INF5620

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Computing with a non-uniform mesh

John-Anders Stende

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Introduction

In this problem I will derive the linear system for the equation $-u''(x) = 2$ on $[0,1]$ with $u(0) = 0$ and $u(1) = 1$, using P1 elements and a non-uniform mesh. Then I wlll compare the equations with the finite difference equations for the same problem.

a)

Because we have a non-zero Dirichlet boundary condition, we have to write u as

$$u(x) = B(x) + \sum_{j \epsilon I_b} c_j \psi_j(x) \tag{1}$$

where $B(x)$ is the boundary function

$$B(x) = \sum_{j \epsilon I_b} U_j \phi_j(x) = 1 \cdot \varphi_{N_n - 1} = \varphi_{N_n - 1} \tag{2}$$

The variational form of this problem is

$$-(u'', v) = (2, v) \tag{3}$$

After partial integration of the u-term we get

$$(u', v') = (2, v) \tag{4}$$

or

$$\left(\sum_{j \epsilon I_b} c_j \psi_j'(x), v' \right) = (2, v) - (B', v') \tag{5}$$

The element matrix mapped to the reference element is thus

$$A^{(e)}_{i-1,j-1} = \int_{\Omega^{(e)}} \varphi_i'(x) \varphi_j'(x) dx \tag{6}$$

$$= \int_{-1}^{1} \frac{d}{dx} \tilde{\varphi}_r(X) \frac{d}{dx} \tilde{\varphi}_s(X) \frac{h_e}{2} dX \tag{7}$$

For P1 elements we have two nodes per element; $r, s = 0, 1$. We have that (from chain rule)

$$\frac{\tilde{\varphi}_r}{dx} = \frac{\tilde{\varphi}_r}{dX} \frac{dX}{dx} = \frac{2}{h_e} \frac{\tilde{\varphi}_r}{dX} \tag{8}$$

so

$$\int_{-1}^{1} \frac{2}{h_e} \frac{d}{dX} \tilde{\varphi}_r(X) \frac{2}{h_e} \frac{d}{dX} \tilde{\varphi}_s(X) \frac{h_e}{2} dX \tag{9}$$

The element vector consists of two integrals. The first is

$$\int_{\Omega^{(e)}} 2\varphi_i(x) dx = \int_{-1}^{1} 2\tilde{\varphi}_r(X) \frac{h_e}{2} dX \tag{10}$$

The second is

$$- \int_{\Omega^{(e)}} \varphi'_{N_n - 1}(x) \varphi_i'(x) dx \tag{11}$$

which contributes $1/h_e$ to the global element vector. We have that

$$\tilde{\varphi}_0(x) = \frac{1}{2}(1 - x), \quad \tilde{\varphi}_1(x) = \frac{1}{2}(1 + x) \tag{12}$$

which have derivatives $1/2$ and $-1/2$ respectively. The element matrix entries can now be computed, the results are

$$\tilde{A}^{(e)}_{0,0} = 1/h_e, \quad \tilde{A}^{(e)}_{0,1} = -1/h_e, \tag{13}$$

$$\tilde{A}^{(e)}_{1,0} = -1/h_e, \quad \tilde{A}^{(e)}_{1,1} = 1/h_e \tag{14}$$

and for the element vector:

$$\tilde{b}^{(e)}_0 = h_e, \quad \tilde{b}^{(e)}_1 = h_e \tag{15}$$

thus

$$\tilde{A}^{(e)} = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{16}$$

and

$$\tilde{b}^{(e)} = h_e \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{17}$$

The element matrix and vector are in general different for the first and last cell due to boundary conditions. Contribution from the first cell:

$$\tilde{A}^{(e)} = \frac{1}{h_e} (1), \quad \tilde{b}^{(e)} = h_e (1) \tag{18}$$

and the last cell:

$$\tilde{A}^{(e)} = \frac{1}{h_e} (1), \quad \tilde{b}^{(e)} = (h_e + 1/h_e) (1) \tag{19}$$

where $1/h_e$ is the contribution from the non-zero Dirichlet condition $u(1) = 1$.

b)

The centered difference formula for a second order derivative can be computed as

$$[D_x(D_x u)]_i = Dx \left(\frac{u_{i+1/2} - u_{i-1/2}}{x_{i+1/2} - x_{i-1/2}} \right) = \frac{1}{x_{i+1/2} - x_{i-1/2}} \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \right) \tag{20}$$

and we see that

$$x_{i+1/2} - x_{i-1/2} = \frac{1}{2}(x_i - x_{i-1}) + \frac{1}{2}(x_{i+1} - x_i) = \frac{1}{2}(x_{i+1} - x_{i-1}) \tag{21}$$

Inserting this in our equation yields

$$u''(x_i) = \frac{2}{x_{i+1} - x_{i-1}} \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \right) = 2 \tag{22}$$

Setting $h_i = x_{i+1} - x_i$ and $h_{i-1} = x_i - x_{i-1}$ leads to

$$u''(x_i) = \frac{1}{h_i + h_{i-1}} \left(\frac{u_{i+1} - u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 1 \tag{23}$$

When the element matrix and element vector in a) is assembled, we get the following general equation for the unknown coefficients

$$-h_{i-1}^{-1} c_{i-1} + (h_{i-1}^{-1} - h_i^{-1}) c_i + h_i^{-1} c_{i+1} = h_{i-1} + h_i \tag{24}$$

Replacing c_i by u_i etc. and arranging yields

$$u''(x_i) = \frac{1}{h_i + h_{i-1}} \left(\frac{u_{i+1} - u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 1 \tag{25}$$

which is the exact same formula as we deduced using finite differences.