Practice questions

Problems provided as practice questions for the final exam. Note that this is only to supplement the practice assignments, and not to be used as the only study guide.

1. Consider the following problem:

$$\begin{aligned} \max \quad & 5x + 3y \\ s.t. \quad & 5x - 2y \geq 0 \\ & x + y \leq 7 \\ & x \leq 5 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

   a. Plot the feasible region and identify the optimal solution.
   b. Transform this problem into a LP with equality constraints.

2. Suppose you have k sorted arrays, each with n elements, and you wantt o combine them into a single sorted array of kn elements.

   a. Here is one strategy: using the merge procedure from merge sort and merge the first two arrays, then merge in the third, then merge in the fourth, and so on. what is the time complexity of this algorithm, in terms of k and n?
   b. Use divide and conquer to design a more efficient solution to this problem.

3. Consider the CLIQUE problem restricted to graphs in which every vertex has degree at most 3. Call this problem CLIQUE-3.

   a. Prove that CLIQUE-3 is in NP.
   b. What is wrong with the following proof of NP-completeness for CLIQUE-3?
      we know that the CLIQUE problem in general graph is NP-complete, so it is enough to present a reduction from CLIQUE-3 to CLIQUE. Given a graph G with vertices of degrees ≤ 3, and a parameter g, the reduction leaves the graph and the parameter unchanged: clearly the output of the reduction is a possible input for the CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, shows that CLIQUE-3 is in NP-complete.
   c. It is true that vertex cover problem remains NP-complete even when restricted to graphs in which every vertex has degree at most 3. Call this problem VC-3. What is wrong with the following proof of NP-completeness for CLIQUE-3?

      We present a reduction from vc-3 to CLIQUE-3. Given a graph G with node degrees bounded by 3, and a parameter b, we create an instance of $CLIQUE - 3$ by leaving the graph unchanged and switching the parameter to $\|V\| - b$. Now, a subset $C \subset V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G. Therefore G has a vertex cover of size $\leq b$ if and only if it has a clique of size $\geq |v| - b$. This proves the correctness of the reduction and consequently the NP-completeness of CLIQUE-3.

4. Prove NP-hard by generalization. Problem A is a generalization of problem B if we can show that problem B can be viewed as a special case of problem A. For each problem below, prove that it is NP-hard by showing that it is a generalization of some NP-complete problem from the given list (3SAT, Independent Set, Vertex cover, CLIQUE, Hamiltonian Cycle, Hamiltonian Path).

   a. Subgraph isomorphism: given as input two undirected graphs G and H, determine if G is a subgraph of H.

   b. SPARSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at most b edges between them.

   c. DENSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at least b edges between them.

5. Consider the following Traveling salesman Problem (TSP): Given a set of n cities, the pairwise distances between them and a budget b, find a tour of all cities (which is a cycle that passes through every vertex exactly once) of total distance b or less. Show that this problem is NP-complete. Hint: build on the fact that hamiltonian cycle is NP-complete.