Mass Properties of a Triangular Pyramid

And more complex meshes such as STL files

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Abstract

Consider a triangle in space defined by three vertices A, B, C and connect them to the origin to create a elemental volume element. The shape is that of a triangular pyramid, or wedge. Such shape may come from any mesh definition consisting of nodes (vertices) and elements (faces). Examples are STL files, OBJ files, SLP files, and VDA files. The purpose of this paper is to analyze each triangular face in the mesh and establish calculations for properties such as area, volume, center of mass and mass moment of inertia. The combined properties of all the triangular faces in a mesh are then used to establish the mass properties of a solid geometry object. The end result is to establish the mass properties of a complex object for use in rigid body simulations.

1 Definitions

A mesh decomposed a shape into multiple spatial triangular faces. Each face is described by three vertices, each with coordinates $\boldsymbol{r}_A$, $\boldsymbol{r}_B$, and $\boldsymbol{r}_C$. Additionally, the uniform density ρ is given for the entire shape. That is all the information needed to extract all the mass properties needed. See Figure 1 for a visualization of each wedge elemental volume described by the triangular face ABC and the origin. The mass properties of each elemental volume is calculated in order to combine them to get the mass properties of the entire shape.

1.1 Parameterization

The goal is to find a way to represent the position of every point inside the volume using three parameters u, v and w that only range between $0 \ldots 1$ in order to cover the entire volume. Start by parameterizing the points between

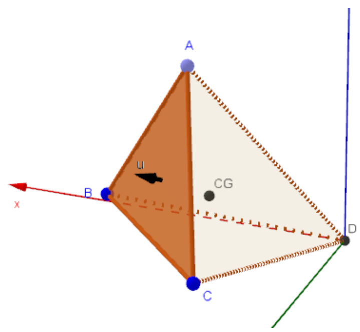

Figure 1 – Basic wedge definition from three vertices A, B and C to form a triangle that is connected to the origin. The properties considered are: a) The area of the triangle, b) The normal direction of the triangle, c) The centroid of the triangle, d) The volume of the wedge, e) The center of mass of the wedge, f) The mass moment of inertia of the wedge.

two points, for example B and C. This is done with $(1-v)\,\boldsymbol{r}_B + v\,\boldsymbol{r}_C$. Apply the same scheme again to parameterize the triangle ABC by

$$\boldsymbol{r}_\triangle = (1-u)\,\boldsymbol{r}_A + u\,\left((1-v)\,\boldsymbol{r}_B + v\,\boldsymbol{r}_C\right) \tag{1}$$

A simple scaling by a parameter w can be used to select which triangle is represented between the origin and ABC. So the volume parameterization is

$$\boldsymbol{r}(u,v,w) = w\,(1-u)\,\boldsymbol{r}_A + w\,u\,\,(1-v)\,\boldsymbol{r}_B + w\,u\,v\,\boldsymbol{r}_C \tag{2}$$

2 Analysis

To analyze a parametric surface or volume, the following partial derivatives are needed

$$\begin{aligned}\frac{\partial\boldsymbol{r}}{\partial u} &= w\,(-\boldsymbol{r}_A + (1-v)\,\boldsymbol{r}_B + v\boldsymbol{r}_C)\\ \frac{\partial\boldsymbol{r}}{\partial v} &= -u\,w\,\,(\boldsymbol{r}_B - \boldsymbol{r}_C)\\ \frac{\partial\boldsymbol{r}}{\partial w} &= (1-u)\,\boldsymbol{r}_A + u\,\,(1-v)\,\boldsymbol{r}_B + u\,v\,\boldsymbol{r}_C\end{aligned} \tag{3}$$

2.1 Area

The surface area of the triangle is only of interest because it is intermediate step to finding the volume. Also after the volume is calculated, the area is used to find the height of the triangle since the well known volume formula connects the base area of a triangular prism with the height of the prism.

$$(\text{volume}) = \frac{1}{3}(\text{height})(\text{base area}) \tag{4}$$

The differential area element for the parametric surface $\boldsymbol{r}(u,v)$ where w is fixed is defined by

$$\mathrm{d}A = \| \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) \| \mathrm{d}u\mathrm{d}v \tag{5}$$

