Lecture 6

Review Quantum Mechanics

QM Linear Algebra in a complex vector space.

State of a system is a vector (ray) in the Complex Vector Space.

$$|\alpha\rangle \text{ - state vector}$$

Linear Superpositions

$$|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle$$

If the $|\alpha' s\rangle$ are vectors and the c's are complex numbers, then $|\psi\rangle$ is another vector in the space.

Dual Space:

For every vector $|\alpha\rangle$ ("ket") there is another vector $\langle\alpha|$ ("bra") in a "dual" space.

Dual space is a mirror image of the ket space.

If, $|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle$, then $\langle\psi| = c_1^* \langle\alpha_1| + c_2^* \langle\alpha_2|$ where the c^* is the complex conjugate.

Inner Product: Dual space allows us to define an inner product between to vectors.

Given 2 vectors can get a "c#"

$\langle\alpha|\beta\rangle$

Properties

1. $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
2. $\langle\alpha|\alpha\rangle \geq 0$
3. $\langle\beta| (c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle) = c_1 \langle\beta|\alpha_1\rangle + c_2 \langle\beta|\alpha_2\rangle$

$|\alpha\rangle$ and $|\beta\rangle$ are orthogonal if $\langle\alpha|\beta\rangle = 0$.

States can be normalized $\langle\alpha|\alpha\rangle = 1$.

Operators: Things that act on a given state, and return another state

$X\,|\alpha\rangle = |\alpha'\rangle$

Product:

$(YX)\,|\alpha\rangle = Y(X\,|\alpha\rangle$

In general, not commutative: $XY \neq YX$, but they are associative.

$\langle\alpha| = \langle\alpha'|\,X^\dagger$

In general, $X^\dagger \neq X$, if so X is said to be "Hermitian".

System characterized by single observable A

eg: position / Momentum / energy

Measurement of A gives possible values

$$a_1, a_2, ...$$

$|\alpha\rangle$ = State for which A has a value of a

$$\sum_{a'} |\alpha'\rangle\,\langle\alpha'| = 1$$

$$\langle\alpha|\alpha'\rangle = \delta_{aa'}$$

Any physical observable corresponds to an operator like

$$A = \sum_{a'} a'\,|\alpha'\rangle\,\langle\alpha'|$$

eg:

$$A|\alpha\rangle = \left(\sum_{a'} a' |\alpha'\rangle\langle\alpha'|\right)|\alpha\rangle = a|\alpha\rangle$$

Physical observables are real numbers, therefore physical operators A are Hermitian

proof:

$$A^\dagger = \sum_a a^* |\alpha\rangle\langle\alpha| = \sum_a a|\alpha\rangle\langle\alpha| = A$$

Probabilities:

Consider a filter $M(a) = |a\rangle\langle a|$ on a general state $|s\rangle$.

$$\begin{aligned} M(a)|s\rangle &= |a\rangle\langle a|s\rangle \\ &= \langle a|s\rangle|a\rangle \end{aligned}$$

where $\langle a|s\rangle$ is a c# that tells you something about what fraction of the time you get through.

$\langle a|s\rangle$ is related to the pass fraction

*But, a) not real, b) not normalized

However, we know that $|\langle a|s\rangle|^2 = \langle a|s\rangle\langle s|a\rangle = \langle s|a\rangle\langle a|s\rangle$ is both real and normalized.

$$\sum_a \langle s|a\rangle\langle a|s\rangle = \langle s|s\rangle = 1$$

Interpretation

$|\langle a|s\rangle|^2$ = Probability that a system prepared in state $|s\rangle$ will be found in a state $|a\rangle$ with value a for observable A after measurement.

Comments on the measurement problem....

Position Operator

$$\vec{X}\,|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

where, $\vec{X}$ is position operator and $\vec{x}$ is position eigenvalue.

