mathematical model and do the converter analysis in the discrete time domain, we can apply a variable change by this equation:

$$t = (p+q)T \quad \text{for} \quad p = 0, 1, 2, \cdots \quad 0 \leq q < 1 \tag{4}$$

By replacing variable above, $f(t)$ converts to p discrete functions with q variable and we will have a single time system. By replacing this variable change in Eq. (3), $f(t)$ can be written as below:

$$\begin{aligned} f(t) &= \sum_{p=0}^{\infty} \left[u\left(pT + qT - t_1 - pT\right) - u(pT + qT - T - pT)\right] = \\ &= \sum_{p=0}^{\infty} \left[u\left(qT - t_1\right) - u(qT - T)\right] = pf(q) \end{aligned} \tag{5}$$

In Eq. (5), $f(q)$ will be equal to:

$$f(q) = u\left(qT - t_1\right) - u(qT - T)$$

,

Also, $u\left(qT - t_1\right)$ and $u(qT - T)$ can be calculated as below:

$$u\left(qT - t_1\right) = \begin{cases} 1 & qT - t_1 \geq 0 \text{ if } q \geq \frac{t_1}{T} \\ 0 & qT - t_1 < 0 \text{ if } q < \frac{t_1}{T} \end{cases}$$

$$u(qT - T) = \begin{cases} 1 & qT - T \geq 0 \text{ if } q \geq 1 \\ 0 & qT - T < 0 \text{ if } q < 1 \end{cases}$$

In these equations D is the converter's duty cycle and can be written as:

$$D = \frac{t_1}{T}$$

By applying this value to $u(qT - t_1)$, we can rewrite it as:

$$f(q) = \begin{cases} 0 & \text{for } 0 \leq q < D \\ 1 & \text{for } D \leq q < 1 \end{cases} \tag{6}$$

Considering (6), it is observed that the interval $[0, 1]$ is classified in two intervals $[0, D]$ and $[D, 1]$. During $[0, D]$, the switch S is on and during $[D, 1]$, the switch S is off

Applying (4) in (1) and (2), and applying the value of $f(q)$ in the obtained equations, we get,

Therefore, when the switch S is ON from [0,D] and f(q)=0 in [0,D], (1) and (2) become :

$$\frac{di_{L,p}}{dq} = \frac{T}{L}V_i - \frac{TR_{net1}}{L}i_{L,p} \tag{7}$$

$$\frac{dv_{o,p}}{dq} = -\frac{T}{RC}v_o, p \tag{8}$$

where $R_{net1} = R_L + R_{ds1on}$ and $dq = Tdt$ from differentiating (4) and $i_{L,p}$ and $v_{o,p}$ represents the inductor current and capacitor voltage in the p^{th} interval.
Similarly, when the switch is OFF from [D,1] and f(q)=1 in [D,1], :

$$\frac{di_{L,p}}{dq} = \frac{T}{L}V_i - \frac{TR_{net2}}{L}i_{L,p} - \frac{T}{L}v_{o,p} \tag{9}$$

$$\frac{dv_{o,p}}{dq} = \frac{T}{C}i_{L,p} - \frac{T}{RC}v_{o,p} \tag{10}$$

These can be written in a matrix form as shown below:

$$\begin{bmatrix} \frac{di_{L,p}}{dq} \\ \frac{dv_{o,p}}{dq} \end{bmatrix} = \begin{bmatrix} -\frac{R_{net1}T}{L} & 0 \\ 0 & -\frac{T}{RC} \end{bmatrix} \begin{bmatrix} i_{L,p} \\ v_{o,p} \end{bmatrix} + \begin{bmatrix} \frac{T}{L} \\ 0 \end{bmatrix} V_i \quad 0 \leq q < D \tag{11}$$

$$i_{L,p}(0) = i_{L0,p}, v_{o,p}(0) = v_{o0,p}$$

$$\begin{bmatrix} \frac{di_{L,p}}{dq} \\ \frac{dv_{o,p}}{dq} \end{bmatrix} = \begin{bmatrix} -\frac{R_{net2}T}{L} & -\frac{T}{L} \\ \frac{T}{C} & -\frac{T}{RC} \end{bmatrix} \begin{bmatrix} i_{L,p} \\ v_{o,p} \end{bmatrix} + \begin{bmatrix} \frac{T}{L} \\ 0 \end{bmatrix} V_i, \quad D \leq q < 1 \tag{12}$$

$$i_{L,p}(D) = i_{L1,p}, v_{o,p}(D) = v_{ol,p}$$

where $R_{net2} = R_L + R_{ds2on}$.

Considering (11) and (12), $i_{L0,p}, v_{o0,p}, i_{L1,p}$ and $v_{o1,p}$ are the initial values of the inductor current and the output voltage in the intervals $[0, D]$ and $[D, 1]$, respectively (to be used in the laplace transform).
As it is observed, the initial values of the output voltage and the inductor current are the functions of p. So the values of each of these parameters will be different in each switching interval.

D Obtaining the Input Current and Capacitor Voltage equations using Laplace Transform

Laplace Transform on (7) gives us:

$$\left(s + \frac{TR_{net1}}{L}\right) I_{L,p}(s) = i_{L0,p} + \frac{T}{Ls}V_i \text{ if } 0 \leq q < D \tag{17}$$

$$I_{L,p}(s) = \frac{1}{s + \frac{TR_{net1}}{L}} \left(i_{L1,p} + \frac{T}{Ls}V_i\right)$$

Laplace Transform on (8) gives us:

$$\left(s+\frac{T}{RC}\right)V_{o,p}(s)=v_{o0,p} \text{ if } 0\le q<D \tag{18}$$
$$V_{o,p}=\frac{1}{s+\frac{T}{RC}}v_{o0,p}$$

Laplace Transform on (9) gives us:
I think here Vo,p(s) is assumed as a constant vo1,p because they used vo1,p in the matrix form.

$$sI_{L,p}(s)-I_{L,p}(D)=\frac{T}{Ls}V_i-\frac{TR_{net2}}{L}I_{L,p}(s)-\frac{T}{L}V_{o,p}(s), i_{L,p}(D)=i_{L1,p}, v_{o,p}(D)=v_{o1,p}$$
$$\left(s+\frac{TR_{net2}}{L}\right)I_{L,p}(s)=i_{L1,p}+\frac{T}{Ls}V_i-\frac{T}{L}V_{o,p}(s), D\le q<1 \tag{19}$$
$$I_{L,p}(s)=\frac{1}{s+\frac{TR_{net2}}{L}}\left(i_{L1,p}+\frac{T}{Ls}V_i-\frac{T}{L}V_{o,p}(s)\right)$$

Laplace Transform on (10) gives us:

$$\left(s+\frac{T}{RC}\right)V_{o,p}(s)=v_{o1,p}+\frac{T}{C}I_{L,p} \text{ if } D\le q<1 \tag{20}$$
$$V_{o,p}=\frac{1}{s+\frac{T}{RC}}\left(v_{o1,p}+\frac{T}{C}I_{L,p}\right)$$

