INF5620

Autumn 2015

Nonlinear diffusion equation

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Date: November 10, 2015

Abstract

The goal of this project is to discuss numerical aspects of a nonlinear diffusjon equation

$$\varrho u_t = \nabla \cdot (\alpha \nabla u) + f(\vec{x}, t) \tag{1}$$

with initial condition $u(\vec{x}, 0) = I(\vec{x})$ and boundary condition $\frac{\partial u}{\partial n} = 0$. ϱ is a constant, $\alpha(u)$ is a known function of u.

a)

Introduce some finite difference approximation in time that leads to an implicit scheme (say a Backward Euler, Crank-Nicolson, or 2-step backward scheme). Derive a variational formulation of the initial condition and the spatial problem to be solved at each time step (and at the first time step in case of a 2-step backward scheme).

A Backward Euler scheme for (1) reads (in operator notation)

$$[\varrho D_t^- u = \nabla \cdot (\alpha(u) \nabla u) + f(\vec{x}, t)]^n \tag{2}$$

Written out,

$$\varrho u^n - \Delta t \big(\nabla \cdot (\alpha(u^n) \nabla u^n) + f(\vec{x}, t_n)\big) = \varrho u^{n-1} \tag{3}$$

This is the time-discrete version of the PDE, but we also need to discretize in space. Variational formulation of the spatial part using Galerkin method (multiplying with a test function v and integrating over the domain) yields

$$\varrho \int_\Omega u^n v dx - \int_\Omega \Delta t (\nabla \cdot (\alpha \nabla u^n)) v dx - \int_\Omega \Delta t f^n v dx = \varrho \int_\Omega u^{n-1} v dx \tag{4}$$

We do integration by parts on the integral involving derivatives of u,

$$\int_\Omega \nabla \cdot (\alpha \nabla u^n) v dx = - \int_\Omega \alpha \nabla u^n \cdot v dx + \int_{\partial\Omega} \alpha \frac{\partial u^n}{\partial n} v dx \tag{5}$$

The last term vanishes because we have the Neumann condition $\partial u^n / \partial n = 0$ for all n. Our discrete problem in space and time then reads

$$\int_\Omega (\varrho u^n v + \Delta t \alpha(u^n) \nabla u^n \cdot \nabla v - \Delta t f^n v - \varrho u^{n-1} v) dx = 0 \tag{6}$$

u is now approximated as a linear combination of basis functions ψ_i, $u = \sum_i c_i \psi_i$. The nonlinear algebraic equation F_i for u^n follow from setting $v = \psi_i$,

$$F_i = \int_\Omega (\varrho u^n \psi_i + \Delta t \alpha(u^n) \nabla u^n \cdot \nabla \psi_i - \Delta t f^n \psi_i - \varrho u^{n-1} \psi_i) dx = 0 \tag{7}$$

This is the equation that needs to be solved for each time step. The variational problem for the initial condition is simply

$$\int_\Omega (u^0 \psi_i - I \psi_i) dx = 0 \tag{8}$$

b)

Formulate a Picard iteration method at the PDE level, using the most recently computed u function in the $\alpha(u)$ coefficient. Derive general formulas for the entries in the linear system to be solved in each Picard iteration. Use the solution at the previous time step as initial guess for the Picard iteration.

Picard iteration linearize the equation by using the most recent approximation u^- to u in the coefficient $\alpha(u)$ (switching to the notation $u^n = u$ and $u^{n-1} = u^{(1)}$),

$$F_i \approx \hat{F}_i = \int_\Omega (\varrho u \psi_i + \Delta t \alpha(u^-) \nabla u \cdot \nabla \psi_i - \Delta t f \psi_i - \varrho u^{(1)} \psi_i) dx \tag{9}$$

The equation $\hat{F}_i = 0$ to be solved for each Picard iteration is now linear and can be translated to a linear system $\sum_j A_{i,j} c_j = b_i$ for the unknown coefficients c_i by inserting $u = \sum_i c_i \psi_i$ and moving all unknown terms to the rhs and known terms to the lhs,

$$A_{i,j} = \int_\Omega (\varrho \psi_j \psi_i + \Delta t \alpha(u^-) \nabla \psi_j \cdot \nabla \psi_i) dx, \quad b_i = \int_\Omega (\Delta t f \psi_i + \varrho u^{(1)} \psi_i) dx \tag{10}$$

The iteration begins with setting $u^- = u^{(1)}$.

c)

Restrict the Picard iteration to a single iteration. That is, simply use a u value from the previous time step in the $\alpha(u)$ coefficient. Implement this method with the aid of the FEniCS software (in a dimension-independent way such that the code runs in 1D, 2D, and 3D).

Implemented in script *diffusion.py*

d)

he first verification of the FEniCS implementation may reproduce a constant solution. Find values of the input data ϱ, α, f and I such that $u(\vec{x},t) = C$ where C is some chosen constant. Write test functions that verify the computation of a constant solution in 1D, 2D, and 3D for P1 and P2 elements (use simple domains: interval, square, box).

Inserting $u(\vec{x},t) = C$ in the PDE, considering that the derivative of u is zero, yields

$$\varrho \cdot 0 = \nabla \cdot (\alpha(C) \cdot 0) + f(\vec{x},t) \tag{11}$$

and we see that f must be zero, while ϱ and α can have any value. Obviously, $I(\vec{x}) = C$.