Gravitation in Astronomy

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1 Gravity

1.1 An Extremely Brief History of Gravity

[1] In 1687, Isaac Newton published *Philosophiae Naturalis Principia Mathematica* describing the universal law of gravitation and laying the groundwork for classical (Newtonian) mechanics. This theory of mechanics laid largely undisturbed for over 200 hundred years until Albert Einstein revolutionized physics with his special and general theories of relativity.

1.2 Newton and Gravity

Isaac Newton's ***universal law of gravitation*** describes the gravitational force between two masses, m_1 and m_2, by,

$$\mathbf{F_g} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}, \tag{1}$$

where $\mathbf{r}$ is the vector connecting the two point masses. This law is often simply written,

$$F_g = -\frac{Gm_1m_2}{r^2}. \tag{2}$$

This law, combined with Newton's three laws of motion, forms the basis of classical mechanics and is still the starting point for any introductory class on classical mechanics. For the sake of completeness, here are Newton's Three Laws of Motion (in *original language*, **concept**, and translation[2])

- *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*

  Inertia : Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by frees impress d thereon.

- *Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.*

  $\vec{\mathbf{F}} = \mathbf{m}\vec{\mathbf{a}} = \frac{\mathbf{d}\vec{\mathbf{p}}}{\mathbf{dt}}$: The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd.

[1] Adapted from Who Discovered Gravity : UniverseToday.com
[2] Motte's 1729 translation of *Philosophiae Naturalis Principia Mathematica*

