Generating, from scratch, the near-field asymptotic forms of scalar resistance functions for two unequal rigid spheres in low-Reynolds-number flow

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When investigating the behaviour of particle suspensions, for example in the Stokesian dynamics simulation technique, it is sometimes necessary to use near-field asymptotic forms of scalar resistance functions for two unequal rigid spheres, commonly notated

$$\begin{aligned}&X_{11}^A, X_{12}^A, Y_{11}^A, Y_{12}^A, Y_{11}^B, Y_{12}^B, X_{11}^C, X_{12}^C, Y_{11}^C, Y_{12}^C,\\ &X_{11}^G, X_{12}^G, Y_{11}^G, Y_{12}^G, Y_{11}^H, Y_{12}^H, X_{11}^M, X_{12}^M, Y_{11}^M, Y_{12}^M, Z_{11}^M, Z_{12}^M.\end{aligned} \tag{1}$$

The required expressions for generating these scalars were initially published in Jeffrey & Onishi (1984) and Jeffrey (1992).

These important papers suffer from a number of small errors, and furthermore, the reader may find it difficult to generate the required expressions (and therefore the value of these functions) independently, given the omission of intermediate formulae.

A partial list of errata has been published by Kengo Ichiki (`http://ryuon.sourceforge.net/twobody/errata.html`), and some of these errors appear to have been noticed by authors using these papers in their extensions. However, I have not found a comprehensive description of how to fully generate, from scratch, expressions for these functions.

This short article is a compilation of the relevant equations, with those originally omitted now added, and with any errors fixed. Equations from Jeffrey & Onishi (1984) are labelled (JO 1.1), those from Jeffrey (1992) are labelled (J 1), and those from the helpful Ichiki *et al.* (2013) are labelled (I 1).

At the end of the article, the questions 'how do we know there are errors?' and 'how do we know they are fixed?' are addressed.

Throughout this article, we use the same notation as these papers. For two spheres of

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radius a_1, a_2 a distance s apart, we define the non-dimensional gap ξ and size ratio λ as

$$\xi = \frac{2s}{a_1 + a_2} - 2, \qquad \lambda = \frac{a_2}{a_1}. \tag{2}$$

The near-field forms are valid for $\xi \ll 1$ and $\xi \ll \lambda$.

In order to match the far-field forms, it is necessary to scale the terms

$$X^A, Y^A, Y^B, X^C, Y^C, X^G, Y^G, Y^H, X^M, Y^M, Z^M \tag{3}$$

by multiplying respectively by

$$1,\ 1,\ \frac{2}{3},\ \frac{4}{3},\ \frac{4}{3},\ \frac{2}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{10}{9},\ \frac{10}{9},\ \frac{10}{9}. \tag{4}$$

X^A terms

Here the X^A formulae are given in full, with changes from the source material when noted. The same directions for alteration, when required, will be given for the other terms in later sections.

Set up the recurrence relations

$$P_{n00} = \delta_{1n}, \tag{5}$$

$$V_{n00} = \delta_{1n}, \tag{6}$$

$$V_{npq} = P_{npq} - \frac{2n}{(n+1)(2n+3)} \sum_{s=1}^{q} \binom{n+s}{n} P_{s(q-s)(p-n-1)}, \tag{7}$$

$$\begin{aligned} P_{npq} = \sum_{s=1}^{q} \binom{n+s}{n} \Bigg(& \frac{n(2n+1)(2ns-n-s+2)}{2(n+1)(2s-1)(n+s)} P_{s(q-s)(p-n+1)} \\ & - \frac{n(2n-1)}{2(n+1)} P_{s(q-s)(p-n-1)} - \frac{n\left(4n^2-1\right)}{2(n+1)(2s+1)} V_{s(q-s-2)(p-n+1)} \Bigg), \end{aligned} \tag{8}$$

