Two things want to discuss before
moving to dynamics

-) (Brief) Acceleration

-) Relation of L.T to the geometric
Analogy.

$\to$ x' t' what about? $a'_x = \dfrac{d\beta'_x}{dt'}$ $a'_y = \dfrac{d\beta'_y}{dt'}$

β'_x β'_y

Know,

$$\beta_x = \frac{\beta'_x + \beta}{1 + \beta'_x \beta} \qquad \beta_y = \frac{\beta'_y}{\gamma(1 + \beta'_x \beta)}$$

Start w/ a_x

$$d\beta_x = \frac{d\beta'_x}{1+\beta'_x\beta} - \frac{\beta'_x + \beta}{(1+\beta'_x\beta)^2}\,\beta\, d\beta'_x$$

$$= \left[\frac{1}{1+\beta'_x\beta} - \frac{\beta'_x\beta + \beta^2}{(1+\beta'_x\beta)^2}\right] d\beta'_x$$

$$= \frac{1 + \beta'_x\beta - \beta'_x\beta - \beta^2}{(1+\beta'_x\beta)^2}\, d\beta'_x = \frac{1-\beta^2}{(1+\beta'_x\beta)^2}\, d\beta'_x$$

$$= \frac{d\beta'_x}{\gamma^2(1+\beta'_x\beta)^2}$$

$$dt = \beta\gamma\, dx' + \gamma\, dt' = \gamma(\beta\beta'_x + 1)\, dt'$$

$$\frac{d\beta_x}{dt} = \frac{1}{\gamma^3(1+\beta'_x\beta)^3}\frac{d\beta'_x}{dt'} = \frac{1-\beta^2}{\gamma(1+\beta'_x\beta)^3}\, a'_x$$

Now a_y

$$d\beta_y = \frac{d\beta_y'}{\gamma(1+\beta_x'\beta)} - \frac{\beta_y'}{\gamma(1+\beta_x'\beta)^2}\,\beta\, d\beta_x'$$

$$= \frac{(1+\beta_x'\beta)\,d\beta_y' - \beta_y'\beta\, d\beta_x'}{\gamma(1+\beta_x'\beta)^2}$$

$$\frac{d\beta_y}{dt} = \frac{(1+\beta_x'\beta)\,d\beta_y' - \beta_y'\beta\, d\beta_x'}{\gamma^2(1+\beta_x'\beta)^3\, dt'} = \frac{a_y'}{\gamma^2(1+\beta_x'\beta)} - \frac{\beta\beta_y' a_x'}{\gamma^2(1+\beta_x'\beta)^3}$$

Not invariant

$\boxed{\text{Compare } a = a' \text{ w } x = x' + vt \text{ GT}}$

Expressions complicated

Different components transform in very different ways.

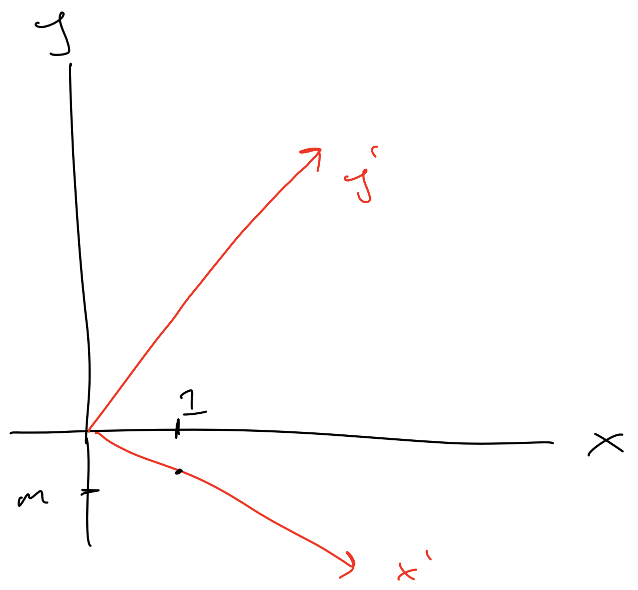
Main lesson a_x a_y a_x' a_y' have limited/questionable value in relativity.