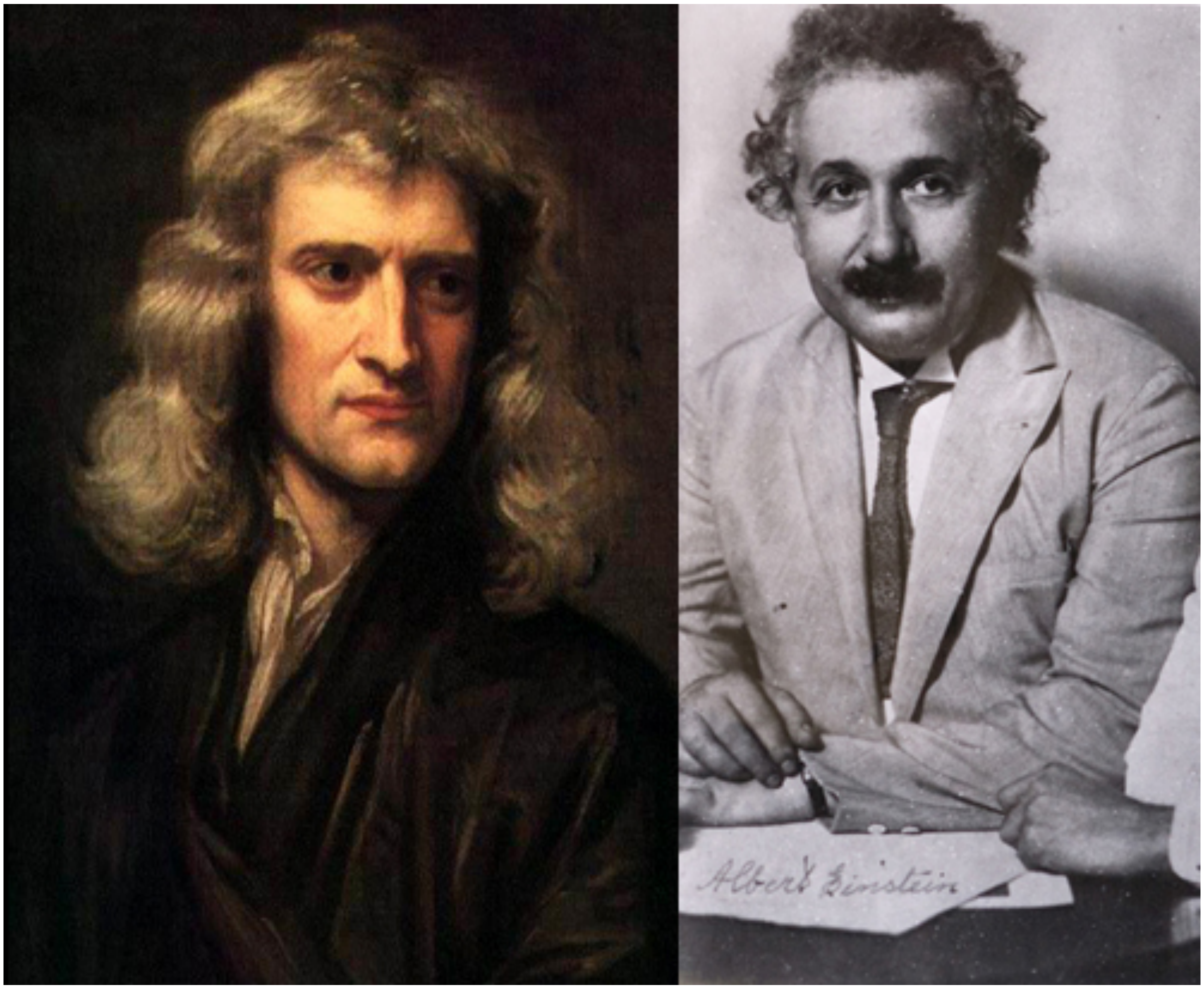


Figure 1: Isaac Newton and Albert Einstein. Arguably two of the most revered physicists in history

- *Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.*

  Equal and Opposite: To every Action there is always opposed an equal Reaction : or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

1.3 Work and Potential Energy

A very basic principle in physics is the conservation of energy in a closed system, thus it is beneficial to understand how energy and forces are related. Consider the application of force to a mass M along some path $\mathbf{r}$. The application of force imparts some kinetic energy $(K = \frac{p^2}{2m})$ to the mass. This energy comes from the application of force across the distance $|\mathbf{r}|$. This change in kinetic energy is called

work[3]. The proof is.

$$W = \int_{r_1}^{r_2} \mathbf{F} \cdot \mathbf{dr} = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}(t)\ dt \tag{3}$$

$$= \int_{t_2}^{t_1} m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\ dt = \frac{1}{2} \int_{t_2}^{t_1} m \frac{d}{dt}\left(v^2\right)\ dt \tag{4}$$

$$= \frac{1}{2} m v(t_2)^2 - \frac{1}{2} m v(t_1)^2 = K_2 - K_1 = \Delta K \tag{5}$$

1.3.1 Conservative Forces and Potential Energy

If W is path independent then $\mathbf{F}$ is the gradient of some scalar function. This condition holds for any physical force, but functions can be thought of for which this does not hold. The condition is equivalent to saying that $\nabla \times \mathbf{F} = 0$ (in other words; $curl(\mathbf{F}) = 0$ or $\mathbf{F}$ is *irrotational*).

Therefore, there exists a function $\mathcal{U}$ for which,

$$\mathbf{F} = \nabla \mathcal{U} \tag{6}$$

We call this function a **potential**. It's magnitude is equal to the kinetic energy a particle would gain if it started falling from some position $\mathbf{r}$ to some fiducial position (perhaps the origin) $\mathcal{O}$. If we consider that energy should be conserved[4], then,

$$\Delta E = \Delta U + \Delta K = 0 \therefore \Delta U = -\Delta K = -W \tag{7}$$

$$-W = \Delta U$$

$$-\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{F} \cdot \mathbf{dr} = U(\mathbf{r}) - U(\mathcal{O}) \tag{8}$$

$$U(\mathbf{r}) = U(\mathcal{O}) - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{F} \cdot \mathbf{dr} \tag{9}$$

The only thing we care about with potentials is the difference in the potential. Taking the gradient of both sides and using the fundamental theorem of calculus[5],

$$\mathbf{F} = -\nabla U(\mathbf{r}) \tag{10}$$

where $U(\mathbf{r})$ is the potential energy for the conservative force.

