Last time

- ψ the thing that is waving is related to ($|\psi|^2 dx$) the probability of finding the particle at x

Today

→ Reminder about basic prob. theory

→ Wave equation that governs ψ.

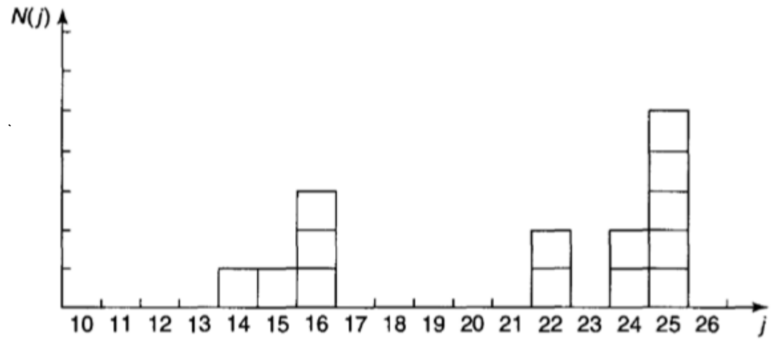
Probability

B/c of the statistical interpretation, Probability plays critical role in QM

(Sure you've all seen this, just so we're all on same page)

Discrete Variables

Example: Ages of People in a room



$N(j)$ - # of people w/ age j

eg $N(16) = 3$ $\quad N(17) = 0$

$$\underset{\text{total}}{N} = \sum_{j=0}^{\infty} N(j) \qquad (= 14 \text{ in our example})$$

Probabilities: Pick one at random, Prob age $= j$?

$$P(j) = \frac{N(j)}{N} \qquad P(16) = \frac{3}{14}$$

$$P(17) = 0/14$$

Note $\sum_j P(j) = 1$ ← Certain to get some age!

What is most probable? 25

$P(j)$ is maximum

What is average age?

$$\frac{14 + 15 + 3(16) + 2(22) + 2(24) + 5(25)}{14} = 21 \quad (P(21) = 0)$$

$$\langle j \rangle = \bar{j} = \frac{\sum j N(j)}{N} = \sum_j j P(j)$$

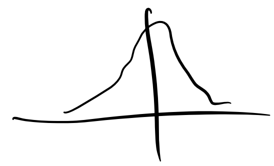
Average age squared

$$\langle j^2 \rangle = \sum_j j^2 P(j)$$

Often use "variance" "standard deviation"

to characterize spread in values

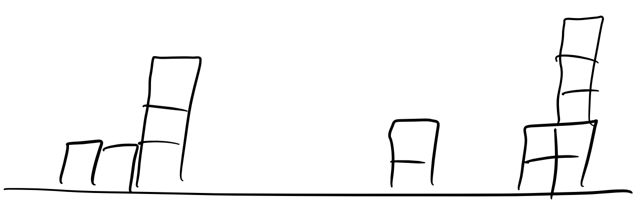
$$\left\langle (j - \langle j \rangle)^2 \right\rangle$$



$$= \langle j^2 \rangle - \langle j \rangle^2 \equiv \sigma^2$$

show this.

Continuous Variables



$N(j)$

$\rho(x)$

$$N(j) \longrightarrow \int_a^b \rho(x)\,dx$$

$$\sum P(j) = 1 \longrightarrow \int_{-\infty}^{\infty} \rho(x)\,dx = 1$$

$$\langle j \rangle = \sum j\,P(j) \longrightarrow \langle x \rangle = \int_{-\infty}^{\infty} x\,\rho(x)\,dx$$

$$\langle f(j) \rangle = \sum f(j)\,P(j) \longrightarrow \langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\,\rho(x)\,dx$$

Back to QM.

$|\Psi(x,t)|^2$ is the probability density for finding particle at x

$$\Rightarrow \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1$$

"Normalization"

"The particle has to be somewhere"

without this constraint on Ψ, the statistical interpretation would be non-sense.

Obvious Question: Is this constraint consistent w/ the wave equation?

Schrodinger Equation

Want a wave eq that reproduces non-relativistic Energy-momentum relation

$$E = \frac{p^2}{2m} + V(x)$$

Wave eq from Maxwell (light)

$$\hbar = 1$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \qquad \text{try} \quad \psi(x,t) = C e^{i(px - Et)}$$

$$(ip)^2 \psi(x,t) = \frac{1}{c^2}(-iE)^2 \psi(x,t)$$

$$-p^2 = \frac{1}{c^2} - E^2 \implies E^2 = p^2c^2$$

$$E = pc$$

Correct expression for γ's!