(JO 3.6–3.9). Then define the formulae

$$f_k(\lambda) = 2^k \sum_{q=0}^{k} P_{1(k-q)q} \lambda^q, \tag{9}$$

$$g_1(\lambda) = 2\lambda^2(1+\lambda)^{-3}, \tag{10}$$

$$g_2(\lambda) = \frac{1}{5}\lambda(1+7\lambda+\lambda^2)(1+\lambda)^{-3}, \tag{11}$$

$$g_3(\lambda) = \frac{1}{42}(1+18\lambda-29\lambda^2+18\lambda^3+\lambda^4)(1+\lambda)^{-3}, \tag{12}$$

$$m_1(m) = -2\delta_{m2} + (m-2)(1-\delta_{m2}), \tag{13}$$

(JO 3.15, 3.19); and

$$A^X_{11} = 1 - \frac{1}{4}g_1 + \sum_{\substack{m=2\\ m \text{ even}}}^{\infty} \left[2^{-m}(1+\lambda)^{-m} f_m - g_1 - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3\right], \quad (14)$$

$$-\frac{1}{2}(1+\lambda)A^X_{12} = \frac{1}{4}g_1 + 2g_2 \log 2 - 2g_3 + \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \left[2^{-m}(1+\lambda)^{-m} f_m - g_1 - 2m^{-1}g_2 + 4m^{-1}(m+2)^{-1}g_3\right], \quad (15)$$

(JO 3.22–3.23), noting the correction from m_1 to $m+2$ in A^X_{12}. Then the resistance scalars are given by

$$X^A_{11} = g_1\xi^{-1} + g_2 \log(\xi^{-1}) + A^X_{11} + g_3\xi \log(\xi^{-1}), \quad (16)$$

$$-X^A_{12} = g_1\xi^{-1} + g_2 \log(\xi^{-1}) - \frac{1}{2}(1+\lambda)A^X_{12} + g_3\xi \log(\xi^{-1}), \quad (17)$$

from (JO 3.17–3.18) up to $O(\xi \log(\xi^{-1}))$: note the different factor on X^A_{12}.

Y^A terms

The recurrence relations are (JO 4.6–4.11), but with V_{npq} corrected to

$$V_{npq} = P_{npq} + \frac{2n}{(n+1)(2n+3)} \sum_{s=1}^{q} \binom{n+s}{n+1} P_{s(q-s)(p-n-1)}, \quad (18)$$

noticing the sign change on the 1 in the last subscript.

The required intermediate formulae for the f, g and m functions are eq. (9), (JO 4.16–4.17), and eq. (13), respectively.

Then the A^Y terms are given by (JO-4.17–4.18), leaving us with the resistance scalar formulae,

$$Y^A_{11} = g_2 \log(\xi^{-1}) + A^Y_{11} + g_3\xi \log(\xi^{-1}), \quad (19)$$

$$-Y^A_{12} = g_2 \log(\xi^{-1}) - \frac{1}{2}(1+\lambda)A^Y_{12} + g_3\xi \log(\xi^{-1}), \quad (20)$$

from (JO 4.15–4.16), with a different factor on Y^A_{12}.

Y^B terms

The recurrence relations are the same as those for the Y^A terms. The required intermediate formulae for the f and g functions are

$$f_k(\lambda) = 2^{k+1} \sum_{q=0}^{k} Q_{1(k-q)q}\lambda^q, \quad (21)$$

and (JO between 5.6 and 5.7), respectively.

The B^Y terms are given by

$$B_{11}^Y = 2g_2 \log 2 - 2g_3 + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \left[2^{-m}(1+\lambda)^{-m} f_m - 2m^{-1} g_2 + 4m^{-1}(m+2)^{-1} g_3\right], \tag{22}$$

$$-\frac{1}{4}(1+\lambda)^2 B_{12}^Y = -g_3 + \sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} \left[2^{-m}(1+\lambda)^{-m} f_m - 2m^{-1} g_2 + 4m^{-1}(m+2)^{-1} g_3\right], \tag{23}$$

having been corrected from (JO 5.7–5.8).

Then the resistance scalars are given by

$$Y_{11}^B = g_2 \log(\xi^{-1}) + B_{11}^Y + g_3 \xi \log(\xi^{-1}), \tag{24}$$

$$-Y_{12}^B = g_2 \log(\xi^{-1}) - \frac{1}{4}(1+\lambda)^2 B_{12}^Y + g_3 \xi \log(\xi^{-1}), \tag{25}$$

from (JO 5.5–5.6), with a different factor on Y_{12}^B.

