$$T_{fi} = \frac{\langle \psi^3\psi^4 | V | \psi^3 \phi_\gamma \psi^2 \rangle \langle \psi^3 \phi_\gamma \psi^2 | V | \psi^1 \psi^2 \rangle}{(E_1 + E_2) - (E_3 + E_4 + E_\gamma)} + 2^{nd} \text{ term}$$

$E_n \neq E_i$ allowed by uncertainty principle QM.

$$\langle \psi^3\psi^4 | V | \psi^3 \phi \psi^2 \rangle = \langle \psi^4 | V | \phi \psi^2 \rangle$$

$$\boxed{\langle \phi^\gamma | \phi(x) | 0 \rangle = e^{-i \vec{p}_\gamma \cdot \vec{x}}}$$

$$= \frac{e}{2} \int d^3x \, \langle \psi^4 \phi | \psi_e(x) \phi(x) \psi_e(x) | \psi^4 \rangle$$

$$= e \int d^3x \, e^{i(p_4 - p_2 - p_\gamma)x} = e(2\pi)^3 \delta(p_4 - p_2 + p_\gamma)$$

Other product

$$\langle \psi^3 \phi | V | \psi^1 \rangle = e(2\pi)^3 \delta(\vec{p}_1 - \vec{p}_3 - p_\gamma)$$

$$T^{(1)}_{fi} \sim \int d^3p_\gamma \, \delta \, \delta \, \frac{e^2}{E_i - E_n} \qquad E_n = E_3 + E_2 + E_\gamma$$

$\hookrightarrow E_1 + E_2$

$$T^{(1)}_{fi} \approx \frac{e^2}{(E_1 + E_2) - (E_3 + E_2 + E_\gamma} = \frac{e^2}{(E_1 - E_3) - E_\gamma}$$

Same logic

$$T^{(2)} \sim \frac{e^2}{(E_2 - E_4) - E_\gamma}$$

End of day, need to add processes

$$E_1 + E_2 = E_3 + E_4$$
$$E_1 - E_3 = E_4 - E_2 = \Delta E$$

$$T^{(1)} + T^{(2)} = \frac{e^2}{\Delta E - E_\gamma} + \frac{e^2}{-\Delta E - E_\gamma} = \frac{2e^2 E_\gamma}{(\Delta E)^2 - E_\gamma^2}$$

define $k^\mu \equiv p_3^\mu - p_1^\mu = (\Delta E, \vec{P}_\gamma)$

← Not the photon momentum!

$k^2 \neq 0$

$= (\Delta E)^2 - E_\gamma^2$

$$T_{fi} = \underline{\underline{2E_\gamma}} \left(\frac{e^2}{k^2} \right)$$

Related to normalization

Summary OFPT

- All states are physical (on-shell)
- Matrix Element V_{ij} 0 unless 3-momentum conserved
- Energy **Not** conserved @ each vertex

Feynman Rules

- Internal lines get "propagators" $\dfrac{i}{p^2 - m^2 + i\varepsilon}$
- Vertices come from interactions in the Lagrangian. They get factors of the coupling constant times i
- Lines connected to external points do not get propagators (Scalars get $\times 1$ / Spinor by u or v / ~~Sp~~ Spin 1 by ε ε^*)
- 4-momenta is conserved @ each vertex
- Integrate over all undetermined 4-mom.
- Sum over all possible diagrams.

Example: $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3$

Consider cross-section for $\phi\phi \to \phi\phi$ scattering.

$$d\sigma = \frac{1}{(2E_1)(2E_2)|v_1 - v_2|}\,|M|^2\, d\Pi_{LIPS} \qquad\qquad P_1 + P_2 \to P_3 + P_4$$

In com frame, $\vec{P}_1 = -\vec{P}_2 \quad \vec{P}_3 = -\vec{P}_4 \qquad E_1 + E_2 = E_3 + E_4 \equiv E_{cm}$

$$d\Pi_{LIPS} = (2\pi)^4\,\delta^4(\Sigma p)\,\frac{d^3p_3}{(2\pi)^3}\,\frac{1}{2E_3}\,\frac{d^3p_4}{(2\pi)^3}\,\frac{1}{2E_4}$$

Integrating over $\vec{P}_4$

$$= \frac{1}{16\pi^2}\,d\Omega \int dp_f\,\frac{p_f^2}{E_3}\,\frac{1}{E_4}\,\delta(E_3 + E_4 - E_{cm})$$

$P_f = |\vec{P}_3| = |\vec{P}_4| \qquad E_3 = \sqrt{m^2 + p_f^2} = E_4 \qquad \int d^3p_3 = \int dp_f\, p_f^2\, d\Omega$