Relation of the L.T. to the Geometric Analogy

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

will try to get same intuition about them and see that there is an alternative way of thinking about them.

How does this relate to Euclidean Analogy?



Characterize orientation by m, slope of x'-axis in x-y frame

$$x'\big|_s = mx$$

$$\hat{x}'\big|_s \propto \begin{pmatrix} 1 \\ m \end{pmatrix} = \frac{1}{\sqrt{1+m^2}} \begin{pmatrix} 1 \\ m \end{pmatrix} \qquad \hat{y}'\big|_s \propto \begin{pmatrix} -m \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1+m^2}} \begin{pmatrix} -m \\ 1 \end{pmatrix}$$

$$\equiv G \begin{pmatrix} 1 \\ B \end{pmatrix} \qquad \hat{x}' \cdot \hat{y}' = 0 \qquad = G \begin{pmatrix} -B \\ 1 \end{pmatrix}$$

So a vector w/ coordinates $\begin{pmatrix} a \\ b \end{pmatrix}$ in S' frame

Appears as $a\,\hat{x}'\big|_s + b\,\hat{y}'_s$ in S frame

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} G & -BG \\ BG & G \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad G = \frac{1}{\sqrt{1+B^2}}$$

Compare

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

The velocity Parameter

- Found the transformation relating the "covariant" coordinates

$$LT \ (x-t) \qquad + \qquad \text{Rotation} \ (x-y)$$

in terms of $\qquad \beta \qquad\qquad\qquad\qquad B \equiv m$

relative velocity $\qquad\qquad\qquad$ relative slope x-axis

Neither of these descriptions provides simplest description of the relation between coordinate systems!

What is the better way?

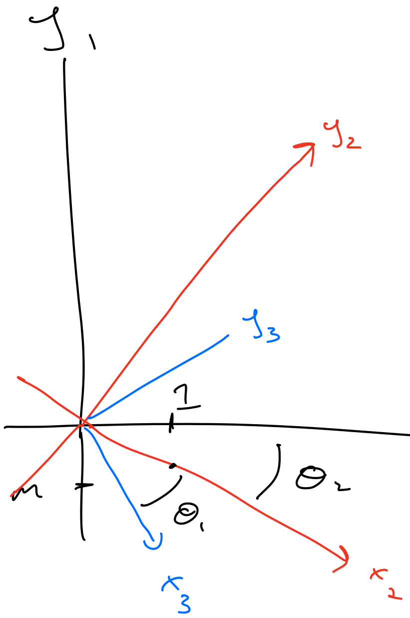
Angle Θ is the best measure of rotation

will see there is an analogue for L.T.

Euclidean Example:

Why are Angles better?

A: Angles are <u>additive</u>, slopes are not.



$$\hat{x}_3\Big|_{S_2} = \frac{1}{\sqrt{1+m_{23}}}\begin{pmatrix}1\\ m_{23}\end{pmatrix}$$

$$\begin{pmatrix}x^2\\ y^2\end{pmatrix} = \begin{pmatrix}G_{23} & -B_{23}G_{23}\\ B_{23} & G_{23}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

"$\hat{x}$" Slope in S^1 is B_{23}

$$\boxed{\text{Sanity check: } \quad \frac{y}{x} = \frac{B_{23}G_{23}}{G_{23}}}$$

