1)

Start w/ a_x

$$d\beta_x = \frac{d\beta_x}{1+\beta_x'\beta} - \frac{\beta_x'+\beta}{(1+\beta_x'\beta)^2}\,\beta\, d\beta_x'$$

$$= \left[\frac{1}{1+\beta_x'\beta} - \frac{\beta_x'\beta+\beta^2}{(1+\beta_x'\beta)^2}\right] d\beta_x'$$

$$= \frac{1+\beta_x'\beta-\beta_x'\beta-\beta^2}{(1+\beta_x'\beta)^2}\,d\beta_x' = \frac{1-\beta^2}{(1+\beta_x'\beta)^2}\,d\beta_x'$$

$$= \frac{d\beta_x'}{\gamma^2(1+\beta_x'\beta)^2}$$

$$dt = \beta\gamma\, dx' + \gamma\, dt' = \gamma(\beta\beta_x'+1)\,dt'$$

$$\frac{d\beta_x}{dt} = \frac{1}{\gamma^3(1+\beta_x'\beta)^3}\frac{d\beta_x'}{dt'} = \frac{1-\beta^2}{\gamma(1+\beta_x'\beta)^3}\,a_x'$$

Now a_y

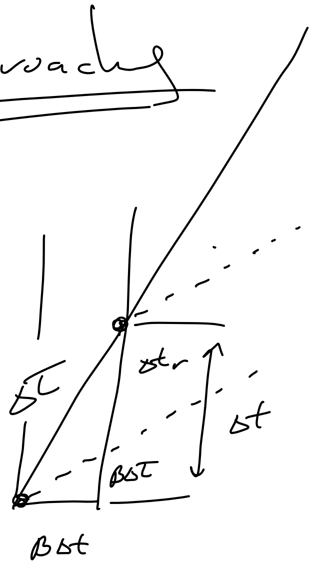
$$d\beta_y = \frac{d\beta_y'}{\gamma(1+\beta_x'\beta)} - \frac{\beta_y'}{\gamma(1+\beta_x'\beta)^2}\,\beta\, d\beta_x'$$

$$= \frac{(1+\beta_x'\beta)\,d\beta_y' - \beta_y'\beta\, d\beta_x'}{\gamma(1+\beta_x'\beta)^2}$$

$$\frac{d\beta_y}{dt} = \frac{(1+\beta_x'\beta)\,d\beta_y' - \beta_y'\beta\, d\beta_x'}{\gamma^2(1+\beta_x'\beta)^3\,dt'} = \frac{a_y'}{\gamma^2(1+\beta_x'\beta)} - \frac{\beta\beta_y' a_x'}{\gamma^2(1+\beta_x'\beta)^3}$$

3 Doppler

Approach



$$\Delta t = \gamma \Delta\tau$$

$$\Delta t_r = \Delta t - \beta\Delta t$$

$$= \Delta t(1-\beta)$$

$$\Delta t_r = \gamma\Delta\tau(1-\beta)$$

$$\frac{\Delta t_r}{\Delta\tau} = \sqrt{\frac{(1-\beta)^2}{(1+\beta)(1-\beta)}} = \sqrt{\frac{1-\beta}{1+\beta}}$$

(4)

$$x = x' \cosh \eta_r + t' \sinh \eta_r$$

$$t = x' \sinh \eta_r + t' \cosh \eta_r$$

$$t^2 - x^2$$

$$= x'^2 \sinh^2 \eta + t'^2 \cosh^2 \eta + 2x't' \sinh \eta \cosh \eta$$

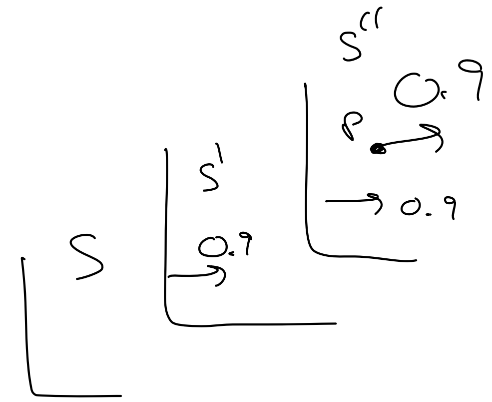
$$- \left(x'^2 \cosh^2 \eta_r + t'^2 \sinh^2 \eta_r + 2x't' \cosh \eta \sinh \eta\right)$$

$$= t'^2 \left(\cosh^2 \eta - \sinh^2 \eta\right) - x'^2 \left(\cosh^2 \eta - \sinh^2 \eta\right)$$

