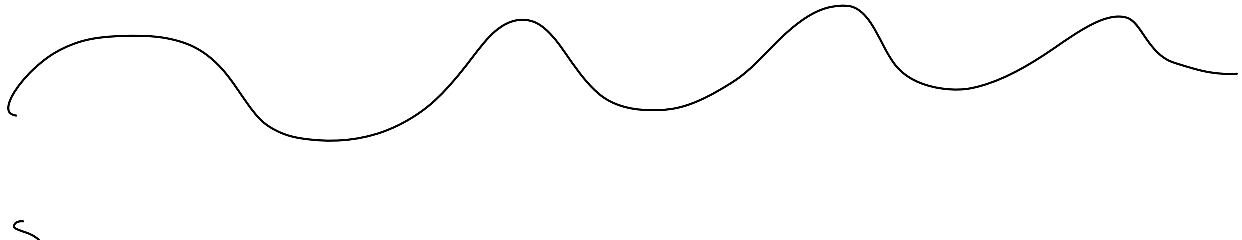
Last time

Saw: $y(x,t) = y_0 e^{i(kx - \omega t)}$

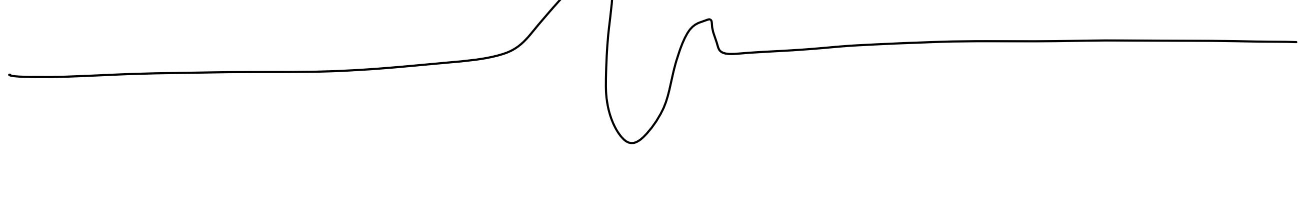
w/ $\omega = k v_p$ "Harmonic Waves"

is a solution to wave eq.

Nice, but not local



vs



"Fourier Anaysis"

Example w/ Basic idea

$$y(x,t) = y_0 e^{i(k_1 x - \omega_1 t)} + y_0 e^{i(k_2 x - \omega_2 t)} \qquad e^{ix} = \cos x + i \sin x$$

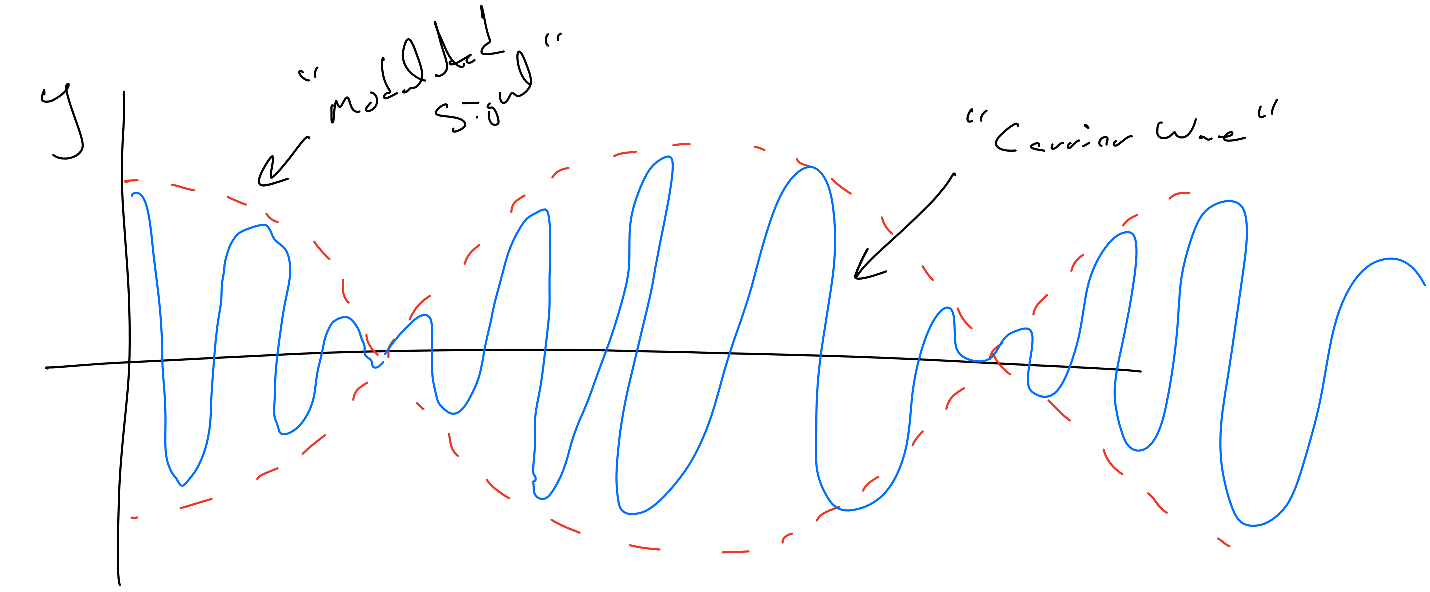
$$\boxed{\begin{aligned} e^{ik_1 x} + e^{ik_2 x} &= e^{i\frac{k_1}{2}x}\left(e^{i\frac{k_1}{2}x} + e^{i(k_2 - \frac{k_1}{2})x}\right) \\ &= e^{i\frac{k_1}{2}x} e^{i\frac{k_2}{2}x}\left(e^{i\frac{k_1-k_2}{2}x} + e^{i\left(\frac{k_2-k_1}{2}\right)x}\right) \\ &= e^{i\bar{k}x}\left(2\cos\frac{\Delta k}{2}x\right) \end{aligned}}$$

