Lecture 17

Lorentz Invariance and "Soft Limits"

Punch line that we've been building to in first part of this course.

Matrix element we would get by scattering external γ.

$$M = \epsilon^\mu M_\mu$$

where ϵ^μ is some linear combination of two photon polarization vector ϵ^1 and ϵ^2

M is Lorentz Invariant, under lorentz transformation

$$M \rightarrow \epsilon'^\mu M'_\mu$$

where $M'_\mu = \Lambda_\mu{}^\nu M_\nu$

However (here comes the major constraint) ϵ is not a full 4-vector. Only has 2 components.

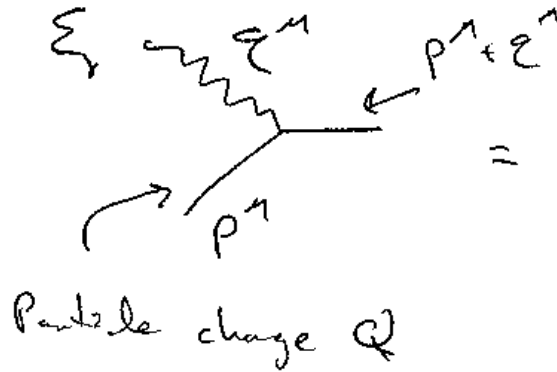Under little group transformations (you will show in your H.W.)

$$\epsilon \rightarrow \underbrace{c_1 \epsilon_1^\mu + c_2 \epsilon_2^\mu}_{\substack{\epsilon' \text{ can only be made} \\ \text{of these pieces}}} + \underbrace{c_3 p^\mu}_{\substack{\text{Not valid} \\ \text{"Not in Hilbert Space"}}}$$

So,

$$
\begin{aligned}
M = \epsilon^\mu M_\mu \;\rightarrow\; & \left(c_1 \epsilon_1^\mu + c_2 \epsilon_2^\mu + c_3 p^\mu \right) M'_\mu \\
= \; & \epsilon'^\mu M'_\mu + \underbrace{c_3 p^\mu M'_\mu}_{\text{Must go to } 0}
\end{aligned}
$$

We will see, this has enourmous implications !!!

Will be considering diagrams with external "γ"s (massless spin 1 particles)

$$\begin{aligned} &= iQ(p^\mu + (p^\mu + q^\mu))\epsilon_\mu \qquad\qquad (q^\mu \epsilon_\mu = 0) \\ &= iQ2p^\mu\epsilon_\mu \end{aligned}$$

this is the most general form in the "soft limit" $q \to 0$

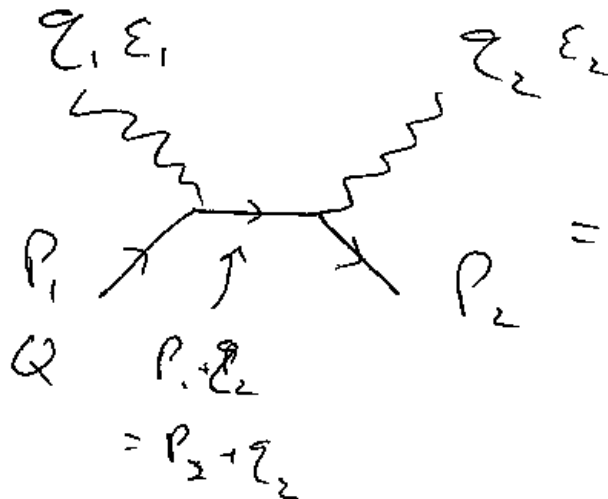$$\Gamma_\mu \sim p_\mu F(q^2, p^2, p \cdot q)$$
By dimensional analyiss $F(q^2, p^2, p \cdot q) \to F(\frac{p \cdot q}{m^2})$

Consider "Compton Scattering"

Start with one type of spin-1 boson and one type of matter particle.

