Relativistic Wave Eqs

Sch eq $\quad i\frac{d}{dt}\psi = \left(-\frac{\nabla^2}{2m} + V\right)\psi$

Problems - Conservation of non-relativistic Energy.

$$E \leftrightarrow i\frac{d}{dt} \qquad p = -i\nabla$$

$\Rightarrow$ **Sch eq** $\Rightarrow E = \frac{p^2}{2m} + V$ ← Not L. I.

- time/position not on equal footing.

Start w/ relativistic E/p relation!

$$E^2 - p^2 - m^2 = 0$$

Klein-Gordon Eq $\quad \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)\phi(x,t) = 0$

Solution

$$\phi(x,t) = e^{i p\cdot x} \qquad \text{"On shell solutions"}$$

everybody is L. I.

$$\boxed{\text{Note two solutions } E>0 + E<0}$$

Alternative Standard Powerful formulation of Physics via Lagrangians

KG eq $\quad \frac{\partial^2}{\partial t^2}\phi = \vec{\nabla}^2\phi - m^2\phi$

$\phi(x,t)$ - parameters time & space

& looks like $\quad \frac{d^2}{dt^2}x(t) = -\frac{\partial U}{\partial X}$ ← Potential Energy

$\frac{\partial^2}{\partial t^2}\phi$ - looks like the "acceleration" of the field

$-m^2\phi$ - effective "force" from the mass.
(harmonic oscillator)

$\nabla^2\phi$ - "shear force" shearing fabric takes force

Now have some intuition about what KG telling us

Integrate force to get the potential

See from finite diff method

$$-\frac{\partial U}{\partial \phi} = \nabla^2\phi - m^2\phi \Rightarrow U = +\frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2$$

Potential energy

K.E. from generalization of $\frac{1}{2}\dot{x}^2$ to fields

$$K = \frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2$$

$$K+U = \frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{m^2}{2}\phi^2$$ ← at the point (x,t)

Total energy from integrating

$$H(t) = \int d\vec{x} \left[\frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 + \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2 \right]$$

$\hookrightarrow$ with this you could go back to 3 lectures ago and replace $\frac{1}{2}\frac{\nabla\phi^\dagger\nabla\phi}{m}$ and do everything again (We won't bother here ...)

However we can use this to formulate the field eq in a totally different, ultimately more useful way...

think of classical mechanics of point particles
Can all be formulated with Lagrangian & principle of least action.

$$L = \frac{1}{2}\dot{x}^2 - U(x)$$

difference KE & Potential

The action $S[x(t)] = \int dt\, L$

you could take this as the starting point & derive Newtons 2^{nd} law by minimizing the action

$$\cancel{\frac{dS}{dx(t)} = \ddot{x} + \frac{\partial U}{\partial x}}$$

Do the same for relativistic fields

$$L = K - U = \int d^3x \left[\frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 - \frac{1}{2}(\nabla\phi)(\nabla\phi) - \frac{m^2}{2}\phi^2 \right]$$

$$S[\phi(t)] = \int d^4x \left[\nearrow \right]$$

K.G. can be found from minimizing action wrt ϕ

$\partial_\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$ $$S[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2\right]$$

manifestly L. I.

This tells us how to construct general L I descriptions of Relativistic QM systems.

- Need L I Lagrangian and we are "done"

Schematically $\partial^\mu \phi \longleftarrow \longrightarrow \partial_\mu \phi$

~ all of particle physics follows from this.

$$\boxed{\begin{array}{l} \text{Two more relativistic systems important for S.M.} \\ \text{Spin 1} \quad \text{"E M"} \qquad \text{K G} \quad \text{Spin 0} \\ \text{Spin } 1/2 \quad \text{dirac equation} \end{array}}$$

K G 2^{nd} order in x^μ. Contains no info about intrinsic angular momentum. (No free Lorentz indices)

ϕ from K G is Spin-0 field

Need relativistic eq. with free Lorentz indices

Can we find 1^{st} order relativistic eq of motion?

Assume there exists a wave eq linear in t + x

$$\left(\alpha \frac{\partial}{\partial t} + \vec{\beta} \cdot \vec{\nabla}\right)\psi = m\psi$$

for some $\alpha, \vec{\beta}, m$

If rel invariant, must imply the KG

Assume that it is the square root

$$\left(\alpha \frac{d}{dt} + \vec{\beta}\cdot\vec{\nabla}\right)\left(\alpha \frac{\partial}{\partial t} + \vec{\beta}\cdot\vec{\nabla}\right)\psi = m^2\psi$$

$$\Rightarrow \left[\alpha^2 \frac{\partial^2}{\partial t^2} + \alpha \vec{\beta}\cdot\nabla \frac{\partial}{\partial t} + \vec{\beta}\cdot\vec{\nabla}\alpha \frac{\partial}{\partial t} + (\beta\cdot\nabla)^2\right]\psi = m^2\psi$$

to get $\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = m^2\psi$

$$\Rightarrow \quad \alpha^2 = -1 \qquad \beta_i\beta_j = \delta_{ij} \qquad \alpha\beta_i + \beta_i\alpha = 0$$

C #   Cannot be #'s   Lets keep going ...

$$\alpha = i\gamma_0 \qquad \beta_i = i\gamma_i$$

$$\gamma_0\gamma_0 = 1 \qquad \gamma_i\gamma_j = -\delta_{ij} \qquad \gamma_0\gamma_i + \gamma_i\gamma_0 = 0$$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = 2\eta_{\mu\nu}$$

$$\{\gamma_\mu, \gamma_\nu\}$$

$$(i\gamma_\mu\partial^\mu - m)\psi = 0$$ ← Dirac equation

γ matrices 4×4 $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$ "Weyl basis" $\sigma = (1\ \sigma_i)$, $\bar{\sigma} = (1\ -\sigma_i)$

Other choices you will work out ...

Dirac eq describes spin 1/2 particles (e)
↳ has the Pauli spin matrices

B/c γ_μ are matrices

the solution ψ will be a 4-component vector.
"Spinor"

Solution to Dirac Eq

Assume of the form $\psi(x) = u(p)\, e^{-ip\cdot x}$

$$(i\gamma_\mu \partial^\mu - m)\psi = (i\gamma_\mu(-ip^\mu) - m)\, u\, e^{-ip\cdot x} = 0$$

$\Rightarrow$ matrix eq $\quad (\gamma\cdot p - m)u = 0$

Massive Particle $\quad p = (E, 0\,0\,0)$

$$\underbrace{\begin{pmatrix} -mI & EI \\ EI & -mI \end{pmatrix}}_{2\times 2 \text{ sub blocks}} u = 0 \qquad u = \begin{pmatrix} C \\ D \end{pmatrix}$$

C & D 2 component spinors

$\Rightarrow \quad -mC + ED = 0 \quad$ & $\quad EC - mD = 0$

$E^2 = m^2 \quad \Rightarrow \quad E = \pm m$ Same as seen in K.G.

$E = +m \Rightarrow C = D \qquad E = -m \Rightarrow C = -D$

$$\psi_+ = \begin{pmatrix} C \\ C \end{pmatrix} e^{-imt} \qquad \psi_- = \begin{pmatrix} C \\ -C \end{pmatrix} e^{imt}$$

C - two components represent spin up or down