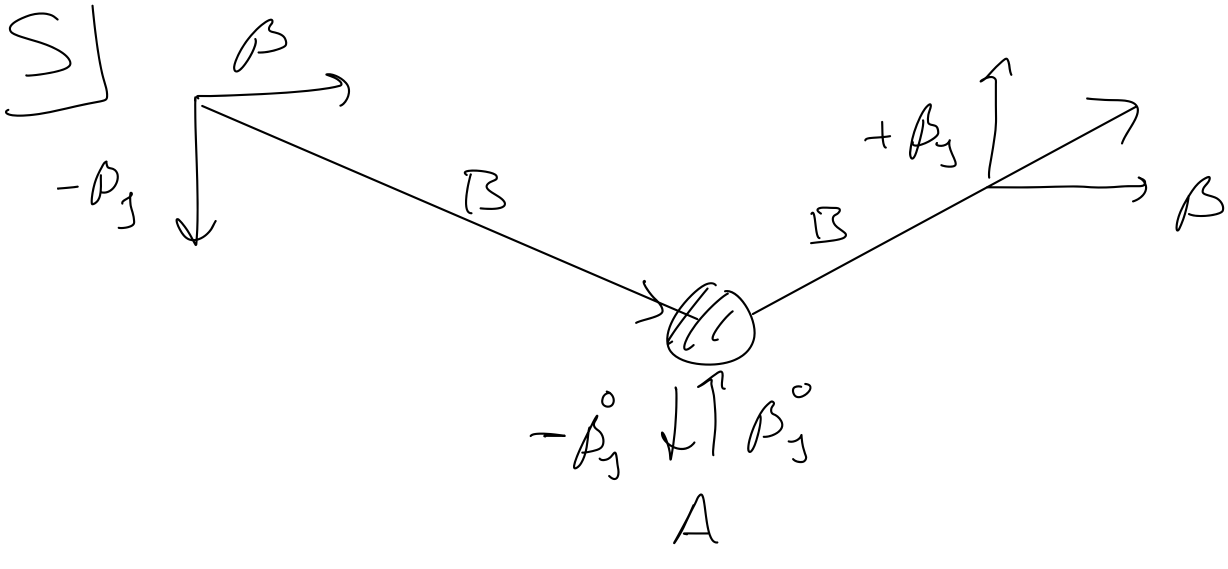
Reminder

- Trying to find Relativistic extension of $\vec{p}$



$$\underbrace{C(m_A,\beta_A)}\vec{\beta}_A + C(m_B,\beta_B)\vec{\beta}_B = C(m_A,\beta_A)\vec{\beta}_A + C(m_B,\beta_B)\vec{\beta}$$

$\hookrightarrow$ C's no longer invariant, but can only be a function of particle mass + speed.

S frame

$$\beta_A = \begin{pmatrix} 0 \\ \beta_y^0 \end{pmatrix} \qquad \bar{\beta}_A = \begin{pmatrix} 0 \\ -\beta_y^0 \end{pmatrix}$$

$$\beta_B = \begin{pmatrix} \beta \\ -\beta_y \end{pmatrix} \qquad \bar{\beta}_B = \begin{pmatrix} \beta \\ +\beta_y \end{pmatrix}$$

Note $\beta_y \neq \beta_y^0$, But we know how to calculate it!

$$\beta_y = \frac{\beta_y'}{\gamma(1+\beta_x'\beta)} = \frac{\beta_y^0}{\gamma(1+0)} = \frac{\beta_y^0}{\gamma}$$

$$\Rightarrow \frac{\beta_y^0}{\beta_y} = \gamma$$

Now let's impose conservation of y-momentum:

$$C(m, \beta_y^0)\,\beta_y^0 - C\left(m, \sqrt{\beta^2 + \beta_y^{0\,2}}\right)\beta_y$$
$$= -C(m, \beta_y^0)\,\beta_y^0 + C\left(m, \sqrt{\beta^2 + \beta_y^{0\,2}}\right)\beta_y$$

or
$$C\left(m, \sqrt{\beta^2 + \beta_y^{0\,2}}\right)\beta_y = C(m, \beta_y^0)\,\beta_y^0$$

or
$$\frac{C\left(m, \sqrt{\beta^2 + \beta_y^{0\,2}}\right)}{C(m, \beta_y^0)} = \frac{\beta_y^0}{\beta_y} = \gamma$$

