(1)

Recap S.R. / Lorentz Invariance

$$(t, x) \xrightarrow[\text{observer}]{\text{another moving}} (t', x')$$

Invariant notion of distance

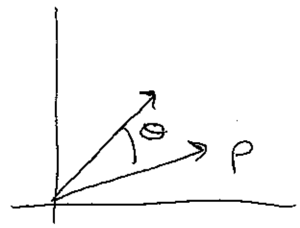
$$t^2 - x^2 = t'^2 - x'^2$$

(this should all be familiar to you)

We will recap this in an adult way ...

Start w/ rotations

Have an invariant (length of $\vec{p}$)



$$x^2 + y^2 = x'^2 + y'^2$$

$$x' = x \cos\theta - y \sin\theta$$
$$y' = y \cos\theta + x \sin\theta$$

Look at this another way ...

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x'^2 + y'^2 = x^2 + y^2$$

After the set of all such matrices that have this property

Start w/ infinitesimal case

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

③ $$(x' \;\; y')\begin{pmatrix} x' \\ y' \end{pmatrix} = (x, y)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = x + \varepsilon a x + \varepsilon b y$$
$$y' = y + \varepsilon c x + \varepsilon d y$$

Keep terms linear in ε

$$x'^2 + y'^2 = x^2 + y^2 + 2\varepsilon(\underbrace{ax^2 + bxy + cxy + dy^2}_{=0 \quad \forall\, x \,\&\, y})$$

$$ax^2 + (b+c)xy + dy^2 = 0$$

$$\Rightarrow a = d = 0 \quad \& \quad b = -c$$

Can rescale ε such that $b = 1$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[\underline{\underline{1}} + \varepsilon\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix}$$

<u>More Sophisticated way:</u>

$x_i \qquad i = 1, 2 \qquad x_1 \equiv x \quad x_2 \equiv y$

$$x_i' = R_{i1} x_1 + R_{i2} x_2 = \sum_{j=1}^{2} R_{ij} x_j = R_{ij} x_j$$

$$x_i' = R_{ij} x_j$$

$$x_i' x_i' = x_i x_i$$

This is what it takes for R to be a rotation.
Need to find special R's such that this is satisfied

④ Identity Matrix

$R_{ij} = \delta_{ij}$ if no rotation @ all.

$$R_{ij} = \delta_{ij} + \varepsilon \omega_{ij}$$

$$x_i' = x_i + \varepsilon \omega_{ij} x_j$$

$$x_i' x_i' = x_i x_i + 2\varepsilon \underbrace{\omega_{ij} x_j x_i}_{=0 \ \forall x} + \mathcal{O}(\varepsilon^2)$$

$\Rightarrow$ ω_{ij} has to be anti-symmetric. $\omega_{ij} = -\omega_{ji}$

Back to previous example, find <u>finite</u> rotations (without any mention of geometry etc)

take infinitesimal rotation & do it n-times

$$\Theta = N\varepsilon \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \Big[\underline{\underline{1}} + \varepsilon \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \Big] \begin{pmatrix} x \\ y \end{pmatrix}$$

call the "I" for the moment

$$(1 + \varepsilon I)(1 + \varepsilon I) \dots (\quad) \dots)$$

$$(1 + I\varepsilon)^N = \left(1 + I\frac{\Theta}{N}\right)^N$$ Now let $N \to \infty$

$$\hookrightarrow e^{I\Theta}$$

Build up finite rotation $R(\Theta) = e^{I\Theta}$

$$x'(\theta) = R(\theta)\, x = e^{I\theta} x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

lets calculate the first terms

$$I^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1}!$$

Just discovered i (following our nose)

$$R(\theta) = \cos\theta + I \sin\theta$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Strategy: first understand the action of the symmetry infinitesimally, then the big symmetry action is obtained by iterating the infinitesimal

Always e^x

Great Strategy for any kind of symmetry.

Now 3D Rotations

Something New happens

(6) 3-parameters associated w/ 3D rotation

Already saw, any rotation is of the form

$$x_i' = x_i + \varepsilon\, \omega_{ij} x_{ij} \quad \text{w/ } \omega_{ij} = -\omega_{ji}$$

Most general 3×3 anti-symmetric matrix

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \qquad \varepsilon\, \omega_{ij} = \varepsilon_{12}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{generator that we just saw}} + \varepsilon_{13}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \varepsilon_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Now have 3 generators corresponding to rotation in 3D

How to get the finite version? Easy just exponentiate

Something new happens in 3D

2D rotations commute.

3D " do not

$$\varepsilon_{12} T_{12} + \varepsilon_{13} T_{13} + \varepsilon_{23} T_{23} \rightarrow i\varepsilon_3 J_3 + i\varepsilon_2 J_2 + i\varepsilon_1 J_1$$

Rotations form a group $\quad e^{\theta_{12}T_{12}}\, e^{\theta_{13}T_{13}}\, e^{{}_{23}T_{23}} = e^{\phi_{12}T_{12} + \phi_{23}T_{23} + \phi_{23}T_{23}}$

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$e^{i\theta_3 J_3} \qquad [J_1, J_2] = i J_3 \quad + \text{cyclic}$$

⑦ Can step back + think about this more abstractly

Matrices we found form a group, but this group exists abstractly independent of these 3x3 matrices. Fully determined by the comm. relations.

(Just like vectors + components)

In general, many matrices that satisfy the algebra. those give different representations

<u>Deep</u> Rotations can act on more than just 3D vectors

$$\begin{bmatrix} (J_i) & 0 \\ 0 & (J_i) \end{bmatrix}$$

officially a representation

"Reducible" Representation

<u>More Generally</u>

$$[J^a, J^b] = i f_{abc} J^c \qquad J^a \quad a = 1, 2 \ldots \text{dim of group}$$

Lie found all the possible symmetries when J is hermitian (there are not many)

One final example w/ Rotations →

⑧ Rotations

Traceless 2x2 hermitian matrices

σ - pauli matrices

$$M = \begin{pmatrix} z & x+iy \\ -x-iy & -z \end{pmatrix} = \vec{\sigma} \cdot \vec{x}$$

$$M' = U^{\dagger} M U$$

U is unitary, any unitary matrix Can be written as a phase $e^{i\theta}$ times 2x2 hermitian matrix w/ det = 1 ("Special Unitary Matrix")

U - unitary + det(1)

$U \in SU(2)$ ← Do this B/c phase cancels

M' - still hermitian, still traceless

$\Rightarrow \quad M' = \vec{\sigma} \cdot \vec{x}_u'$ ← this $\vec{x}$ depends on U

$$\mathrm{Det}(M) = -z^2 - x^2 - y^2 = -\vec{x}^2$$

$$\mathrm{Det}(M') = \mathrm{Det}(M) \qquad \vec{x}_u'^2 = \vec{x}^2$$

→ direct correspondence between 2x2 hermitian matrices + Rotations

2D action of rotations

Now easy to generalize all of this to Lorentz group