Lecture 7

Review Quantum Mechanics (Dynamics)

$$|\alpha, t_0\rangle \rightarrow |\alpha, t\rangle$$

This is what we mean by time evolution.

In QM, then there has to be an operator associated with taking the first state to the second.

Time Evolution Operator: $U(t, t_0)$

$$|\alpha, t\rangle = U(t, t_0)\, |\alpha, t_0\rangle$$

$U(t, t_0)$ Properties

1. $U^\dagger(t, t_0)U(t, t_0) = 1$ Unitary See this from $\langle \alpha t_0 | \alpha t_0 \rangle = \langle \alpha t | \alpha t \rangle$
2. $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$ Composition Rule
3. $U(t_0, t_0) = 1$

How can we possibly determine what the time operator is ???? (Should be getting old by now...)

Start with infinitesimal time evolution

$$U(t + \epsilon, t) = 1 - i\Omega\epsilon$$

where Ω is a Hermitian operator (b/c) U is unitary

So,

$$|\alpha, t + \epsilon\rangle = (1 - i\epsilon\Omega)\, |\alpha, t\rangle$$

OR,

$$\Omega\, |\alpha, t\rangle = i\frac{|\alpha, t + \epsilon\rangle - |\alpha, t\rangle}{\epsilon} = i\frac{\partial}{\partial t}\, |\alpha, t\rangle$$

in limit $\epsilon \rightarrow 0$

Physical Meaning of Ω:

As before to get the generate for the finite movement you have to exponetiate

$$U(t) = e^{-i\Omega t}$$

Note that Ω has units 1/[time]. Just like energy.

Identify $\Omega = \frac{1}{\hbar}H$ where H is the Hamiltonian operator.

$$i\frac{\partial}{\partial t}|\psi\rangle = \Omega|\psi\rangle = \frac{E}{\hbar}|\psi\rangle$$

Schrodinger Equation

$$i\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle$$

Non-relativistically: $H \sim \frac{p^2}{2m} = -\frac{1}{2m}\frac{\partial^2}{\partial^2 x}$

Time Evolution

$$|\alpha, t\rangle = U(t, t_0)|\alpha, t_0\rangle$$

Case I: H is time independent

$$U(t, t_0) = \lim_{N\to\infty}\prod_i^N e^{-\frac{i}{\hbar}H\Delta t} = e^{\frac{-iH(t-t_0)}{\hbar}}$$

Case II: H is time dependent, but $[H(t), H(t')] = 0$

$$U(t, t_0) = e^{\frac{-i}{\hbar}\int_{t_0}^{t} H(t')dt'}$$

Case III: H is time dependent, and $[H(t), H(t')] \neq 0$

(Note: in general $e^A e^B \neq e^{A+B}$ if $[A, B] \neq 0$)

$$U(t,t_0) = T\left[e^{\frac{-i}{\hbar}\int_{t_0}^{t} H(t')dt'}\right]$$

"Time-ordered product" - Power series expansion with earlier terms on the right (Will come back to this later)

Example

For a time independent Hamiltonian.

$$U(t,0) = e^{-iHt}$$

Choose a basis of eigenstates of H

$$H|n\rangle = E_n|n\rangle$$

and

$$H = \sum_n E_n |n\rangle\langle n|$$

Then

$$U(t,0) = \sum_n e^{-iE_nt}|n\rangle\langle n|$$

Now some arbitrary state:

$$\begin{aligned} |\psi(t)\rangle = U(t,0)|\psi(0)\rangle &= \sum_n U(t,0)|n\rangle\langle n|\psi\rangle \\ &= \sum_n |n\rangle e^{-iE_nt} \underbrace{\langle n|\psi\rangle}_{\text{time independent}} \end{aligned}$$

Now lets talk about the expectation values of an observable and how they change with time....

$$\begin{aligned}
\langle A\rangle(t) = \langle\psi(t)|A|\psi(t)\rangle &= \langle\psi(0)|U^\dagger(t,0)AU(t,0)|\psi(0)\rangle \\
&= \underbrace{\left(\langle\psi(0)|\,U^\dagger(t,0)\right)A\,(U(t,0)\,|\psi(0)\rangle)}_{\text{“Schrodinger Picture”}} \\
&= \underbrace{\langle\psi(0)|\left(U^\dagger(t,0)AU(t,0)\right)|\psi(0)\rangle}_{\text{“Heisenberg Picture”}}
\end{aligned}$$

<u>Schrodinger Picture</u>

- $|\psi(t)\rangle$s move through Hilbert Space guided by U(t)
- Operators are independent of time
- Basis kets (eigenstates of observables) eg: $|x\rangle$ and $|p\rangle$ are <u>independent</u> of time.

