Quantum Dynamics

$$|\alpha, t_0\rangle \xrightarrow[\text{time evolution}]{} |\alpha, t\rangle$$ time evolution operator $U(t, t_0)$

$$|\alpha, t\rangle = U(t, t_0)\, |\alpha\, t_0\rangle$$

$U(t, t_0)$ Properties

1. $U^\dagger(t, t_0)\, U(t, t_0) = I$ Unitary $\langle \alpha\, t_0 | \alpha\, t_0\rangle = \langle \alpha\, t | \alpha\, t\rangle$

2. $U(t_2\, t_0) = U(t_2\, t_1)\, U(t_1\, t_0)$ Composition Rule

3. $U(t_0, t_0) = I$

How can we possibly determine the time evolution operator???
(should be getting old by now...)

Infinitesimal time evolution

$$U(t+\varepsilon, t) = I - i\Omega\, \varepsilon$$ (Ω ← operator) Ω - hermitian B/c U unitary

$$|\alpha, t+\varepsilon\rangle = (1 - i\varepsilon\Omega)\, |\alpha\, t\rangle$$

$$\Omega |\alpha\, t\rangle = i\, \frac{|\alpha, t+\varepsilon\rangle - |\alpha\, t\rangle}{\varepsilon} = i\, \frac{\partial}{\partial t} |\alpha\, t\rangle$$

$$\lim_{\varepsilon \to 0} = \frac{\partial}{\partial t} |\alpha\, t\rangle$$

~~Make~~ Show Physical Meaning of Ω:

$$U(t) = e^{-i\Omega t}$$

$$\cancel{\psi(x,t) = \langle x | U(t) | \psi \rangle = e^{i(xk - \Omega t)} \psi(0)}$$

$$\boxed{E = \hbar\omega \quad \Omega \sim \frac{1}{[\text{time}]}}$$

$$\cancel{i \frac{\partial}{\partial t} \psi(x,t) = \omega \psi(x,t)}$$

Identify

$$i \frac{\partial}{\partial t} |\psi\rangle = \frac{E}{\hbar} |\psi\rangle = \Omega |\psi\rangle \qquad \Omega = \frac{1}{\hbar} H$$

Hamiltonian Operator

Schrodinger Equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H(t) |\psi\rangle$$

$$\boxed{\text{N.R.} \quad H \sim \frac{p^2}{2m} = -\frac{1}{2m} \frac{\partial^2}{\partial x^2}}$$

<u>Time Evolution</u>

$$|\alpha\, t\rangle = U(t, t_0) |\alpha\, t_0\rangle$$

Case I: H is time independent

$$U(t,t_0) = \lim_{N \to \infty} \prod_i^N e^{-\frac{i}{\hbar} H \Delta t} = e^{-\frac{i H (t - t_0)}{\hbar}}$$

Case II: H is time dependent, but $[H(t), H(t')] = 0$

$$U(t,t_0) = e^{-i/\hbar \int_{t_0}^{t} H(t')\, dt'}$$

Case III: H is time dependent & $[H(t), H(t')] \neq 0$

$$\left[\text{in general } e^A e^B \neq e^{A+B} \text{ if } [A,B] \neq 0 \right]$$

$$U(t,t_0) = T\left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'} \right]$$

Power Series Expansion
Earliest term on right

$T \longrightarrow$ time ordered product
(Come back to this later...)

Example

For a time independent hamiltonian

$$U(t,0) = e^{-iHt}$$

Choose a basis of eigenstates of H

$$H|n\rangle = E_n|n\rangle \qquad H = \sum_n E_n |n\rangle\langle n|$$

$$U(t,0) = \sum_n e^{-iE_nt} |n\rangle\langle n|$$

$$|\psi(t)\rangle = U(t,0)|\psi(0)\rangle = \sum_n U(t,0)|n\rangle\langle n|\psi\rangle$$

$$= \sum |n\rangle e^{-\frac{i}{\hbar}E_nt} \underbrace{\langle n|\psi_0\rangle}_{\text{time independent.}}$$

Expectation values of an observable

$$\langle A \rangle(t) = \langle \psi(t)|A|\psi(t)\rangle = \langle \psi(0)|U^{\dagger}(t,0)\, A\, U(t,0)|\psi(0)\rangle$$

$$= \left(\langle \psi(0)|U^{\dagger}(t,0)\right) A \left(U(t,0)|\psi(0)\rangle\right)$$

"Schroding Picture"

$$= \langle \psi(0)|\left(U^{\dagger}(t,0)\, A\, U(t,0)\right)|\psi(0)\rangle$$

"Heisenberg Picture"

Schrodinger Picture

-) $|\psi(t)\rangle_S$ moves through Hilbert Space guided by $U(t)$

-) Operators are independent of time

-) Basis kets (eigenstates of observables)
eg $|x\rangle$ $|p\rangle$ are independent of time.