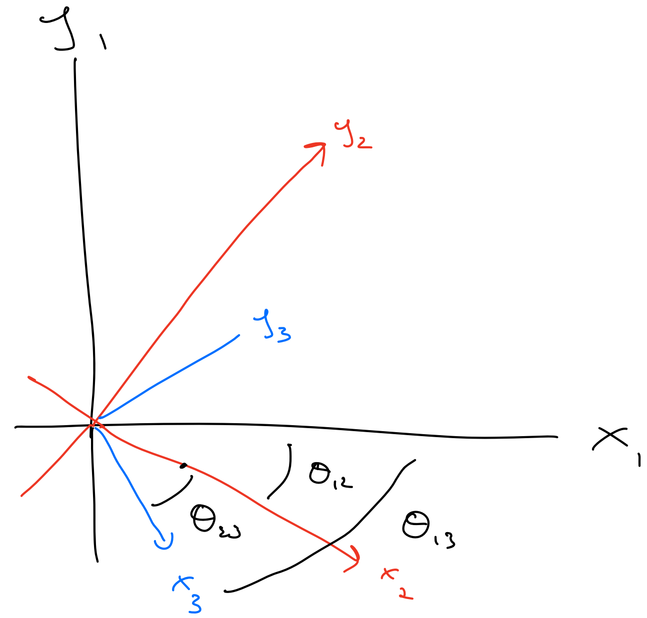
"$\hat{x}$" Slope in S

$$\begin{pmatrix}x\\ y\end{pmatrix} = \underbrace{\begin{pmatrix}G_{12} & -B_{12}G_{12}\\ B_{12}G_{12} & G_{12}\end{pmatrix}}_{\hat{x}^3:\, S^2\to S^1}\underbrace{\begin{pmatrix}G_{23} & -B_{23}G_{23}\\ B_{23}G_{23} & G_{23}\end{pmatrix}}_{\hat{x}^3:\, S^3\to S^2}\begin{pmatrix}1\\ 0\end{pmatrix}$$

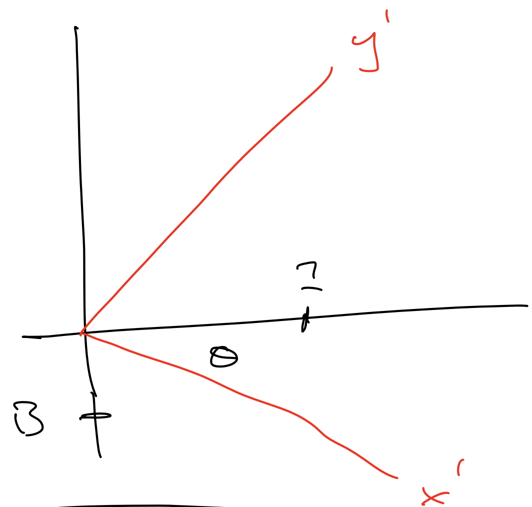
$$= G_{12}G_{23}\begin{pmatrix}1 & -B_{12}\\ B_{12} & 1\end{pmatrix}\begin{pmatrix}1\\ B_{23}\end{pmatrix} = G_1 G_2\begin{pmatrix}1 - B_{12}B_{23}\\ B_{12}+B_{23}\end{pmatrix}$$

$$B_{comb} = \frac{y}{x} = \frac{B_{12}+B_{23}}{1 - B_{12}B_{23}} \quad \leftarrow \text{Not } B_{12}+B_{23}!$$

However Angles do add



$$\theta_{13} = \theta_{12} + \theta_{23}$$



$$\tan\theta = \beta$$

$$\boxed{\begin{array}{c} \hat{x}'\big|_\beta \propto \begin{pmatrix} 1 \\ \beta \end{pmatrix} \\ \tan\theta = \dfrac{y}{x} \end{array}}$$

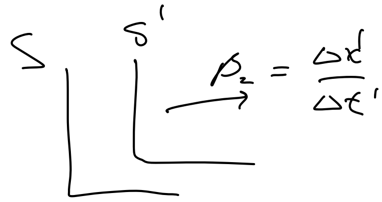
Another view of $\hat{x}^3\big|_{S'}$

$$\tan(\theta_{13}) = \tan(\theta_{12} + \theta_{23})$$

$$= \frac{\tan\theta_{12} + \tan\theta_{23}}{1 - \tan\theta_{12}\tan\theta_{23}}$$

trig ID!

$$\Rightarrow \beta_{13} = \frac{\beta_{12} + \beta_{23}}{1 - \beta_{12}\beta_{23}} \quad \checkmark$$