By sorting Eq. (17) to Eq. (20), in the form of a matrix, the relation between input current and capacitor voltage in the Laplace domain can be written as:

if $0\le q<D$,

$$\begin{bmatrix} I_{L,p}(s) \\ V_{o,p}(s) \end{bmatrix}=\begin{bmatrix} \frac{1}{s+(\alpha-\gamma)T}\left(i_{L0,p}+\frac{T}{L}\frac{V_i}{s}\right) \\ \frac{v_{o0,p}}{s+(\alpha+\gamma)T} \end{bmatrix} \tag{21}$$

if $D\le q<1$,

$$\begin{bmatrix} I_{L,p}(s) \\ V_{o,p}(s) \end{bmatrix}=\frac{1}{s^2+2\alpha Ts+T^2\left(\alpha^2+\omega^2\right)}\begin{bmatrix} (\alpha+\gamma)T+s & (\alpha+\gamma)T+s & -\frac{T}{L} \\ \frac{T}{C} & 0 & s+T(\alpha-\gamma) \end{bmatrix}\begin{bmatrix} i_{L1,p} \\ \frac{TV_i}{Ls} \\ v_{ol,p} \end{bmatrix} \tag{22}$$

In these equations, α,ω and γ can be calculated as shown below:
$\alpha_1=\frac{\left(\frac{R_{net1}}{L}+\frac{1}{RC}\right)}{2}, \omega_1=\sqrt{\frac{1}{LC}\left(\frac{R_{net1}}{R}\right)-\alpha_1^2}, \gamma_1=\frac{\left(\frac{1}{RC}-\frac{R_{net1}}{L}\right)}{2}$,
$\alpha_2=\frac{\left(\frac{R_{net2}}{L}+\frac{1}{RC}\right)}{2}, \omega_2=\sqrt{\frac{1}{LC}\left(\frac{R_{net2}}{R}\right)-\alpha_2^2}, \gamma_2=\frac{\left(\frac{1}{RC}-\frac{R_{net2}}{L}\right)}{2}$,
Let us perform Inverse Laplace Transform on current equation of (21):

$$\mathcal{L}^{-1}\left\{\frac{1}{s+(\alpha_1-\gamma_1)T}\left(i_{L0,p}+\frac{T}{L}\frac{V_i}{s}\right)\right\}$$

first term:

$$i_{L0,p}e^{-(\alpha_1-\gamma_1)qT}$$

second term:

$$\begin{aligned}
\frac{TV_i}{Ls(s+(\alpha_1-\gamma_1)T)} &= \frac{A}{Ls}+\frac{B}{s+\frac{R_{net1}}{L}T}\\
&= \frac{\frac{V_i}{R_{net1}}L}{Ls}+\frac{\frac{-V_i}{R_{net1}}}{s+\frac{R_{net1}}{L}T}\\
&= \frac{V_i}{R_{net1}}\left(\frac{1}{s}-\frac{1}{s+\frac{R_{net1}}{L}T}\right)\\
&= \frac{V_i}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)qT}\right)
\end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+(\alpha_1-\gamma_1)T}\left(i_{L0,p}+\frac{T}{L}\frac{V_i}{s}\right)\right\}=i_{L0,p}e^{-(\alpha_1-\gamma_1)qT}+\frac{V_i}{R_{net1}}\left(1-e^{-(\alpha_1-\gamma_1)qT}\right) \tag{23}$$

Similarly, inverse laplace transform of voltage equation of (21),

$$\mathcal{L}^{-1}\left\{\frac{v_{o0,p}}{s+(\alpha_1+\gamma_1)T}\right\}=v_{o0,p}e^{-(\alpha_2+\gamma_2)qT} \tag{24}$$

Let us perform Inverse Laplace Transform on current equation of (22):

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{s^2+2\alpha_2Ts+T^2\left(\alpha_2^2+\omega_2^2\right)}\left[\left((\alpha_2+\gamma_2)T+s)(i_{L1,p}+\frac{TV_i}{Ls}\right)-\frac{T}{L}v_{o1,p}\right]\right\}\\
&\frac{((\alpha_2+\gamma_2)T+s)i_{L1,p}+((\alpha_2+\gamma_2)T+s)\frac{TV_i}{Ls}-\frac{T}{L}v_{o1,p}}{s^2+2\alpha_2Ts+T^2\left(\alpha_2^2+\omega_2^2\right)}
\end{aligned} \tag{25}$$

Laplace transform of first term of (25):

$$\begin{aligned}
&\frac{((\alpha_2+\gamma_2)T+s)i_{L1,p}}{s^2+2\alpha_2Ts+T^2\left(\alpha_2^2+\omega_2^2\right)}\\
&\left(\frac{((\alpha_2+\gamma_2)T+s)}{(s+T\alpha_2)^2+T^2\omega_2^2}\right)i_{L1,p}\\
&\left(\frac{((s+T\alpha_2)+T\gamma_2}{(s+T\alpha_2)^2+T^2\omega_2^2}\right)i_{L1,p}\\
&\left(\frac{(s+T\alpha_2)}{(s+T\alpha_2)^2+T^2\omega_2^2}+\frac{\gamma_2}{\omega_2}\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2}\right)i_{L1,p}
\end{aligned}$$

$$i_{L1,p}e^{-\alpha_2(qT-t_1)}\left(\cos\omega_2(qT-t1)+\frac{\gamma_2}{\omega_2}\sin\omega_2(qT-t_1)\right) \tag{26}$$

Laplace transform of second term of (25):

$$\frac{((\alpha_2+\gamma_2)T+s)\frac{TV_i}{Ls}}{s^2+2\alpha_2 Ts+T^2\left(\alpha_2^2+\omega_2^2\right)}$$
$$\left(\frac{s+\alpha_2 T}{(s+T\alpha_2)^2+T^2\omega_2^2}\right)\frac{TV_i}{Ls}+\frac{\frac{T^2\gamma_2 V_i}{Ls}}{(s+T\alpha_2)^2+T^2\omega_2^2} \tag{27}$$

partial fraction of first term of (27):

$$\left(\frac{s+\alpha_2 T}{(s+T\alpha_2)^2+T^2\omega_2^2}\right)\frac{TV_i}{Ls}=\frac{A}{Ls}+\frac{Bs+C}{(s+T\alpha_2)^2+T^2\omega_2^2}$$
$$=\frac{\alpha_2 V_i}{L(\alpha_2+\omega_2^2)}\frac{1}{s}+\frac{-\alpha_2 V_i}{L(\alpha_2^2+\omega_2^2)}\frac{s}{(s+T\alpha_2)^2+T^2\omega_2^2}+\frac{TV_i}{T\omega_2 L}\left(1-\frac{2\alpha_2^2}{\alpha_2^2+\omega_2^2}\right)\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2} \tag{28}$$

partial fraction of second term of (27):

$$\frac{\frac{T^2\gamma_2 V_i}{Ls}}{(s+T\alpha_2)^2+T^2\omega_2^2}=\frac{A}{Ls}+\frac{Bs+C}{(s+T\alpha_2)^2+T^2\omega_2^2}$$
$$=\frac{\gamma_2 V_i}{L(\alpha_2^2+\omega_2^2)}\frac{1}{s}+\frac{-\gamma_2 V_i}{L(\alpha_2^2+\omega_2^2)}\frac{s}{(s+T\alpha)^2+T^2\omega_2^2}+\frac{-2\alpha_2\gamma_2 TV_i}{T\omega_2 L(\alpha_2^2+\omega_2^2)}\frac{T\omega_2}{(s+T\alpha)^2+T^2\omega_2^2} \tag{29}$$

