

# Homework Set #4

## Solutions

### 1) Find the generators of the “Little Group” for Massive particles

(5 points)

The little group equation is:

$$W \cdot k = k \Rightarrow \omega \cdot k = 0$$

where

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & A & B \\ -b & -A & 0 & C \\ -c & -B & -C & 0 \end{bmatrix}$$

For a massive particle we can take  $k$  to be  $k = (m, 0, 0, 0)$ .

So  $\omega \cdot k = (0, -ma, -mb, -mc)$ . So the little group generators for massive particles are

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & -A & 0 & C \\ 0 & -B & -C & 0 \end{bmatrix}$$

which are the rotation matrices.

### 2) Heisenberg Equation of Motion

(5 points)

a)

$$\begin{aligned} \frac{dA(t)_H}{dt} &= \frac{d}{dt} (e^{iHt} A_s e^{-iHt}) \\ &= iH (e^{iHt} A_s e^{-iHt}) - i (e^{iHt} A_s H e^{-iHt}) \\ &= -i \left( \underbrace{e^{iHt} A_s e^{-iHt}}_{A_H(t)} H - H \underbrace{e^{iHt} A_s H e^{-iHt}}_{A_H(t)} \right) \\ &= -i [A_H(t), H] \end{aligned}$$

b)  $\phi_H(x, t) = e^{-iE_p t} \phi_S(x)$

$$\frac{d\phi_H(x, t)}{dt} = -iE_p \phi_H(x, t)$$

and

$$\begin{aligned}
[\phi_H(x, t), H] &= [\int \not{d}p e^{ip \cdot x} a^\dagger, \int dp' E_{p'} a^\dagger a] \\
&= \int \not{d}p e^{ip \cdot x} \int dp' E_{p'} [a^\dagger, a^\dagger a] \\
&= \int \not{d}p e^{ip \cdot x} \int dp' E_{p'} a^\dagger [a^\dagger, a] \\
&= \int \not{d}p e^{ip \cdot x} \int dp' E_{p'} a^\dagger \delta^3(\vec{p} - \vec{p}') \\
&= \int \not{d}p e^{ip \cdot x} E_p a^\dagger \\
&= E_p \int \not{d}p e^{ip \cdot x} a^\dagger = E_p \phi_H(x, t)
\end{aligned}$$

So

$$\frac{d\phi_H(x, t)}{dt} = -i [\phi_H(x, t), H]$$

**3) Show that  $\int \not{d}^3 p \equiv \int \frac{d^3 \vec{p}}{2E_p}$  is Lorentz invariant. (2 points)** We will start with a manifestly Lorentz invariant integral and show that it is the same as  $\int \not{d}^3 p$ .

$$\int d^4 p \delta(E^2 - (|\vec{p}|^2 + m^2)) = \int dE d^3 p \delta(E^2 - (|\vec{p}|^2 + m^2))$$

Now,  $dE^2 = 2EdE$  or  $dE = dE^2/2E$ , so

$$= \int \frac{dE^2}{2E} d^3 p \delta(E^2 - (|\vec{p}|^2 + m^2))$$

Can now do the integral over  $dE^2$

$$= \int \frac{d^3 p}{2\sqrt{|\vec{p}|^2 + m^2}} = \int \not{d}^3 p$$

#### 4) Anti-Particles

(5 points)

- a) Expand  $\Phi^{\dagger 2} \Phi^2$  in terms of  $a$ ,  $a^\dagger$ ,  $b$ , and  $b^\dagger$  (Ignore the exponentials and integrals)

$$\begin{aligned}\Phi^{\dagger 2} \Phi^2 &= (\phi_-^a + \phi_+^b)^2 (\phi_+^a + \phi_-^b)^2 \sim (a + b^\dagger)^2 (a^\dagger + b)^2 \\ &= (aa + 2ab^\dagger + b^\dagger b^\dagger) (a^\dagger a^\dagger + 2a^\dagger b + bb) \\ &= (aaa^\dagger a^\dagger + 2aaa^\dagger b + aabb + 2ab^\dagger a^\dagger a^\dagger + 4ab^\dagger a^\dagger b + 2ab^\dagger bb + b^\dagger b^\dagger a^\dagger a^\dagger + 2b^\dagger b^\dagger a^\dagger b + b^\dagger b^\dagger bb)\end{aligned}$$

- b) See figure.

- c) Let the charge ( $Q$ ) of particle  $a$  be  $q_a$  and the charge of particle  $b$  be  $q_b$ . Calculate  $\Delta Q$  for each process.

See figure each term goes like  $(q_a + q_b)$

- d) What happens if you take  $q_a = -q_b$  ? When  $q_a = -q_b$ , all processes conserve charge.

$$\frac{a \ a \ a^+ \ a^+}{a \quad a}$$

$$\Delta Q = 0$$

$$\frac{a \ a \ a^+ \ b}{a \quad a \quad b}$$

$$\Delta Q = -z_a - z_b = -(z_a + z_b)$$

$$\frac{a \ a \ b \ b}{a \quad a \quad b \quad b}$$

$$\Delta Q = -2z_a - 2z_b = -2(z_a + z_b)$$

$$\frac{a \ b \ a^+ \ a^+}{b \quad a \quad a}$$

$$\Delta Q = (z_a + z_b)$$

$$\frac{a \ b \ a^+ \ b}{a \quad b}$$

$$\Delta Q = 0$$

$$\frac{a \ b^+ \ b \ b}{b \quad b \quad a}$$

$$\Delta Q = -(z_a + z_b)$$

$$\frac{b^+ \ b^+ \ a^+ \ a^+}{b \quad b \quad a \quad a}$$

$$\Delta Q = 2(z_a + z_b)$$

$$\frac{b^+ \ b^+ \ a^+ \ b}{b \quad b \quad a \quad b}$$

$$\Delta Q = (z_a + z_b)$$

$$\frac{b^+ \ b^+ \ b \ b}{b \quad b}$$

$$\Delta Q = 0$$