Lecture 15

From Last time...

Example of scattering of 2 electrons

$$|i\rangle = |e_1, e_2\rangle \qquad\qquad |f\rangle = |e_3, e_4\rangle$$

$$T_{fi} = V_{fi} + \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni} + ...$$

- E_i- is the initial (= final) energy
- E_f - is the energy of the intermediate state

Now, the $\sum_n$ runs over energy thing in the Hilbert (Fock) space, however only certain states will be non-0.

In relativistic theory, the action-at-a-distance of EM is replaced by a process where 2 electrons interact with a γ which travels at c. (This tells us there should be γ in the intermediate state)

$$V \sim e \int d^3x \psi\phi\psi \qquad \text{(Ignoring spin)}$$

this operator will have terms that go like ($\sim a^\dagger_{e_3}\, a^\dagger_\gamma\, a_{e_1}$)

However, here all terms involve $a^\dagger_\gamma$.
Because $|i\rangle$ and $|f\rangle$ do not contain a γ, $V_{fi} = 0$

to get a non-zero term, we need $|n\rangle$ with a photon.

$$T_{fi} = \frac{\langle e_3 e_4|V|e_3 \gamma e_2\rangle \langle e_3 \gamma e_2|V|e_1 e_2\rangle}{(E_1 + E_2) - (E_3 + E_4 + E_\gamma)} + \text{(2nd term)}$$

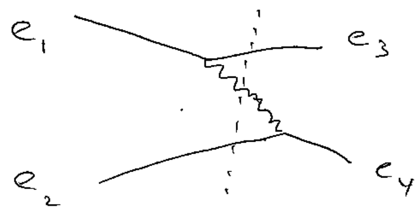
Note: $E_n \neq E_i$ which is allowed by uncertainty principle.

Look at

$$\langle e_3 e_4|V|e_3 \gamma e_2\rangle = \langle e_4|V|\gamma e_2\rangle$$

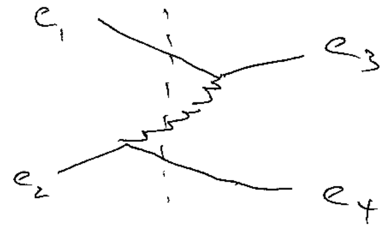
(up to overall normalization from $\langle e_3|e_3\rangle$.

2 ways time $\rightarrow$



$|n\rangle = |\psi_e^3 \phi_\gamma^{\gamma} \psi_e^2\rangle$

(1)



$|n\rangle = |\psi_e^1 \phi_\gamma \psi_e^4\rangle$

(2)

$$\boxed{\langle\gamma|\phi_\gamma(x)|0\rangle = e^{-ip_\gamma\cdot x}}$$

$$\begin{aligned}
\langle e_4|V|\gamma e_2\rangle &= \frac{e}{2}\int d^3x\,\langle e_2\gamma|\psi_e\phi_\gamma\psi_e|e_4\rangle \\
&= e\int d^3x e^{-(p_4-p_2-p_\gamma)x} = e(2\pi)^3\delta(p_4-p_2-p_\gamma)
\end{aligned}$$

Other product

$$\langle e_3\gamma|V|e_1\rangle = e(2\pi)^3\delta(p_4-p_2-p_\gamma)$$

Combining this gives

$$T^1_{fi} \sim \int d^3p_\gamma\,\delta\delta\frac{e^2}{E_i-E_n}$$

where $E_i = (E_1+E_2)$ and $E_n = (E_3+E_2+E_\gamma)$.

$$T^1_{fi} \sim \frac{e^2}{(E_1+E_2)-(E_3+E_2+E_\gamma)} = \frac{e^2}{(E_1-E_3)-E_\gamma}$$

Same logic for 2nd term leads to

$$T^2_{fi} \sim \frac{e^2}{(E_2-E_4)-E_\gamma}$$

End of the day, need to add the two processes.

