(4b) Talk about all these particle states in a more convenient way.

For every given momenta we have stacks of Hilbert Space
- ∞ many possibilities for Bosons $0 \to N$
- fixed # depending on the Spin for fermions.

Ultimately interested in interactions between particles.
- Defined by some hamiltonian.
- Could specify the ~~the~~ hamiltonian by describing how it acts on all states in the hilbert space.
- Instead for our convenience introduce creation + annihilation operators.
  ↳ keeps track of states in a simple way.

vacuum state

$$|0\rangle$$

These are primary

$$|p,\sigma\rangle \equiv a^{+}_{p\sigma}|0\rangle$$

↪ defines a^{+}

$$|p_1\sigma_1, p_2\sigma_2\rangle = a^{+}_{p_1\sigma_1} a^{+}_{p_2\sigma_2}|0\rangle$$

$\vdots$

$a_{p\sigma}|0\rangle = 0$ a - removes states.

$\vdots$

Can encode boson/fermion statistics in a + a^{+}.

$$[a^{+}_{p_1\sigma_1}\ a^{+}_{p_2\sigma_2}] = 0 \qquad \{a^{+}_{p_1\sigma_1}, a^{+}_{p_2\sigma_2}\} = 0$$

Bosons   Fermions

$$[a_{p_1\sigma_1}\ a_{p_2\sigma_2}] = 0 \qquad \{a_{p_1\sigma_1}, a_{p_2\sigma_2}\} = 0$$

Normalization

$$\langle p', \sigma' | p, \sigma \rangle = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}')$$ ↙ use compressed Notation

$$= \delta_{\sigma\sigma'} \delta_{pp'}$$

Normalization is our convience.

$\langle p'\sigma' | p\sigma \rangle = \delta\delta$
$\langle 0 | a_{p'\sigma'} a^+_{p\sigma} | 0 \rangle$
$= \langle 0 | a^+ a + [a\, a^+] | 0 \rangle$
$= 0 + \langle 0 | [a\, a^+] | 0 \rangle = \delta\delta$

$$[a_{p\sigma}, a^+_{p'\sigma'}] = \delta_{\sigma\sigma'} \delta_{pp'} \quad \underline{\text{Bosons}}$$

$$\{a_{p\sigma}, a^+_{p'\sigma'}\} = \delta_{\sigma\sigma'} \delta_{pp'} \quad \text{Fermions}$$

Note, these are operators that We have defined for our convience. Makes it easier to talk about the states.

Usualy you see it presented as,
- Start w/ fields, quantize, then find these commutation relations

But really the fields are secondary concepts, what comes first is the particles. "Not one drop of this thing going on here"

What would the free hamiltonian be?

$$H = \sum_{p\sigma} E_p a^+_{p\sigma} a_{p\sigma}$$ — Labeling states by 4-momenta, but the energy is constrained ($E^2 = p^2 + m^2$)

really an integral.

Can label by the 3-momentum

$$\langle \vec{p}'\sigma' | \vec{p}\sigma \rangle = (2\pi)^3\, 2E_p\, \delta^3(\vec{p} - \vec{p}')$$

Note d^3p is not Lorentz invariant, but $\frac{d^3p}{2E_p}$ is Lorentz invariant.

$$\int \frac{d^3p}{(2\pi)^3 2E_p}$$

$$\int d^4p\, \delta(p^2 - m^2) = \int dE\, d^3\vec{p}\, \delta(E^2 - (\vec{p}^2 + m^2))$$

$$= \int \frac{dE^2}{2E} d^3\vec{p}\, \delta(E^2 - (\vec{p}^2 + m^2))$$

Can do $\int dE$

$$= \int \frac{d^3\vec{p}}{2E}$$

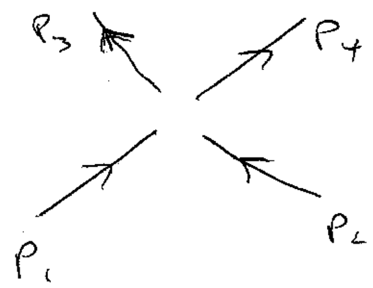
Make my life eisier by defing.

