MATH 233 - Linear Algebra I

Lecture Notes

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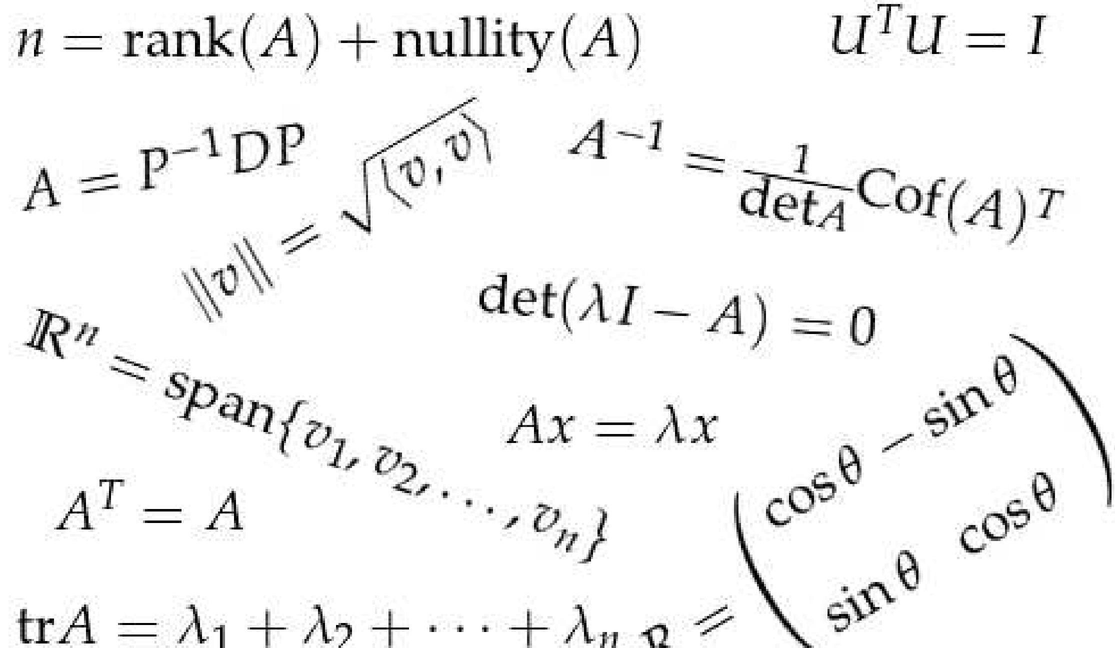

Contents

Lecture 1

Systems of Linear Equations

In this lecture, we will introduce linear systems and the method of row reduction to solve them. We will introduce matrices as a convenient structure to represent and solve linear systems. Lastly, we will discuss geometric interpretations of the solution set of a linear system in 2- and 3-dimensions.

1.1 What is a system of linear equations?

Definition 1.1: A **system of m linear equations** in n unknown variables $x_1, x_2, \ldots, x_n$ is a collection of m equations of the form

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1.1}$$

The numbers a_{ij} are called the **coefficients** of the linear system; because there are m equations and n unknown variables there are thefore $m \times n$ coefficients. The main problem with a linear system is of course to solve it:

Problem: Find a list of n numbers $(s_1, s_2, \ldots, s_n)$ that satisfy the system of linear equations (1.1).

In other words, if we substitute the list of numbers $(s_1, s_2, \ldots, s_n)$ for the unknown variables $(x_1, x_2, \ldots, x_n)$ in equation (1.1) then the left-hand side of the ith equation will equal b_i. We call such a list $(s_1, s_2, \ldots, s_n)$ **a solution** to the system of equations. Notice that we say "a solution" because there may be more than one. The **set** of all solutions to a linear system is called its **solution set**. As an example of a linear system, below is a linear system consisting of $m = 2$ equations and $n = 3$ unknowns:

$$\begin{aligned} x_1 - 5x_2 - 7x_3 &= 0 \\ 5x_2 + 11x_3 &= 1 \end{aligned}$$

Here is a linear system consisting of $m = 3$ equations and $n = 2$ unknowns:

$$\begin{aligned} -5x_1 + x_2 &= -1 \\ \pi x_1 - 5x_2 &= 0 \\ 63x_1 - \sqrt{2}x_2 &= -7 \end{aligned}$$

