

Homework Set Lagrangians

Due Date: Before class Friday February 22nd

2) Use the Weyl basis of the γ matrices to solve the Dirac equation for a spinor ψ (5 points)

See Text Section 2.2

3) Clifford Algebra (5 points)

In writing the Dirac equation, we chose a particular representation of the γ matrices that satisfied $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, which is called the Clifford algebra. The choice we used in class is called the Weyl basis. In this problem, we will study the Clifford algebra and the Weyl basis.

a) Will calculate:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu$$

Three cases to consider:

Case 1) $\mu = \nu = 0$

$$\gamma_0 \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So,

$$\gamma_0 \gamma_0 + \gamma_0 \gamma_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Case 2) $\mu = 0, \nu = i$ (and reversed)

$$\gamma_0 \gamma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

$$\gamma_i \gamma_0 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

So,

$$\gamma_0 \gamma_i + \gamma_i \gamma_0 = \gamma_i \gamma_0 + \gamma_0 \gamma_i = 0$$

Case 3) $\mu = i, \nu = j$

$$\gamma_i \gamma_j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i \sigma_j & 0 \\ 0 & -\sigma_i \sigma_j \end{pmatrix} = \begin{pmatrix} -\delta_{ij} & 0 \\ 0 & -\delta_{ij} \end{pmatrix}$$

So,

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}$$

b)

$$\gamma_\mu = S \gamma_\mu^{\text{Weyl}} S^\dagger,$$

Multiply both sides on left by S.

$$\gamma_\mu S = S \gamma_\mu^W$$

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are 2x2 matrices.

Consider $\mu = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

so, $a = b, c = -d$

Now require $S S^\dagger = 1$

$$\begin{pmatrix} a & a \\ c & -c \end{pmatrix} \begin{pmatrix} a^\dagger & c^\dagger \\ a^\dagger & -c^\dagger \end{pmatrix} = \begin{pmatrix} 2aa^\dagger & ac^\dagger - ac^\dagger \\ ca^\dagger - ca^\dagger & 2cc^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So a and c are hermitian. The only 2x2 matrices that are hermitian are I or σ_k , so $\sqrt{2}a = \pm I$ or $\pm\sigma_a$ and same for c.

Now Consider $\mu = i$

$$\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} a & a \\ c & -c \end{pmatrix} = \begin{pmatrix} a & a \\ c & -c \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\begin{pmatrix} \sigma_i c & -\sigma_i c \\ -\sigma_i a & -\sigma_i a \end{pmatrix} = \begin{pmatrix} -a\sigma_i & a\sigma_i \\ c\sigma_i & c\sigma_i \end{pmatrix}$$

Assume both a and c are pauli matrices, then, $\sigma_i \sigma_c + \sigma_a \sigma_i = \delta_{ic} + \delta_{ai} = 0$, which cannot hold when $i = a$ or c . So a and c cannot both be pauli matrices.

Assume one (a) is a pauli matrices and c is I then, $\sigma_i + \sigma_a \sigma_i = \sigma_i + \delta_{ai} = 0$, which also cannot be impossible.

So both A and C are $\pm I$ and above implies that $a = -c$

$$\text{So } S = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \text{ and } S^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

c) Applying the similarity transformation to the Dirac equation, gives

$$(iS\gamma_\mu S^\dagger \partial^\mu - m)\psi = S(i\gamma \cdot \partial - m)S^\dagger \psi = 0.$$

$$S(i\gamma \cdot \partial - m)S^\dagger \psi = 0 \Rightarrow (i\gamma \cdot \partial - m)S^\dagger \psi = 0$$

Where here ψ is the solution in the new basis, and $S^\dagger \psi$ is the solution in the old basis.

$$\text{We know from class that } S^\dagger \psi = \psi_+ = \begin{pmatrix} C \\ C \end{pmatrix} e^{-imt} \text{ or } \psi_- = \begin{pmatrix} C \\ -C \end{pmatrix} e^{imt}$$

$$\text{So } \psi = S\psi_+ = \sqrt{2} \begin{pmatrix} C \\ 0 \end{pmatrix} e^{-imt} \text{ or } S\psi_- = -\sqrt{2} \begin{pmatrix} 0 \\ C \end{pmatrix} e^{imt} \text{ which is either}$$

4) Show that $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu$ is gauge invariant **(5 points)**

Gauge invariance implies that the \mathcal{L}_{EM} is unchanged under $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$

Look at,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Under a gauge transformation

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &\rightarrow \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + (\partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}$$

because the partial derivatives commute.

So the $F_{\mu\nu}$ term is gauge invariant, which means that the first term in \mathcal{L}_{EM} is also gauge invariant.

The second term transforms as:

$$J^\mu A_\mu \rightarrow J^\mu (A_\mu + \partial_\mu \lambda) = J^\mu A_\mu + J^\mu \partial_\mu \lambda$$

Because \mathcal{L} lives in an integral over all space, we can integrate by parts to move the derivative from λ to J^μ

$$J^\mu A_\mu \rightarrow J^\mu (A_\mu + \partial_\mu \lambda) = J^\mu A_\mu - (\partial_\mu J^\mu) \lambda$$

Conservation of charge implies that $\partial_\mu J^\mu = 0$, which means that $J^\mu A_\mu$ is also gauge invariant.

5) Maxwell's Equations. **(5 points)**

- a) In class, I mentioned that Gauss's law follows from taking the 0 component of the equations of motion of the electromagnetic Lagrangian $\partial_\mu F^{\mu\nu} = J^\nu$.

$$J^\mu = (\rho, \vec{J})$$

Lets look at $\nu = 0$,

$$\partial_\mu F^{\mu 0} = J^0 \quad (1)$$

$$\partial_0 \underbrace{F^{00}}_0 - \partial_i(-E_i) = \rho \quad (2)$$

$$\partial_i E_i = \rho \quad (3)$$

$$\vec{\nabla} \cdot \vec{E} = \rho \quad (4)$$

$$(5)$$

- b) Show that the three other of Maxwell's equations follow from the equations of motion and the Bianchi identity ($\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$). Lets look at $\nu = i$

$$\partial_\mu F^{\mu i} = J^i \quad (6)$$

$$\partial_0 F^{0i} - \partial_j F^{ji} = J^i \quad (7)$$

$$\partial_t(-E_i) - \partial_j F^{ji} = J^i \quad (8)$$

$$(9)$$

Now,

$$\partial_j F^{ji} = -\vec{\nabla} \times \vec{B}$$

So,

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J} \quad (10)$$

now taking $\mu = 0, \nu = i$ and $\rho = j$ in the bianchi identity imples:

$$\partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} = 0$$

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0$$

choosing $i = j = k$ gives

$$\vec{\nabla} \cdot \vec{B} = 0$$