Now some actual Solutions

Saw before, Solving the Sch. amounts to solving the "time-independent" Sch Eq & tacking on $e^{-i\frac{Et}{\hbar}}$

time-independent Sch Eq:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi = E\psi$$

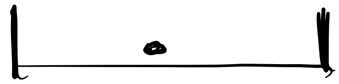
Different Solutions depending on $V(x)$

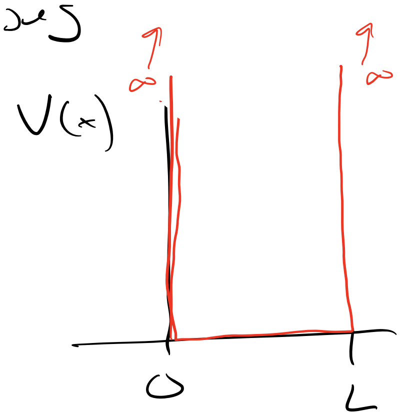
Today will work through one example in detail.
Next week will talk through other solutions

- $V(x)$ infinite square well.
- $V(x)$ finite square well.
- "Harmonic Oscillator
- Double square well

All 1D for now

Infinite Square Well ("Particle in a Box")

- Classical Example 
- Easy to devise actual physical realizations



$$V(x) = 0 \quad \text{for} \quad 0 < x < L$$

$$V(x) = \infty \quad \text{otherwise}$$

Note - Not as stupid as it looks
- Simple model for thinking about more complex V's

Ok, Outside the well, $\psi(x) = 0$
(would require $E = \infty$ to satisfy Sch Eq)

(the particle cannot exist in region w/ $V(x) = \infty$)

→ Makes sense classically.

So, only need to solve Schr. Eq $0 < x < L$

or $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

rewrite as $\frac{d^2\psi}{dx^2} = -k^2\psi$ $\quad k \equiv \frac{\sqrt{2mE}}{\hbar}$

$(E>0)$

Should be familiar. Simple Harmonic Oscillator

$$\psi(x) = A\sin kx + B\cos kx$$

where A & B are arbitrary.

Solving Sch Eq reduced to finding A, B & k

Typically fixed by "Boundary Conditions" (Norm)

- ψ & $\frac{d\psi}{dx}$ - continuous

$\hookrightarrow$ Only if V finite (relax here)

Now ψ continuity $\Rightarrow$ $\psi(0) = \psi(L) = 0$

$$\psi(0) = B \quad \Rightarrow \quad B = 0$$

$$\psi(L) = A\sin kL = 0$$

$\Rightarrow \psi = 0$

$\sin(-\theta) = -\sin\theta$

Only true if $kL = \cancel{0}, \pm\pi, \pm 2\pi \ldots$

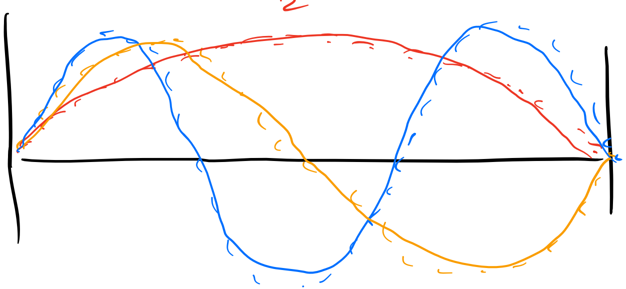
$$k_n = \frac{n\pi}{L} \quad n = 1, 2, \ldots$$

Now, $k = \frac{\sqrt{2mE}}{\hbar} \Rightarrow E = \frac{k^2\hbar^2}{2m}$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$\boxed{\underline{\text{Note}} \quad \Delta x \sim L \Rightarrow p \sim \frac{\hbar}{L} \Rightarrow E = \frac{p^2}{2m} \sim \frac{\hbar^2}{2mL^2}}$$

Those boundary conditions are the same as vibrating string with ends at 0 + L fixed
$\Rightarrow$ Only certain allowed wavelengths to fit in L
(frequencies)



Allowed solutions have

$$n\frac{\lambda}{2} = L$$

$n = 1, 2, 3 \ldots$

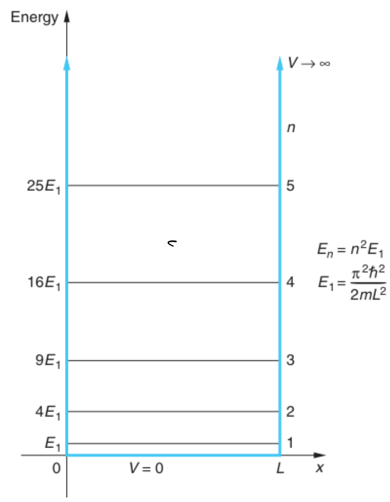
A can be fixed w/ $\int |\psi|^2 dx = 1$

$$a = \sqrt{\frac{2}{a}}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

-) infinite number of solutions

ψ_n w/ $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$



$n = 1$ "Ground State"

$n > 1$ "Excited States"