The diagram:
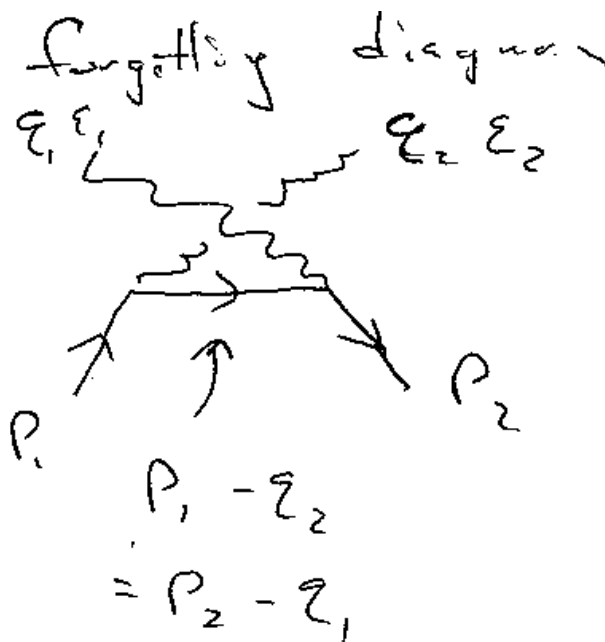

$$
\begin{aligned}
&= (iQ)\epsilon_\mu^1(2p_1^\mu)\frac{i}{(p_1+q_1)^2-m^2}(iQ)\epsilon_\nu^2(2p_2^\nu)\\
&= (-iQ^2)4\frac{(p_1\cdot\epsilon^1)(p_2\cdot\epsilon^2)}{m^2+2p_1\cdot q_1-m^2} = \epsilon_1^\mu\epsilon_2^\nu\underbrace{\left(\frac{(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1\cdot q_1}\right)}_{M_{\mu\nu}}
\end{aligned}
$$

As we said above, Lorentz Invariant $\Rightarrow q_1^\mu q_2^\nu M_{\mu\nu} = 0$

But here, $q_1^\mu q_2^\nu M_{\mu\nu} = (-iQ^2)2(p_2\cdot q_2) \neq 0$!

Looks like we're dead...

_____

However we are forgetting a diagram.

$$= (iQ)\epsilon_\mu^1(2p_2^\mu)\frac{i}{(p_2-q_1)^2-m^2}(iQ)\epsilon_\nu^2(2p_1^\nu)$$

$$= \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iQ^2)4p_{2\mu}p_{1\nu}}{-2p_2\cdot q_1}\right)\underbrace{\sim}_{\text{Soft limit } p_1\,=\,p_2}\epsilon_1^\mu\epsilon_2^\nu\left(\frac{-(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1\cdot q_1}\right)$$
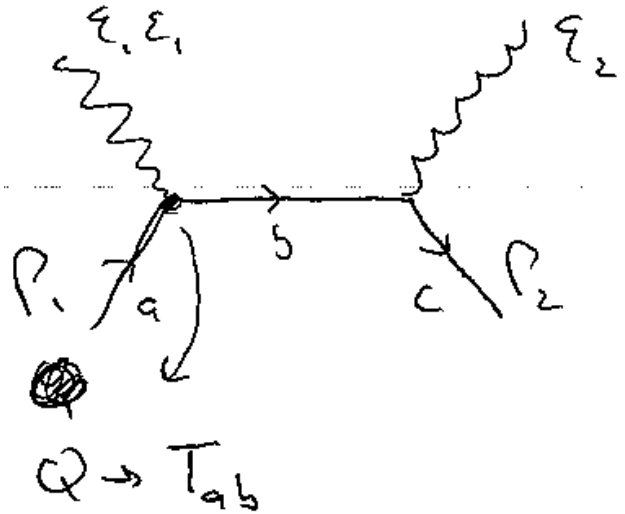
and for this diagram, $q_1^\mu q_2^\nu M_{\mu\nu} = -(-iQ^2)2(p_2\cdot q_2)$

So the sum $M_{\mu\nu}^A + M_{\mu\nu}^B$ is Lorentz Invariant. (Residual non Lorentz Invariant pieces of each diagram cancel)
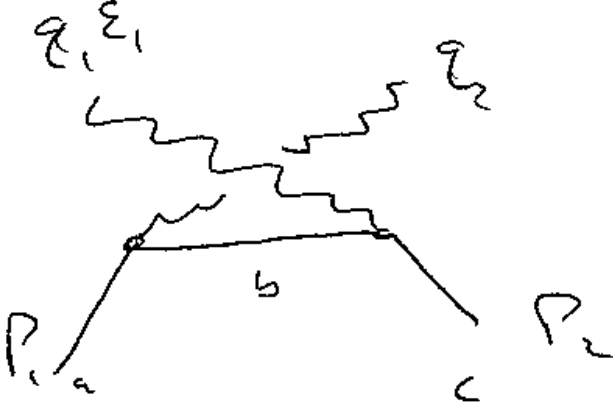
Very good.