$$\int đ^3 p = \int \frac{d^3 p}{(2\pi)^3\, 2E}$$

$$H^{Free} = \sum_{p\sigma} E_p \, a^\dagger_{p\sigma} a_{p\sigma}$$

$$= \int đ^3 p \, E_p \, a^\dagger_{p\sigma} a_{p\sigma}$$

Now lets imagine building interactions. Add interaction hamiltonian.



$$+ \int đ^3 p_1 \, đ^3 p_2 \, đ^3 p_3 \, đ^3 p_4 \, \delta(\vec{P}_1 \cdots \vec{P}_4) \, \delta(E_1 \cdots E_4) \, \underbrace{a^\dagger_{p_4\sigma_4} a^\dagger_{p_3\sigma_3} a_{p_2\sigma_2} a_{p_1\sigma_1}} \, V^{\sigma_1\sigma_2\sigma_3\sigma_4}(P_1 P_2 P_3 P_4)) + H.C.$$

↳ Acts on the initial state and gives the final state.

Interactions made out of
 Strings of a's and $a^\dagger$'s.

Also easy in this picture to talk about creation and destruction of particles (Not just scattering)

Very convient to use this to map between different particle states

So far we havnt said the word "feild."

Now comes the challange;
- these are interactions between momentum eigenstates.
- mom eigen states are like big plane waves.

A totally generic coefficient $(\delta(\vec{P}_1 \cdots \vec{P}_4)\,\delta(E_1 \ldots E_4)\, V^{\sigma_1 \ldots \sigma_4}(P_1 \ldots P_4))$ is not going to correspond to point-like local interactions.

Would like to come up with some engine to allow us to build interaction Hamiltonians where we can just see explicitly that the interactions are local.

↳ this is where the utility of the field concept comes in.

The states that we defined act very nicely under the translation operator, (just pick up phase) but to get interactions local in space need x to make an appearance.

Build out of these other operators

$$\phi(x) \qquad T: \phi(x) \rightarrow \phi(\vec{x}+\vec{a})$$

Very nice way of doing this, fourier transforms.

<u>Define</u>

$$\phi_+(\vec{x}) = \int đ^3p \; a^\dagger_{\vec{p}} \, e^{i\vec{p}\cdot\vec{x}} \qquad \phi_-(x) = \phi_+^\dagger(x)$$

$$\phi_-(\vec{x}) = \int đ^3p \; a_{\vec{p}} \, e^{-i\vec{p}\cdot\vec{x}}$$

Indeed ϕ's behave as above under translations.

Now can go back to the free hamiltonian and write it very simply using ϕ's

H^{free} <u>non relativistic for the moment.</u> $E_p = \frac{p^2}{2m}$

Same as $H^{free} = \sum_{p\sigma} E_p \, a^\dagger_{p\sigma} a_{p\sigma}$

$$H^{free} = \int d^3x \; \frac{(\vec{\nabla}\phi_+)^\dagger(\nabla\phi_+)}{2m}$$

Now its clear how you could write down interactions that take place locally.

$$H^{free} + \int d^3x \; C \left[\phi_+(x) \, \phi_+(x) \, \phi_-(x) \, \phi_-(x) \right] + \dots$$

Its totally clear now that this is local in space.

Taking this an expanding out gives something like we just talked about with 2 a's and 2 $a^\dagger$'s. All the rest comes along for the ride.

