Soft Photon and Gluon Theorems

Lorentz Invariance and "Soft Limits"

Punch line that we've been building to in first part of this course.

Matrix element we would get by scattering external γ.

$$M = \epsilon^{\mu} M_{\mu}$$

where ϵ^{μ} is some linear combination of two photon polarization vector ϵ^1 and ϵ^2

M is Lorentz Invariant, under Lorentz transformation

$$M \to \epsilon'^{\mu} M'_{\mu}$$

where $M'_{\mu} = \Lambda_{\mu}{}^{\nu} M_{\nu}$

However (here comes the major constraint) ϵ is not a full 4-vector. Only has 2 components.

Under little group transformations (you will show in your H.W.)

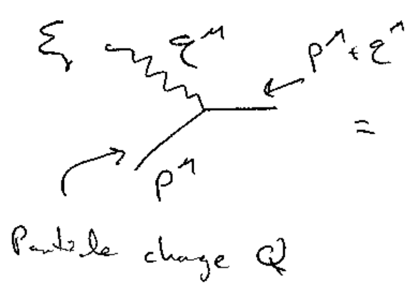
$$\epsilon \to \underbrace{c_1\epsilon_1^{\mu} + c_2\epsilon_2^{\mu}}_{\substack{\epsilon' \text{ can only be made} \\ \text{of these pieces}}} + \underbrace{c_3 p^{\mu}}_{\substack{\text{Not valid} \\ \text{"Not in Hilbert Space"}}}$$

So,

$$\begin{aligned} M = \epsilon^{\mu} M_{\mu} &\to \left(c_1\epsilon_1^{\mu} + c_2\epsilon_2^{\mu} + c_3 p^{\mu}\right) M'_{\mu} \\ &= \epsilon'^{\mu} M'_{\mu} + \underbrace{c_3 p^{\mu} M'_{\mu}}_{\text{Must go to 0}} \end{aligned}$$

We will see, this has enormous implications !!!

Will be considering diagrams with external "γ"s (mass-less spin 1 particles)



$$\begin{aligned} &= iQ(p^{\mu} + (p^{\mu} + q^{\mu}))\epsilon_{\mu} \qquad (q^{\mu}\epsilon_{\mu} = 0) \\ &= iQ2p^{\mu}\epsilon_{\mu} \end{aligned}$$

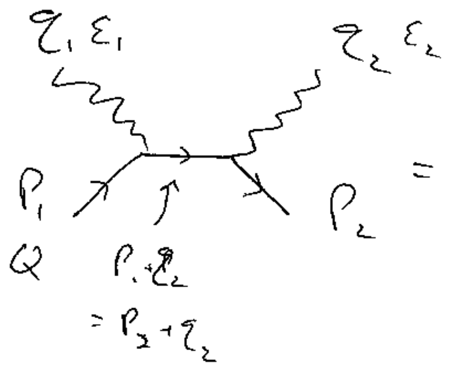
This is the most general form in the "soft limit" $q \to 0$

$$\boxed{\begin{array}{c} \Gamma_\mu \sim p_\mu F(q^2, p^2, p \cdot q) \\ \text{By dimensional analysis } F(q^2, p^2, p \cdot q) \to F(\frac{p \cdot q}{m^2}) \end{array}}$$

Consider "Compton Scattering"

Start with one type of spin-1 boson and one type of matter particle.

The diagram:



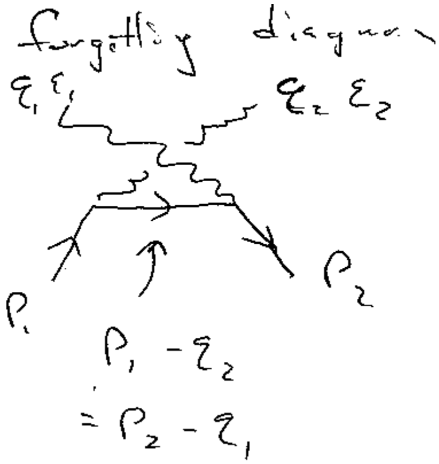
$$\begin{aligned} &= (iQ)\epsilon^1_\mu(2p_1^\mu)\frac{i}{(p_1+q_1)^2 - m^2}(iQ)\epsilon^2_\nu(2p_2^\nu) \\ &= (-iQ^2)4\frac{(p_1 \cdot \epsilon^1)(p_2 \cdot \epsilon^2)}{m^2 + 2p_1 \cdot q_1 - m^2} = \epsilon_1^\mu \epsilon_2^\nu \underbrace{\left(\frac{(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1 \cdot q_1}\right)}_{M_{\mu\nu}} \end{aligned}$$

As we said above, Lorentz Invariant $\Rightarrow q_1^\mu q_2^\nu M_{\mu\nu} = 0$

But here, $q_1^\mu q_2^\nu M_{\mu\nu} = (-iQ^2)2(p_2 \cdot q_2) \neq 0$!

