Lecture 4

Special Relativity

Recap from Last Time

Talking about relativity means talking about Lorentz invariance.

If there is a point in space time:

$$(t, x) \xrightarrow[\text{observer}]{\text{another moving}} (t', x')$$

Invariant notion of distance:

$$t^2 - x^2 = t'^2 - x'^2$$

Started with Rotations Where we have an invariant notion of length of $\vec{p}$

$$x^2 + y^2 = x'^2 + y'^2$$

Started in 2D, looking at the infinitesimal rotation This lead to what is called "generator"

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Form a group: A group is a set of actions (in our case rotations) that multiply (or compose) with operation denoted by $\cdot$.

Four criteria for group:

- Have Identity element: (no rotation or by 360)
- Every element of the group has an inverse (for us, rotate by $360 - \theta$)
- The group is closed: for any elements thier product is also in the group.
- The multiplication is associative.

If commute: $a \cdot b = b \cdot a$ say that the group is "Abelian".

Last lecture studied 3D rotations and thier group "SO(2)" refers to matriecies studied (S = Special (det =1) / O = Orthogonal (preseves length) / 2 = 2x2)

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

We also looked at 3D rotations, theier group was "SO(3)", what do you think this stands for ???
We saw at the end of the lecture that $SO(3) \simeq SU(2)$ SU(2) = (Special / Unitary / 2x2 matricies.

(You will show in for homework that this is equavilant to another group U(1).)

All the machinery in place to look at Lorentz Transformations...

Well almost all the machinery... Remember invariant need to preserve is $t^2 - |\vec{x}|^2$

Lets deal with this minus sign...

4-vectors: $x^\mu = (t, \vec{x})$

Encode the minus sign in matrix:

$$\eta_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1, 2, 3\ (i) \\ 0 & \text{otherwise} \end{cases}$$

Can now write: $x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})$

And finally: $x_\mu x^\mu = t^2 - |\vec{x}|^2$

(Comment on $\eta^{\mu\nu} = \eta_{\mu\nu}$ and $\eta^\mu_\nu = \delta^\mu_\nu$)

Now we after all the transformations that leave $x_\mu x^\mu$ (Lorentz transformations)

$$x'^\mu = \underbrace{\Lambda^\mu_\nu}_{\text{4x4 matrix}} x^\nu$$

Same thing as before Start with the infintesimal transofmations.

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \epsilon\omega^{\mu}_{\nu}$$

$$x'^{\mu} = x^{\mu} + \epsilon\omega^{\mu}_{\nu}x^{\nu}$$

Require $x'^{\mu}x'_{\mu} = x^{\mu}x_{\mu}$ or

$$\begin{aligned} \eta_{\mu\nu}x'^{\mu}x'^{\nu} &= \eta_{\mu\nu}x^{\mu}x^{\nu} \\ = \eta_{\mu\nu}x^{\mu}x^{\nu} &+ 2\epsilon\eta_{\mu\nu}x^{\nu}\omega^{\mu}_{\rho}x^{\rho} \end{aligned}$$

So, $\underbrace{\eta_{\mu\nu}\omega^{\mu}_{\rho}}_{\equiv\omega_{\mu\rho}} x^{\nu}x^{\rho} = 0$

After all 4x4 anti-symmetric matricies $\omega_{\mu\rho} = -\omega_{\rho\mu}$

Lets just be damn explicit

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & A & B \\ -b & -A & 0 & C \\ -c & -B & -C & 0 \end{bmatrix}$$

However, the think that enters the transformation is ω^{μ}_{ν}.

Now raising a 0-compentent is free, $\omega_{0i} = \omega^{0}_{i}$
raising a i-compentent costs a -1, $\omega_{i0} = -\omega^{i}_{0}$
But also know $\omega_{0i} = -\omega_{i0} \Rightarrow \omega^{0}{}_{i} = +\omega^{i}{}_{0}$

Same argument shows: $\omega^{i}{}_{j} = -\omega^{j}{}_{i}$

$$\omega^{\mu}_{\nu} = \begin{bmatrix} 0 & a & b & c \\ a & 0 & A & B \\ b & -A & 0 & C \\ c & -B & -C & 0 \end{bmatrix}$$

Generators of the Lorentz Group. We have 6 “generators” abcABC.

3-rotations (guess which) / 3-boosts

Crucial sign differnece between boosts/rotations

$$\omega^1{}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to e^{\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \cos\theta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sin\theta$$

$$\omega^0{}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to e^{\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \cosh\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sinh\eta$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = 1 \text{ whereas } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -1$$

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} =$$

Can (but I wont) work out the Lie Algrabra....Homework.