$$\boxed{\text{But } \cosh^2 \eta - \sinh^2 \eta = 1}$$

$$\Rightarrow \; = t'^2 - x'^2$$

5



$$\tanh x = 0.9 \Rightarrow x = 1.47$$

$$x_s = x_p + x_{s''} + x_{s'}$$

$$= 3 \cdot 1.47$$

$$\tanh(x_s) = 0.9997$$

Which agrees w/ last HW

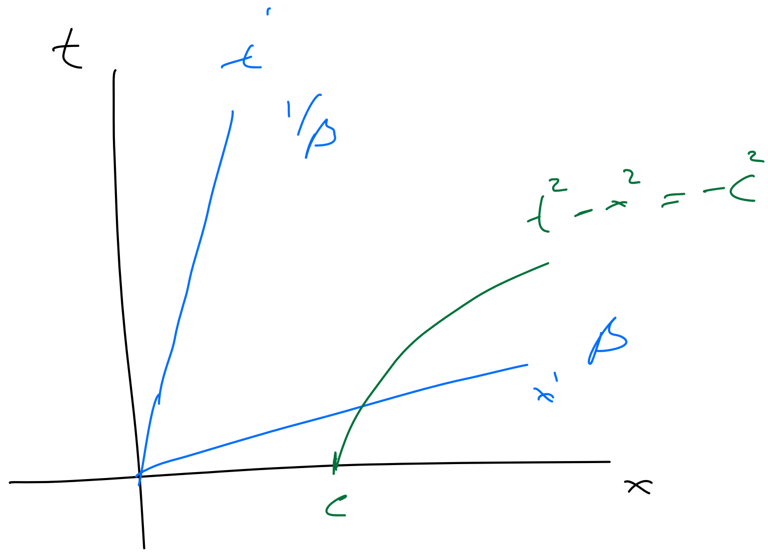
and is much easier!

⑥

$$t^2 - x^2 = -c^2$$

$$\Rightarrow t\,dt - x\,dx = 0$$

or $\dfrac{dt}{dx} = \dfrac{x}{t}$



when the hyperbola intersects the x' axis, $t' = 0$

when $t' = 0 \Rightarrow t = \beta x$

Intersection when both equations satisfied

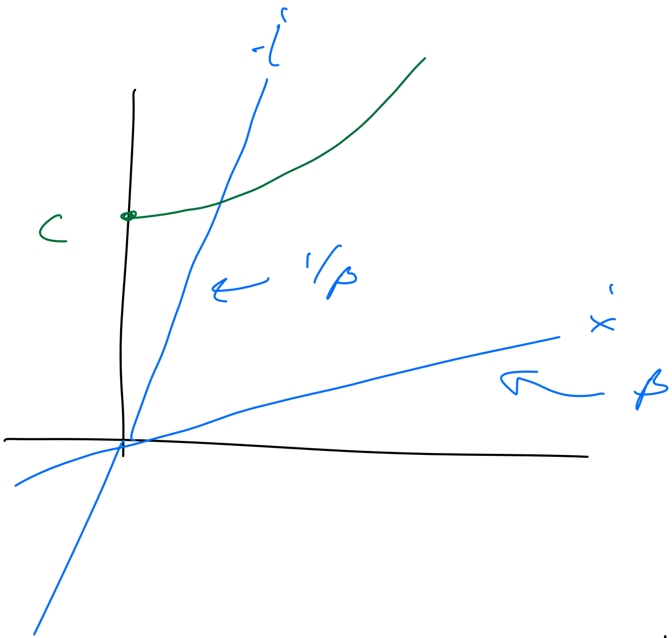
$$(\beta x)^2 - x^2 = -c^2$$

$$-x^2(1-\beta^2) = -c^2$$

or $x = \gamma c$ & $t = \beta\gamma c$

$$\frac{dt}{dx} = \frac{x}{t} = \frac{\gamma c}{\beta\gamma c} = \frac{1}{\beta}$$ which has same slope as t' axis

Bonus Intersection of t' axis also parallel to x' axis



$$t^2 - x^2 = c^2$$

when hyperbola intersects t' axis, $x' = 0$

when $x' = 0$ $\qquad t = \frac{1}{\beta} x$

Intersection $\Rightarrow$ Both satisfied

$$\frac{x^2}{\beta^2} - x^2 = c^2 \qquad \text{or} \qquad \frac{x^2}{\beta^2}(1 - \beta^2) = c$$