X^C terms

Expressions for the resistance scalars can be expressed directly as

$$X_{11}^C = \frac{\lambda^3}{(1+\lambda)^3} \zeta\left(3, \frac{\lambda}{1+\lambda}\right) - \frac{\lambda^2}{4(1+\lambda)} \xi \log(\xi^{-1}), \tag{26}$$

$$X_{12}^C = -\frac{\lambda^3}{(1+\lambda)^3} \zeta(3,1) + \frac{\lambda^2}{4(1+\lambda)} \xi \log(\xi^{-1}), \tag{27}$$

(JO 6.9–6.10), where X_{12}^C has been divided by $8/(1+\lambda)^3$, and where $\zeta(z,a)$ is the Hurwitz zeta function,

$$\zeta(z,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^z}. \tag{28}$$

Y^C terms

The recurrence relations are the same as those for the Y^A terms except the initial conditions are replaced by (JO 7.3–7.5).

The intermediate formula for the f function is

$$f_k(\lambda) = 2^k \sum_{q=0}^{k} Q_{1(k-q)q} \lambda^{q+(k \bmod 2)}, \tag{29}$$

with the g formula given by (JO between 7.10 and 7.11), with the correction to g_5 of

$$g_5(\lambda) = \frac{2}{125} \lambda(43 - 24\lambda + 43\lambda^2)(1+\lambda)^{-4}. \tag{30}$$

Then the C^Y terms are

$$C_{11}^Y = 1 - g_3 + \sum_{\substack{m=2\\ m \text{ even}}}^{\infty} \left[2^{-m}(1+\lambda)^{-m} f_m - 2m^{-1} g_2 + 4m^{-1}(m+2)^{-1} g_3\right], \tag{31}$$

$$C_{12}^Y = 2g_4 \log 2 - 2g_5 + \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \left[2^{3-m}(1+\lambda)^{-3-m} f_m - 2m^{-1} g_4 + 4m^{-1}(m+2)^{-1} g_5\right], \tag{32}$$

noting the corrections to both of (JO 7.11-7.12).

The resistance scalars are finally

$$Y_{11}^C = g_2 \log(\xi^{-1}) + C_{11}^Y + g_3 \xi \log(\xi^{-1}), \tag{33}$$

$$\frac{8}{(1+\lambda)^3} Y_{12}^C = g_4 \log(\xi^{-1}) + C_{12}^Y + g_5 \xi \log(\xi^{-1}), \tag{34}$$

noting the different factor on Y_{12}^C from (JO 7.9–7.10).

X^G terms

The recurrence relations are the same as those for X^A, and the f and g functions are (I 94) and (J between 19b and 20a). The G^X terms are given by (J 21), noting that in their notation, $\widetilde{f}(\lambda) = 2^{-m} f(\lambda)$. This gives us expressions for X^G of

$$X_{11}^G = g_1 \xi^{-1} + g_2 \log(\xi^{-1}) + G_{11}^X + g_3 \xi \log(\xi^{-1}), \tag{35}$$

$$X_{12}^G = -g_1 \xi^{-1} - g_2 \log(\xi^{-1}) + \frac{1}{4}(1+\lambda)^2 G_{12}^X - g_3 \xi \log(\xi^{-1}), \tag{36}$$

from (J 19) with a different factor on the X_{12}^G.

Y^G terms

The recurrence relations are the same as those for Y^A, and the f and g functions are (I 115) and (J between 27b and 28a). The G^Y terms are given by (J 29), giving us expressions for Y^G of

$$Y_{11}^G = g_2 \log(\xi^{-1}) + G_{11}^Y + g_3 \xi \log(\xi^{-1}), \tag{37}$$

$$Y_{12}^G = -g_2 \log(\xi^{-1}) + \frac{1}{4}(1+\lambda)^2 G_{12}^Y - g_3 \xi \log(\xi^{-1}), \tag{38}$$

from (J 27) with a different factor on the Y_{12}^G.

Y^H terms

The recurrence relations are the same as those for Y^C, and the f and g functions are (I 120) and (J between 35b and 36a). The G^Y terms are given by (J 37), giving us expressions for Y^H of

$$Y_{11}^H = g_2 \log(\xi^{-1}) + H_{11}^Y + g_3 \xi \log(\xi^{-1}), \tag{39}$$

$$Y_{12}^H = g_5 \log(\xi^{-1}) + \frac{1}{8}(1+\lambda)^3 H_{12}^Y + g_6 \xi \log(\xi^{-1}), \tag{40}$$

from (J 35) with a different factor on the Y_{12}^H.

