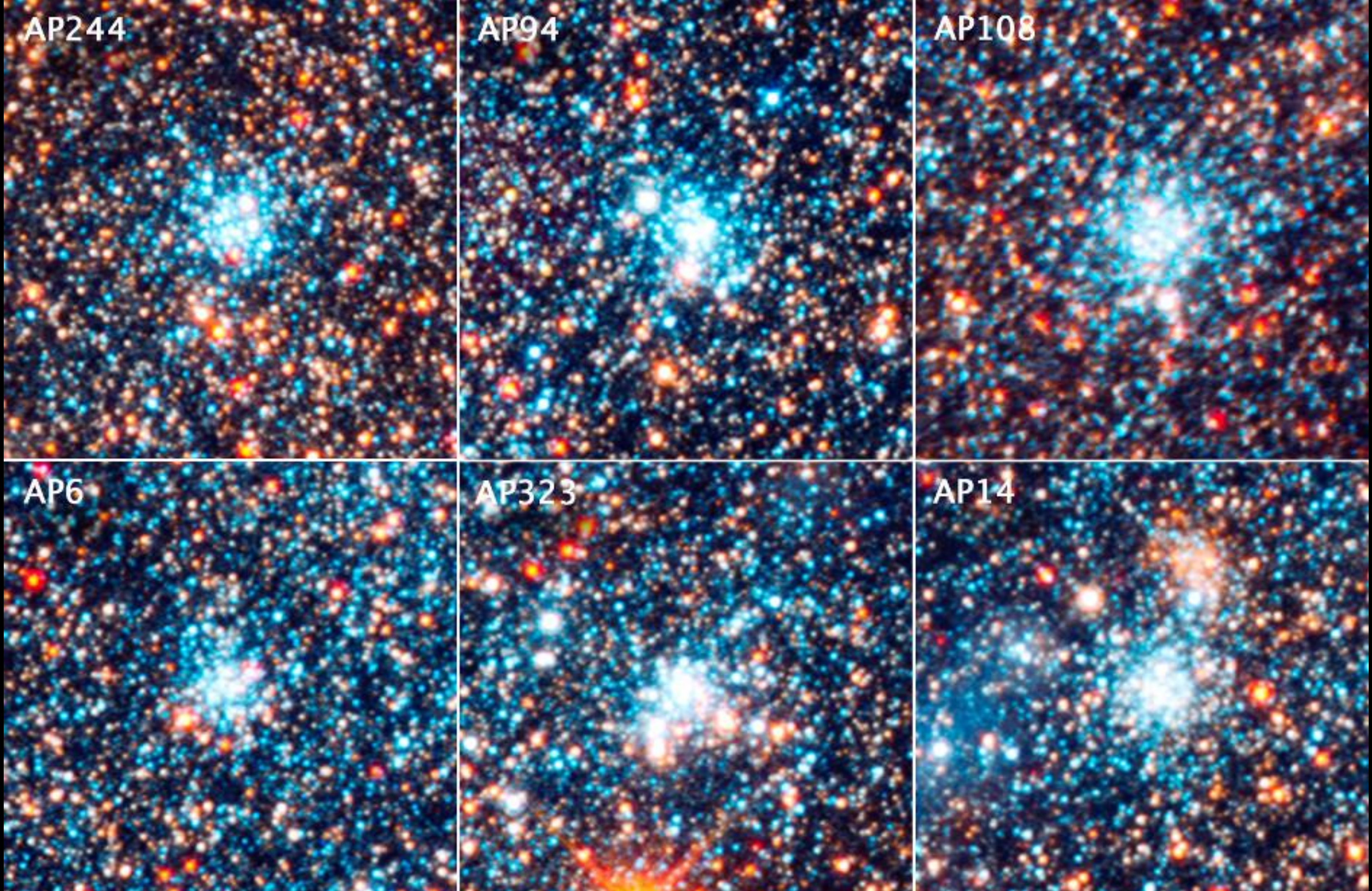

AP244
AP94
AP108
AP6
AP323
AP14

Stellar Interiors

Outline and Goals

- Day 1: Introduction & History
    - Stars - What are they
    - The Harvard Computers
    - Stellar Classification & HR Diagram
- Day 2: Hydrostatic Equilibrium
    - Simple Derivation from Classical Physics
    - Polytropes & the Lane-Emden Equation
- Day 3: Energy Transfer
    - Random Motion
- Day 4: Homology Relations
    - Stellar Scaling Relations

Learning is a lifestyle

Fixed Mindset

- Intelligence is fixed
- Problems are too hard
- Leaning on others = weakness
- Not knowing = failure

Growth Mindset

- Intelligence grows and is multifaceted
- There are problems you haven't solved yet
- Leaning on others = humility & wisdom
- Not knowing = opportunity to learn