Adding (28) and (29),

$$\frac{((\alpha_2+\gamma_2)T+s)\frac{TV_i}{Ls}}{s^2+2\alpha_2 Ts+T^2\left(\alpha_2^2+\omega_2^2\right)}=\frac{\alpha_2 V_i}{L(\alpha_2+\omega_2^2)}\frac{1}{s}+\frac{-\alpha_2 V_i}{L(\alpha_2^2+\omega_2^2)}\frac{s}{(s+T\alpha_2)^2+T^2\omega_2^2}+\frac{V_i}{\omega_2 L}\left(1-\frac{2\alpha_2^2}{\alpha_2^2+\omega_2^2}\right)\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2}$$
$$+\frac{\gamma_2 V_i}{L(\alpha_2^2+\omega_2^2)}\frac{1}{s}+\frac{-\gamma_2 V_i}{L(\alpha_2^2+\omega_2^2)}\frac{s}{(s+T\alpha_2)^2+T^2\omega_2^2}+\frac{-2\alpha_2\gamma_2 V_i}{\omega_2 L(\alpha_2^2+\omega_2^2)}\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2}$$

Taking common terms,

$$=\frac{(\alpha_2+\gamma_2)V_i}{L(\alpha_2+\omega_2^2)}\frac{1}{s}-\frac{(\alpha_2+\gamma_2)V_i}{L(\alpha_2+\omega_2^2)}\frac{s}{(s+T\alpha_2)^2+T^2\omega_2^2}+\left(\frac{-2\alpha_2\gamma_2 V_i+V_i(\omega_2^2-\alpha_2^2)}{\omega L(\alpha_2^2+\omega_2^2)}\right)\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2} \tag{30}$$

Lets perform inverse laplace transform on each term of (30) separately.
First term:

$$\frac{(\alpha_2+\gamma_2)V_i}{L(\alpha_2^2+\omega_2^2)}=\frac{V_i}{R_{net2}+R}$$
$$\mathcal{L}^{-1}\left\{\frac{(\alpha_2+\gamma_2)V_i}{L(\alpha_2^2+\omega_2^2)}\frac{1}{s}\right\}=\frac{V_i}{R_{net2}+R}\cdot 1 \tag{31}$$

Second term:

$$\frac{(\alpha_2+\gamma_2)V_i}{L(\alpha_2^2+\omega_2^2)}\cdot\frac{s}{(s+T\alpha_2)^2+T^2\omega_2^2}$$
$$\frac{V_i}{R_{net2}+R}\cdot\frac{s+T\alpha_2-T\alpha_2}{(s+T\alpha_2)^2+T^2\omega_2^2}$$
$$\frac{V_i}{R_{net2}+R}\left(\frac{s+T\alpha_2}{(s+T\alpha_2)^2+T^2\omega_2^2}-\frac{T\alpha_2}{(s+T\alpha_2)^2+T^2\omega_2^2}\cdot\frac{T\omega_2}{T\omega_2}\right)$$
$$\frac{V_i}{R_{net2}+R}e^{-\alpha_2(qT-t_1)}\left(\cos\omega_2(qT-t1)+\frac{\alpha_2}{\omega_2}\sin\omega_2(qT-t1)\right) \tag{32}$$

Third term:
Lets simplify,

$$\begin{aligned}
\frac{-2\alpha_2\gamma_2 V_i + V_i(\omega_2^2-\alpha_2^2)}{\omega L(\alpha_2^2+\omega_2^2)} &= \frac{V_i}{L}\left(\frac{\alpha_2^2-\alpha_2\gamma_2-\alpha_2\gamma_2-\alpha_2^2}{\omega_2(\alpha_2^2+\omega_2^2)}\right)\\
&= \frac{V_i}{L}\left(\frac{\omega_2^2-\alpha_2\gamma_2-\alpha_2(\gamma_2+\alpha_2)}{\omega_2(\alpha_2^2+\omega_2^2)}\right)\\
&= \frac{V_i}{L}\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2^2+\omega_2^2)}-\frac{\alpha_2(\gamma_2+\alpha_2)}{\omega_2(\alpha_2^2+\omega_2^2)}\right)\\
&= \frac{V_i(\gamma_2+\alpha_2)}{L(\alpha_2^2+\omega_2^2)}\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}-\frac{\alpha_2}{\gamma_2}\right)
\end{aligned}$$

Therefore,

$$\left(\frac{-2\alpha_2\gamma_2 V_i + V_i(\omega_2^2-\alpha_2^2)}{\omega_2 L(\alpha_2^2+\omega_2^2)}\right)\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2} =$$
$$\frac{V_i(\gamma_2+\alpha_2)}{L(\alpha_2^2+\omega_2^2)}\left[\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}\right)\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2}-\frac{\alpha_2}{\omega_2}\frac{T\omega_2}{(s+T\alpha_2)^2+T^2\omega_2^2}\right]$$

$$= \frac{V_i}{R_{net2}+R}\left(\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}\right)e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1)-\frac{\alpha_2}{\omega_2}\sin\omega_2(qT-t1)\right) \tag{33}$$

Substituting (31),(32), and (33) in (30), we get

$$\frac{V_i}{R_{net2}+R}\left[1-e^{-\alpha_2(qT-t_1)}\cos\omega_2(qT-t1)+\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}\right)e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1)\right] \tag{34}$$

Inverse Laplace transform of third term of (25):

$$\begin{aligned}
&-\frac{\frac{T}{L}v_{o1,p}}{s^2+2\alpha Ts+T^2(\alpha^2+\omega^2)}\cdot\frac{\omega_2}{\omega_2}\\
&-\frac{v_{o1,p}}{L\omega_2}e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1)
\end{aligned} \tag{35}$$

Combining (23),(26),(34), and (35), we get

$$i_{L,p}(q) = \begin{cases} i_{L0,p}e^{-(\alpha_1-\gamma_1)qT+\frac{V_i}{R_{net2}}\left(1-e^{-(\alpha-\gamma)qT}\right)} \quad \text{if} \quad 0\le q<D,\\ \frac{V_i}{R_{net2}+R}\left[1-e^{-\alpha_2(qT-t_1)}\cos\omega_2(qT-t1)+\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}\right)e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1)\right]\\ +i_{L1,p}e^{-\alpha_2(qT-t_1)}\left(\cos\omega_2(qT-t1)+\frac{\gamma_2}{\omega_2}\sin\omega_2(qT-t_1)\right)\\ -\frac{v_{o1,p}}{L\omega_2}e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1) \quad \text{if} \quad D\le q<1 \end{cases} \tag{36}$$

Similarly, the inverse laplace transform of $V_{o0,p}(s)$ is

$$v_{o,p}(q) = \begin{cases} v_{o0,p}e^{-(\alpha_1+\gamma_1)qT} \quad \text{if} \quad 0\le q<D,\\ \frac{i_{L1,p}}{\omega C}e^{-\alpha(qT-t_1)}\sin\omega\left(qT-t_1\right)\\ +v_{o1,p}e^{\alpha(t_1-qT)}\begin{bmatrix}\cos\omega\left(qT-t_1\right)+\\ \frac{\gamma}{\omega}\sin\omega\left(t_1-qT\right)\end{bmatrix} \text{ if } D\le q<1, \end{cases} \tag{37}$$

Considering which the functions $i_L(t)$ and $v_o(t)$ are continuous, we have:

$$\lim_{q\to(D)^-} i_{L,p}(q) = \lim_{q\to(D)^+} i_{L,p}(q)$$

Therefore, $i_{L1,p}$ can be obtained in terms of $i_{L0,p}$ in the following way:

$$i_{L1,p} = i_{L0,p}e^{-(\alpha_1-\gamma_1)t_1} + \frac{V_i}{R_{net1}}\left[1-e^{-(\alpha_1-\gamma_1)t_1}\right], \tag{38}$$

