

Lecture 17

Lorentz Invariance and “Soft Limits”

Punch line that we’ve been building to in first part of this course.

Matrix element we would get by scattering external γ .

$$M = \epsilon^\mu M_\mu$$

where ϵ^μ is some linear combination of two photon polarization vector ϵ^1 and ϵ^2

M is Lorentz Invariant, under lorentz transformation

$$M \rightarrow \epsilon'^\mu M'_\mu$$

where $M'_\mu = \Lambda_\mu{}^\nu M_\nu$

However (here comes the major constraint) ϵ is not a full 4-vector. Only has 2 components.

Under little group transformations (you will show in your H.W.)

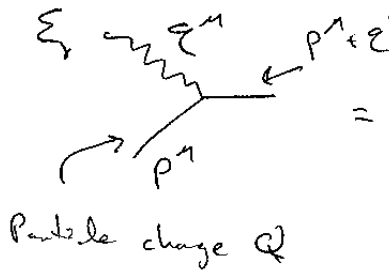
$$\epsilon \rightarrow \underbrace{c_1 \epsilon_1^\mu + c_2 \epsilon_2^\mu}_{\epsilon' \text{ can only be made of these pieces}} + \underbrace{c_3 p^\mu}_{\substack{\text{Not valid} \\ \text{“Not in Hilbert Space”}}}$$

So,

$$\begin{aligned} M = \epsilon^\mu M_\mu &\rightarrow (c_1 \epsilon_1^\mu + c_2 \epsilon_2^\mu + c_3 p^\mu) M_\mu \\ &= \epsilon'^\mu M'_\mu + \underbrace{c_3 p^\mu M_\mu}_{\text{Must go to 0}} \end{aligned}$$

We will see, this has enormous implications !!!

Will be considering diagrams with external “ γ ”s (massless spin 1 particles)



$$\begin{aligned}
 &= iQ(p^\mu + (p^\mu + q^\mu))\epsilon_\mu \quad (q^\mu \epsilon_\mu = 0) \\
 &= iQ2p^\mu \epsilon_\mu
 \end{aligned}$$

Particle charge Q

This is the most general form in the “soft limit” $q \rightarrow 0$

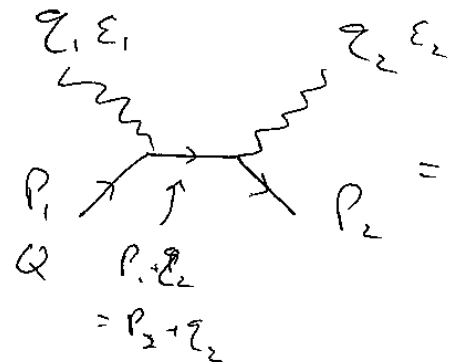
$$\Gamma_\mu \sim p_\mu F(q^2, p^2, p \cdot q)$$

By dimensional analysis $F(q^2, p^2, p \cdot q) \rightarrow F(\frac{p \cdot q}{m^2})$

Consider “Compton Scattering”

Start with one type of spin-1 boson and one type of matter particle.

The diagram:



$$\begin{aligned}
 &= (iQ)\epsilon_\mu^1(2p_1^\mu) \frac{i}{(p_1 + q_1)^2 - m^2} (iQ)\epsilon_\nu^2(2p_2^\nu) \\
 &= (-iQ^2)4 \frac{(p_1 \cdot \epsilon^1)(p_2 \cdot \epsilon^2)}{m^2 + 2p_1 \cdot q_1 - m^2} = \epsilon_1^\mu \epsilon_2^\nu \underbrace{\left(\frac{(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1 \cdot q_1} \right)}_{M_{\mu\nu}}
 \end{aligned}$$

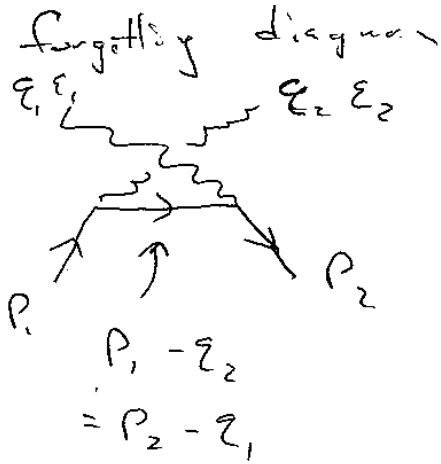
$= p_2 + q_2$

As we said above, Lorentz Invariant $\Rightarrow q_1^\mu q_2^\nu M_{\mu\nu} = 0$

But here, $q_1^\mu q_2^\nu M_{\mu\nu} = (-iQ^2)2(p_2 \cdot q_2) \neq 0 !$

Looks like we're dead...

However we are forgetting a diagram.



$$\begin{aligned}
 &= (iQ)\epsilon_\mu^1(2p_2^\mu) \frac{i}{(p_2 - q_1)^2 - m^2} (iQ)\epsilon_\nu^2(2p_1^\nu) \\
 &= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iQ^2)4p_{2\mu}p_{1\nu}}{-2p_2 \cdot q_1} \right) \\
 &\stackrel{\sim}{\sim} \epsilon_1^\mu \epsilon_2^\nu \left(\frac{-(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1 \cdot q_1} \right) \\
 &\text{Soft limit } p_1 = p_2
 \end{aligned}$$

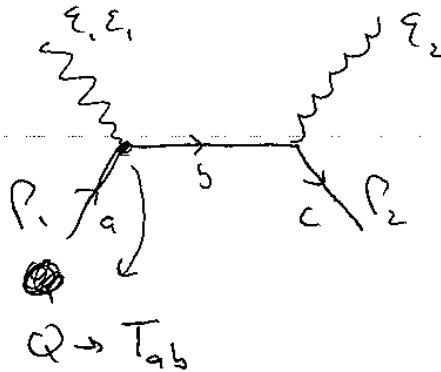
and for this diagram, $q_1^\mu q_2^\nu M_{\mu\nu} = -(-iQ^2)2(p_2 \cdot q_2)$

So the sum $M_{\mu\nu}^A + M_{\mu\nu}^B$ is Lorentz Invariant. (Residual non Lorentz Invariant pieces of each diagram cancel)

Very good!