And finally, below is a linear system consisting of $m = 4$ equations and $n = 6$ unknowns:

$$\begin{aligned} -5x_1 + x_3 - 44x_4 - 55x_6 &= -1 \\ \pi x_1 - 5x_2 - x_3 + 4x_4 - 5x_5 + \sqrt{5}x_6 &= 0 \\ 63x_1 - \sqrt{2}x_2 - \tfrac{1}{5}x_3 + \ln(3)x_4 + 4x_5 - \tfrac{1}{33}x_6 &= 0 \\ 63x_1 - \sqrt{2}x_2 - \frac{1}{5}x_3 - \tfrac{1}{8}x_4 - 5x_6 &= 5 \end{aligned}$$

Example 1.2. Verify that $(1, 2, -4)$ is a solution to the system of equations

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 2 \\ x_1 + 3x_2 - x_3 &= 11. \end{aligned}$$

Is $(1, -1, 2)$ a solution to the system?

Solution. The number of equations is $m = 2$ and the number of unknowns is $n = 3$. There are $m \times n = 6$ coefficients: $a_{11} = 2$, $a_{12} = 1$, $a_{13} = 1$, $a_{21} = 1$, $a_{22} = 3$, and $a_{23} = -1$. And $b_1 = 0$ and $b_2 = 11$. The list of numbers $(1, 2, -4)$ is a solution because

$$2 \cdot (1) + 2(2) + (-4) = 2$$

$$(1) + 3 \cdot (2) - (-4) = 11$$

On the other hand, for $(1, -1, 2)$ we have that

$$2(1) + 2(-1) + (2) = 2$$

but

$$1 + 3(-1) - 2 = -4 \neq 11.$$

Thus, $(1, -1, 2)$ is not a solution to the system. $\square$

A linear system may not have a solution at all. If this is the case, we say that the linear system is **inconsistent**:

INCONSISTENT $\Leftrightarrow$ NO SOLUTION

A linear system is called **consistent** if it has at least one solution:

CONSISTENT $\Leftrightarrow$ AT LEAST ONE SOLUTION

We will see shortly that a consistent linear system will have either just one solution or infinitely many solutions. For example, a linear system cannot have just 4 or 5 solutions. If it has multiple solutions, then it will have infinitely many solutions.

Example 1.3. Show that the linear system does not have a solution.

$$\begin{aligned} -x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1. \end{aligned}$$

Solution. If we add the two equations we get

$$0 = 4$$

which is a contradiction. Therefore, there does not exist a list (s_1, s_2) that satisfies the system because this would lead to the contradiction $0 = 4$. $\square$

Example 1.4. Let t be an arbitrary real number and let

$$\begin{aligned} s_1 &= -\tfrac{3}{2} - 2t \\ s_2 &= \tfrac{3}{2} + t \\ s_3 &= t. \end{aligned}$$

Show that for any choice of the parameter t, the list (s_1, s_2, s_3) is a solution to the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 3x_2 - x_3 &= 3. \end{aligned}$$

Solution. Substitute the list (s_1, s_2, s_3) into the left-hand-side of the first equation

$$\left(-\tfrac{3}{2} - 2t\right) + \left(\tfrac{3}{2} + t\right) + t = 0$$

and in the second equation

$$\left(-\tfrac{3}{2} - 2t\right) + 3\left(\tfrac{3}{2} + t\right) - t = -\tfrac{3}{2} + \tfrac{9}{2} = 3$$

Both equations are satisfied for any value of t. Because we can vary t arbitrarily, we get an infinite number of solutions parameterized by t. For example, compute the list (s_1, s_2, s_3) for $t = 3$ and confirm that the resulting list is a solution to the linear system. $\square$

