Lecture 3

Special Relativity

Talking about relativity means talking about Lorentz invariance.

If there is a point in space time:

$$(t, x) \xrightarrow[\text{observer}]{\text{another moving}} (t', x')$$

Invariant notion of distance:

$$t^2 - x^2 = t'^2 - x'^2$$

(This should all be familar to you.)

We will recap this in an adult way...

Start with Rotations

Have an invariant notion of length of $\vec{p}$

$$x^2 + y^2 = x'^2 + y'^2 \qquad\qquad x' = x\,cos(\theta) - y\,sin(\theta)$$
$$x' = y\,cos(\theta) + x\,sin(\theta)$$

Look at this another way ...

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We are after the set of all of matrices such that $x^2 + y^2 = x'^2 + y'^2$

Start with the infitesimal case

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We require: $\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Multiplying through:

$$\begin{aligned} x' &= x + \epsilon a x + \epsilon b y \\ y' &= y + \epsilon c x + \epsilon d y \end{aligned}$$

Keeping terms linear in ϵ.

$$x'^2 + y'^2 = x^2 + y^2 + 2\epsilon \underbrace{(ax^2 + bxy + cxy + dy^2)}_{=0 \quad \forall x \& y}$$

so

$$ax^2 + bxy + cxy + dy^2 = 0$$

$$\Rightarrow a = d = 0 \quad \& \qquad \underbrace{b = -c}_{\text{Can rescale } \epsilon \text{ such that b=1}}$$

So

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\mathbb{1} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

More Sophisticated Way:

x_i where $i = 1, 2$ $x_1 = x$ and $x_2 = y$

The rotation can now be written as:

$$x'_i = R_{i1}x_1 + R_{i2}x_2 = \sum_{j=1}^{2} R_{ij}x_j \equiv R_{ij}x_j$$

In the last expression, the sum is implied by the repeated indices (known as Einstein notation).

$$x'_i = R_{ij}x_j$$

$x'_i x'_i = x_i x_i$ is what it takes for R to be a rotation. Need to find the special Rs such that this is satisfied.

The identity matrix is written as δ_{ij}, where $\delta_{ij} = 1$ if $i = j$, 0 otherwise.

If no rotation at all: $R_{ij} = \delta_{ij}$, of an infinitesimal rotation:

$$R_{ij} = \delta_{ij} + \epsilon w_{ij}$$

$$x'_i = x_i + \epsilon w_{ij}x_j$$

$$x'_i x'_i = x_i x_i + 2\epsilon \underbrace{w_{ij}x_j x_i}_{=0 \ \forall x} + O(\epsilon^2)$$

$\Rightarrow w_{ij}$ has to be anti-symmetric $w_{ij} = -w_{ji}$.

Back to previous example, find finite rotations (without any mention of geometry etc.)

Take the infinitesimal rotation and do it n-times.

Define $\theta = N\epsilon$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\mathbb{1} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} \equiv \left[\mathbb{1} + \epsilon \mathrm{I} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

So the finite rotation ($R(\theta)$) given by,

$$R(\theta) = (1 + \epsilon I)(1 + \epsilon I) \dots (1 + \epsilon I) \cdots = (1 + \epsilon I)^N = \left(1 + \frac{\theta}{N} I\right)^N$$

Now let $N \to \infty$, $R(\theta) = e^{I\theta}$.

Built up finite rotation from the infinitesimal rotations.

$x'(\theta) = R(\theta)x = e^{I\theta}$

The meaning of e^X when X is a matrix is simply the expansion.

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

$I^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1}$! We have just discovered i following our nose.

$R(\theta) = cos(\theta) + Isin(\theta) = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$

Strategy: First understand the action of the symmetry infinitesimally, then the big symmetry action is obtained by iterating the infinitesimal. Always e^X where X is the generator. This is a great strategy for any kind of symmetry.

Will now do 3D rotations... Something new happends.

3D Rotations

3-parameters associated with a 3D rotation.

Already saw, any rotation is of the form