For output voltage we can also write:

$$\lim_{q\to(D)^-} v_{o,p}(q) = \lim_{q\to(D)^+} v_{o,p}(q)$$

And, $v_{o1,p}$ based on $v_{o0,p}$ can be calculated as:

$$v_{o1,p} = v_{o0,p}e^{-(\alpha_1-\gamma_1)t_1} \tag{39}$$

Substituting (38) and (39) in (36) and (37),

$$i_{L,p}(q) = \begin{cases} i_{L0,p}e^{-(\alpha_1-\gamma_1)qT+\frac{V_i}{R_{net2}}\left(1-e^{-(\alpha_1-\gamma_1)qT}\right)} \quad \text{if} \quad 0 \le q < D, \\ \frac{V_i}{R_{net2}+R}\left[1-e^{-\alpha_2(qT-t_1)}\cos\omega_2(qT-t1)+\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}\right)e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1)\right] \\ +\left(i_{L0,p}e^{-(\alpha_1-\gamma_1)t_1}+\frac{V_i}{R_{net1}}\left[1-e^{-(\alpha_1-\gamma_1)t_1}\right]\right)e^{-\alpha_2(qT-t_1)}\left(\cos\omega_2(qT-t1)+\frac{\gamma_2}{\omega_2}\sin\omega_2(qT-t_1)\right) \\ -\frac{v_{o0,p}e^{-(\alpha_1-\gamma_1)t_1}}{L\omega_2}e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1) \quad \text{if} \quad D \le q < 1 \end{cases}$$

$$v_{o,p}(q) = \begin{cases} v_{o0,p}e^{-(\alpha_1+\gamma_1)qT} \quad \text{if} \quad 0 \le q < D, \\ \frac{\left(i_{L0,p}e^{-(\alpha_1-\gamma_1)t_1}+\frac{V_i}{R_{net1}}\left[1-e^{-(\alpha_1-\gamma_1)t_1}\right]\right)}{\omega C}e^{-\alpha(qT-t_1)}\sin\omega\left(qT-t_1\right) \\ +\left(v_{o0,p}e^{-(\alpha_1-\gamma_1)t_1}\right)e^{\alpha(t_1-qT)}\begin{bmatrix}\cos\omega\left(qT-t_1\right)+ \\ \frac{\gamma}{\omega}\sin\omega\left(t_1-qT\right)\end{bmatrix} \text{ if } D \le q < 1, \end{cases}$$

Simplifying,

$$i_{L,p}(q) = \begin{cases} i_{L0,p}e^{-(\alpha_1-\gamma_1)qT+\frac{V_i}{R_{net2}}\left(1-e^{-(\alpha_1-\gamma_1)qT}\right)} \quad \text{if} \quad 0 \le q < D, \\ \frac{V_i}{R_{net2}+R}\left[1-e^{-\alpha_2(qT-t_1)}\cos\omega_2(qT-t1)+\left(\frac{\omega_2^2-\alpha_2\gamma_2}{\omega_2(\alpha_2+\gamma_2)}\right)e^{-\alpha_2(qT-t_1)}\sin\omega_2(qT-t_1)\right] \\ +\left[\left(i_{L0,p}-\frac{V_i}{R_{net1}}\right)e^{\gamma_1t_1-\alpha_2qT+\beta t_1}+\frac{V_i}{R_{net1}}e^{-\alpha_2(qT-t_1)}\right]\times\left(\cos\omega_2(t_1-qT)+\frac{\gamma_2}{\omega_2}\sin\omega_2(qT-t_1)\right) \\ -\frac{1}{L\omega_2}v_{o0,p}e^{(\gamma_1t_1-\alpha_2qT+\beta t_1)}\sin\omega_2(qT-t_1) \quad \text{if} \quad D \le q < 1 \end{cases} \tag{40}$$

$$v_{o,p}(q) = \begin{cases} v_{o0,p}e^{-(\alpha_1+\gamma_1)qT} \quad \text{if} \quad 0 \le q < D, \\ \frac{1}{\omega_2C}\left[\left(i_{L0,p}-\frac{V_i}{R_{net1}}\right)e^{\gamma_1t_1-\alpha_2qT+\beta t_1}+\frac{V_i}{R_{net1}}e^{-\alpha_2(qT-t_1)}\right]\times\sin\omega_2\left(qT-t_1\right) \\ +v_{o0,p}e^{(\gamma_1t_1-\alpha_2qT+\beta t_1)}\begin{bmatrix}\cos\omega_2\left(qT-t_1\right)+ \\ \frac{\gamma}{\omega_2}\sin\omega_2\left(t_1-qT\right)\end{bmatrix} \text{ if } D \le q < 1, \end{cases}$$

References

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Testing

The inductor current equation in (40) was plotted in Matlab with different initial values since we did not know what the actual value was. $i_{L0,p}$ is the initial value of inductor current in the first interval of the time period and $v_{o0,p}$ is the initial capacitor voltage in the first interval of the time period. The small ripple approximation plot is blue and the laplace transform plot is orange.

