Homework Set #4

Solutions

1) Find the generators of the "Little Group" for Massive particles

(5 points)

The little group equation is:

$$W \cdot k = k \Rightarrow \omega \cdot k = 0$$

where

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & A & B \\ -b & -A & 0 & C \\ -c & -B & -C & 0 \end{bmatrix}$$

For a massive particle we can take k to be k = (m,0,0,0).

So $\omega \cdot k = (0, -ma, -mb, -mc)$. So the little group generators for massive particles are

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & -A & 0 & C \\ 0 & -B & -C & 0 \end{bmatrix}$$

which are the rotation matrices.

2) Heisenberg Equation of Motion

(5 points)

a)

$$\frac{dA(t)_{H}}{dt} = \frac{d}{dt} \left(e^{iHt} A_{s} e^{-iHt} \right)$$

$$= iH \left(e^{iHt} A_{s} e^{-iHt} \right) - i \left(e^{iHt} A_{s} H e^{-iHt} \right)$$

$$= -i \left(\underbrace{e^{iHt} A_{s} e^{-iHt}}_{A_{H}(t)} H - H \underbrace{e^{iHt} A_{s} H e^{-iHt}}_{A_{H}(t)} \right)$$

$$= -i \left[A_{H}(t), H \right]$$

b)
$$\phi_H(x,t) = e^{-iE_p t} \phi_S(x)$$

$$\frac{d\phi_H(x,t)}{dt} = -iE_p\phi_H(x,t)$$

and

$$\begin{split} [\phi_H(x,t),H] &= [\int dp e^{ip\cdot x} a^\dagger, \int dp' E_{p'} a^\dagger a] \\ &= \int dp e^{ip\cdot x} \int dp' E_{p'} [a^\dagger, a^\dagger a] \\ &= \int dp e^{ip\cdot x} \int dp' E_{p'} a^\dagger [a^\dagger, a] \\ &= \int dp e^{ip\cdot x} \int dp' E_{p'} a^\dagger \delta^3 (\vec{p} - \vec{p'}) \\ &= \int dp e^{ip\cdot x} E_p a^\dagger \\ &= E_p \int dp e^{ip\cdot x} a^\dagger = E_p \phi_H(x,t) \end{split}$$

So

$$\frac{d\phi_H(x,t)}{dt} = -i\left[\phi_H(x,t), H\right]$$

3) Show that $\int d^3p = \int \frac{d^3\vec{p}}{2E_p}$ is Lorentz invariant. (2 points) We will start with a manifestly Lorentz invariant integral and show that it is the same as $\int d^3p$.

$$\int d^4p \; \delta(E^2 - (|\vec{p}|^2 + m^2)) = \int dE d^3p \delta(E^2 - (|\vec{p}|^2 + m^2))$$

Now, $dE^2 = 2EdE$ or $dE = dE^2/2E$, so

$$= \int \frac{dE^2}{2E} d^3p \delta(E^2 - (|\vec{p}|^2 + m^2))$$

Can now do the integral over dE^2

$$= \int \frac{d^3p}{2\sqrt{|\vec{p}|^2 + m^2}} = \int d^3p$$

4) Anti-Particles (5 points)

a) Expand $\Phi^{\dagger^2}\Phi^2$ in terms of a, a^{\dagger} , b, and b^{\dagger} (Ignore the exponentials and integrals)

$$\begin{split} \Phi^{\dagger^2} \Phi^2 &= \left(\phi_-^a + \phi_+^b \right)^2 \left(\phi_+^a + \phi_-^b \right)^2 \sim \left(a + b^\dagger \right)^2 \left(a^\dagger + b \right)^2 \\ &= \left(aa + 2ab^\dagger + b^\dagger b^\dagger \right) \left(a^\dagger a^\dagger + 2a^\dagger b + bb \right) \\ &= \left(aaa^\dagger a^\dagger + 2aaa^\dagger b + aabb + 2ab^\dagger a^\dagger a^\dagger + 4ab^\dagger a^\dagger b + 2ab^\dagger bb + b^\dagger b^\dagger a^\dagger a^\dagger + 2b^\dagger b^\dagger a^\dagger b + b^\dagger b^\dagger bb \right) \end{split}$$

- b) See figure.
- c) Let the charge (Q) of particle a be q_a and the charge of particle b be q_b . Calculate ΔQ for each process.
 - See figure each term goes like $(q_a + q_b)$
- d) What happens if you take $q_a = -q_b$? When $q_a = -q_b$, all processes conserve charge.

