Lecture 14

From Last time...

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v_1} - \vec{v_2}|}dP$$

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} \underbrace{d\Pi}_{\substack{\text{Region of final state momenta} \\ \text{we are considering}}}$$

On interval of size L, the momenta of available states are $P_n = \frac{2\pi n}{L}$ (from particle in a box).

$\Rightarrow$ throughout a volume V

$$N = \int \frac{V}{(2\pi)^3}d^3p$$

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}d^3p_j$$

where j runs over final state particles.

OK, lets deal with the normalization factors.

Note, $\langle f|f\rangle$ and $\langle i|i\rangle \neq 1$ (The inner products are not Lorentz invariant...)

$$\begin{aligned} \langle p'|p\rangle &= (2\pi)^3\, 2E\, \delta^3(p'-p) \\ \langle p|p\rangle &= (2\pi)^3\, 2E_p\, \delta^3(0) \\ &= 2E_pV \end{aligned}$$

$\Rightarrow$

$$\langle i|i\rangle = \langle p_1p_2|p_1p_2\rangle = 2E_1V\, 2E_2V$$

$$\langle f|f\rangle = \prod_j (2E_jV)$$

Now have to deal with $\langle f|S|i\rangle$

S elements always calculated pertubatively

$$S = \underbrace{1}_{\text{free theory}} + \underbrace{iT}_{\text{perturbatively small}}$$

Know that S matrix should vanish if momentum not conserved

$$\langle f|T|i\rangle = (2\pi)^4\delta^4(\sum p) \underbrace{M}_{\text{``}MatrixElement''}$$

Now, might worry that we have to square the δ function

$$\begin{aligned} |\langle f|T|i\rangle|^2 &= (2\pi)^8\, \delta^4\left(\sum p\right)\, \delta^4(0)\, |M|^2 \\ &= (2\pi)^4\, \delta^4\left(\sum p\right)\, TV|M|^2 \end{aligned}$$

So,

$$\begin{aligned} dP &= \frac{(2\pi)^4\, \delta^4\,(\sum p)\;\, TV}{(2E_1V)(2E_2V)} \frac{1}{\prod_j (2E_jV)} |M|^2 \prod_j \frac{V}{(2\pi)^3} d^3p_j \\ &= \frac{T}{V}\frac{1}{(2E_1)(2E_2)}|M|^2 \qquad \underbrace{d\Pi_{\text{LIPS}}}_{\substack{\text{L.I. Phase space} \\ =(2\pi)^4\;\delta^4(\sum p)\;\prod_j \frac{d^3p}{(2\pi)^3 2E_p}}} \end{aligned}$$

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v_1} - \vec{v_2}|}|M|^2 d\Pi_{\text{LIPS}}$$

where $\vec{v} = \vec{p}/p_0$

known as “Fermis Golden Rule”

Decay rate probability that a one-particle state turns into a multi-particle state over time T.

$$p_1 \to \{P_j\}$$

thing of it as $1 \to N$ scattering.

follow same steps as above

$$d\Gamma = \frac{1}{2E_1}|M|^2 d\Pi_{\text{LIPS}}$$

"Feynman Diagrams"

Last piece we need is the LI matrix element

$$M(1+2 \to 3+4+...n) = \langle \{p_j\}_{\text{out}} | p_1 p_2 \rangle_{\text{in}}$$

This represents an element of the S-matrix.

QFT + Lagrangian gives a procedure (recipe) for calculating M

Very nice interpretation in terms of picture "Feynman diagrams"

QM Perturbation Theory

$H = H_0 + V$

Remember we are interested in how some free state at early times ($-\infty$) evolve to some (potentially) other free state at late times.

At early times have a state with a given energy E, which is an eigenstate of H_0

$$H_0 |\phi\rangle = E |\phi\rangle$$

Including the interaction piece, will also be eigenstate of the full Hamiltonian with the same energy.

$$H |\psi\rangle = E |\psi\rangle$$

Now we can formally write,

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} V |\psi\rangle$$

which can be verified by multiplying through by $(E - H_0)$.

Whats happening here is that the interaction at intermediate times is inducing transitions among the states $|\phi\rangle$, which are non-interacting at early (and late) times.

So the full state $|\psi\rangle$ is given by the free state $|\phi\rangle$ plus a scattering term.

Really want to express the full state $|\psi\rangle$ entirely in terms of $|\phi\rangle$.

We do this by defining operator T: $V |\psi\rangle = T |\phi\rangle$.

This gives us

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} T |\phi\rangle$$

or

$$\begin{aligned} V|\psi\rangle &= V|\phi\rangle + V\frac{1}{E - H_0} T |\phi\rangle \\ &= T|\phi\rangle \end{aligned}$$

So we get a nice iterate equation for T

$$T = V + V\frac{1}{E - H_0} T$$

which we can solve perturbatively in V.

eg

$$T = V + V\frac{1}{E - H_0} V + V\frac{1}{E - H_0} V\frac{1}{E - H_0} V + ...$$

Of course, we are interested in inner products of these with the initial/final states

$$\underbrace{\langle \phi_f | T | \phi_i \rangle}_{T_{fi}} = \underbrace{\langle \phi_f | V | \phi_i \rangle}_{V_{fi}} + \sum_j \frac{\langle \phi_f | V | \phi_j \rangle \langle \phi_j | V | \phi_i \rangle}{E - H_0} + ...$$

<u>Example of scattering of 2 electrons</u>

$$|i\rangle = |e_1, e_2\rangle \qquad\qquad |f\rangle = |e_3, e_4\rangle$$

$$T_{fi} = V_{fi} + \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni} + ...$$

- E_i- is the initial (= final) energy
- E_f - is the energy of the intermediate state

Now, the $\sum_n$ runs over energy thing in the Hilbert (Fock) space, however only certain states will be non-0.

In relativistic theory, the action-at-a-distance of EM is replaced by a process where 2 electrons interact with a γ which travels at c. (This tells us there should be γ in the intermediate state)

$$V \sim e \int d^3x \psi \phi \psi \qquad \text{(Ignoring spin)}$$

this operator will have terms that go like ($\sim a^\dagger_{e_3}\, a^\dagger_\gamma\, a_{e_1}$)

However, here all terms involve $a^\dagger_\gamma$.
Because $|i\rangle$ and $|f\rangle$ do not contain a γ, $V_{fi} = 0$

to get a non-zero term, we need $|n\rangle$ with a photon.