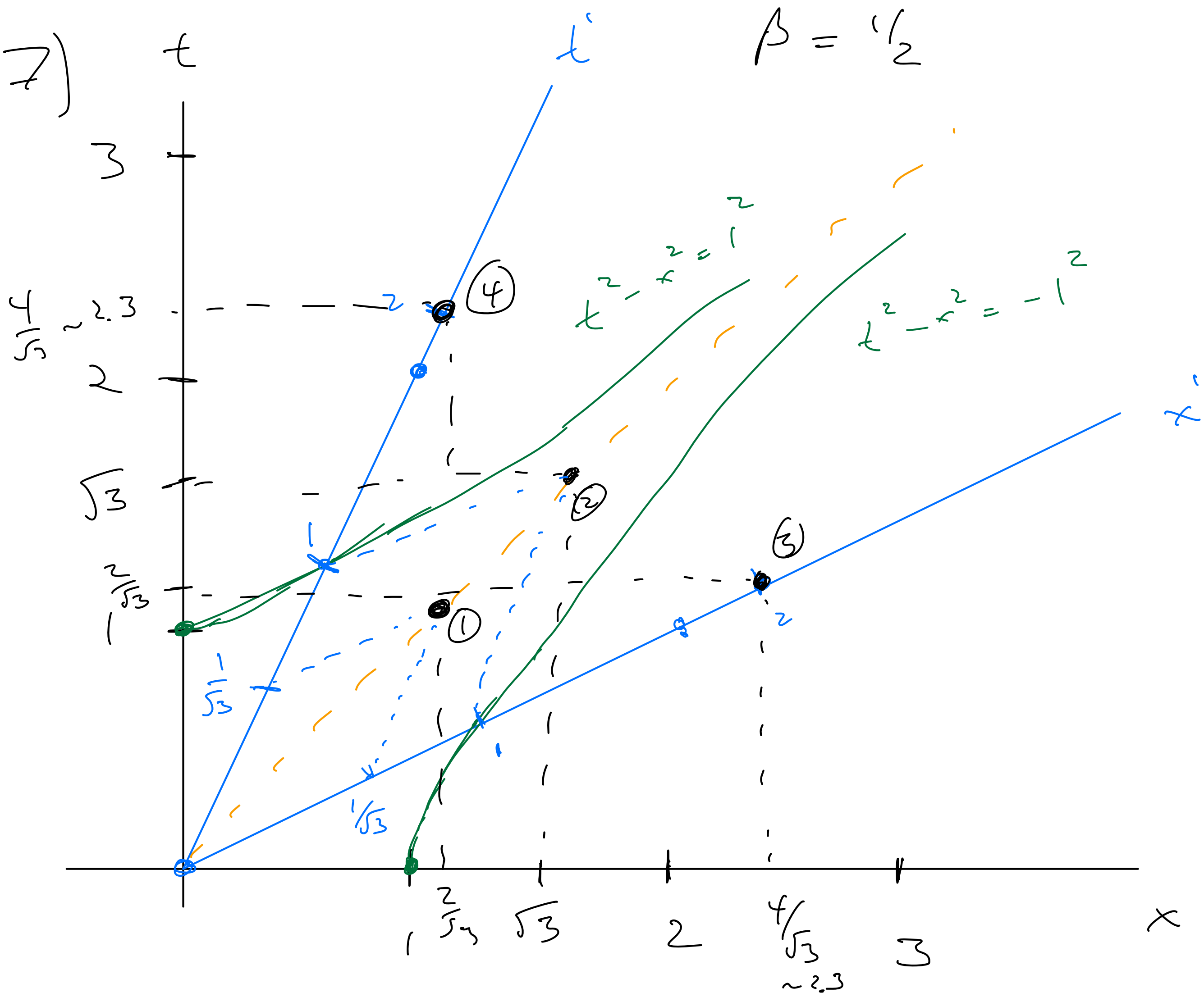
$$x = \beta \gamma c$$

$$\& \quad t = \gamma c$$

$$\frac{dt}{dx} = \frac{x}{t} = \frac{\beta\gamma c}{\gamma c} = \beta$$

$\leftarrow$ Same slope as x-axis.

7)



1) $(x, t) = (1, 1)$

$$\beta = 1/2$$

$$\gamma = \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}}$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x' = \gamma - \beta\gamma = \gamma(1-\beta) = \frac{1}{\sqrt{3}} < 1$$

$$t' = -\beta\gamma + \gamma = \gamma(1-\beta) = \frac{1}{\sqrt{3}} < 1$$

2)
$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x = \gamma(1+\beta) = \frac{2}{\sqrt{3}}\left(\frac{3}{2}\right) = \sqrt{3}$$

$$t = \gamma(1+\beta) = \frac{2}{\sqrt{3}}\ \frac{3}{2} = \sqrt{3}$$

3)
$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$x = 2\gamma = 4/\sqrt{3}$$