$$y(x,t) = 2 y_0 \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(\bar{k}x - \bar{\omega}t\right)$$

Consider the case $k_1 \sim k_2 \sim \bar{k}$, $\omega_1 \sim \omega_2 \sim \bar{\omega}$

And $\frac{\Delta k}{2} \ll \bar{k}$ $\quad \frac{\Delta\omega}{2} \ll \bar{\omega}$

We then get a "modulated" carrier wave w/ $(\bar{k}, \bar{\omega})$ that has a much slower modulation $(\Delta k, \Delta\omega)$ on top



Carrier Waves move w/ phase velocity $\frac{\omega}{k}$

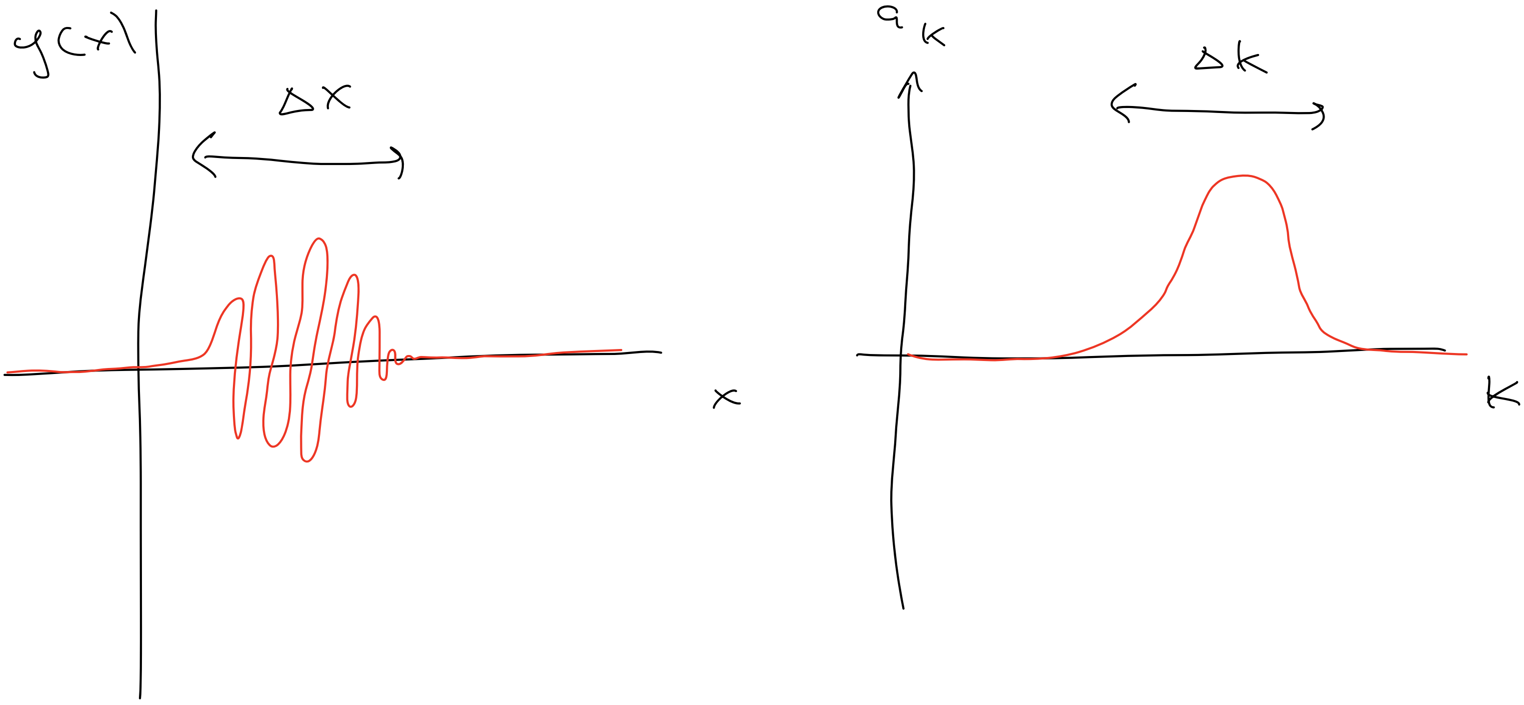
Modulation moves w/ "group velocity" $\frac{\Delta \omega}{\Delta k}$

