Blackbodies : Planck, Wein, & Rayleigh Jean

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1 Spherical Coordinates

$$dV = r^2 \sin\phi dr d\phi d\theta$$

$$dA = r^2 \sin\phi d\phi d\theta$$

$$d\Omega = \sin\phi d\phi d\theta$$

$d\Omega$ is the differential solid angle

$$\Omega = \int d\Omega = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi d\phi = 4\pi$$

2 The Planck Function

Perhaps the best form in which to introduce the blackbody form is as an energy density,

$$u_\nu(T)d\nu = \frac{8\pi}{c}\frac{h\nu^3}{c^2}\frac{1}{\exp\frac{h\nu}{kT}-1}d\nu$$

, where u_ν is called the *spectral energy density* and has units of $ergs\ cm^{-3}\ Hz^{-1}$. There is another quantity called specific intensity, $I_n u$, with units of $ergs\ cm^{-2}\ Hz^{-1}\ ster^{-1}$.

2.1 Relating I_ν and u_ν

2.1.1 Method 1

The energy being described by the Planck distribution is electromagnetic energy, therefore it can be described by rays. [1] Specific intensity $I_\nu(\Omega)$ is

> The amount of energy crossing, in a time dt, an area dA perpendicular to $\mathbf{\Omega}$, within a differential solid angle $d\Omega$ n a frequency range $d\nu$

In other words,

$$I_\nu(\Omega) = \frac{dE}{dA\ d\Omega\ dt\ d\nu}$$

The spectral energy density is the energy per unit volume per unit frequency range. Since we need to look in a specific direction, we need to use the energy density angular distribution $u_\nu(\Omega)$, which has units of specific energy density per solid angle. They are related by the integral $u_\nu = \int u_\nu(\Omega)d\nu$. The spectral energy density per unit solid angle is the amount traveling along $\mathbf{\Omega}$, through an area dA, within differential solid angle $d\Omega$, in time dt, so that the differential volume element is $dV = dAcdt$. In other words,

$$u_\nu(\Omega) = \frac{dE}{dA\ c\ dt\ d\Omega\ d\nu}.$$

In the differential limit, we can assume the energy that goes in, is the energy that flows out, so that we can find that

$$u_\nu(\Omega) = \frac{I_\nu(\Omega)}{c}.$$

[1] Technically, the flow of energy can be described by something called the Poynting vector (Carrol & Ostlee 3.3), but that will not be discussed here. What is important is that it shows that the electromagnetic waves carry energy in the direction of propagation along (generally)straight lines.

We can recover the angle independent spectral energy density by integrating over solid angle.

$$u_\nu = \int u_\nu(\Omega)d\Omega = \int I_\nu(\Omega)d\Omega.$$

When integrated over the surface of a sphere $\int d\Omega = \int \sin\phi \; d\phi \; d\theta = 4\pi$. So

$$u_\nu = \frac{4\pi}{c} I_\nu.$$

2.1.2 Method 2

This method makes use of a more intuitive argument. The energy density is energy per unit volume per unit range in frequency. What we want is the energy flow through a surface per unit area per unit solid angle per unit time per unit range in frequency. We will call this quantity the specific intensity (brightness).

If we have a region of energy density u_ν, the amount of energy flowing through a bounding surface is $u_\nu \; c\Delta t$, the energy flux through the surface during a time Δt. Since the surface completely surrounds the region, the solid angle through which the energy is flowing is 4π. Putting the argument together we get,

$$\frac{u_\nu \; c \; \Delta t}{4\pi \; \Delta t},$$

which has dimensions.