Day 1

Introduction to stellar interiors and a dasch of history

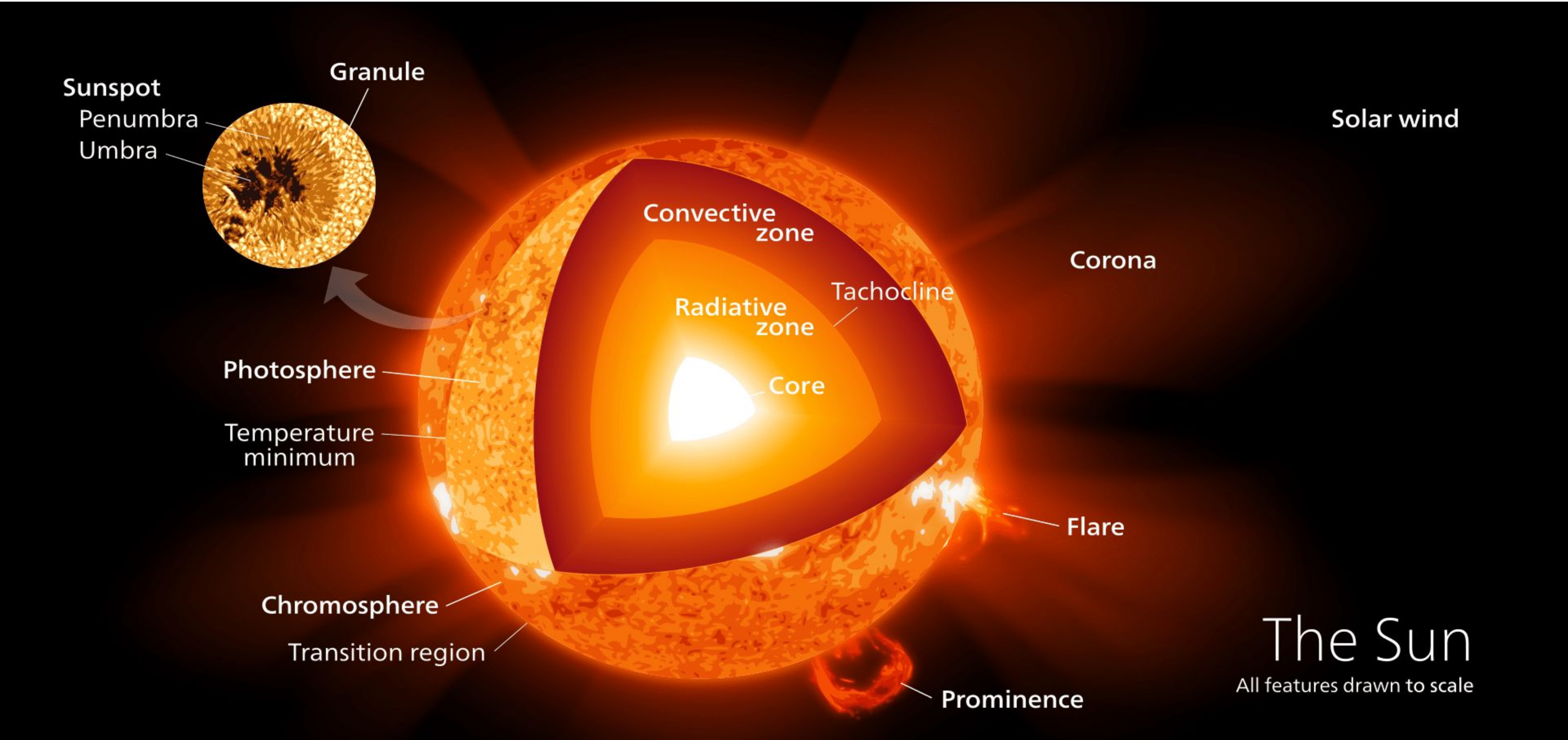
the Sun: vital statistics

Mass : 2×10^{33} g (3.33 x 10^5 earth mass)

Luminosity = 4×10^{33} erg/s

Radius = 7×10^{10} cm (100 earth radii)

Distance = 1.5×10^{13} cm (215 solar radii)