> Note:
> $$E_1 + E_2 = E_3 + E_4$$
> $$E_1 - E_3 = E_4 - E_2 \equiv \Delta E$$

$$T^1 + T^2 = \frac{e^2}{\Delta E - E_\gamma} + \frac{e^2}{-\Delta E - E_\gamma} = \frac{2e^2 E_\gamma}{(\Delta E)^2 - E_\gamma^2}$$

define $k^\mu \equiv p_3^\mu - p_1^\mu = (\Delta E, \vec{p_\gamma})$

Note k^μ is not the photon momentum! $k^2 \neq 0 (= (\Delta E)^2 - E_\gamma^2)$

$$T_{fi} = \underbrace{2E_\gamma}_{\text{Related to normalization}} \frac{e^2}{k^2}$$

<u>Summary</u> Standard "old-fashion" perturbation theory

- All states are physical (on-shell)
- Matrix element V_{ij} vanishes unless 3-momentum conserved
- Energy <u>not</u> conserved at each vertex
- Add all time orderings

Modern way to interpret same thing "Feynam rules"

<u>Summary</u> Feynman Rules

- Draw diagrams ignoring time ordering
- Vertices come from interactions in Lagrangian: factor of i times coupling constant
- Internal lines get "propagators" = $\frac{i}{p^2-m^2}$
- Lines connected to external points do not get propagators (scalars × 1 / spinors × u or v / spin-1 $\times\epsilon$)
- Four momenta is conserved at each vertex
- Integrate over all undetermined 4-momenta

Example:

$$L = -\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3$$

Consider cross-section for $\phi\phi \to \phi\phi$ scattering.

$$d\sigma = \frac{1}{(2E_1)(2E_2)|v_1 - v_2|}|M|^2 d\Pi_{LIPS}$$

$p_1 + p_2 \to p_3 + p_4$

In COM frame, $\vec{p}_1 = -\vec{p}_2$ and $\vec{p}_3 = -\vec{p}_4$
Also, $E_1 + E_2 = E_3 + E_4 = E_{CM}$

$$d\Pi_{LIPS} = (2\pi)^4 \delta^4(\sum p)\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}$$

(integrating over $\vec{p}_4$)

$$= \frac{1}{16\pi^2}d\Omega \int dp_f \frac{p_f^2}{E_3}\frac{1}{E_4}\delta(E_3 + E_4 - E_{CM})$$

where $p_f = |\vec{p}_3| = |\vec{p}_4|$, $E_3 = \sqrt{m^2 + p_f^2}$, and $\int d^3p_f = \int dp_f p_f^2 d\Omega$

Now change variables,

$p_f \to x = E_3 + E_4 - E_{CM}$

$$dx = \frac{d}{dp_f}(E_3 + E_4 - E_{CM})dp_f = \frac{p_f}{E_3} + \frac{p_f}{E_4} = \frac{E_3 + E_4}{E_3E_4}p_f dp_f$$

$\Rightarrow$

$$\frac{dp_f p_f^2}{E_3E_4} = \frac{dx p_f}{E_{CM}}$$

$$
\begin{aligned}
d\Pi_{LIPS} &= \frac{1}{16\pi^2} d\Omega \int_{m_3+m_4-E_{CM}}^{\infty} dx \frac{p_f}{E_{CM}} \delta(x) \\
&= \frac{1}{16\pi^2} d\Omega \frac{p_f}{E_{CM}} \text{if } E_{CM} > m_3 + m_4 \text{ else } 0
\end{aligned}
$$

$$
d\sigma = \frac{1}{(2E_1)(2E_2)|v_1 - v_2|} \frac{1}{16\pi^2} d\Omega \frac{p_f}{E_{CM}} |M|^2
$$

$$
|v_1 - v_2| = \left| \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_2|}{E_2} \right| = p_i \frac{E_{CM}}{E_1 E_2}
$$