Homework Set #1

Solutions

2) Radius of Planets *(5 points)*

(a) A planet is an object whose internal pressure coming from it being a solid made up of atoms is balanced by the gravitational pressure it feels.

Lets work out the internal pressure of a solid first:

In class we worked out that $r_{\text{atom}} \sim \frac{1}{Z\alpha m_e}$. (using $E \sim -\frac{Z\alpha}{r} + \frac{p^2}{m_e}$ and $p \times r \sim 1$)

It follows from this that $E_{atom} \sim Z^2\alpha^2 m_e$ and $V_{atom} \sim r^3_{\text{atom}}$.

The atomic pressure (Units F/area or (better!) E/volume) is then $P_{solid} \sim E_{atom}/V_{atom} \sim \frac{Z^2\alpha^2 m_e}{(Z\alpha m_e)^{-3}} \sim\sim Z^5\alpha^5 m_e^4 \sim Z\alpha r^{-4}_{\text{atom}}$

Now lets do P_{Grav}. Here we need E_{Grav} and V_{Grav}, the energy of the planet from the gravitational force and the volume over which it acts.

V_{Grav} is the volume of the planet $\sim R^3_{Planet}$

The gravitational energy of a sphere is given by $E_{Grav} \sim G_N \frac{M^2_{Planet}}{R_{Planet}}$

So, $P_{Grav} \sim G_N \frac{M^2_{Planet}}{R^4_{Planet}}$.

Assuming the planet is a solid made of atoms, we can write $M_{Planet} \sim \rho_{solid} R^3_{Planet}$, where $\rho_{solid} \sim \frac{Zm_p}{r^3_{\text{atom}}}$

Then, $P_{Grav} \sim G_N \frac{Z^2 m_p^2 R^6_{Planet}}{r^6_{\text{atom}} R^4_{Planet}} \sim (G_N m_p^2) Z^2 \left(\frac{R_{Planet}}{r_{\text{atom}}}\right)^2 \frac{1}{r^4_{\text{atom}}} \sim \alpha_G Z^2 \left(\frac{R_{Planet}}{r_{\text{atom}}}\right)^2 r^{-4}_{\text{atom}}$.

Setting $P_{Grav} \sim P_{Solid}$ gives

$$\alpha_G Z^2 \left(\frac{R_{Planet}}{r_{\text{atom}}}\right)^2 r^{-4}_{\text{atom}} \sim Z\alpha r^{-4}_{\text{atom}}$$

or

$$\left(\frac{R_{Planet}}{r_{\text{atom}}}\right)^2 \sim Z^{-1}\frac{\alpha}{\alpha_G} \Rightarrow R_{Planet} \sim \sqrt{\frac{\alpha}{Z\alpha_G}}\frac{1}{Z\alpha m_e}$$

(b)

$$R_{Planet} \sim \sqrt{\frac{\alpha}{Z\alpha_G}} r_{\text{atom}}$$

(c) Lets take $r_{\text{atom}} \sim 10^{-10}$ m, $Z \sim 10^2$, $\alpha \sim 10^{-2}$, and $\alpha_G \sim 10^{-38}$

$$R_{Planet} \sim \sqrt{\frac{10^{-2}}{10^2 10^{-38}}} 10^{-10} m \sim 10^{17} 10^{-10} m \sim 10^7 m$$

vs actual $6.37 \times 10^6 m$ (Pretty Good!)

3) Solid State Physics *(5 points)*

(a) We worked out in class $r_{\text{atom}} \sim \frac{1}{Z\alpha m_e}$. (using $E \sim -\frac{Z\alpha}{r} + \frac{p^2}{m_e}$ and $p \times r \sim 1$) So a solid has a spacing of r_{atom}.

(b) To probe distances of order r_{atom} need to photons with Energy $\sim \frac{1}{r_{\text{atom}}} \sim Z\alpha m_{\text{electron}}$. Assuming $Z \sim 10$,
Energy $\sim 10 \cdot 10^{-2} \cdot 10^{-3}$ GeV $\sim 10^{-4}$GeV $\sim 10^2$ keV

(c) 10^2 keV photons are x-rays.

