

## Homework Set #2

**Due Date:** Friday February 11th

### 1) Work out the commutation relation among the $\vec{X}$ and $\vec{P}$ operators:

ie:  $[\vec{X}, \vec{X}]$ ,  $[\vec{P}, \vec{P}]$ , and  $[\vec{X}, \vec{P}]$

(5 points)

Hint: for  $[\vec{X}, \vec{P}]$  work out how the commutator  $[\vec{X}, T(\vec{d})]$  acts on position eigenstate for generic translation. Then apply the result to the infinitesimal case.

### 2) Harmonic Oscillator

(10 points)

The 1D Harmonic oscillator has Hamiltonian:

$$H = \frac{P^2}{2m} + \frac{1}{2}mw^2X^2$$

where P and X are position and momentum operators

a Define “raising” and “lowering” operators as

$$a = \sqrt{\frac{mw}{2}} \left( X + i \frac{P}{mw} \right) \quad a^\dagger = \sqrt{\frac{mw}{2}} \left( X - i \frac{P}{mw} \right)$$

What are the position and momentum operators in terms of the raising and lowering operators?

b Find  $[a, a^\dagger]$

c What is the Hamiltonian in terms of  $a$  and  $a^\dagger$ ?

d Define the “Number” operator  $N$  as  $a^\dagger a$ . What is the Hamiltonian in terms of the number operator?

e Work out the commutation relations:  $[N, a^\dagger]$  and  $[N, a]$ .

f Show that the eigenvalues of  $N$  ( $n$ ) are real and satisfy  $n \geq 0$ . (Hint: consider  $\langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle$ , where  $|n\rangle$  are eigenkets of  $N$ )

g Show that  $a |n\rangle$  is an eigenstate of  $N$ , with eigenvalue  $(n-1)$ . This implies  $a |n\rangle \propto |n-1\rangle$  and justifies calling  $a$  the lower operator.

h Show that  $a^\dagger |n\rangle$  is an eigenstate of  $N$ , with eigenvalue  $(n+1)$ . This implies  $a^\dagger |n\rangle \propto |n+1\rangle$  and justifies calling  $a^\dagger$  the raising operator.

i Find  $c_n$  such that  $|n+1\rangle = c_n a^\dagger |n\rangle$  is normalized.

j Since  $n \geq 0$ , there must be a state  $|0\rangle$  which satisfies  $a |0\rangle = 0$  and  $n$  must be an integer. What is the general state  $|n\rangle$  in terms of  $|0\rangle$  and  $a^\dagger$ ? What is the energy associated to this state?