(Note this is done non-relativistically for the moment to stress that this has nothing to do w/ relativity. This is about making interactions local)

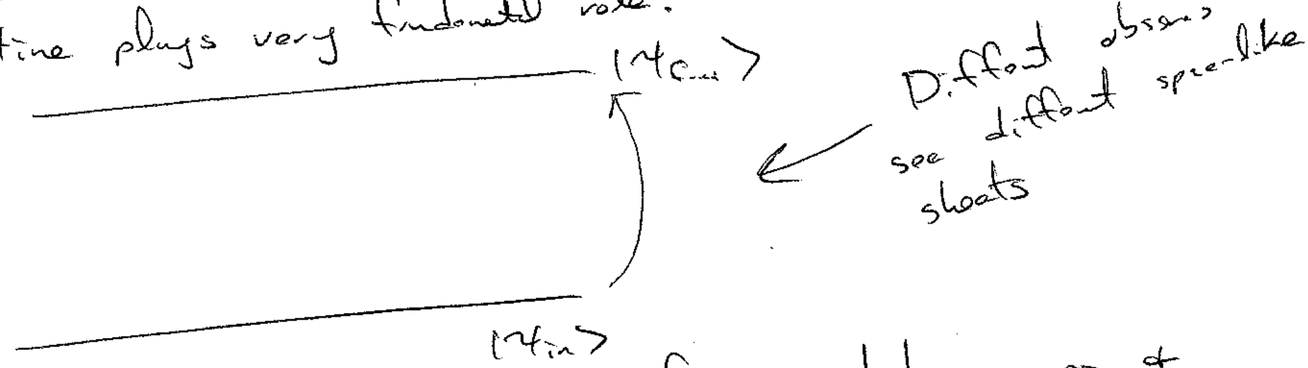
Why we use fields. Makes local interactions of particles manifest.

Hardwired into the description of particles.
locality

Where does relativity come in? What is the dificulty?

Time evolution

QM - time plays very fundamental role.



Have a hope of Lorentz invariance if we start a $-\infty$ + go to $+\infty$.

Throw particles in from ∞ let them scatter + go back out to ∞. S-matrix

$$t=-\infty \quad |p_1\sigma_1 \dots p_n\sigma_n\rangle \longrightarrow S|\vec{p}_1\sigma_1 \dots \vec{p}_n\sigma_n\rangle \quad t=+\infty$$

$\hookrightarrow$ might be (at least a hope) Lorentz invariant.

Figure out what S is in a totally generic theory, then see what it would take to make it Lorentz Invariant.

Sure doesn't look like it will be L.I. S is the only object that you could even have a hope to make L.I.

We will see that for <u>very special</u> choices of the interaction it will barely be possible for it to be Lorentz Invariant.

↳ Those choices force on us anti-particles and the connection between spin & statistics.

Something annoying that we should get rid of right away. Free evolution, just evolves w/ phase. totally irrelevant part.

Standard way of removing the free evolution "<u>Interaction Representation</u>"

$$H = H_{free} + H_{Int}$$

$$i \frac{d|\psi\rangle}{dt} = (H_{free} + H_{int})|\psi\rangle$$

$\underline{H_{int}=0}$

$$|\psi\rangle = e^{-iH_f t}|\psi_{in}\rangle$$

We don't have free theory, but if the interaction is small going to be pretty close to evolving like this.

$$|\psi\rangle = e^{-iH_f t}\underbrace{|\psi_I\rangle}_{\text{definition}}$$

If, $H_{int}=0$, $|\psi_{int}\rangle$ doesn't evolve at all.

B/c there is H_{int} $|\psi_I\rangle$ will evolve

$$i \frac{d}{dt}|\psi\rangle = H_f|\psi\rangle + e^{-iH_f t}\, i\frac{d}{dt}|\psi_{Int}\rangle$$

$$= (H_f + H_{int})\, e^{-iH_f t}|\psi_{int}\rangle$$

$$i\frac{d}{dt}|\psi_{Int}\rangle = \underbrace{\left[e^{iH_0 t} H_{int}\, e^{-iH_0 t}\right]}|\psi_{int}\rangle$$

H_I - interaction hamiltonian in the interaction representation.

$$i\frac{d}{dt}|\psi_{int}\rangle = H_I|\psi_{int}\rangle \quad (\text{Use } \psi_I \text{ for } \psi_{int})$$

can be time dependent.