If $\beta_y^0 \ll 1$ $\quad C(m, \beta_y^0) = m$ (Newtonian Limit)

$$\beta_y < \beta_y^0 \ll \sqrt{\beta^2 + \beta_y^{0\,2}}$$

$$\Rightarrow \sqrt{\beta^2 + \beta_y^{0\,2}} \sim \beta$$

Then,

$$\frac{c\left(m, \sqrt{\beta^2 + \beta_y^{o2}}\right)}{c(m, \beta_y^o)} = \frac{c(m, \beta)}{m} = \gamma$$

So $c(m, \beta) = m\gamma = \dfrac{m}{\sqrt{1-\beta^2}}$

$\Rightarrow$ Relativistic Momentum (i.e. the thing that is conserved + reduces to $m\beta$ at β small)

By construction

is given By

$$\vec{p} = m\gamma\vec{\beta}$$

Comments: the γ means that a particle can transport an arbitrarily great amt of p if $\beta \sim 1$

Not expected from the Newtonian formula

Many different ways to view this

η Classical $m\beta = m \tanh \eta$

Relativistic $m\beta\gamma = m \sinh \eta$

Note η small

$\tanh \eta \sim \sinh \eta \sim \eta$

$$\vec{p} = \underbrace{m\gamma}\,\vec{\beta}$$

"Bad way" $\quad m \longrightarrow m_r = \dfrac{m_0}{\sqrt{1-\beta^2}} \qquad \vec{p} = m_r \vec{\beta}$

Sometimes see this in old books.

Useful if you like "$\vec{p} \equiv m\vec{\beta}$"

Not useful if you want to avoid confusion

Nicer to keep mass as an invariant property

Better way of thinking about it

$$\vec{p} = m\gamma\vec{\beta}$$

$$= m\gamma\frac{d\vec{x}}{dt} = m\gamma\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{pmatrix} = \overbrace{m}^{\text{Invariant}}\gamma\overbrace{\frac{d}{dt}}^{\text{Not Invariant, } t=\gamma\tau}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

But note, $t = \gamma\tau \leftarrow$ Proper time (Invariant)

$$\Rightarrow dt = \gamma\, d\tau \quad \& \quad \gamma\frac{d}{dt} = \frac{d}{d\tau}$$

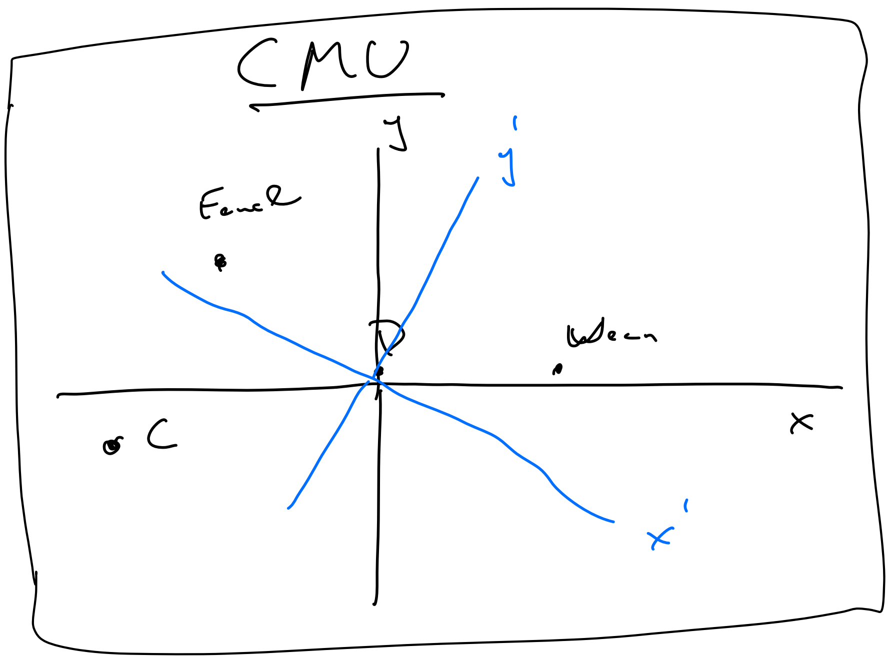
So, $$\vec{p} = m\frac{d\vec{x}}{d\tau} = \underbrace{m\frac{d}{d\tau}}_{\text{Both invariant!}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

