House keeping: HW# Posted today
Due Friday

**Recap** Transformations

Analogy w/ geometry

## Why aren't G.T. good enough?

Bunch of good reasons

- E&M (last time) ✓
- Determinism  (1st day) ✓
- Gen'l Coordinate Transformations
- Philosophy (Einstein's main motivation) (+ EM) (Next time)
- Experiment!

Experiment "Michelson & Morely"

Early $20^{th}$ C: All waves known to propogate in something "medium" (water, sound, etc)

Light is a wave! (eg Double Split exp / Maxwell equs)
There must be an associated media "Ether"

Maxwell's eq $\Rightarrow$ light waves move at $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \sim 3\,10^8\ m/s$
$\left( 1\,ft/ns \quad \text{or } 30\,cm/ns \right)$ wrt the ether

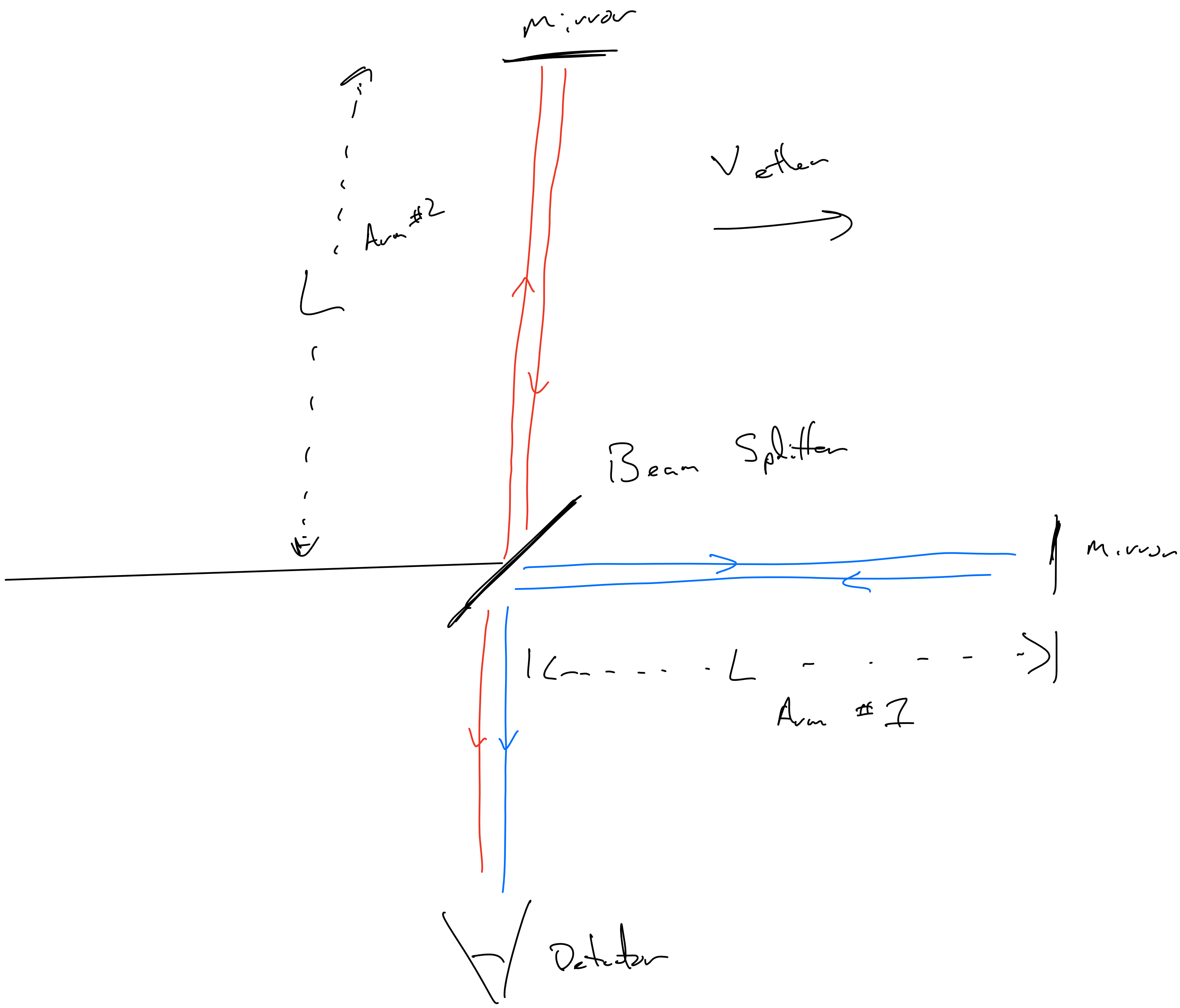
Earth moving $\sim 30\,\frac{km}{s}$, B/c moving in circle, cant always align w/ ether