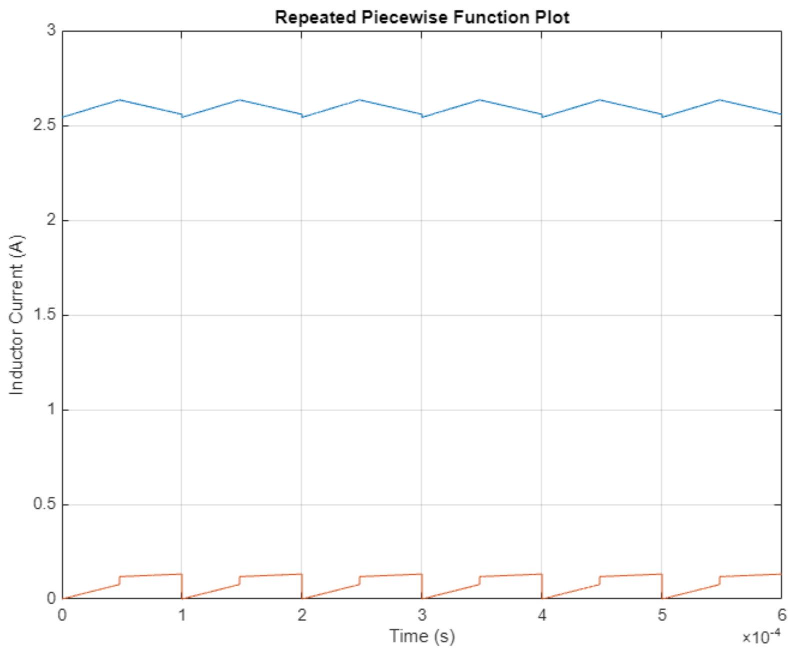

Figure 4: $i_{L0,p} = 0$ and $v_{o0,p} = 0$

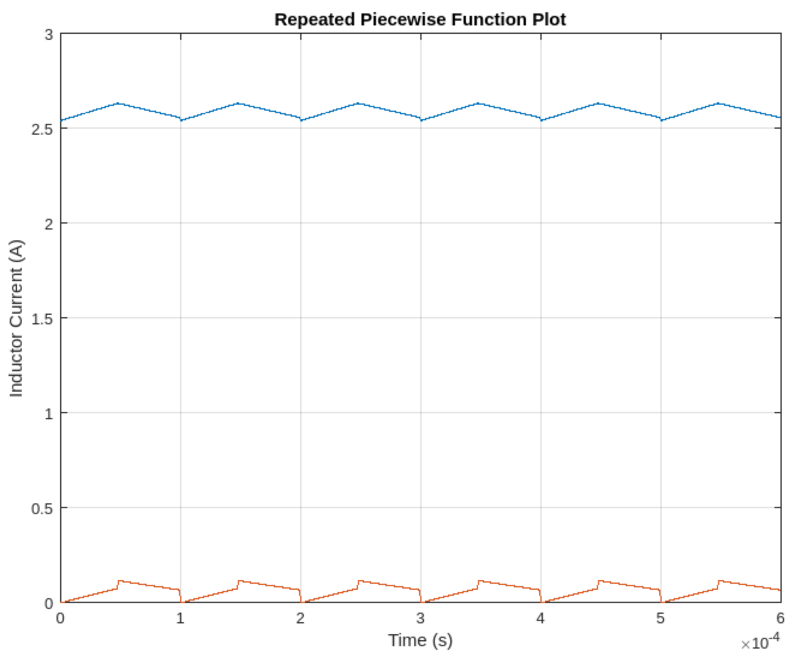

Figure 5: $i_{L0,p} = 0$ and $v_{o0,p} = 5$

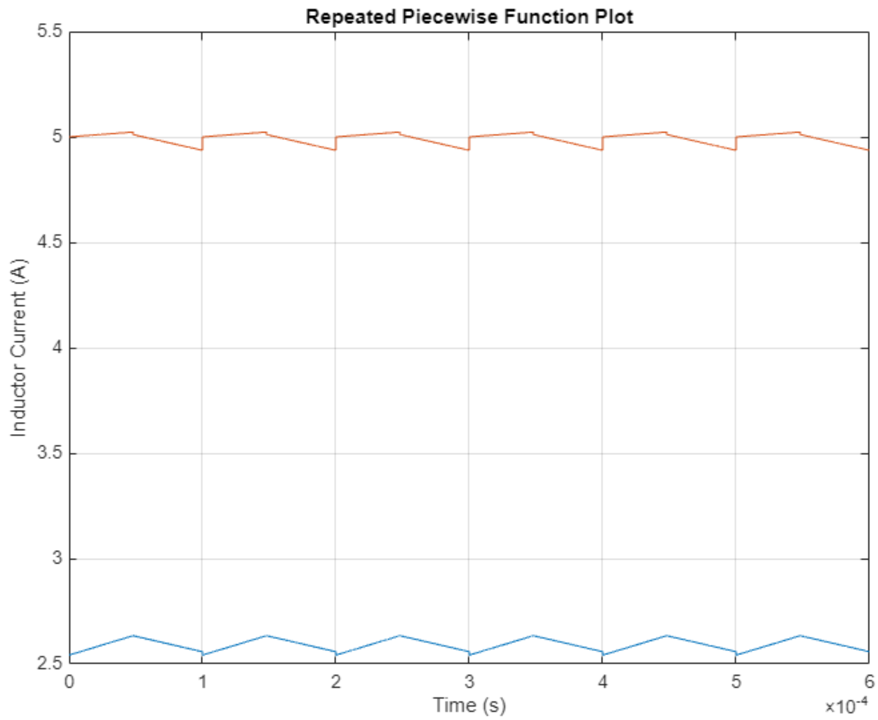

Figure 6: $i_{L0,p} = 5$ and $v_{o0,p} = 0$

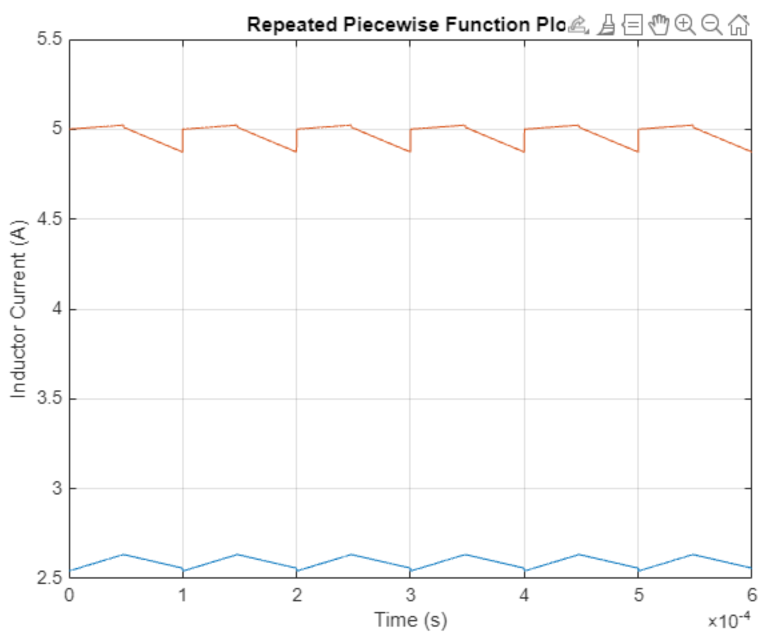

Figure 7: $i_{L0,p} = 5$ and $v_{o0,p} = 5$

We saw unexpected plots that didnt match the plot of the small ripple approximation. This was because the initial values of the intervals play a crucial role in where the plot intervals start and end. So finding the right initial value is key.

Next, we tried plotting graphs using the formulae derived in the CRADA report that used small-ripple approximation for the equations.

$$i_L(t) \approx \left(\frac{(-D'R_{ds1\_on} + D'R_{ds2\_on} + D'^2R)V_g}{(R_L + DR_{ds1\_on} + D'R_{ds2\_on} + D'^2R)L}\right)t + I - \Delta i_L, \qquad 0 \le t \le DT \qquad (15)$$

Figure 8: The inductor equation from the CRADA report

$$V = \frac{D'RV_g}{R_L + DR_{ds1\_on} + D'R_{ds2\_on} + D'^2R}$$

Figure 9: The average capacitor voltage equation from the CRADA report

The initial value(i.e., when t=0) using equation (15) would be $I - \Delta i_L$. Hence we assigned $i_{L0,p} = I - \Delta i_L$. We assigned $v_{o1,p} = D * V$ because $v_{o1,p}$ was the initial value of output/capacitor voltage in the second interval of the time period and V was the average value of the capacitor voltage derived from the CRADA report as well. I chose $D * V$ because I assumed it would give us the voltage at the point in time where $\frac{t_1}{T} = D$. This assumption does not have a detailed explanation as to how it works and that is why it reduces this method's credibility. However, I tried to plot this in MATLAB, I got these plots:

