Lecture 11

Relativistic Wave Equations

Lets look at the Schrodinger Equation

$$i\frac{d}{dt}\psi = -\left(\frac{\nabla^2}{2m} + V\right)\psi$$

Problems - Conservation of non-relativistic energy.

$E \leftrightarrow i\frac{d}{dt}$ and $p \leftrightarrow -i\nabla$

$\Rightarrow$ Schrodinger Equation $E = \frac{p^2}{2m} + V$

- time/space not on equal footing.

Start w/relativistic E/p relation.

$$E^2 - p^2 - m^2 = 0$$

Leads us to the Klein-Gordon equation

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)\phi(x,t) = 0$$

Solution,

$$\phi(x,t) = e^{ipx}$$

Called "On shell-solutions".

Everything is Lorentz Invariant.

Note Two solutions E>0 and E<0.

Alternative Standard Powerful formulation of Physics via Lagrangians.

KG eq

$$\frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

where $\phi(x,t)$ permeates space and time

<u>Looks like</u>

$$\frac{d^2}{dt^2}x(t) = -\frac{\partial U}{\partial x}$$

Where U is a potential energy.

- $\frac{\partial^2}{\partial t^2}\phi$ - looks like the "acceleration" of the field
- $-m^2\phi$ - effective "Force" for the mass (harmonic oscillator)
- $\nabla^2\phi$ - shear force (shearing fabric takes force)

Now have some intuition about what the KG equation is telling us.

Integrate force to get the potential

$$-\frac{\partial U}{\partial \phi} = \nabla^2\phi - m^2\phi \Rightarrow U = \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2$$

KE form generalization of $1/2\dot{x}^2$ to fields.

$$K = 1/2(\frac{\partial}{\partial t}\phi)^2$$

so,

$$K + U = \frac{1}{2}(\frac{\partial}{\partial t}\phi)^2 + \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2$$

Get total energy from integrating,

$$H(t) = \int d\vec{x}\left[\frac{1}{2}(\frac{\partial}{\partial t}\phi)^2 + \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2\right]$$

with this you could go back to three lectures ago and replace $\frac{\nabla\phi^\dagger\nabla\phi}{2m}$ and do everything again (we wont bother here...)

However we can use this to formulate the field eq in a totally different, ultimately more useful way...

Think of classical mechanics of point partilces. Can all be formulated with Lagrangians and the principle of least action.

$$L = \frac{1}{2}\dot{x}^2 - U(x)$$

difference in KE and potential

The action

$$S[x(t)] = \int dtL$$

you could take this as the starting point and derive Newton's 2nd law by minimizing the action.

We will do the same for relativistic fields.

$$L = K - U = \int d^3x \left[\frac{1}{2}(\frac{\partial}{\partial t}\phi)^2 - \frac{1}{2}(\nabla\phi)(\nabla\phi) - \frac{m^2}{2}\phi^2\right]$$

$$S[\phi(t)] = \int d^4x \,[(\text{Above})]$$

KG can be found from minimizing the action wrt ϕ.

$$\partial_\mu = (\frac{\partial}{\partial t}, -\vec{\nabla})$$

$$S[\phi(t)] = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2\right]$$

manifestly L.I

This tells us how to construct general LI descriptions of relativistic QM systems. Need a LI Lagrangian and we are "done" (meaning $\sim$ all of physics follows from this)

KG 2nd order in x^μ. Contains no info about intrinsic angular momentum. (No free Lorentz indices) ϕ from KG is Spin-0 field.

Two more relativistic systems important for SM

- Spin-1 "EM"
- Spin-1/2 "Dirac Equation"

Start with the the "Dirac Equation".

Can we find a 1st order relativistic eq of motion?

Assume there exists a wave eq. linear in x and t.

$$\left(\alpha \frac{\partial}{\partial t} + \vec{\beta}\vec{\nabla}\right)\psi = m\psi$$

for some $\alpha, \vec{\beta}$, and m.

If relativistic invariant, must imply the KG.

Assume that it is the square root

$$\left(\alpha \frac{\partial}{\partial t} + \vec{\beta}\vec{\nabla}\right)\left(\alpha \frac{\partial}{\partial t} + \vec{\beta}\vec{\nabla}\right)\psi = m^2\psi$$

$\Rightarrow$

$$\left[\alpha^2 \frac{\partial^2}{\partial t^2} + \alpha\vec{\beta}\vec{\nabla}\frac{\partial}{\partial t} + \vec{\beta}\vec{\nabla}\alpha\frac{\partial}{\partial t} + (\beta \cdot \nabla)^2\right]\psi = m^2\psi$$

to get $(-\frac{\partial^2}{\partial t^2} + \nabla^2)\psi = m^2\psi$

$\Rightarrow$

- $\alpha^2 = -1$ - can be a complex number
- $\beta_i\beta_j = \delta_{ij}$ - cannot be numbers
- $\alpha\beta_i + \beta_i\alpha = 0$

Lets keep going...

Define

$\alpha = i\gamma_0$ $\qquad\qquad$ $\beta_i = i\gamma_i$

$\gamma_0\gamma_0 = 1$ $\qquad\qquad$ $\gamma_i\gamma_j = -\delta_{ij}$ $\qquad\qquad$ $\gamma_0\gamma_i + \gamma_i\gamma_0 = 0$

$$\underbrace{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}_{\{\gamma_\nu,\gamma_\mu\}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = 2\eta_{\mu\nu}$$

$$(i\gamma_\mu\partial^\mu - m)\psi = 0$$

"Dirac Equation" γ matrices are 4x4.

Can use ("Weyl basis")

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$$

where

$\sigma = (1, \vec{\sigma})$ $\qquad\qquad$ $\bar{\sigma} = (1, -\vec{\sigma})$

There are other choices you can work out...

Dirac equation describes spin 1/2 partilces (e)

B/c γ_μ are matrices, the ψ solution will be a 4-component vector "spinor".

<u>Solution to DE</u> work out in homework ...

Solutions are four component objeccts spinors ψ (complex)

Use $\bar{\psi}$ to denote the complex conjugate of the spinor.

The dirac equation can be formulated from Lagrangian principle via

$$S[\psi, \bar{\psi}] = \int d^4x\, \bar{\psi}[i\gamma_\mu\partial^\mu - m]\psi$$

Varrying the action with respect to $\bar{\psi}$ gives the dirac equation.