Lorentz Invariance & "Soft Limits"

Punch line that we've been building to...

Matrix Element we would get by scattering external γ

$$\mathcal{M} = \varepsilon^{\mu} \mathcal{M}_{\mu}$$

↪ where ε^{μ} is some linear combination of two photon polarization vectors ε^{1} & ε^{2}

$\mathcal{M}$ is L.I., under Lorentz transformation

$$\mathcal{M} \to \varepsilon'^{\mu} \mathcal{M}'_{\mu}$$

where $\mathcal{M}'_{\mu} = \Lambda_{\mu}^{\ \nu} \mathcal{M}_{\nu}$

Hence ε is not really a full 4-vector
Only 2 components

Under little group transformations (you will show in your H.W.)

~~$\varepsilon^{\mu} \to \varepsilon^{\mu} + C p^{\mu}$ ← photon momentum~~

$$\varepsilon^{\mu} \to \underbrace{c_1 \varepsilon_1^{\mu} + c_2 \varepsilon_2^{\mu}}_{\text{only this piece } \varepsilon'^{\mu}} + c_3 p^{\mu}$$

↑ Not valid "Not in Hilbert Space"

So,

~~$\mathcal{M} = \varepsilon^{\mu} \mathcal{M}_{\mu} \to (\varepsilon^{\mu} + C p^{\mu}) \mathcal{M}'_{\mu}$ → Not a valid polarization vector~~

Only L.I. if we have $p^{\mu} \mathcal{M}'_{\mu} = 0$

or $p^{\mu} \mathcal{M}_{\mu} = 0$ ← (By Little group $p' = p$ & L I dropping ''s)

So, $\mathcal{M} = \varepsilon^{\mu} \mathcal{M}_{\mu} \to (c_1 \varepsilon_1^{\mu} + c_2 \varepsilon_2^{\mu} + c_3 p^{\mu}) \mathcal{M}'_{\mu}$

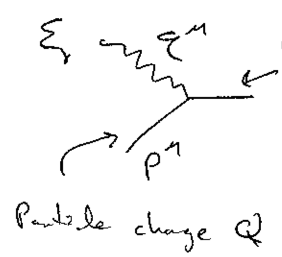
$$= \varepsilon'^{\mu} \mathcal{M}'_{\mu} + c_3 \underline{p^{\mu} \mathcal{M}'_{\mu}}$$

↪ must go to 0

We will see, has enormous implications!!!...

will be considering diagrams with external "γ"s

$\hookrightarrow$ massless spin 1 particles



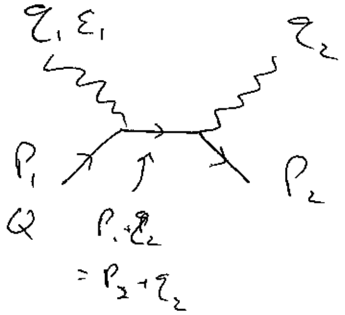
$$= i Q \left(p^\mu + (p^\mu + q^\mu) \right) \varepsilon_\mu \qquad \varepsilon^\mu \varepsilon_\mu = 0$$

$$= i Q \, 2 \, p^\mu \varepsilon_\mu$$

$$\boxed{\begin{array}{l} \text{most general form in} \\ \text{"soft limit" } q \to 0 \\ \Gamma_\mu(p, q) \sim p_\mu F(q^2 \, p^2 \, p\cdot q) \\ \qquad \text{dimensional analysis} \to F\left(\frac{p\cdot q}{--}\right) \end{array}}$$

Consider "Compton Scattering"

Start w/ 1-spin-1 Boson & 1 matter Particle



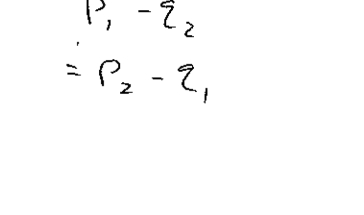
$$= (iQ) \varepsilon^1_\mu (2 p_1^\mu) \frac{i}{(p_1 + q_1)^2 - m^2} (iQ) \varepsilon^2_\nu (2 p_2^\nu)$$

$$= (-i Q^2) \, 4 \, \frac{(p_1 \cdot \varepsilon_1)(p_2 \cdot \varepsilon_2)}{m^2 + 2 p_1 \cdot q_1 - m^2} = \varepsilon_1^\mu \varepsilon_2^\nu \underbrace{\left(\frac{(-iQ^2) \, 2 \, p^1_\mu p^2_\nu}{p_1 \cdot q_1} \right)}_{M^{A}_{\mu\nu}}$$