The cross product is expanded

$$\begin{aligned}
&\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \\
&= \left(-w\boldsymbol{r}_A + w\,(1-v)\,\boldsymbol{r}_B + w\,v\boldsymbol{r}_C\right) \times \left(-u\,w\,\left(\boldsymbol{r}_B - \boldsymbol{r}_C\right)\right) \\
&= \left(u\,w^2\right)\left(\boldsymbol{r}_A - (1-v)\,\boldsymbol{r}_B - v\boldsymbol{r}_C\right) \times \left(\boldsymbol{r}_B - \boldsymbol{r}_C\right) \\
&= \left(u\,w^2\right)\left(\left(\boldsymbol{r}_A - v\boldsymbol{r}_C\right) \times \boldsymbol{r}_B - \left(\boldsymbol{r}_A - (1-v)\,\boldsymbol{r}_B\right) \times \boldsymbol{r}_C\right) \\
&= \left(u\,w^2\right)\left(\left(\boldsymbol{r}_A \times \boldsymbol{r}_B\right) - v\left(\boldsymbol{r}_C \times \boldsymbol{r}_B\right) - \left(\boldsymbol{r}_A \times \boldsymbol{r}_C\right) + (1-v)\left(\boldsymbol{r}_B \times \boldsymbol{r}_C\right)\right) \\
&= \left(u\,w^2\right)\left(\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A\right)
\end{aligned} \tag{6}$$

The last part with all the cross products defines a constant vector $\boldsymbol{n}$ which happens to be normal to the triangle.

$$\mathrm{d}A = \|\boldsymbol{n}\| \left(u\,w^2\right) \mathrm{d}u\mathrm{d}v \tag{7}$$

where

$$\boldsymbol{n} = \boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A \tag{8}$$

point to the surface normal of the triangle ABC. You can prove at the above by defining the surface normal as the cross product of two sides

$$\begin{aligned}
\boldsymbol{n} &= \left(\boldsymbol{r}_B - \boldsymbol{r}_A\right) \times \left(\boldsymbol{r}_C - \boldsymbol{r}_A\right) \\
&= \boldsymbol{r}_B \times \boldsymbol{r}_C - \boldsymbol{r}_A \times \boldsymbol{r}_C - \boldsymbol{r}_B \times \boldsymbol{r}_A
\end{aligned} \tag{9}$$

So the area of the the triangle with constant w is

$$\begin{aligned}
A(w) &= \int \mathrm{d}A = w^2 \left(\iint_0^1 u\,\mathrm{d}u\mathrm{d}v \right) \|\boldsymbol{n}\| \\
A(w) &= \tfrac{w^2}{2} \|\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A\|
\end{aligned} \tag{10}$$

This confirms the well known formula for the triangle ABC, by using $w = 1$ above

$$A_{\triangle} = \tfrac{1}{2} \|\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A\| \tag{11}$$

2.2 Volume & Mass

The differential volume element for the parametric triangular pyramid $\boldsymbol{r}(u,v,w)$ is defined by

$$\mathrm{d}V = \frac{\partial \boldsymbol{r}}{\partial w} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) \mathrm{d}u\mathrm{d}v\mathrm{d}w \tag{12}$$

The cross product is already simplified above in the area calculation, and so the volume element is

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}u\mathrm{d}v\mathrm{d}w} &= \frac{\partial \boldsymbol{r}}{\partial w} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) \\
&= \frac{\partial \boldsymbol{r}}{\partial w} \cdot \left(\left(u\, w^2 \right) \boldsymbol{n} \right) \\
&= \left(u\, w^2 \right) \left((1-u)\, \boldsymbol{r}_A \cdot \boldsymbol{n} + u\, (1-v)\, \boldsymbol{r}_B \cdot \boldsymbol{n} + u\, v\, \boldsymbol{r}_C \cdot \boldsymbol{n} \right) \\
&= w^2 u\, (1-u)\, \boldsymbol{r}_A \cdot \left(\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A \right) \\
&\quad + w^2 u^2\, (1-v)\, \boldsymbol{r}_B \cdot \left(\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A \right) \\
&\quad + w^2 u^2\, v\, \boldsymbol{r}_C \cdot \left(\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A \right) \\
&= w^2 u\, (1-u)\, \boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) + w^2 u^2\, (1-v)\, \boldsymbol{r}_B \cdot (\boldsymbol{r}_C \times \boldsymbol{r}_A) + w^2 u^2\, v\, \boldsymbol{r}_C \cdot (\boldsymbol{r}_A \times \boldsymbol{r}_B) \\
&= w^2 u\, (1-u)\, \boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) + \left(w^2 u^2\, (1-v) + w^2 u^2\, v \right) \boldsymbol{r}_C \cdot (\boldsymbol{r}_A \times \boldsymbol{r}_B) \\
&= w^2 u\, (1-u)\, \boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) + \left(w^2 u^2 \right) \boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) \\
&= \left(w^2 u\, (1-u) + w^2 u \right) \left(\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) \right)
\end{aligned} \tag{13}$$