1.2 Matrices

We will use **matrices** to develop systematic methods to solve linear systems and to study the properties of the solution set of a linear system. Informally speaking, a **matrix** is an array or table consisting of *rows* and *columns*. For example,

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 7 & 11 & -5 \end{bmatrix}$$

is a matrix having $m = 3$ rows and $n = 4$ columns. In general, a matrix with m rows and n columns is a $m \times n$ matrix and the set of all such matrices will be denoted by $M_{m\times n}$. Hence, $\mathbf{A}$ above is a 3×4 matrix. The entry of $\mathbf{A}$ in the ith row and jth column will be denoted by a_{ij}. A matrix containing only one column is called a **column vector** and a matrix containing only one row is called a **row vector**. For example, here is a row vector

$$\mathbf{u} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}$$

and here is a column vector

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

We can associate to a linear system three matrices: (1) the coefficient matrix, (2) the output column vector, and (3) the augmented matrix. For example, for the linear system

$$\begin{aligned} 5x_1 - 3x_2 + 8x_3 &= -1 \\ x_1 + 4x_2 - 6x_3 &= 0 \\ 2x_2 + 4x_3 &= 3 \end{aligned}$$

the coefficient matrix $\mathbf{A}$, the output vector $\mathbf{b}$, and the augmented matrix $[\mathbf{A}\ \mathbf{b}]$ are:

$$\mathbf{A} = \begin{bmatrix} 5 & -3 & 8 \\ 1 & 4 & -6 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 5 & -3 & 8 & -1 \\ 1 & 4 & -6 & 0 \\ 0 & 2 & 4 & 3 \end{bmatrix}.$$

If a linear system has m equations and n unknowns then the coefficient matrix $\mathbf{A}$ must be a $m \times n$ matrix, that is, $\mathbf{A}$ has m rows and n columns. Using our previously defined notation, we can write this as $\mathbf{A} \in M_{m\times n}$.

If we are given an augmented matrix, we can write down the associated linear system in an obvious way. For example, the linear system associated to the augmented matrix

$$\begin{bmatrix} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \end{bmatrix}$$

is

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7. \end{aligned}$$

We can study matrices without interpreting them as coefficient matrices or augmented matrices associated to a linear system. **Matrix algebra** is a fascinating subject with numerous applications in every branch of engineering, medicine, statistics, mathematics, finance, biology, chemistry, etc.

1.3 Solving linear systems

In algebra, you learned to solve equations by first "simplifying" them using operations that do not alter the solution set. For example, to solve $2x = 8 - 2x$ we can add to both sides $2x$ and obtain $4x = 8$ and then multiply both sides by $\frac{1}{4}$ yielding $x = 2$. We can do similar operations on a linear system. There are three basic operations, called **elementary operations**, that can be performed:

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another.

These operations do not alter the solution set. The idea is to apply these operations iteratively to simplify the linear system to a point where one can easily write down the solution set. It is convenient to apply elementary operations on the augmented matrix $[\mathbf{A}\ \mathbf{b}]$ representing the linear system. In this case, we call the operations **elementary row operations**, and the process of simplifying the linear system using these operations is called **row reduction**. The goal with row reducing is to transform the original linear system into one having a **triangular structure** and then perform **back substitution** to solve the system. This is best explained via an example.

Example 1.5. Use back substitution on the augmented matrix

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

to solve the associated linear system.

Solution. Notice that the augmented matrix has a triangular structure. The third row corresponds to the equation $x_3 = 1$. The second row corresponds to the equation

$$x_2 - x_3 = 0$$

and therefore $x_2 = x_3 = 1$. The first row corresponds to the equation

$$x_1 - 2x_3 = -4$$

and therefore

$$x_1 = -4 + 2x_3 = -4 + 2 = -2.$$

Therefore, the solution is $(-2, 1, 1)$. $\square$

Example 1.6. Solve the linear system using elementary row operations.

