Homework Set #3

Due Date: Before 5pm Friday January 7th

1) Work out the commutation relation among the $\vec{X}$ and $\vec{P}$ operators: ie: $[\vec{X}, \vec{X}]$, $[\vec{P}, \vec{P}]$, and $[\vec{X}, \vec{P}]$ *(5 points)*

Hint: for $[\vec{X}, \vec{P}]$ work out how the commutator $[\vec{X}, T(\vec{a})]$ acts on position eigenstate for generic translation. Then apply the result to the infinitesimal case.

2) Harmonic Oscillator *(10 points)*

The 1D Harmonic oscillator has Hamiltonian:

$$H = \frac{P^2}{2m} + \frac{1}{2} m w^2 X^2$$

where P and X are position and momentum operators

a Define "raising" and "lowering" operators as

$$a = \sqrt{\frac{mw}{2}} \left(X + i \frac{P}{mw} \right) \qquad a^\dagger = \sqrt{\frac{mw}{2}} \left(X - i \frac{P}{mw} \right)$$

What are the position and momentum operators in terms of the raising and lowering operators?

b Find $[a, a^\dagger]$

c What is the Hamiltonian in terms of a and $a^\dagger$?

d Define the "Number" operator N as $a^\dagger a$. What is the Hamiltonian in terms of the number operator?

e Work out the commutation relations: $[N, a^\dagger]$ and $[N, a]$.

f Show that the eigenvalues of N (n) are real and satisfy $n \geq 0$. (Hint: consider $\langle n| N |n\rangle = \langle n| a^\dagger a |n\rangle$, where $|n\rangle$ are eigenkets of N)

g Show that $a|n\rangle$ is an eigenstate of N, with eigenvalue (n-1). This implies $a|n\rangle \propto |n-1\rangle$ and justifies calling a the lower operator.

h Show that $a^\dagger |n\rangle$ is an eigenstate of N, with eigenvalue (n+1). This implies $a^\dagger |n\rangle \propto |n+1\rangle$ and justifies calling $a^\dagger$ the raising operator.

i Find c_n such that $|n+1\rangle = c_n a \dagger |n\rangle$ is normalized.

j Since $n \geq 0$, there must be a state $|0\rangle$ which satisfies $a|0\rangle = 0$ and n must be an integer. What is the general state $|n\rangle$ in terms of $|0\rangle$ and $a^\dagger$? What is the energy associated to this state ?