Homework #4

MATH 568 – Advanced Calculus
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Problem

Consider the wave equation $u_{xx} = u_{tt}/c^2$ on the interval $x \in [0, L]$ and $t \geq 0$ with the usual boundary conditions $u(0,t) = u(L,t) = 0$ $0 < a < q < b < L$

a

Find the solution $u_{exact}(x,t)$ to the wave equation as described above for the following initial conditions:

$$u(x,0) = \begin{cases} 0 & \text{if } 0 \leq x < a, \\ \dfrac{x-a}{q-a} & \text{if } a \leq x < q, \\ 1 - \dfrac{x-q}{b-q} & \text{if } a \leq x < b, \\ 0 & \text{if } b \leq x < L \end{cases}$$

$$u_t(x,0) = 0, (\forall)x \in [0, L]$$

ans

Let $u(x,t) = f(x+ct) + g(x-ct)$, then:

$$u_t(x,t) = cf'(x+ct) - cg'(x-ct)$$

$$u_x(x,t) = f'(x+ct) + g'(x-ct)$$
$$u_{tt}(x,t) = c^2 f''(x+ct) + c^2 g''(x-ct)$$
$$u_{xx}(x,t) = f''(x+ct) + g''(x-ct)$$

Applying the initial conditions;

$$f(x) + g(x) = \begin{cases} 0 & \text{if } 0 \leq x < a, \\ \dfrac{x-a}{q-a} & \text{if } a \leq x < q, \\ 1 - \dfrac{x-q}{b-q} & \text{if } a \leq x < b, \\ 0 & \text{if } b \leq x < L \end{cases}$$

$$cf'(x) - cg'(x) = 0 \implies f'(x) = g'(x)$$

and by integration; we have:

$$f(x) = g(x)$$

hence

$$f(x) = g(x) = \frac{1}{2}\begin{cases} 0 & \text{if } 0 \leq x < a, \\ \dfrac{x-a}{q-a} & \text{if } a \leq x < q, \\ 1 - \dfrac{x-q}{b-q} & \text{if } a \leq x < b, \\ 0 & \text{if } b \leq x < L \end{cases}$$

$$f(x+ct) = \frac{1}{2}\begin{cases} 0 & \text{if } 0 \leq x+ct < a, \\ \dfrac{x+ct-a}{q-a} & \text{if } a \leq x+ct < q, \\ 1 - \dfrac{x+ct-q}{b-q} & \text{if } a \leq x+ct < b, \\ 0 & \text{if } b \leq x+ct < L \end{cases} \quad g(x-ct) = \frac{1}{2}\begin{cases} 0 & \text{if } 0 \leq x-ct < a, \\ \dfrac{x-ct-a}{q-a} & \text{if } a \leq x-ct < q, \\ 1 - \dfrac{x-ct-q}{b-q} & \text{if } a \leq x-ct < b, \\ 0 & \text{if } b \leq x-ct < L \end{cases}$$

b

Write the solution under (a) as a Fourier series.

ans

Assume the solution is separable, $u(x,t) = X(x)T(t)$

$$u_{xx} = u_{tt}/c^2$$

$$X''T = XT''/c^2 \implies \frac{X''}{X} = \frac{T''}{c^2T}$$

$$\frac{X''}{X} = \frac{T''}{c^2T} = -\lambda^2 \implies X = \varphi_1 \sin(\lambda x) + \varphi_2 \cos(\lambda x), \quad T = \psi_1 \sin(c\lambda t) + \psi_2 \cos(c\lambda t)$$

using the boundary conditions;

$$\varphi_2 = 0, \lambda = \frac{n\pi}{L}$$

$$u(x,t) = \phi_1 \sin\left(\frac{n\pi}{L}x\right) \sin\left(c\frac{n\pi}{L}t\right) + \phi_2 \sin\left(\frac{n\pi}{L}x\right) \cos\left(c\frac{n\pi}{L}t\right)$$

we can use the trig product identity to get the form of de Alambert solution.