As we said above, LI $\Rightarrow$ $q_1^\mu q_2^\nu M_{\mu\nu} = 0$

But $q_1^\mu q_2^\nu M_{\mu\nu} = (-iQ^2) 2 (p_2 \cdot q_2) \neq 0$! Looks like we're dead...

forgetting diagram



$$= (iQ) \varepsilon_1^\mu (2 p_{2\mu}) \frac{i}{(p_2 - q_1)^2 - m^2} (iQ) \varepsilon_2^\nu (2 p_{1\nu})$$

$$= \varepsilon_1^\mu \varepsilon_2^\nu \left(\frac{(-iQ^2) \, 4 \, p^2_{2\mu} p_{1\nu}}{-2 p_2 \cdot q_1} \right) \approx \underbrace{\varepsilon_1^\mu \varepsilon_2^\nu \left(\frac{-(-iQ^2) \, 2 \, p^1_{\mu} p^2_\nu}{p_1 \cdot q_1} \right)}_{M^{B}_{\mu\nu}}$$

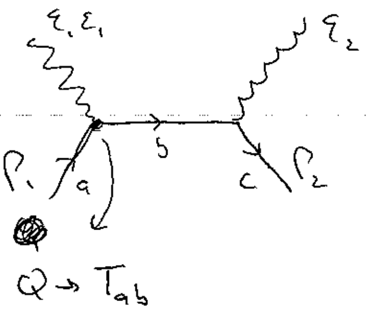
soft limit
$p_1 = p_2$

$$\varepsilon_1^{\mu} \varepsilon_2^{\nu} M_{\mu\nu}^{B} = -(iQ^2)\, 2\,(p_2 \cdot \varepsilon_2)$$

So sum $M_{\mu\nu}^{A} + M_{\mu\nu}^{B}$ is L.I. (residual $\overset{\text{non L.I}}{\text{pieces}}$ of (A) & (B) cancel.)

Same as before but with many diff't matter particles

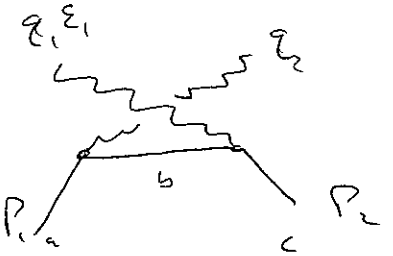
$i = 1, \ldots N_{\text{matter}}$



$Q \to T_{ab}$

$$= \frac{(i T_{ab})\, \varepsilon_\mu^1 (2 p_1^{\mu}) \quad i \quad (i T_{bc})\, \varepsilon_\nu^2 (2 p_2^{\nu})}{\underbrace{(p_1 + q_1)^2 - m_b^2}_{m_a^2 + 2 p_1 \cdot q_1 - m_b^2}}$$

$$\varepsilon_\mu^1 \varepsilon_\nu^2 \left[\frac{(-i T_{ab} T_{bc})\, 4\, p^{1\mu} p^{2\nu}}{m_a^2 + 2 p_1 \cdot q_1 - m_b^2} \right] = M^{\mu\nu}_{(A)}$$



$$\Rightarrow \quad \varepsilon_\mu^1 \varepsilon_\nu^2 \left[\frac{(-i T_{ab} T_{bc})\, 4\, p^{2\mu} p^{1\nu}}{m_a^2 - 2 p_2 \cdot q_1 - m_b^2} \right] \equiv M^{\mu\nu}_{(B)}$$

If $m_a = m_b = m_c$ $\quad \varepsilon_\mu^1 \varepsilon_\nu^2 \left(M^{\mu\nu}_{(A)} + M^{\mu\nu}_{(B)} \right) = 0$ as above

