

Homework Set #1

Solutions

2) Solid State Physics

(5 points)

- (a) We worked out in class $r_{\text{atom}} \sim \frac{1}{Z\alpha m_e}$. (using $E \sim -\frac{Z\alpha}{r} + \frac{p^2}{m_e}$ and $p \times r \sim 1$) So a solid has a spacing of r_{atom} .
- (b) To probe distances of order r_{atom} need to photons with Energy $\sim \frac{1}{r_{\text{atom}}} \sim Z\alpha m_{\text{electron}}$. Assuming $Z \sim 10$,
 Energy $\sim 10 \cdot 10^{-2} \cdot 10^{-3} \text{ GeV} \sim 10^{-4} \text{ GeV} \sim 10^2 \text{ keV}$
- (c) 10^2 keV photons are x-rays.

3) Strength of Gravity on Earth

(5 points)

- (a) From class, $R_{\text{Planet}} \sim \sqrt{\frac{\alpha}{\alpha_G}} \times r_{\text{atom}}$, $\rho_{\text{solid}} \sim \frac{Zm_{\text{proton}}}{r_{\text{atom}}^3}$, and $M_{\text{Planet}} \sim \rho_{\text{Solid}} \times R_{\text{Planet}}^3$

$$g_{\text{local}} \sim G_N \frac{M_{\text{Planet}}}{R_{\text{Planet}}^2} \sim \frac{G_N Z m_{\text{proton}} R_{\text{Planet}}}{r_{\text{atom}}^3} \sim \sqrt{\alpha_G \alpha} \frac{Z}{m_{\text{proton}} r_{\text{atom}}^2}$$

- (b)

$$\begin{aligned} g_{\text{local}} &\sim (\alpha_G \alpha)^{1/2} \cdot Z \cdot r_{\text{atom}}^{-2} \cdot m_{\text{proton}}^{-1} \\ &\sim (10^{-39} 10^{-2})^{1/2} \cdot 10 \cdot 10^{-8} \text{ GeV}^{-2} \cdot \text{GeV} \\ &\sim 10^{-27.5} \text{ GeV} \end{aligned}$$

Need to convert GeV to $\frac{m}{s^2}$ which has units of [distance]×[time]⁻². c has units of [distance]×[time]⁻¹, h has units of [energy]×[time]. So, can convert from [energy] to [distance]×[time]⁻² by multiplying by c/h. c = 10⁸ m/s, h = 10⁻¹⁵ eV·s = 10⁻²⁴ GeV·s. So, c/h = 10⁸ · 10²⁴ = 10³² $\frac{m}{\text{GeV} s^2}$

$$\begin{aligned} g_{\text{local}} &\sim 10^{-27.5} \text{ GeV} \times (1) \\ &\sim 10^{-27.5} \text{ GeV} \times \frac{c}{h} \\ &\sim 10^{-27.5} \text{ GeV} \times 10^{32} \frac{m}{\text{GeV} s^2} \\ &\sim 10^4 \frac{m}{s^2} \end{aligned}$$

- (c) Not so close to $10 \frac{m}{s^2}$, If we has used $r_{\text{atom}} \sim 10^{-10} m$ instead of 10^{-12} which accounts for the screening of multiple electrons in the atom. Would have calculated factor of 10^{-4} smaller or $g_{\text{local}} \sim 1$, which is pretty close.

4) Neutron Stars

(5 points)

- (a) Neutron star will be stable when $P_{Neutron} \sim P_{Grav}$. $P_{Neutron} \sim \frac{E_{Neutron}}{R_{Neutron}^3} \sim m_{proton}^4$

$$P_{Grav} \sim \frac{G_N \frac{M_{NS}^2}{R_{NS}}}{R_{NS}^3} \sim \frac{G_N M_{NS}^2}{R_{NS}^4}$$

$$M_{NS} \sim \rho_{Neutron} \times R_{NS}^3$$

$$\rho_{Neutron} \sim m_{proton}^4$$

$$P_{Grav} \sim P_{Neutron} \Rightarrow G_N m_{proton}^8 R_{NS}^2 \sim m_{proton}^4$$

$$R_{NS} \sim \sqrt{\frac{1}{\alpha_G} \frac{1}{m_{proton}}} \sim \sqrt{\frac{1}{\alpha_G}} r_{proton}$$

$$\Rightarrow M_{NS} \sim \left(\frac{1}{\alpha_G} \right)^{3/2} m_{proton}$$

$$\text{Speed of sound } v_{NS} \sim \sqrt{\frac{P_{NS}}{\rho_{NS}}} \sim 1$$

- (b) $R_{NS} \sim 10^{19} 10^{-15} \text{m} \sim 10^4 \text{m} \sim 10 \text{ km}$

$$M_{NS} \sim 10^{58} 10^{-27} \text{kg} \sim 10^{31} \text{kg} \sim 10 M_{\odot}$$

$$v_{NS} \sim c$$

- (c) “Neutron stars have a radius of the order of 10 kilometres and a mass lower than a 2.16 solar masses.” -wikipedia

- (d) $m_{proton} = 938.27 \text{ MeV}$

$$m_{neutron} = 939.56 \text{ MeV} \sim 1.001 m_{proton}$$

5) 2D Rotations

(3 points)

(a)

$$e^{I\theta} = 1 + I\theta + \frac{I^2\theta^2}{2!} + \frac{I^3\theta^3}{3!} + \frac{I^4\theta^4}{4!} + \dots$$

$$I^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Show that $R(\Theta) = e^{I\Theta} = \cos(\Theta) + I\sin(\Theta)$

$$e^{I\theta} = I \left(\theta + \frac{I^2\theta^3}{3!} + \frac{I^4\theta^5}{5!} + \dots \right) + \left(1 + \frac{I^2\theta^2}{2!} + \frac{I^4\theta^4}{4!} + \dots \right)$$

$$e^{I\theta} = I \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \right) + \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \right) = I\sin(\theta) + \cos(\theta)$$

(b)

$$\begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & \sin(\theta_2) \\ -\sin(\theta_2) & \cos(\theta_2) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2) \\ -\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

which is clearly symmetric $\theta_1 \leftrightarrow \theta_2$

6) 3D Rotations

(5 points)

(a)

$$J_1 J_2 - J_2 J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -iJ_3$$

etc.

(b) $\text{tr}(M') = \text{tr}(U^\dagger M U) = \text{tr}(U U^\dagger M) = \text{tr}(M)$

$$(M')^\dagger = (U^\dagger M U)^\dagger = (U M^\dagger U^\dagger) = (U^\dagger M U) = M'$$

$$\det(M') = \det(U^\dagger) \det(M) \det(U) = 1 \times \det(M) \times 1$$