$$\begin{aligned} -3x_1 + 2x_2 + 4x_3 &= 12 \\ x_1 - 2x_3 &= -4 \\ 2x_1 - 3x_2 + 4x_3 &= -3 \end{aligned}$$

Solution. Our goal is to perform elementary row operations to obtain a triangular structure and then use back substitution to solve. The augmented matrix is

$$\begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix}.$$

Interchange Row 1 (R_1) and Row 2 (R_2):

$$\begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{bmatrix}$$

As you will see, this first operation will simplify the next step. Add $3R_1$ to R_2:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{3R_1 + R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{bmatrix}$$

Add $-2R_1$ to R_3:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix}$$

Multiply R_2 by $\frac{1}{2}$:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix}$$

Add $3R_2$ to R_3:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix} \xrightarrow{3R_2 + R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

Multiply R_3 by $\frac{1}{5}$:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{\frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We can continue row reducing but the row reduced augmented matrix is in triangular form. So now use back substitution to solve. The linear system associated to the row reduced augmented matrix is

$$\begin{aligned} x_1 - 2x_3 &= -4 \\ x_2 - x_3 &= 0 \\ x_3 &= 1 \end{aligned}$$

The last equation gives that $x_3 = 1$. From the second equation we obtain that $x_2 - x_3 = 0$, and thus $x_2 = 1$. The first equation then gives that $x_1 = -4 + 2(1) = -2$. Thus, the solution to the original system is $(-2, 1, 1)$. You should verify that $(-2, 1, 1)$ is a solution to the original system. $\square$

The original augmented matrix of the previous example is

$$\mathbf{M} = \begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix} \qquad \rightarrow \qquad \begin{aligned} -3x_1 + 2x_2 + 4x_3 &= 12 \\ x_1 - 2x_3 &= -4 \\ 2x_1 - 3x_2 + 4x_3 &= -3. \end{aligned}$$

After row reducing we obtained the row reduced matrix

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \rightarrow \qquad \begin{aligned} x_1 - 2x_3 &= -4 \\ x_2 - x_3 &= 0 \\ x_3 &= 1. \end{aligned}$$

Although the two augmented matrices $\mathbf{M}$ and $\mathbf{N}$ are clearly distinct, it is a fact that they have the same solution set.

Example 1.7. Using elementary row operations, show that the linear system is inconsistent.

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + x_3 &= 0 \\ 2x_1 + 4x_3 &= 1 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$$

Perform the operation $-2R_1 + R_3$:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The last row of the simplified augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 = -1$$

Obviously, there are no numbers x_1, x_2, x_3 that satisfy this equation, and therefore, the linear system is inconsistent, i.e., it has no solution. In general, if we obtain a row in an **augmented matrix** of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}$$

where c is a **nonzero** number, then the linear system is inconsistent. We will call this type of row an **inconsistent row**. However, a row of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

corresponds to the equation $x_2 = 0$ which is perfectly valid. □

1.4 Geometric interpretation of the solution set

The set of points (x_1, x_2) that satisfy the linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \tag{1.2}$$

is the intersection of the two lines determined by the equations of the system. The solution for this system is $(3, 2)$. The two lines intersect at the point $(x_1, x_2) = (3, 2)$, see Figure 1.1.

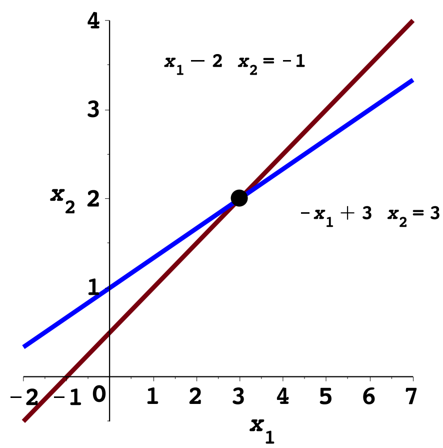

Figure 1.1: The intersection point of the two lines is the solution of the linear system (1.2)

Similarly, the solution of the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \tag{1.3}$$

is the intersection of the three planes determined by the equations of the system. In this case, there is only one solution: $(29, 16, 3)$. In the case of a consistent system of two equations, the solution set is the line of intersection of the two planes determined by the equations of the system, see Figure 1.2.

