Practice questions

Problems provided as practice questions for the final exam. Note that this is only to supplement the practice assignments, and not to be used as the only study guide.

1. Consider the following problem:

$$\begin{aligned} \max \quad & 5x + 3y \\ s.t. \quad & 5x - 2y \geq 0 \\ & x + y \leq 7 \\ & x \leq 5 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

   a. Plot the feasible region and identify the optimal solution.
   b. Transform this problem into a LP with equality constraints.

   The optimal solution is achieved in the upper right corner of the feasible region, i.e. at the point (5; 2), and has value $5x + 3y = 31$.

2. Suppose you have k sorted arrays, each with n elements, and you wantt o combine them into a single sorted array of kn elements.

   a. Here is one strategy: using the merge procedure from merge sort and merge the first two arrays, then merge in the third, then merge in the fourth, and so on. what is the time complexity of this algorithm, in terms of k and n?
   b. Use divide and conquer to design a more efficient solution to this problem.

   a. Let $T(i)$ *be the time to merge arrays 1 to* i. *This requires merging an array of size* $n(i-1)$ *with another array of size* n. *So we can express this with* $T(i) = T(i-1) + O(ni)$. *Solving this leads to* $T(k) = O(nk^2)$.
   b. Divide the arrays into two sets. Recursively merge each set and the merge the resulting two sorted arrays into one. The run time is given by $T(k) = 2T(k/2) + O(nk)$. *Solving this leads to* $T(k) = O(nk \log k)$.

3. Consider the CLIQUE problem restricted to graphs in which every vertex has degree at most 3. Call this problem CLIQUE-3.

   a. Prove that CLIQUE-3 is in NP.
   b. What is wrong with the following proof of NP-completeness for CLIQUE-3?
   we know that the CLIQUE problem in general graph is NP-complete, so it is enough to present a reduction from CLIQUE-3 to CLIQUE. Given a graph G with vertices of degrees ≤ 3, and a parameter g, the reduction leaves the graph and the parameter unchanged: clearly the output of the reduction is a possible input for the CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, shows that CLIQUE-3 is in NP-complete.

c. It is true that vertex cover problem remains NP-complete even when restricted to graphs in which every vertex has degree at most 3. Call this problem VC-3. What is wrong with the following proof of NP-completeness for CLIQUE-3?

We present a reduction from vc-3 to CLIQUE-3. Given a graph G with node degrees bounded by 3, and a parameter b, we create an instance of $CLIQUE - 3$ by leaving the graph unchanged and switching the parameter to $\|V\| - b$. Now, a subset $C \subset V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G. Therefore G has a vertex cover of size $\leq b$ if and only if it has a clique of size $\geq |v| - b$. This proves the correctness of the reduction and consequently the NP-completeness of CLIQUE-3.

a. *Given a proposed solution/clique in the graph, it is easy to verify in polynomial time that there is an edge between every pair of vertices. Hence a solution to CLIQUE-3 can be checked in polynomial time. THis shows htat CLIQUE-3 is NP.*

b. *The reduction is in the wrong direction. We must reduce CLIQUE to CLIQUE-3, if we intend to show that CLIQUE-3 is at least as hard as CLIQUE.*

c. *The statement a subset $C \subset V$ is a vertex cover in G is and only if the complimentary set VC is a clique in G used in the reduction is false. C is a vertex cover if and only if VC is an independent set in G.*

4. Prove NP-hard by generalization. Problem A is a generalization of problem B if we can show that problem B can be viewed as a special case of problem A. For each problem below, prove that it is NP-hard by showing that it is a generalization of some NP-complete problem from the given list (3SAT, Independent Set, Vertex cover, CLIQUE, Hamiltonian Cycle, Hamiltonian Path).

   a. Subgraph isomorphism: given as input two undirected graphs G and H, determine if G is a subgraph of H. *This is a generalization of CLIQUE. CLIQUE with graph H and parameter b is a special case subgraph isomorphism problem with input G and H, where G is set to be a fully connected graph of b nodes.*

   b. SPARSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at most b edges between them. *This is a generalization of Independent Set. Independent set is a sparse subgraph problem with $b = 0$.*

   c. DENSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at least b edges between them. *This is another generalization of CLIQUE. CLIQUE is a dense subgraph problem with $b = a(a+1)/2$.*

5. Consider the following Traveling salesman Problem (TSP): Given a set of n cities, the pairwise distances between them and a budget b, find a tour of all cities (which is a cycle that passes through every vertex exactly once) of total distance b or less. Show that this problem is NP-complete. Hint: build on the fact that hamiltonian cycle is NP-complete.

   To show that TSP is NP-complete, we need to show that it is NP, and it is NP-hard.

   To show it's NP, we note that a proposed solution/tour is a ordered list of nodes that specifies the order of the tour, starting and ending at the same node. We just need to traverse through this list and verify that each vertex of the graph is visited exactly once except for the start and end node, and add up the distance between neighboring nodes to check if the total sum is b or less. This clearly can be done in polynomial time, proving that TSP is NP.

   To show it is NP-hard, we will reduce hamiltonian cycle to TSP. Given an instance of the HC problem, which is a graph G, we will create a TSP problem by creating one city for each vertex in G, and assign the distance between two cities to be 1 if there exists an edge between the two vertices in G, and $1 + c$ otherwise, where c is a large constant, and setting the budget b to be $|V|$. It is easy to see that if there is a hamiltonian cycle, that cycle provides a tour within the budget. Conversely, if G has no hamiltonian *cycle, then the cheapest possible TSP tour will have a cost at least $|V| + c$ (because at least one edge in the tour is not present in G and thus has distance $1 + c$). This shows that this is a correct reduction and TSP is NP-hard.*

This completes the proof that TSP is NP-complete.