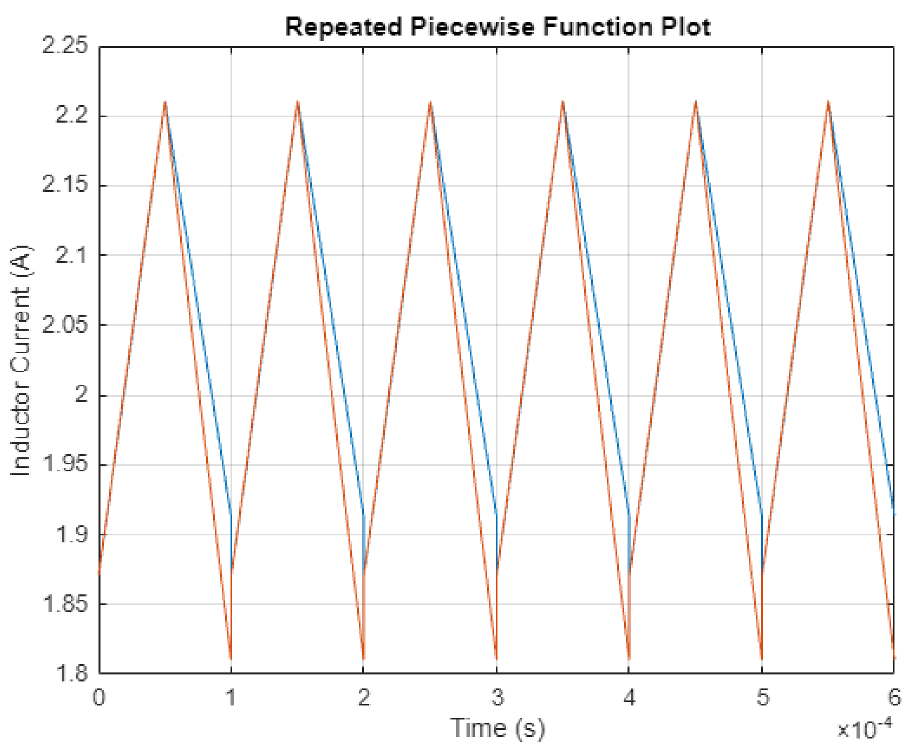

Figure 10: D =0.5

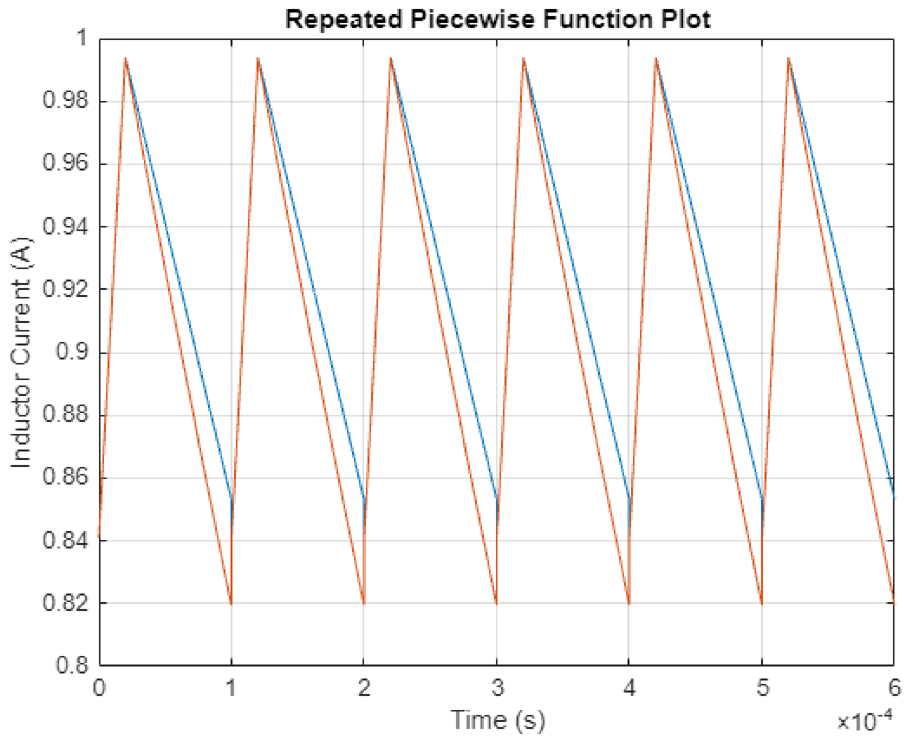

Figure 11: D =0.2

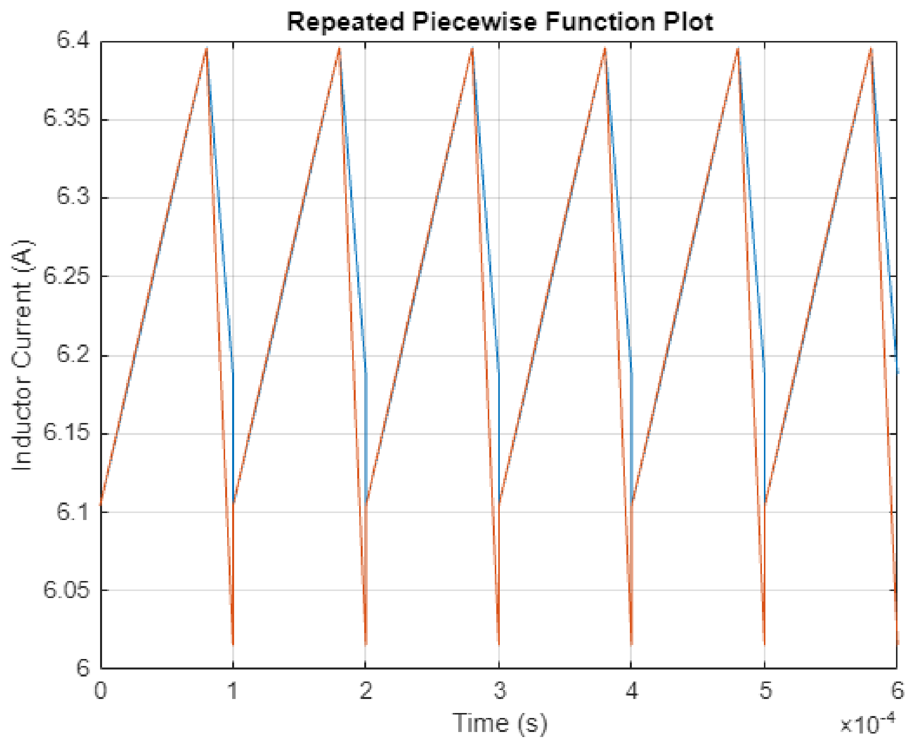

Figure 12: D=0.8

We wanted to know why the minimum changed when the value for D changed. After experimenting with different parameter values, I found that the closeness between the 2 plots is inversely proportional to the value of R_{ds2on}. As R_{ds2on} value increases, the closeness decreases (i.e. the plots have bigger gaps). When R_{ds2on} value decreases, the closeness increases (i.e. the plots come close to each other). For eg: When R_{ds1on}=0.4 and R_{ds2on}=0.1, we get the below graph (almost overlap each other).