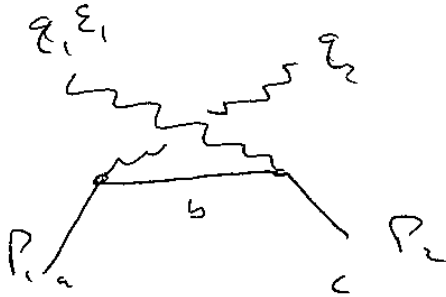
Now lets do the same thing as before, but with many different possible matter particles.

$i = 1, \dots, N_{\text{matter}}$



$$\begin{aligned}
 &= (iT_{ab})\epsilon_\mu^1(2p_1^\mu) \frac{i}{\underbrace{(p_1 + q_1)^2 - m^2}_{m_a^2 + 2p_1 \cdot q_1 - m_b^2}} (iT_{bc})\epsilon_\nu^2(2p_2^\nu) \\
 &= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}T_{bc})4p_{1\mu}p_{2\nu}}{m_a^2 + 2p_1 \cdot q_1 - m_b^2} \right) \equiv M_A^{\mu\nu}
 \end{aligned}$$

Other diagram:



$$= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}T_{bc})4p_{2\mu}p_{1\nu}}{m_a^2 - 2p_2 \cdot q_1 - m_b^2} \right) \equiv M_B^{\mu\nu}$$

Now, if $m_a = m_b = m_c$ then, $q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = 0$ as above $\left| \begin{array}{l} m_a^2 - m_b^2 = 0 \\ \text{relative - size} \end{array} \right.$

However if $m_a \neq m_b$, in soft limit

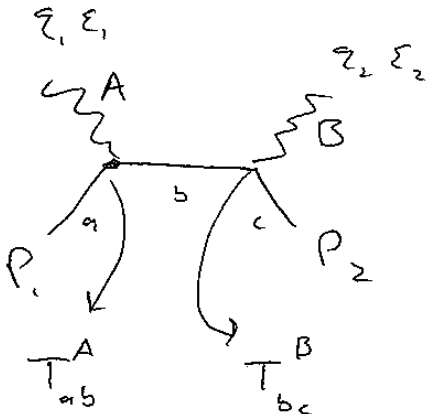
$$q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = - \left[\frac{(-iT_{ab}T_{bc})4}{m_a^2 - m_b^2} (2p_1^\mu p_1^\nu) \right] \neq 0$$

Massless spin-1 particles can only interact with particles of the same mass!

Now allow many different matter fields (but same mass!) and many force carriers “gluons”

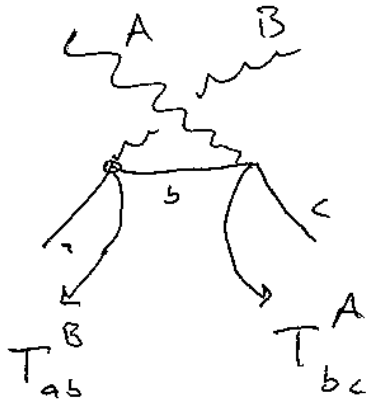
$i = 1, \dots, N_{\text{matter}}$

$I = 1, \dots, N_{\text{gluons}}$



$$= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}^A T_{bc}^B)4p_{1\mu}p_{2\nu}}{2p_1 \cdot q_1} \right) \equiv M_A^{\mu\nu}$$

and



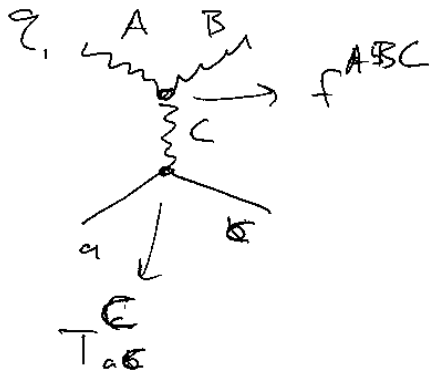
$$\underbrace{=}_{\text{"soft limit"}} \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}^B T_{bc}^A) 4p_{1\mu} p_{2\nu}}{-2p_1 \cdot q_1} \right) \equiv M_B^{\mu\nu}$$

Now,

$$\begin{aligned} q_{1\mu} q_{2\nu} (M_A^{\mu\nu} + M_B^{\mu\nu}) &= 2(-i)(p_2 \cdot q_2)(T_{ab}^A T_{bc}^B - T_{ab}^B T_{bc}^A) \\ &= 2(-i)(p_2 \cdot q_2)[T^A, T^B] \end{aligned}$$

$[T^A, T^B]$ not 0 for random Ts.

In fact, another diagram we are missing:



$$= +2i(q \cdot q) i f^{ABC} T_{ac}^C$$

Sum of all three only lorentz invariant if

$$[T^A, T^B] = i f^{ABC} T^C$$

“gluons” (or any other group of interacting massless spin-1 particles) must transform as a Lie group !

Only question is which group, there are only a finite handful of possibilities

“Yang-Mills” Interaction.