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Recap S.R. / Lorentz Invariance

$(t, x) \xrightarrow[\text{observer}]{\text{another moving}} (t', x')$

Invariant notion of distance

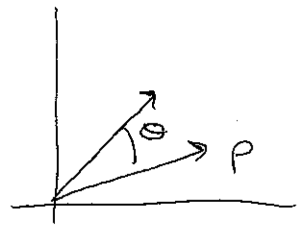
$t^2 - x^2 = t'^2 - x'^2$

(this should all be familiar to you)

We will recap this in an adult way ...

<u>Start w/ rotations</u>

Have an invariant (length of $\vec{p}$)



$x^2 + y^2 = x'^2 + y'^2$

$x' = x\cos\theta - y\sin\theta$

$y' = y\cos\theta + x\sin\theta$

Look at this another way ...

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$x'^2 + y'^2 = x^2 + y^2$

After the set of all such matrices that have this property

Start w/ infinitesimal case

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(3) $(x' \;\; y')\begin{pmatrix} x' \\ y' \end{pmatrix} = (x, y)\begin{pmatrix} x \\ y \end{pmatrix}$

$$x' = x + \varepsilon a x + \varepsilon b y$$
$$y' = y + \varepsilon c x + \varepsilon d y$$

<u>Keep terms linear in ε</u>

$$x'^2 + y'^2 = x^2 + y^2 + 2\varepsilon(\underbrace{ax^2 + bxy + cxy + dy^2}_{=0 \quad \forall\, x \,\&\, y})$$

$$ax^2 + (b+c)xy + dy^2 = 0$$

$$\Rightarrow a = d = 0 \quad \& \quad b = -c$$

Can rescale ε such that $b = 1$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\mathbb{1} + \varepsilon\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix}$$

<u>More Sophisticated way:</u>

$x_i \qquad i = 1, 2 \qquad x_1 \equiv x \quad x_2 \equiv y$

$$x_i' = R_{i1} x_1 + R_{i2} x_2 = \sum_{j=1}^{2} R_{ij} x_j = R_{ij} x_j$$

$x_i' = R_{ij} x_j$

$x_i' x_i' = x_i x_i$ this is what it takes for R to be a rotation.

Need to find special R's such that this is satisfied

(4) Identity Matrix

$R_{ij} = \delta_{ij}$ if no rotation @ all.

$$R_{ij} = \delta_{ij} + \varepsilon \omega_{ij}$$

$$x_i' = x_i + \varepsilon \omega_{ij} x_j$$

$$x_i' x_i' = x_i x_i + 2\varepsilon \underbrace{\omega_{ij} x_j x_i}_{=0 \ \forall x} + \mathcal{O}(\varepsilon^2)$$

$\Rightarrow$ ω_{ij} has to be anti-symmetric. $\omega_{ij} = -\omega_{ji}$

Back to previous example, find <u>finite</u> rotations (without any mention of geometry etc)

take infinitesimal rotation & do it n-times

$\Theta = N\varepsilon$ $\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\underline{\underline{1}} + \varepsilon \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{} \right] \begin{pmatrix} x \\ y \end{pmatrix}$

call the "I" for the moment

$$(1 + \varepsilon \underline{I})(1 + \varepsilon \underline{I}) \cdots (\quad) \cdots)$$

$$(1 + I\varepsilon)^N = \left(1 + I\frac{\Theta}{N}\right)^N \quad \text{Now let } N \to \infty$$

$$\hookrightarrow e^{I\Theta}$$

Built up finite rotation $\quad R(\Theta) = e^{I\Theta}$

⑤

$$x'(\theta) = R(\theta)\,x = e^{I\theta}x$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$ ← lets calculate the first terms

$$I^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{1}!$$ Just discovered i (following our nose)

$$R(\theta) = \cos\theta + I\sin\theta$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\boxed{\begin{aligned} e^{I\theta} &= 1 + \frac{I^2}{2}\theta^2 + \frac{I^4\theta^4}{4!}\dots \\ &\quad + I\theta + \frac{I^3\theta^3}{3!}\dots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots\right)}_{\cos\theta} \\ &\quad + \underbrace{I\left(\theta - \frac{\theta^3}{3!} + \dots\right)}_{I\sin\theta} \end{aligned}}$$

Strategy: first understand the action of the symmetry infinitesimally, then the big symmetry action is obtained by iterating the infinitesimal

Always e^{x} — Great Strategy for any kind of symmetry.

