Homework Set #2

Due Date: Before class Friday February 1st

1) Show that SO(2) ≃ U(1) *(2 points)*

(a) Consider the complex plane and independent variables z and z^* where $z = x + iy$ and z^* is the complex conjugate. What is zz^* in terms of x and y ?

(b) Consider the action of the operation: $z \rightarrow e^{i\theta} z, z^* \rightarrow e^{-i\theta} z^*$ Show that these satisfy same multiplication law as we found for SO(2), namely: $M(\theta_1)M(\theta_2) = M(\theta_1 + \theta_2)$. The group of transformations $e^{i\theta}$ is referred to as $U(1)$, for Unitary and 1 dimensional.

2) Work out the algebra of the generators of the Lorentz group *(5 points)*

Define the Lorentz generators as:

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

3) Connection to βs and γs *(5 points)*

In class we showed the T_1 generator from above lead to the following transformation:

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

(a) Show that $B(\eta) = e^{I_B \eta} = \cosh(\eta) + I_B \sinh(\eta)$, where, $I_B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) Derive the relationship of $\cosh(\eta)$ and $\sinh(\eta)$ to $\beta = v$ and $\gamma = \frac{1}{\sqrt{1-v^2}}$
(Hint: consider the primed reference frame moving at velocity v with respect to the unprimed reference frame)

4) Z Boson decays *(5 points)*

The Z Boson is a force carrier of the weak interaction (along with the W^+ and W^-). You can think of it as a massive version of the photon. The Z Boson has a mass of 90 GeV. An important way the Z-Boson can decay is to charged leptons (eg: $Z \to e^+e^-$ and $Z \to \mu^+\mu^-$). As we will discuss later in the course, the signatures of electrons and muons are easy to detect experimentally. In fact, the Z Boson was discovered by analyzing the invariant mass spectra of charged leptons.

a) Analyze the $Z \to ee$ decay assuming electron (and positron) are massless. What are the energies and momenta of the outgoing electrons. Ignore angular momentum.

b) Analyze the $Z \to ee$ decay as above including the effect of the electron and positron masses. What is the size of the correction to the electron momentum ?

c) b-quarks are the heaviest thing that the Z-boson can decay into. The b-quark has a mass of ~5 GeV. What is the size of the correction assuming a finite b-quark mass ?

5) GZK cutoff energy *(5 points)*

In the 1960s, Kenneth Greisen, Vadim Kuzmin, and Georgiy Zatsepin predicted that through interactions with the cosmic microwave background (CMB), extra-galactic cosmic rays would have an upper bound to their energy.[1] They proposed that cosmic rays (high energy protons), would lose energy by interacting with the CMB photons by producing a neutral pion:

$$p + \gamma_{\mathrm{CMB}} \to p + \pi_0.$$

The proton energy at which this process can occur is called the GZK cutoff.

a) Estimate the GZK cutoff energy. The energy of CMB photons is 3×10^{-13} GeV (which follows from the measurement of CMB temperature of about 2.7 K). (Hints: The sum of the initial momentum four-vectors must be equal to the sum of the final four-vectors. The simplest way to analyze this reaction is to express the reaction exclusively in Lorentz-invariant four-vector dot products. This should greatly simplify the analysis.)

b) What is your answer in Joules?

Extra-galactic protons above this energy can lose energy by interaction with the CMB. Because the CMB exists throughout the visible universe, this suggests that there can not be protons with energies larger than this value, hence the name "GZK cutoff." Figure 1 shows the distribution of cosmic ray energies from numerous experiments, tabulated in the Particle Data Group. The horizontal axis is the energy of cosmic rays in eV, and the vertical axis is a measure of the number of cosmic rays with that energy. The labeled

[1] K. Greisen, "End to the cosmic ray spectrum?," Phys. Rev. Lett. 16, 748 (1966); G. T. Zatsepin and V. A. Kuzmin, "Upper limit of the spectrum of cosmic rays," JETP Lett. 4, 78 (1966) [Pisma Zh. Eksp. Teor. Fiz. 4, 114 (1966)].

"Knee," "2nd Knee," and "Ankle" are features in the distribution that correspond to different astrophysical sources of cosmic rays.

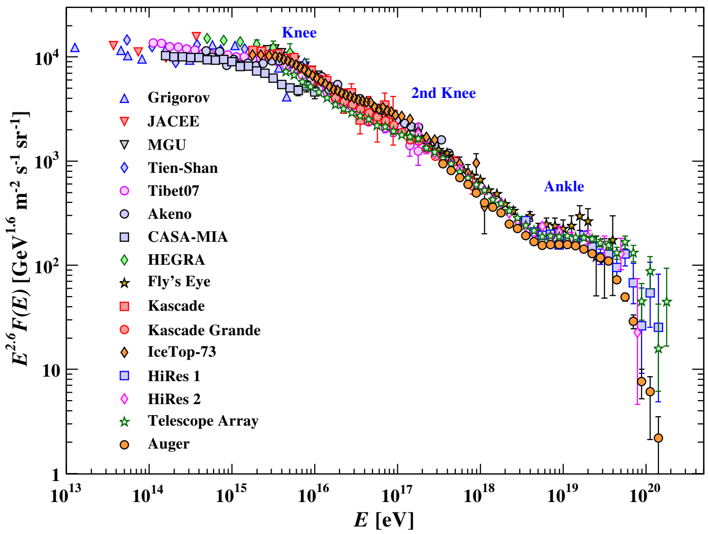

Figure 1: Observed energy distribution of cosmic rays (extra-galactic protons) from various experiments. The data are determined by measuring the energy of particles from air showers due cosmic rays hitting the upper atmosphere of Earth. From K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38, 010009 (2014) and 2015 update.

Compare your estimated cut off to this cosmic ray data. Are there any cosmic rays observed with energies above the GZK cutoff?