[3] W stands for *work* which is a term I have never liked as it hides the fundamental fact that it is a change in energy.
[4] which is true for any closed system
[5] $\frac{d}{dx} \int f(x) dx = f(x)$

1.4 Gravitational Potential Energy

The ***gravitational acceleration vector*** $\mathbf{g}$ represents the force per unit mass on a particle in the gravitational field of a particle with mass M [6],

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}. \tag{11}$$

The ***gravitational potential*** Φ_g is the potential energy per unit mass of a particle in the gravitational field of a particle with mass M.

$$\Phi_g = \frac{U}{m} \tag{12}$$

What we currently know is [7]

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} \tag{13}$$

$$\mathbf{g} = -\nabla\Phi_g \tag{14}$$

The gravitational potential of a point particle,

$$F_g(r) = -\frac{dU}{dr} \tag{15}$$

$$-\frac{GMm}{r^2} = -\frac{dUt}{dr}$$

$$\int \frac{GMm}{r^2}\,dr = \int dU = U(r) + C_1$$

$$-\frac{GMm}{r} + C_2 = U(r) + C_1$$

$$U(r) = -\frac{GMm}{r} + C \tag{16}$$

If we use the generic condition that $\lim\limits_{r\to\infty} U(r) = 0$, then we get the general expression,

$$U(r) = -\frac{GMm}{r} \tag{17}$$

1.5 The Gravitational Field and Potential

1.5.1 Of a Point mass

Consider the strength gravitation field and potential at $\mathbf{r}$ at of a point mass located at $\mathbf{r_0}$. If we consider $\mathbf{r_0}$ to be the origin, then $|\mathbf{r_0} - \mathbf{r}| = r$ The Gravitational Potential

$$\Phi_g = -\frac{GM}{r} \tag{18}$$

The Gravitational Field

$$\begin{aligned}\mathbf{g} &= -\nabla\Phi_g \\ &= -\frac{GM}{r^2}\hat{\mathbf{r}}\end{aligned} \tag{19}$$

[6] Marion & Thornton, *Classical Dynamics*

[7] Note: We are technically working in spherical coordinates when treating a point particle. Fortunately the gravitational force has no ϕ or θ dependence, it's spherically symmetric for a point particle, and the r derivative behaves as we are used to from Cartesian coordinates

1.5.2 Continuous Matter Distribution

Let us consider two general, continuous distributions of matter. Here $r = |\mathbf{r_1} - \mathbf{r_2}|$.

$$\Phi_g = -G \int \frac{\rho(\mathbf{r_1})}{r}\ d^3 r_1 \tag{20}$$

$$\mathbf{g} = -G \int \frac{\rho(\mathbf{r_1})}{r^2} \hat{\mathbf{r}}\ d^3 r_1 \tag{21}$$

Note that $\mathbf{r}$ and similarly r vary as a function of $\mathbf{r_1}$.

Consider the gravitational force and potential energy

$$U_g = -G \int \frac{\rho(\mathbf{r_1})\rho(\mathbf{r_2})}{|\mathbf{r_1} - \mathbf{r_2}|}\ d^3 r_1 d^3 r_2 \tag{22}$$

$$\mathbf{F_g} = -G \int \frac{\rho(\mathbf{r_1})\rho(\mathbf{r_2})}{|\mathbf{r_1} - \mathbf{r_2}|^2} \hat{\mathbf{r}}\ d^3 r_1 d^3 r_2 \tag{23}$$

Let us write the integral in spherical coordinates, and assume the distributions are independent of the polar angle ϕ. If this is true, then the vector separation $|\mathbf{r_1} - \mathbf{r}|$ can be expressed using the law of cosines

1.6 Total Gravitational Potential (Binding) Energy

The total potential energy of an object (aka **binding energy**) is the amount of energy required pull a system apart, out to infinity. Basically, if we didn't give it enough energy to go to infinity, then it would eventually decelerate, stop, and finally accelerate back to where it was and hence would not have escaped the gravitational potential. So how tightly bound is a distribution of matter. Let's consider this for a spherical matter distribution with constant density. some tiny $dm = \rho\ dV$[8]on the surface on the sphere if radius R is bound to the system by the gravitational potential generated by the mass beneath it, $M(< R) = \frac{4}{3}\pi R^3 \rho$. Since we are dealing with a spherically symmetric system, we can get rid of the angular dependence by integrating or just noting that dm is the mass contained in a spherical shell with radius dr such that $dm = 4\pi\ \rho\ r^2 dr$. So we can say that the potential energy holding that shell in place is

$$dU = \frac{GM(< R)}{R} dm$$

So if we put dU into the surface of the sphere a tiny amount will shove off to infinity. But we want to unbind all the mass, all the way form the surface at $r = R$ to the center at $r = 0$. We want to add up

