

# Homework Set #3

## Solutions

1) Work out the commutation relation among the  $\vec{X}$  and  $\vec{P}$  operators:  
ie:  $[\vec{X}, \vec{X}]$ ,  $[\vec{P}, \vec{P}]$ , and  $[\vec{X}, \vec{P}]$

(5 points)

On one hand,

$$\vec{X}T(\vec{a})|\vec{x}\rangle = (\vec{x} + \vec{a})|\vec{x} + \vec{a}\rangle$$

on the other,

$$T(\vec{a})\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x} + \vec{a}\rangle$$

So

$$[\vec{X}, T(\vec{a})] = \vec{a}T(\vec{a})$$

Consider the infinitesimal translation ( $\vec{a} \rightarrow \vec{\epsilon}$ ). Then,

$$[\vec{X}, 1 - i\vec{\epsilon} \cdot \vec{P}] = \vec{\epsilon} \cdot (1 - i\vec{\epsilon} \cdot \vec{P})$$

Keeping terms to first order in  $\epsilon$ .

$$\begin{aligned} [X_i, 1 - i\epsilon_j P_j] &= \epsilon_i \\ -i\epsilon_j [X_i, P_j] &= \epsilon_i \end{aligned}$$

or

$$[X_i, P_j] = i\delta_{ij}$$

2) Harmonic Oscillator

(10 points)

The 1D Harmonic oscillator has Hamiltonian:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

where P and X are position and momentum operators

a Define “raising” and “lowering” operators as

$$a = \sqrt{\frac{mw}{2}} \left( X + i \frac{P}{mw} \right) \quad a^\dagger = \sqrt{\frac{mw}{2}} \left( X - i \frac{P}{mw} \right)$$

What are the position and momentum operators in terms of the raising and lowering operators?

*Solution:*

$$x = \frac{x_0}{\sqrt{2}} (a + a^\dagger) \quad p = \frac{-i}{x_0 \sqrt{2}} (a - a^\dagger)$$

where  $x_0 = \sqrt{\frac{1}{mw}}$ .

b

$$\begin{aligned} [a, a^\dagger] &= \frac{mw}{2} \left[ x + \frac{ip}{mw}, x - \frac{ip}{mw} \right] \\ &= \frac{mw}{2} \frac{-i}{mw} [x, p] + \frac{mw}{2} \frac{i}{mw} [p, x] \\ &= \frac{1}{2} (-i[x, p] + i[p, x]) = 1 \end{aligned}$$

c What is the Hamiltonian in terms of  $a$  and  $a^\dagger$ ?

$$H = \omega(a^\dagger a + \frac{1}{2})$$

d Define the “Number” operator  $N$  as  $a^\dagger a$ . What is the Hamiltonian in terms of the number operator?

$$H = \omega(N + \frac{1}{2})$$

e Work out the commutation relations:  $[N, a^\dagger]$  and  $[N, a]$ .

$$[N, a^\dagger] = a^\dagger \quad [N, a] = -a$$

f Show that the eigenvalues of  $N$  ( $n$ ) are real and satisfy  $n \geq 0$ .

$$n = \langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle = \langle an | \underbrace{|an\rangle}_{|n'\rangle} = \langle n' | n' \rangle \geq 0$$

- g Show that  $a|n\rangle$  is an eigenstate of  $N$ , with eigenvalue  $(n-1)$ . This implies  $a|n\rangle \propto |n-1\rangle$  and justifies calling  $a$  the lower operator.

$$|n'\rangle = a|n\rangle$$

$$N|n'\rangle = Na|n\rangle = (aN + [N, a])|n\rangle = (aN - a)|n\rangle = (n - 1)|n'\rangle$$

- h Show that  $a^\dagger|n\rangle$  is an eigenstate of  $N$ , with eigenvalue  $(n+1)$ . This implies  $a^\dagger|n\rangle \propto |n+1\rangle$  and justifies calling  $a^\dagger$  the raising operator.

$$|n'\rangle = a^\dagger|n\rangle$$

$$N|n'\rangle = (a^\dagger N + [N, a^\dagger])|n\rangle = (n + 1)|n'\rangle$$

- i Find  $c_n$  such that  $|n+1\rangle = c_n a^\dagger|n\rangle$  is normalized.

$$|n+1\rangle = c_n a^\dagger|n\rangle$$

$$1 = \langle n+1|n+1\rangle = |c_n|^2 \langle n|aa^\dagger|n\rangle = |c_n|^2(n+1)$$

$$c_n = \frac{1}{\sqrt{n+1}}$$

- j Since  $n \geq 0$ , there must be a state  $|0\rangle$  which satisfies  $a|0\rangle = 0$  and  $n$  must be an integer. What is the general state  $|n\rangle$  in terms of  $|0\rangle$  and  $a^\dagger$ ? What is the energy associated to this state?

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

The energy associated to  $|n\rangle$  is  $\omega(n + 1/2)$