Lecture 17

Lorentz Invariance and "Soft Limits"

Punch line that we've been building to in first part of this course.

Matrix element we would get by scattering external γ.

$$M = \epsilon^{\mu} M_{\mu}$$

where ϵ^{μ} is some linear combination of two photon polarization vector ϵ^1 and ϵ^2

M is Lorentz Invariant, under Lorentz transformation

$$M \to \epsilon'^{\mu} M'_{\mu}$$

where $M'_{\mu} = \Lambda_{\mu}{}^{\nu} M_{\nu}$

However (here comes the major constraint) ϵ is not a full 4-vector. Only has 2 components.

Under little group transformations (you will show in your H.W.)

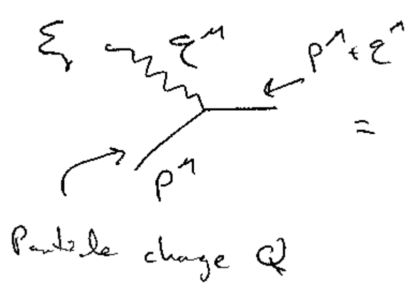
$$\epsilon \to \underbrace{c_1\epsilon_1^{\mu} + c_2\epsilon_2^{\mu}}_{\substack{\epsilon' \text{ can only be made} \\ \text{of these pieces}}} + \underbrace{c_3 p^{\mu}}_{\substack{\text{Not valid} \\ \text{"Not in Hilbert Space"}}}$$

So,

$$\begin{aligned} M = \epsilon^{\mu} M_{\mu} &\to \left(c_1\epsilon_1^{\mu} + c_2\epsilon_2^{\mu} + c_3 p^{\mu}\right) M'_{\mu} \\ &= \epsilon'^{\mu} M'_{\mu} + \underbrace{c_3 p^{\mu} M'_{\mu}}_{\text{Must go to 0}} \end{aligned}$$

We will see, this has enormous implications !!!

Will be considering diagrams with external "γ"s (mass-less spin 1 particles)



$$\begin{aligned} = \quad &= iQ(p^{\mu} + (p^{\mu} + q^{\mu}))\epsilon_{\mu} \qquad (q^{\mu}\epsilon_{\mu} = 0) \\ &= iQ2p^{\mu}\epsilon_{\mu} \end{aligned}$$

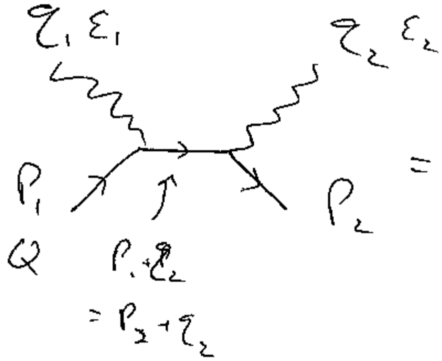
This is the most general form in the “soft limit” $q \to 0$

$$\boxed{\begin{array}{c} \Gamma_\mu \sim p_\mu F(q^2, p^2, p \cdot q) \\ \text{By dimensional analysis } F(q^2, p^2, p \cdot q) \to F(\frac{p \cdot q}{m^2}) \end{array}}$$

Consider “Compton Scattering”

Start with one type of spin-1 boson and one type of matter particle.

The diagram:



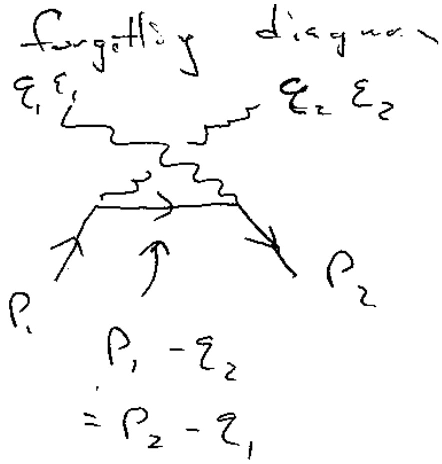
$$\begin{aligned} = &\; (iQ)\epsilon^1_\mu(2p_1^\mu)\frac{i}{(p_1+q_1)^2 - m^2}(iQ)\epsilon^2_\nu(2p_2^\nu) \\ = &\; (-iQ^2)4\frac{(p_1 \cdot \epsilon^1)(p_2 \cdot \epsilon^2)}{m^2 + 2p_1 \cdot q_1 - m^2} = \epsilon_1^\mu \epsilon_2^\nu \underbrace{\left(\frac{(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1 \cdot q_1}\right)}_{M_{\mu\nu}} \end{aligned}$$

As we said above, Lorentz Invariant $\Rightarrow q_1^\mu q_2^\nu M_{\mu\nu} = 0$

But here, $q_1^\mu q_2^\nu M_{\mu\nu} = (-iQ^2)2(p_2 \cdot q_2) \neq 0$!

Looks like we’re dead...

However we are forgetting a diagram.



$$
\begin{aligned}
&= \quad (iQ)\epsilon^1_\mu(2p_2^\mu)\frac{i}{(p_2-q_1)^2-m^2}(iQ)\epsilon^2_\nu(2p_1^\nu) \\
&= \quad \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iQ^2)4p_{2\mu}p_{1\nu}}{-2p_2\cdot q_1}\right) \\
&\underbrace{\sim}_{\text{Soft limit } p_1 = p_2} \quad \epsilon_1^\mu\epsilon_2^\nu\left(\frac{-(-iQ^2)2p_{1\mu}p_{2\nu}}{p_1\cdot q_1}\right)
\end{aligned}
$$

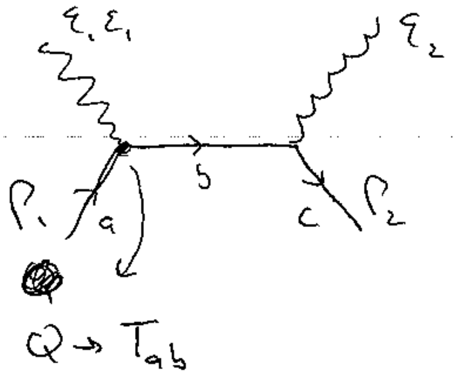
and for this diagram, $q_1^\mu q_2^\nu M_{\mu\nu} = -(-iQ^2)2(p_2\cdot q_2)$

So the sum $M^A_{\mu\nu} + M^B_{\mu\nu}$ is Lorentz Invariant. (Residual non Lorentz Invariant pieces of each diagram cancel)

Very good!

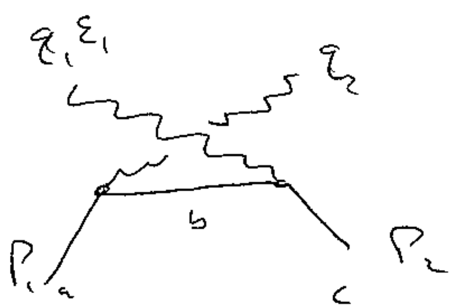
Now lets do the same thing as before, but with many different possible matter particles.

$i = 1, ...N_{\text{matter}}$



$$
\begin{aligned}
&= (iT_{ab})\epsilon^1_\mu(2p_1^\mu)\frac{i}{\underbrace{(p_1+q_1)^2-m^2}_{m_a^2+2p_1\cdot q_1-m_b^2}}(iT_{bc})\epsilon^2_\nu(2p_2^\nu) \\
&= \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iT_{ab}T_{bc})4p_{1\mu}p_{2\nu}}{m_a^2+2p_1\cdot q_1-m_b^2}\right) \equiv M_A^{\mu\nu}
\end{aligned}
$$