The simplifications above are due to the possible permutations of the vector triple product $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$.

$$\mathrm{d}V = w^2 u\, \left(\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) \right) \mathrm{d}u\mathrm{d}v\mathrm{d}w \tag{14}$$

The total volume of the triangular pyramid is calculated as

$$\begin{aligned}
V = \int \mathrm{d}V = \iiint_0^1 w^2 u\, \left(\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) \right) \mathrm{d}u\mathrm{d}v\mathrm{d}w \\
V = \tfrac{1}{6} \left(\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C) \right)
\end{aligned} \tag{15}$$

The above constant $6V = \boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C)$ is used in the volume integrals below to calculate the mass properties of the pyramid volume

$$m = 6 \iiint_0^1 \left(w^2 u \right) \rho V\, \mathrm{d}u\mathrm{d}v\mathrm{d}w = \rho V \tag{16}$$

2.3 Height

The height of the pyramid is calculated from the volume and surface area of triangle base

$$h = \frac{3V}{A} = \frac{\frac{3}{6}\left(\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C)\right)}{\frac{1}{2}\|\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A\|}$$

$$h = \frac{\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C)}{\|\boldsymbol{r}_A \times \boldsymbol{r}_B + \boldsymbol{r}_B \times \boldsymbol{r}_C + \boldsymbol{r}_C \times \boldsymbol{r}_A\|} \tag{17}$$

2.4 Center of Mass

The center of mass is defined as the weighted sum of each location within the volume.

$$\begin{aligned} \boldsymbol{c} &= \tfrac{1}{m} 6 \iiint_0^1 \left(w^2 u\right) \boldsymbol{r}(u,v,w) \rho V \,\mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= 6 \iiint_0^1 \left(w^2 u\right) \left(w\,(1-u)\,\boldsymbol{r}_A + w\,u\;(1-v)\,\boldsymbol{r}_C + w\,u\,v\,\boldsymbol{r}_C\right) \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= 6 \int_0^1 \tfrac{w^3}{6} \left(\boldsymbol{r}_A + \boldsymbol{r}_C + \boldsymbol{r}_C\right) \mathrm{d}w \end{aligned} \tag{18}$$

The result is consistent with the well known formula for center of mass of a triangular pyramid

$$\boldsymbol{c} = \tfrac{1}{4}\left(\boldsymbol{r}_A + \boldsymbol{r}_C + \boldsymbol{r}_C\right) \tag{19}$$

2.5 Mass Moment of Inertia

Mass moment of inertia about the origin is defined from the integral form of the parallel axis theorem

$$\begin{aligned} \mathbf{I} &= \int \rho \left(-\boldsymbol{r} \times \boldsymbol{r}\times\right) \mathrm{d}V \\ &= 6m \iiint_0^1 \left(w^2 u\right) \left(-\boldsymbol{r}(u,v,w) \times \boldsymbol{r}(u,v,w)\times\right) \mathrm{d}u\mathrm{d}v\mathrm{d}w \end{aligned} \tag{20}$$

here $\mathbf{cross}\left(\boldsymbol{r}\right) = \boldsymbol{r}\times$ represents a 3×3 skew symmetric matrix called the cross product operator matrix, and it is defined as

$$\mathbf{cross}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times = \begin{vmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{vmatrix} \tag{21}$$