Looks like we're dead...

However we are forgetting a diagram.



$$
\begin{aligned}
&= \quad (iQ)\epsilon^1_\mu(2p_2^\mu)\frac{i}{(p_2-q_1)^2-m^2}(iQ)\epsilon^2_\nu(2p_1^\nu) \\
&= \quad \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iQ^2)4p_{2_\mu}p_{1_\nu}}{-2p_2\cdot q_1}\right) \\
&\underbrace{\sim}_{\text{Soft limit } p_1\,=\,p_2} \quad \epsilon_1^\mu\epsilon_2^\nu\left(\frac{-(-iQ^2)2p_{1_\mu}p_{2_\nu}}{p_1\cdot q_1}\right)
\end{aligned}
$$

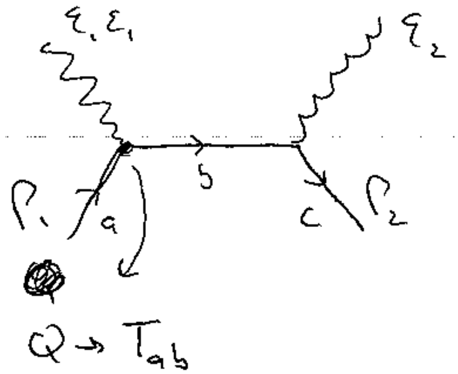
and for this diagram, $q_1^\mu q_2^\nu M_{\mu\nu} = -(-iQ^2)2(p_2\cdot q_2)$

So the sum $M^A_{\mu\nu} + M^B_{\mu\nu}$ is Lorentz Invariant. (Residual non Lorentz Invariant pieces of each diagram cancel)

Very good!

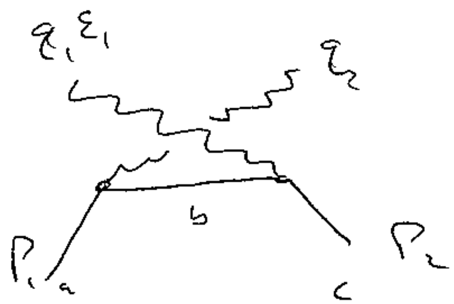
Now lets do the same thing as before, but with many different possible matter particles.

$i = 1, ...N_{\text{matter}}$



$$
\begin{aligned}
&= (iT_{ab})\epsilon^1_\mu(2p_1^\mu)\frac{i}{\underbrace{(p_1+q_1)^2-m^2}_{m_a^2+2p_1\cdot q_1-m_b^2}}(iT_{bc})\epsilon^2_\nu(2p_2^\nu) \\
&= \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iT_{ab}T_{bc})4p_{1\mu}p_{2\nu}}{m_a^2+2p_1\cdot q_1-m_b^2}\right) \equiv M_A^{\mu\nu}
\end{aligned}
$$

Other diagram:



$$
= \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iT_{ab}T_{bc})4p_{2\mu}p_{1\nu}}{m_a^2-2p_2\cdot q_1-m_b^2}\right) \equiv M_B^{\mu\nu}
$$

Now, if $m_a = m_b = m_c$ then, $q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = 0$ as above | $m_a^2 - m_b^2 = 0$ relative - size

However if $m_a \neq m_b$, in soft limit

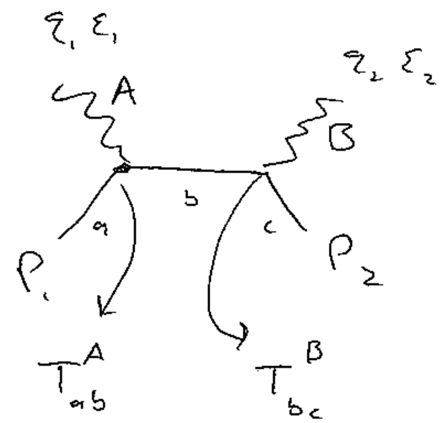
$q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = -\left[\frac{(-iT_{ab}T_{bc})4}{m_a^2-m_b^2}(2p_1^\mu p_1^\nu)\right] \neq 0$

Mass-less spin-1 particles can only interact with particles of the same mass!