4) 2D Rotations *(5 points)*

(a)

$$e^{I\theta} = 1 + I\theta + \frac{I^2\theta^2}{2!} + \frac{I^3\theta^3}{3!} + \frac{I^4\theta^4}{4!} + \ldots$$

$I^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Show that $R(\Theta) = e^{I\Theta} = cos(\Theta) + Isin(\Theta)$

$$e^{I\theta} = I\left(\theta + \frac{I^2\theta^3}{3!} + \frac{I^4\theta^5}{5!} + \ldots\right) + \left(1 + \frac{I^2\theta^2}{2!} + \frac{I^4\theta^4}{4!} + \ldots\right)$$

$$e^{I\theta} = I\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right) + \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) = Isin(\theta) + cos(\theta)$$

(b)

$$\begin{pmatrix} cos(\theta_1) & sin(\theta_1) \\ -sin(\theta_1) & cos(\theta_1) \end{pmatrix} \begin{pmatrix} cos(\theta_2) & sin(\theta_2) \\ -sin(\theta_2) & cos(\theta_2) \end{pmatrix} =$$

$$\begin{pmatrix} cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) & cos(\theta_1)sin(\theta_2) + sin(\theta_1)cos(\theta_2) \\ -sin(\theta_1)cos(\theta_2) - cos(\theta_1)sin(\theta_2) & -sin(\theta_1)sin(\theta_2) + cos(\theta_1)cos(\theta_2) \end{pmatrix} =$$

$$\begin{pmatrix} cos(\theta_1+\theta_2) & sin(\theta_1+\theta_2) \\ -sin(\theta_1+\theta_2) & cos(\theta_1+\theta_2) \end{pmatrix}$$

which is clearly symmetric $\theta_1 \leftrightarrow \theta_2$

(c)

$$zz^* = (x+iy)(x-iy) = x^2 + y^2$$

So, given a vector in the complex plain specified by (x,y), zz^* gives the length of the vector.

Under the action of: $z \rightarrow e^{i\theta} z, z^* \rightarrow e^{-i\theta} z^*$

$$M(\theta_1) : z \rightarrow e^{i\theta_1} z \text{ (+ complex conjugate)}$$
$$M(\theta_2) : z \rightarrow e^{i\theta_2} z \text{ (+ complex conjugate)}$$
$$M(\theta_1)M(\theta_2) : z \rightarrow e^{i\theta_1} e^{i\theta_2} z = e^{i(\theta_1+\theta_2)} z = M(\theta_1+\theta_2)$$

And $M(\theta_1)M(\theta_2) = M(\theta_1+\theta_2) = M(\theta_2+\theta_1) = M(\theta_2)M(\theta_1)$, because addition commutes.

5) 3D Rotations *(5 points)*

(a) $[\boldsymbol{J}_{23}, \boldsymbol{J}_{13}] = \boldsymbol{J}_{12}, \quad [\boldsymbol{J}_{12}, \boldsymbol{J}_{23}] = \boldsymbol{J}_{13}, \quad [\boldsymbol{J}_{13}, \boldsymbol{J}_{12}] = \boldsymbol{J}_{23}$

(b) The most general 2x2 trace-less and hermitian matrix can be written as

$$M = \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix} \equiv \vec{\sigma} \cdot \vec{r}$$

The determinant is given by $-z^2 - x^2 - y^2 = -r^2$.

Consider $M' = U^\dagger M U$, where U is unitary.

Now, $tr(M') = tr(U^\dagger M U) = tr(UU^\dagger M) = tr(M)$

And $(M')^\dagger = (U^\dagger M U)^\dagger = (UM^\dagger U^\dagger) = (U^\dagger M U) = M'$

So M' is also a 2x2 trace-less and hermitian matrix which can be written as $M = \vec{\sigma} \cdot \vec{r'}$

The determinant of M' is given by $det(M') = det(U^\dagger) det(M) det(U) = 1 \times det(M) \times 1 = -r^2$.

This implies that the Unitary transform performs a transformation that preserves the length of the vector r defined by $\vec{\sigma} \cdot \vec{r}$. The action of U preforms a rotation on the vector defined by M.

6) Lorentz Transformations *(5 points)*

(a)

$$e^{I\eta} = 1 + I\eta + \frac{I^2\eta^2}{2!} + \frac{I^3\eta^3}{3!} + \frac{I^4\eta^4}{4!} + \ldots$$

$$I_B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{I\eta} = I\left(\eta + \frac{I^2\eta^3}{3!} + \frac{I^4\eta^5}{5!} + \ldots\right) + \left(1 + \frac{I^2\eta^2}{2!} + \frac{I^4\eta^4}{4!} + \ldots\right)$$

$$e^{I\eta} = I\left(\eta + \frac{\eta^3}{3!} + \frac{\eta^5}{5!} + \ldots\right) + \left(1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \ldots\right) = I\sinh(\eta) + \cosh(\eta)$$

(b) The origin of the primed frame is at $x' = 0$ in the prime frame and at $x = vt$ in the unprimed frame (assuming the origins coincided at t=0)

$$x = t'\sinh(\eta) \text{ and } t = t'\cosh(\eta)$$

$$v = \frac{x}{t} = \tanh(\eta) \text{ and } \cosh^{-2} = 1 - \tanh^2$$

$$\Rightarrow \cosh(\eta) = \frac{1}{\sqrt{1 - v^2}} \equiv \gamma$$

$$\sinh(\eta) = \frac{v}{\sqrt{1 - v^2}} = \beta\gamma$$