Can we detect the change in speed of light due to earths motion

$v_e \sim \frac{30\,km}{s} \qquad c \sim 3\,10^8\ m/s \sim 3\,10^5\ \frac{km}{s}$

$$\frac{v_e}{c} \sim 10^{-4}$$

**Idea** "Interferometer" Compare time of flight for light traveling in different directions




$$t_1 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{L}{c}\left[\frac{1}{1+v/c} + \frac{1}{1-v/c}\right] \qquad \beta \equiv \frac{v}{c} \ll 1$$

$$= \frac{L}{c}\left[\frac{1}{1+\beta} + \frac{1}{1-\beta}\right]$$

$$= \frac{L}{c}\left[1 - \beta + \beta^2 + O(\beta^3) + (1 + \beta + \beta^2 + O(\beta^3)\right]$$

Or

$$t_1 = \frac{2L}{c}\left[1 + \beta^2 + \mathcal{O}(\beta^3)\right]$$

## Arm # 2 Different

$$t_2 = \frac{L}{\sqrt{c^2 - v^2}} + \frac{L}{\sqrt{c^2 - v^2}} = \frac{2L}{c}\left(1 + \frac{1}{2}\beta^2 + \mathcal{O}(\beta^3)\right)$$

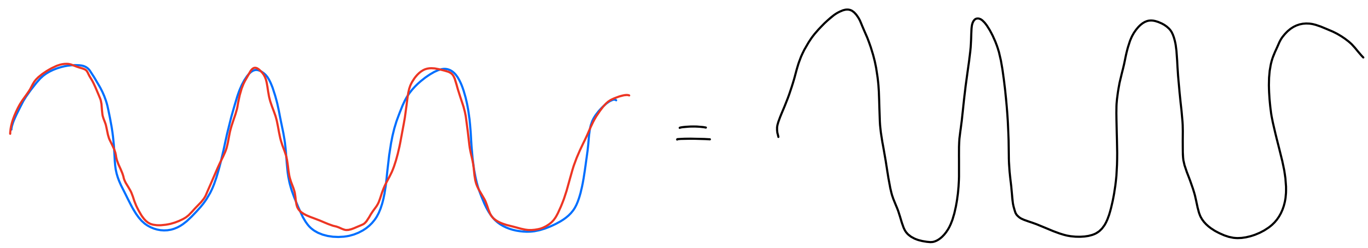
## Rel difference

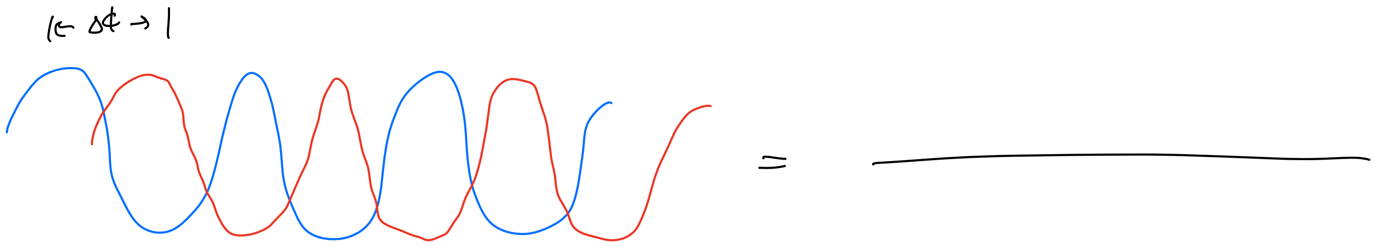
$$\Delta t = t_2 - t_1 = \frac{2L}{c}\left[\left(1 + \beta^2 + \mathcal{O}(\beta^3)\right) - \left(1 + \frac{1}{2}\beta^2 + \mathcal{O}(\beta^3)\right)\right]$$

$$= \frac{2L}{c}\left[\frac{1}{2}\beta^2\right] + \mathcal{O}(\beta^3)$$

$$= \frac{L v^2}{c^3}$$

Tiny wavelength of light allows us to probe tiny $\Delta t$ by looking for interference






$$\Delta\phi \sim \frac{c\,\Delta t}{\lambda} \sim \frac{L}{\lambda}\left(\frac{v}{c}\right)^2$$

$$= 4\pi \frac{10\,\mathrm{m}}{590\,\mathrm{nm}}\left(10^{-4}\right)^2 \simeq 2.1 = 0.3 \; 2\pi$$

1/3 of fringe

Easily observed, but not seen!

