Homework Set #1

Solutions

2) Solid State Physics *(5 points)*

(a) We worked out in class $r_{\text{atom}} \sim \frac{1}{Z\alpha m_e}$. (using $E \sim -\frac{Z\alpha}{r} + \frac{p^2}{m_e}$ and $p \times r \sim 1$) So a solid has a spacing of r_{atom}.

(b) To probe distances of order r_{atom} need to photons with Energy $\sim \frac{1}{r_{\text{atom}}} \sim Z\alpha m_{\text{electron}}$. Assuming $Z \sim 10$,
Energy $\sim 10 \cdot 10^{-2} \cdot 10^{-3}$ GeV $\sim 10^{-4}$GeV $\sim 10^{2}$ keV

(c) 10^2 keV photons are x-rays.

3) Strength of Gravity on Earth *(5 points)*

(a) From class, $R_{\text{Planet}} \sim \sqrt{\frac{\alpha}{\alpha_G}} \times r_{\text{atom}}$, $\rho_{\text{solid}} \sim \frac{Zm_{\text{proton}}}{r^3_{\text{atom}}}$, and $M_{\text{Planet}} \sim \rho_{\text{Solid}} \times R^3_{\text{Planet}}$

$$g_{\text{local}} \sim G_N \frac{M_{\text{Planet}}}{R^2_{Planet}} \sim \frac{G_N Z m_{\text{proton}} R_{\text{Planet}}}{r^3_{\text{atom}}} \sim \sqrt{\alpha_G \alpha} \frac{Z}{m_{\text{proton}} r^2_{\text{atom}}}$$

(b)

$$\begin{aligned} g_{\text{local}} \sim & (\alpha_G \alpha)^{1/2} \cdot Z \cdot r^{-2}_{\text{atom}} \cdot m^{-1}_{\text{proton}} \\ \sim & (10^{-39} 10^{-2})^{1/2} \cdot 10 \cdot 10^{-8} \text{ GeV}^{-2} \cdot \text{GeV} \\ \sim & 10^{-27.5} \text{GeV} \end{aligned}$$

Need to convert GeV to $\frac{m}{s^2}$ which has units of [distance]×[time]$^{-2}$. c has units of [distance]×[time]$^{-1}$, h has units of [energy]×[time]. So, can convert from [energy] to [distance]×[time]$^{-2}$ by multiplying by c/h. c = 10^8 m/s, h = 10^{-15} eV·s = 10^{-24} GeV· s. So, $c/h = 10^8 \cdot 10^{24} = 10^{32} \frac{m}{\text{GeV} s^2}$

$$\begin{aligned} g_{\text{local}} \sim & 10^{-27.5} \text{GeV} \times (1) \\ \sim & 10^{-27.5} \text{GeV} \times \frac{c}{h} \\ \sim & 10^{-27.5} \text{GeV} \times 10^{32} \frac{m}{\text{GeV} s^2} \\ \sim & 10^4 \frac{m}{s^2} \end{aligned}$$

(c) Not so close to $10\frac{m}{s^2}$, If we has used $r_{\text{atom}} \sim 10^{-10} m$ instead of 10^{-12} which accounts for the screening of multple electrons in the atom. Would have calculated factor of 10^{-4} smaller or g_{local} 1, which is pretty close.

4) Neutron Stars *(5 points)*

(a) Neutron star will be stable when $P_{Neutron} \sim P_{Grav}$. $P_{Neutron} \sim \frac{E_{Neutron}}{R^3_{Neutron}} \sim m^4_{\text{proton}}$

$$P_{\text{Grav}} \sim \frac{G_N \frac{M^2_{NS}}{R_{NS}}}{R^3_{NS}} \sim \frac{G_N M^2_{NS}}{R^4_{NS}}$$

$$M_{\text{NS}} \sim \rho_{Neutron} \times R^3_{NS}$$

$$\rho_{Neutron} \sim m^4_{\text{proton}}$$

$$P_{\text{Grav}} \sim P_{Neutron} \Rightarrow G_N m^8_{\text{proton}} R^2_{NS} \sim m^4_{\text{proton}}$$

$$R_{NS} \sim \sqrt{\frac{1}{\alpha_G}} \frac{1}{m_{\text{proton}}} \sim \sqrt{\frac{1}{\alpha_G}} r_{\text{proton}}$$

$$\Rightarrow M_{NS} \sim \left(\frac{1}{\alpha_G}\right)^{3/2} m_{\text{proton}}$$

Speed of sound $v_{NS} \sim \sqrt{\frac{P_{NS}}{\rho_{NS}}} \sim 1$

(b) $R_{NS} \sim 10^{19} 10^{-15}\text{m} \sim 10^4\text{m} \sim 10\ \text{km}$
$M_{NS} \sim 10^{58} 10^{-27}\text{kg} \sim 10^{31}\text{kg} \sim 10\ M_\odot$
$v_{NS} \sim c$

(c) "Neutron stars have a radius of the order of 10 kilometres and a mass lower than a 2.16 solar masses." -wikipedia

(d) $m_{\text{proton}} = 938.27$ MeV
$m_{\text{neutron}} = 939.56 \text{ MeV} \sim 1.001 m_{\text{proton}}$

5) 2D Rotations *(3 points)*

(a)

$$e^{I\theta} = 1 + I\theta + \frac{I^2\theta^2}{2!} + \frac{I^3\theta^3}{3!} + \frac{I^4\theta^4}{4!} + \dots$$

$I^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Show that $R(\Theta) = e^{I\Theta} = cos(\Theta) + I sin(\Theta)$

$$e^{I\theta} = I\left(\theta + \frac{I^2\theta^3}{3!} + \frac{I^4\theta^5}{5!} + \dots\right) + \left(1 + \frac{I^2\theta^2}{2!} + \frac{I^4\theta^4}{4!} + \dots\right)$$

$$e^{I\theta} = I\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) + \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) = I sin(\theta) + cos(\theta)$$

(b)

$$\begin{pmatrix} cos(\theta_1) & sin(\theta_1) \\ -sin(\theta_1) & cos(\theta_1) \end{pmatrix}\begin{pmatrix} cos(\theta_2) & sin(\theta_2) \\ -sin(\theta_2) & cos(\theta_2) \end{pmatrix} =$$

$$\begin{pmatrix} cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) & cos(\theta_1)sin(\theta_2) + sin(\theta_1)cos(\theta_2) \\ -sin(\theta_1)cos(\theta_2) - cos(\theta_1)sin(\theta_2) & -sin(\theta_1)sin(\theta_2) + cos(\theta_1)cos(\theta_2) \end{pmatrix} =$$

$$\begin{pmatrix} cos(\theta_1 + \theta_2) & sin(\theta_1 + \theta_2) \\ -sin(\theta_1 + \theta_2) & cos(\theta_1 + \theta_2) \end{pmatrix}$$

which is clearly symmetric $\theta_1 \leftrightarrow \theta_2$

6) 3D Rotations *(5 points)*

(a)

$$J_1 J_2 - J_2 J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -iJ_3$$

etc.

(b) $tr(M') = tr(U^\dagger M U) = tr(U U^\dagger M) = tr(M)$

$(M')^\dagger = (U^\dagger M U)^\dagger = (U M^\dagger U^\dagger) = (U^\dagger M U) = M'$

$det(M') = det(U^\dagger) det(M) det(U) = 1 \times det(M) \times 1$