Now allow many different matter fields (but same mass!) and many force carriers "gluons"

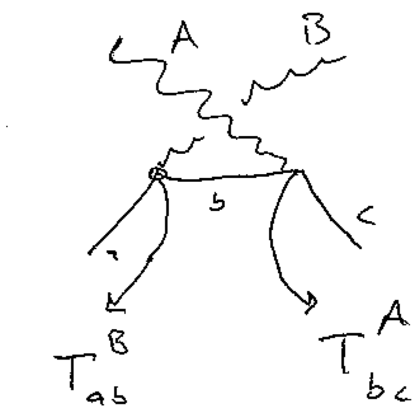
$i = 1, \ldots N_{\text{matter}}$

$I = 1, \ldots N_{\text{gluons}}$



$$= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT^A_{ab}T^B_{bc})4p_{1\mu}p_{2\nu}}{2p_1 \cdot q_1} \right) \equiv M_A^{\mu\nu}$$

and



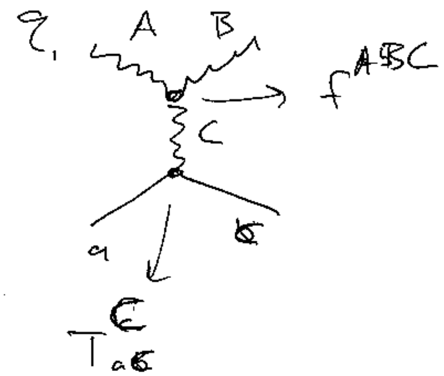
$$\underbrace{=}_{\text{“soft limit”}} \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT^B_{ab}T^A_{bc})4p_{1\mu}p_{2\nu}}{-2p_1 \cdot q_1} \right) \equiv M_B^{\mu\nu}$$

Now,

$$\begin{aligned} q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) &= 2(-i)(p_2 \cdot q_2)(T^A_{ab}T^B_{bc} - T^B_{ab}T^A_{bc}) \\ &= 2(-i)(p_2 \cdot q_2)[T^A, T^B] \end{aligned}$$

$[T^A, T^B]$ not 0 for random Ts.

In fact, another diagram we are missing:



$$= +2i(q \cdot q)if^{ABC}T^C_{ac}$$

Sum of all three only Lorentz invariant if

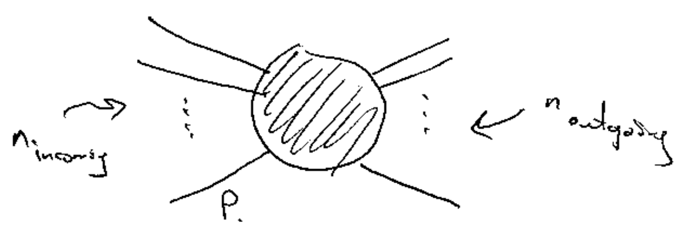
$$[T^A, T^B] = if^{ABC}T^C$$

"gluons" (or any other group of interacting mass-less spin-1 particles) must transform as a Lie group !

Only question is which group, there are only a finite handful of possibilities

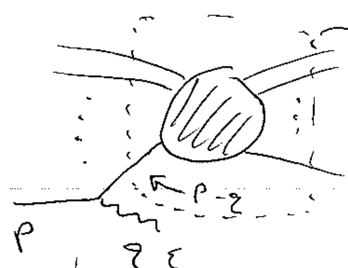
"Yang-Mills" Interaction.

Now do the same thing to a more general interaction



$$\equiv iM_a(p_i)$$

Consider what happens if we attach a "photon" to an incoming leg

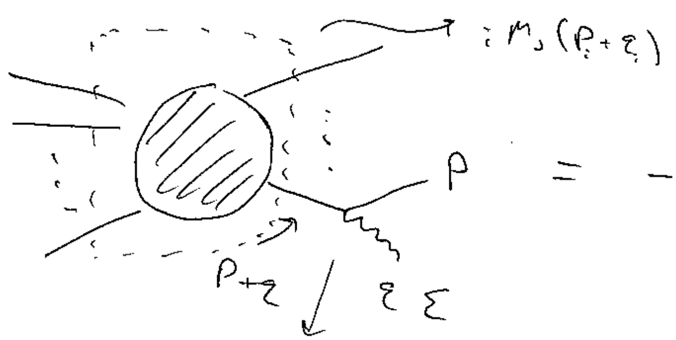


$$i M_0 (p_i - q)$$

$$\cancel{-Q\frac{(p\cdot\varepsilon)}{-(p\cdot q)}} = Q\,\frac{(p\cdot\varepsilon)}{(p\cdot q)}\, i M_0 (p_i - q)$$

