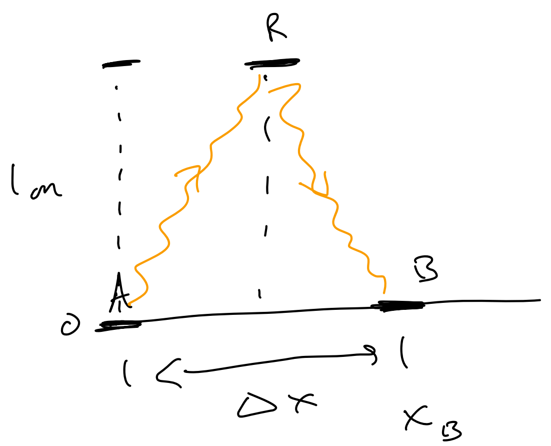
So far have been focused on the interval, not the coordinates.

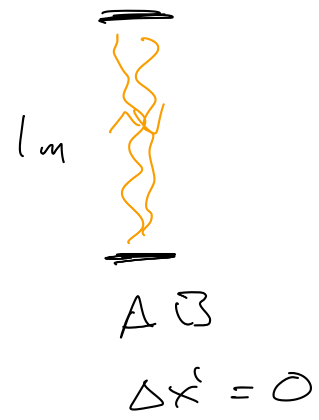
Will now talk about the coordinates themselves.

- Arbitrary, but important. These are what we measure!

Lab frame (S)



Rocket Frame (S')



Lets work out what the coordinates of B are in the Lab frame.

Know

$$t_B = 2\sqrt{1^2 + \left(\frac{x_B}{2}\right)^2} = 2\sqrt{\left(\frac{t'_B}{2}\right)^2 + \left(\frac{x_B}{2}\right)^2}$$

$$x_B = v_{m/s} \times t_{seconds} \qquad t_{seconds} = \frac{t}{c}$$

$$= \frac{v_{m/s}}{c}\, t \equiv \beta t$$

Characterize velocities w/ β dimensionless

$$v = 0 \Rightarrow \beta = 0$$

$$v = c \Rightarrow \beta = 1$$

"Dimensions are dirty!"

Ok (Now dropping the "B" s)

$$t = 2\sqrt{\left(\tfrac{t'}{2}\right)^2 + \left(\tfrac{\beta t}{2}\right)^2} \quad \text{or} \quad t^2 = 4\left(\frac{t'^2}{4} + \frac{\beta^2 t^2}{4}\right)$$

$$t^2 = t'^2 + \beta^2 t^2$$

$$t^2(1-\beta^2) = t'$$

$$t = \underbrace{\frac{1}{\sqrt{1-\beta^2}}}\, t'$$

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \begin{array}{ccc} \gamma = 1 & \rightarrow & \infty \\ \nearrow & & \nearrow \\ \beta = 0 & & \beta \rightarrow c \end{array}}$$

$$x = \beta t = \beta \gamma t'$$

So,

$$x_B = \beta\gamma t'_B$$

$$t_B = \gamma t'_B$$

Note this simple example not enough to fully specify the coordinate transformation B/c $x'_B = 0$

Can write

$$x_B = A x'_B + \beta\gamma t'_B$$

$$t_B = B x'_B + \gamma t'_B$$

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} A & \beta\gamma \\ B & \gamma \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

Not complex enough to constrain A & B.

Degenerate B/c $x'_B = 0$

Pick some other point in S' $(x',t') \neq 0$

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} A & \beta\gamma \\ B & \gamma \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$ A & B unknown

$$x = Ax' + \beta\gamma t'$$

$$t = Bx' + \gamma t'$$

What else could we impose?

What else do we know about

(x,t) & (x',t')?

The Invariant

Whatever x,t are

$$t^2 - x^2 = t'^2 - x'^2$$

Lets impose this

$$x^2 = A^2 x'^2 + 2A\beta\gamma x' t' + \beta^2\gamma^2 t'^2$$

$$t^2 = B^2 x'^2 + 2B\gamma x' t' + \gamma^2 t'^2$$

Already fixed! Better $= 1$

$$\left(\frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = 1 \right)$$

Now

$$t^2 - x^2 = \overbrace{(\gamma^2 - \beta^2\gamma^2)}\, t'^2 +$$

$$\underbrace{2\gamma(B - \beta A)}_{\text{Need } = 0}\, x' t' +$$

$$\underbrace{(B^2 - A^2)}_{\text{Needs } = -1}\, x'^2$$

$$B - \beta A = 0 \quad \Rightarrow \quad B = \beta A$$

$$B^2 - A^2 = -1 \quad \Rightarrow \quad \beta^2 A^2 - A^2 = -1$$

$$A^2(1 - \beta^2) = 1$$

$$A = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

$$B = \beta\gamma$$

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

$$x = \gamma x' + \beta\gamma t'$$

$$t = \beta\gamma x' + \gamma t'$$

Or in crappy units

$$x = \gamma x' + \gamma v t'_{\text{seconds}}$$

$$t_{\text{seconds}} = \frac{v}{c^2}\gamma x + \gamma t'$$

$$\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \quad \text{vs} \quad \begin{pmatrix} \gamma & \gamma v \\ \gamma \frac{v}{c^2} & \gamma \end{pmatrix}$$

Sanity Check Newton works when v small

$\Rightarrow$ Better get G.T. when $v \ll c$

$\beta \ll 1$

when $\beta \ll 1$ $\quad \gamma \to 1$

$$\frac{v}{c} \to 0 \qquad \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

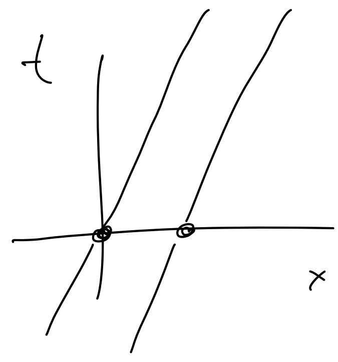
Note inverse transformation w/ $\beta \to -\beta$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Length Contraction

at rest in S'

$$L' = x_2' - x_1'$$



$$L = x_2 - x_1 \quad (\text{at } t_1 = t_2)$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$x_2' - x_1' = \gamma(x_2 - \beta t_2) - \gamma(x_1 - \beta t_1)$$

$$= \gamma(x_2 - x_1) - \gamma\beta\cancelto{0}{(t_2 - t_1)}$$

$$L' = \gamma L \implies L = \frac{L'}{\gamma}$$

N.t.

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

$$x_2 = \gamma x_2' + \beta\gamma t_2'$$

$$x_1 = \gamma x_1' + \beta\gamma t_1'$$

$$L = x_2 - x_1 = \gamma \underbrace{(x_2' - x_1')}_{L'} + \beta\gamma \underbrace{(t_2' - t_1')}_{\Delta t'}$$

$$t_2 - t_1 = \beta\gamma (x_2' - x_1') + \gamma (t_2' - t_1')$$

$$0 = \beta\gamma L' + \gamma \Delta t'$$

$$\Rightarrow \quad \Delta t' = -\beta L'$$

$$L = \gamma L' + \beta\gamma(-\beta L')$$

$$= \gamma(1-\beta^2) L'$$

$$= \gamma \frac{1}{\gamma^2} L' \Rightarrow L = \frac{L'}{\gamma}$$