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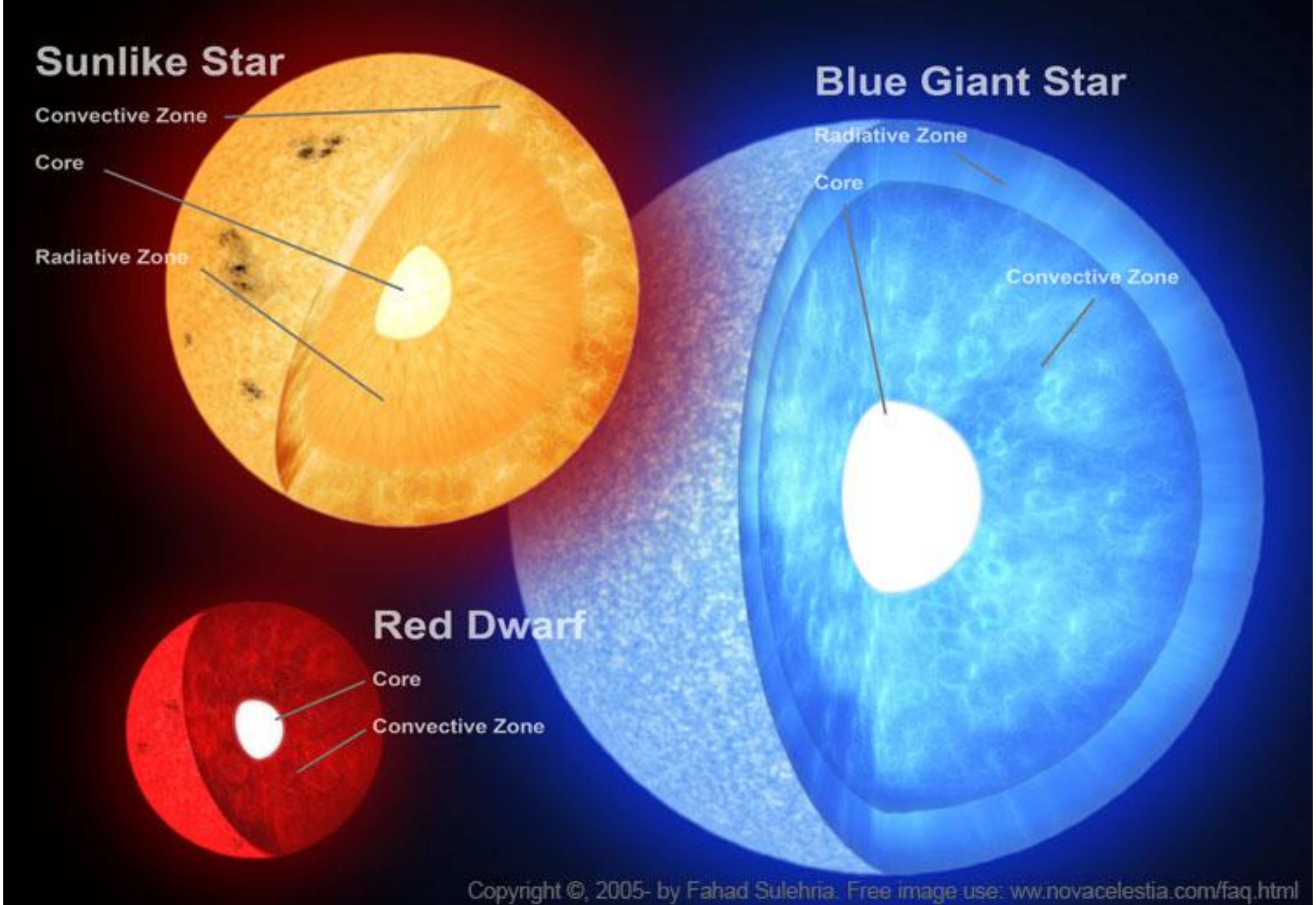