Now to velocity

$\beta_2 = \frac{\Delta x'}{\Delta t'}$ S' moves w/ β_1

What is β_{comb}?

$$\begin{pmatrix} \Delta x \\ \Delta t \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta_1 \\ \beta_1 & 1 \end{pmatrix} \begin{pmatrix} \Delta x' \\ \Delta t' \end{pmatrix}$$

$$\beta_c = \frac{\Delta x}{\Delta t} = \frac{\Delta x' + \beta_1 \Delta t'}{\beta_1 \Delta x' + \Delta t'} = \cancel{\frac{\Delta t'}{\Delta t'}} \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$

Not additive

$$\boxed{\beta_c = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}}$$

What does G.T give?

$$v' = \frac{\Delta x'}{\Delta t'}$$

$$v = \frac{\Delta x}{\Delta t} \qquad \begin{pmatrix} \Delta x \\ \Delta t \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x' \\ \Delta t' \end{pmatrix}$$

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v \Delta t'}{\Delta t'} = v' + v$$

as expected.

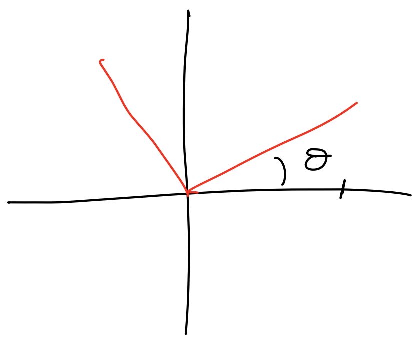
Propose to find new measure of velocity "velocity parameter"

η which is additive (Analog to θ)

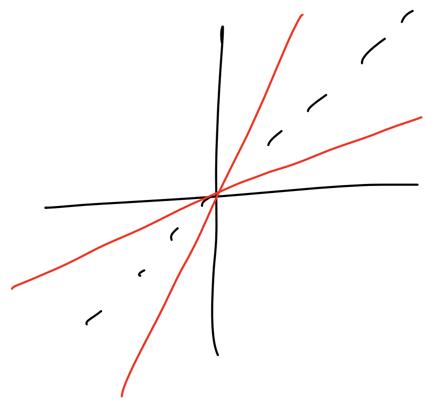
ie)

$$\eta = \eta' + \eta_r$$

Note η cannot be a simple angle



vs



Euclidean

Lorentz

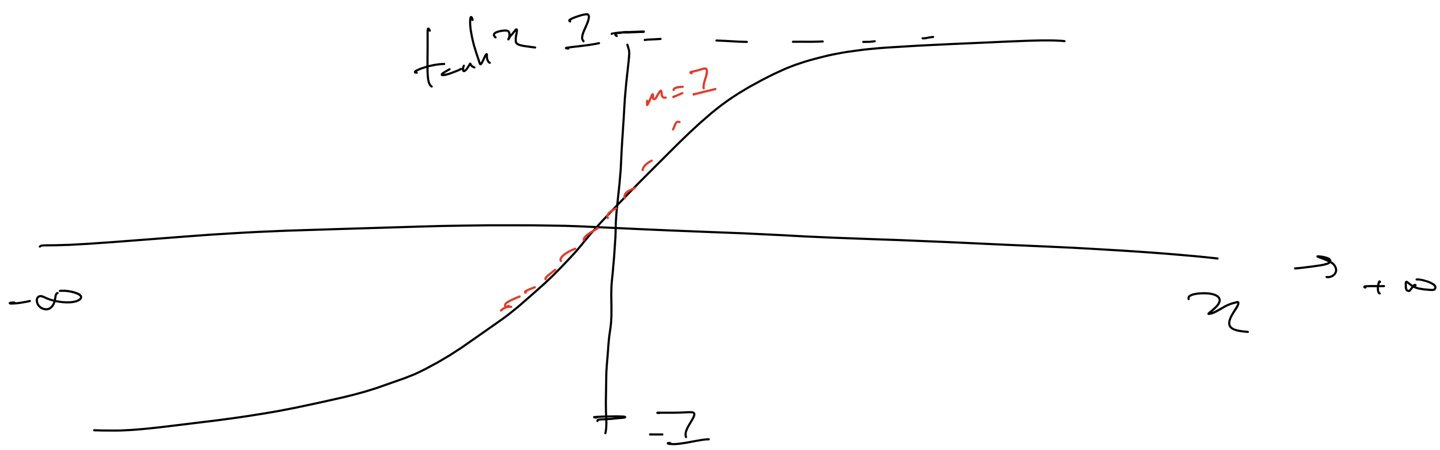
$$\tan\theta = \frac{\Delta y}{\Delta x} = \beta$$