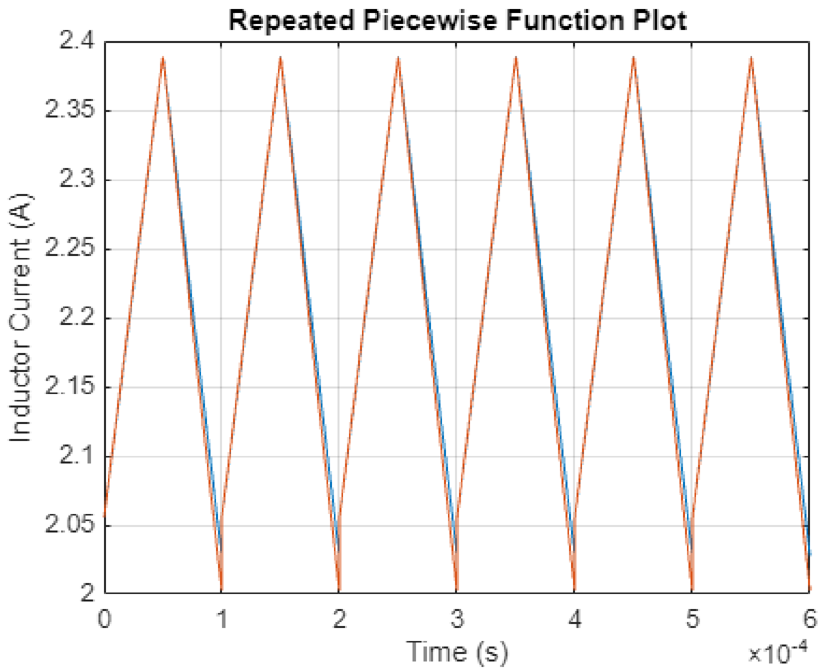

Figure 13: When R_{ds1on}=0.4 and R_{ds2on}=0.1

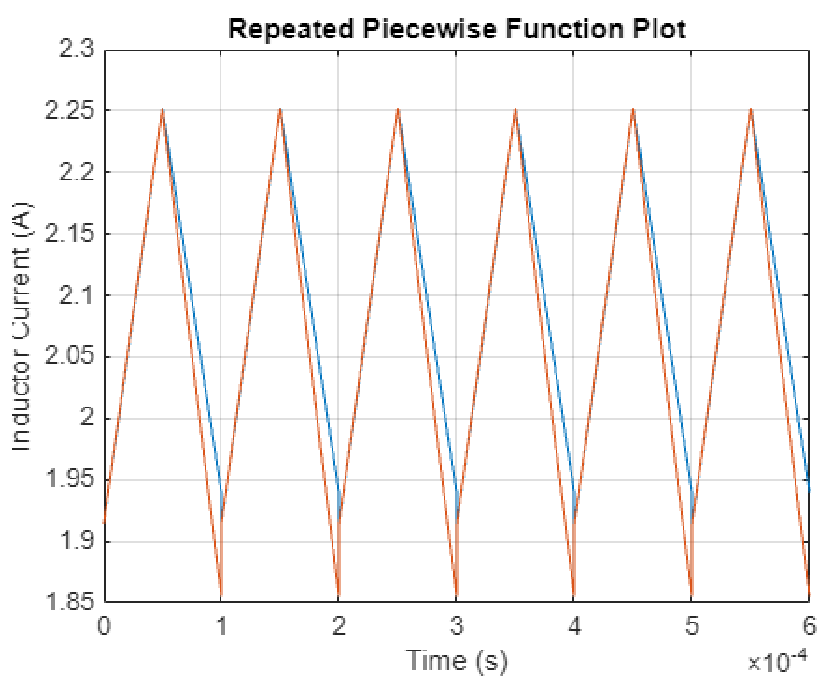

Figure 14: When R_{ds1on}=0.4 and R_{ds2on}=0.4

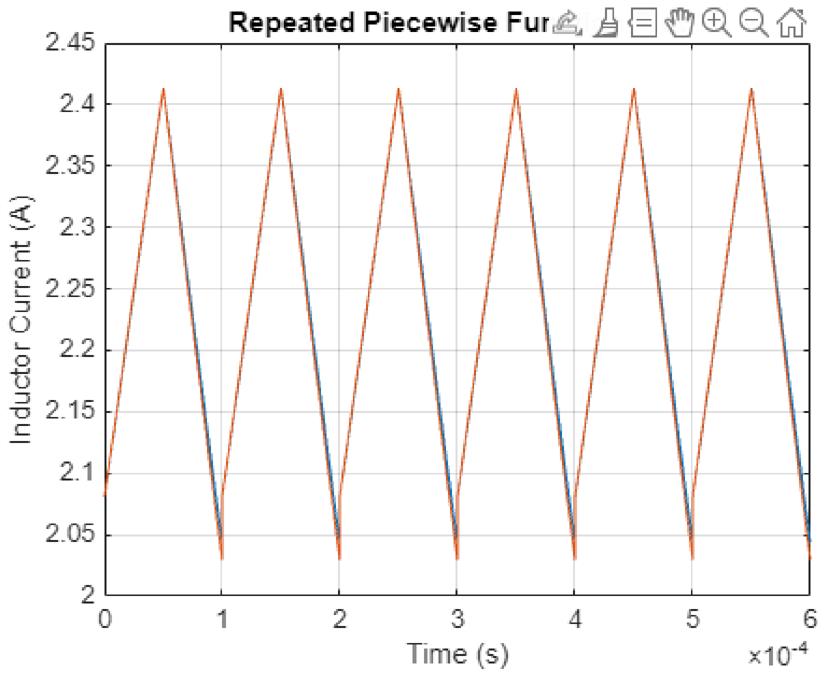

Figure 15: When R_{ds1on}=0.4 and R_{ds2on}=0

When we zoomed into one time period of Fig.15, we saw a 1% difference in minimums.

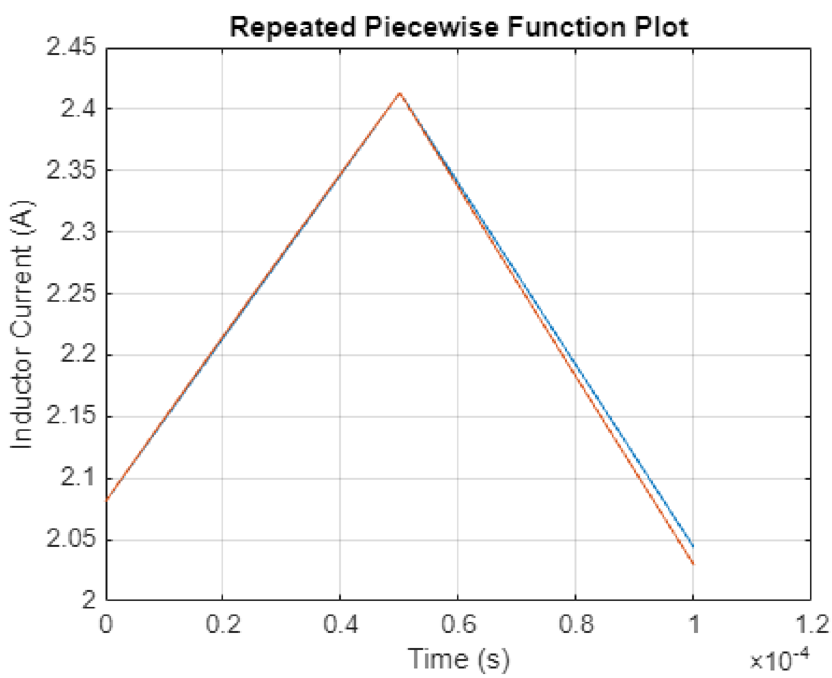

Figure 16: One time period when R_{ds1on}=0.4 and R_{ds2on}=0

It was at this point of looking for an explanation as to why $v_{o1,p} = D * V$ and why R_{ds2on}=0 gives us such approximate plots that I found the error in equation (22) of this approach and had to stop using it.

The research papers I was using as guides to solving these equations had 2 major errors:
1) The first one was that their matrices didnt represent the equations correctly.

$$\left(s+\frac{T}{RC}\right)V_{o,p}(s)=v_{o1,p}+\frac{T}{C}I_{L,p} \quad \text{if} \quad D\le q<1, \qquad (20)$$

By sorting Eq. (17) to Eq. (20), in the form of a matrix, the relation between input current and capacitor voltage in the Laplace domain can be written as:

$$\begin{bmatrix} I_{L,p}(s) \\ V_{o,p}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+(\alpha-\gamma)T}\left(i_{L0,p}+\frac{T}{L}\frac{V_i}{s}\right) \\ \frac{v_{o0,p}}{s+(\alpha+\gamma)T} \end{bmatrix} \quad \text{if } 0\le q<D, \quad (21)$$

$$\begin{bmatrix} I_{L,p}(s) \\ V_{o,p}(s) \end{bmatrix} = \frac{1}{s^2+2\alpha Ts+T^2(\alpha^2+\omega^2)} \times \begin{bmatrix} (\alpha+\gamma)T+s & -\frac{T}{L} \\ \frac{T}{C} & s+T(\alpha-\gamma) \end{bmatrix} \begin{bmatrix} i_{L1,p}+\frac{TV_i}{Ls} \\ v_{o1,p} \end{bmatrix} \quad \text{if } D\le q<1, \qquad (22)$$

Figure 17: Error 1

The matrix multiplication product of the circled elements should give us equation (20) in Fig.17. But if you performed matrix multiplication on the circled elements, you will obtain an equation that has an extra term in it ($\frac{T^2 V_i}{LCs}$). This term should not be a part of the equation.