Other diagram:



$$
= \epsilon_1^\mu\epsilon_2^\nu\left(\frac{(-iT_{ab}T_{bc})4p_{2\mu}p_{1\nu}}{m_a^2-2p_2\cdot q_1-m_b^2}\right) \equiv M_B^{\mu\nu}
$$

Now, if $m_a = m_b = m_c$ then, $q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = 0$ as above | $m_a^2 - m_b^2 = 0$ relative - size

However if $m_a \neq m_b$, in soft limit

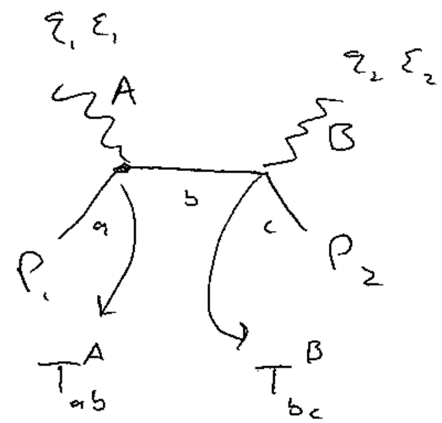
$q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) = -\left[\frac{(-iT_{ab}T_{bc})4}{m_a^2-m_b^2}(2p_1^\mu p_1^\nu)\right] \neq 0$

Mass-less spin-1 particles can only interact with particles of the same mass!

Now allow many different | matter fields (but same mass!) | and many force carriers "gluons"

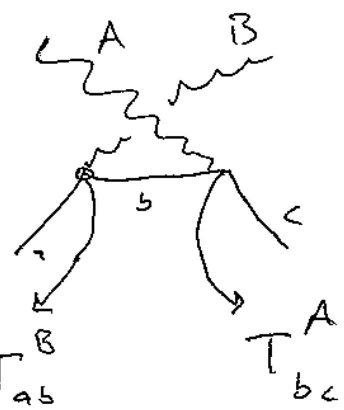
$i = 1, \ldots N_{\text{matter}}$

$I = 1, \ldots N_{\text{gluons}}$



$$= \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT^A_{ab}T^B_{bc})4p_{1\mu}p_{2\nu}}{2p_1 \cdot q_1} \right) \equiv M_A^{\mu\nu}$$

and



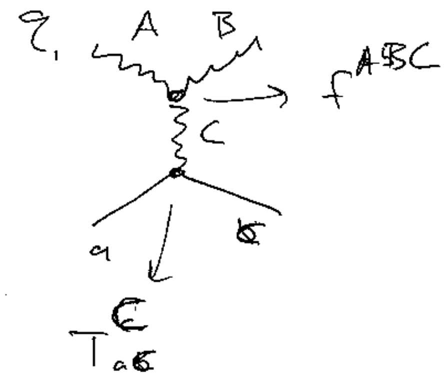
$$\underbrace{=}_{\text{"soft limit"}} \epsilon_1^\mu \epsilon_2^\nu \left(\frac{(-iT^B_{ab}T^A_{bc})4p_{1\mu}p_{2\nu}}{-2p_1 \cdot q_1} \right) \equiv M_B^{\mu\nu}$$

Now,

$$\begin{aligned} q_{1\mu}q_{2\nu}(M_A^{\mu\nu} + M_B^{\mu\nu}) &= 2(-i)(p_2 \cdot q_2)(T^A_{ab}T^B_{bc} - T^B_{ab}T^A_{bc}) \\ &= 2(-i)(p_2 \cdot q_2)[T^A, T^B] \end{aligned}$$

$[T^A, T^B]$ not 0 for random Ts.

In fact, another diagram we are missing:



$$= +2i(q \cdot q) i f^{ABC} T^{C}_{ac}$$

Sum of all three only Lorentz invariant if

$$[T^A, T^B] = i f^{ABC} T^C$$

"gluons" (or any other group of interacting mass-less spin-1 particles) must transform as a Lie group !

Only question is which group, there are only a finite handful of possibilities

"Yang-Mills" Interaction.