

Now it's clear how you could write down interactions that take place locally.

$$H^{\text{free}} + \int d^3x \left[\phi_+(x) \phi_+(x) \phi_-(x) \phi_-(x) \right] + \dots$$

It's totally clear now that this is local in space.

Taking this and expanding out gives something like we just talked about with 2 a's and 2 a's. All the rest comes along for the ride.

(Note this is done non-relativistically for the moment to stress that this has nothing to do w/ relativity. This is about making interactions local)

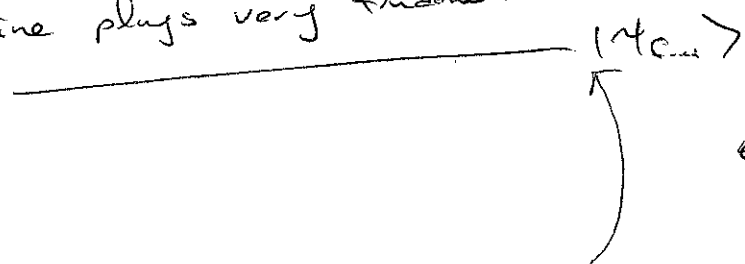
Why we use fields. Makes local interactions of particles manifest. Hardwired into the description of particles.

↑
locality

Where does relativity come in? What is the difficulty?

Time evolution

QM - time plays very fundamental role.



Diff. obs. see diff. spec-like sheets

Have a hope of Lorentz invariance if we start at $-\infty$ & go to $+\infty$.

Throw particles in from $-\infty$ let them scatter & go back out to $+\infty$. S-matrix $t=+\infty$

$$t=-\infty |p_1, \sigma_1, \dots, p_n, \sigma_n\rangle \longrightarrow S | \vec{p}_1, \sigma_1, \dots, \vec{p}_n, \sigma_n \rangle$$

↳ might be (at least a hope) Lorentz invariant.

Figure out what S is in a totally generic theory,
then see what it would take to make it Lorentz Invariant.

Sure doesn't look like it will be L.I. S is the only
object that you could even have a hope to make L.I.

We will see that for very special choices of the interaction
it will barely be possible for it to be Lorentz Invariant.

↳ Those choices force on us anti-commutators and the connection
between spin & statistics.

Something annoying that we should get rid of right away.
Free evolution, just evolves w/ phase. totally invariant part.

Standard way of removing the free evolution
"Interaction Representation"

$$H = H_{\text{free}} + H_{\text{int}}$$

$$i \hbar \frac{d|\psi\rangle}{dt} = (H_{\text{free}} + H_{\text{int}})|\psi\rangle$$

$$\frac{H_{\text{int}}=0}{|\psi\rangle} = e^{-iH_{\text{free}}t} |\psi_{\text{in}}\rangle$$

We don't have free theory, but
if the interaction is small going
to be pretty close to evolving like
this.

$$|\psi\rangle = e^{-iH_{\text{free}}t} |\psi_{\text{I}}\rangle$$

↗
definition

If, $H_{\text{int}}=0$, $|\psi_{\text{int}}\rangle$ doesn't
evolve at all.

B/c the H_{int} $|\psi_{\text{I}}\rangle$ will evolve

$$\begin{aligned} i \hbar \frac{d}{dt} |\psi\rangle &= H_{\text{f}} |\psi\rangle + e^{-iH_{\text{free}}t} i \hbar \frac{d}{dt} |\psi_{\text{int}}\rangle \\ &= (H_{\text{f}} + H_{\text{int}}) e^{-iH_{\text{free}}t} |\psi_{\text{int}}\rangle \end{aligned}$$

$$i \frac{d}{dt} |\psi_{\text{int}}\rangle = \underbrace{\begin{bmatrix} e^{iH_0 t} & 0 \\ 0 & e^{-iH_0 t} \end{bmatrix}}_{H_I} |\psi_{\text{int}}\rangle$$

H_I - interaction hamiltonian in the interaction representation.

$$i \frac{d}{dt} |\psi_{\text{int}}\rangle = H_I |\psi_{\text{int}}\rangle \quad (\text{Use } \psi_I \text{ for } \psi_{\text{int}})$$

↑
can be time dependent.