$$\frac{\mathrm{E \; T}}{\mathrm{L}^3} \frac{\mathrm{L \; T}}{\Omega \; \mathrm{T}^2} = \frac{\mathrm{E \; T}}{\mathrm{L}^2 \; \mathrm{T} \; \Omega}$$

which in physical units is,

$$\frac{energy}{area \; time \; solid \; angle \; frequency},$$

which matches out units for specific intensity,

$$I_\nu = \frac{dE}{dA \; dt \; d\Omega \; d\nu},$$

so,

$$I_\nu = \frac{4\pi}{c} u_\nu$$

2.1.3 The Blackbody Function

Now we can get the standard form of the blackbody equation used most by astronomers, just replacing, $I_\nu(T)$ with $B_\nu(T)$ for conventions sake,

$$B_\nu(T)d\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp\frac{h\nu}{kT} - 1} d\nu.$$

If we want this in terms of λ, we make the substitution $\nu = c/\lambda$, and remembering that $d\nu = \frac{-c}{\lambda^2}$,

$$B_\lambda(T)d\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\frac{hc}{\lambda kT} - 1} d\lambda.$$

2.2 Flux from an Isotropic Sphere

Imagine a sphere that radiates $B_\nu(T)$ isotropically form every point on its surface. The energy flowing through a surface area element dA along the line of sight into a solid angle $d\Omega$ is $B_\nu(T)$ from the point attenuated by $\cos\phi$.

$$F_\nu = \int I_\nu(\Omega)\cos\phi d\Omega = \int I_\nu(\Omega)\ \cos\phi\ \sin\phi\ d\phi\ d\theta.$$

Because the energy flow is outward only, the ϕ integral i only over $[0, \pi/2]$.

$$F_\nu = \int\limits_0^{2\pi} d\theta \int\limits_0^{\frac{\pi}{2}} B_\nu(T)\ \cos\phi\ \sin\phi\ d\phi = \pi B_\nu(T)$$

3 Wien's Displacement Law

Wien's law was discovered a while before the Planck Distribution was known. It predicts the wavelength at which the intensity of blackbody will peak for a given temperature.

The extrema of any function is located where the derivative of the function equals zero. Let's do this for the Planck distribution. We want to find λ_{max} for a given T. It would be useful here to simplify the problem by making the substitution $x = \frac{hc}{\lambda kT}$. This gives use the Planck distribution in x,

$$B(x) = \frac{2(kT)^5}{h^4c^3}\frac{x^5}{e^x - 1}.$$

$$\frac{dB(x)}{dx} = \frac{2(kT)^5}{h^4c^3}\frac{d}{dx}\left[\frac{x^5}{e^x-1}\right] = 0$$

$$\frac{d}{dx}\left[\frac{x^5}{e^x-1}\right] = 0$$

$$\frac{5x^4}{e^x-1} + \frac{-x^5e^x}{(e^x-1)^2} = 0$$

$$\frac{5x^4}{e^x-1} = \frac{x^5e^x}{(e^x-1)^2}$$

$$5 = \frac{xe^x}{e^x-1}$$

One might think that this is the end of what we can do analytically, but let's think about this function. It would be incredibly easy if $\frac{e^x}{e^x-1} = 1$. Exponentials grow fast, this means if $x > 1 \Rightarrow \frac{e^x}{e^x-1} \approx 1$. Since the Wein approximation is for $hc/\lambda >> kT \therefore x >> 1$, we can use this approximation and find that

$$x\left(\frac{e^x}{e^x-1}\right) = 5$$

$$x = 5$$
$$\frac{hc}{\lambda_{max}kT} = 5$$
$$\lambda_{max}T = \frac{hc}{k5}$$

This gets you to within 0.4% of the correct answer. Solving numerically we find,

$$x = 4.96511.$$

Substituting back in $x = \frac{hc}{\lambda kT}$ and solving, we get,

$$\lambda_{max}T = 2897.8\mu m \cdot K$$

An easier set of numbers (and in more useful units) to remember is,

$$\lambda_{max}T \approx 3000\mu m \cdot K$$

3.1 Alternative with frequency

We can find the Wein's law for frequency as well, using the frequency form of the equation