$$t_1 = \frac{L}{c}\left[\frac{1}{1+\beta} + \frac{1}{1-\beta}\right]$$

$$= \frac{2L_1}{c}\left[1 + \beta^2\right]$$

$$t_2 = \frac{2L_2}{c}\left[1 + \frac{1}{2}\beta^2\right]$$

$$\Delta t = t_1 - t_2 = \frac{2L_1}{c}\left[1 + \beta^2\right] - \frac{2L_2}{c}\left[1 + \frac{1}{2}\beta^2\right]$$

$$L_1 = L_2 \Rightarrow \quad = \frac{2L}{c}\left[1 + \beta^2 - 1 - \frac{1}{2}\beta^2\right]$$

$$= \frac{L\beta^2}{c} = \frac{Lv^2}{c^3}$$

Or

$$\Delta L = L_1 - L_2 \qquad \frac{2}{c}\left[\Delta L + L_1\beta^2 - L_2\frac{\beta^2}{2}\right]$$

$$L_2 = L_1 - \Delta L$$

$$= \frac{2}{c}\left[\Delta L + \frac{L_1\beta^2}{2} + \cancel{\frac{\Delta L\,\beta^2}{2}}\right]$$

# General Coordinate Transformations

-) Assume transform is linear — Between reference frame w/ relative velocity $v$

$$x = ax' + bt'$$
$$t = ex' + ft'$$

or

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} a & b \\ e & f \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix}$$

-) Impose when

$x' = 0 \qquad x = vt$

$x = 0 \qquad x' = -vt'$

Impose group

$$T(v_2)T(v_1) = T(v_c)$$

$$T(-v)T(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

HW

what you find ...



$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\sqrt{1-\left(\frac{v}{v_*}\right)^2}} \begin{pmatrix} 1 & v \\ \frac{v}{v_*^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

Will show $v_*$ | maximum speed!
| coordinate invariant

Meaning if $x' = v_* t'$ (Something moving at speed $v_*$ in primed frame)

Also has speed $v_*$ in unprimed frame!

when $v_* \to \infty$ recover G.T. $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$

# Bonus Derivation General Coordinate Transform (C.T)

1) Assume transform is linear Between reference frame w/ relative velocity $v$

$$x = ax' + bt'$$
$$t = ex' + ft'$$

or $\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} a & b \\ e & f \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

w/ $(a, b, e, f)$ unknown functions of $v$

2) Now when $x' = 0$ $x = vt$ ← Assumption of relative $v$

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} a & b \\ e & f \end{pmatrix} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} bt' \\ ft' \end{pmatrix}$$

$$\frac{x}{t} = \frac{bt'}{ft'} = v \quad \text{or} \quad \begin{pmatrix} x \\ t \end{pmatrix} = f \begin{pmatrix} \frac{a}{f} & v \\ \frac{e}{f} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

3) When $x = 0$ $x' = -vt'$ (view from other frame)

$$\begin{pmatrix} 0 \\ t \end{pmatrix} = f \begin{pmatrix} \frac{a}{f} & v \\ e/f & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} ax' + fvt' \\ ex' + ft' \end{pmatrix}$$

$$= \begin{pmatrix} ax' - fx' \\ ex' - \frac{f}{v}x' \end{pmatrix}$$

$\Rightarrow a = f$

So General C.T.

$$\begin{pmatrix} x \\ t \end{pmatrix} = a \begin{pmatrix} 1 & v \\ \frac{e}{a} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

3) Now impose C.T. form a group

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} \xrightarrow[v_1]{} \begin{pmatrix} x' \\ t' \end{pmatrix} \xrightarrow[v_2]{} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} x \\ t \end{pmatrix} = a(v_2) \begin{pmatrix} 1 & v_2 \\ \frac{e}{a}(v_2) & 1 \end{pmatrix} a(v_1) \begin{pmatrix} 1 & v_1 \\ \frac{e}{a}(v_1) & 1 \end{pmatrix} \begin{pmatrix} x'' \\ t'' \end{pmatrix}$$

$$= a(v_2)\, a(v_1) \begin{pmatrix} 1 + \frac{e}{a}(v_1) \cdot v_2 & v_1 + v_2 \\ \frac{e}{a}(v_1) + \frac{e}{a}(v_2) & 1 + \frac{e}{a}(v_2) \cdot v_1 \end{pmatrix} \begin{pmatrix} x'' \\ t'' \end{pmatrix}$$

Now action of 2 C.T's must be described by the action of single combined C.T.