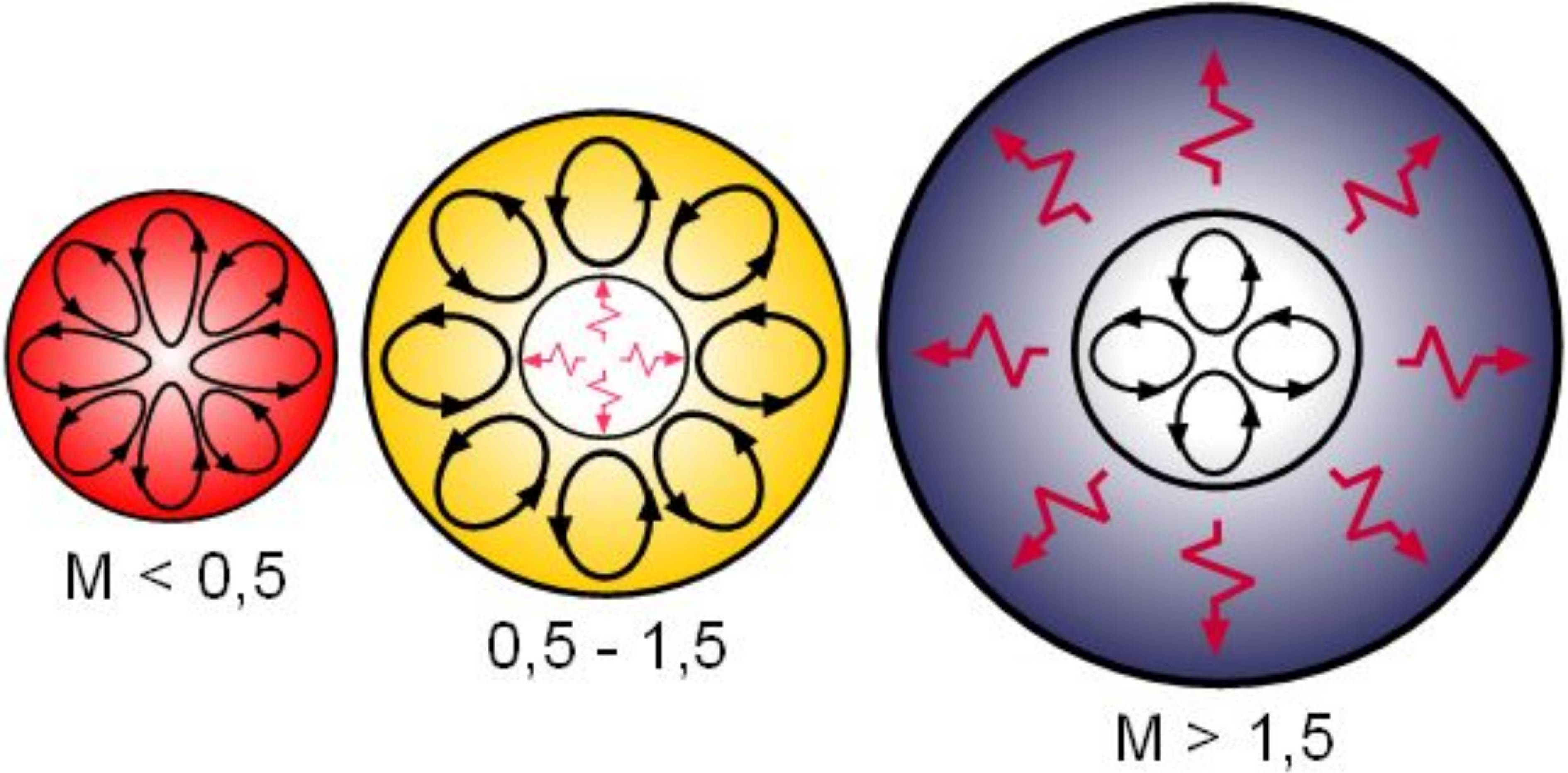
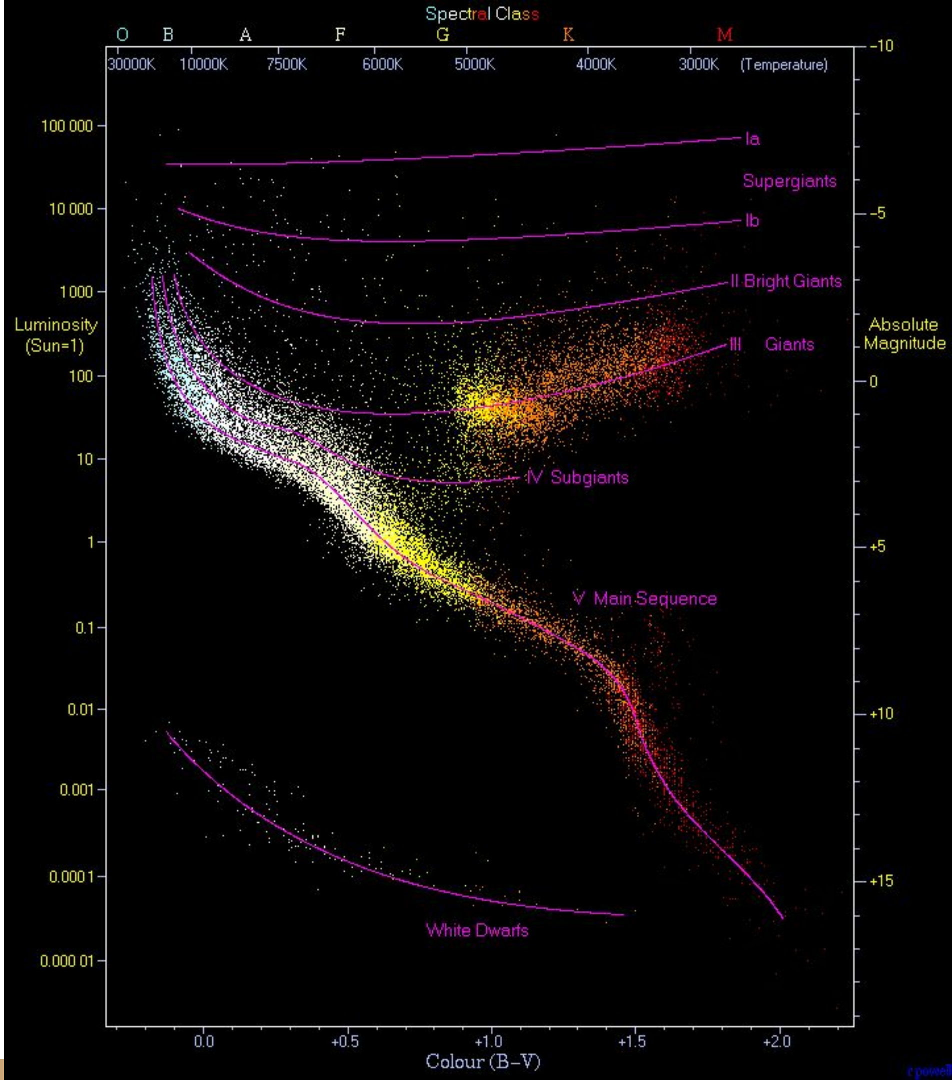
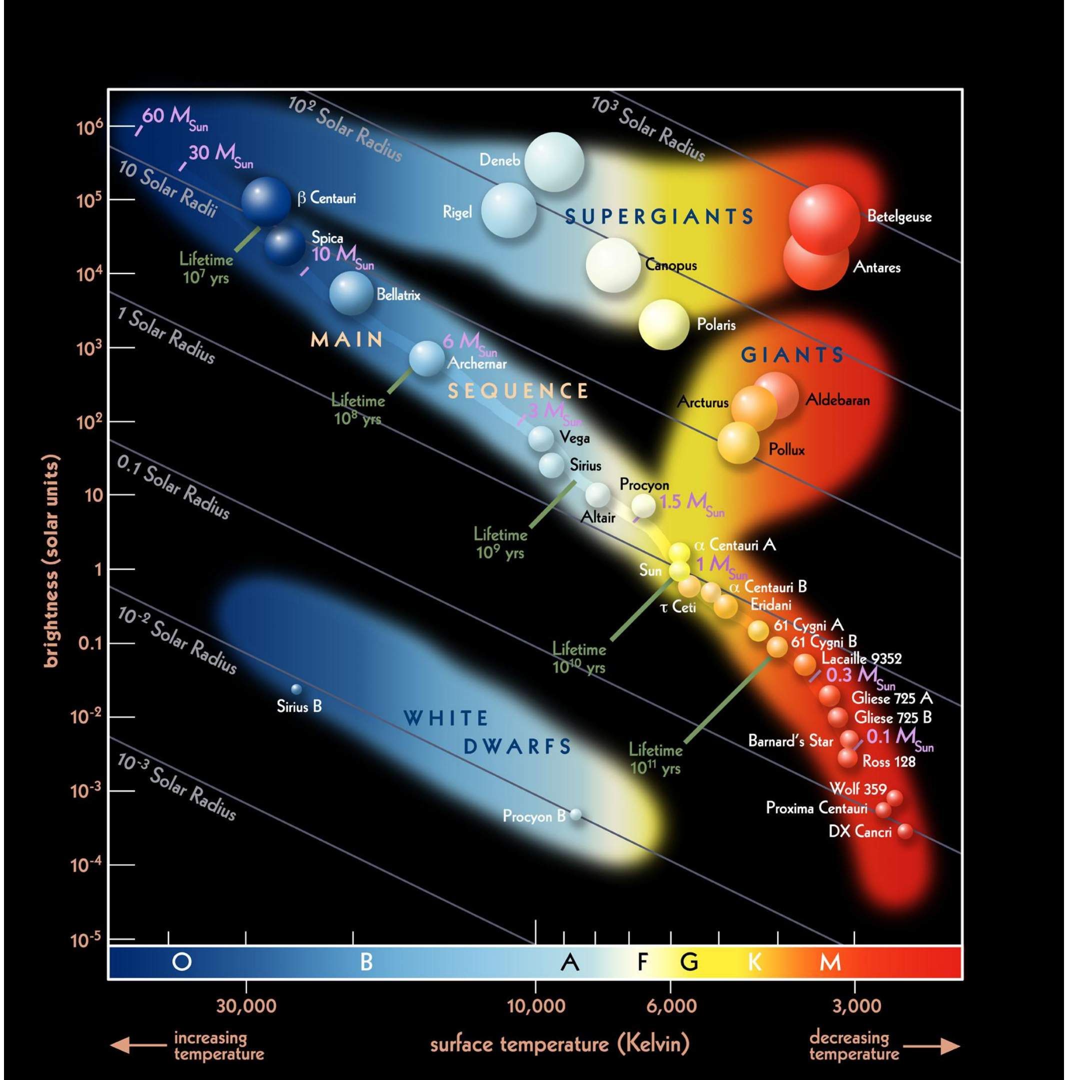

