

Lorentz Invariance & "Soft Limits"

①

Push the limit we were been building to...

Matrix Element we would get by scattering external γ

$$M = \epsilon^\mu M_\mu$$

→ where ϵ^μ is some linear combination of two photon polarization vectors $\epsilon^1 + \epsilon^2$

M is L.I., under Lorentz transformation

$$M \rightarrow \epsilon'^\mu M'_\mu \quad \text{where } M'_\mu = \Lambda_\mu^\nu M_\nu$$

hence ϵ is not really a full 4-vector
only 2 components

Under little group transformations (you will show in your h.w.)

$$\epsilon^\mu \rightarrow \epsilon^\mu + c_1 \epsilon^\mu + c_2 \epsilon^\mu + c_3 P^\mu$$

only this piece ϵ^μ is not valid "not in H.L. space"

So,

$$M = \epsilon^\mu M_\mu \rightarrow (\epsilon^\mu + c_1 \epsilon^\mu + c_2 \epsilon^\mu + c_3 P^\mu) M'_\mu$$

Not a valid polarization vector

Only L.I. if we have $P^\mu M'_\mu = 0$

$$\text{or } P^\mu M_\mu = 0 \leftarrow (\text{By L.I. group } P' = P + \text{L.I. drop "1"})$$

$$\text{So, } M = \epsilon^\mu M_\mu \rightarrow (c_1 \epsilon^\mu + c_2 \epsilon^\mu + c_3 P^\mu) M'_\mu$$

$$= \epsilon'^\mu M'_\mu + c_3 \overline{P^\mu M'_\mu}$$

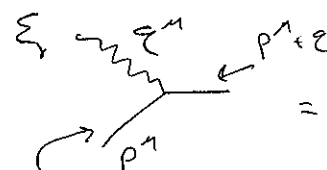
→ must go to 0

We will see, have

enormous implication !!!

will be considering diagrams with external "g"s

massless spin 1 particle



$$= i Q \not{p} (\not{p} + (\not{p} + \not{q})) \epsilon_\mu \quad \epsilon^\mu \epsilon_\mu = 0$$

$$= i Q 2 \not{p} \epsilon_\mu$$

Particle charge Q

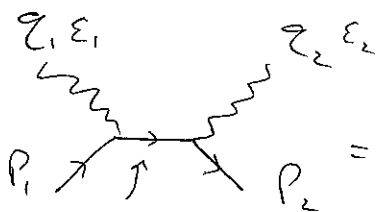
Consider "Compton Scattering"

Start w/ 1-spin-1 Boson & 1 matter Particle

most general form in "soft limit" $q \rightarrow 0$

$$\Gamma_\mu(p, q) \sim p_\mu F(q^2 p^2 p \cdot q)$$

dimensional analysis $\rightarrow F(\frac{p \cdot q}{-})$



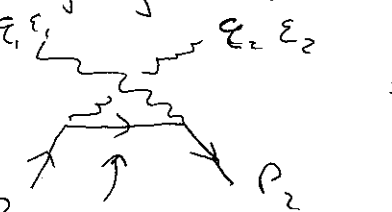
$$= (iQ) \epsilon_\mu^\dagger (2 p_1^\mu) \frac{i}{(p_1 + q_1)^2 - m^2} (iQ) \epsilon_\nu^\dagger (2 p_2^\nu)$$

$$= (-iQ^2) + \frac{(p_1 \cdot \epsilon_1)(p_2 \cdot \epsilon_2)}{m^2 + 2 p_1 \cdot q_1 - m^2} = \epsilon_1^\mu \epsilon_2^\nu \underbrace{\left(\frac{(-iQ^2) 2 p_1^\mu p_2^\nu}{p_1 \cdot \epsilon_1} \right)}_{M_{\mu\nu}^{AA}}$$

As we said above, $L I \Rightarrow \epsilon_1^\mu \epsilon_2^\nu M_{\mu\nu} = 0$

But $\epsilon_1^\mu \epsilon_2^\nu M_{\mu\nu} = (-iQ^2) 2 (p_2 \cdot q_2) \neq 0$! Looks like we're dead...

forgetting diagram



$$= (iQ) \epsilon_1^\mu (2 p_{2\mu}) \frac{i}{(p_2 - q_1)^2 - m^2} (iQ) \epsilon_2^\nu (2 p_{1\nu})$$

$$= \cancel{(-iQ^2)} \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iQ^2) + 2 p_{2\mu} p_{1\nu}}{-2 p_2 \cdot q_1} \right) \approx \epsilon_1^\mu \epsilon_2^\nu \underbrace{\left(\frac{(-iQ^2) 2 p_{2\mu} p_{1\nu}}{p_1 \cdot \epsilon_1} \right)}_{M_{\mu\nu}^B}$$

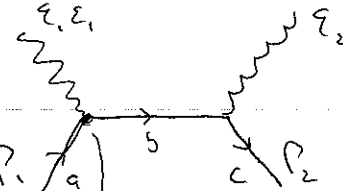
soft limit $p_1 \rightarrow p_2$

(3)

$$\epsilon_1^\mu \epsilon_2^\nu M_{\mu\nu}^B = -(\epsilon_1 \cdot \epsilon_2) 2(P_1 \cdot P_2)$$

So sum $M_{\mu\nu}^A + M_{\mu\nu}^B$ is L.T. (residual pieces of (A) & (B) cancel.) ^{non LI}