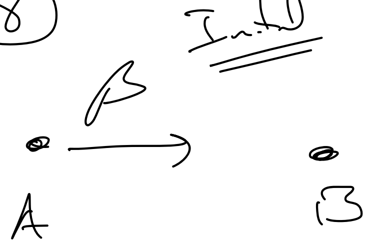
$$t = 2\beta\gamma = \frac{2}{\sqrt{3}}$$

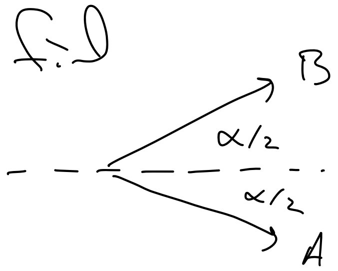
4)
$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$x = 2\beta\gamma$$

$$t = 2\gamma$$

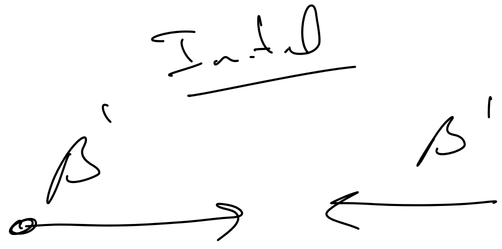
8

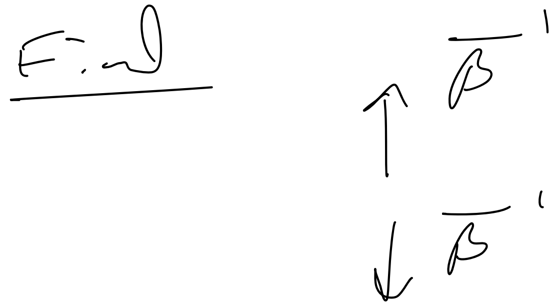




S = lab frame

Define S' frame such that





What is β'? $\beta' = \tanh \eta'$

$\boxed{|\bar{\beta}'| = |\beta'| \text{ By E conservation}}$

$\eta - \eta' = \eta' \Rightarrow \eta' = \frac{\eta}{2}$

$\tanh \eta' = \tanh\left(\frac{\eta}{2}\right) = \frac{\cosh \eta - 1}{\sinh \eta} = \frac{\sqrt{1-\beta^2}}{\beta}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)$

trig ID

$= \frac{1}{\beta}\left(1 - \sqrt{1-\beta^2}\right)$

when $\beta \ll 1$

$\beta' \sim \frac{1}{\beta}\left(1 - \left(1 - \frac{1}{2}\beta^2\right)\right) = \frac{1}{\beta}\frac{1}{2}\beta^2 = \frac{1}{2}\beta$ Makes sense

After Collision

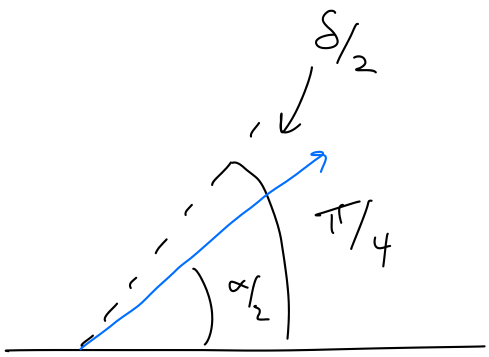
In lab frame

$$\beta^x = \beta'$$

$$\beta^y = \frac{\bar{\beta}'}{\gamma} = \bar{\beta}'\sqrt{1-\beta'^2}$$

$$\tan\frac{\alpha}{2} = \frac{\beta^y}{\beta^x} = \frac{\bar{\beta}'\sqrt{1-\beta'^2}}{\beta'} = \sqrt{1-\beta'^2}$$

$$|\bar{\beta}'| = |\beta'|$$



$$\tan\frac{\delta}{2} = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\alpha}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\alpha}{2}}$$

$$\boxed{\begin{aligned} &\tan\frac{\pi}{4} = 1 \\ &\tan\frac{\alpha}{2} = \sqrt{1-\beta'^2} \end{aligned}} \longrightarrow \quad = \frac{1 - \sqrt{1-\beta'^2}}{1 + \sqrt{1-\beta'^2}}$$

$$\simeq \frac{1 - \left(1 - \frac{1}{2}\beta'^2\right)}{1 + \left(1 - \frac{1}{2}\beta'^2\right)}$$

$$= \frac{\frac{1}{2}\beta'^2}{2} \sim \frac{\beta'^2}{4} \Rightarrow \delta = \frac{\beta'^2}{2}$$

$$\delta \sim 10^{-2} \Rightarrow \beta'^2 \sim \frac{1}{50} \Rightarrow \beta' \sim \frac{1}{7}$$