Now lets do the same thing as before, but with many different possible matter particles.

$i = 1, ... N_{\text{matter}}$

$$
= (iT_{ab})\epsilon^1_\mu(2p_1^\mu)\frac{i}{\underbrace{(p_1 + q_1)^2 - m^2}_{m_a^2 + 2p_1 \cdot q_1 - m_b^2}}(iT_{bc})\epsilon^2_\nu(2p_2^\nu)
$$

$$
= \epsilon^\mu_1\epsilon^\nu_2\left(\frac{(-iT_{ab}T_{bc})4p_{1\mu}p_{2\nu}}{m_a^2 + 2p_1 \cdot q_1 - m_b^2}\right) \equiv M^{\mu\nu}_A
$$

Other diagram

$$= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}T_{bc})4p_{2\mu}p_{1\nu}}{m_a^2 - 2p_2 \cdot q_1 - m_b^2} \right) \equiv M_B^{\mu\nu}$$

Now, if $m_a = m_b = m_c$ then, $q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = 0$ as above $\left|\begin{array}{l} m_a^2 - m_b^2 = 0 \\ \text{relative - size} \end{array}\right.$
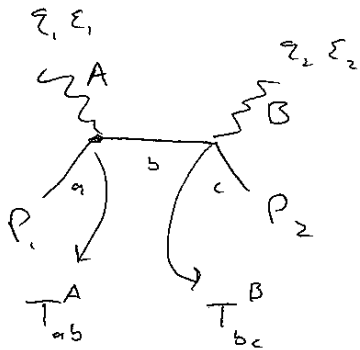
However if $m_a \neq m_b$, in soft limit

$$q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = -\left[\frac{(-iT_{ab}T_{bc})4}{m_a^2 - m_b^2}(2p_1^\mu p_1^\nu) \right] \neq 0$$

Massless spin-1 particles can only interact with particles of the same mass!

Now allow many differnt | matter fields (but same mass!) | and many force carriers "gluons"
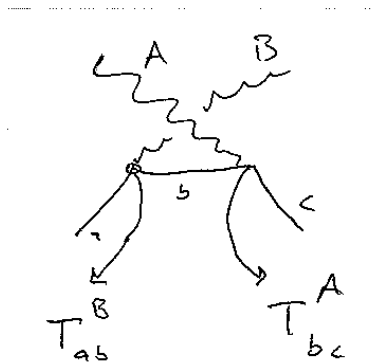
$i = 1, ...N_{\text{matter}}$

$I = 1, ...N_{\text{gluons}}$

$$= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}^A T_{bc}^B)4p_{1\mu}p_{2\nu}}{2p_1 \cdot q_1} \right) \equiv M_A^{\mu\nu}$$
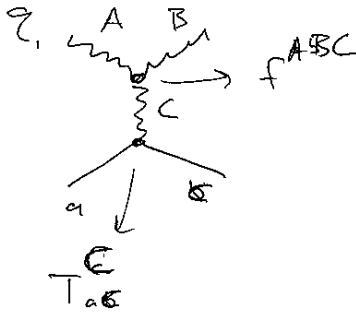
and

$$\underbrace{=}_{\text{"soft limit"}} \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT_{ab}^B T_{bc}^A)4p_{1\mu}p_{2\nu}}{-2p_1 \cdot q_1} \right) \equiv M_B^{\mu\nu}$$

Now,

$$
\begin{aligned}
q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) &= 2(-i)(p_2 \cdot q_2)(T_{ab}^A T_{bc}^B - T_{ab}^B T_{bc}^A) \\
&= 2(-i)(p_2 \cdot q_2)[T^A, T^B]
\end{aligned}
$$

$[T^A, T^B]$ not 0 for random Ts.

In fact

$$
= +2i(q \cdot q) i f^{ABC} T_{ac}^C
$$

Sum of all three only lorentz invariant if

$$
[T^A, T^B] = i f^{ABC} T^C
$$

"gluons" (or any other group of interacting massless spin-1 particles) must transform as a Lie group !

Only question is which group, there are only a finite handful of possibilities

"Yang-Mills" Interaction.