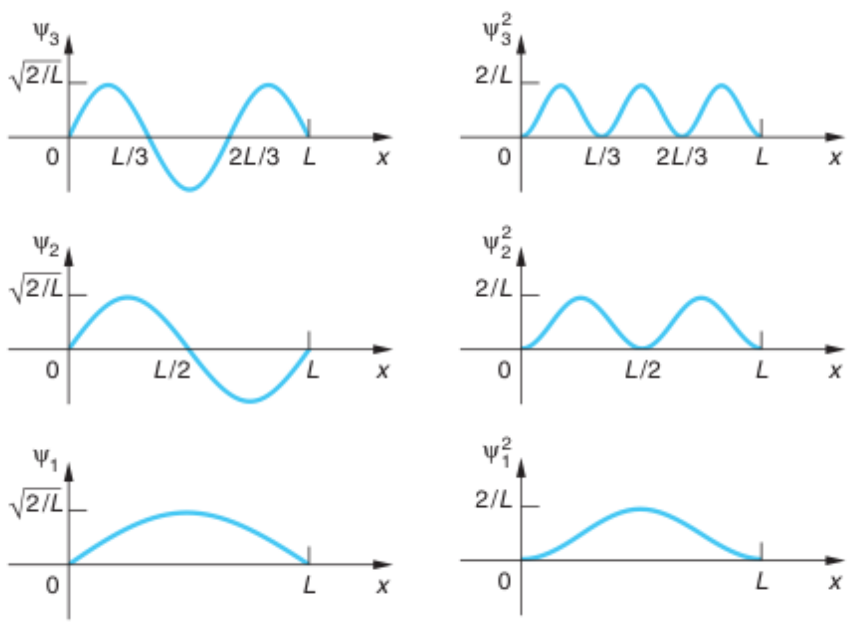
n - "quantum number" $\psi(0) = \psi(L) = 0$

Note: quantum number occurs B/c of boundary conditions

will see later, in 3D need 3 quantum numbers.

-) ψ_n even or odd wrt center of well.

-) As E increases, so does $\frac{\partial\psi}{\partial x}$ (more nodes)

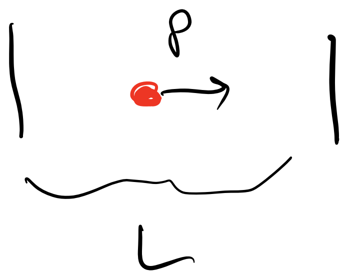


Compare with Classical Result

$F = 0$, between the walls

$F = \infty$, if colliding w/ wall

$\Rightarrow$ elastic collisions with wall



$$\frac{d|p|}{dt} = 0$$

The ball keeps bouncing back and forth with whatever $|\vec{p}|$ it started with

<u>Note</u> Any speed, momentum or E allowed

E can be zero, $x(t) = x_0$

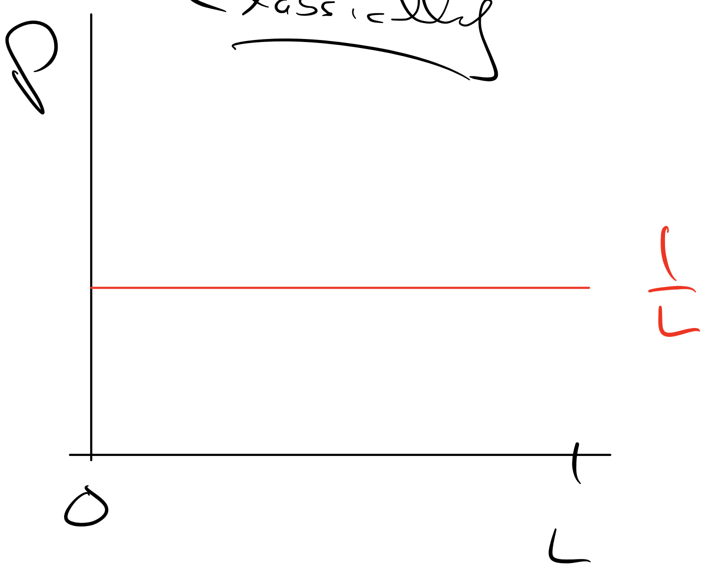
"degenerate solutions" $\begin{cases} \vec{P}(t) \\ -p(t) \end{cases}$ same E

Measuring Position

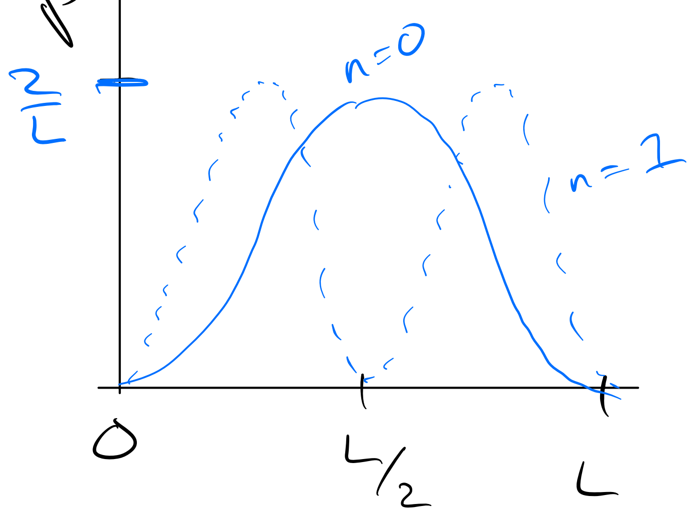
Classically: Prob. of finding particle in some region dx scales w/ time spent in that region

$$P \sim \frac{dx}{v} \qquad v \text{ constant } (v \neq 0) \qquad P_{cl}(x) = \frac{1}{L}$$

Classically



QM



- when n small, very different in QM.

- when n large, (E large) QM $\rightarrow$ Classical.