Lecture 13

Noether's Theorm

Lagrangian may be invariant under some type of transformation (variation)

eg: $\phi \to \phi + \delta$

This transformation is a symmerty of the Lagrangian

Say ϕ is complex: 2 DoF ϕ and ϕ^*.

And you have a Lagrangian given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*$$

symmetry

$$\phi \to e^{-i\alpha}\phi \qquad\qquad \phi^* \to e^{i\alpha}\phi^*$$

Whenever we have a continous symmetry (meaning there is a continious limit)

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta \alpha} = 0 &= \sum_n \left[\frac{\partial \mathcal{L}}{\partial \phi_n} \frac{\delta \phi_n}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta(\partial_\mu \phi_n)}{\delta \alpha} \right] \\
&= \sum_n \left[\underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right]}_{=0 \text{ Euler Lagrange}} \frac{\delta \phi_n}{\delta \alpha} + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \right] \right]
\end{aligned}$$

$$\phi_n = \{\phi, \phi^*\}$$

$\Rightarrow$

$$\partial_\mu J^\mu = 0$$

with

$$J^{\mu} = \sum_n \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \right]$$

J^{μ} is a conserved current. "Noethers Current"

<u>total "charge"</u>

$$Q \equiv \int d^3x \, J_0$$

$$\partial_t Q = \int d^3x \, \partial_t J_0 = \underbrace{\int d^3x \, \vec{\nabla} \cdot \vec{J}}_{\text{Vanishes on the Boundary}} = 0$$

Q does not change with time!

Very gerneral and important theorm <u>"Noethers Theorem"</u>

If $\mathcal{L}$ has ("enjoys") a continous symmetry, there exists an associated current that is conserved.

$$\phi(x) \to \phi(x + \epsilon) = \phi(x) + \epsilon^{\mu} \partial_\mu \phi(x)$$

This leaves $\mathcal{L}$ and $\mathcal{S}$ invariant and gives a four vector of noether currents.

$\Rightarrow$ Noether's theorm tells us why energy and momentum are conserved.