Lecture 4

Special Relativity

Re-cap from Last Time

Talking about relativity means talking about Lorentz invariance.

If there is a point in space time:

$$(t, x) \xrightarrow[\text{observer}]{\text{another moving}} (t', x')$$

Invariant notion of distance:

$$t^2 - x^2 = t'^2 - x'^2$$

Started with Rotations Where we have an invariant notion of length of $\vec{p}$

$$x^2 + y^2 = x'^2 + y'^2$$

Started in 2D, looking at the infinitesimal rotation This lead to what is called "generator"

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Form a group: A group is a set of actions (in our case rotations) that multiply (or compose) with operation denoted by $\cdot$.

Four criteria for group:

- Have Identity element: (no rotation or by 360)
- Every element of the group has an inverse (for us, rotate by $360 - \theta$)
- The group is closed: for any elements their product is also in the group.
- The multiplication is associative.

If commute: $a \cdot b = b \cdot a$ say that the group is "Abelian". If they do not commute say that the group is "non-Abelian". We have already seen examples of both of these, both types are also critical for the structure of the SM.

Last lecture studied 2D rotations and their group "SO(2)" refers to matrices studied (S = Special (det =1) / O = Orthogonal (preserves length) / 2 = 2x2)

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

We also looked at 3D rotations, their group was "SO(3)", what do you think this stands for ???
We saw at the end of the lecture that $SO(3) \simeq SU(2)$ SU(2) = (Special / Unitary / 2x2 matrices.

(You will show in for homework that this is equivalent to another group U(1).)

All the machinery in place to look at Lorentz Transformations...

Well almost all the machinery... Remember invariant need to preserve is $t^2 - |\vec{x}|^2$

Lets deal with this minus sign...

4-vectors: $x^\mu = (t, \vec{x})$

Encode the minus sign in matrix:

$$\eta_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1, 2, 3 \ (i) \\ 0 & \text{otherwise} \end{cases}$$

Can now write: $x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})$

And finally: $x_\mu x^\mu = t^2 - |\vec{x}|^2$

(Comment on $\eta^{\mu\nu} = \eta_{\mu\nu}$ and $\eta^\mu_\nu = \delta^\mu_\nu$)

Now we after all the transformations that leave $x_\mu x^\mu$ (Lorentz transformations)

$$x'^\mu = \underbrace{\Lambda^\mu_\nu}_{\text{4x4 matrix}} x^\nu$$

Same thing as before Start with the infinitesimal transformations.

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \epsilon\omega^{\mu}_{\nu}$$

$$x'^{\mu} = x^{\mu} + \epsilon\omega^{\mu}_{\nu}x^{\nu}$$

Require $x'^{\mu}x'_{\mu} = x^{\mu}x_{\mu}$ or

$$\begin{aligned} \eta_{\mu\nu}x'^{\mu}x'^{\nu} &= \eta_{\mu\nu}x^{\mu}x^{\nu} \\ &= \eta_{\mu\nu}x^{\mu}x^{\nu} + 2\epsilon\eta_{\mu\nu}x^{\nu}\omega^{\mu}_{\rho}x^{\rho} \end{aligned}$$

So, $\underbrace{\eta_{\mu\nu}\omega^{\mu}_{\rho}}_{\equiv\omega_{\mu\rho}} x^{\nu}x^{\rho} = 0$

After all 4x4 anti-symmetric matrices $\omega_{\mu\rho} = -\omega_{\rho\mu}$

Lets just be damn explicit

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & A & B \\ -b & -A & 0 & C \\ -c & -B & -C & 0 \end{bmatrix}$$

However, the think that enters the transformation is ω^{μ}_{ν}.

Now raising a 0-component is free, $\omega_{0i} = \omega^{0}_{i}$
raising a i-component costs a -1, $\omega_{i0} = -\omega^{i}_{0}$
But also know $\omega_{0i} = -\omega_{i0} \Rightarrow \omega^{0}{}_{i} = +\omega^{i}{}_{0}$

Same argument shows: $\omega^{i}{}_{j} = -\omega^{j}{}_{i}$

$$\omega^{\mu}_{\nu} = \begin{bmatrix} 0 & a & b & c \\ a & 0 & A & B \\ b & -A & 0 & C \\ c & -B & -C & 0 \end{bmatrix}$$

Generators of the Lorentz Group. We have 6 “generators” abcABC.

3-rotations (guess which) / 3-boosts

Crucial sign difference between boosts/rotations

$$\omega^1{}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to e^{\theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \cos\theta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sin\theta$$

$$\omega^0{}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to e^{\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \cosh\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sinh\eta$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = 1$ whereas $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -1$

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

Can (but I wont) work out the Lie Algrabra....Homework.

Group is called $SO^+(1,3)$

Lets look at general 2×2 hermitian matrices, (not restricted to traceless as before). Any 2×2, hermitian matrix can be written as:

$$M = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \equiv \sigma_\mu x^\mu$$

where $\sigma_\mu = (1, \sigma_i)$ and σ_i are the Pauli matrices.

Note that $\det(M) = x^\mu x_\mu$

Consider the action,

$$M' = L^\dagger M L$$

of L -any 2x2 complex matrix w/det(L) = 1. This set of matrices is called SL(2,c) = Special / Linear/ 2x2 / Complex.

M’ is still Hermitian $\Rightarrow M' = \sigma_\mu x^\mu_{(L)}$

And crucially, det(M) = det(M’) $\Rightarrow x_{(L)\mu} x^{\mu}_{(L)} = x_\mu x^\mu$

So the L’s give (“furnish”) a representation of the Lorentz group.

We see here $SO^+(1,3) \simeq SL(2,c)$