Position of course is a continuous observable so "sums go to integrals" etc.

eg: (Completeness and Orthogonality)

$$\sum_{a'} |\alpha'\rangle \langle\alpha'| = 1 \Rightarrow \int d^3x\,|x\rangle\langle x| = 1$$

$$\langle\alpha'|\alpha'\rangle = \delta_{\alpha,\alpha'} \Rightarrow \langle\vec{x}|\vec{x}'\rangle = \delta^3(\vec{x}-\vec{x}')$$

Wave function

$$|\psi\rangle = \int d^3x\,|x\rangle\langle x|\psi\rangle = \int d^3x\psi(x)\,|x\rangle$$

where $\psi(x) = \langle x|\psi\rangle$ is called the Position-space wave-function.

$$\langle\psi|\psi\rangle = 1 \Rightarrow \int d^3x\psi(x)\,\langle\psi|x\rangle = \int d^3x\psi^*(x)\psi(x) = 1$$

Interpretation

$|\psi(x)|^2 d^3x$ is the probability to find the particle in volume d^3x around $\vec{x}$.

Translation Operator "The operator that moves you over"

$$T(\vec{a})\,|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$$

What is $T^\dagger(\vec{a})$???

Well,

$$\langle\vec{x'}|\left(T(\vec{a})\,|\vec{x}\rangle\right) = \delta^3((\vec{x}+\vec{a})-\vec{x'})$$

or

$$\left(\langle\vec{x'}|\,T(\vec{a})\right)|\vec{x}\rangle = \delta^3(\vec{x}-(\vec{x'}-\vec{a}))$$

$$\Rightarrow \langle x'|\,T(a) = \langle x-a|$$

which says that $T^\dagger(a)\,|\vec{x}\rangle = |\vec{x}-\vec{a}\rangle$

So,

$$T^\dagger(\vec{a}) = T(-\vec{a}) = T^{-1}(\vec{a})$$

Properties of T

1. Unitary $T^\dagger T = 1$
2. $T(\vec{a})T(\vec{b}) = T(\vec{a}+\vec{b}) = T(\vec{b})T(\vec{a})$, Translations commute ($[T(\vec{a}), T(\vec{b})] = 0$)
3. $T(0) = 1$

Infinitesimal Translations

consider $\vec{a} = N\vec{\epsilon}$

$$T(\vec{\epsilon}) = 1 - i\vec{\epsilon}\cdot\vec{k}$$

where

- $\vec{\epsilon}$is a three vector
- $\vec{k}$ is a vector of operators $\vec{k} = (k_x, k_y, k_z)$

Now, we know $T^\dagger T = 1$, or

$$\left(1 + i\vec{\epsilon} \cdot \vec{k}^\dagger\right)\left(1 - i\vec{\epsilon} \cdot \vec{k}\right) = 1$$

$$1 + \underbrace{i\vec{\epsilon}(\vec{k}^\dagger - \vec{k})}_{=0} + O(\epsilon^2) = 1$$

$\Rightarrow k^\dagger = k$, or k is some hermitian operator.

Note: if the i wasn't there $T(\vec{a})$ would not be hermitian.

Just like with SR, any finite translation can be built out of infinitesimal translations

And $\vec{k}$ is the "generator" of translations.

Lets build a finite translation...

$$\underbrace{T(\vec{a})}_{finite} = \lim_{N\to\infty}\left[1 - i\vec{\epsilon} \cdot \vec{k}\right]^N = \underbrace{\lim_{N\to\infty}\left[1 - i\frac{\vec{a}}{N} \cdot \vec{k}\right]^N}_{e^{-i\vec{k}\cdot\vec{a}}}$$

Can see explicitly that T is unitary and k is hermitian.

Eigenstates of $\vec{k}$ (and of course automatically of T)

$$\vec{K}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle$$

$$T(\vec{a})\,|\vec{k}\rangle = e^{-i\vec{k}\cdot\vec{a}}\,|\vec{k}\rangle$$

Eigenstates of $\vec{k}$ behave nicely under translations (they pick up a phase).