Formally solve this

$$|\psi_I(t_2)\rangle = |\psi_I(t_1)\rangle - i \int_{t_1}^{t_2} dt H_I |\psi_I(t)\rangle$$

just integrating.

Now can keep going

$$= |\psi_I(t_1)\rangle - i \int_{t_1}^{t_2} dt H_I^{(1)}(|\psi_I(t_1)\rangle) - i \int_{t_1}^{t_2} dt' H_I(t') |\psi_I(t')\rangle$$

$$= |\psi_I(t_1)\rangle - i \int_{t_1}^{t_2} dt H_I(t) |\psi_I(t)\rangle - (-i)^2 \int_{t_1}^{t_2} dt \int_{t_1}^t dt' H_I(t) H_I(t') |\psi_I(t')\rangle$$

Pattern is clear, & can keep on going

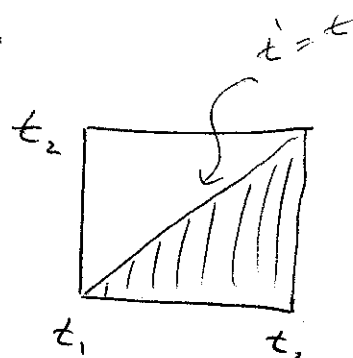
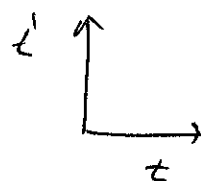
$$|\psi_I(t_2)\rangle = \left[\mathbb{I} + (-i) \int_{t_1}^{t_2} dt H_I(t) + (-i)^2 \int_{t_1}^{t_2} dt \int_{t_1}^t dt' H_I(t) H_I(t') + (-i)^3 \int_{t_1}^{t_2} dt \int_{t_1}^t dt' \int_{t_1}^{t'} dt'' H_I(t) H_I(t') H_I(t'') + \dots \right] |\psi_I(t_1)\rangle$$

If H_I is small this

is giving us some nice perturbation theory.

Look @ the second term.

$$\int_{t_1}^{t_2} dt \int_{t_1}^t dt'$$



Nice to write it over the whole region:

$$\int_{t_1}^{t_2} dt \int_{t_1}^t dt' H_I(t) H_I(t') = \frac{1}{2} \int_{t_1}^{t_2} dt \int_{t_1}^t dt' T(H_I(t) H_I(t'))$$

time ordered product

$$T(A(t) B(t')) = \begin{cases} A(t) B(t') & t > t' \\ B(t') A(t) & t < t' \end{cases}$$

$$\left[1 + (-i) \int_{t_1}^{t_2} dt T(H_I(t)) + \frac{(-i)^2}{2} \int_{t_1}^{t_2} dt \int_{t_1}^t dt' T(H_I(t) H_I(t')) + \frac{(-i)^3}{3!} \int_{t_1}^{t_2} dt \int_{t_1}^t dt' \int_{t_1}^{t'} dt'' T(H_I(t) H_I(t') H_I(t'')) + \dots \right]$$

$$|\psi_I(t_2)\rangle = T \left(e^{-i \int_{t_1}^{t_2} dt H_I(t)} \right) |\psi_I(t_1)\rangle$$

Now,

$$|\psi_I(+\infty)\rangle = T \left(e^{-i \int_{-\infty}^{\infty} dt H_I(t)} \right) |\psi_I(-\infty)\rangle$$

Lets go back to field theory.

$$\phi_+(x) = \int d^3p e^{i\vec{p}\cdot\vec{x}} a_{\vec{p}}^\dagger$$

$$\phi_+(x) = \int d^3p e^{i\vec{p}\cdot\vec{x}} a_{p0} C_0(p)$$

← use scalars for the moment.

