

# Homework Set #2

## Solutions

1) Show that  $SO(2) \simeq U(1)$

(2 points)

(a)

$$zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Given a vector in the complex plane specified by (x,y),  $zz^*$  gives the length of the vector.

(b)

$$M(\theta_1) : z \rightarrow e^{i\theta_1} z \text{ (+ complex conjugate)}$$

$$M(\theta_2) : z \rightarrow e^{i\theta_2} z \text{ (+ complex conjugate)}$$

$$M(\theta_1)M(\theta_2) : z \rightarrow e^{i\theta_1} e^{i\theta_2} z = e^{i(\theta_1 + \theta_2)} z = M(\theta_1 + \theta_2)$$

2) Work out the algebra of the generators of the Lorentz group

(5 points) Assuming:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T_i, T_i] = 0$$

$$[T_1, T_2] = J_3$$

$$[T_1, T_3] = J_2$$

$$[T_2, T_3] = J_1$$

OR

$$[T_i, T_j] = \epsilon_{ijk} J_k \quad J_2 \rightarrow -J_2$$

$$[T_1, J_1] = 0$$

$$[T_1, J_2] = -T_3$$

$$[T_1, J_3] = T_2$$

$$[T_2, J_1] = T_3$$

$$[T_2, J_2] = 0$$

$$[T_2, J_3] = -T_1$$

$$[T_3, J_1] = -T_2$$

$$[T_3, J_2] = T_1$$

$$[T_3, J_3] = 0$$

OR

$$[T_i, J_j] = -\epsilon_{ijk} T_k \quad J_2 \rightarrow -J_2$$

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

### 3) Connection to $\beta$ s and $\gamma$ s

(5 points)

(a)

$$e^{I\eta} = 1 + I\eta + \frac{I^2\eta^2}{2!} + \frac{I^3\eta^3}{3!} + \frac{I^4\eta^4}{4!} + \dots$$

$$I_B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Show that  $B(\eta) = e^{I_B\eta} = \cosh(\eta) + I_B \sinh(\eta)$

$$e^{I\eta} = I \left( \eta + \frac{I^2\eta^3}{3!} + \frac{I^4\eta^5}{5!} + \dots \right) + \left( 1 + \frac{I^2\eta^2}{2!} + \frac{I^4\eta^4}{4!} + \dots \right)$$

$$e^{I\eta} = I \left( \eta + \frac{\eta^3}{3!} + \frac{\eta^5}{5!} + \dots \right) + \left( 1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \dots \right) = I \sinh(\eta) + \cosh(\eta)$$

- (b) The origin of the primed frame is at  $x' = 0$  in the prime frame and at  $x = vt$  in the unprimed frame (assuming the origins coincided at  $t=0$ )

$$x = t' \sinh(\eta) \text{ and } t = t' \cosh(\eta)$$

$$v = \frac{x}{t} = \tanh(\eta) \text{ and } \cosh^{-2} = 1 - \tanh^2$$

$$\Rightarrow \cosh(\eta) = \frac{1}{\sqrt{1 - v^2}} \equiv \gamma$$

$$\sinh(\eta) = \frac{v}{\sqrt{1 - v^2}} = \beta\gamma$$

#### 4) Z Boson decays

(5 points)

- (a) Same derivation as we did for the  $\pi \rightarrow \gamma\gamma$  decay in class gives

$$p_{e_1} = (m_Z/2, 0, 0, m_Z/2)$$

$$p_{\gamma_2} = (m_Z/2, 0, 0, -m_Z/2)$$

- (b) Including the mass term gives...

$$p_{e_1} = (E_1, 0, 0, P_1)$$

$$p_{\gamma_2} = (E_2, 0, 0, -P_2)$$

Momentum conservation implies  $P_1 = -P_1 \equiv P$

Energy conservation implies  $M_Z = E_1 + E_2$

So,

$$p_{e_1} = (E, 0, 0, P)$$

$$p_{\gamma_2} = (m_Z - E, 0, 0, -P)$$

Imposing  $P_1^2 = m_e^2$  gives:  $E = \sqrt{m_e^2 + P^2}$

Imposing  $P_2^2 = m_e^2$  gives:  $P^2 = (m_Z - E)^2 - m_e^2$

Combining implies,

$$E = \frac{m_Z}{2}$$

and

$$P = \frac{m_Z}{2} \sqrt{1 - 4 \frac{m_e^2}{m_Z^2}}$$

So no correction to the electron Energies.

The the electron momentum is  $P \simeq \frac{m_Z}{2} \left(1 - \frac{2m_e^2}{m_Z^2}\right)$ , which gives a correction of order  $\left(\frac{m_e}{m_Z}\right)^2 \sim \left(\frac{10^{-3} \text{ GeV}}{100 \text{ GeV}}\right)^2 \sim 10^{-10}$

(b) Including the mass term for the b-quark gives...

No correction to the energies.

The b-quark momentum has a correction of order  $\left(\frac{m_b}{m_Z}\right)^2 \sim \left(\frac{10 \text{ GeV}}{100 \text{ GeV}}\right)^2 \sim 10^{-2}$  about 1%.

### 5) GZK cutoff energy

(5 points)

(a)  $(P + \gamma) \rightarrow (p + \pi_0)$

The final momentum in the center of mass frame is:

$$P_F^\mu = (m_p + m_\pi, 0, 0, 0)$$

$$P_F^2 = (m_p + m_\pi)^2 = m_p^2 + 2m_p m_\pi + m_\pi^2$$

The initial four vector is given by the sum of the proton and CMB photon four vectors.

$$P_I = (P_p^\mu + P_\gamma^\mu)$$

$$P_I^2 = \underbrace{P_p^2}_{=m_p^2} + 2P_p \cdot P_\gamma + \underbrace{P_\gamma^2}_{=0}$$

To evaluate  $P_p \cdot P_\gamma$ , can use reference frame where the photon and proton are colliding head on.

(Assume for the moment that we can neglect the proton mass compared to it momentum ...)

$$P_p = (p_p, 0, 0, p_p) \quad P_\gamma = (E_{CMB}, 0, 0, -E_{CMB})$$

Where  $E_{CMB} = 3 \cdot 10^{-13} \text{ GeV}$ .

$$P_p \cdot P_\gamma = 2p_p E_{CMB}$$

Imposing  $P_I^2 = P_F^2$ , allows us to solve for  $p_p$ .

$$m_p^2 + 4p_p E_{CMB} = m_p^2 + 2m_p m_\pi + m_\pi^2$$

or

$$p_p = \frac{2m_p m_\pi + m_\pi^2}{4E_{CMB}} = \frac{2 \cdot 1 \cdot 0.14 + 0.14^2}{10 \cdot 10^{-13}} \text{ GeV} \sim 3 \times 10^{11} \text{ GeV} \sim 10^{20} \text{ eV}$$

(b)  $\sim 30 \text{ J}$