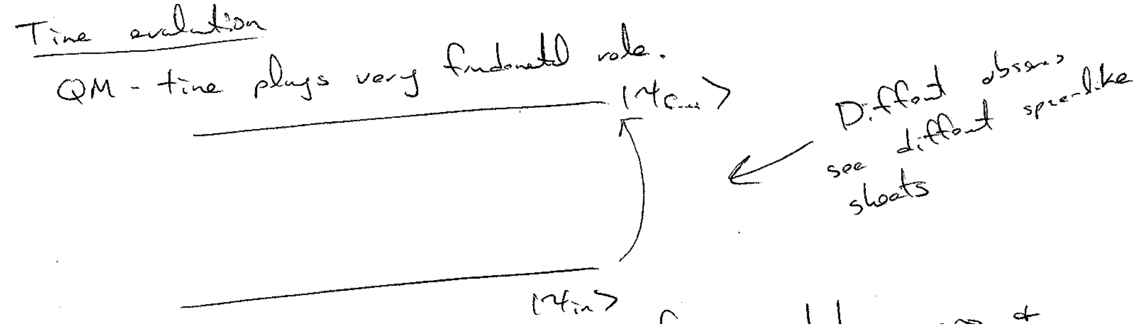
Lecture 10

QFT Continued...

Summary From Last Time



Only have a hope of Lorentz invariance if we start at -∞ and go to +∞.

Throw particles in from ∞ let them scatter & go back out to ∞.

Define S-Matrix

$$\underbrace{|p_1\sigma_1, ...p_n\sigma_n\rangle}_{t=-\infty} \rightarrow \underbrace{\mathcal{S}\,|p_1\sigma_1, ...p_n\sigma_n\rangle}_{t=+\infty}$$

$\mathcal{S}$ might be (at least a hope) Lorentz Invariant.

Big Picture: The plan is to Figure out what $\mathcal{S}$ is in a tottaly generic theory, then see what it would take to make it Lorentz Invariant.

Sure doesnt look like ti will be L.I. $\mathcal{S}$ is the only object that you could even have a hope to make L.I.

We will see that for very special choices of the interaction it will barely be possible for it to be Lorentz Invariant. These choices force on us anti-particle and the connection between spin and statistics.

Something annoying that we should get rid of right away. Free evolution, just evolves w/phase. Totally irrelevant part.

Standard way of removing the free evolution <u>"Interaction Representation"</u>.

$$H = H_{\text{free}} + H_{\text{Int}}$$

$$i\frac{d\left|\psi\right\rangle}{dt} = \left(H_{\text{free}} + H_{\text{Int}}\right)\left|\psi\right\rangle$$

<u>For $H_{\text{Int}} = 0$</u>

$$\left|\psi\right\rangle = e^{-iH_f t}\left|\psi_{in}\right\rangle$$

Now, we dont have a free theory, but if the interaction is small going to be pretty close to evolving like this.

$$\left|\psi\right\rangle = e^{-iH_f t}\underbrace{\left|\psi_I\right\rangle}_{\text{definition}} \tag{1}$$

If $H_{\text{Int}} = 0, \left|\psi_{\text{Int}}\right\rangle$ doesnt evolve at all. Bc there is H_{Int}, $\left|\psi_{\text{Int}}\right\rangle$ will evolve.

$$\begin{aligned} i\frac{d}{dt}\left|\psi\right\rangle &= H_{\text{f}}\left|\psi\right\rangle + e^{-iH_f t} i\frac{d}{dt}\left|\psi_I\right\rangle \\ &= (H_f + H_{\text{int}})e^{-iH_f t}\left|\psi_I\right\rangle \end{aligned}$$

Note: first line from derivitive of 1, the second from the Schrodinger Equation.

The RHSs imply,

$$i\frac{d}{dt}\left|\psi_I\right\rangle = \underbrace{e^{-iH_f t}H_{\text{Int}}e^{-iH_f t}}_{\text{Interaction Hamiltonian in the interaction representation}}\left|\psi_I\right\rangle$$

So,

$$i\frac{d}{dt}\left|\psi_I\right\rangle = H_I\left|\psi_I\right\rangle$$

where H_I can be time dependent.

