An Intro to Hydrostatic Equilibrium

A Hydrostatic Haiku:
Gravity Pulls Down
Down, Inexorably Down
P briefly impedes

Consider the Earth's atmosphere by assuming the constituent particles comprise an ideal gas, such that $P = nk_{\rm B}T$, where n is the number density of particles (with units cm^{-3}), $k = 1.4 \times 10^{-16}$ erg K^{-1} is the Boltzmann constant. We'll use this ideal gas law in just a bit, but first

1. Think of a small, cylindrical parcel of gas, with the axis running vertically in the Earth's atmosphere. The parcel sits a distance r from the Earth's center, and the parcel's size is defined by a height $\Delta r \ll r$ and a circular cross-sectional area A (it's okay to use r here, because it is an intrinsic property of the atmosphere). The parcel will feel pressure pushing up from gas below ($P_{\rm up} = P(r)$) and down from above ($P_{\rm down} = P(r + \Delta r)$).

   Make a drawing of this, and discuss the situation and the various physical parameters with your group.

2. What other force will the parcel feel, assuming it has a density $\rho(r)$ and the Earth has a mass $M_\oplus$?

3. If the parcel is not moving, give a mathematical expression relating the various forces, remembering that force is a vector and pressure is a force per unit area.

4. Give an expression for the gravitational acceleration, g, at at a distance r above the Earth's center in terms of the physical variables of this situation.

5. Show that

$$\frac{dP(r)}{dr} = -g\rho(r) \tag{1}$$

   This is the equation of hydrostatic equilibrium.

6. Now go back to the ideal gas law described above. Derive an expression describing how the density of the Earth's atmosphere varies with height, $\rho(r)$? (HINT: It may be useful to recall that $dx/x = d\ln x$.)

7. Show that the height, H, over which the density falls off by an factor of $1/e$ is given by

$$H = \frac{kT}{\bar{m}g} \tag{2}$$

   where $\bar{m}$ is the mean (average) mass of a gas particle. This is the "scale height." First, check the units. Then do the math. Then make sure it makes physical sense, e.g. what do you think *should* happen when you increase $\bar{m}$? Finally, pat yourselves on the back for solving a first-order differential equation and finding a key physical result!

8. What is the Earth's scale height, $H_\oplus$? The mass of a proton is 1.7×10^{-24} g, and the Earth's atmosphere is mostly molecular nitrogen, N_2, where *atomic* nitrogen has 7 protons, 7 neutrons.