Heisenberg Picture

- $|\psi(t)\rangle = |\psi\rangle_H$ is fixed & independent of time.

- Operators in Heisenberg Picture are time dependent

$$A_H(t) = U^{\dagger}(t)\, A_S\, U(t) = e^{iHt} A_S e^{-iHt}$$

Time dependent Perturbation theory

$$H(t) = H_0 + V(t) \leftarrow \text{small}$$

this will always be the case for us.

$$H_0|n\rangle = E_n|n\rangle$$

A general state @ $t=0$

$$|\psi(0)\rangle = \sum_n c_n|n\rangle$$

For $V=0$

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_nt}|n\rangle$$

For $V \neq 0$

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_nt}|n\rangle$$

time dep. in $c_n(t)$ due only to V.

Interaction Picture

$$|\psi(t)\rangle_I \equiv e^{+iH_0t}|\psi(t)\rangle_S$$

* clear that

$${}_I\langle\psi(t)|A_I|\psi t\rangle_I = {}_S\langle\psi(t)|A_S|\psi(t)\rangle_S$$

$$A_I = e^{iH_0t}A_s e^{-iH_0t}$$

When $V=0$, Heisenberg & Interaction Pictures coincide.

$$i\hbar \frac{d}{dt}|\psi t\rangle_I = i\hbar \frac{\partial}{\partial t} e^{iH_0 t}|\psi t\rangle_s = e^{iH_0 t}\left(-H_0 + i\hbar \frac{d}{dt}|\psi t\rangle_s\right)$$

$$= e^{+iH_0 t}\left(-H_0 + (H_0 + V_s)\right)|\psi t\rangle_s$$

$$= e^{iHt} V_s e^{-iH_0 t} e^{iH_0 t}|\psi t\rangle_s$$

$$= V_I(t)|\psi(t)\rangle_I$$

Hybrid of S & H Pictures

time evolution of state kets + operators depend on different parts of H.

$$|\psi(t)\rangle_s = \sum_n c_n(t) e^{-iE_n t}|n\rangle = e^{iH_0 t}\sum_n c_n(t)|n\rangle$$

$$|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_s = \sum_n c_n(t)|n\rangle$$

$c_n(t) = \langle n|\psi(t)\rangle_I$ ← Once we have $|\psi(t)\rangle_I$ we are done.

Solve the "Schrodinger Eq" Iteratively

$$i\frac{d}{dt}|\psi\rangle_I = V_I(t)|\psi\rangle_I$$

$$|\psi(t)\rangle_I = |\psi t_0\rangle_I + \int_{t_0}^{t} dt' \underbrace{\left[-i V_I(t')|\psi t'\rangle\right]}_{\frac{d}{dt}|\psi, t'\rangle}$$

this term is of order V, which is small.

$$= |\psi\, t_0\rangle - i\int_{t_0}^{t} dt'\, V(t')\left[|\psi\, t_0\rangle - i\int_{t_0}^{t'} dt''\, V(t'')\, |\psi\, t''\rangle\right]$$

3rd order

$$= |\psi\, t_0\rangle - i\int_{t_0}^{t} V(t')\,|\psi\, t_0\rangle + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, V(t')\, V(t'')\,|\psi\, t_0\rangle$$

$$+ (-i)^3 \iiint V(t')\, V(t'')\, V(t''')\, |\psi\, t_0\rangle$$

$$|\psi\, t\rangle = \underline{U_I(t, t_0)}\, |\psi\, t_0\rangle$$

$$\longrightarrow I + (-i)\int V(t')\,dt' + (-i)^2 \iint dt'\,dt''\, V(t')\, V(t'') + \dots$$

"Dyson Series"

$$= T\left[e^{-i\int_{t_0}^{t} V_I(t')\,dt'}\right]$$

"time ordered Product"