Homework Set #2

Solutions

1) Work out the commutation relation among the $\vec{X}$ and $\vec{P}$ operators:
ie: $[\vec{X},\vec{X}]$, $[\vec{P},\vec{P}]$, and $[\vec{X},\vec{P}]$ *(5 points)*

On one hand,

$$\vec{X}T(\vec{a})\,|\vec{x}\rangle = (\vec{x}+\vec{a})\,|\vec{x}+\vec{a}\rangle$$

on the other,

$$T(\vec{a})\vec{X}\,|\vec{x}\rangle = \vec{x}|\vec{x}+\vec{a}\rangle$$

So

$$[\vec{X}, T(\vec{a})] = \vec{a}T(\vec{a})$$

Consider the infinitesimal translation ($\vec{a} \to \vec{\epsilon}$). Then,

$$[\vec{X}, 1 - i\vec{\epsilon}\cdot\vec{P}] = \vec{\epsilon}\cdot(1 - i\vec{\epsilon}\cdot\vec{P})$$

Keeping terms to first order in ϵ.

$$\begin{aligned}[X_i, 1 - i\epsilon_j P_j] &= \epsilon_i \\ -i\epsilon_j[X_i, P_j] &= \epsilon_i\end{aligned}$$

or

$$[X_i, P_j] = i\delta_{ij}$$

2) Harmonic Oscillator *(10 points)*

The 1D Harmonic oscillator has Hamiltonian:

$$H = \frac{P^2}{2m} + \frac{1}{2}mw^2X^2$$

where P and X are position and momentum operators a Define "raising" and "lowering" operators as

$$a = \sqrt{\frac{mw}{2}}\left(X + i\frac{P}{mw}\right) \qquad a^\dagger = \sqrt{\frac{mw}{2}}\left(X - i\frac{P}{mw}\right)$$

What are the position and momentum operators in terms of the raising and lowering operators?

Solution:

$$x = \frac{x_0}{\sqrt{2}}(a + a^\dagger) \qquad p = \frac{-i}{x_0\sqrt{2}}(a - a^\dagger)$$

where $x_0 = \sqrt{\frac{1}{mw}}$.

b

$$\begin{aligned} [a, a^\dagger] = & \tfrac{mw}{2}[x + \tfrac{ip}{mw}, x - \tfrac{ip}{mw}] \\ = & \tfrac{mw}{2}\tfrac{-i}{mw}[x, p] + \tfrac{mw}{2}\tfrac{i}{mw}[p, x] \\ = & \tfrac{1}{2}(-i[x, p] + i[p, x]) = 1 \end{aligned}$$

c What is the Hamiltonian in terms of a and $a^\dagger$?

$$H = \omega(a^\dagger a + \frac{1}{2})$$

d Define the "Number" operator N as $a^\dagger a$. What is the Hamiltonian in terms of the number operator?

$$H = \omega(N + \frac{1}{2})$$

e Work out the commutation relations: $[N, a^\dagger]$ and $[N, a]$.

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a$$

f Show that the eigenvalues of N (n) are real and satisfy $n \geq 0$.

$$n = \langle n| N |n\rangle = \langle n| a^\dagger a |n\rangle = \langle an| \underbrace{|an\rangle}_{|n'\rangle} = \langle n'|n'\rangle \geq 0$$

g Show that $a|n\rangle$ is an eigenstate of N, with eigenvalue (n-1). This implies $a|n\rangle \propto |n-1\rangle$ and justifies calling a the lower operator.

$$|n'\rangle = a|n\rangle$$

$$N|n'\rangle = Na|n\rangle = (aN + [N,a])|n\rangle = (aN - a)|n\rangle = (an - a)|n\rangle = (n-1)|n'\rangle$$

h Show that $a^\dagger|n\rangle$ is an eigenstate of N, with eigenvalue (n+1). This implies $a^\dagger|n\rangle \propto |n+1\rangle$ and justifies calling $a^\dagger$ the raising operator.

$$|n'\rangle = a^\dagger|n\rangle$$
$$N|n'\rangle = (a^\dagger N + [N,a^\dagger])|n\rangle = (n+1)|n'\rangle$$

i Find c_n such that $|n+1\rangle = c_n a \dagger |n\rangle$ is normalized.

$$|n+1\rangle = c_n a^\dagger |n\rangle$$

$$1 = \langle n+1|n+1\rangle = |c_n|^2 \langle n| aa^\dagger |n\rangle = |c_n|^2 (n+1)$$

$$c_n = \frac{1}{\sqrt{n+1}}$$

j Since $n \geq 0$, there must be a state $|0\rangle$ which satisfies $a|0\rangle = 0$ and n must be an integer. What is the general state $|n\rangle$ in terms of $|0\rangle$ and $a^\dagger$? What is the energy associated to this state ?

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle$$

The energy associated to $|n\rangle$ is $\omega(n+1/2)$.