Back to momentum . . .

Here

$$\vec{p} = m\,\frac{d\vec{x}}{d\tau} = m\,\frac{d}{d\tau}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

← Nearly have 4-vector! missing "time" compont

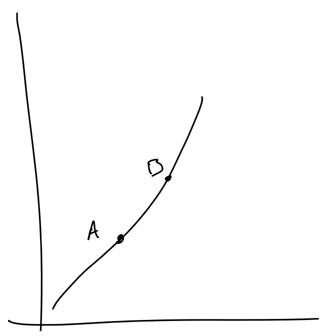
Put it in by hand

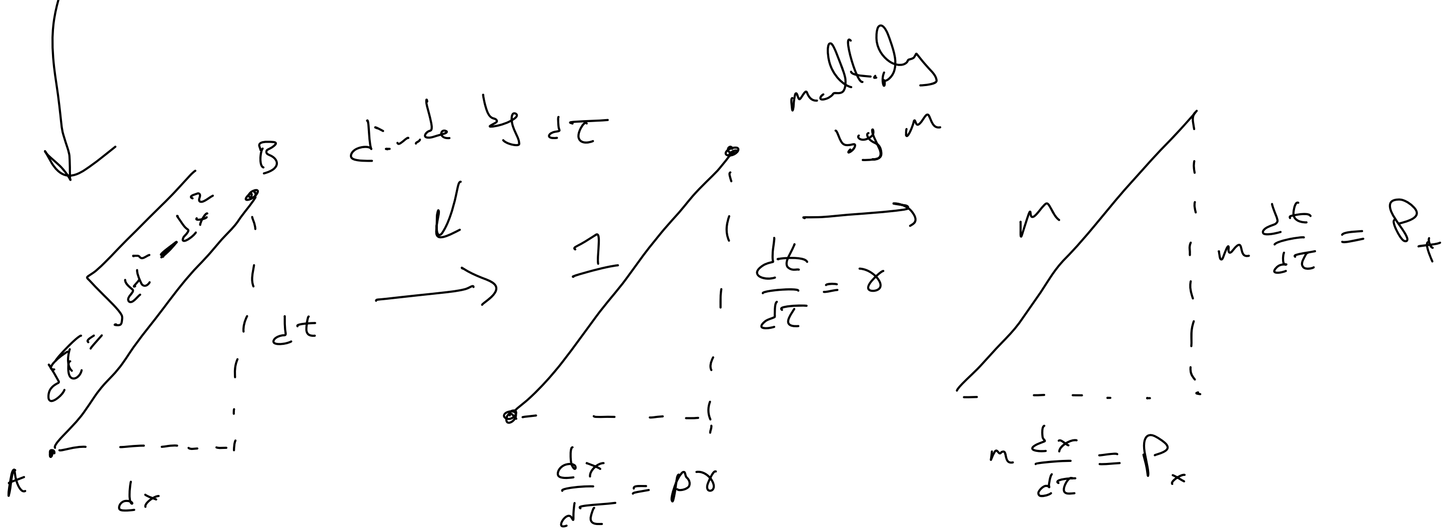
$$p = \begin{pmatrix} p_t \\ \vec{p} \end{pmatrix} = m\,\frac{d}{d\tau}\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \frac{m}{d\tau}\underbrace{\begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}}_{\text{Clearly a 4-vector}}$$

Is a 4-vector!

Multiply by invariants

$$d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$





$$d\tau^2 = dt^2 - dx^2 \longrightarrow m^2 = p_t^2 - p_x^2$$

Now what is P_t?

Units of mass, But not mass (invariant)

not momentum $m \frac{d\vec{x}}{d\tau}$

What else do we know about it?