Now $\quad p_f \to x = E_3 + E_4 - E_{cm}$

$$dx = \frac{d}{dp}(E_3 + E_4 - E_{cm})\,dp_f = \frac{p_f}{E_3} + \frac{p_f}{E_4} = \frac{\overbrace{E_3 + E_4}^{E_{cm}}}{E_3 E_4}\,p_f\,dp_f$$

$$\Rightarrow \quad \frac{dp_f\,p_f^2}{E_3 E_4} = \frac{dx\,p_f}{E_{cm}}$$

$$d\Pi_{LIPS} = \frac{1}{16\pi^2}\,d\Omega \int_{m_3 + m_4 - E_{cm}}^{\infty} dx\,\frac{p_f}{E_{cm}}\,\delta(x) = \frac{1}{16\pi^2}\,d\Omega\,\frac{p_f}{E_{cm}} \begin{cases} \text{if } E_{cm} > m_3 + m_4 \\ 0 \text{ otherwise} \end{cases}$$

$$d\sigma = \frac{1}{(2E_1)(2E_2)|v_1 - v_2|}\,\frac{1}{16\pi^2}\,d\Omega\,\frac{p_f}{E_{cm}}\,|M|^2$$

$$|v_1 - v_2| = \left|\frac{|\vec{P}_1|}{E_1} + \frac{|\vec{P}_2|}{E_2}\right| = P_i\,\frac{E_{cm}}{E_1 E_2} \qquad \boxed{|\vec{P}_i| = |\vec{P}_1| = |\vec{P}_2|}$$

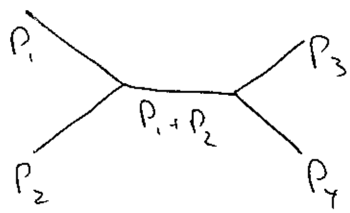
$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{64\pi^2 E_{cm}^2}\frac{P_f}{P_i}|M|^2$$

if masses are equal $P_f = P_i$

$$= \frac{1}{64\pi^2 E_{cm}^2}|M|^2$$

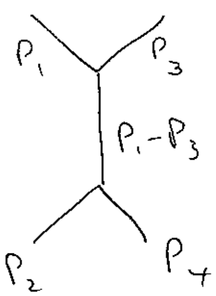
<u>Now to M</u>

"s-channel" diagram



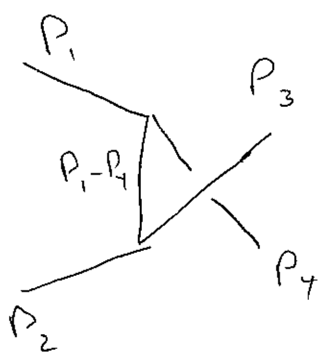
$$= (ig)\frac{i}{\underbrace{(P_1+P_2)^2}_{\equiv s} - m^2}(ig) = \frac{-ig^2}{s-m^2}$$

<u>t-channel</u>



$$= (ig)\frac{i}{\underbrace{(P_1-P_3)^2}_{\equiv t} - m^2}\, ig = \frac{-ig^2}{t-m^2}$$

<u>u-channel</u>

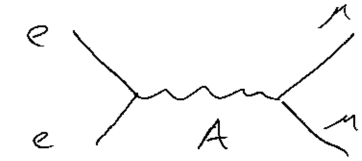


$$= (ig)\frac{i}{(P_1-P_4)^2 - m^2}\, ig = \frac{-ig^2}{u-m^2}$$

$$\frac{d\sigma}{d\Omega}(\phi\phi\to\phi\phi) = \frac{g^4}{64\pi^2 E_{cm}^2}\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2$$

$s+t+u = \sum_i m_i^2$ $\quad$ s, t, u are L.I.

Example 2a



$\mathcal{M}$ - dimensionless
given by appropriate spin projections.

~~think~~ focus on projections of initial spins to γ polarizations
then γ polarization to final μ-spins

$$\mathcal{M}(s_1 s_2 \to s_3 s_4) = \sum_{\underset{\text{photon polarizations}}{\varepsilon}} \underbrace{\langle s_3 s_4 |}_{\text{spins of outgoing } \mu} \varepsilon \rangle \langle \varepsilon | \underbrace{s_1 s_2 \rangle}_{\text{spins of incoming } e}$$

At high-energies take e & μ massless

$P_1 = (E, 0, 0, E)$ $\qquad$ $P_2 = (E, 0, 0, -E)$

In this limit think of electron as having helicity

Linear Basis (you will do circular in HW)

$|s_1 s_2\rangle = |\leftrightarrow\leftrightarrow\rangle$ (along x), $|\updownarrow\updownarrow\rangle$ (along y), $|\updownarrow\leftrightarrow\rangle$ or $|\leftrightarrow\updownarrow\rangle$ $\quad$ Spin 1/2

only $|\leftrightarrow\leftrightarrow\rangle$ & $|\updownarrow\updownarrow\rangle$ can project on to a spin = 1 state

photon polarizations $\quad \varepsilon^1 = (0, 1, 0, 0)$ or $\varepsilon^2 = (0, 0, 1, 0)$

$|\leftrightarrow\leftrightarrow\rangle$ gives ε^1
$|\updownarrow\updownarrow\rangle$ " ε^2

Now μ's are also spin 1/2 (Also have ~~two~~ 4 spin states)

In general, μ not moving along $\quad P_3 = E(1, 0, \sin\theta, \cos\theta)$

$P_4 = E(1, 0, -\sin\theta, -\cos\theta)$

Also azimuthal angle ϕ can be set to 0 by cylindrical symmetry.

