QM Linear Algebra in a complex vector space.

State of a system is a vector (ray) in the Complex Vector Space

$|\alpha\rangle$

state vector

Linear Superpositions

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle$$

complex #'s

is another vector.

Dual Space:

for every vector $|\alpha\rangle$ (ket) there is another vector $\langle\alpha|$ (bra) in a "dual" space.

Mirror image of the ket space.

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle \quad \text{then} \quad \langle\psi| = c_1^*\langle\alpha_1| + c_2^*\langle\alpha_2|$$

Complex Conj.

Inner Product - given 2 vectors can get a "c#"

$\langle\alpha|\beta\rangle$

Properties

1. $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
2. $\langle\alpha|\alpha\rangle \geq 0$
3. $\langle\beta|(c_1|\alpha_1\rangle + c_2|\alpha_2\rangle) = c_1\langle\beta|\alpha_1\rangle + c_2\langle\beta|\alpha_2\rangle$

$|\alpha\rangle$ + $|\beta\rangle$ are <u>orthogonal</u> if $\langle\alpha|\beta\rangle = 0$

States can be <u>normalized</u> $\langle\alpha|\alpha\rangle = 1$

<u>Operators</u> $X|\alpha\rangle = |\alpha'\rangle$ In general, not commutative $XY \neq YX$

Product $(YX)|\alpha\rangle = Y(X|\alpha\rangle)$

But they <u>are</u> associative. $\langle\alpha'| = \langle\alpha| X^\dagger$

$X^\dagger \neq X$ in general
Hermitian if so

System characterized by single observable A
eg: Position / Momentum / Energy

Measurement of A gives possible values

$a_1, a_2, \ldots$

$|a\rangle$ = State for which A has value a

$$\sum_a |a\rangle\langle a| = 1 \qquad \langle a|a'\rangle \; \delta_{aa'}$$

Any physical observable corresponds to an operator like

$$A = \sum_{a'} a' |a'\rangle\langle a'|$$

$$\boxed{\begin{aligned} A|a\rangle &= \left(\sum a'|a'\rangle\langle a'|\right)|a\rangle \\ &= a|a\rangle \end{aligned}}$$

<u>A is Hermitian</u>

$$A^\dagger = \sum_a a^* |a\rangle\langle a| = \sum a|a\rangle\langle a| = A$$

B/c a is real

Physical Observables are Hermitian Operators

Probabilities

Consider a filter $M(a) = |a\rangle\langle a|$ on general states $|s\rangle$

$$M(a)|s\rangle = |a\rangle\langle a|s\rangle$$
$$= \langle a|s\rangle |a\rangle$$

$\hookrightarrow$ c # $\rightarrow$ tells you something about what fraction of the time you get through

$\langle a|s\rangle$ related to pass fraction

* But, a) not real b) not normalized

However, $|\langle a|s\rangle|^2 = \langle a|s\rangle\langle s|a\rangle = \langle s|a\rangle\langle a|s\rangle$

1. real
2. normalized $\sum_a \langle s|a\rangle\langle a|s\rangle = \langle s|s\rangle = 1$

Interpretation

$|\langle a|s\rangle|^2 =$ Probability that a system prepared in state $|s\rangle$ will be found in state $|a\rangle$ with value a for an observable A after measurement.

~~Example~~

Comments on the measurement Problem

Position Operator

$$\vec{X}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}\rangle$$

Position operator — Position eigenvalue

Completeness & Orthogonality

$$\sum_a |a\rangle\langle a| = 1 \Rightarrow \int d^3x\, |x\rangle\langle x| = \mathbb{1}$$

$$\langle a'|a\rangle = \delta_{aa'} \Rightarrow \langle x|x'\rangle = \delta^3(\vec{x}-\vec{x}')$$

Wavefunction

$$|\psi\rangle = \int d^3x\, |x\rangle\langle x|\psi\rangle = \int d^3x\, \psi(x)\,|x\rangle$$

*where $\psi(x) = \langle x|\psi\rangle$
Position space wavefunction

$$\langle\psi|\psi\rangle = 1 \Rightarrow \int d^3x\, \psi(x)\langle\psi|x\rangle = \int d^3x\, \psi^*(x)\psi(x) = \underline{1}$$

Interpretation $|\psi(x)|^2 d^3x$ = Probability to find particle in volume d^3x around $\vec{x}$.