X^M terms

The recurrence relations are the same as those for X^A, but with the different initial conditions (J 44). The f and g functions are given by (I 105) and (J between 48b and 49a). The M^X terms are given by (J 50), giving us expressions for X^M of

$$X_{11}^M = g_1 \xi^{-1} + g_2 \log(\xi^{-1}) + M_{11}^X + g_3 \xi \log(\xi^{-1}), \tag{41}$$

$$X_{12}^M = g_4 \xi^{-1} + g_5 \log(\xi^{-1}) + \frac{1}{8}(1+\lambda)^3 M_{12}^X + g_6 \xi \log(\xi^{-1}), \tag{42}$$

from (J 48) with a different factor on the X_{12}^M.

Y^M terms

The recurrence relations are the same as those for Y^A, but with the different initial conditions (J 58). The f and g functions are given by (I 125) and (J between 64b and 65a). The M^Y terms are given by (J 66), giving us expressions for Y^M of

$$Y_{11}^M = g_2 \log(\xi^{-1}) + M_{11}^Y + g_3 \xi \log(\xi^{-1}), \tag{43}$$

$$Y_{12}^M = g_5 \log(\xi^{-1}) + \frac{1}{8}(1+\lambda)^3 M_{12}^Y + g_6 \xi \log(\xi^{-1}), \tag{44}$$

from (J 64) with a different factor on the Y_{12}^M.

Z^M terms

The recurrence relations are (J 73–76). The f and g functions are given by (I 131) and (J between 79b and 80a). The M^Z terms are given by (J 81), giving us expressions for Z^M of

$$Z_{11}^M = M_{11}^Z + g_3 \xi \log(\xi^{-1}), \tag{45}$$

$$Z_{12}^M = \frac{1}{8}(1+\lambda)^3 M_{12}^Z - g_3 \xi \log(\xi^{-1}), \tag{46}$$

from (J 79) with a different factor on the Z_{12}^M.

How do we know there are mistakes?

The original articles provide tabulated values of the intermediate scalars A_{11}^X, etc. The easiest way for the reader to confirm mistakes in the formulae is to confirm that the values computed from these formulae do not match those tabulated. For example:

	value from (JO) formulae	value in (JO) tables	correct value
$A^X_{12}(\lambda=1)$	-0.24300	-0.35022	-0.35022
$B^Y_{11}(\lambda=1)$	-0.8355	-0.2390	-0.2390

Alternatively, these can be spotted by either deriving the equations independently, or observing that the values provided do not match those in the mid-field; both methods are described below.

How do we know they are fixed?

The reader is invited to derive and confirm the above formulae themselves, should they wish. The method is perhaps best explained in Jeffrey (1992), sections II (starting at the paragraph containing the definition of ξ), III B & III C, where X^G is used as an example.

We can also confirm that the formulae produce values of the resistance scalars which match those in the mid-field. Figures 1 to 3 demonstrate the near-field values matching to the mid-field values, which have been computed independently for $\xi \gtrsim 0.014$ using the two-sphere method of Wilson (2013), based on the solution to Stokes flow given by Lamb (1932). Recall that the near-field equations are valid only for $\xi \ll \lambda$.