The parallel axis part is simplified a little by collecting like terms

$$\begin{aligned}
&-\boldsymbol{r}(u,v,w)\times\boldsymbol{r}(u,v,w)\times= \\
&\quad-\left(w\,(1-u)\,\boldsymbol{r}_A\times+w\,u\,(1-v)\,\boldsymbol{r}_B\times+w\,u\,v\,\boldsymbol{r}_C\times\right)\left(w\,(1-u)\,\boldsymbol{r}_A\times\right) \\
&\quad-\left(w\,(1-u)\,\boldsymbol{r}_A\times+w\,u\,(1-v)\,\boldsymbol{r}_B\times+w\,u\,v\,\boldsymbol{r}_C\times\right)\left(w\,u\,(1-v)\,\boldsymbol{r}_B\times\right) \\
&\quad-\left(w\,(1-u)\,\boldsymbol{r}_A\times+w\,u\,(1-v)\,\boldsymbol{r}_B\times+w\,u\,v\,\boldsymbol{r}_C\times\right)\left(w\,u\,v\,\boldsymbol{r}_C\times\right) \\
&=-w^2\,(1-u)\left((1-u)\,\boldsymbol{r}_A\times\boldsymbol{r}_A\times+u\,(1-v)\,\boldsymbol{r}_B\times\boldsymbol{r}_A\times+u\,v\,\boldsymbol{r}_C\times\boldsymbol{r}_A\times\right) \\
&-w^2\,u\,(1-v)\left((1-u)\,\boldsymbol{r}_A\times\boldsymbol{r}_B\times+u\,(1-v)\,\boldsymbol{r}_B\times\boldsymbol{r}_B\times+u\,v\,\boldsymbol{r}_C\times\boldsymbol{r}_B\times\right) \\
&-w^2\,u\,v\,\left((1-u)\,\boldsymbol{r}_A\times\boldsymbol{r}_C\times+u\,(1-v)\,\boldsymbol{r}_B\times\boldsymbol{r}_C\times+u\,v\,\boldsymbol{r}_C\times\boldsymbol{r}_C\times\right)
\end{aligned} \tag{22}$$

The integration is carried out in three parts

$$\begin{aligned}
\mathbf{I}&=6m\iiint_0^1\left(w^2u\right)\left(-\boldsymbol{r}(u,v,w)\times\boldsymbol{r}(u,v,w)\times\right)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&=-6m\iiint_0^1\left(w^2u\right)\left(w^2\,(1-u)\left((1-u)\,\boldsymbol{r}_A\times\boldsymbol{r}_A\times+u\,(1-v)\,\boldsymbol{r}_B\times\boldsymbol{r}_A\times+u\,v\,\boldsymbol{r}_C\times\boldsymbol{r}_A\times\right)\right)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&-6m\iiint_0^1\left(w^2u\right)\left(w^2\,u\,(1-v)\left((1-u)\,\boldsymbol{r}_A\times\boldsymbol{r}_B\times+u\,(1-v)\,\boldsymbol{r}_B\times\boldsymbol{r}_B\times+u\,v\,\boldsymbol{r}_C\times\boldsymbol{r}_B\times\right)\right)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&-6m\iiint_0^1\left(w^2u\right)\left(w^2\,u\,v\,\left((1-u)\,\boldsymbol{r}_A\times\boldsymbol{r}_C\times+u\,(1-v)\,\boldsymbol{r}_B\times\boldsymbol{r}_C\times+u\,v\,\boldsymbol{r}_C\times\boldsymbol{r}_C\times\right)\right)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&=-6m\left(\frac{2\boldsymbol{r}_A\times\boldsymbol{r}_A\times+\boldsymbol{r}_B\times\boldsymbol{r}_A\times+\boldsymbol{r}_C\times\boldsymbol{r}_A\times}{120}+\right. \\
&\qquad+\frac{\boldsymbol{r}_A\times\boldsymbol{r}_B\times+2\boldsymbol{r}_B\times\boldsymbol{r}_B\times+\boldsymbol{r}_C\times\boldsymbol{r}_B\times}{120} \\
&\qquad\left.+\frac{\boldsymbol{r}_A\times\boldsymbol{r}_C\times+\boldsymbol{r}_B\times\boldsymbol{r}_C\times+2\boldsymbol{r}_C\times\boldsymbol{r}_C\times}{120}\right) \\
&=-6m\frac{\left[(\boldsymbol{r}_A+\boldsymbol{r}_B)\times\right]^2+\left[(\boldsymbol{r}_B+\boldsymbol{r}_A)\times\right]^2+\left[(\boldsymbol{r}_C+\boldsymbol{r}_A)\times\right]^2}{120}
\end{aligned} \tag{23}$$

with some work collecting like terms the mass moment of inertia of the triangular pyramid is