<u>Heisenberg Picture</u>

- $|\psi(t)\rangle = |\psi\rangle_H$ is fixed and independent of time
- Operators in Heisenberg picture <u>are</u> time dependent

$$A_H(t) = U^\dagger(t)A_S U(t) = e^{iHt}A_s e^{-iHt}$$

<u>Time dependent perturbation theory</u>

$$H(t) = H_0 + V(t)$$

where V(t) is small (this will always be the case for us)

$$H_0\,|n\rangle = E_n\,|n\rangle$$

Now, a general state at t=0

$$|\psi(0)\rangle = \sum_n c_n\,|n\rangle$$

For V = 0

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |n\rangle$$

For V ≠ 0

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t} |n\rangle$$

where time dependence in $c_n(t)$ due to only V.

Interaction Picture

Define...

$$|\psi(t)\rangle_I \equiv e^{+iH_0 t} |\psi(t)\rangle_S$$

$$A_I \equiv e^{+iH_0 t} A_S e^{-iH_0 t}$$

From these definitions, its clear that

$${}_I\langle\psi(t)|A_I|\psi(t)\rangle_I = {}_S\langle\psi(t)|A_S|\psi(t)\rangle_S$$

When V=0, Heisenberg and Interaction Picture Coincide.

Ok, here's why we care about the interaction picture....

$$\begin{aligned}
i\frac{d}{dt}|\psi(t)\rangle_I &= i\tfrac{d}{dt} e^{iH_0 t}|\psi(t)\rangle_S = e^{iH_0 t}\left(-H_0|\psi(t)\rangle_S + i\tfrac{d}{dt}|\psi(t)\rangle_S\right) \\
&= e^{iH_0 t}\left(-H_0 + (H_0 + V_S)\right)|\psi(t)\rangle_S \\
&= e^{iH_0 t} V_S e^{-iH_0 t} e^{iH_0 t}|\psi(t)\rangle_S \\
&= V_I(t)|\psi(t)\rangle_I
\end{aligned}$$

The interaction picture is a hybrid of the Schrodinger and Heisenberg pictures.

Time evolution of state kets and operators depend on different parts of H.

$$|\psi(t)\rangle_S = \sum_n c_n(t) e^{-iE_n t} |n\rangle = e^{-iH_0 t} \sum_n c_n(t) |n\rangle$$

$$|\psi(t)\rangle_I = e^{iH_0 t} |\psi(t)\rangle_S = \sum_n c_n(t) |n\rangle$$

where,

$$c_n(t) = \langle n|\psi(t)\rangle_I$$

so once we have $\langle\psi(t)\rangle_I$ we are done.

Solve the "Schrodinger Eq" iteratively

$$i\frac{d}{dt} |\psi(t)\rangle_I = V_I(t) |\psi(t)\rangle_I$$

integrate and get

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle + \int_{t_0}^{t} dt' \left[-iV_I(t') |\psi(t')\rangle\right]$$

where the second term is of order V_0 which is small.

Now we keep iterating

$$= |\psi(t_0)\rangle - i \int_{t_0}^{t} dt' V_I(t') \left[|\psi(t_0)\rangle - i \int_{t_0}^{t'} dt'' V_I(t'') |\psi(t'')\rangle \right]$$

at 3rd order (iterate again...)

$$\begin{aligned} = |\psi(t_0)\rangle - & \; i \int_{t_0}^{t} dt' V_I(t') |\psi(t_0)\rangle + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V(t')V(t'') |\psi(t_0)\rangle \\ + & \; (-i)^3 \int \int \int V(t')V(t'')V(t''') |\psi(t_0)\rangle \end{aligned}$$

$$|\psi(t)\rangle = U_I(t, t_0) |\psi(t_0)\rangle$$

where

$$U_I(t, t_0) = 1 + (-i) \int V(t')dt' + (-i)^2 \int \int V(t')V(t'') + ...$$

"Dyson Series", can be written in slick form (by doing the sum)

$$U_I(t, t_0) = T \left[e^{-i \int_{t_0}^{t} V_I(t')dt'} \right]$$

Where the "T" stands for a "time ordered product"