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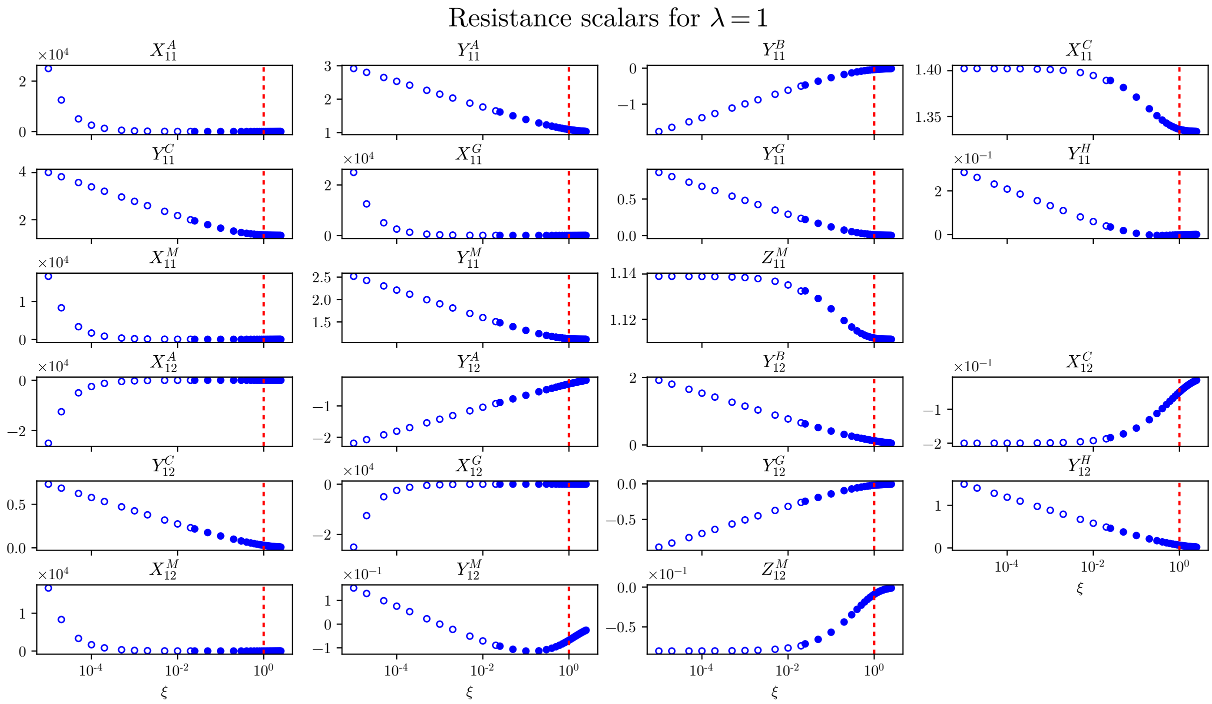

Figure 1: Values of the scalar resistance functions over non-dimensional gap, ξ, for size ratio $\lambda = 1$. Those generated from the near-field formulae are represented by hollow circles (○), and those generated from Lamb's solution (Wilson, 2013) are filled circles (●). The dashed vertical line appears at $\xi = \lambda$, recalling that the near-field formulae are only valid for $\xi \ll \lambda$.

