Homework Set #4

Solutions

1) Find the generators of the "Little Group" for Massive particles *(5 points)*

The little group equation is:

$$W \cdot k = k \Rightarrow \omega \cdot k = 0$$

where

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & A & B \\ -b & -A & 0 & C \\ -c & -B & -C & 0 \end{bmatrix}$$

For a massive particle we can take k to be k = (m,0,0,0).

So $\omega \cdot k = (0, -ma, -mb, -mc)$. So the little group generators for massive particles are

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & -A & 0 & C \\ 0 & -B & -C & 0 \end{bmatrix}$$

which are the rotation matrices.

2) Heisenberg Equation of Motion *(5 points)*

a)

$$\begin{aligned} \frac{dA(t)_H}{dt} &= \frac{d}{dt}\left(e^{iHt}A_s e^{-iHt}\right) \\ &= iH\left(e^{iHt}A_s e^{-iHt}\right) - i\left(e^{iHt}A_s H e^{-iHt}\right) \\ &= -i\left(\underbrace{e^{iHt}A_s e^{-iHt}}_{A_H(t)} H - H\,\underbrace{e^{iHt}A_s H e^{-iHt}}_{A_H(t)}\right) \\ &= -i\left[A_H(t), H\right] \end{aligned}$$

b) $\phi_H(x,t) = e^{-iE_p t}\phi_S(x)$

$$\frac{d\phi_H(x,t)}{dt} = -iE_p\phi_H(x,t)$$

and

$$
\begin{aligned}
[\phi_H(x,t), H] &= [\int \not{d}\not{p} e^{ip\cdot x} a^\dagger, \int dp' E_{p'} a^\dagger a] \\
&= \int \not{d}\not{p} e^{ip\cdot x} \int dp' E_{p'} [a^\dagger, a^\dagger a] \\
&= \int \not{d}\not{p} e^{ip\cdot x} \int dp' E_{p'} a^\dagger [a^\dagger, a] \\
&= \int \not{d}\not{p} e^{ip\cdot x} \int dp' E_{p'} a^\dagger \delta^3(\vec{p} - \vec{p}') \\
&= \int \not{d}\not{p} e^{ip\cdot x} E_p a^\dagger \\
&= E_p \int \not{d}\not{p} e^{ip\cdot x} a^\dagger = E_p \phi_H(x,t)
\end{aligned}
$$

So

$$\frac{d\phi_H(x,t)}{dt} = -i\,[\phi_H(x,t), H]$$

3) Show that $\int \not{d}^3\not{p} \equiv \int \frac{d^3\vec{p}}{2E_p}$ is Lorentz invariant. ***(2 points)*** We will start with a manifestly Lorentz invariant integral and show that it is the same as $\int \not{d}^3\not{p}$.

$$\int d^4p\ \delta(E^2 - (|\vec{p}|^2 + m^2)) = \int dE d^3p \delta(E^2 - (|\vec{p}|^2 + m^2))$$

Now, $dE^2 = 2EdE$ or $dE = dE^2/2E$, so

$$= \int \frac{dE^2}{2E} d^3p \delta(E^2 - (|\vec{p}|^2 + m^2))$$

Can now do the integral over dE^2

$$= \int \frac{d^3p}{2\sqrt{|\vec{p}|^2 + m^2}} = \int \not{d}^3\not{p}$$

4) Anti-Particles ***(5 points)***

a) Expand ${\Phi^{\dagger}}^2\Phi^2$ in terms of a, $a^\dagger$, b, and $b^\dagger$ (Ignore the exponentials and integrals)

$$
\begin{aligned}
{\Phi^{\dagger}}^2\Phi^2 &= \left(\phi_-^a + \phi_+^b\right)^2 \left(\phi_+^a + \phi_-^b\right)^2 \sim \left(a + b^\dagger\right)^2 \left(a^\dagger + b\right)^2 \\
&= \left(aa + 2ab^\dagger + b^\dagger b^\dagger\right)\left(a^\dagger a^\dagger + 2a^\dagger b + bb\right) \\
&= \left(aaa^\dagger a^\dagger + 2aaa^\dagger b + aabb + 2ab^\dagger a^\dagger a^\dagger + 4ab^\dagger a^\dagger b + 2ab^\dagger bb + b^\dagger b^\dagger a^\dagger a^\dagger + 2b^\dagger b^\dagger a^\dagger b + b^\dagger b^\dagger bb\right)
\end{aligned}
$$

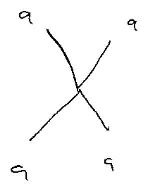
b) See figure.

c) Let the charge (Q) of particle a be q_a and the charge of particle b be q_b. Calculate ΔQ for each process.

See figure each term goes like $(q_a + q_b)$

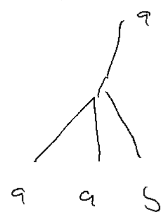
d) What happens if you take $q_a = -q_b$? When $q_a = -q_b$, all processes conserve charge.

$\underline{a\,a\,a^{+}a^{+}}$



$\Delta Q = 0$

$\underline{a\,a\,a^{+}b}$



$\Delta Q = -q_a - q_b$
$= -(q_a + q_b)$

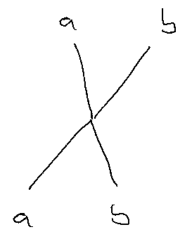
$\underline{a\,a\,b\,b}$

$\Delta Q = -2q_a - 2q_B$
$= -2(q_a + q_B)$

$\underline{a\,b^{+}a^{+}a^{+}}$

$\Delta Q = (q_a + q_B)$

$\underline{a\,b^{+}a^{+}b}$



$\Delta Q = 0$

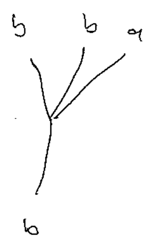
$\underline{a\,b^{+}b\,b}$

$\Delta Q = -(q_a + q_B)$

$\underline{b^{+}b^{+}a^{+}a^{+}}$

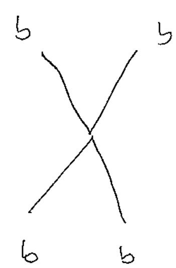
$\Delta Q = 2(q_a + q_B)$

$\underline{b^{+}b^{+}a^{+}b}$



$\Delta Q = (q_a + q_B)$

$b^{+}b^{+}b\,b$



$\Delta Q = 0$