

## Lecture 11

### Relativistic Wave Equations

Lets look at the Schrodinger Equation

$$i\frac{d}{dt}\psi = -\left(\frac{\nabla^2}{2m} + V\right)\psi$$

Problems - Conservation of non-relativistic energy.

$$E \leftrightarrow i\frac{d}{dt} \text{ and } p \leftrightarrow -i\nabla$$

$$\Rightarrow \text{Schrodinger Equation } E = \frac{p^2}{2m} + V$$

- time/space not on equal footing.

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Start w/relativistic E/p relation.

$$E^2 - p^2 - m^2 = 0$$

Leads us to the Klein-Gordon equation

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)\phi(x, t) = 0$$

Solution,

$$\phi(x, t) = e^{ipx}$$

Called “On shell-solutions”.

Everything is Lorentz Invariant.

Note Two solutions  $E>0$  and  $E<0$ .

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Alternative Standard Powerful formulation of Physics via Lagrangians.

KG eq

$$\frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

where  $\phi(x, t)$  permeates space and time

Looks like

$$\frac{d^2}{dt^2}x(t) = -\frac{\partial U}{\partial x}$$

Where U is a potential energy.

- $\frac{\partial^2}{\partial t^2}\phi$  - looks like the “acceleration” of the field
- $-m^2\phi$  - effective “Force” for the mass (harmonic oscillator)
- $\nabla^2\phi$  - shear force (shearing fabric takes force)

Now have some intuition about what the KG equation is telling us.

Integrate force to get the potential

$$-\frac{\partial U}{\partial \phi} = \nabla^2\phi - m^2\phi \Rightarrow U = \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2$$

KE form generalization of  $1/2\dot{x}^2$  to fields.

$$K = 1/2\left(\frac{\partial}{\partial t}\phi\right)^2$$

so,

$$K + U = \frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 + \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2$$

Get total energy from integrating,

$$H(t) = \int d\vec{x} \left[ \frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 + \frac{1}{2}(\nabla\phi)(\nabla\phi) + \frac{m^2}{2}\phi^2 \right]$$

with this you could go back to three lectures ago and replace  $\frac{\nabla\phi^\dagger\nabla\phi}{2m}$  and do everything again (we won't bother here...)

However we can use this to formulate the field eq in a totally different, ultimately more useful way...

Think of classical mechanics of point particles. Can all be formulated with Lagrangians and the principle of least action.

$$L = \frac{1}{2}\dot{x}^2 - U(x)$$

difference in KE and potential

The action

$$S[x(t)] = \int dt L$$

you could take this as the starting point and derive Newton's 2nd law by minimizing the action.

We will do the same for relativistic fields.

$$L = K - U = \int d^3x \left[ \frac{1}{2} \left( \frac{\partial}{\partial t} \phi \right)^2 - \frac{1}{2} (\nabla \phi)(\nabla \phi) - \frac{m^2}{2} \phi^2 \right]$$

$$S[\phi(t)] = \int d^4x [(\text{Above})]$$

KG can be found from minimizing the action wrt  $\phi$ .

$$\partial_\mu = \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

$$S[\phi(t)] = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2} \phi^2 \right]$$

manifestly L.I

This tells us how to construct general LI descriptions of relativistic QM systems. Need a LI Lagrangian and we are "done" (meaning ~ all of physics follows from this)

KG 2nd order in  $x^\mu$ . Contains no info about intrinsic angular momentum. (No free Lorentz indices)  $\phi$  from KG is Spin-0 field.

Two more relativistic systems important for SM

- Spin-1 “EM”
- Spin-1/2 “Dirac Equation”

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Start with the “Dirac Equation”.

Can we find a 1st order relativistic eq of motion?

Assume there exists a wave eq. linear in  $x$  and  $t$ .

$$\left( \alpha \frac{\partial}{\partial t} + \vec{\beta} \vec{\nabla} \right) \psi = m \psi$$

for some  $\alpha, \vec{\beta}$ , and  $m$ .

If relativistic invariant, must imply the KG.

Assume that it is the square root

$$\left( \alpha \frac{\partial}{\partial t} + \vec{\beta} \vec{\nabla} \right) \left( \alpha \frac{\partial}{\partial t} + \vec{\beta} \vec{\nabla} \right) \psi = m^2 \psi$$

$\Rightarrow$

$$\left[ \alpha^2 \frac{\partial^2}{\partial t^2} + \alpha \vec{\beta} \vec{\nabla} \frac{\partial}{\partial t} + \vec{\beta} \vec{\nabla} \alpha \frac{\partial}{\partial t} + (\vec{\beta} \cdot \vec{\nabla})^2 \right] \psi = m^2 \psi$$

to get  $(-\frac{\partial^2}{\partial t^2} + \nabla^2) \psi = m^2 \psi$

$\Rightarrow$

- $\alpha^2 = -1$  - can be a complex number
- $\beta_i \beta_j = \delta_{ij}$  - cannot be numbers
- $\alpha \beta_i + \beta_i \alpha = 0$

Lets keep going...

Define

$$\alpha = i\gamma_0$$

$$\beta_i = i\gamma_i$$

$$\gamma_0\gamma_0 = 1$$

$$\gamma_i\gamma_j = -\delta_{ij}$$

$$\gamma_0\gamma_i + \gamma_i\gamma_0 = 0$$

$$\underbrace{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}_{\{\gamma_\nu, \gamma_\mu\}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = 2\eta_{\mu\nu}$$

$$(i\gamma_\mu\partial^\mu - m)\psi = 0$$

“Dirac Equation”  $\gamma$  matrices are 4x4.

Can use (“Weyl basis”)

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$$

where

$$\sigma = (1, \vec{\sigma})$$

$$\bar{\sigma} = (1, -\vec{\sigma})$$

There are other choices you can work out...

Dirac equation describes spin 1/2 particles (e)

B/c  $\gamma_\mu$  are matrices, the  $\psi$  solution will be a 4-component vector “spinor”.

Solution to DE work out in homework ...

Solutions are four component objects spinors  $\psi$  (complex)

Use  $\bar{\psi}$  to denote the complex conjugate of the spinor.

The Dirac equation can be formulated from Lagrangian principle via

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}[i\gamma_\mu\partial^\mu - m]\psi$$

Varying the action with respect to  $\bar{\psi}$  gives the Dirac equation.