

## Lecture 7

### Review Quantum Mechanics (Dynamics)

$$|\alpha, t_0\rangle \rightarrow |\alpha, t\rangle$$

This is what we mean by time evolution.

In QM, then there has to be an operator associated with taking the first state to the second.

Time Evolution Operator:  $U(t, t_0)$

$$|\alpha, t\rangle = U(t, t_0) |\alpha, t_0\rangle$$

$U(t, t_0)$  Properties

1.  $U^\dagger(t, t_0)U(t, t_0) = 1$  Unitary See this from  $\langle \alpha t_0 | \alpha t_0 \rangle = \langle \alpha t | \alpha t \rangle$
2.  $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$  Composition Rule
3.  $U(t_0, t_0) = 1$

How can we possibly determine what the time operator is ??? (Should be getting old by now...)

Start with infinitesimal time evolution

$$U(t + \epsilon, t) = 1 - i\Omega\epsilon$$

where  $\Omega$  is a Hermitian operator (b/c)  $U$  is unitary

So,

$$|\alpha, t + \epsilon\rangle = (1 - i\epsilon\Omega) |\alpha, t\rangle$$

OR,

$$\Omega |\alpha, t\rangle = i \frac{|\alpha, t + \epsilon\rangle - |\alpha, t\rangle}{\epsilon} = i \frac{\partial}{\partial t} |\alpha, t\rangle$$

in limit  $\epsilon \rightarrow 0$

### Physical Meaning of $\Omega$ :

As before to get the generate for the finite movement you have to exponentiate

$$U(t) = e^{-i\Omega t}$$

Note that  $\Omega$  has units 1/[time]. Just like energy.

Identify  $\Omega = \frac{1}{\hbar} H$  where  $H$  is the Hamiltonian operator.

$$i \frac{\partial}{\partial t} |\psi\rangle = \Omega |\psi\rangle = \frac{E}{\hbar} |\psi\rangle$$

### Schrodinger Equation

$$i \frac{\partial}{\partial t} |\psi\rangle = H(t) |\psi\rangle$$

Non-relativistically:  $H \sim \frac{p^2}{2m} = -\frac{1}{2m} \frac{\partial^2}{\partial^2 x}$

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### Time Evolution

$$|\alpha, t\rangle = U(t, t_0) |\alpha, t_0\rangle$$

**Case I:**  $H$  is time independent

$$U(t, t_0) = \lim_{N \rightarrow \infty} \prod_i^N e^{-\frac{i}{\hbar} H \Delta t} = e^{-\frac{iH(t-t_0)}{\hbar}}$$

**Case II:**  $H$  is time dependent, but  $[H(t), H(t')] = 0$

$$U(t, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^t H(t') dt'}$$

**Case III:**  $H$  is time dependent, and  $[H(t), H(t')] \neq 0$

(Note: in general  $e^A e^B \neq e^{A+B}$  if  $[A, B] \neq 0$ )

$$U(t, t_0) = T \left[ e^{\frac{-i}{\hbar} \int_{t_0}^t H(t') dt'} \right]$$

“Time-ordered product” - Power series expansion with earlier terms on the right (Will come back to this later)

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### **Example**

For a time independent Hamiltonian.

$$U(t, 0) = e^{-iHt}$$

Choose a basis of eigenstates of H

$$H |n\rangle = E_n |n\rangle$$

and

$$H = \sum_n E_n |n\rangle \langle n|$$

Then

$$U(t, 0) = \sum_n e^{-iE_n t} |n\rangle \langle n|$$

Now some arbitrary state:

$$\begin{aligned} |\psi(t)\rangle &= U(t, 0) |\psi(0)\rangle = \sum_n U(t, 0) |n\rangle \langle n|\psi\rangle \\ &= \sum_n |n\rangle e^{-iE_n t} \underbrace{\langle n|\psi\rangle}_{\text{time independent}} \end{aligned}$$


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Now lets talk about the expectation values of an observable and how they change with time....

$$\begin{aligned}
 \langle A \rangle(t) = \langle \psi(t) | A | \psi(t) \rangle &= \langle \psi(0) | U^\dagger(t, 0) A U(t, 0) | \psi(0) \rangle \\
 &= \underbrace{\left( \langle \psi(0) | U^\dagger(t, 0) \right) A \left( U(t, 0) | \psi(0) \rangle \right)}_{\text{"Schrodinger Picture"}} \\
 &= \underbrace{\langle \psi(0) | \left( U^\dagger(t, 0) A U(t, 0) \right) | \psi(0) \rangle}_{\text{"Heisenberg Picture"}}
 \end{aligned}$$

### Schrodinger Picture

- $|\psi(t)\rangle$ s move through Hilbert Space guided by  $U(t)$
- Operators are independent of time
- Basis kets (eigenstates of observables) eg:  $|x\rangle$  and  $|p\rangle$  are independent of time.