↪ Not for non-relativistic e^-'s

Let's try to work backwards...

Assume

$$\psi(x,t) \sim e^{i(Px - Et)}$$

or equivalently

$$\frac{\partial \psi}{\partial x} = i p \psi \qquad \frac{\partial \psi}{\partial t} = -i E \psi$$

$$p\,\psi(x,t) = -i \frac{\partial \psi}{\partial x} \qquad E\,\psi(x,t) = i \frac{\partial \psi}{\partial t}$$

① ②

orbits e don't have const p.

Lets try ②, and use ① to define p

$$E(p)\,\psi(x,t) = i \frac{\partial \psi}{\partial t} \qquad E(p) = \frac{1}{2m} p^2 + V$$

what to use for p? try ① $\Rightarrow p \to -i \frac{\partial}{\partial x}$

$$E\left(-i\frac{\partial}{\partial x}\right) \psi(x,t) = i \frac{\partial \psi}{\partial t}$$

$$\left(\frac{1}{2m}\left(-i\frac{\partial}{\partial x}\right)\left(-i\frac{\partial}{\partial x}\right) + V\right) \psi = i \frac{\partial \psi}{\partial t}$$

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V\right] \psi = i \frac{\partial \psi}{\partial t}$$

("Schrodinger Eq")

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right] \psi = i\hbar \frac{\partial \psi}{\partial t}$$

Comments

- Now need complex solutions! "i" in the eq. cos / sin don't work on their own.

- $\Psi(x,t)$ will be complex!

- Solutions harmonic waves if $V=0$ (Also V_0 $H(\omega)$) $V(x)$ will give more complicated Ψ's

- Sch 1st order in $\frac{\partial}{\partial t}$. $\Psi(x,0)$ fixes $\Psi(x,t)$! (2nd order in $\frac{\partial}{\partial x}$)

  $\longrightarrow$ Problem for relativity!

-) If $\Psi(x,t)$ is a solution so is
$A\,\Psi(x,t)$ A - constant

-) have to pick A such that Ψ - normalized

<u>Obvious Next Question ...</u>

If pick A at one time, will Sch Eq change it? eg will Ψ stay normalized?

Important. A has to be a constant (independent of t)
$A(t)\,\Psi(x,t)$ <u>not</u> a solution to Sch Eq.

<u>Turns Out</u> (Hint that Statistical Interpretation Right?)
Sch. Eq automatically preserves normalization

Proof

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2\,dx$$

$$|\Psi|^2 = \Psi^*\Psi \qquad \frac{\partial}{\partial t}|\Psi|^2 = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

Sch. Eq

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi$$

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

$$\frac{\partial |\Psi|^2}{\partial t^2} = \frac{i\hbar}{2m}\Big(\underbrace{\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi}_{=}\Big)$$

$$\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) = \cancel{\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}} + \Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi - \cancel{\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi|^2\,dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)dx$$

$$= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty} = 0$$

$$\boxed{\Psi(\pm\infty)=0}$$

If you normalize Ψ, it stays normalized

Position & Momentum

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi|^2 dx$$

Note: Not the average of measuring particle many times!

Average x if you measured many particles w/ $\Psi(x,t)$

How does $\langle x \rangle$ change w/ Sch Eq?

$$\frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} |\Psi|^2 dx$$

$$= \frac{i\hbar}{2m} \int x \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx$$

Integration By Parts $\boxed{\text{H.W.}}$

$$= -\frac{i\hbar}{2m}\int\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx$$

More Integration By Parts

$$= -\frac{i\hbar}{m}\int\Psi^*\frac{\partial\Psi}{\partial x}\,dx = \frac{d\langle x\rangle}{dt}$$

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx$$

or

$$\boxed{\begin{aligned}\langle x\rangle &= \int\Psi^*(x)\,\Psi\,dx\\ \langle p\rangle &= \int\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx\end{aligned}}$$

The average of $\frac{\partial}{\partial x}$ is telling you how much momentum ($\sim$ KE) the particle has.

All classical dynamical variables can be expressed in terms of x & p.

eg) $KE = \frac{p^2}{2m}$

To calculate the QM expectation value

-) $f(x,p) \longrightarrow f\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

-) Integrate with respect to ψ^* & ψ

eg

$$\langle KE \rangle = -\frac{\hbar^2}{2m}\int \psi^* \frac{\partial^2 \psi}{\partial x^2}\, dx$$