Practice Assignment 1

Due: Tuesday, Jan 12th at 2pm to Canvas

To get credit, each student must submit their own solutions (no need to be typeset, but must be submitted as a pdf) to CANVAS by the due date above – no late submissions are allowed. Note that while you must each submit solutions individually, you are free to work through the problem sets in groups. Solutions will be posted shortly after class and may be discussed in class. These will be graded on the basis of *completion* alone, not *correctness*.

The footnotes are included as an extra guide – they are not required for a correct solution.

1. Sort the following terms from *slowest* growing to *fastest* growing. If two terms are equivalent asymptotically, draw a circle around them in your ordering.

$$(\log n + 1)^3 \quad 7^{2n} \quad n^{1/2} \quad n^{\log_3 7} \quad 2^{7n} \quad 1000(\log n)^3 \quad 2^{\log_2 n} \quad n \log n \quad 5^{\log_3 n}$$

$$\left\{ \begin{array}{c} (\log n + 1)^3 \\ 1000(\log n)^3 \end{array} \right\}, \quad n^{1/2}, \quad n, \quad n \log n, \quad 5^{\log_3 n} = n^{\log_3 5}, \quad n^{\log_3 7}, \quad 7^{2n} = 49^n, \quad 2^{7n} = 128^n$$

Useful facts: $a^{log_b n} = n^{log_b a}$

2. For each of the following, indicate whether $f = O(g)$, $f = \Omega(g)$, and/or $f = \Theta(g)$.

	$f(n)$	$g(n)$
(a)	$3n + 6$	$10000n - 500$
(b)	$n^{1/2}$	$n^{2/3}$
(c)	$\log(7n)$	$\log(n)$
(d)	$n^{1.5}$	$n \log n$
(e)	$\sqrt{n}$	$(\log n)^3$
(f)	$n2^n$	3^n

(a) $f = \Theta(g)$, $f = O(g)$, $f = \Omega(g)$
(b) $f = O(g)$
(c) $f = \Theta(g)$
(d) $f = \Omega(g)$

(e) *$f = \Omega(g)$ lesson: n^c for any constant $c > 0$ always dominates $(\log n)^d$ for any constant $d > 0$ no matter how c and d compare*

(f) *$f = O(g)$ lesson: d^n for any constant $d > 1$ always dominates n^c for any constant $c > 0$*

3. Show that, if c is a positive real number, then $g(n) = 1 + c + c^2 + \cdots + c^n$ is

   (a) $\Theta(1)$ if $c < 1$

   (b) $\Theta(n)$ if $c = 1$

   (c) $\Theta(c^n)$ if $c > 1$

   The moral: in big-Θ terms, the sum of a geometric series is simply the first term if the series is strictly decreasing, the last term if the series is strictly increasing, of the number of terms if the series is unchanging.

   If $c = 1$, $c^j = 1\ \forall j$ and $g(n) = 1 + \cdots + 1$, n times, so $g(n) = n = \Theta(n)$.

   One Answer: $g(n) = \sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}$. *If $c > 1$, $\frac{c^{n+1}-1}{c-1} = \Theta(c^n)$ and if $c < 1$, $\frac{c^{n+1}-1}{c-1} = \Theta(1)$.*

   Another Answer: *For the remaining two cases, let's define a function $f(x) = c^{\lfloor x \rfloor}$. Then*[1] *$g(n) = \int_0^{n+1} f(x)dx$. It should be easy to see*[2] *that $c^x \leq f(x) \leq c^{x+1}$ and so*

$$\int_0^{n+1} c^x dx \leq g(n) \leq \int_0^{n+1} c^{x+1} dx$$

   integrating,

$$\frac{1}{\ln c}(c^{n+1} - 1) \leq g(n) \leq c(\frac{1}{\ln c}(c^{n+1} - 1))$$

   If $c > 1$, both the lower bound gives us $g(n) = \Omega(\frac{1}{\ln c}(c^{n+1} - 1)) = \Omega(c^n)$ and the upper bound gives us $g(n) = O(c(\frac{1}{\ln c}(c^{n+1} - 1))) = O(c^n)$. Together this gives us $g(n) = \Theta(c^n)$.
   If $c < 1$, $c^{n+1} = \Theta(1)$, giving us that $g(n) = \Theta(1)$ too.

4. Show that $\log(n!) = \Theta(n \log n)$.

   Using Creativity: *Notice that $n! = 1 \times 2 \times 3 \ldots \times n \leq (n \times n \times n \ldots \times n) = n^n$. Taking the log of both sides of this equation, we get*

$$\log(n!) \leq \log(n^n) = n \log n = O(n \log n)$$

   By ignoring the first half of the terms in the product $n! = 1 \times 2 \times 3 ... \times n$ and keeping only the larger $\lfloor n/2 \rfloor$ items, we have

$$n! \geq \lceil n/2 \rceil \times (\lceil n/2 \rceil + 1) \times (\lceil n/2 \rceil + 2) \times \cdots \times n \geq \lceil n/2 \rceil \times \lceil n/2 \rceil \times \cdots \times \lceil n/2 \rceil \geq \lceil n/2 \rceil^{\lfloor n/2 \rfloor}$$

   Therefore

$$\log(n!) \geq \log((\lceil n/2 \rceil)^{\lfloor n/2 \rfloor}) = \lfloor n/2 \rfloor \log \lceil n/2 \rceil = \Omega(n \log n)$$

[1] because $f(x)$ is a histogram of the terms of $g(n)$, and adding the terms of $g(n)$ is the same as determining the area under the histogram

[2] it may be helpful to sketch a picture of this

Using Limits: *You have to be very careful here, because you may be tempted to simplify too early, canceling out logs that can't be canceled.*

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log(n!)}{n\log n} &= \lim_{n\to\infty} \frac{\log(n\times(n-1)\times(n-2)\times\cdots\times 3\times 2\times 1)}{n\log n} \\
&= \lim_{n\to\infty} \frac{\log(n)+\log(n-1)+\log(n-2)+\cdots+\log(3)+\log(2)+\log(1)}{n\log n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=1}^{n}\log i}{n\log n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} 1/i}{\log n+1} \text{(by l'Hôpital's rule)} \\
&= \lim_{n\to\infty} \frac{\log n+\gamma}{\log n+1} \\
&= 1.
\end{aligned}$$

In the second last line, $\sum_{i=1}^{n} 1/i$ *is the* n^{th} *Harmonic number and equals* $\log n+\gamma$ *where* γ *is the EulerMascheroni constant.*

In the above, I assumed $\log$ *is the natural logarithm.*