$$\mathbf{I}=\tfrac{m}{20}\left(\mathbf{parallel}\left(\boldsymbol{r}_A+\boldsymbol{r}_B\right)+\mathbf{parallel}\left(\boldsymbol{r}_B+\boldsymbol{r}_C\right)+\mathbf{parallel}\left(\boldsymbol{r}_C+\boldsymbol{r}_A\right)\right) \tag{24}$$

where $\mathbf{parallel}(\boldsymbol{r})=-\left[\boldsymbol{r}\times\right]^2=-\boldsymbol{r}\times\boldsymbol{r}\times=\begin{vmatrix}y^2+z^2 & -xy & -xz\\ -xy & x^2+z^2 & -yz\\ -xz & -yz & x^2+y^2\end{vmatrix}$ is the parallel axis theorem in matrix form.

Finally, the mass moment of inertia at the center of mass is calculated yet again with the parallel axis theorem

$$\mathbf{I}_C=\mathbf{I}-m\left(-\boldsymbol{c}\times\boldsymbol{c}\times\right)=\mathbf{I}-m\,\mathbf{parallel}\left(\boldsymbol{c}\right) \tag{25}$$

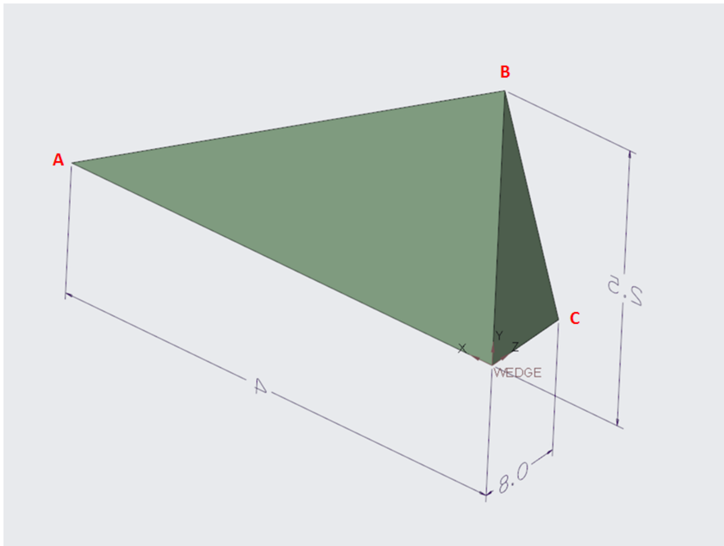

Figure 2 – Simple wedge definition from three points ABC. The volume and mass properties are calculated in CAD and compared to the values calculated in this paper.

3 Example Calculation

A simple wedge is defined in Figure 2 with sides 40mm, 25mm and 8mm. The three vertices defining the driving triangle are defined as

$$\boldsymbol{r}_A = \begin{pmatrix} 40 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{r}_B = \begin{pmatrix} 0 \\ 25 \\ 0 \end{pmatrix} \quad \boldsymbol{r}_C = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} \tag{26}$$

The values calculated in CAD are $V = 1333.33\overline{3}$ and with density of $\rho = 10\,10^{-6}$ the mass is $m = 0.0133\overline{3}$. The center of mass is at $\boldsymbol{c} = \begin{pmatrix} 10.0 & 6.25 & 2.0 \end{pmatrix}$and the mass moment of inertia components at the origin and center of mass are

$$\mathbf{I} = \begin{vmatrix} 0.9186\overline{6} & -0.66\overline{6} & -0.213\overline{3} \\ & 0.22186\overline{6} & -0.133\overline{3} \\ \text{symmetry} & & 2.96\overline{6} \end{vmatrix} \quad \mathbf{I}_C = \begin{vmatrix} 0.3445 & 0.166\overline{6} & 0.0533\overline{3} \\ & 0.8320 & 0.33\overline{3} \\ \text{symmetry} & & 1.1125 \end{vmatrix} \tag{27}$$