Now 3D Rotations

Something New happens

(6) 3-parameters associated w/ 3D rotation

Already saw, any rotation is of the form

$$x_i' = x_i + \varepsilon\, \omega_{ij}\, x_{ij} \quad \text{w/ } \omega_{ij} = -\omega_{ji}$$

Most general 3×3 anti-symmetric matrix

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \qquad \varepsilon\, \omega_{ij} = \varepsilon_{12}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + \varepsilon_{13}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \varepsilon_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Now have 3 generators corresponding to rotation in 3D

generator that we just saw

How to get the finite version? Easy just exponentiate

Something new happens in 3D

2D rotations commute.

3D " do not

$$\varepsilon_{12} T_{12} + \varepsilon_{13} T_{13} + \varepsilon_{23} T_{23} \rightarrow i\varepsilon_3 J_3 + i\varepsilon_2 J_2 + i\varepsilon_1 J_1$$

Rotations form a group

$$e^{\theta_{12} T_{12}}\, e^{\theta_{13} T_{13}}\, e^{{}_{23} T_{23}} = e^{\phi_{12} T_{12} + \phi_{23} T_{23} + \phi_{23} T_{23}}$$

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$e^{i\theta_3 J_3} \qquad [J_1, J_2] = i J_3 \quad + \text{cyclic}$$

⑦ Can step back + think about this more abstractly

Matrices we found form a group, but this group exists abstractly independent of these 3×3 matrices. Fully determined by the comm. relations.
(Just like vectors + components)

In general, many matrices that satisfy the algebra. those give different representations

Deep Rotations can act on more than just 3D vectors

$$\left[\begin{array}{c|c} (J_i) & 0 \\ \hline 0 & (J_j) \end{array}\right]$$

officially a representation

"Reducible" Representation

More Generally

$$[J^a, J^b] = i f_{abc} J^c \qquad J^a \quad a = 1, 2 \ldots \dim \text{ of group}$$

Lie found all the possible symmetries when J is hermitian (there are not many)

One final example w/ Rotations →

8 Rotations

Traceless 2x2 hermitian matrices

σ - pauli matrices

$$M = \begin{pmatrix} z & x+iy \\ -x-iy & -z \end{pmatrix} = \vec{\sigma} \cdot \vec{x}$$

$$M' = U^{\dagger} M U$$

U is unitary, any unitary matrix Can be written as a phase $e^{i\theta}$ times 2x2 hermitian matrix w/ det = 1 ("Special Unitary Matrix")

U - unitary + det(1)

$U \in SU(2)$ ← Do this B/c phase cancels

M' - still hermitian, still traceless

$\Rightarrow \quad M' = \vec{\sigma} \cdot \vec{x}_u'$ ← this $\vec{x}$ depends on U

$$Det(M) = -z^2 - x^2 - y^2 = -\vec{x}^2$$

$$Det(M') = Det(M) \qquad \vec{x}_u'^2 = \vec{x}^2$$

→ direct correspondence between 2x2 hermitian matrices + Rotations

2D action of rotations

Now easy to generalize all of this to Lorentz group

9 Recap S.R.

$(t, x) \quad (t', x') \qquad t^2 - x^2 = t'^2 - x'^2$

$x^{\pm} = t \pm x \qquad x^+ x^- = t^2 - x^2$

$\qquad\qquad\qquad\qquad = x'^+ x'^-$

$x^+ \longrightarrow e^{\eta} x^+$

$x^- \longrightarrow e^{-\eta} x^-$

$t' + x' = e^{\eta}(t + x)$

$t' - x' = e^{-\eta}(t - x)$

$$\Rightarrow \boxed{\begin{aligned} t' &= \cosh\eta \, t + \sinh\eta \, x \\ x' &= \cosh\eta \, x + \sinh\eta \, t \end{aligned}}$$

