

## Lecture 6

### Review Quantum Mechanics

QM Linear Algebra in a complex vector space.

State of a system is a vector (ray) in the Complex Vector Space.

$|\alpha\rangle$  - state vector

### Linear Superpositions

$$|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle$$

If the  $|\alpha's\rangle$  are vectors and the  $c$ 's are complex numbers, then  $|\psi\rangle$  is another vector in the space.

### Dual Space:

For every vector  $|\alpha\rangle$  (“ket”) there is another vector  $\langle\alpha|$  (“bra”) in a “dual” space.

Dual space is a mirror image of the ket space.

If,  $|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle$ , then  $\langle\psi| = c_1^* \langle\alpha_1| + c_2^* \langle\alpha_2|$  where the  $c^*$  is the complex conjugate.

**Inner Product:** Dual space allows us to define an inner product between two vectors.

Given 2 vectors can get a “c#”

$$\langle\alpha|\beta\rangle$$

### Properties

1.  $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
2.  $\langle\alpha|\alpha\rangle \geq 0$
3.  $\langle\beta|(c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle) = c_1 \langle\beta|\alpha_1\rangle + c_2 \langle\beta|\alpha_2\rangle$

$|\alpha\rangle$  and  $|\beta\rangle$  are orthogonal if  $\langle\alpha|\beta\rangle = 0$ .

States can be normalized  $\langle\alpha|\alpha\rangle = 1$ .

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**Operators:** Things that act on a given state, and return another state

$$X|\alpha\rangle = |\alpha'\rangle$$

Product:

$$(YX)|\alpha\rangle = Y(X|\alpha\rangle)$$

In general, not commutative:  $XY \neq YX$ , but they are associative.

$$\langle\alpha| = \langle\alpha'|X^\dagger$$

In general,  $X^\dagger \neq X$ , if so  $X$  is said to be “Hermitian”.

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System characterized by single observable  $A$

eg: position / Momentum / energy

Measurement of  $A$  gives possible values

$$a_1, a_2, \dots$$

$|\alpha\rangle$  = State for which  $A$  has a value of  $a$

$$\sum_{a'} |\alpha'\rangle \langle\alpha'| = 1$$

$$\langle\alpha|\alpha'\rangle = \delta_{aa'}$$

Any physical observable corresponds to an operator like

$$A = \sum_{a'} a' |\alpha'\rangle \langle\alpha'|$$

eg:

$$A |\alpha\rangle = \left( \sum_{a'} a' |\alpha'\rangle \langle \alpha'| \right) |\alpha\rangle = a |\alpha\rangle$$

Physical observables are real numbers, therefore physical operators A are Hermitian proof:

$$A^\dagger = \sum_a a^* |\alpha\rangle \langle \alpha| = \sum_a a |\alpha\rangle \langle \alpha| = A$$

### Probabilities:

Consider a filter  $M(a) = |a\rangle \langle a|$  on a general state  $|s\rangle$ .

$$\begin{aligned} M(a) |s\rangle &= |a\rangle \langle a|s\rangle \\ &= \langle a|s\rangle |a\rangle \end{aligned}$$

where  $\langle a|s\rangle$  is a c# that tells you something about what fraction of the time you get through.

$\langle a|s\rangle$  is related to the pass fraction

\*But, a) not real, b) not normalized

However, we know that  $|\langle a|s\rangle|^2 = \langle a|s\rangle \langle s|a\rangle = \langle s|a\rangle \langle a|s\rangle$  is both real and normalized.

$$\sum_a \langle s|a\rangle \langle a|s\rangle = \langle s|s\rangle = 1$$

### Interpretation

$|\langle a|s\rangle|^2 =$  Probability that a system prepared in state  $|s\rangle$  will be found in a state  $|a\rangle$  with value  $a$  for observable A after measurement.

Comments on the measurement problem....

## Position Operator

$$\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

where,  $\vec{X}$  is position operator and  $\vec{x}$  is position eigenvalue.