$$i(iQ)\frac{\left(\varepsilon_\mu\left(p_i^\mu + (p_i^\mu - q^\mu)\right)\right)}{(p_i - q)^2 - m^2}$$

Can also attach photon to outgoing leg



$$= -Q \frac{(p\cdot\epsilon)}{(p\cdot q)}\, i M(p_i+q)$$

$$(-Q)\,\frac{\epsilon_\mu\left(p_i^\mu + (p_i^\mu + q^\mu)\right)}{(p+q)^2 - m^2}$$

Total Amplitude is then given by

$$M = \sum_{\text{incoming}} Q_i \frac{(p\cdot\epsilon)}{(p\cdot q)}\, iM_0(p-q) + \sum_{\text{outgoing}} -Q_i \frac{(p\cdot\epsilon)}{(p\cdot q)}\, iM_0(p+q)$$

Take soft limit: $M_0(p \pm q) \to M_0(p)$

$$M = iM_0 \left(\sum_{\text{incoming}} Q_i \frac{(p\cdot\epsilon)}{(p\cdot q)} + \sum_{\text{outgoing}} -Q_i \frac{(p\cdot\epsilon)}{(p\cdot q)} \right)$$

Now as before $\epsilon_\mu \to \epsilon'_\mu + q_\mu$ means that M must vanish when $\epsilon_\mu \to q_\mu$.

OR under a Lorentz Transform

$$\epsilon_\mu \cdot M \to \epsilon'_\mu \cdot M' + iM_0 \underbrace{\left(\sum_{\text{incoming}} Q_i + \sum_{\text{outgoing}} -Q_i \right)}_{\substack{=0 \text{ only if} \\ \sum_{\text{incoming}} Q_i = \sum_{\text{outgoing}} Q_i}}$$

Charge has to be conserved!

Now same logic for Spin-2 (describes interaction w/Gravitons)

Same as above except 2-component polarization vector.

$$\epsilon_{\mu\nu} \underbrace{\rightarrow}_{\text{under little group}} \epsilon_{\mu\nu} + \underbrace{A_\mu q_\nu + B_\mu q_\mu + C q_\mu q_\nu}_{\text{effect from all of these need to be 0 as before}}$$

where A, B C's are non-zero and depend on the particular little group transformation done.

$$= i(iK_i)\epsilon_{\mu\nu}\frac{(p^\mu p^\nu)}{-p \cdot q}$$

(Same idea with the outgoing leg)

Now, (lets focus on piece that goes like $\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + q_\mu B_\nu$

$$\begin{aligned} \epsilon_{\mu\nu} \to \epsilon'_{\mu\nu} M'^{\mu\nu} \ &+ \ M\left(\sum_{\text{incoming}} K_i B_\nu p^\nu - \sum_{\text{outgoing}} K_i B_\nu p^\nu\right) \\ &+ \ M B_\nu \left(\sum_{\text{incoming}} K_i p^\nu - \sum_{\text{outgoing}} K_i p^\nu\right) \end{aligned}$$

$\Rightarrow K_i p_i^\nu$ is $\underline{\underline{\text{conserved}}}$

We know that p_i^ν is conserved by E and momentum conservation.

Only way can have nontrivial solutions is if $k_i = k$ for all i

All particles interact with gravity with the same strength.

Gravitational interaction is Universal !

Discovered the "Principle of Equivalence" that is the starting point of General Relativity!

Can keep going...

For a massless spin-3 particle we would do the same exercise.

We would find we need

$$\sum_{\text{incoming}} \beta_i p_i^\mu p_i^\nu = \sum_{\text{outgoing}} \beta_i p_i^\mu p_i^\nu$$

eg: $\mu\nu = 0$

$$\sum_{\text{incoming}} \beta_i E_i^2 = \sum_{\text{outgoing}} \beta_i E_i^2$$

Way too constraining.

Only way if $\beta_i = 0$

There can be no interacting theories of massless particles of Spin greater than 2 !