[8]For a sphere $dV = r^2 \sin(\Phi) dr d\theta d\Phi$

all of that energy, which sounds like a job for an integral.

$$\text{Binding Energy} = \int_0^R \frac{GM(<r)}{r}\, dm \tag{24}$$

$$= \int_0^R \frac{G\frac{4}{3}\pi r^3 \rho}{r} 4\pi\rho r^2\, dr \tag{25}$$

$$= \int_0^R \frac{16 G\pi^2\rho^2}{3} r^4\, dr$$

$$= \frac{16}{3} G\pi^2\rho^2 \frac{R^5}{5}$$

$$= \frac{4}{5} G\ \frac{4}{3}\pi R^3 \rho\pi\rho R^2$$

$$= \frac{4}{5} G\ \left(\frac{4}{3}\pi R^3 \rho\right)\left(\frac{4}{3}\pi R^3 \rho\right)\frac{3}{4R}$$

$$= \frac{3}{5}\frac{GM^2}{R} \tag{26}$$

1.7 Kepler's Law

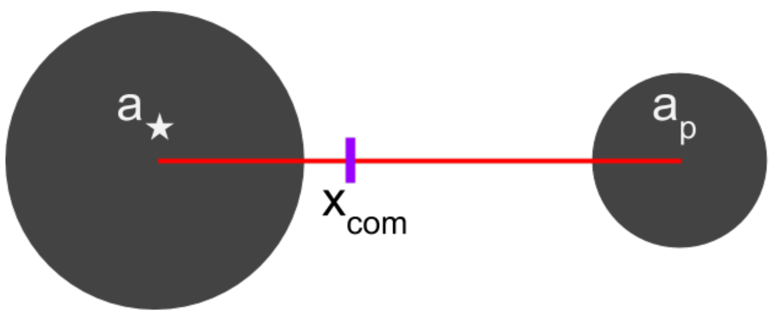


Figure 2: Objects orbit each other around a common center of mass (COM). We call the distance from the COM to that star and planet, $\mathrm{a}_\star$ and $\mathrm{a_p}$ respectively.

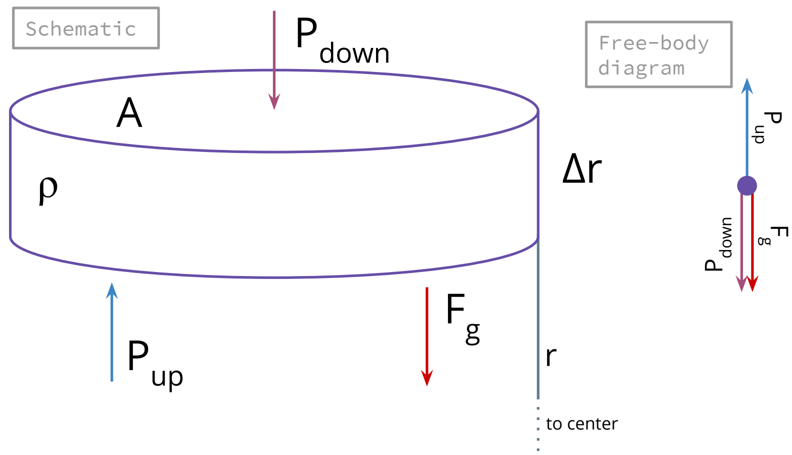
$$m$$

1.8 Hydrostatic Equilibrium

The first equation of stellar structure is **hydrostatic equilibrium**.

> A parcel of fluid is said to be in hydrostatic equilibrium occurs when it has no net acceleration. This occurs when external forces are balanced by pressure.