Position of course is a continuous observable so “sums go to integrals” etc.

eg: (Completeness and Orthogonality)

$$\sum_{\alpha'} |\alpha'\rangle \langle \alpha'| = 1 \Rightarrow \int d^3x |x\rangle \langle x| = 1$$

$$\langle \alpha' | \alpha' \rangle = \delta_{\alpha, \alpha'} \Rightarrow \langle \vec{x} | \vec{x}' \rangle = \delta^3(\vec{x} - \vec{x}')$$

## Wave function

$$|\psi\rangle = \int d^3x |x\rangle \langle x|\psi\rangle = \int d^3x \psi(x) |x\rangle$$

where  $\psi(x) = \langle x|\psi\rangle$  is called the Position-space wavefunction.

$$\langle \psi | \psi \rangle = 1 \Rightarrow \int d^3x \psi(x) \langle \psi | x \rangle = \int d^3x \psi^*(x) \psi(x) = 1$$

## Interpretation

$|\psi(x)|^2 d^3x$  is the probability to find the particle in volume  $d^3x$  around  $\vec{x}$ .

## Translation Operator “The operator that moves you over”

$$T(\vec{a})|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$$

What is  $T^\dagger(\vec{a})$  ???

Well,

$$\langle \vec{x}' | (T(\vec{a})|\vec{x}\rangle) = \delta^3((\vec{x} + \vec{a}) - \vec{x}')$$

or

$$(\langle \vec{x}' | T(\vec{a}))|\vec{x}\rangle = \delta^3(\vec{x} - (\vec{x}' - \vec{a}))$$

$$\Rightarrow \langle \vec{x}' | T(\vec{a}) = \langle \vec{x} - \vec{a} |$$

which says that  $T^\dagger(\vec{a})|\vec{x}\rangle = |\vec{x} - \vec{a}\rangle$

So,

$$T^\dagger(\vec{a}) = T(-\vec{a}) = T^{-1}(\vec{a})$$

Properties of T

1. Unitary  $T^\dagger T = 1$
2.  $T(\vec{a})T(\vec{b}) = T(\vec{a} + \vec{b}) = T(\vec{b})T(\vec{a})$ , Translations commute ( $[T(\vec{a}), T(\vec{b})] = 0$ )
3.  $T(0) = 1$

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## Infinitesimal Translations

consider  $\vec{a} = N\vec{\epsilon}$

$$T(\vec{\epsilon}) = 1 - i\vec{\epsilon} \cdot \vec{k}$$

where

- $\vec{\epsilon}$  is a three vector
- $\vec{k}$  is a vector of operators  $\vec{k} = (k_x, k_y, k_z)$

Now, we know  $T^\dagger T = 1$ , or

$$(1 + i\vec{\epsilon} \cdot \vec{k}^\dagger)(1 - i\vec{\epsilon} \cdot \vec{k}) = 1$$

$$1 + \underbrace{i\vec{\epsilon}(\vec{k}^\dagger - \vec{k})}_{=0} + O(\epsilon^2) = 1$$

$\Rightarrow k^\dagger = k$ , or  $k$  is some hermitian operator.

Note: if the  $i$  wasnt there  $T(\vec{d})$  would not be hermitian.

Just like with SR, any finite translation can be built out of infinitesimal translations

And  $\vec{k}$  is the “generator” of translations.

Lets build a finite translation...

$$\underbrace{T(\vec{d})}_{finite} = \lim_{N \rightarrow \infty} [1 - i\vec{\epsilon} \cdot \vec{k}]^N = \lim_{N \rightarrow \infty} \underbrace{\left[1 - i\frac{\vec{d}}{N} \cdot \vec{k}\right]^N}_{e^{-i\vec{k} \cdot \vec{d}}}$$

Can see explicitly that  $T$  is unitary and  $k$  is hermitian.

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Eigenstates of  $\vec{k}$  (and of course automatically of  $T$ )

$$\vec{K} |\vec{k}\rangle = \vec{k} |\vec{k}\rangle$$

$$T(\vec{d}) |\vec{k}\rangle = e^{-i\vec{k} \cdot \vec{d}} |\vec{k}\rangle$$

Eigenstates of  $\vec{k}$  behave nicely under translations (they pick up a phase).