$$u(x,t) = \frac{\phi_1}{2}\left(\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right) + \frac{\phi_2}{2}\left(\sin\left(\frac{n\pi}{L}(x-ct)\right) + \sin\left(\frac{n\pi}{L}(x+ct)\right)\right)$$

but equating to the initial conditions;

$$\phi_1 = 0; \quad u(x,t) = \phi_2 \sin\left(\frac{n\pi}{L}x\right) = \begin{cases} 0 & \text{if } 0 \le x < a, \\ \dfrac{x-a}{q-a} & \text{if } a \le x < q, \\ 1 - \dfrac{x-q}{b-q} & \text{if } a \le x < b, \\ 0 & \text{if } b \le x < L \end{cases}$$

$$\phi_n = \frac{\int_a^q \sin\left(\frac{n\pi}{L}x\right)\left(\frac{x-a}{q-a}\right)dx + \int_q^b \sin\left(\frac{n\pi}{L}x\right)\left(1 - \frac{x-q}{b-q}\right)dx}{\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx}$$

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(c\frac{n\pi}{L}t\right)$$

c

The Numerical solution is shown in the code below:

C:\Users\Windows 10\OneDrive\Documents\MATLAB\MTH 568\Solve_wave_equation.m

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```
function Solve_wave_equation
clc; close all
global dx dt Fn Gn c fl fr mp1 mm1 m
M = 200; L = 10; c = 1; a = 2; mp1 = 3:M; mm1 = 1:M-2; m = 2:M-1; b = 3;
q = 2.75; dt = 1/30; n = 1; X = linspace(0,L,M)'; t = 0; dx = X(2) - X(1);
opts  = optimset('Display','off'); G = zeros(M-2,1); Fl = @(t) 0*t;
U  = (X-a).*(a<=X&X<q)/(q-a) + (1 - (X-q)/(b-q)).*(q<=X&X<b); Fr = @(t) 0*t;
wave = plot(X,U,'k'); drawnow; axis([0,L,-1.1,1.1]); Sol = [];
while (n < 420)
    np1  = n + 1;          tnp1  = t + dt;
    fl   = Fl(tnp1);       fr    = Fr(tnp1);
    Fn   = U(2:M - 1)';    Gn    = G';
    x1   = [Fn;Gn];        x0    = x1(:);
    x    = fsolve(@res,x0,opts);
    U    = [fl;x(1:2:end);fr]; G  = x(2:2:end);
    wave.YData = U; drawnow; n = np1; t = tnp1;
    if (min(abs(tnp1 - [2.5,5,10,13.3333])) < 0.5*dt)
        Sol = [Sol, U];
    end
end
figure('Position', [200 100 900 700],'Color', 'w'); plot(X, Sol, 'linewidth',2);
legend('$t = 2.5$','$t = 5$','$t = 10$','$t = 13.33$', 'FontSize',12, ...
                    'interpreter','latex', 'location', 'northeastoutside');
ylabel('$u$', 'Interpreter', 'Latex'); xlabel('$x$','Interpreter', 'Latex');
ax = gca; ax.TickLabelInterpreter = 'Latex'; ax.FontSize = 12;
txt1 = 'Solution to wave equation';
txt2 = '$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$';
title({txt1,txt2}, 'FontSize',15, 'interpreter','latex');
frame = getframe(gcf); imwrite(frame.cdata, 'Solution.png');

function y =  res(y0)
global dx dt Fn Gn c fl fr mp1 mm1 m
F = [fl;y0(1:2:end);fr]; G = y0(2:2:end);
y = [(F(m) - Fn(mm1)')/dt - G(mm1),...
     (G(mm1) - Gn(mm1)')/dt - c^2*(F(mp1) - 2*F(m) + F(mm1))/dx^2]';

```

d

The plots for the time $t = [2.5, 5, 10, 13.33]$