Adding more terms suppresses other modulation peaks
(Figure in Book)

$V_p = \frac{\omega}{k}$ group velocity $\rightarrow$ $V_g = \frac{d\omega}{dk}$

$$y_{packet}(x,t) = \sum_{\vec{k}} a_k e^{i(kx - \omega t)}$$

$\Rightarrow \omega = v_p k$



Claim: in order to construct packet w/ spatial extension Δx

Need sum of "harmonic" waves from a region of k w/ extension Δk

Such that

$$\Delta x \cdot \Delta k \gtrsim 1$$

Spatial Extent

frequency Range

$$\omega = v_p k$$

$$\Delta k = \frac{\Delta \omega}{v_p}$$

$$\Delta x = v_p \Delta t$$

&

$$\Delta t \cdot \Delta \omega \gtrsim 1$$

temporal

frequency

"Uncertainty Relations" (Horrible Name!)

- Telling you that these concepts are mutually exclusive.
  "Incompatible / "Orthogonal" / Opposite / Complementary

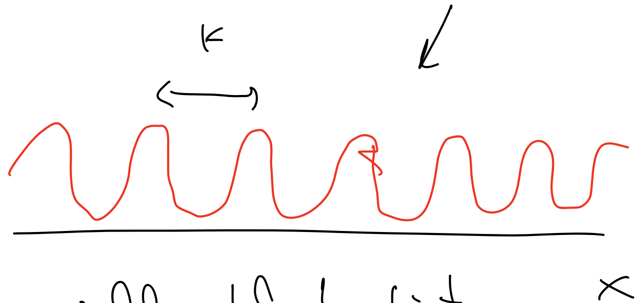
Δx - Small

requires many k terms

Only one peak

$\Rightarrow$ No well-defined notion of "Distance between peaks"

Only one k term (Delocalized)



Well-defined dist between peaks

$\Rightarrow$ Many peaks!

Well defined k $\Rightarrow$ Non-localized wave!

localized wave $\Rightarrow$ No well defined k!

Property of all wave packets. (Not specific to Quantum Theory)

Has nothing to do w/ your measurement of the wave.
An inherent constraint on the definition of the wave.

What speed does the wave packet move?

$$\frac{\Delta\omega}{\Delta k} \to \frac{d\omega}{dk}$$

Individual harmonic terms move w/ $v_p = \frac{\omega}{k}$ $\quad \omega = k v_p$

$$\frac{d\omega}{dk} = v_p + k\frac{dv_p}{dk}$$

"Group Velocity"

Non-zero in general

If $\frac{dv_p}{dk} = 0$ then $v_p = v_g$

Apply De Broglie Relations

$E = \hbar\omega \qquad p = \hbar k$

Represent particle by localized wave-packets $\psi(x,t)$

Immediately follows that there are corresponding "uncertainty relations"

$$\Delta x \, \Delta k \gtrsim 1 \implies \Delta x \, \Delta p \gtrsim \hbar$$

$$\Delta t \, \Delta\omega \geq 1 \implies \Delta t \, \Delta E \gtrsim \hbar$$

"Heisenberg Uncertainty Principle"

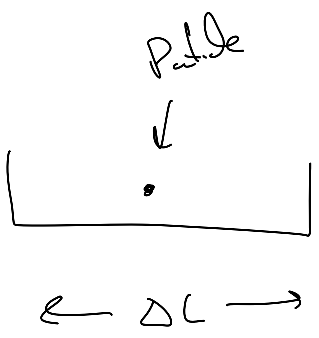
Deep Zeroth order statement of what QM is

Has major implications

Intuitive path to the implications of QM.