-) $m_a^2 - m_b^2 = 0$
-) Relative − sign

However if $m_a \neq m_b$, soft limit

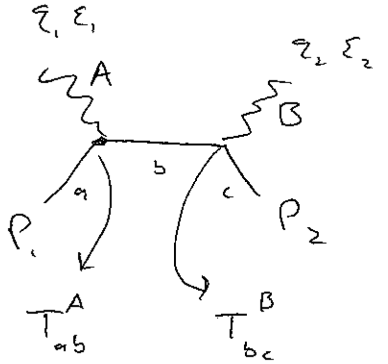
$$\varepsilon_\mu^1 \varepsilon_\nu^2 \left(M^{\mu\nu}_{(A)} + M^{\mu\nu}_{(B)} \right) = \varepsilon_\mu \varepsilon_\nu \left[\frac{(-i T_{ab} T_{bc})\, 4}{m_a^2 - m_b^2} \left(2 p^{1\mu} p^{2\nu} \right) \right] \neq 0$$

Massless Spin-1 particles can only interact w/ Particles of Same mass!

Now allow many different | matter fields (but same mass!) & many force carriers "gluons" (4)

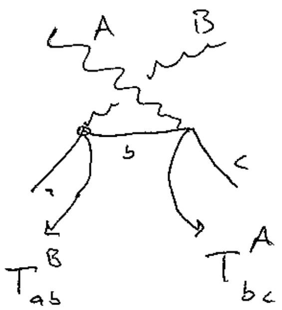
$i = 1, \ldots N_{matter}$

$I = 1, \ldots N_{gluons}$



$$= \underbrace{\frac{(-i\, T^A_{ab}\, T^B_{bc})\, 4 p_{1\mu} p_{2\nu}}{2\, p_1 \cdot q_1}}_{M^{(A)}_{\mu\nu}} \varepsilon_1^\mu \varepsilon_2^\nu$$

soft limit



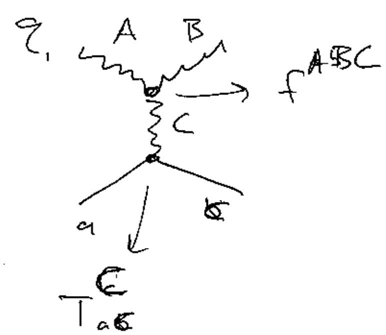
$$= \underbrace{\frac{(-i\, T^B_{ab}\, T^A_{bc})\, 4 p_{1\mu} p_{\nu}}{-2\, p_1 \cdot q_1}}_{M^{(B)}_{\mu\nu}} \varepsilon_1^\mu \varepsilon_2^\nu$$

$$q_1^\mu q_2^\nu \left(M^{(A)}_{\mu\nu} + M^{(B)}_{\mu\nu}\right) = 2(-i)(p_2\cdot q_2)\left(T^A_{ab} T^B_{bc} - T^B_{ab} T^A_{bc}\right)$$

$$= -2i\,(p_2\cdot q_2)\,\left[T^A, T^B\right]$$

Not $= 0$ for random T^A & T^B s

In fact, missing diagram



$$= +2i\,(p\cdot q)\; i f^{ABC}\, T^C_{ab}$$

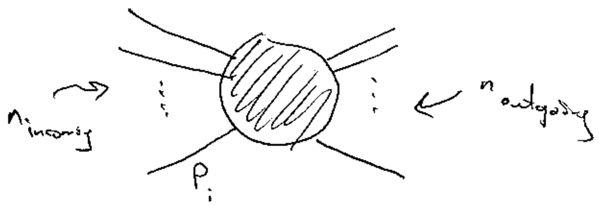
"gluons must transform as Lie group"

Only LI if

$$\left[T^A, T^B\right] = i f^{ABC}\, T^C$$

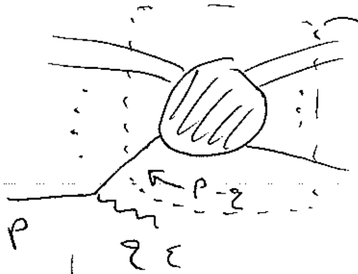
Yang-Mills Interaction

Now do the same thing to a more general interaction (5)



$$\equiv i M_0(p_i)$$

Consider what happens if we attach "photon" to incoming leg

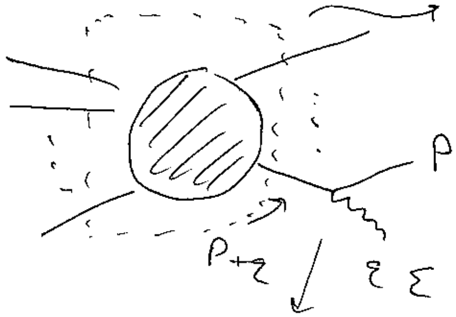


$$\to i M_0(p_i - q)$$

$$\cancel{-Q\frac{(p\cdot\epsilon)}{-(p\cdot q)}} = Q\,\frac{(p\cdot\epsilon)}{(p\cdot q)}\, i M_0(p_i - q)$$

$$\to i(iQ)\frac{\left(\epsilon_\mu\left(p_i^\mu + (p_i^\mu - q^\mu)\right)\right)}{(p_i - q)^2 - m^2}$$