Sunlike Star
Convective Zone
Core
Radiative Zone
Blue Giant Star
Radiative Zone
Core
Convective Zone
Red Dwarf
Core
Convective Zone
Copyright ©, 2005- by Fahad Sulehria. Free image use: ww.novacelestia.com/faq.html


M < 0,5
0,5 - 1,5
M > 1,5

HR Diagram

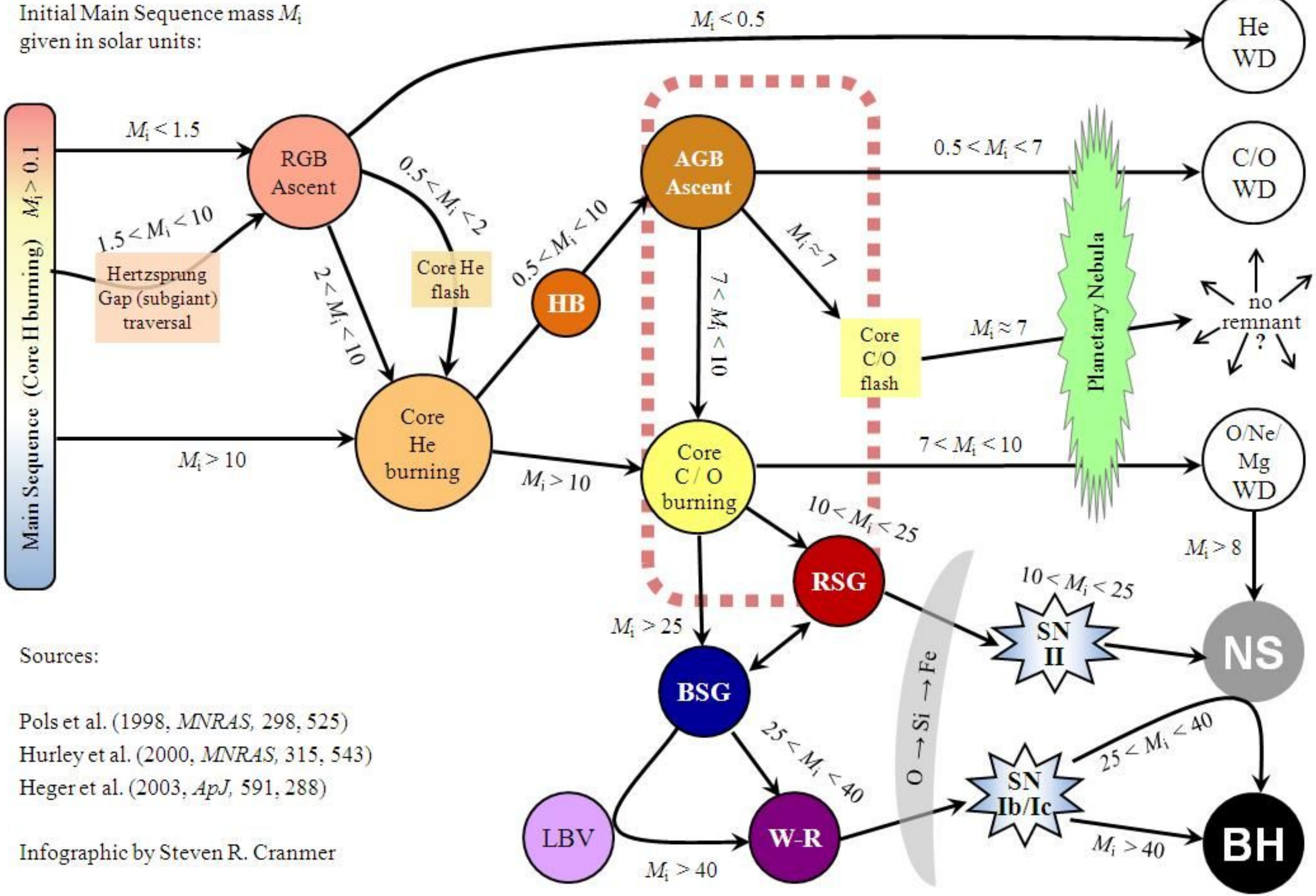

Spectral Class
O B A F G K M
30000K 10000K 7500K 6000K 5000K 4000K 3000K (Temperature)
Luminosity (Sun=1)
100 000
10 000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.000 01
Absolute Magnitude
-10
-5
0
+5
+10
+15
Ia
Supergiants
Ib
II Bright Giants
III Giants
IV Subgiants
V Main Sequence
White Dwarfs
0.0 +0.5 +1.0 +1.5 +2.0
Colour (B–V)



brightness (solar units)
surface temperature (Kelvin)
increasing temperature
decreasing temperature
30,000
10,000
6,000
3,000
O
B
A
F
G
K
M
10^6
10^5
10^4
10^3
10^2
10
1
0.1
10^-2
10^-3
10^-4
10^-5
10^2 Solar Radius
10^3 Solar Radius
10 Solar Radii
1 Solar Radius
0.1 Solar Radius
10^-2 Solar Radius
10^-3 Solar Radius
60 M Sun
30 M Sun
10 M Sun
6 M Sun
3 M Sun
1.5 M Sun
1 M Sun
0.3 M Sun
0.1 M Sun
Lifetime 10^7 yrs
Lifetime 10^8 yrs
Lifetime 10^9 yrs
Lifetime 10^10 yrs
Lifetime 10^11 yrs
SUPERGIANTS
GIANTS
MAIN
SEQUENCE
WHITE
DWARFS
Deneb
Rigel
Betelgeuse
Antares
Canopus
Polaris
Arcturus
Aldebaran
Pollux
β Centauri
Spica
Bellatrix
Archernar
Vega
Sirius
Procyon
Altair
α Centauri A
Sun
α Centauri B
τ Ceti
Eridani
61 Cygni A
61 Cygni B
Lacaille 9352
Gliese 725 A
Gliese 725 B
Barnard's Star
Ross 128
Wolf 359
Proxima Centauri
DX Cancri
Sirius B
Procyon B

Single Star Evolution Flowchart (*Solar metallicity*)

Initial Main Sequence mass M_i given in solar units:



Sources:

Pols et al. (1998, *MNRAS*, 298, 525)

Hurley et al. (2000, *MNRAS*, 315, 543)

Heger et al. (2003, *ApJ*, 591, 288)