$$\tanh\eta = \beta$$

$$\tanh\eta = \frac{\sinh\eta}{\cosh\eta}$$

$$\tanh(\eta_1 + \eta_2) = \frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2}$$

$$\tanh\eta \approx \eta \text{ for small } \eta$$



Clarifies how c can be the same in all reference frames.

$$c \longleftrightarrow \eta = \infty$$

So, $\eta = \eta' + \eta_{a}$ if $\eta' = \infty$

$$\eta = \infty + (\text{finite}) = \infty \Rightarrow \eta = \infty$$

η is the natural way to think about velocities.

Intuition

θ's we deal w/ θ's of all size. (large & small)

eg no one would ever add slope $B = 1$ ($\theta = 45°$) to another slope $B = 1$ to get $B = 2$ ($\theta = 64°$)

you add angles $45 + 45 = 90$ $B = \infty$

η: we only deal w/ β small!

w/ θ's Euclidean transformations Become

$$x = x' \cos\theta_r + y' \sin\theta_r$$

$$y = -y' \sin\theta_r + y' \cos\theta_r$$

w/ $\cos^2\theta + \sin^2\theta = 1$ directly implies:

$$x^2 + y^2 = x'^2 + y'^2$$

Similarly, w/ η L.T Become

$$x = x' \cosh\eta_r + t' \sinh\eta_r$$

$$t = x' \sinh\eta_r + t' \sinh\eta_r$$

w/ $\cosh^2\eta - \sinh^2\eta = 1$ directly implies

$$t^2 - x^2 = t'^2 - x'^2$$

Note

$\cosh\eta_r = \gamma \qquad \sinh\eta_r = \beta\gamma \qquad \tanh\eta_r = \dfrac{\sinh\eta}{\cosh\eta} = \beta$

$$\sin i\theta = \frac{e^{-\theta} - e^{+\theta}}{2i} = -i\left(\frac{e^{+\theta} - e^{-\theta}}{2}\right) = -i\sinh\theta$$

$$\cos i\theta = \frac{e^{-\theta} + e^{+\theta}}{2} = \cosh\theta$$

$$\tan i\theta = \frac{\sin i\theta}{\cos i\theta} = \frac{-i\sinh\theta}{\cosh\theta} = -i\tanh\theta$$

$x^2 + y^2$ invariant $\Rightarrow$ $\begin{matrix} x \\ y \end{matrix} = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}$

$t^2 - x^2$ let $t = iT$

$T^2 + x^2$ invariant $\Rightarrow \begin{pmatrix} T \\ x \end{pmatrix} = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} T' \\ x' \end{pmatrix}$

$$T = \cos\theta\, T' + \sin\theta\, x'$$

$$x = -\sin\theta\, T' + \cos\theta\, x'$$

or

$it = i\cos\theta\, t + \sin\theta\, x'$ $\Rightarrow$ $t = \cos\theta\, t' - i\sin\theta\, x'$

$x = -i\sin\theta\, t + \cos\theta\, x'$ $\qquad$ $x = -i\sin\theta\, t' + \cos\theta\, x'$

$\hookrightarrow$ Real if $\sin\theta$ imaginary

$\Rightarrow \theta = i\eta$

Then

$$t = \cosh \eta \, t' + \sinh \eta \, x'$$

$$x = \sinh \eta \, t' + \cosh \eta \, x'$$