What is  $\psi_{\vec{k}}(x) \equiv \langle \vec{x} | \vec{k} \rangle$ ?

$$\begin{aligned}\langle \vec{x} | T(\vec{a}) | \vec{k} \rangle &= e^{-i\vec{k} \cdot \vec{a}} \langle \vec{x} | \vec{k} \rangle = e^{-i\vec{k} \cdot \vec{a}} \psi_{\vec{k}}(x) \\ \text{"T on x"} &= \langle \vec{x} - \vec{a} | \vec{k} \rangle = \psi_{\vec{k}}(x - a)\end{aligned}$$

or

$$\psi_{\vec{k}}(\vec{x} - \vec{a}) = e^{-i\vec{k} \cdot \vec{a}} \psi_{\vec{k}}(\vec{x})$$

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Turns out that  $\vec{k}$  is just the momentum operator

$$\vec{p} = \hbar \vec{k} \text{ and } T(\vec{a}) = e^{-i\vec{p} \cdot \vec{a}}$$

Couple of ways to see this. (One is the De Brojlie wavelength, we will see another shortly)

“Momentum is the generator of translations”

Note Translations for a group:

1. Closure
2. Identity
3. Inverse
4. Associative

(Can show that htis is an abealian group (HW))

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Another look at the  $\vec{p}$  operator...

$$\begin{aligned}
\langle y| T(\epsilon) |\psi\rangle &= \int d^3x \langle y| \underbrace{T(\epsilon) |x\rangle}_{|x+\epsilon\rangle} \langle x|\psi\rangle \\
&= \int d^3x \langle y|x\rangle \langle x-\epsilon|\psi\rangle \\
&= \langle y-\epsilon|\psi\rangle \\
&= \psi(y-\epsilon) \\
&= \psi(y) - \epsilon \frac{\partial}{\partial y} \psi(y)
\end{aligned}$$

(using change of variables  $x \rightarrow x - \epsilon$  and Taylor expansion)

Now, on the otherhand...

$$\begin{aligned}
\langle y| T(\epsilon) |\psi\rangle &= \langle y| \left(1 - \frac{i\vec{p}\vec{\epsilon}}{\hbar}\right) |\psi\rangle \\
&= \int d^3x \langle y|x\rangle \langle x| \left(1 - \frac{i\vec{p}\vec{\epsilon}}{\hbar}\right) |\psi\rangle \\
&= \int d^3x \langle y|x\rangle \left(\langle x|\psi\rangle - \langle x| \frac{i\vec{p}\vec{\epsilon}}{\hbar} |\psi\rangle\right) \\
&= \langle y|\psi\rangle - \langle y| \frac{i\vec{p}\vec{\epsilon}}{\hbar} |\psi\rangle \\
&= \psi(y) - \epsilon \langle y| \frac{i\vec{p}}{\hbar} |\psi\rangle
\end{aligned}$$

$$\Rightarrow \vec{p} = i\hbar \frac{\partial}{\partial x} \text{ (This will be critical later)}$$

Now lets look at how momentum eigenstates behave.

$$\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$$

we already saw that  $\langle x|p\rangle \sim e^{i\frac{p \cdot x}{\hbar}}$  “Momentum Space wave function”

$$\phi(p) \equiv \langle p|\psi\rangle$$

Can show (HW) that  $\phi(p) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\frac{p \cdot x}{\hbar}}$

Then,



$$\langle x|\psi\rangle = \int d^3p \langle x|p\rangle \langle p|\psi\rangle$$

or,

$$\psi(x) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int d^3p e^{i\frac{px}{\hbar}} \phi(p)$$

Similarly,

$$\langle p|\psi\rangle = \int d^3x \langle p|x\rangle \langle x|\psi\rangle$$

or,

$$\phi(p) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int d^3x e^{-i\frac{px}{\hbar}} \psi(x)$$

$\Rightarrow \psi(x)$  and  $\phi(p)$  are Fourier transforms of each other...