Infographic by Steven R. Cranmer

History

Williamina Fleming

Annie Jump Cannon

Cecilia Payne-Gaposchkin

History

Williamina Fleming

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Cecilia Payne-Gaposchkin

History

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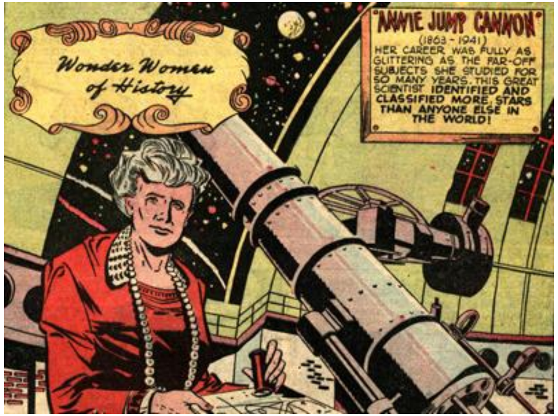
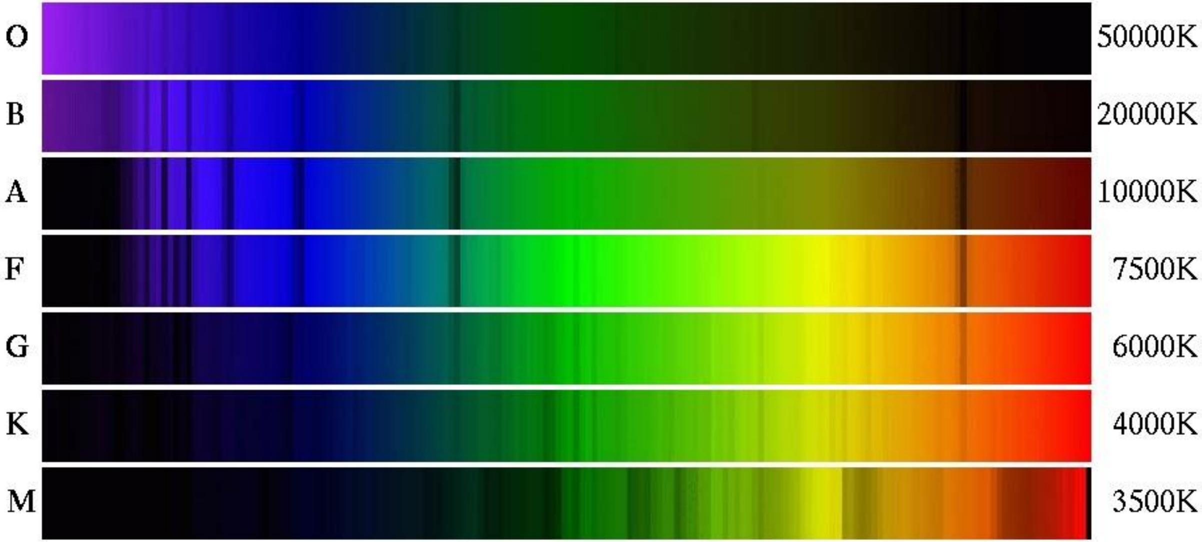




History

Williamina Fleming

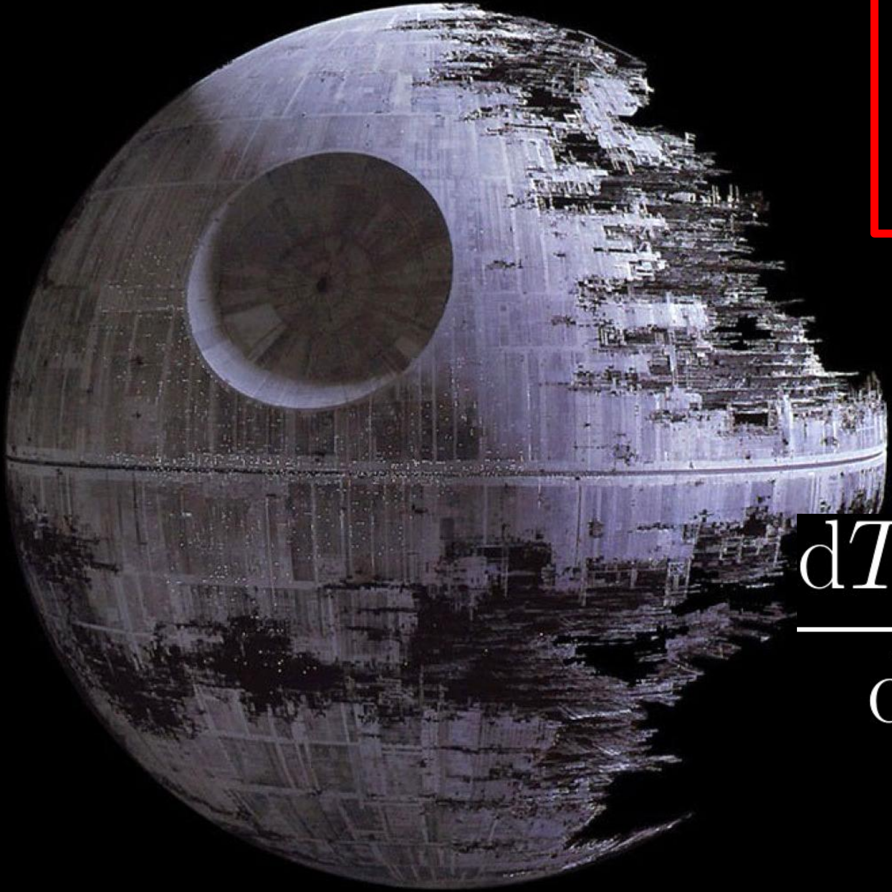
Annie Jump Cannon

Cecilia Payne-Gaposchkin

Yeah, so turns out the sun isn't an iron ball;
it is made mostly of hydrogen

Day 2

Hydrostatic Equilibrium



$$\boxed{\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{\rho(r)GM(r)}{r^2}}$$

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2\rho(r)$$

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{3\kappa\rho(r)}{64\pi r^2\sigma T^3}L(r)$$