These the calculations in this paper are confirmed from the values above

$$V = \tfrac{1}{6}\begin{pmatrix}40\\0\\0\end{pmatrix}\cdot\left(\begin{pmatrix}0\\25\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\8\end{pmatrix}\right) = \tfrac{1}{6}\begin{pmatrix}40\\0\\0\end{pmatrix}\cdot\begin{pmatrix}200\\0\\0\end{pmatrix} = \frac{8000}{6} = 1333.3\overline{3}\ \checkmark \tag{28}$$

$$m = \rho V = 10\,10^{-6}\,1333.3\overline{3} = 0.01333\overline{3}\ \checkmark \tag{29}$$

$$\boldsymbol{c} = \tfrac{1}{4}\left(\begin{pmatrix}40\\0\\0\end{pmatrix}+\begin{pmatrix}0\\25\\0\end{pmatrix}+\begin{pmatrix}0\\0\\8\end{pmatrix}\right) = \tfrac{1}{4}\begin{pmatrix}40\\25\\8\end{pmatrix} = \begin{pmatrix}10\\6.25\\2\end{pmatrix}\ \checkmark \tag{30}$$

$$\mathbf{I} = \tfrac{m}{20}\left(\textbf{parallel}\begin{pmatrix}40\\25\\0\end{pmatrix}+\textbf{parallel}\begin{pmatrix}0\\25\\8\end{pmatrix}+\textbf{parallel}\begin{pmatrix}40\\0\\8\end{pmatrix}\right) \tag{31}$$

$$= m\begin{vmatrix}68.9 & -50 & -16\\ & 166.4 & -10\\ \text{symmetry} & & 22.5\end{vmatrix} = \begin{vmatrix}0.9186\overline{6} & -0.66\overline{6} & -0.213\overline{3}\\ & 0.22186\overline{6} & -0.133\overline{3}\\ \text{symmetry} & & 2.96\overline{6}\end{vmatrix}\ \checkmark \tag{32}$$

$$\mathbf{I}_C = \begin{vmatrix}0.9186\overline{6} & -0.66\overline{6} & -0.213\overline{3}\\ & 0.22186\overline{6} & -0.133\overline{3}\\ \text{symmetry} & & 2.96\overline{6}\end{vmatrix} - m\,\textbf{parallel}\begin{pmatrix}10\\6.25\\2\end{pmatrix} = \tag{33}$$

$$= \begin{vmatrix}0.9186\overline{6} & -0.66\overline{6} & -0.213\overline{3}\\ & 0.22186\overline{6} & -0.133\overline{3}\\ \text{symmetry} & & 2.96\overline{6}\end{vmatrix} - m\begin{vmatrix}40.0625 & -62.5 & -20.0\\ & 104.0 & -12.5\\ \text{symmetry} & & 139.0625\end{vmatrix} \tag{34}$$

$$= \begin{vmatrix}0.3445 & 0.166\overline{6} & 0.0533\overline{3}\\ & 0.8320 & 0.33\overline{3}\\ \text{symmetry} & & 1.1125\end{vmatrix}\ \checkmark \tag{35}$$

4 Complex Geometry

To handle a complex mesh, such as an STL file, then each triangle defines volume V_i, mass m_i, center of mass $\boldsymbol{c}_i$ and mass moment of inertia $\mathbf{I}_i$. All of these are combined to form the overall properties. Note that the volume might be either a positive or a negative value depending if the triangle nodes define a clockwise sequence or a counter-clockwise sequence which results the contact normal pointing away from the origin in the first case, and towards the origin in the second case. This happens naturally in the calculation of $\boldsymbol{r}_A \cdot (\boldsymbol{r}_B \times \boldsymbol{r}_C)$ for each triangle, and the result is that for *any* closed shape that the sum of all

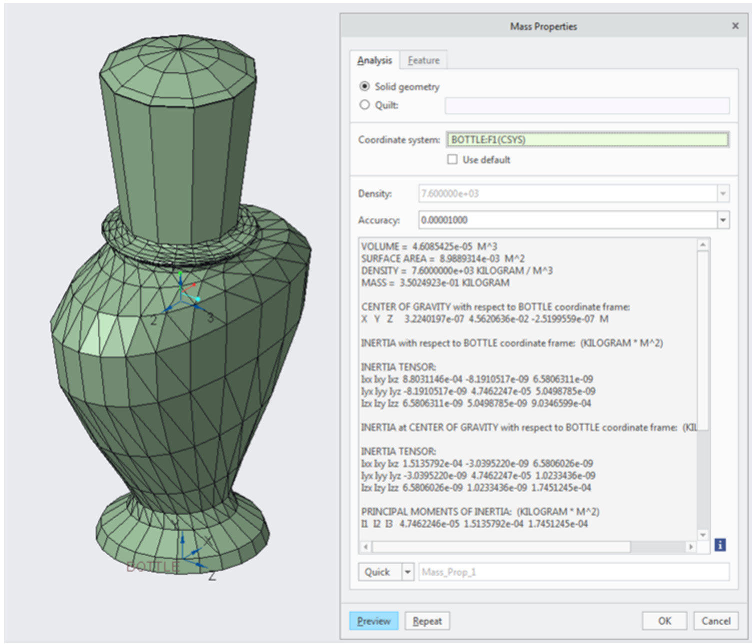

Figure 3 – Complex shape defined as an STL file with mass properties analyzed in CAD. The file contains 5088 vertices and 1696 triangular faces.

