Homework Set #2

Solutions

1) Show that SO(2) ≃ U(1) *(2 points)*

(a)

$$zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Given a vector in the complex plain specified by (x,y), zz^* gives the length of the vector.

(b)

$$\begin{aligned} M(\theta_1) &: z \to e^{i\theta_1} z \text{ (+ complex conjugate)} \\ M(\theta_2) &: z \to e^{i\theta_2} z \text{ (+ complex conjugate)} \\ M(\theta_1)M(\theta_2) &: z \to e^{i\theta_1} e^{i\theta_2} z = e^{i(\theta_1+\theta_2)} z = M(\theta_1 + \theta_2) \end{aligned}$$

2) Work out the algebra of the generators of the Lorentz group *(5 points)* Assuming:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} [T_i, T_i] &= 0 \\ [T_1, T_2] &= J_3 \\ [T_1, T_3] &= J_2 \\ [T_2, T_3] &= J_1 \end{aligned}$$

OR

$$[T_i, T_j] = \epsilon_{ijk} J_k \qquad J_2 \to -J_2$$

$$
\begin{aligned}
[T_1, J_1] &= 0 \\
[T_1, J_2] &= -T_3 \\
[T_1, J_3] &= T_2 \\
\\
[T_2, J_1] &= T_3 \\
[T_2, J_2] &= 0 \\
[T_2, J_3] &= -T_1 \\
\\
[T_3, J_1] &= -T_2 \\
[T_3, J_2] &= T_1 \\
[T_3, J_3] &= 0 \\
&\text{OR} \\
[T_i, J_j] = -\epsilon_{ijk} T_k \qquad J_2 &\to -J_2
\end{aligned}
$$

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

3) Connection to βs and γs *(5 points)*

(a)

$$e^{I\eta} = 1 + I\eta + \frac{I^2\eta^2}{2!} + \frac{I^3\eta^3}{3!} + \frac{I^4\eta^4}{4!} + \dots$$

$$I_B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Show that $B(\eta) = e^{I_B\eta} = \cosh(\eta) + I_B \sinh(\eta)$

$$e^{I\eta} = I\left(\eta + \frac{I^2\eta^3}{3!} + \frac{I^4\eta^5}{5!} + \dots\right) + \left(1 + \frac{I^2\eta^2}{2!} + \frac{I^4\eta^4}{4!} + \dots\right)$$

$$e^{I\eta} = I\left(\eta + \frac{\eta^3}{3!} + \frac{\eta^5}{5!} + \dots\right) + \left(1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \dots\right) = I\sinh(\eta) + \cosh(\eta)$$

(b) The origin of the primed frame is at $x' = 0$ in the prime frame and at $x = vt$ in the unprimed frame (assuming the origins coincided at t=0)

$$x = t' \sinh(\eta) \text{ and } t = t' \cosh(\eta)$$

$$v = \frac{x}{t} = \tanh(\eta) \text{ and } \cosh^{-2} = 1 - \tanh^2$$

$$\Rightarrow \cosh(\eta) = \frac{1}{\sqrt{1 - v^2}} \equiv \gamma$$

$$\sinh(\eta) = \frac{v}{\sqrt{1 - v^2}} = \beta\gamma$$

4) Z Boson decays *(5 points)*

(a) Same derivation as we did for the $\pi \to \gamma\gamma$ decay in class gives

$$p_{e_1} = (m_Z/2, 0, 0, m_Z/2) \qquad p_{\gamma_2} = (m_Z/2, 0, 0, -m_Z/2)$$

(b) Including the mass term gives...

$$p_{e_1} = (E_1, 0, 0, P_1) \qquad p_{\gamma_2} = (E_2, 0, 0, -P_2)$$

Momentum conservation implies $P_1 = -P_1 \equiv P$
Energy conservation implies $M_Z = E_1 + E2$
So,

$$p_{e_1} = (E, 0, 0, P) \qquad p_{\gamma_2} = (m_Z - E, 0, 0, -P)$$

Imposing $P_1^{e\,2} = m_e$ gives: $E = \sqrt{m_e^2 + P^2}$
Imposing $P_2^{e\,2} = m_e$ gives: $P^2 = (m_Z - E)^2 - m_e^2$
Combining implies,

$$E = \frac{m_Z}{2} \qquad \text{and} \qquad P = \frac{m_Z}{2}\sqrt{1 - 4\frac{m_e^2}{m_Z^2}}$$

So no correction to the electron Energies.
The the electron momentum is $P \simeq \frac{m_Z}{2}\left(1 - \frac{2m_e^2}{m_Z^2}\right)$, which gives a correction of order $\left(\frac{m_e}{m_Z}\right)^2 \sim \left(\frac{10^{-3}\text{ GeV}}{100\text{ GeV}}\right)^2 \sim 10^{-10}$

(b) Including the mass term for the b-quark gives...
No correction to the energies.
The b-quark momentum has a correction of order $\left(\frac{m_b}{m_Z}\right)^2 \sim \left(\frac{10\text{ GeV}}{100\text{ GeV}}\right)^2 \sim 10^{-2}$ about 1%.

5) GZK cutoff energy *(5 points)*

(a) $(P + \gamma) \to (p + \pi_0)$

The final momentum in the center of mass frame is:

$$P_F^\mu = (m_p + m_\pi, 0, 0, 0)$$

$$P_F^2 = (m_p + m_\pi)^2 = m_p^2 + 2m_p m_\pi + m_\pi^2$$

The initial four vector is given by the sum of the proton and CMB photon four vectors.

$$P_I = (P_p^\mu + P_\gamma^\mu)$$

$$P_I^2 = \underbrace{P_p^2}_{=m_p^2} + 2P_p \cdot P_\gamma + \underbrace{P_\gamma^2}_{=0}$$

To evaluate $P_p \cdot P_\gamma$, can use reference frame where the photon and proton are colliding head on.

(Assume for the moment that we can neglect the proton mass compared to it momentum ...

$$P_p = (p_p, 0, 0, p_p) \qquad P_\gamma = (E_{CMB}, 0, 0, -E_{CMB})$$

Where $E_{CMB} = 3 \cdot 10^{-13}$GeV.

$$P_p \cdot P_\gamma = 2p_p E_{CMB}$$

Imposing $P_I^2 = P_F^2$, allows us to solve for p_p.

$$m_p^2 + 4p_p E_{CMB} = m_p^2 + 2m_p m_\pi + m_\pi^2$$

or

$$p_p = \frac{2m_p m_\pi + m_\pi^2}{4E_{CMB}} = \frac{2 \cdot 1 \cdot 0.14 + 0.14^2}{10 \cdot 10^{-13}}\text{GeV} \sim 3 \times 10^{11}\text{GeV} \sim 10^{20}\text{eV}$$

(b) ~ 30 J