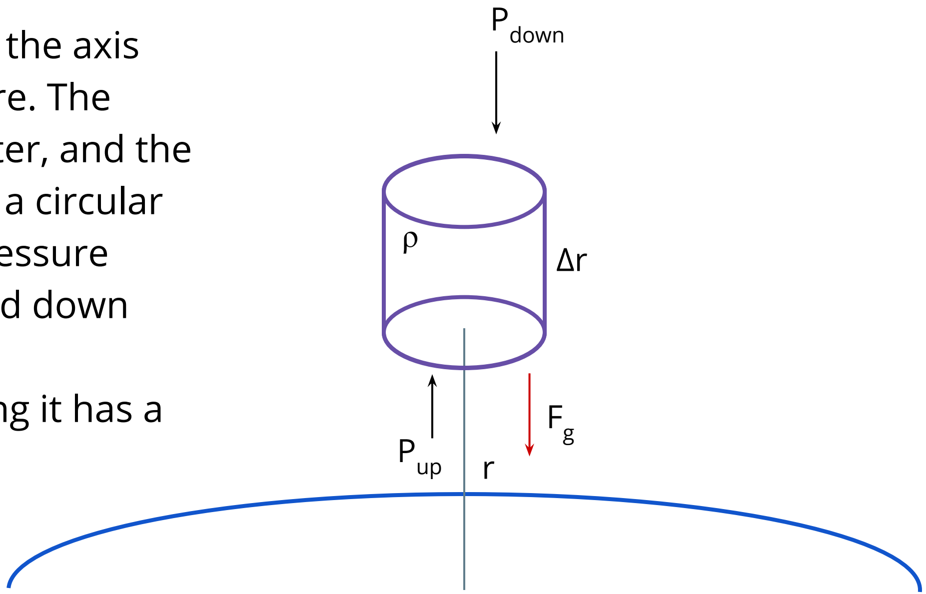
$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2\rho(r)\varepsilon(r)$$


DON’T PANIC

Hydrostatic Equilibrium

Consider the Earth's atmosphere by assuming the constituent particles comprise an ideal gas.

- Think of a small cylindrical parcel of gas with the axis pointing vertically into the Earth's atmosphere. The parcel sits a distance, r, from the Earth's center, and the parcel's size is defined by a height $\Delta r \ll r$ and a circular cross-sectional area A. The parcel will feel pressure pushing up from the gas below (Pup=P(r)) and down from above (Pdown=P(r+Δr))
- What other force will the parcel feel, assuming it has a density, ρ, and the Earth has a mass, $M_{🌎}$?



Hydrostatic Equilibrium

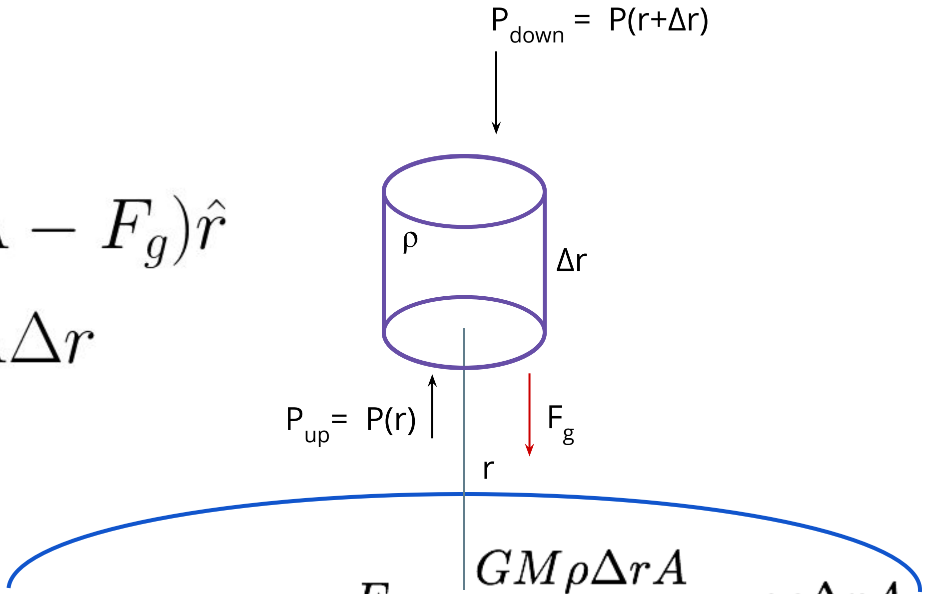
The solution arises from force balance.
 In the limit of a small cylinder, the equation becomes a differential equation.

$$\sum \vec{F} = 0 = (P_{\mathrm{up}}\,\mathrm{A} - P_{\mathrm{down}}\,\mathrm{A} - F_g)\hat{r}$$

$$\mathrm{A}[P(r+\Delta r) - P(r)] = -g\rho \mathrm{A}\Delta r$$

$$P(r+\Delta r) - P(r) = -g\rho\Delta r$$

$$\frac{dP(r)}{dr} = -g\rho(r)$$



$$F_g = \frac{GM\rho\Delta r A}{r^2} = g\rho\Delta r A$$

Day 3

Radiative transport in stars

or

How to run in a random directions for 11,000 years

Random Walk

Random Walk

```
# only necessary if you're running Python 2.7 or lower
from __future__ import print_function
from __builtin__ import range

import numpy as np

# some code to randomly generate a series
# of -1 or 1 and store them in an array

N = 1000                                          # let's just make the list
steps = np.random.choice([-1,1],size=(N,))  # huge and trim later

i = -1 # initialize to -1 so that it will increment to 0. see below

# for each step enter commands below
#--------------------
i+=1;print(steps[i])
#--------------------

steps = steps[:i+1] # slice or array arr[x:y] is x/y in/ex-clusive

# what stats do we want to do now
```

Photons Random Walking Out of a Star

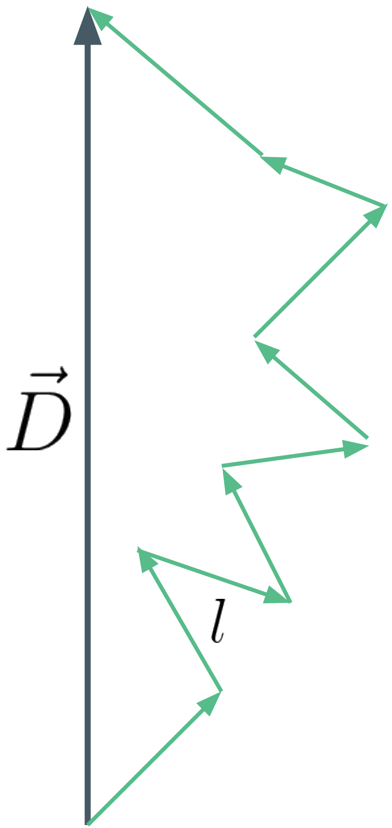
Courtesy of ASTRON 16 Worksheets

A photon does not travel freely from the Sun's center to the surface. Instead it random walks, one collision/scattering at a time. Each step traverses an average distance l, also known as the *mean free path*. On average, how many steps does the photon take to travel a distance Δr?