Formally solve this — just integrating.

$$|\psi_I(t_2)\rangle = |\psi_I(t_1)\rangle - i\int_{t_1}^{t_2} dt\; H_I|\psi_I(t)\rangle$$

Now can keep going

$$= |\psi_I(t_1)\rangle - i\int_{t_1}^{t_2} dt\, H_I^{(1)}\left(|\psi_I(t_1)\rangle - i\int_{t'}^{t'} dt'\; H_I(t')\,|\psi(t')\rangle\right)$$

$$= |\psi_I(t_1)\rangle - i\int_{t'}^{t_2} dt\; H_I(t)\,|\psi_I(t_1)\rangle$$

$$(-i)^2\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt'\; H_I(t)\,H_I(t')\,|\psi_I(t')\rangle$$

Pattern is clear, & can keep on going

$$|\psi_I(t_2)\rangle = \Bigg[1 + (-i)\int_{t_1}^{t_2} dt\; H_I(t)$$

$$+ (-i)^2\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt'\; H_I(t)\, H_I(t')$$

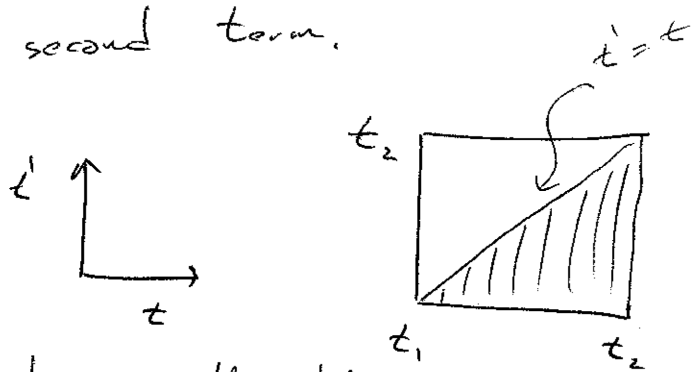
$$+ -i\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt'\int_{t_1}^{t'} dt''\; H_I(t)\, H_I(t')\, H_I(t'') \Bigg]|\psi_I(t_1)\rangle$$

$$+ \cdots$$

If H_I is small this is giving us some nice perturbation theory.

Look @ the second term.

$$\int_{t_1}^{t_2} dt \int_{t_1}^{t} dt'$$



Nice to write it over the whole region:

$$\int_{\text{▨}} H_I(t)\, H_I(t') = \frac{1}{2} \int_{\text{▨}} T\left(H_I(t)\, H_I(t')\right)$$

↳ time ordered product

$$T\left(A(t)\,B(t')\right) = \begin{cases} A(t)\, B(t') & t > t' \\ B(t')\, A(t) & t < t' \end{cases}$$

$$\Big[\, 1 + (-i)\int_{t_1}^{t} dt\; T\left(H_I(t)\right)$$

$$+ \frac{(-i)^2}{2} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt'\; T\left(H_I(t)\, H_I(t')\right)$$

$$+ \frac{(-i)^3}{3!} \int_{t_1}^{t_2} dt\, dt'\, dt''\; T\left(H_I(t)\, H_I(t')\, H_I(t'')\right)$$

$$+ \cdots \Big]$$

$$|\Psi_I(t_2)\rangle = T\left(e^{-i\int_{t_1}^{t_2} dt\, H_I(t)}\right) |\Psi_I(t_1)\rangle$$

Now,

$$|\Psi_I(+\infty)\rangle = T\left(e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)}\right) |\Psi_I(-\infty)\rangle$$

Lets go back to field theory.

$$\phi_+(x) = \int d\vec{p}\; e^{i\vec{p}\cdot\vec{x}}\, a^+_{\vec{p}}$$

$$\boxed{\phi_+(\vec{x}) = \int d^3p\; e^{i\vec{p}\vec{x}}\, a_{p\sigma}\, C_\sigma(p)}$$

← use scalars for the moment.

