

Homework Set #2

Solutions

1) Show that $SO(2) \simeq U(1)$

(2 points)

(a)

$$zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Given a vector in the complex plane specified by (x,y), zz^* gives the length of the vector.

(b)

$$M(\theta_1) : z \rightarrow e^{i\theta_1} z \text{ (+ complex conjugate)}$$

$$M(\theta_2) : z \rightarrow e^{i\theta_2} z \text{ (+ complex conjugate)}$$

$$M(\theta_1)M(\theta_2) : z \rightarrow e^{i\theta_1} e^{i\theta_2} z = e^{i(\theta_1+\theta_2)} z = M(\theta_1 + \theta_2)$$

2) Work out the algebra of the generators of the Lorentz group

(5 points) Assuming:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T_i, T_i] = 0$$

$$[T_1, T_2] = J_3$$

$$[T_1, T_3] = J_2$$

$$[T_2, T_3] = J_1$$

OR

$$[T_i, T_j] = \epsilon_{ijk} J_k$$

$$[T_1, J_1] = 0$$

$$[T_1, J_2] = T_3$$

$$[T_1, J_3] = T_2$$

$$[T_2, J_1] = T_3$$

$$[T_2, J_2] = 0$$

$$[T_2, J_3] = -T_1$$

$$[T_3, J_1] = -T_2$$

$$[T_3, J_2] = -T_1$$

$$[T_3, J_3] = 0$$

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

3) Connection to β s and γ s

(5 points)

(a)

$$e^{I\eta} = 1 + I\eta + \frac{I^2\eta^2}{2!} + \frac{I^3\eta^3}{3!} + \frac{I^4\eta^4}{4!} + \dots$$

$$I_B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Show that $B(\eta) = e^{I_B\eta} = \cosh(\eta) + I_B \sinh(\eta)$

$$e^{I\eta} = I \left(\eta + \frac{I^2\eta^3}{3!} + \frac{I^4\eta^5}{5!} + \dots \right) + \left(1 + \frac{I^2\eta^2}{2!} + \frac{I^4\eta^4}{4!} + \dots \right)$$

$$e^{I\eta} = I \left(\eta + \frac{\eta^3}{3!} + \frac{\eta^5}{5!} + \dots \right) + \left(1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \dots \right) = I \sinh(\eta) + \cosh(\eta)$$

(b)

4) Z Boson decays

(5 points)

(a) Same derivation as we did for the $\pi \rightarrow \gamma\gamma$ decay in class gives

$$p_{e_1} = (m_Z/2, 0, 0, m_Z/2) \quad p_{\gamma_2} = (m_Z/2, 0, 0, -m_Z/2)$$

(b) Including the mass term gives...

$$p_{e_1} = (E_1, 0, 0, P_1) \quad p_{\gamma_2} = (E_2, 0, 0, -P_2)$$

Momentum conservation implies $P_1 = -P_2 \equiv P$

Energy conservation implies $M_Z = E_1 + E_2$

So,

$$p_{e_1} = (E, 0, 0, P) \quad p_{\gamma_2} = (m_Z - E, 0, 0, -P)$$

Imposing $P_1^2 = m_e^2$ gives: $E = \sqrt{m_e^2 + P^2}$

Imposing $P_2^2 = m_e^2$ gives: $P^2 = (m_Z - E)^2 - m_e^2$

Combining implies,

$$E = \frac{m_Z}{2} \quad \text{and} \quad P = \frac{m_Z}{2} \sqrt{1 - 4 \frac{m_e^2}{m_Z^2}}$$

So no correction to the electron Energies.

The the electron momentum is $P \simeq \frac{m_Z}{2} \left(1 - \frac{2m_e^2}{m_Z^2}\right)$, which gives a correction of order $\left(\frac{m_e}{m_Z}\right)^2 \sim \left(\frac{10^{-3} \text{ GeV}}{100 \text{ GeV}}\right)^2 \sim 10^{-10}$

(b) Including the mass term for the b-quark gives...

No correction to the energies.

The b-quark momentum has a correction of order $\left(\frac{m_b}{m_Z}\right)^2 \sim \left(\frac{10 \text{ GeV}}{100 \text{ GeV}}\right)^2 \sim 10^{-2}$ about 1%.

5) GZK cutoff energy

(5 points)

(a)

(b)