Because $P = \begin{pmatrix} P_t \\ \vec{p} \end{pmatrix}$ is a 4-vector know how it transforms to different frame

$$\begin{pmatrix} P_t \\ P_x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} P_t' \\ P_x' \end{pmatrix}$$

Now lets look at our collision from two <u>different</u> frames

$(P_x^1 + P_x^2)$ - total x momentum Before collision in S-frame

By L.T.

$$(P_x^1 + P_x^2) = \gamma(P_x^{1'} + P_x^{2'}) + \beta\gamma(P_t^{1'} + P_t^{2'})$$

|| By mom cons S || By mom cons S' $\Rightarrow$ these must be =

$$(\overline{P_x^1} + \overline{P_x^2}) = \gamma(\overline{P_x^{1'}} + \overline{P_x^{2'}}) + \beta\gamma(\overline{P_t^{1'}} + \overline{P_t^{2'}})$$

If $\vec{p}$ conserved all frames, then P_t also conserved

Know one more thing.

$$P_t = m\frac{dt}{d\tau} = m\gamma\frac{dt}{dt} = m\gamma = \frac{m}{\sqrt{1-\beta^2}}$$

Small $\beta \ll 1$

$$P_t = m\left(1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + \dots\right) \sim C + \frac{1}{2}m\beta^2 + O(\beta^4)$$

Classical KE!

Summary P_t -
- Has units of E
- Total P_t sum of individual particle P_t's
- Is conserved in collisions
- Reduces to classical form $\beta \ll 1$

↳ All the properties we want in

Relativistic Energy!

(13)

$$E = m\frac{dt}{d\tau} = m\gamma = m\cosh\eta$$

Note Relativistic Energy Not KE.

$\beta \ll 1$

$$E = m + \frac{1}{2}m\beta^2$$

Related to KE

"Rest energy of particle"

Energy that the particle has when at rest $\beta = 0$

$$E_{\text{conventional}} = mc^2$$

Rest Energy can be ignored in Newtonian Physics

Dynamics only depends on ΔE's

Overall constants make have no impact.

Rest Energy essential to Relativistic Physics

Cannot have conservation P & E w/o it.

$E = \gamma m$ Becomes ∞ as $\beta \to c$

Cannot accelerate a particle to c even if we have ∞ Energy.

From the 4-vector invariant

$$m^2 = E^2 - p^2$$

or

$$E^2 = m^2 + \vec{p}^{\,2} \qquad E = \sqrt{m^2 + \vec{p}^{\,2}}$$

Note $p = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

$$\begin{aligned} p^2 &= E^2 - |\vec{p}|^2 \\ &= m^2 + |\vec{p}|^2 - |\vec{p}|^2 \\ &= m^2 \end{aligned}$$

Limits

$$E = \sqrt{m^2 + p^2}$$

"Non relativistic"

$p \ll m \qquad \beta \ll 1$

$$E = m\left(1 + \frac{p^2}{m^2}\right)^{1/2} \sim m + \frac{p^2}{2m} + \cdots$$

"Ultra Relativistic"

$p \gg m \qquad \beta \sim 1$

$$E = p\left(1 + \frac{m^2}{p^2}\right)^{1/2} \sim p + \frac{m^2}{2p} + \cdots$$

$$\sim p$$

Other comments

$$p = m\gamma\beta \qquad E = m\gamma$$

$$p = \beta E \qquad \beta = \frac{p}{E}$$

As $p \to E$ $\beta \to 1$

Classically

$$p = m\beta$$

"transport of mass"

Relativistically

$$p = E\beta$$

"transport of mass-Energy"

$$\gamma = \frac{E}{m} \qquad \beta = \frac{p}{E}$$

$$= \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{p^2}{E^2}}} = \frac{E}{\sqrt{E^2-p^2}} = \frac{E}{m}$$

Cool trick when $\beta \sim 1$

$$1-\beta^2 = (1+\beta)(1-\beta) \sim 2(1-\beta)$$

$$= \frac{1}{\gamma^2} = \frac{m^2}{E^2}$$

$$\Rightarrow \quad 1-\beta \sim \frac{m^2}{2E^2} \qquad \beta \sim 1$$