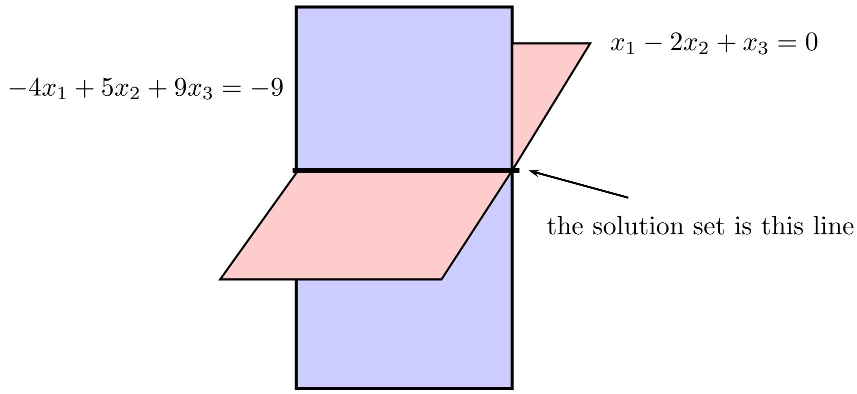

Figure 1.2: The intersection of the two planes is the solution set of the linear system (1.3)

After this lecture you should know the following:

- what a linear system is
- what it means for a linear system to be consistent and inconsistent
- what matrices are
- what are the matrices associated to a linear system
- what the elementary row operations are and how to apply them to simplify a linear system
- what it means for two matrices to be row equivalent
- how to use the method of back substitution to solve a linear system
- what an inconsistent row is
- how to identify using elementary row operations when a linear system is inconsistent
- the geometric interpretation of the solution set of a linear system

Lecture 2

Row Reduction and Echelon Forms

In this lecture, we will get more practice with row reduction and in the process introduce two important types of matrix forms. We will also discuss when a linear system has a unique solution, infinitely many solutions, or no solution. Lastly, we will introduce a convenient parameter called the rank of a matrix.

2.1 Row echelon form (REF)

Consider the linear system

$$\begin{aligned} x_1 + 5x_2 - 2x_4 - x_5 + 7x_6 &= -4 \\ 2x_2 - 2x_3 + 3x_6 &= 0 \\ -9x_4 - x_5 + x_6 &= -1 \\ 5x_5 + x_6 &= 5 \\ 0 &= 0 \end{aligned}$$

having augmented matrix

$$\begin{bmatrix} 1 & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & 2 & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -9 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above augmented matrix has the following properties:

P1. All nonzero rows are above any rows of all zeros.

P2. The leftmost nonzero entry of a row is to the right of the leftmost nonzero entry of the row above it.

Any matrix satisfying properties P1 and P2 is said to be in **row echelon form** (**REF**). In REF, the leftmost nonzero entry in a row is called a **leading entry**:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \mathbf{-9} & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \mathbf{5} & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A consequence of property P2 is that every entry below a leading entry is zero:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -4 & -1 & -7 \\ \mathbf{0} & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{-9} & -1 & 1 & -1 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{5} & 1 & 5 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix}$$

We can perform elementary row operations, or **row reduction**, to transform a matrix into REF.