Consider a parcel of air in the Earth's atmosphere with surface area A, density ρ, and thickness Δr. The forces that need to be taken into consideration are the forces dues to pressure and gravity. In a



fluid, absent some flow in the horizontal direction or a change in viscosity, the only forces will be due to pressure and gravity. Pressure is defined as force per unit area. So listing the forces

$$\mathbf{F}_{\text{up}} = P_{\text{up}}\ \text{A}\ \hat{\mathbf{r}} \tag{27}$$
$$\mathbf{F}_{\text{down}} = -P_{\text{down}}\ \text{A}\ \hat{\mathbf{r}} \tag{28}$$
$$\mathbf{F}_{\text{g}} = -m\ \mathbf{g} = -m\frac{GM(<r)}{r^2}\hat{\mathbf{r}} \tag{29}$$

We can rewite the pressures as,

$$P_{\text{up}} = P(r) \tag{30}$$
$$P_{\text{down}} = P(r+\Delta r) \tag{31}$$

Using force balance, we want the pressure gradient to balance the gravitational force. Assume the gravitational acceleration is due to the mass beneath the slab, $M(<r)$.

$$\sum \mathbf{F} = 0 \tag{32}$$
$$\mathbf{F}_{\text{up}} + \mathbf{F}_{\text{down}} + \mathbf{F}_{\text{g}} = 0 \tag{33}$$
$$\tag{34}$$

Since everything is along the $\hat{\mathbf{r}}$ direction, we can take the magnitude and drop the vector. Substituting and rearranging, we get

$$P(r+\Delta r)A - PA = -\frac{GM(<r)}{r^2}\rho A\Delta r \tag{35}$$
$$\frac{P(r+\Delta r) - P}{\Delta r} = -\frac{GM(<r)}{r^2}\rho \tag{36}$$
$$\frac{dP}{dr} = -\frac{GM(<r)}{r^2}\rho \tag{37}$$

or,

$$\frac{dP}{dr} = -\rho g \tag{38}$$

1.9 Collapse

1.9.1 Jean's Collapse of a GMC

Star formation is one of the most important topics in astrophysics. How they form, and basic models of star formation can be derived with what we have so far. Stars form when giant molecular clouds collapse under the force of gravity. In general a cloud is supported by its internal pressure. This relationship can be derived using the viral theorem by making a few assumptions about the support and equation of state of the gas.

In a molecular cloud it is possible to say the the primary source of energy support in the cloud is thermal energy. This is because the motion of the particles is so small, that the primary source of kinetic energy comes from the Maxwell-Boltzmann distribution with describes the energy states occupied by a gas in thermal equilibrium (as a cloud of molecular gas is in). It can be shown, via the derivation of the Equipartition Theorem[9], that for each degree of freedom, a particle has a thermal energy, $E = \frac{1}{2}kT$. In a giant molecular in LTE, every particle has the same temperature on average, there for the total thermal support in the cloud is

$$E = \frac{f}{2}NkT = \frac{1}{2}\frac{M}{\mu m_H}kT, \tag{39}$$

where we have f degrees of freedom for a gas with mean molecular mass μm_H. Using the Virial theorem, we know that for a stable bound system

$$K = -\frac{1}{2}U.$$

For a giant molecular, we can assume the total gravitational potential for the cloud is

$$U = -A\frac{GM^2}{R},$$

where A is some geometry dependent factor of order unity. For the cloud to collapse to form a star, we require that the gravity over powers the available pressure (kinetic energy) support,

$$\begin{aligned}
-\frac{1}{2}U &> K \qquad (40)\\
A\frac{GM^2}{2R} &> \frac{f}{2}NkT \qquad (41)\\
A\frac{GM}{R} &> f\frac{kT}{\mu m_H}\\
A\frac{GM}{\left(\frac{3M}{4\pi\rho}\right)^{1/3}} &> f\frac{kT}{\mu m_H}\\
A\,G\left(\frac{4\pi\rho}{3}\right)^{1/3} M^{2/3} &> f\frac{kT}{\mu m_H}\\
M^{2/3} &> \frac{f}{A}\frac{kT}{\mu m_H G}\left(\frac{3}{4\pi\rho}\right)^{1/3}\\
M &> \left(\frac{fk}{A\,G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\rho^{-1/2}T^{3/2} = M_J
\end{aligned}$$