$ds^2 = dt^2 - dx^2$

$\quad = dx^+ dx^-$

$x^+ = \rho e^{\tau}$

$x^- = \rho e^{-\tau}$

$\rho \to \rho^*$

$\tau \to \tau + \eta$

$ds^2 = \rho^2 d\tau^2 - d\rho^2$

(10) Lorentz Transformations

Start by thinking of the action on 4-vectors
(Many other representations)

$$x^\mu = (t, \vec{x})$$

look for transformation such that invariant length $(t^2 - \vec{x}^2)$ is preserved.

$$x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{4\times 4\text{-matrix}} x^\nu$$

This length is $\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$

$$\eta_{\mu\nu} = \begin{cases} 1 & \mu=\nu=0 \\ -1 & \mu=\nu=1,2,3 \\ 0 & \text{otherwise} \end{cases} \qquad \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$x_\mu = \eta_{\mu\nu} x^\nu$$

$$\Rightarrow \eta_{\mu\nu} x^\mu x^\nu = x_\nu x^\nu \qquad x_\mu = (t, -\vec{x})$$

Same thing as before: Get the infinitesimal versions

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon \omega^\mu{}_\nu \qquad \boxed{\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu=\nu \\ 0 & \text{otherwise} \end{cases}}$$

$$x'^\mu = x^\mu + \varepsilon \omega^\mu{}_\nu x^\nu$$

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

$$= \eta_{\mu\nu} x^\mu x^\nu + 2\varepsilon \eta_{\mu\nu} x^\nu \omega^\mu{}_\rho x^\rho$$

So, $\underbrace{\eta_{\mu\nu}\omega^\mu{}_\rho}\, x^\nu x^\rho = 0$

$\equiv \omega_{\nu\rho} x^\nu x^\rho = 0$ ← All 4×4 anti-symmetric matrices $\omega_{\mu\rho} = -\omega_{\rho\mu}$

(11)

$$\omega_{\mu\nu} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & a & b & c \\ 1 & -a & 0 & A & B \\ 2 & -b & -A & 0 & C \\ 3 & -c & -B & -C & 0 \end{array}$$

* However the thing that enters the transformation is $\omega^\mu{}_\nu$

$$= \eta_{\mu\nu}\, \omega^\mu{}_\nu$$

$$\omega_{0i} = \omega^0{}_i \qquad \text{But } \omega_{0i} = -\omega_{i0}$$

$$\omega_{i0} = -\omega^i{}_0 \qquad \Downarrow \qquad \omega^0{}_i = +\omega^i{}_0$$

$$\omega^\mu{}_\nu \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & a & b & c \\ 1 & a & 0 & A & B \\ 2 & b & -A & 0 & C \\ 3 & c & -B & -C & 0 \end{array}$$

Generators of the Lorentz group (6, 3-rotations, 3 Boosts)
(Crucial "- sign" difference between boosts / ~~rotations~~

$$\omega^1{}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \omega^0{}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \underline{\mathbb{1}}$$

$$e^{\theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \cos\theta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin\theta \qquad e^{\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \cosh\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sinh\eta$$

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Can work out the Lie Algebra

ect.

(12) Spinor Representation

$x^0, x^1, x^2, x^3 \xrightarrow[2\times 2]{\text{Hermitian}} M = \begin{pmatrix} x^0 + x^3 & x^1 + i x^2 \\ x^1 - i x^2 & x^0 - x^3 \end{pmatrix} = \sigma_\mu x^\mu$

$\sigma_0 = 1$
$\sigma_i = \sigma_i$ - pauli

$M' = L^+ M L$ L any 2×2 complex Matrix
w/ $\det(L) = 1$

M' - still hermitian

$SL(2, C)$

$\Rightarrow M' = \sigma_\mu x^\mu_{(L)}$ $\det(M) = x^\mu x_\mu$