Example 2.1. Explain why the following matrices are not in REF. Use elementary row operations to put them in REF.

$$\mathbf{M} = \begin{bmatrix} 3 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} 7 & 5 & 0 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & 6 & -5 & 2 \end{bmatrix}$$

Solution. Matrix $\mathbf{M}$ fails property P1. To put $\mathbf{M}$ in REF we interchange R_2 with R_3:

$$\mathbf{M} = \begin{bmatrix} 3 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 3 & -1 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{N}$ fails property P2. To put $\mathbf{N}$ in REF we perform the operation $-2R_2 + R_3 \rightarrow R_3$:

$$\begin{bmatrix} 7 & 5 & 0 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & 6 & -5 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 7 & 5 & 0 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

□

Why is REF useful? Certain properties of a matrix can be easily deduced if it is in REF. For now, REF is useful to us for solving a linear system of equations. If an augmented matrix is in REF, we can use **back substitution** to solve the system, just as we did in Lecture 1. For example, consider the system

$$\begin{aligned} 8x_1 - 2x_2 + x_3 &= 4 \\ 3x_2 - x_3 &= 7 \\ 2x_3 &= 4 \end{aligned}$$

whose augmented matrix is already in REF:

$$\begin{bmatrix} 8 & -2 & 1 & 4 \\ 0 & 3 & -1 & 7 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

From the last equation we obtain that $2x_3 = 4$, and thus $x_3 = 2$. Substituting $x_3 = 2$ into the second equation we obtain that $x_2 = 3$. Substituting $x_3 = 2$ and $x_2 = 3$ into the first equation we obtain that $x_1 = 1$.

2.2 Reduced row echelon form (RREF)

Although REF simplifies the problem of solving a linear system, later on in the course we will need to completely row reduce matrices into what is called **reduced row echelon form (RREF)**. A matrix is in RREF if it is in REF (so it satisfies properties P1 and P2) and in addition satisfies the following properties:

P3. The leading entry in each nonzero row is a 1.

P4. All the entries above (and below) a leading 1 are all zero.

A leading 1 in the RREF of a matrix is called a **pivot**. For example, the following matrix in RREF:

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

has three pivots:

$$\begin{bmatrix} \mathbf{1} & 6 & \mathbf{0} & 3 & \mathbf{0} & 0 \\ \mathbf{0} & 0 & \mathbf{1} & -4 & \mathbf{0} & 5 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{1} & 7 \end{bmatrix}$$

Example 2.2. Use row reduction to transform the matrix into RREF.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution. The first step is to make the top leftmost entry nonzero:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Now create a leading 1 in the first row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create zeros under the newly created leading 1:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create a leading 1 in the second row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create zeros under the newly created leading 1:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

We have now completed the top-to-bottom phase of the row reduction algorithm. In the next phase, we work bottom-to-top and create zeros **above** the leading 1's. Create zeros above the leading 1 in the third row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create zeros above the leading 1 in the second row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{3R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This completes the row reduction algorithm and the matrix is in RREF. □

Example 2.3. Use row reduction to solve the linear system.

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 8 \\ x_1 + 2x_2 + 4x_3 &= 8 \\ 3x_1 + 6x_2 + 9x_3 &= 12 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Create a leading 1 in the first row:

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Create zeros under the first leading 1:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, however, there are only 2 nonzero rows but 3 unknown variables. This means that the solution set will contain $3 - 2 = 1$ **free parameter**. The second row in the augmented matrix is equivalent to the equation:

$$x_3 = 4.$$

The first row is equivalent to the equation:

$$x_1 + 2x_2 + 3x_3 = 4$$

and after substituting $x_3 = 4$ we obtain

$$x_1 + 2x_2 = -8.$$

We now must choose one of the variables x_1 or x_2 to be a parameter, say t, and solve for the remaining variable. If we set $x_2 = t$ then from $x_1 + 2x_2 = -8$ we obtain that

$$x_1 = -8 - 2t.$$

We can therefore write the solution set for the linear system as

$$\begin{aligned} x_1 &= -8 - 2t \\ x_2 &= t \\ x_3 &= 4 \end{aligned} \tag{2.1}$$

where t can be any real number. If we had chosen x_1 to be the parameter, say $x_1 = t$, then the solution set can be written as

$$\begin{aligned} x_1 &= t \\ x_2 &= -4 - \tfrac{1}{2}t \\ x_3 &= 4 \end{aligned} \tag{2.2}$$

Although (2.1) and (2.2) are two different parameterizations, they both give the same solution set. □