Can also attach photon to outgoing leg



$$\to i M_0(p_i + q)$$

$$= -Q\,\frac{(p\cdot\epsilon)}{(p\cdot q)}\, i M(p_i + q)$$

$$\frac{(-Q)\,\epsilon_\mu\left(p_i^\mu + (p_i^\mu + q^\mu)\right)}{(p+q)^2 - m^2}$$

Total Amplitude

$$M = \sum_{\text{incoming}} Q_i\,\frac{(p\cdot\epsilon)}{(p\cdot q)}\, i M_0(p - q) + \sum_{\text{outgoing}} -Q_i\,\frac{(p\cdot\epsilon)}{(p\cdot q)}\, i M(p_i + q)$$

Soft limit $\quad M_0(p \mp q) \to M_0(p)$

$$= i M_0\left(\sum_{\text{incoming}} Q\,\frac{(p\cdot\epsilon)}{(p\cdot q)} + \sum_{\text{outgoing}} -Q\,\frac{(p\cdot\epsilon)}{(p\cdot q)}\right)$$

now as before $\varepsilon_\mu \to \varepsilon'_\mu + q_\mu$ means that M must vanish when $\varepsilon_\mu \to q_\mu$

OR under LT

$$\varepsilon \cdot M \to \varepsilon' \cdot M' + \underbrace{i M \left(\sum_{\text{incoming}} Q - \sum_{\text{outgoing}} Q \right)}$$

$\hookrightarrow = 0$ only if $\sum_{\text{incoming}} Q = \sum_{\text{outgoing}} Q$ (charge conserved!)

Now same logic for Spin-2 (describes interaction w/ Gravitons)

Same as above except 2-component polarization vector

$$\underset{\substack{\nearrow \\ \text{under little group}}}{\varepsilon_{\mu\nu} \longrightarrow} \varepsilon_{\mu\nu} + \underbrace{\lambda_\mu q_\nu + \lambda_\nu q_\mu + \lambda q_\mu q_\nu}$$

$\hookrightarrow$ effect from all these needs be 0 as before.

$= i(i\kappa)\, \varepsilon_{\mu\nu} \dfrac{(2 p^\mu p^\nu)}{-p \cdot q}$ (Same idea with outgoing leg)

$$\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + q_\mu \lambda_\nu$$

$$\varepsilon_{\mu\nu} M^{\mu\nu} \to \varepsilon'_{\mu\nu} M'^{\mu\nu} + M \left(\sum_{\text{incoming}} K_i \lambda_\nu p^\nu - \sum_{\text{outgoing}} K_o \lambda_\nu p^\nu \right)$$

$$+ M \lambda_\nu \left(\sum_{\text{in}} K_i p^\nu - \sum_{\text{out}} K_{o} p^\nu \right) \Rightarrow K_i p_i^\nu \text{ is } \underline{\underline{\text{conserved}}}$$

We know that p_i^ν is conserved by E & mom.

Only way can have nontrivial solutions is if

$$k_i = k \text{ for all } i$$

All particles interact w/ gravity with the same strength

Gravitational interaction is universal!

"Principle of Equivalence"

Can keep going ...

For massless spin 3 we would need

$$\sum_{\text{incoming}} \beta_i \, p_\nu^i \, p_\mu^i = \sum_{\text{outgoing}} \beta_i \, p_\nu^i \, p_\mu^i$$

eg $\mu\nu = 0$ $\quad \displaystyle\sum_{\text{incoming}} \beta_i E_i^2 = \sum_{\text{outgoing}} \beta_i E_i^2$

way too ~~..~~ constraining only way if $\beta_i = 0$

No interacting theories of massless particles of Spin greater than 2