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} \xrightarrow[v_c]{} \begin{pmatrix} x \\ t \end{pmatrix}$$

For this to be the case, the diagonal elements must be equal

$$1 + \frac{e}{a}(v_1) \cdot v_2 = 1 + \frac{e}{a}(v_2)\, v_1$$

$$\Rightarrow \frac{1}{v_1} \frac{e}{a}(v_1) = \frac{1}{v_2} \frac{e}{a}(v_2)$$

B/c we have separation of variables, both sides must be equal to a constant (See this By applying $\frac{\partial}{\partial v_1}$ & $\frac{\partial}{\partial v_2}$)

$$\Rightarrow \frac{1}{v} \frac{e}{a}(v) = g \quad \Rightarrow \frac{e}{a}(v) = g\, v$$

free constant (independent of $v$)

So, General C.T.

$$\begin{pmatrix} x \\ t \end{pmatrix} = a \begin{pmatrix} 1 & v \\ gv & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

<u>Note</u>

$[t] = [g][v][x]$

or $[g] = \frac{[t]}{[x][v]} = \frac{1}{[v]^2}$

$g \to \frac{1}{v_*^2}$

4) find $a$, by imposing $\begin{pmatrix} x \\ t \end{pmatrix} \xrightarrow[v]{} \begin{pmatrix} x' \\ t' \end{pmatrix} \xrightarrow[-v]{} \begin{pmatrix} x \\ t \end{pmatrix}$

$$\begin{pmatrix} x \\ t \end{pmatrix} = a(-v) \begin{pmatrix} 1 & -v \\ -\frac{v}{v_*^2} & 1 \end{pmatrix} a(v) \begin{pmatrix} 1 & v \\ \frac{v}{v_*^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$= \underbrace{a(-v)\, a(v) \begin{pmatrix} 1 - \frac{v^2}{v_*^2} & 0 \\ 0 & 1 - \frac{v^2}{v_*^2} \end{pmatrix}}_{\text{must } = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\Rightarrow \quad a(-v)\, a(v) = \frac{1}{1 - \frac{v^2}{v_*^2}}$$

Symmetry of Space

$\Rightarrow a(v) = a(|v|)$

or $a(-v)\, a(v) = (a(v))^2$

$$\Rightarrow \quad a(v) = \frac{1}{\sqrt{1 - \frac{v^2}{v_*^2}}}$$

General C.T. parametrized by $v_*$ given by

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{v_*^2}}} \begin{pmatrix} 1 & v \\ \frac{v}{v_*^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

What is $v_*$? Assume something has velocity $v_*$ in primed coordinates

$$x' = v_* t'$$

In unprimed coordinates,

$$\begin{pmatrix}x\\ t\end{pmatrix} = \gamma_{v_*}\begin{pmatrix}1 & v\\ \frac{v}{v_*^2} & 1\end{pmatrix}\begin{pmatrix}v_* t'\\ t'\end{pmatrix} = \gamma_{v_*}\begin{pmatrix} v_* t' + v t' \\ \frac{v}{v_*}t' + t'\end{pmatrix}$$

$$\frac{x}{t} = \frac{v_*+v}{\frac{v}{v_*}+1} = \frac{v_*+v}{\frac{v+v_*}{v_*}} = v_*$$

$v_*$ | maximum speed!
| coordinate invariant

Finally Note

$$v_*^2 t^2 - x^2 = v_*^2\left(\frac{\frac{v}{v_*^2}x'+t'}{\sqrt{1-\left(\frac{v}{v_*}\right)^2}}\right)^2 - \left(\frac{x'+vt'}{\sqrt{1-\left(\frac{v}{v_*}\right)^2}}\right)^2$$

$$= \frac{\left(\frac{v}{v_*}x' + v_* t'\right)^2 - (x'+vt')^2}{1-\left(\frac{v}{v_*}\right)^2}$$

$$= \frac{1}{1-\left(\frac{v}{v_*}\right)^2}\left(\frac{v^2}{v_*^2}x'^2 + 2\cancel{vx'}t' + v_*^2 t'^2 - x'^2 - 2\cancel{vx'}t' - v^2 t'^2\right)$$

$$= \frac{1}{1-\left(\frac{v}{v_*}\right)^2}\left[(v_*^2 - v^2)t'^2 + \left(\frac{v^2}{v_*^2}-1\right)x'^2\right]$$

$$= \frac{1}{1-\left(\frac{v}{v_*}\right)^2}\left[\left(1-\left(\frac{v}{v_*}\right)^2\right) v_*^2 t'^2 - \left(1-\left(\frac{v}{v_*}\right)^2\right) x'^2\right]$$

$$= v_*^2 t'^2 - x'^2$$

<u>PPS</u> when $v_* \to \infty$ we recover Galilean Transfms

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{v_*^2}}}\begin{pmatrix} 1 & v \\ \frac{v}{v_*^2} & 1 \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix}$$

$$\xrightarrow[v_* \to \infty]{} = 1 \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix}$$

or

$$x = x' + vt'$$

$$t = t'$$