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\frac{h\nu}{kT} - 1}.$$

Making a similar substitution as before, $x = \frac{h\nu}{kT}$, we get

$$B(x) = \frac{2(kT)^3}{h^2c^2}\frac{x^3}{e^x-1}.$$

To find the peak, we take the derivative w.r.t. x and set it equal to 0,

$$\frac{dB(x)}{dx} = \frac{2(kT)^3}{h^2c^2}\frac{d}{dx}\left[\frac{x^3}{e^x-1}\right] = 0$$

Which means

$$\frac{d}{dx}\left[\frac{x^3}{e^x-1}\right] = 0$$
$$\frac{3x^2}{e^x-1} + \frac{-x^3e^x}{(e^x-1)^2} = 0$$
$$\frac{3x^2}{e^x-1} = \frac{x^3e^x}{(e^x-1)^2}$$

If we assume $x \neq 0$, which is true here, then

$$\frac{xe^x}{e^x - 1} = x\left(\frac{1}{1-e^{-x}}\right) = 3$$

$$3(e^x - 1) = xe^x$$

$$(x-3)e^x + 3 = 0$$

How do we solve this? Well, when in the form $3 = \frac{xe^x}{e^x-1}$, we can do the same thing as we did above and get an approximate form and solve, getting $x = 3$, however, we see in the simplified form $(x-3)e^x + 3 = 0$, that $x = 3$ is in fact not a valid solution (which of course it shouldn't be, as it is the result of an approximation). Given an approximate solution, a, we can use a Taylor series expansion around a to get a more precise answer. The 1st order Taylor expansion around (a) is

$$f(x \approx a) \approx f(a) + f'(a)(x-a) = [e^a(a-3)+3] + [(e^a + e^a(a-3))(x-a)]$$

The closed form full Taylor series is

$$f(x) = 3 + \sum_{n=0}^{\infty} e^3 \frac{(x-3)^p}{(p-1)!}$$

If we plug in our approximate solution $a = 3$ we get

$$e^3(x-3) + 3$$

, So our equation, expanded around our approximate solution is

$$(x-3)e^x + 3 \approx e^3(x-3) + 3 = 0$$

Solving for x we get

$$x \to 3 - \frac{3}{e^3} = 2.8506...$$

Which is very close to the correct answer (which can be found numerically) x = 2.82144

4 Rayleigh-Jeans Tail

It can be shown that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \text{ where } f^{(n)}(a) \equiv \left(\frac{d^n f(x)}{dx^n}\right)_{x=a}$$

. This is called a Taylor Expansion around a. The function e^x can be expanded, as

$$e^x = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$$

which to first order is

$$e^x \approx e^a(1 + (x-a)).$$

If x is small, we can expand around 0 and get that

$$e^x \approx 1 + x,$$

a brilliantly elegant and simple result.

Looking at the blackbody function, we see that $\frac{h\nu}{kT}$ is small means that $h\nu \ll kT$. This is the Rayleigh Jeans tail of the blackbody function. Using our approximation, we get,

$$B_\nu(T) \approx \frac{2kT}{c^2}\nu^2,$$

and for λ,

$$B_\lambda(T) \approx \frac{2kTc}{\lambda^4}.$$

The first expression is the most commonly used form of the expression. We use the Rayleigh-Jeans approximation for determining the intensity of long wavelength (low frequency) objects.

4.1 Bolometric Blackbody Flux

To determine the total amount of energy being radiated by a blackbody at temperature T, we need to integrate the equation of all frequencies (or wavelengths). For a star, recall,

$$F_\nu = \int I_\nu \cos\phi d\Omega$$

. For a star approximated as a blackbody, the intensity is isotropic and independent of angle so $F_\nu = \pi I_\nu$ and the bolometric flux is

$$F = \int\limits_0^\infty \pi B_\nu(T) d\nu.$$

$$= \pi \int \frac{2h\nu^3}{c^2} \frac{1}{\exp\frac{h\nu}{kT} - 1} d\nu$$

substitute $x = \frac{h\nu}{kT},\ dx = \frac{h}{kT}d\nu$,

$$= \pi \int \frac{2h}{c^2} \left(\frac{kTx}{h}\right)^3 \frac{1}{e^x - 1} \left(\frac{kT}{h}\right) dx$$

$$\pi \frac{2k^4T^4}{c^2h^3} \int \frac{x^3}{e^x - 1} dx$$

Here it is useful to know the $\int_0^\infty \frac{x^3}{e^x-1} dx = \frac{\pi^4}{15}$ [2],

$$F = \frac{2\pi^5 k^4}{15c^2h^3} T^4$$

$$F = \sigma T^4$$

$$\sigma = 5.67 \times 10^{-5} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ K}^{-4}$$

