Lecture 3

Special Relativity

Talking about relativity means talking about Lorentz invariance.

If there is a point in space time:

$$(t, x) \xrightarrow[\text{observer}]{\text{another moving}} (t', x')$$

Invariant notion of distance:

$$t^2 - x^2 = t'^2 - x'^2$$

(This should all be familiar to you.)

We will re-cap this in an adult way...

Start with Rotations

Have an invariant notion of length of $\vec{p}$

$$x^2 + y^2 = x'^2 + y'^2 \qquad\qquad \begin{aligned} x' &= x\ cos(\theta) - y\ sin(\theta) \\ x' &= y\ cos(\theta) + x\ sin(\theta) \end{aligned}$$

Look at this another way ...

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We are after the set of all of matrices such that $x^2 + y^2 = x'^2 + y'^2$

Start with the infinitesimal case

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We require: $\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Multiplying through:

$$\begin{aligned} x' &= x + \epsilon a x + \epsilon b y \\ y' &= y + \epsilon c x + \epsilon d y \end{aligned}$$

Keeping terms linear in ϵ.

$$x'^2 + y'^2 = x^2 + y^2 + 2\epsilon \underbrace{(ax^2 + bxy + cxy + dy^2)}_{=0 \quad \forall x \& y}$$

so

$$ax^2 + bxy + cxy + dy^2 = 0$$

$$\Rightarrow a = d = 0 \quad \& \qquad \underbrace{b = -c}_{\text{Can re-scale } \epsilon \text{ such that b=1}}$$

So

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\mathbb{1} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

More Sophisticated Way:

x_i where $i = 1, 2$ $x_1 = x$ and $x_2 = y$

The rotation can now be written as:

$$x'_i = R_{i1}x_1 + R_{i2}x_2 = \sum_{j=1}^{2} R_{ij}x_j \equiv R_{ij}x_j$$

In the last expression, the sum is implied by the repeated indices (known as Einstein notation).

$$x'_i = R_{ij}x_j$$

$x'_i x'_i = x_i x_i$ is what it takes for R to be a rotation. Need to find the special Rs such that this is satisfied.

The identity matrix is written as δ_{ij}, where $\delta_{ij} = 1$ if $i = j$, 0 otherwise.

If no rotation at all: $R_{ij} = \delta_{ij}$, of an infinitesimal rotation:

$$R_{ij} = \delta_{ij} + \epsilon w_{ij}$$

$$x'_i = x_i + \epsilon w_{ij}x_j$$

$$x'_i x'_i = x_i x_i + 2\epsilon \underbrace{w_{ij}x_j x_i}_{=0\ \forall x} + O(\epsilon^2)$$

$\Rightarrow w_{ij}$ has to be anti-symmetric $w_{ij} = -w_{ji}$.

Back to previous example, find finite rotations (without any mention of geometry etc.)

Take the infinitesimal rotation and do it n-times.

Define $\theta = N\epsilon$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\mathbb{1} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} \equiv \left[\mathbb{1} + \epsilon \mathrm{I} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

So the finite rotation ($R(\theta)$) given by,

$$R(\theta) = (1 + \epsilon I)(1 + \epsilon I) \dots (1 + \epsilon I) \cdots = (1 + \epsilon I)^N = \left(1 + \frac{\theta}{N} I\right)^N$$

Now let $N \to \infty$, $R(\theta) = e^{I\theta}$.

Built up finite rotation from the infinitesimal rotations.

$x'(\theta) = R(\theta)x = e^{I\theta}$

The meaning of e^X when X is a matrix is simply the expansion.

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

$I^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1}$! We have just discovered i following our nose.

$R(\theta) = cos(\theta) + Isin(\theta) = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$ (You will show in homework...)

Strategy: First understand the action of the symmetry infinitesimally, then the big symmetry action is obtained by iterating the infinitesimal. Always e^X where X is the generator. This is a great strategy for any kind of symmetry.

Will now do 3D rotations... Something new happens.

3D Rotations

3-parameters associated with a 3D rotation.

Already saw, any rotation is of the form

$$x_i' = x_i + \epsilon w_{ij} x_j \qquad \text{with} \qquad w_{ij} = -w_{ji}$$

Most general 3 anti-symmetric matrix: $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, Now have 3 generators corresponding to the rotations in 3D.

$$\epsilon w_{ij} = \epsilon_{12} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{generator we just saw}} + \epsilon_{13} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \epsilon_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

How to get the finite version ? Easy, just exponentiate.

Something new happens in 3D:

- 2D rotations commute
- 3D rotations do not

Define $J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $J_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$, and $J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$

Rotations form a group. Any three rotations give something that is also a rotation:

$$e^{i\theta_3 J_3} e^{i\theta_2 J_2} e^{i\theta_1 J_1} = e^{\phi_3 J_3 + \phi_2 J_2 + \phi_1 J_1}$$

This can only be possible if

$$[J_1, J_2] = iJ_3 + \text{ cyclic}$$

Can step back and think about this more abstractly. The matrices we found form a group, but this group exists abstractly independent of these 3×3 matrices. Fully determined by the commutation relations. (Just like vectors and components)

In general, many matrices that satisfy the algebra (the commutation relations). These give different representations.

<u>Deep:</u> rotations can act on more than just 3D vectors.

$$\begin{bmatrix} (J_i) & 0 \\ 0 & \left(J_j\right) \end{bmatrix}$$

this is officially a representation. Its called a “Reducible” Representation.

More Generally

$$[J_a, J_b] = i f_{abc} J_c \qquad \text{where } J_a,\, a = 1, 2, \text{ ... dim. of the group}$$

Lie found all the possible symmetries when J is hermitian (there are not many).

One final example with rotations

Lets look at traceless 2×2 hermitian matrices. Any 2×2, traceless, hermitian matrix can be written as:

$$M = \begin{pmatrix} z & x + iy \\ x - iy & -z \end{pmatrix} \equiv \vec{\sigma} \cdot \vec{x}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and σ_i are the Pauli matrices.

Note that $\det(M) = -\vec{x} \cdot \vec{x} = -|\vec{x}|$.

Consider $M' = U^\dagger M U$, where U is unitary. Any unitary matrix can be written as a phase $e^{i\theta}$ times a 2×2 hermitian matrix with det = 1. (“Special Unitary Matrix”) Because the phase cancels in M' we will only consider U as unitary and det = 1. $U \in SU(2)$

If M is hermitian and traceless, then M’ is still hermitian and traceless.

$$M' = \sigma \cdot \vec{x'}_u$$

$\vec{x'}_u$ depends on U. det(M’) = det(M) $\Rightarrow \vec{x'}^2_u = \vec{x}^2$direct correspondence between 2×2 hermitian matrices & rotations.

This is a 2D action of rotations.

Now easy to generalize all of this to the Lorentz group...