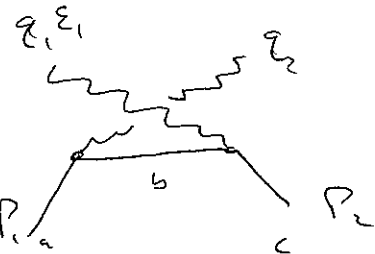
Same as before but with many diff't matter particles
 $i = 1, \dots, N_{\text{matter}}$



$$= (i T_{ab}) \epsilon_1^\mu (2 P_1^\mu) : (i T_{bc}) \epsilon_2^\nu (2 P_2^\nu)$$

$$\underbrace{(P_1 + \epsilon_1)^2 \neq m_b^2}_{m_a^2 + 2 P_1 \cdot \epsilon_1 \neq m_b^2}$$

$$\epsilon_1^\mu \epsilon_2^\nu \left[\frac{(-i T_{ab} T_{bc}) 4 P_1^\mu P_2^\nu}{m_a^2 + 2 P_1 \cdot \epsilon_1 - m_b^2} \right] = M^{\mu\nu} \textcircled{A}$$



$$\Rightarrow \epsilon_1^\mu \epsilon_2^\nu \left[\frac{(-i T_{cb} T_{ba}) 4 P_2^\mu P_1^\nu}{m_a^2 - 2 P_2 \cdot \epsilon_1 - m_b^2} \right] \equiv M^{\mu\nu} \textcircled{B}$$

If $m_a = m_b = m_c$ $\epsilon_1^\mu \epsilon_2^\nu (M^{\mu\nu} \textcircled{A} + M^{\mu\nu} \textcircled{B}) = 0$ as above

However if $m_a \neq m_b$, soft limit \Rightarrow

-) $m_c^2 - m_b^2 = 0$
 -) Relative - sign

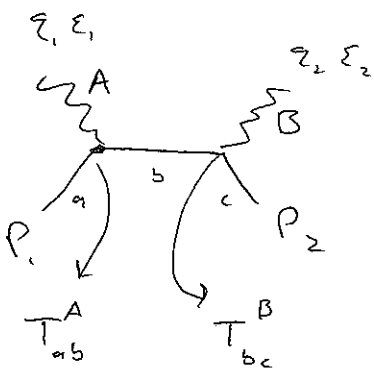
$$\epsilon_1^\mu \epsilon_2^\nu (M^{\mu\nu} \textcircled{A} + M^{\mu\nu} \textcircled{B}) = \epsilon_1^\mu \epsilon_2^\nu \left[\frac{(-i T_{ab} T_{bc}) 4 (2 P_1^\mu P_2^\nu)}{m_a^2 - m_b^2} \right] \neq 0$$

Massless Spin-7 particles can only interact
 w/ particles of same mass.

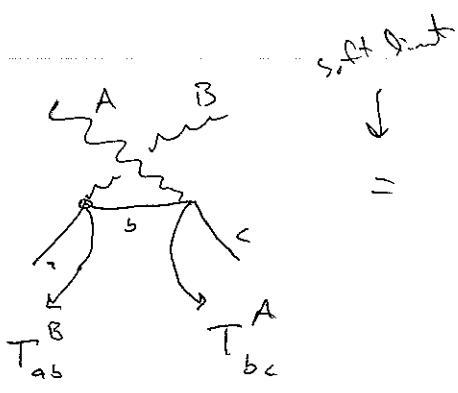
Now allow many different matter fields + many free (adj.) "gluons" (4)

$$i = 1, \dots, N_{\text{matter}}$$

$$I = 1, \dots, N_{\text{gluons}}$$



$$= \frac{(-i T_{ab}^A T_{bc}^B) 4 p_1 \cdot p_2 \epsilon_1^\mu \epsilon_2^\nu}{2 p_1 \cdot \epsilon_1} M_{\mu\nu}^{(A)}$$



$$\downarrow \text{soft limit}$$

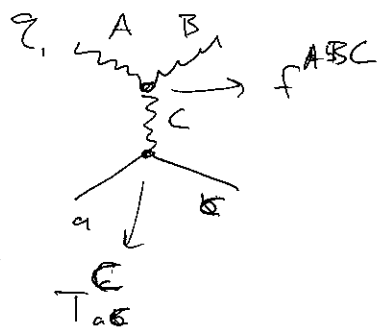
$$= \frac{(-i T_{ab}^B T_{bc}^A) 4 p_1 \cdot p_2 \epsilon_1^\mu \epsilon_2^\nu}{-2 p_1 \cdot \epsilon_1} M_{\mu\nu}^{(B)}$$

$$\epsilon_1^\mu \epsilon_2^\nu (M_{\mu\nu}^{(A)} + M_{\mu\nu}^{(B)}) = 2(-i)(p_2 \cdot \epsilon_2) \left[T_{ab}^A T_{bc}^B - T_{cb}^B T_{ba}^A \right]$$

$$= -2i(p_2 \cdot \epsilon_2) [T^A, T^B]$$

Not = 0 for random T^A, T^B

In fact, missing diagram



$$= +2i(p \cdot \epsilon) \frac{1}{2} i f^{ABC} T_{ab}^C$$

Only $L I = f$

$$[T^A, T^B] = i f^{ABC} T^C$$

"gluons must transform as Lie group"

Yang-Mills Interaction