the volumes yields *the volume of the enclosed space* only.

$$V = \sum_i V_i \tag{36}$$

$$m = \sum_i \rho_i V_i \tag{37}$$

$$\boldsymbol{c} = \tfrac{1}{m} \sum_i \boldsymbol{c}_i \rho_i V_i \tag{38}$$

$$\mathbf{I} = \tfrac{1}{m} \sum_i \rho_i V_i \mathbf{I}_i \tag{39}$$

$$\mathbf{I}_C = \mathbf{I} - m\, \mathbf{parallel}\left(\boldsymbol{c}\right) \tag{40}$$

The surprising thing is that the above works very well. In figure 3 a complex shape defined in an STL file is imported into CAD and the mass properties analyzed. The same STL file was processed using the calculations in this paper in a Fortran program. The results are summarized in the table below using a density of $\rho = 7600\,\mathrm{kg/m^3}$:

Property	Symbol	Component	CAD	Fortran	Units	Check
Mass	m		0.35025	0.35025	kg	✓
Volume	V		46085.4	46086.1	mm^3	✓
Center of Mass	$\boldsymbol{c}$	x	0.000	0.000	mm	✓
		y	45.6206	45.621	mm	✓
		z	-0.000	-0.000	mm	✓
MMOI at COM	$\mathbf{I}_C$	x	151.36	151.36	kg mm^2	✓
		y	47.463	47.462	kg mm^2	✓
		z	174.52	174.51	kg mm^2	✓

4.1 Fortran Code

Below is a sample of Fortran code used to generate the results above

```fortran
subroutine calc_properties(mesh,mass,V,cg,Ic)
class(mesh3), intent(in) :: mesh
real(8), intent(in) :: mass
real(8), intent(out) :: V
type(vector3), intent(out) :: cg
type(symmatrix3), intent(out) :: Ic
real(8) :: rho, i_V
type(vector3) :: i_cg, A,B,C
type(symmatrix3) :: i_mmoi, I0
integer :: i
    V = 0d0
    cg = 0d0
    I0 = 0d0
    do i=1, mesh%n_faces
        A = mesh%nodes( mesh%faces(1, i) )
        B = mesh%nodes( mesh%faces(2, i) )
        C = mesh%nodes( mesh%faces(3, i) )
        i_V = dot(A, cross(B, C))/6d0
        V = V + i_V
        i_cg = (A+B+C)/4d0
        cg = cg + i_cg*i_V
        i_mmoi = (parallel(A+B) + parallel(B+C) + parallel(C+A))/20d0
        I0 = I0 + i_mmoi*i_V
    end do
! Find the center of mass
    cg = cg/V
! I0 now contains the MMOI at the origin (divided by density)
! We need to transfer it to the center of mass
    rho = mass/V
    Ic = rho*I0 - mass*parallel(cg)
end subroutine
```

```fortran

function parallel(pos) result(r)
class(vector3), intent(in) :: pos
type(symmatrix3) :: r
    r%A11 = (pos%y**2 + pos%z**2)
    r%A22 = (pos%x**2 + pos%z**2)
    r%A33 = (pos%x**2 + pos%y**2)
    r%A12 = -(pos%x*pos%y)
    r%A23 = -(pos%y*pos%z)
    r%A13 = -(pos%x*pos%z)
end function

function dot(this,other) result(r)
class(vector3), intent(in) :: this, other
real(8) :: r
    r = this%x*other%x + this%y*other%y + this%z*other%z
end function

function cross(this,other) result(r)
class(vector3), intent(in) :: this, other
type(vector3) :: r
    r = vector3( &
        this%y*other%z - this%z*other%y, &
        this%z*other%x - this%x*other%z, &
        this%x*other%y - this%y*other%x )
end function
```