Now take a slight detour & pick up this thread in a few minutes

Vectors in Space-time

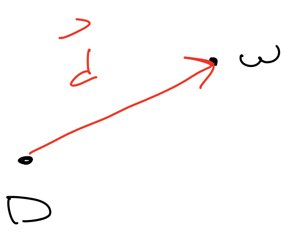
When introducing Space-time Events.

Draw on Geometric analogy | Made point that the coordinates weren't "Real" but the locations of points were

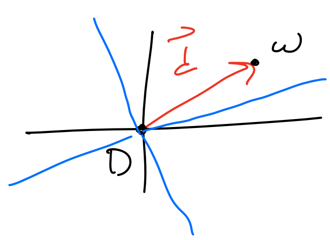


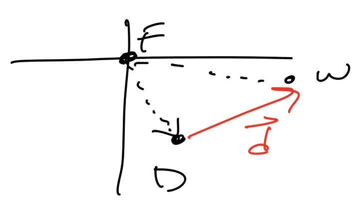
Bit of a simplification | - Points real given a definition for the origin.

What is actually real is the "vector" from the location of the origin to the location of the point



This vector $\vec{d}$ exists in the world independent of axis orientation + Origin definition.





Vectors have independent existance:

eg length $|d|^2 = \vec{d}\cdot\vec{d} = \underbrace{x^2 + y^2 + z^2}_{\text{each term covariant}}$

the sum is <u>invariant</u>

$$\vec{d}' = R\vec{d} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

$$\vec{d}^T\vec{d}$$

under rotation $\vec{d}' \to R\vec{d}$

$$\vec{d}'^T \to \vec{d}^T R^T$$

$$\vec{d}'^2 = d'^T d' = d^T R^T R d = d^T d = \vec{d}^2$$

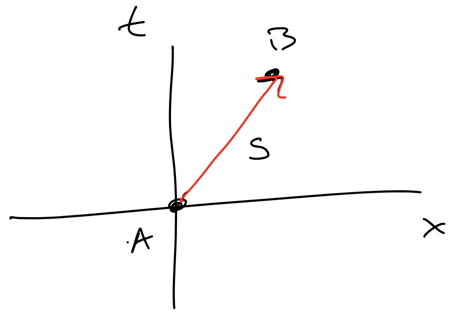
Similarly things in space-time that are real are the vectors between events $A = (t, x, y, z)$

$B = (t, x, y, z)$

Because the events are 4-dimensional entities, the corresponding space-time vectors called "4-vectors"

Defined such that

$$S = (t, x, y, z)$$



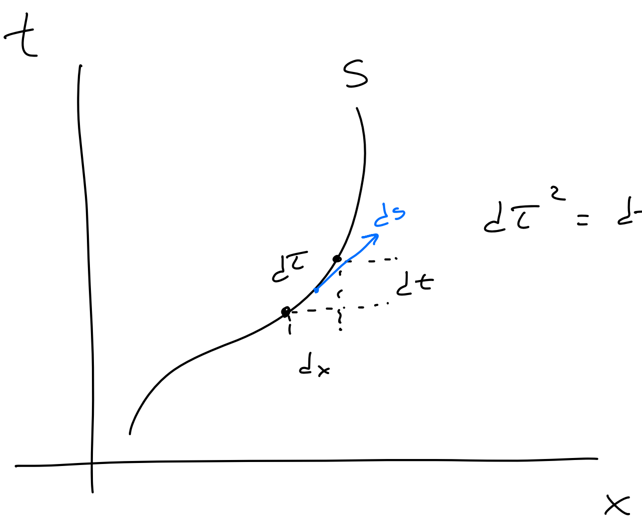
$$S^2 = \underbrace{t^2 - (x^2 + y^2 + z^2)}$$

As above each term coordinate dependent
But sum invariant.