Translation Operator "operator that moves you over"

$$T(\vec{a})|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$$

What is $T^\dagger(\vec{a})$?

$$\langle\vec{x}'|\Big(T(\vec{a})|\vec{x}\rangle\Big) = \delta^3((\vec{x}+\vec{a})-\vec{x}')$$

or

$$\Big(\langle\vec{x}'|T(\vec{a})\Big)|\vec{x}\rangle = \delta^3(\vec{x}-(\vec{x}'-\vec{a}))$$

$$\Rightarrow \langle x'|T(a) = \langle x'-a|$$

* which says that $T^\dagger(a)|\vec{x}'\rangle = |\vec{x}'-a\rangle$

$T^{\dagger}(\vec{a}) = T(-a) = T^{-1}(\vec{a})$ $\quad T^{\dagger}(\vec{a})\,T(\vec{a}) = \mathbb{1}$

Properties of T

1. Unitary $T^{\dagger}T = 1$
2. $T(\vec{a})\,T(\vec{b}) = T(\vec{a}+\vec{b}) = T(\vec{b})\,T(\vec{a})$
   $\Rightarrow [T(\vec{a}), T(\vec{b})] = 0$
3. $T(0) = \mathbb{1}$

Infinitesimal Translations

$\vec{a} = N\vec{\varepsilon}$

$T(\vec{\varepsilon}) = \mathbb{1} - i\,\vec{\varepsilon}\cdot\vec{K}$

3-vector, Operator $\vec{K} = (K_x, K_y, K_z)$

$T^{\dagger}T = \mathbb{1}$ or $(1 + i\,\vec{\varepsilon}\cdot\vec{K}^{\dagger})(1 - i\,\vec{\varepsilon}\cdot K) = \mathbb{1}$

$1 + i\,\vec{\varepsilon}\,\underbrace{(\vec{K}^{\dagger} - \vec{K})}_{=0} + O(\varepsilon^2) = \mathbb{1}$

$\Rightarrow K^{\dagger} = K$ $\quad K$ hermitian

* if the i wasn't there $T(\vec{a})$ would not be hermitian

Just like with SR.

Any finite translation can be built out of infinitesimal translations

* K is "generator" of translations.

$$T(\vec{a}) = \lim_{N \to \infty} \left(1 - i\,\vec{\varepsilon}\cdot\vec{K}\right)^N = \lim_{N\to\infty}\left(1 - i\,\frac{\vec{a}}{N}\cdot K\right)^N$$

$\vec{a}$ → finite

$$e^{-i\vec{k}\cdot\vec{a}}$$

$\vec{a} = N\vec{\varepsilon}$

T - unitary $\quad$ K - hermitian

Eigenstates of $\vec{K}$ (and automatically of T)

$$\vec{K}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle$$

$$T(\vec{a})|\vec{k}\rangle = e^{-i\vec{k}\cdot\vec{a}}|\vec{k}\rangle$$

Momentum Operator

$\vec{P} = \hbar\vec{K}$

what is $\psi_{\vec{k}}(x) = \langle \vec{x}|\vec{k}\rangle$?

$$\langle x|T(\vec{a})|\vec{k}\rangle = e^{-ika}\langle x|k\rangle = e^{-ika}\psi_k(x)$$

T on x $\quad = \langle x-a|\vec{k}\rangle = \psi_k(\vec{x}-\vec{a})$

<u>or</u> $\quad \psi_k(\vec{x}-\vec{a}) = e^{-i\vec{k}\vec{a}}\,\psi(x)$