Need to build H_I out of $\phi_{+/-}$ in the interaction representation.

$$\phi^I_+(x,t) = e^{-iH_0t}\,\phi(x)\, e^{iH_0t}$$

$$\boxed{*\; e^{-iH_0t}\,\phi(x)\, e^{iH_0t} \to e^{-iE_pt}\phi(x)}$$

$$= \int d^3\vec{p}\; e^{i\vec{p}\vec{x}}\, e^{-iE_pt}\, a^+_p$$

$$= \int d^3p\; e^{-ip^\mu x_\mu}\, a^+_p$$

Behaves nicely under Lorentz transforms.

$$\phi_*(\Lambda x) = \phi_+(x)$$

We seem to be in awesome shape, Lets write down an interaction.

$$H^I = \int d^3x \left[k\phi^I_+(x) \ldots \phi^I_-(x) \ldots \right]$$

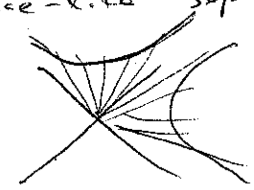
$$T\, e^{-i\int \underbrace{dt\, d^3x}_{d^4x} \left[k\phi^I_+(x) \ldots \phi^I_-(x) \ldots \right.}$$

$$T\, e^{-i\underbrace{\int d^4x \left[k\phi_+(x) \ldots \phi_-(x) \right]}_{\text{All of this is Lorentz invariant!}}}$$

Seems like we are done.

Problem is the time ordering. Only thing that is not necessarily L. I.

Write in a form: $T\, e^{-i\int d^4x\, \mathcal{H}_I(x)}$ this would be LI if T was Lorentz I. But space-like separated objects are not time-ordered in a LI way.



Only way to be L. I. if $\mathcal{H}_I(x)$ & $\mathcal{H}_I(x')$ commute when x & x' space-like separated.

Lorentz Invariance $\Rightarrow$ $[\mathcal{H}_I(x), \mathcal{H}(x')] = 0$ if $x-x' < 0$ spacelike.

Can ask if the $\phi^\dagger$'s or the ϕ's commute outside the light cone
→ they do Not

However can find new combination which does.

Striking Results

Turns out $[\phi_+(x), \phi_+^\dagger(x')] \neq 0$ for $(x-x') < 0$

$$\Phi(x) = \phi_+(x) + \phi_-(x) = \int d^3p \left(a^\dagger e^{-ipx} + a e^{ipx}\right)$$ Scalars have to Bosons!

$[\Phi(x), \Phi(x')] = 0$ for $(x-x') < 0$ $\boxed{\text{Only if } [a, a^\dagger] = 0 \text{ not if } \{a, a^\dagger\} = 0}$

↳ tells us something quite significant.

Build $\mathcal{H}_I$ out of Φ. (Not ϕ_+ + ϕ_- separately)

$$\mathcal{H}_I = \lambda \Phi^4(x)$$

$$= \lambda(\phi_+ + \phi_-)^4 = \lambda\left[\phi_+^2\phi_-^2 + \underbrace{\phi_+^4 + \phi_+\phi_-^3}_{\text{New terms}} + \cdots\right]$$

term we saw before

2 go in 2 go out   e.g. 1 go in, 3 go out

No charges associated with this scalar.
Can create it or destroy it @ will (eg $\phi_+\phi_-^3$)

How do we talk about particles w/ conserved charge?
This will not work! Only have one choice, introduce another operator. a's + b's

$\phi^a \quad \int a_p^\dagger e^{i\vec{p}\cdot\vec{x}}$

$\phi^b \quad \int b_p^\dagger e^{i\vec{p}\cdot\vec{x}}$

$$\Phi = \phi_+^a + \phi_-^b$$ Not hermitian

$$= \int d^3p \left[a_p e^{ipx} + b_p^\dagger e^{-ipx}\right]$$

$$[\Phi(x), \Phi(x')] = 0 \quad (x-x') < 0$$

$$\int d^3x \; \Phi^{\dagger}(x)^2 \; \Phi(x)^2$$ Hermitian

Expand this out and find that everything would conserve charge provided that particles of type b have opposite charge to a.

If you want to talk about particles that carry some well defined charge, you must have <u>anti-particles</u>.

If you put this together with s=½ (putting back the c_σ) you find that they have to have {, }=0 for the hamiltonian to vanish outside the light cone.