2)The second error was when they changed $V_{o,p}(s)$ to $v_{o1,p}$.

$$\left(s+\frac{TR_L}{L}\right)I_{L,p}(s)=i_{L0,p}+\frac{T}{LS}V_i \quad \text{if} \quad 0\le q<D, \qquad (17)$$

$$\left(s+\frac{T}{RC}\right)V_{o,p}(s)=v_{o0,p} \quad \text{if} \quad 0\le q<D, \qquad (18)$$

$$\left(s+\frac{TR_L}{L}\right)I_{L,p}(s)=+i_{L1,p}+\frac{T}{LS}V_i-\frac{T}{L}V_{o,p}(s), \; D\le q<1, \quad (19)$$

$$\left(s+\frac{T}{RC}\right)V_{o,p}(s)=v_{o1,p}+\frac{T}{C}I_{L,p} \quad \text{if} \quad D\le q<1, \qquad (20)$$

By sorting Eq. (17) to Eq. (20), in the form of a matrix, the relation between input current and capacitor voltage in the Laplace domain can be written as:

$$\begin{bmatrix} I_{L,p}(s) \\ V_{o,p}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+(\alpha-\gamma)T}\left(i_{L0,p}+\frac{T}{L}\frac{V_i}{s}\right) \\ \frac{v_{o0,p}}{s+(\alpha+\gamma)T} \end{bmatrix} \quad \text{if } 0\le q<D, \quad (21)$$

$$\begin{bmatrix} I_{L,p}(s) \\ V_{o,p}(s) \end{bmatrix} = \frac{1}{s^2+2\alpha Ts+T^2(\alpha^2+\omega^2)} \times \begin{bmatrix} (\alpha+\gamma)T+s & -\frac{T}{L} \\ \frac{T}{C} & s+T(\alpha-\gamma) \end{bmatrix} \begin{bmatrix} i_{L1,p}+\frac{TV_i}{Ls} \\ v_{o1,p} \end{bmatrix} \quad \text{if } D\le q<1, \qquad (22)$$

Figure 18: Error 2

There is no explanation as to why they changed $V_{o,p}(s)$ which is a function of s to a constant $v_{o1,p}$.

Below is the code that was used to plot Figs 13-16 including the parameter values.

```
% Define constants
Vi = 10;            % Example input voltage, modify as needed
R_L = 0.5;
Rds1on = 0.4;
Rds2on = 0;
Rnet1 = R_L + Rds1on;         % Example resistance value, modify as needed
Rnet2 = R_L + Rds2on;         % Example resistance value, modify as needed
R = 10;             % Example resistance value, modify as needed
L = 0.0012;          % Example inductance, modify as needed
C = 0.033;
T = 1/10000;        % Example period, modify as needed
D = 0.5;
D1 = 1-D;

V=D1*R*Vi / (R_L+D*Rds1on+D1*Rds2on + D1^2*R);
I = V/(D1*R);

delta_iL = (-D1*Rds1on + D1*Rds2on + D1^2*R)*Vi*D*T / (2*(R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L);

delta_v = D1*Vi*D*T/(2*(R_L+D*Rds1on+D1*Rds2on + D1^2*R)*C);

delta_iL2 = (-D*Rds1on + D*Rds2on + D1*D*R)*Vi*D1*T / (2*(R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L);

delta_v2 = D1*Vi*D*T/(2*(R_L+D*Rds1on+D1*Rds2on + D1^2*R)*C);



val1 = @(x) ((-D1*Rds1on + D1*Rds2on + D1^2*R)*Vi / ((R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L))*x + I - delta_iL;

val2 = @(x) ((D*Rds1on - D*Rds2on + D1*D*R)*Vi / ((R_L + D*Rds1on + D1*Rds2on + D1^2*R)*L))*(-x+D*T) + I + delta_iL2;


% Define the piecewise function as a regular MATLAB function
piecewise_func = @(x) (x <= D * T) .* val1(x) + (x > D * T) .* val2(x);

% Define time vector for one period
t_single_period = linspace(0, T, 1000); % 1000 points in one period
y_single_period = piecewise_func(t_single_period);

% Repeat the single period values
num_periods = 1;
y6 = repmat(y_single_period, 1, num_periods);
t6 = linspace(0, T * num_periods, length(y6));

% Plot the repeated signal
figure;
plot(t6, y6);
title('Repeated Piecewise Function Plot');
xlabel('Time (s)');
ylabel('Inductor Current (A)');
grid on;
%set(gca,'ytick', 0:1:10);
hold on;
```

Figure 19: Code part 1/2

```
i_L0_p = I-delta_iL;
vo_0_p =V+delta_v;
t1 = D*T;
alpha1 = ((Rnet1 / L) + (1 / (R * C))) / 2;       % Example value, modify as needed
gamma1 = ((1 / (R * C)) - (Rnet1 / L)) / 2;       % Example value, modify as needed
alpha2 = ((Rnet2 / L) + (1 / (R * C))) / 2;       % Example value, modify as needed
gamma2 = ((1 / (R * C)) - (Rnet2 / L)) / 2;       % Example value, modify as needed
omega2 = sqrt((1 / (L * C)) * (Rnet2 / R) - alpha2^2);      % Example angular frequency, modify as needed
beta = alpha2 - alpha1;        % Example value, modify as needed

t2_end = T-t1;



i_L1 =I-delta_iL;
vo_1 =D*V;

disp(['i_L0 = ', num2str(i_L0_p)]);
disp(['v_o0 = ', num2str(vo_1)]);
disp(['I = ', num2str(I-delta_iL)]);
disp(['V = ', num2str(V-delta_v)]);


% Define val1 and val2 expressions
val1 = @(x) i_L0_p * exp(-(alpha1 - gamma1) .* x) + (Vi / Rnet1) .* (1 - exp(-(alpha1 - gamma1) .* x));

i_L_DT = (Vi / Rnet1) + exp(-(Rnet1 / L) * D * T) * (i_L0_p - (Vi / Rnet1));
%i_L_DT=I+delta_iL2;

val2 = @(x) (Vi / (Rnet2 + R)) .* ...
      (1 - exp(-alpha2 .* (x - t1)) .* cos(omega2 .* (x - t1)) ...
      + ((omega2^2 - alpha2 * gamma2) / (omega2 * (alpha2 + gamma2))) .* exp(-alpha2 .* (x - t1)) .* sin(omega2 .* (x - t1))) ..
      + i_L_DT*exp(-alpha2 .* (x-t1)).* ...
        (cos(omega2 .* (t1 - x)) + (gamma2 / omega2) .* sin(omega2 .* (x - t1))) ...
      - (1 / (L * omega2)) * vo_1 .* exp(- alpha2 .* (x-t1)) .* sin(omega2 .* (x - t1));

% Define the piecewise function as a regular MATLAB function
piecewise_func2 = @(x) (x < D * T) .* val1(x) + (x >= D * T) .* val2(x);

% Define time vector for one period
t_single_period = linspace(0, T, 1000); % 1000 points in one period
y_single_period = piecewise_func2(t_single_period);

% Repeat the single period values
num_periods = 1;
y6 = repmat(y_single_period, 1, num_periods);
t6 = linspace(0, T * num_periods, length(y6));

% Plot the repeated signal
plot(t6, y6);
grid on;
```

Figure 20: Code part 2/2