Lecture 6

Review Quantum Mechanics

QM Linear Algebra in a complex vector space.

State of a system is a vector (ray) in the Complex Vector Space.

$$|\alpha\rangle \text{ - state vector}$$

Linear Superpositions

$$|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle$$

If the $|\alpha' s\rangle$ are vectors and the c's are complex numbers, then $|\psi\rangle$ is another vector in the space.

Dual Space:

For every vector $|\alpha\rangle$ ("ket") there is another vector $\langle\alpha|$ ("bra") in a "dual" space.

Dual space is a mirro image of the ket space.

If, $|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle$, then $\langle\psi| = c_1^* \langle\alpha_1| + c_2^* \langle\alpha_2|$ where the c^* is the complex conjugate.

Inner Product: Dual space allows us to define an inner product between to vectors.

Given 2 vectors can get a "c#"

$\langle\alpha|\beta\rangle$

Properties

1. $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
2. $\langle\alpha|\alpha\rangle \geq 0$
3. $\langle\beta| (c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle) = c_1 \langle\beta|\alpha_1\rangle + c_2 \langle\beta|\alpha_2\rangle$

$|\alpha\rangle$ and $|\beta\rangle$ are orthogonal if $\langle\alpha|\beta\rangle = 0$.

States can be normalized $\langle\alpha|\alpha\rangle = 1$.

Operators: Thigs that act on a given state, and return another state

$X|\alpha\rangle = |\alpha'\rangle$

Product:

$(YX)|\alpha\rangle = Y(X|\alpha\rangle$

In general, not commutative: $XY \neq YX$, but they are associative.

$\langle\alpha| = \langle\alpha'|X^\dagger$

In general, $X^\dagger \neq X$, if so X is said to be "Hermitian".

System characheterized by single observable A

eg: position / Momentum / energy

Measuremnet of A gives possible values

$$a_1, a_2, ...$$

$|\alpha\rangle$ = State for which A has a value of a

$$\sum_{a'} |\alpha'\rangle\langle\alpha'| = 1$$

$$\langle\alpha|\alpha'\rangle = \delta_{aa'}$$

Any physical observable corrisponds to an operator like

$$A = \sum_{a'} a' |\alpha'\rangle\langle\alpha'|$$

eg:

$$A|\alpha\rangle = \left(\sum_{a'} a' |\alpha'\rangle\langle\alpha'|\right)|\alpha\rangle = a|\alpha\rangle$$

Physical observables are real numbers, therefore physical operators A are Hermitian

proof:

$$A^\dagger = \sum_a a^* |\alpha\rangle\langle\alpha| = \sum_a a|\alpha\rangle\langle\alpha| = A$$

Probabilities:

Condsider a filter $M(a) = |a\rangle\langle a|$ on a general state $|s\rangle$.

$$\begin{aligned} M(a)|s\rangle &= |a\rangle\langle a|s\rangle \\ &= \langle a|s\rangle|a\rangle \end{aligned}$$

where $\langle a|s\rangle$ is a c# that tells you something about what fraction of the time you get through.

$\langle a|s\rangle$ is related to the pass fraction

*But, a) not real, b) not normalized

However, we know that $|\langle a|s\rangle|^2 = \langle a|s\rangle\langle s|a\rangle = \langle s|a\rangle\langle a|s\rangle$ is both real and normalized.

$$\sum_a \langle s|a\rangle\langle a|s\rangle = \langle s|s\rangle = 1$$

Interpretation

$|\langle a|s\rangle|^2$ = Probability that a system prepared in state $|s\rangle$ will be found in a state $|a\rangle$ with value a for observable A after measurement.

Comments on the measurement problem....

Position Operator

$$\vec{X}\,|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

where, $\vec{X}$ is position operator and $\vec{x}$ is position eigenvalue.

Position of course is a continous observable so "sums go to intergals" etc.

eg: (Completnees and Orthogonality)

$$\sum_{\alpha'} |\alpha'\rangle\,\langle\alpha'| = 1 \Rightarrow \int d^3x\,|x\rangle\,\langle x| = 1$$

$$\langle\alpha'|\alpha'\rangle = \delta_{\alpha,\alpha'} \Rightarrow \langle\vec{x}|\vec{x}'\rangle = \delta^3(\vec{x}-\vec{x}')$$

Wave function

$$|\psi\rangle = \int d^3x\,|x\rangle\,\langle x|\psi\rangle = \int d^3x\psi(x)\,|x\rangle$$

where $\psi(x) = \langle x|\psi\rangle$ is called the Position-space wavefunction.

$$\langle\psi|\psi\rangle = 1 \Rightarrow \int d^3x\psi(x)\,\langle\psi|x\rangle = \int d^3x\psi^*(x)\psi(x) = 1$$

Interpretation

$|\psi(x)|^2 d^3x$ is the probability to find the particle in volume d^3x around $\vec{x}$.

Translation Operator "The operator that moves you over"

$$T(\vec{a})\,|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$$

What is $T^\dagger(\vec{a})$???

Well,

$$\langle\vec{x}'|\left(T(\vec{a})\,|\vec{x}\rangle\right) = \delta^3((\vec{x}+\vec{a})-\vec{x}')$$

or

$$\left(\langle\vec{x}'|\,T(\vec{a})\right)|\vec{x}\rangle = \delta^3(\vec{x}-(\vec{x}'-\vec{a}))$$

$$\Rightarrow \langle x'|\,T(a) = \langle x-a|$$

which says that $T^\dagger(a)\,|\vec{x}\rangle = |\vec{x}-\vec{a}\rangle$

So,

$$T^\dagger(\vec{a}) = T(-\vec{a}) = T^{-1}(\vec{a})$$

Properties of T

1. Unitary $T^\dagger T = 1$
2. $T(\vec{a})T(\vec{b}) = T(\vec{a}+\vec{b}) = T(\vec{b})T(\vec{a})$, Translations commute ($[T(\vec{a}), T(\vec{b})] = 0$)
3. $T(0) = 1$

Infintesimal Translations

consider $\vec{a} = N\vec{\epsilon}$