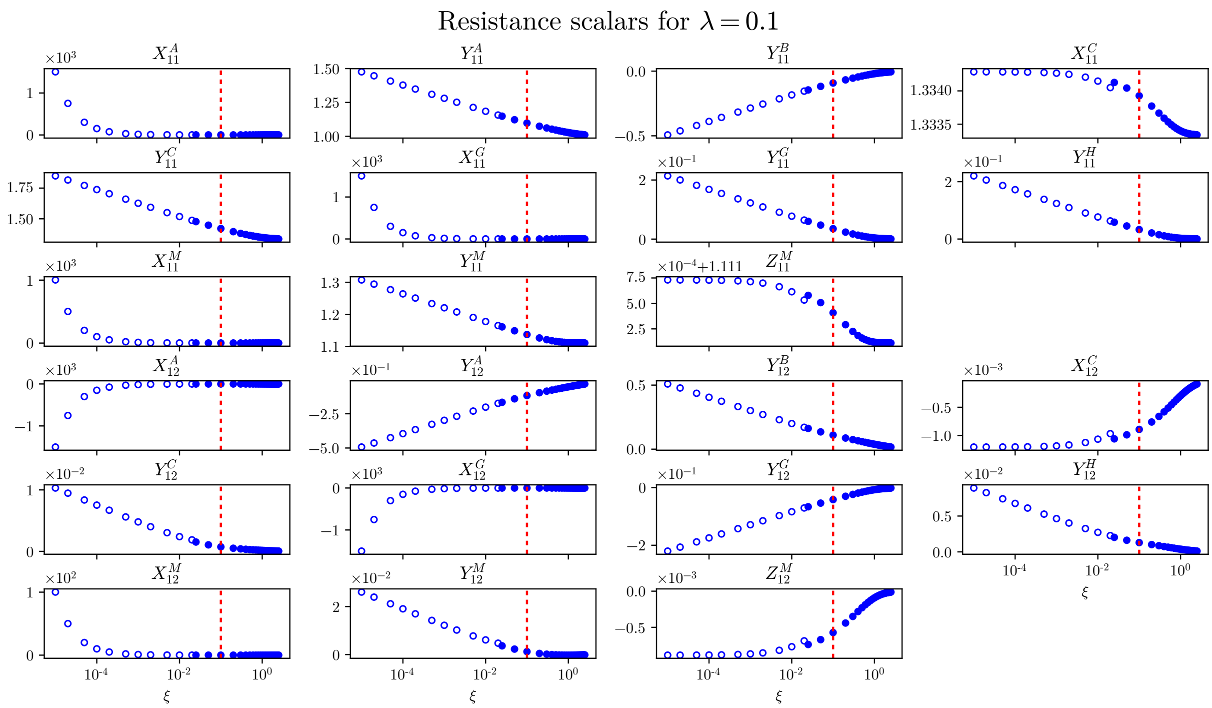

Figure 2: Values of the scalar resistance functions over non-dimensional gap, ξ, for size ratio $\lambda = 0.1$. Those generated from the near-field formulae are represented by hollow circles (○), and those generated from Lamb's solution (Wilson, 2013) are filled circles (●). The dashed vertical line appears at $\xi = \lambda$, recalling that the near-field formulae are only valid for $\xi \ll \lambda$.

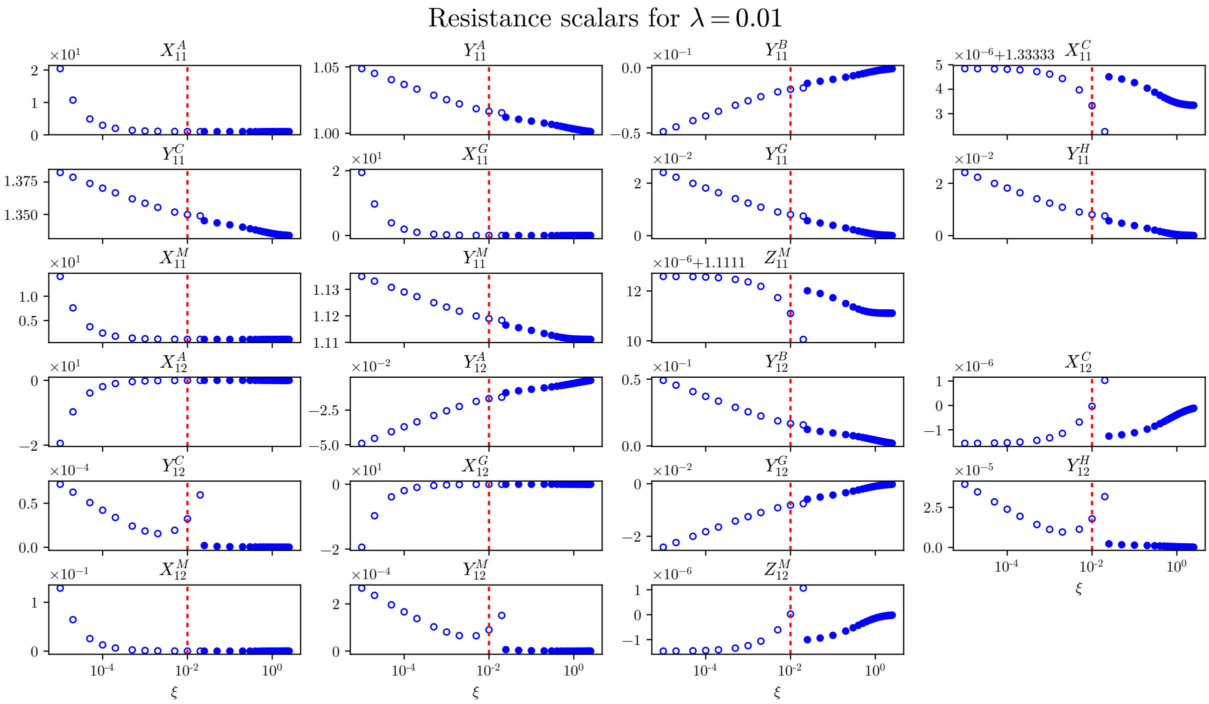

Figure 3: Values of the scalar resistance functions over non-dimensional gap, ξ, for size ratio $\lambda = 0.01$. Those generated from the near-field formulae are represented by hollow circles (○), and those generated from Lamb's solution (Wilson, 2013) are filled circles (●). The dashed vertical line appears at $\xi = \lambda$, recalling that the near-field formulae are only valid for $\xi \ll \lambda$.