Position Operator

$$\vec{X}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}\rangle$$

Position operator — Position eigenvalue

Completeness & Orthogonality

$$\sum_a |a\rangle\langle a| = 1 \;\Rightarrow\; \int d^3x\, |x\rangle\langle x| = 1$$

$$\langle a'|a\rangle = \delta_{aa'} \;\Rightarrow\; \langle x|x'\rangle = \delta^3(\vec{x}-\vec{x}')$$

Wavefunction

$$|\psi\rangle = \int d^3x\, |x\rangle\langle x|\psi\rangle = \int d^3x\, \psi(x)\,|x\rangle$$

* where $\psi(x) = \langle x|\psi\rangle$
Position space wavefunction

$$\langle\psi|\psi\rangle = 1 \;\Rightarrow\; \int d^3x\, \psi(x)\langle\psi|x\rangle = \int d^3x\, \psi^*(x)\psi(x) = \underline{1}$$

Interpretation $|\psi(x)|^2 d^3x$ = Probability to find particle in volume d^3x around $\vec{x}$.

Translation Operator "operator that moves you over"

$$T(\vec{a})|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$$

What is $T^\dagger(\vec{a})$?

$$\langle\vec{x}'|\big(T(\vec{a})|\vec{x}\rangle\big) = \delta^3((\vec{x}+\vec{a})-\vec{x}')$$

or

$$\big(\langle\vec{x}'|T(\vec{a})\big)|\vec{x}\rangle = \delta^3(\vec{x}-(\vec{x}'-\vec{a}))$$

$$\Rightarrow \langle x'|T(a) = \langle x'-a|$$

* which says that $T^\dagger(a)|\vec{x}'\rangle = |\vec{x}'-a\rangle$

$T^\dagger(\vec{a}) = T(-a) = T^{-1}(\vec{a}) \qquad T^\dagger(\vec{a})\,T(\vec{a}) = \mathbb{1}$

Properties of T

1. Unitary $T^\dagger T = 1$
2. $T(\vec{a})\,T(\vec{b}) = T(\vec{a}+\vec{b}) = T(\vec{b})\,T(\vec{a})$
   $\Rightarrow [T(\vec{a}), T(\vec{b})] = 0$
3. $T(0) = \mathbb{1}$

Infinitesimal Translations

$\vec{a} = N\vec{\varepsilon}$

$$T(\vec{\varepsilon}) = \mathbb{1} - i\,\vec{\varepsilon}\cdot\vec{K}$$

$\vec{\varepsilon}$: 3-vector; $\vec{K}$: Operator $\vec{K} = (K_x, K_y, K_z)$

$T^\dagger T = \mathbb{1}$ or $(1 + i\,\vec{\varepsilon}\cdot\vec{K}^\dagger)(1 - i\,\vec{\varepsilon}\cdot K) = \mathbb{1}$

$$1 + i\,\vec{\varepsilon}\,\underbrace{(\vec{K}^\dagger - \vec{K})}_{=0} + O(\varepsilon^2) = \mathbb{1}$$

$\Rightarrow K^\dagger = K \quad K$ hermitian

* if the i wasn't there $T(\vec{a})$ would not be hermitian

Just like with SR.

Any finite translation can be built out of infinitesimal translations

* K is "generator" of translations.

$$T(\vec{a}) = \lim_{N \to \infty} \left(1 - i\,\vec{\varepsilon}\cdot\vec{K}\right)^N = \lim_{N\to\infty}\left(1 - i\,\frac{\vec{a}}{N}\cdot K\right)^N$$

finite

$$e^{-i\vec{k}\cdot\vec{a}}$$

$\vec{a} = N\vec{\varepsilon}$

T - unitary K - hermitian

Eigenstates of $\vec{K}$ (and automatically of T)

$$\vec{K}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle$$

$$T(\vec{a})\,|\vec{k}\rangle = e^{-i\vec{k}\cdot\vec{a}}\,|\vec{k}\rangle$$

$$\boxed{\text{Momentum Operator} \quad \vec{P} = \hbar\vec{K}}$$

What is $\psi_{\vec{k}}(x) = \langle \vec{x} | \vec{k} \rangle$?

$$\langle x | T(\vec{a}) | \vec{k} \rangle = e^{-ika}\langle x | k \rangle = e^{-ika}\psi_k(x)$$

T on x $= \langle x - a | \vec{k} \rangle = \psi_k(\vec{x} - \vec{a})$

or $\psi_k(\vec{x} - \vec{a}) = e^{-i\vec{k}\cdot\vec{a}}\,\psi_k(x)$

$$\boxed{\text{Momentum is the generator of translations}}$$

Couple ways to see this
- De Broglie wavelength

Note translations form a group: "translation group"

1) Closure
2) Identity
3) Inverse
4) Associative

Can show they Abelian

$$\boxed{T(\vec{a}) = e^{-i\vec{p}\cdot\vec{a}}}$$

Another look @ $\vec{P}$ operator

$$T(a)|\psi\rangle = \int d^3x \, \underbrace{T(a)|x\rangle}_{|x+a\rangle}\langle x|\psi\rangle$$

$$= \int d^3x \, |x\rangle\langle x-a|\psi\rangle$$

$$\psi(x-a) = \psi(x) - a\frac{\partial}{\partial x}\psi(x)$$

$$T(a)|\psi\rangle = \left(1 - \frac{i\vec{p}\cdot\vec{a}}{\hbar}\right)|\psi\rangle \qquad \boxed{\vec{p} = -i\hbar\frac{\partial}{\partial x}}$$

Critical result!

$$\vec{P}|p\rangle = \vec{p}|\vec{p}\rangle \qquad \langle x|p\rangle \sim e^{i\frac{p\cdot x}{\hbar}}$$

Momentum Space Wave function

$$\phi(p) \equiv \langle p|\psi\rangle \qquad \left(\begin{array}{l}\text{Can show that} \\ \phi(p) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}}\end{array}\right)$$

$$\langle x|\psi\rangle = \int d^3p \, \langle x|p\rangle\langle p|\psi\rangle$$

$$\psi(x) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p \, e^{\frac{ipx}{\hbar}}\phi(p)$$

$$\langle p|\psi\rangle = \int d^3x \, \langle p|x\rangle\langle x|\psi\rangle$$

$$\phi(p) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3x \, e^{-\frac{ipx}{\hbar}}\psi(x)$$

$\psi(x)$ + $\phi(p)$ fourier transforms of each other