[2] For a demonstration of this see http://web.mit.edu/8.03-esg/watkins/8.03/qthd.pdf

The total energy radiated is

$$u = \frac{4}{c}\sigma T^4$$

. We recall that

$$u = \frac{4\pi}{c} I,$$

where

$$u = \int u_\nu d\nu$$

$$I = \int I_\nu d\nu$$

The luminosity of a star modeled as a blackbody is

$$L = 4\pi R^2 F$$

and the flux receive by an observers some distance d away from a star or radius $R_\star$ and temperature $T_\star$ is

$$F = \sigma T^4 \left(\frac{R}{d}\right)^2$$

5 Planetary Equilibrium Temperature

Image and planet around a star with luminosity, $L_\star$, radius $R_\star$, and temperature $T_\star$. What is the equilibrium temperature of a planet with albedo A and distance r.

First, we assume energy radiated equals the energy received,

$$F_{\text{in}} = F_{\text{out}}$$

(flux from star at planet) $\times$ (area receiving flux) $\times$ (fraction of energy absorbed) $=$ (Blackbody flux from planet)

and let us assume both radiate as black bodies,

$$\left(\frac{L_\star}{4\pi d^2}\right)\left(\pi R_p^2\right)(1-A) = 4\pi R_p^2 \sigma_{SB} T_p^4$$

$$T_p^4 = \frac{L_\star(1-A)}{16\pi\sigma_{SB} d^2}$$

$$T_p = \frac{1}{2d}$$

6 Appendix

This method is not for the faint of heart, isn't written pedagogically, and doesn't explain how it got to the first step. A scheme for approximating a better answer for the Wein approximation can be determined by looking at limits.

$$\text{Let } f(x_0) = 0$$

$$f(x) \approx x - 5 \text{ for } x >> 1$$
$$f(x) < 0 \text{ for } x < x_0$$
$$f(x) > 0 \text{ for } x > x_0$$

If we allow for our error margin to be about 10%, then we can allow $f(x) = x \text{ for } x > 2$.

One might notice that for the limit of large x, the zero appears to be close to 5. Depending on how sharply the function diverges from a straight line for small x, this could be a good starting point for out iteration. A sample iteration scheme would be to test $x = 1 \text{ and } 10$.

$$f(1) = \frac{e}{e-1} - 5 \approx \frac{3}{3-1} - 5 = -3.5$$
$$f(10) \approx 5.00045$$

By checking our limit, we know that 1 is too low and 10 is too high. We can take two directions. Since $x > 1$ we take $f(x) = x - 5$ and live with the approximate answer of $x = 5$ being the location of the maxima. This good to first order. If you want to get a better answer, then the key is to iterate down to the correct answer, which we can get to a couple digits of precision quite simply. So we know a maximum range. We also know that in the limit of large x that $f(x) \approx x - 5$, so we can check for an upper range of 6 and lower value of 4. We can then plug in 5, and see that is is a little to low. Start checking numbers close to 5 (in the range of 4.9-5). The correct answer is

$$x = 4.96511$$
$$\frac{hc}{\lambda_{max} kT} = 4.96511$$
$$\lambda_{max} T = \frac{hc}{k 4.96511}$$
$$\lambda_{max} T = 2897.8 \mu m \cdot K \approx 3000 \mu m \cdot K$$

An easy way to remember this is that room temperature $(300K)$ peaks at $10\mu m$