1D

$$S^2 = t^2 - x^2$$

$$S' = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$



$$d\tau^2 = dt^2 - dx^2$$

$$ds = \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

Calculate proper time along a world line by integrating 4 vectors ds in tangent space

$$S = \int_S ds$$

Back to momentum ...

Here

$$\vec{p} = m\frac{d\vec{x}}{d\tau} = m\frac{d}{d\tau}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$ ← Nearly here 4-vector! missing "time" component

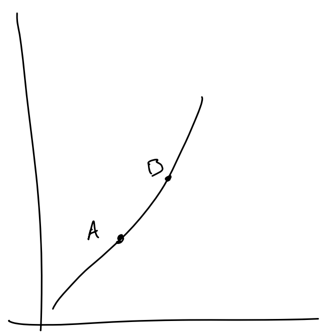
Put it in by hand

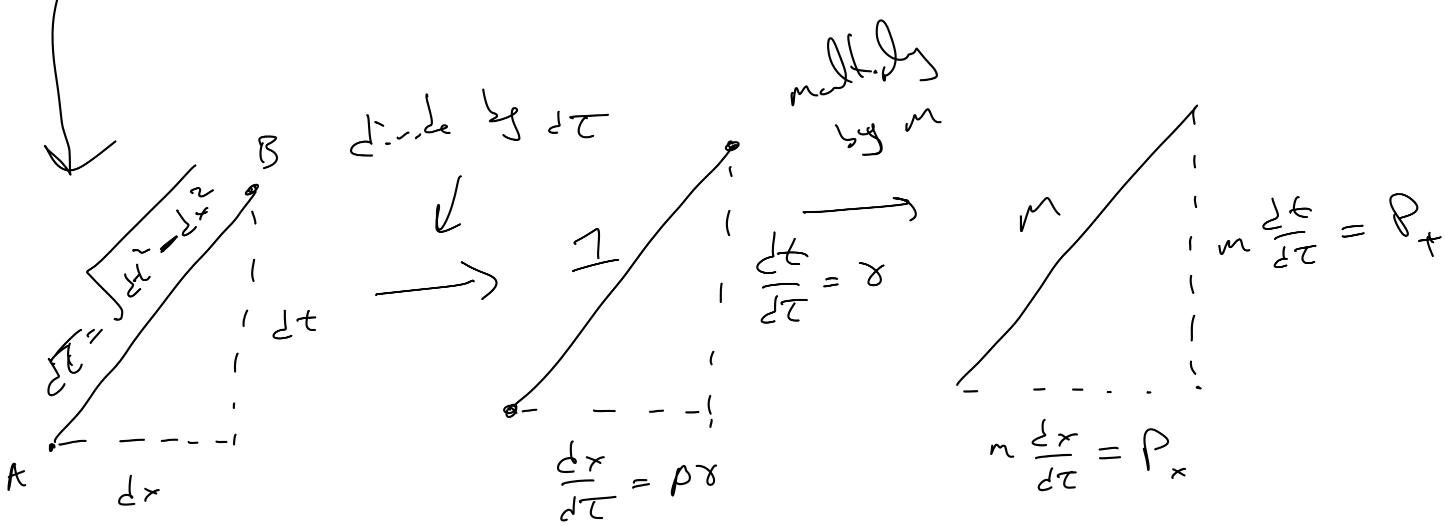
$$p = \begin{pmatrix} p_t \\ \vec{p} \end{pmatrix} = m\frac{d}{d\tau}\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \frac{m}{d\tau}\underbrace{\begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}}_{\text{Clearly a 4-vector}}$$

Is a 4-vector!

Multiply by invariants

$$d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$





$$d\tau^2 = dt^2 - dx^2 \longrightarrow m^2 = p_t^2 - p_x^2$$

Now what is P_t ?

Units of mass, But not mass (invariant)

not momentum $m \frac{d\vec{x}}{d\tau}$

What else do we know about it?