Need to build H_I out of $\phi_{+/-}$ in the interaction representation.

$$\phi_+^I(x,t) = e^{-iH_0 t} \phi_+(x) e^{iH_0 t}$$

$$* \begin{matrix} e^{-iH_0 t} & e^{iH_0 t} & \rightarrow & e^{-iE_0 t} \\ \phi(x) & & & \phi(x) \end{matrix}$$

$$= \int d^3p e^{i\vec{p}\cdot\vec{x}} e^{-iE_0 t} a_{\vec{p}}^\dagger$$

$$= \int d^3p e^{-i\vec{p}\cdot\vec{x}_n} a_{\vec{p}}^\dagger$$

Behaves nicely under Lorentz transforms.

$$\phi_*(1x) = \phi_+(x)$$

We seem to be in a awesome shape, Lets write down an interaction.

$$H^I = \int d^3x \left[k \phi_+^I(x) \dots \phi_-^I(x) \dots \right]$$

$$T e^{-i \underbrace{\int dt d^3x}_{d^4x} [k \phi_+^I(x) \dots \phi_-^I(x) \dots]}$$

$$T e^{-i \int d^4x [k \phi_+(x) \dots \phi_-(x)]}$$

All of this is Lorentz invariant!

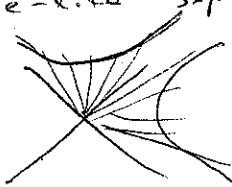
Seems like we are done.

Problem is the time ordering. Only thing that is not necessarily L.I.

write in a form: $T e^{-i \int d^4x \mathcal{H}_I(x)}$ this would be L.I. if T was Lorentz I. But space-like separated objects are not time-ordered in a L.I. way.

Only way to be L.I. if $\mathcal{H}_I(x) \neq$

$\mathcal{H}_I(x')$ commutes when x & x' space-like separated.



Lorentz Invariance $\Rightarrow [\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0$ if $x-x' < 0$
 space-like.

Can ask if the ϕ^\dagger 's or the ϕ 's commute outside the light cone
 \rightarrow they do Not

However can find new combination which does.

Scaling Results

Turns out $[\phi_+(x), \phi_+^\dagger(x')] \neq 0$ for $(x-x') < 0$

$$\Phi(x) = \phi_+(x) + \phi_-(x) = \int \frac{d^3p}{(2\pi)^3} (a_p^\dagger e^{-ipx} + a_p e^{ipx})$$

Scalars have to be Bosons!

$$[\Phi(x), \Phi(x')] = 0 \text{ for } (x-x') < 0$$

Only if $[a, a^\dagger] = 0$
 not if $[a, a^\dagger] \neq 0$

\hookrightarrow tells us something quite significant.

Build \mathcal{H}_I out of Φ . (Not ϕ_+ & ϕ_- separately)

$$\mathcal{H}_I = \lambda \Phi^4(x)$$

term we saw before

$$= \lambda (\phi_+ + \phi_-)^4 = \lambda \left[\phi_+^2 \phi_-^2 + \underbrace{\phi_+^4 + \phi_+^3 \phi_- + \dots}_{\text{New terms}} \right]$$

2 go in 2 out 1 go in, 3 go out

No charges associated with this scalar.

Can create it or destroy it @ will (eg $\phi_+ \phi_-^3$)

How do we talk about particles w/ conserved charge?

This will not work! Only have one choice, introduce another operator. a 's & b 's

$$\begin{aligned} \phi^a &= \int \frac{d^3p}{(2\pi)^3} a_p^\dagger e^{i\vec{p}\cdot\vec{x}} \\ \phi^b &= \int \frac{d^3p}{(2\pi)^3} b_p^\dagger e^{i\vec{p}\cdot\vec{x}} \end{aligned} \quad \Phi = \phi^a + \phi^b$$

Not hermitian

$$= \int \frac{d^3p}{(2\pi)^3} [a_p e^{ipx} + b_p^\dagger e^{-ipx}]$$

$$[\Phi(x), \Phi(x')] = 0 \text{ for } (x-x') < 0$$

$$\int d^3x \bar{\Phi}^\dagger(x)^2 \Phi(x)^2 \quad \checkmark \text{ Hermitian}$$

~~***~~ Expand this out and find that every thing would conserve charge provided that particles of type b have opposite charge to a.

If you want to talk about particles that carry some well defined charge, you must have anti-particles.

If you put this together with $s=1/2$ (putting back the c_0) you find that they have to have $\xi^2=0$ for the Hamiltonian to vanish outside the light cone.