Practice Assignment 5

Due: Tuesday March 1st at 2PM to Canvas

To get credit, each student must submit their own solutions (which need not be typeset) to Canvas by the due date above – no late submissions are allowed. Note that while you must each submit solutions individually, you are free to work through the problem sets in groups. Solutions will be posted shortly after class and may be discussed in class. These will be graded on the basis of *completion* alone, not *correctness*.

The following questions are about linear programming. Section 2 (pages 21–24) of `http://web.williams.edu/Mathematics/sjmiller/public_html/416/currentnotes/LinearProgramming.pdf` will be very helpful in solving these problems.

1. Suppose, in a linear program, my goal is to *maximize* a linear function, but I only know how to minimize a function. What do I do?

   Negate the objective.

2. Consider the following problem:

$$\begin{aligned} \max \quad & x - y \\ s.t. \quad & x + y \leq 5 \\ & |x + 2y| \leq 10 \end{aligned}$$

   Can I solve this problem with a linear program? If so, how? What is the solution?

$$\begin{aligned} \max \quad & x - y \\ s.t. \quad & x + y \leq 5 \\ & x + 2y \leq 10 \\ & x + 2y \geq -10 \end{aligned}$$

   The solution is $x = 20$, $y = -15$ for an objective value of 35.

3. Consider the following problem:

$$\min \ \max\{x, y\} \quad s.t. \ x + 2y \geq 2$$

   Can I solve this problem with a linear program? If so, how? What is the solution?

   Solution when $x = y$ and $x + 2y = 2$, so $x = y = 2/3$

   (a) *Replace $\max\{x, y\}$ with t. The problem becomes:*

$$\begin{aligned} \min \quad & t \\ s.t. \quad & x + 2y \geq 2 \\ & t = \max\{x, y\} \end{aligned}$$

   (b) *$t = \max\{x, y\}$ is equivalent to t is the smallest value such that $t \geq x$ and $t \geq y$ or:*

$$\begin{aligned} \min \quad & t \\ s.t. \quad & t \geq x \\ & t \geq y \end{aligned}$$

replacing $t = \max\{x, y\}$ with this mini-LP gives the LP:

$$\begin{aligned} \min \quad & t \\ s.t. \quad & x + 2y \geq 2 \\ & \min t \\ & s.t.\ t \geq x \\ & t \geq y \end{aligned}$$

but the two $\min t$'s are redundant; dropping the second one gives:

$$\begin{aligned} \min \quad & t \\ s.t. \quad & x + 2y \geq 2 \\ & t \geq x \\ & t \geq y \end{aligned}$$

(c) Simplifying gives:

$$\begin{aligned} \min \quad & t \\ s.t. \quad & x + 2y \geq 2 \\ & t - x \geq 0 \\ & t - y \geq 0 \end{aligned}$$

an LP!

4. You are given a set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the plane. You want to find a line $y = mx + b$ that comes close to each point. You probably learnt the method of least squares to find a line of best fit in your past, but we want to find the line of best fit that minimizes the maximum absolute deviation. That is, you want to find the values of m and b that minimizes:

$$\max_{1 \leq i \leq n} |y_i - mx_i - b|$$

Model this *general problem* as a linear program. If you have time and the means, use the linear program to find the line of minimum-maximum-absolute-deviation for *the instance* given by the following points:

$$(1, 3), (2, 5), (3, 7), (5, 11), (7, 14), (8, 15), (10, 19)$$

For full credit, you must show that you thought about how to solve this problem.

We can solve this by solving the following LP with $2n$ inequalities, two for each data point:

$$\begin{aligned} \min \quad & t \\ s.t. \quad & y_i - mx_i - b \leq t \ \forall i \\ & y_i - mx_i - b \geq -t \ \forall i \end{aligned}$$

For the particular instance above, we can solve using GLPK and Python as at the end of the document.

```
>>> from pulp import *
>>> prob = LpProblem("min abs dev", LpMinimize)
>>> mvar = LpVariable("mvar")
>>> bvar = LpVariable("bvar")
>>> tvar = LpVariable("tvar")
>>> prob += tvar
>>> points = [[1,3],[2,5],[3,7],[5,11],[7,14],[8,15],[10,19]]
>>> for P in points:
...     prob += P[1]-mvar*P[0]-bvar <= tvar
...     prob += P[1]-mvar*P[0]-bvar >= -tvar
...
>>> status = prob.solve()
GLPSOL: GLPK LP/MIP Solver, v4.45
Parameter(s) specified in the command line:
 --cpxlp /tmp/17214-pulp.lp -o /tmp/17214-pulp.sol
Reading problem data from '/tmp/17214-pulp.lp'...
14 rows, 3 columns, 42 non-zeros
23 lines were read
GLPK Simplex Optimizer, v4.45
14 rows, 3 columns, 42 non-zeros
Preprocessing...
14 rows, 3 columns, 42 non-zeros
Scaling...
 A: min|aij| =  1.000e+00  max|aij| =  1.000e+01  ratio =  1.000e+01
Problem data seem to be well scaled
Constructing initial basis...
Size of triangular part = 14
      0: obj =   0.000000000e+00  infeas =  7.400e+01 (0)
*     5: obj =   1.000000000e+00  infeas =  1.332e-15 (0)
*     6: obj =   5.714285714e-01  infeas =  0.000e+00 (0)
OPTIMAL SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.0 Mb (49504 bytes)
Writing basic solution to '/tmp/17214-pulp.sol'...
>>> value(prob.objective)
0.571429
>>> value(bvar)
1.85714
>>> value(mvar)
1.71429
```

The optimal line is $y = 1.71429x + 1.85714$.