Because $P = \begin{pmatrix} P_t \\ \vec{p} \end{pmatrix}$ is a 4-vector know how it transforms to different frame

$$\begin{pmatrix} P_t \\ P_x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} P_t' \\ P_x' \end{pmatrix}$$

Now lets look at our collision from two different frames

$(P_x^1 + P_x^2)$ - total x momentum Before collision in S-frame

By L.T.

$$(P_x^1 + P_x^2) = \gamma (P_x^{1'} + P_x^{2'}) + \beta\gamma (P_t^{1'} + P_t^{2'})$$

$\|$ By mom cons S $\quad\quad \|$ By mom cons S' $\Rightarrow$ $\updownarrow$ these must be =

$$(\overline{P_x^1} + \overline{P_x^2}) = \gamma (\overline{P_x^{1'}} + \overline{P_x^{2'}}) + \beta\gamma (\overline{P_t^{1'}} + \overline{P_t^{2'}})$$

If $\vec{p}$ conserved all frames, then P_t also conserved

Know one more thing.

$$P_t = m\frac{dt}{d\tau} = m\gamma\frac{dt}{dt} = m\gamma = \frac{m}{\sqrt{1-\beta^2}}$$

Small $\beta \ll 1$

$$P_t = m\left(1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + \dots\right) \sim C + \frac{1}{2}m\beta^2 + \mathcal{O}(\beta^4)$$

Classical KE!

Summary P_t -

- Has units of E
- Total P_t sum of individual particle P_t's
- Is Conserved in Collisions
- Reduces to classical form $\beta \ll 1$

↳ All the properties we want in **Relativistic Energy**!

$$E = m\frac{dt}{d\tau} = m\gamma = m\cosh\eta$$

Note Relativistic Energy <u>Not</u> KE.

<u>$\beta \ll 1$</u>

$$E = m + \frac{1}{2}m\beta^2$$ Related to KE

"Rest energy of particle"

Energy that the particle has when at rest $\beta = 0$

$$E_{\text{conventional}} = mc^2$$

Rest Energy can be ignored in Newtonian Physics

Dynamics only depends on ΔE's

Overall constants make have no impact.

Rest Energy essential to Relativistic Physics

Cannot have conservation P & E w/o it.

$E = \gamma m$ Becomes ∞ as $\beta \to c$

Cannot accelerate a particle to c even if we have ∞ Energy.

from the 4-vector invariant

$$m^2 = E^2 - p^2$$

or

$$E^2 = m^2 + \vec{p}^2 \qquad E = \sqrt{m^2 + \vec{p}^2}$$

Note $P = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

$$\begin{aligned} P^2 &= E^2 - |\vec{p}|^2 \\ &= m^2 + |\vec{p}|^2 - |\vec{p}|^2 \\ &= m^2 \end{aligned}$$

Limits

$$E = \sqrt{m^2 + p^2}$$

"Non relativistic"

$p \ll m \qquad \beta \ll 1$

$$E = m\left(1 + \frac{p^2}{m^2}\right)^{1/2} \simeq m + \frac{p^2}{2m} + \cdots$$

"Ultra Relativistic"

$p \gg m \qquad \beta \sim 1$

$$E = p\left(1 + \frac{m^2}{p^2}\right)^{1/2} \simeq p + \frac{m^2}{2p} + \cdots$$

$$\simeq p$$

Other comments

$$p = m\gamma\beta \qquad E = m\gamma$$

$$p = \beta E \qquad \beta = \frac{p}{E}$$

As $p \to E$ $\quad \beta \to 1$

Classically

$$p = m\beta$$

"transport of mass"

Relativistically

$$p = E\beta$$

"transport of mass-Energy"

$$\gamma = \frac{E}{m} \qquad \beta = \frac{p}{E}$$

$$= \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{p^2}{E^2}}} = \frac{E}{\sqrt{E^2-p^2}} = \frac{E}{m}$$

<u>Cool trick</u> when $\beta \sim 1$

$$1-\beta^2 = (1+\beta)(1-\beta) \sim 2(1-\beta)$$

$$= \frac{1}{\gamma^2} = \frac{m^2}{E^2}$$

$$\Rightarrow \quad 1-\beta \sim \frac{m^2}{2E^2} \qquad \beta \sim 1$$