Lets formally sovle this

Just intergrating gives,

$$|\psi_I(t_2)\rangle = |\psi_I(t_1)\rangle - i\int_{t_1}^{t_2} dt H_I \,|\psi_I(t)\rangle$$

Now we can keep iterating,

$$= |\psi_I(t_1)\rangle - i\int_{t_1}^{t_2} dt H_I(t)\left(|\psi_I(t_1)\rangle - i\int_{t_1}^{t} dt' H_I(t')\,|\psi_I(t')\rangle\right)$$

or

$$= |\psi_I(t_1)\rangle - i\int_{t_1}^{t_2} dt H_I\,|\psi_I(t_1)\rangle + (-i)^2\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt' H_I(t)H_I(t')\,|\psi_I(t')\rangle$$

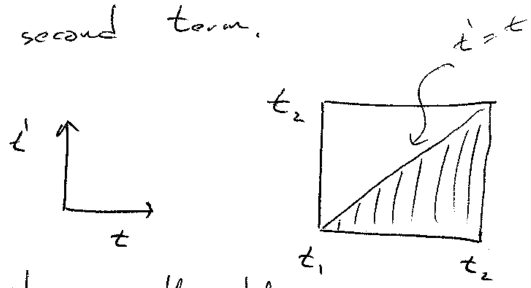
Pattern is clear, can keep going...

$$\begin{aligned}
|\psi_I(t_2)\rangle = [\quad & 1 \quad +(-i)\int_{t_1}^{t_2} dt H_I(t) \\
+ \quad & (-i)^2\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt' H_I(t)H_I(t') \\
+ \quad & (-i)^3\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt'\int_{t}^{t'} dt'' H_I(t)H_I(t')H_I(t''') \\
+ \quad & ...]\,|\psi_I(t_1)\rangle
\end{aligned}$$

If H_I is small this is giving us some nice perturbation theory.

Look @ the second term.

$$\int_{t_1}^{t_2} dt \int_{t_1}^{t} dt'$$



Nice to write it over the whole region!

$$\int H_I(t)\, H_I(t') = \frac{1}{2}\int \top\left(H_I(t)\, H_I(t')\right)$$

⟶ time ordered product

$$\top\left(A(t)\, B(t')\right) = \begin{cases} A(t)\, B(t') & t > t' \\ B(t')\, A(t) & t < t' \end{cases}$$

$$\begin{aligned}
|\psi_I(t_2)\rangle = [\quad & 1 \quad +(-i)\int_{t_1}^{t_2} dt \top(H_I(t)) \\
+ \quad & \frac{(-i)^2}{2!}\int_{t_1}^{t_2} dt \int_{t_1}^{t} dt' \top(H_I(t)H_I(t')) \\
+ \quad & \frac{(-i)^3}{3!}\int_{t_1}^{t_2} dt \int_{t_1}^{t} dt' \int_{t}^{t'} dt'' \top(H_I(t)H_I(t')H_I(t''')) \\
+ \quad & ...]\, |\psi_I(t_1)\rangle
\end{aligned}$$

$$|\psi_I(t_2)\rangle = \top\left(e^{(-i)\int_{t_1}^{t_2} dt H_I(t)}\right)|\psi_I(t_1)\rangle$$

Now, let t_1 and t_2 go to ∞,

$$|\psi_I(+\infty)\rangle = \top\left(e^{(-i)\int_{-\infty}^{+\infty} dt H_I(t)}\right)|\psi_I(-\infty)\rangle$$

OK, Lets go back to feild theory....

$$\phi_+(\vec{x}) = \int d\!\!\!{}^-{}^3 p \; e^{i\vec{p}\vec{x}} \; a^\dagger_{\vec{p}}$$

(use scalars for the moment)

Need to build H_I out of $\phi_{+/-}$ in the interaction representation.

$$\begin{aligned}
\phi^I_+(x,t) \quad &= e^{-iH_f t}\phi(x)e^{iH_f t} \\
&= \int d\!\!\!{}^-{}^3 p \; e^{i\vec{p}\vec{x}} \; e^{-iE_p t} \; a^\dagger_{\vec{p}} \\
&= \int d\!\!\!{}^-{}^3 p \; e^{-ip^\mu x_\mu} \; a^\dagger_{\vec{p}}
\end{aligned}$$

this behaves nicely under Lorentz Transforms $\phi(\Lambda x) = \phi(x)$