<u>Example</u> Energy of a localized particle

"Particle in a Box"



If we know the particle is in the Box

$\Delta x \sim \Delta L \Rightarrow \Delta p \sim \dfrac{\hbar}{\Delta L}$

What exactly does Δp mean? Choices. Obvious one standard Deviation

Average (time)

$$(\Delta p)^2 = \overline{(p - \bar{p})^2} = \overline{(p^2 - 2p\bar{p} + \bar{p}^2)}$$

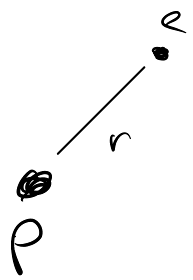
If Box Symmetric $\bar{p} = 0$, $\Rightarrow (\Delta p)^2 = \overline{p^2}$

$$\bar{E} = \frac{\overline{p^2}}{2m} = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2mL^2}$$ ← Non-Zero!

Particle Confined to some region of Space <u>cannot</u> have zero KE.

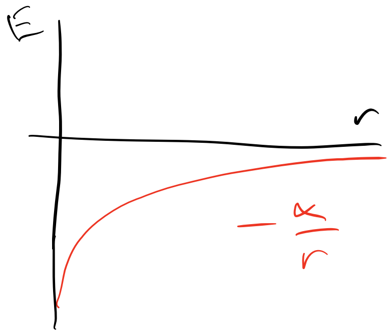
The smaller the confinement distance, the larger the E. (Particle Physics Monologue)

Example: Stability of Matter



$$E = \frac{p^2}{2m} - \frac{\alpha}{r}$$

Classically Electron wants to minimize E by being as close as possible to proton $(r \sim 0)$ **AND** not moving $(p \sim 0)$

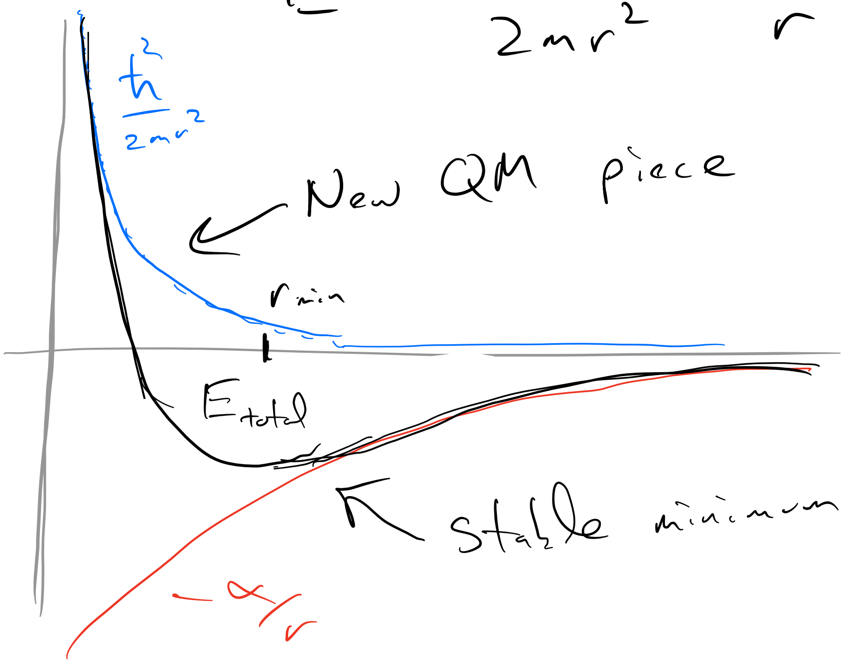


Quantum Mechanically the **AND** not possible

$$\Delta x \sim r \implies \Delta p \sim \frac{\hbar}{r}$$

So now,

$$E = \frac{\hbar^2}{2mr^2} - \frac{\alpha}{r}$$



Stable minimum $r \neq 0$

$$r_{min} \sim \frac{\hbar^2}{m\alpha}$$

$$E_{min} \sim \frac{\alpha^2}{\hbar^2} m \sim 13.7\,eV$$

$$\boxed{\text{Good test Q: Spectral lines}}$$