[9] Schroeder, *Thermal Physics*, §1.2 & 1.3

The corresponding radius is,

$$R^3 = \frac{3}{4\pi}\frac{M}{\rho} > \left(\frac{3}{4\pi}\frac{1}{\rho}\right)\left(\frac{f\ k}{A\ G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\frac{1}{\rho^{1/2}}\,T^{3/2}$$
$$R^3 > \left(\frac{f\ k}{A\ G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi}\frac{T}{\rho}\right)^{3/2}$$
$$R > \left(\frac{3f\ k\ T}{A\ G\ \mu m_H\ 4\pi\ \rho}\right)^{1/2} = \lambda_J \tag{42}$$

Assuming the cloud that collapses to a star is spherical, the constant $A = \frac{3}{5}$, and $f = 3$ for the three degrees of freedom. This is the Jean's mass, M_J,

$$M_J = \left(\frac{5k}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\ \rho^{-1/2}\,T^{3/2} \tag{43}$$
$$s\lambda_J = \sqrt{\frac{15kT}{4\pi G\mu m_H\rho}} \tag{44}$$

We can now simply write the Jeans criterion for collapse,

$$M > M_J$$
$$R > \lambda_J$$

This leads to fragmentation during collapse. When $R \Rightarrow \frac{1}{n}R$ then $R_J \Rightarrow \frac{1}{n\sqrt{n}}R_J$. Therefore the Jeans length shrinks faster than the cloud. This means that there are now multiple regions within the cloud with $R > R_J$ that can collapse.

1.9.2 Free Fall Time

Need to show method using integration of equation of motion

It is a simple matter to also determine how long it takes for a cloud to collapse. This time scale, t_{ff}, can be determined with a neat little trick. If we consider a particle falling towards the center of mass to be on an $e \to 1$[10] Keplerian orbit, we can skip all the dynamics and get straight to the period. If the cloud has radius R, then the semi major axis of the orbit is $R/2$.

$$P^2 = \frac{4\pi^2}{GM}a^3 \tag{45}$$
$$(2\ t_{ff})^2 = \frac{4\pi^2}{G\frac{4}{3}\pi R^3\rho}\left(\frac{R}{2}\right)^3 \tag{46}$$
$$4(t_{ff})^2 = \frac{3\pi^2}{G4\pi\rho}\frac{1}{2} \tag{47}$$
$$t_{ff} = \sqrt{\frac{3\pi}{32G\rho}} \tag{48}$$

[10]Consider that $e = 0$ is circle and $e = 1$ is "maximally" "elliptical"

2 Derivation of the Virial Theorem

Derivation taken from Ben Cook (bcook@cfa.harvard.edu) Take the moment of inertia

$$I = \sum_i m_i r_i^2 = \sum_i m_i \mathbf{r_i} \cdot \mathbf{r_i}$$

$$\dot{I} = \frac{d}{dt} \sum_i m_i r_i^2 \tag{49}$$

$$= \sum m_i \frac{dr_i^2}{dt} \tag{50}$$

$$= \sum m_i 2 r_i \dot{r}_i \tag{51}$$

$$= 2 \sum p_i r_i \tag{52}$$

Now let's look at the second derivative of the moment of inertia

$$\frac{d^2 I}{dt^2} = 2 \sum \frac{d}{dt}(p_i r_i) \tag{53}$$

$$= 2 \sum r_i \frac{d}{dt} p_i + p_i \frac{d}{dt} r_i \tag{54}$$

$$= 2 \sum F_i r_i + 2 \sum m_i v_i^2 \tag{55}$$

$$= 2 \sum F_i r_i + 4K \tag{56}$$

$$\tag{57}$$

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2K + \sum F_i r_i$$

You can go through a proof using Newton's third law to show

$$\sum_i F_i r_i = -(n+1)U \text{ , where } F_i \propto -r^n$$

In equilibrium if

- I not changing
- Average over long enough τ
- Average over a period

2.1 Reducing the second derivative of moment of inertia