For muons 2 possible directions of photon polarizations

$$\vec{\varepsilon}^{\,1} = (0,1,0,0) \qquad \vec{\varepsilon}^{\,2} = (0,0,\cos\theta,-\sin\theta)$$

(Can check they are $\perp$ to P_3 & P_4)

In general hard to measure spins. Sum over all μ spins.

Must sum over all possible combinations of initial / final polarization

For us only $M_1 = M(|\leftrightarrow\leftrightarrow\rangle \to |\bar{\varepsilon}_1\rangle) = \varepsilon^1 \bar{\varepsilon}_1 = -1$

& $M_2 = M(|\updownarrow\updownarrow\rangle \to |\bar{\varepsilon}_2\rangle) = \varepsilon^2 \bar{\varepsilon}_2 = -\cos\theta$

are non-zero.

If our initial beams are unpolarized, sum over initial spins

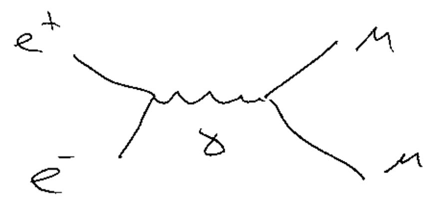
$$|M|^2 = |M_1|^2 + |M_2|^2 = 1 + \cos^2\theta$$

$$\alpha = \frac{e^2}{4\pi} \qquad \alpha^2 = \frac{e^4}{16\pi^2}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E_{cm}^2}(1+\cos^2\theta)$$

$$= \frac{\alpha^2}{4\,E_{cm}^2}(1+\cos^2\theta)$$

2b) Now directly using Feynman diagrams

$\mathcal{M}(e^+e^- \to \mu^+\mu^-) =$ 

Assume all external particles are massless.

-) Massless solutions to Dirac eq for external e's + μ's

$$i\gamma\cdot\partial\psi = 0 \quad \swarrow \; 4 \; 1^{st} \text{ order diff eqs, expect 4-solutions.}$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar\sigma_\mu & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$i\sigma\cdot\partial\psi_R = 0 \qquad \& \qquad i\bar\sigma\cdot\partial\psi_L = 0$$

$$\psi_R = U_R\, e^{-ip\cdot x} \qquad \text{D.E.} \Rightarrow (\sigma\cdot p)U_R = 0 \qquad p = (E, 0, 0, E)$$

↳ some 2 component spinor

$$(\sigma\cdot p) = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}$$

$U_R \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "spin up"

$$= \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_R = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ip\cdot x}$$

*Note for general $P_\mu = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$E\begin{pmatrix} 1-\cos\theta & -e^{-i\phi}\sin\theta \\ -e^{i\phi}\sin\theta & 1+\cos\theta \end{pmatrix} U_R(p) = 0$$

$$\Rightarrow U_R(p) = \sqrt{2E}\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}$$

check the limit $\theta + \phi \to 0$

Also solution w/ $E \to -E$ (antiparticles)

$$v_R(p) = \sqrt{2E}\begin{pmatrix} e^{-i\phi/2}\sin\frac{\theta}{2} \\ -e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

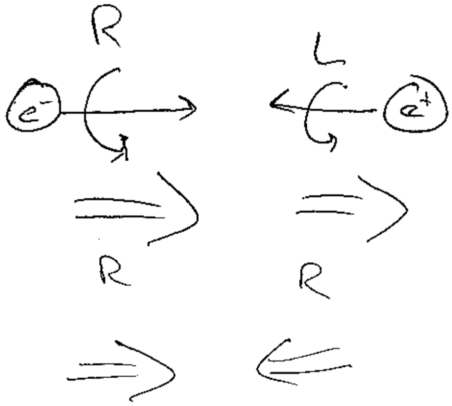
Do the same for the L hand guys

$$\Rightarrow \quad u_R(p) = v_L(p)$$

$$u_L(p) = v_R(p)$$

Aside

$$\boxed{\begin{aligned} u_R^\dagger u_R = u_L^\dagger u_L = v_R^\dagger v_R = v_L^\dagger v_L = 2E \\ u_R^\dagger u_L = v_L^\dagger v_R = 0 \end{aligned}}$$



Allowed

So only diagrams we need are

$$e^+_L e^-_R \to \mu^+_L \mu^-_R$$

L	R	R	L
R	L	L	R
R	L	R	L

Another trick

$M(e^+_L e^-_R \to \mu^+_L$