### Heisenberg Picture

- $|\psi(t)\rangle = |\psi\rangle_H$  is fixed and independent of time
- Operators in Heisenberg picture are time dependent

$$A_H(t) = U^\dagger(t) A_S U(t) = e^{iHt} A_S e^{-iHt}$$

### Time dependent perturbation theory

$$H(t) = H_0 + V(t)$$

where  $V(t)$  is small (this will always be the case for us)

$$H_0 |n\rangle = E_n |n\rangle$$

Now, a general state at  $t=0$

$$|\psi(0)\rangle = \sum_n c_n |n\rangle$$

For  $V = 0$

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |n\rangle$$

For  $V \neq 0$

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t} |n\rangle$$

where time dependence in  $c_n(t)$  due to only  $V$ .

### Interaction Picture

Define...

$$|\psi(t)\rangle_I \equiv e^{+iH_0 t} |\psi(t)\rangle_S$$

$$A_I \equiv e^{+iH_0 t} A_S e^{-iH_0 t}$$

From these definitions, its clear that

$${}_I \langle \psi(t) | A_I | \psi(t) \rangle_I = {}_S \langle \psi(t) | A_S | \psi(t) \rangle_S$$

When  $V=0$ , Heisenberg and Interaction Picture Coincide.

Ok, here's why we care about the interaction picture....

$$\begin{aligned} i \frac{d}{dt} |\psi(t)\rangle_I &= i \frac{d}{dt} e^{iH_0 t} |\psi(t)\rangle_S = e^{iH_0 t} \left( -H_0 |\psi(t)\rangle_S + i \frac{d}{dt} |\psi(t)\rangle_S \right) \\ &= e^{iH_0 t} (-H_0 + (H_0 + V_S)) |\psi(t)\rangle_S \\ &= e^{iH_0 t} V_S e^{-iH_0 t} e^{iH_0 t} |\psi(t)\rangle_S \\ &= V_I(t) |\psi(t)\rangle_I \end{aligned}$$

The interaction picture is a hybrid of the Schrodinger and Heisenberg pictures.

Time evolution of state kets and operators depend on different parts of  $H$ .

$$|\psi(t)\rangle_S = \sum_n c_n(t) e^{-iE_n t} |n\rangle = e^{-iH_0 t} \sum_n c_n(t) |n\rangle$$

$$|\psi(t)\rangle_I = e^{iH_0 t} |\psi(t)\rangle_S = \sum_n c_n(t) |n\rangle$$

where,

$$c_n(t) = \langle n | \psi(t) \rangle_I$$

so once we have  $\langle \psi(t) \rangle_I$  we are done.

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Solve the “Schrodinger Eq” iteratively

$$i \frac{d}{dt} |\psi(t)\rangle_I = V_I(t) |\psi(t)\rangle_I$$

integrate and get

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle + \int_{t_0}^t dt' [-iV_I(t') |\psi(t')\rangle]$$

where the second term is of order  $V_0$  which is small.

Now we keep iterating

$$= |\psi(t_0)\rangle - i \int_{t_0}^t dt' V_I(t') \left[ |\psi(t_0)\rangle - i \int_{t_0}^{t'} dt'' V_I(t'') |\psi(t'')\rangle \right]$$

at 3rd order (iterate again...)

$$= |\psi(t_0)\rangle - i \int_{t_0}^t dt' V_I(t') |\psi(t_0)\rangle + (-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V(t') V(t'') |\psi(t_0)\rangle \\ + (-i)^3 \int \int \int V(t') V(t'') V(t''') |\psi(t_0)\rangle$$

$$|\psi(t)\rangle = U_I(t, t_0) |\psi(t_0)\rangle$$

where

$$U_I(t, t_0) = 1 + (-i) \int V(t') dt' + (-i)^2 \int \int V(t') V(t'') + \dots$$

“Dyson Series”, can be written in slick form (by doing the sum)

$$U_I(t, t_0) = T \left[ e^{-i \int_{t_0}^t V_I(t') dt'} \right]$$

Where the “T” stands for a “time ordered product”