What is $\psi_{\vec{k}}(x) \equiv \langle\vec{x}|\vec{k}\rangle$?

$$\begin{aligned}
\langle\vec{x}|T(\vec{a})|\vec{k}\rangle = &\ e^{-i\vec{k}\cdot\vec{a}}\langle\vec{x}|\vec{k}\rangle = e^{-i\vec{k}\cdot\vec{a}}\psi_{\vec{k}}(x) \\
\text{“T on x”} = &\ \ \langle\vec{x}-\vec{a}|\vec{k}\rangle = \psi_{\vec{k}}(x-a)
\end{aligned}$$

or

$$\psi_{\vec{k}}(\vec{x}-\vec{a}) = e^{-i\vec{k}\cdot\vec{a}}\psi_{\vec{k}}(\vec{x})$$

Turns out that $\vec{k}$ is just the momentum operator

$$\vec{p} = \hbar\vec{k} \text{ and } T(\vec{a}) = e^{-i\vec{p}\vec{a}}$$

Couple of ways to see this. (One is the De Brojlie wavelength, we will see another shortly)

“Momentum is the generator of translations”

Note Translations for a group:

1. Closure
2. Identity
3. Inverse
4. Associative

(Can show that this is an abealian group (HW)

Another look at the $\vec{p}$ operator...

$$
\begin{aligned}
\langle y| T(\epsilon) |\psi\rangle &= \int d^3x \, \langle y| \underbrace{T(\epsilon) |x\rangle}_{|x+\epsilon\rangle} \langle x|\psi\rangle \\
&= \int d^3x \, \langle y|x\rangle \langle x-\epsilon|\psi\rangle \\
&= \langle y-\epsilon|\psi\rangle \\
&= \psi(y-\epsilon) \\
&= \psi(y) - \epsilon \tfrac{\partial}{\partial y}\psi(y)
\end{aligned}
$$

(using change of variables $x \to x - \epsilon$ and Taylor expansion)

Now, on the other-hand...

$$
\begin{aligned}
\langle y| T(\epsilon) |\psi\rangle &= \langle y| \left(1 - \tfrac{i\vec{p}\vec{\epsilon}}{\hbar}\right) |\psi\rangle \\
&= \int d^3x \, \langle y|x\rangle \langle x| \left(1 - \tfrac{i\vec{p}\vec{\epsilon}}{\hbar}\right) |\psi\rangle \\
&= \int d^3x \, \langle y|x\rangle \left(\langle x|\psi\rangle - \langle x| \tfrac{i\vec{p}\vec{\epsilon}}{\hbar} |\psi\rangle\right) \\
&= \langle y|\psi\rangle - \langle y| \tfrac{i\vec{p}\vec{\epsilon}}{\hbar} |\psi\rangle \\
&= \psi(y) - \epsilon \langle y| \tfrac{i\vec{p}}{\hbar} |\psi\rangle
\end{aligned}
$$

$\Rightarrow \vec{p} = i\hbar\frac{\partial}{\partial x}$ (This will be critical later)

Now lets look at how momentum eigenstates behave.

$$\vec{P}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$$

we already saw that $\langle x|p\rangle \sim e^{i\frac{p\cdot x}{\hbar}}$ "Momentum Space wave function"

$$\phi(p) \equiv \langle p|\psi\rangle$$

Can show (HW) that $\phi(p) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\frac{px}{\hbar}}$

Then,

$$\langle x|\psi\rangle = \int d^3p\,\langle x|p\rangle\,\langle p|\psi\rangle$$

or,

$$\psi(x) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int d^3p e^{i\frac{px}{\hbar}}\phi(p)$$

Similarly,

$$\langle p|\psi\rangle = \int d^3x\,\langle p|x\rangle\,\langle x|\psi\rangle$$

or,

$$\phi(p) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int d^3x\, e^{-i\frac{px}{\hbar}}\psi(x)$$

$\Rightarrow$ $\psi(x)$ and $\phi(p)$ are Fourier transforms of each other...