(Hint: Be sure to draw a picture. If each step is a vector $\vec{r}_i$*, then* $\vec{D} = \sum \vec{r}_i$*. However, this displacement is zero for a large number of random-walk journeys. Instead, calculate* $\vec{D}^2$ *and take the square root of the result to be* $\Delta r = (\vec{D}^2)^{1/2}$*. This is the root–mean–squared displacement, which is a scalar rather than vector quantity. Note that 'multiplying' two vectors isn't as simple as multiplying two scalars. You must take the dot product, which looks like* $\vec{A} \cdot \vec{B} = AB\cos\theta$*, where* θ *is the angle between the two vectors.)*

Photons Random Walking Out of a Star

A photon does not travel freely from the Sun's center to the surface. Instead it random walks, one collision/scattering at a time. Each step traverses an average distance l, also known as the *mean free path*. On average, how many steps does the photon take to travel a distance Δr?



the Maths....

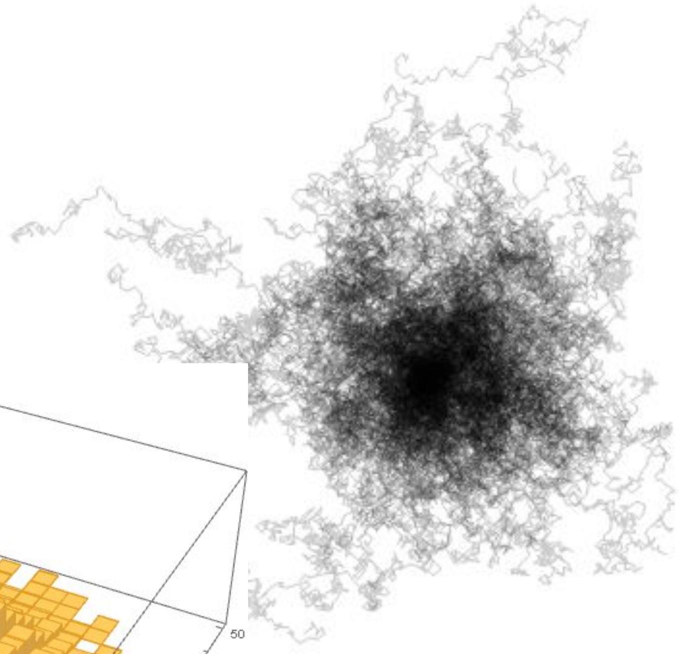
The total distance traveled will just be the sum of the vectors. We would like to know, on average how far does a photon travel. The average is either over an ensemble of photons or for many repeats of the experiment.

We know from experimentation and some simple reasoning that the expectation value (average) of **D** is 0. If you sum a bunch of [+1,-1] steps you get nowhere.

$$\vec{D} = \sum_{i=1}^{N} \ell_i$$

$$\langle \vec{D} \rangle = 0$$

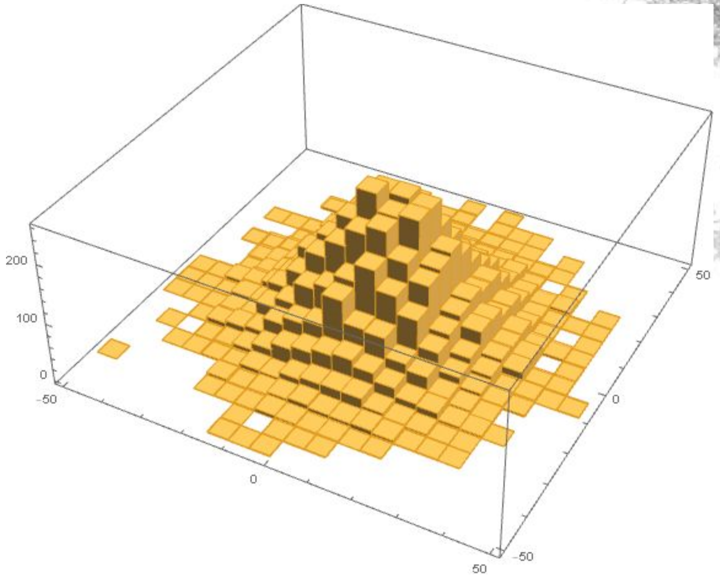
So on average you go nowhere



Super encouraging... I know

The total distance traveled will just be the sum of the vectors. We would like to know, on average how far does a photon travel. The average is either **over an ensemble of photons** or for many repeats of the experiment.

We know from experimentation and some simple reasoning that the expectation value (average) of **D** is 0. If you sum a bunch of [+1,-1] steps you get nowhere.



the Maths....

$$\vec{D} = \sum_{i=1}^{N} \ell_i$$

The total distance traveled will just be the sum of the vectors. However, since the direction is random, on average $\|\boldsymbol{D}\|=0$ (i.e. $\langle\|\boldsymbol{D}\|\rangle=0$).

What we are really interested in is the root-mean-square displacement

$$\Delta r = \sqrt{\langle|\vec{D}|^2\rangle}$$

the Maths....

So why waste time asking what the average vector is, when we are really curious what the average magnitude of the vector is. So we are really interested in the expectation of **D**^{2}.

$$\langle \vec{D}^2 \rangle = (\Delta r)^2$$

the Maths....

Knowing the two equations at the right calculate,

$$\Delta r = (\vec{D}^2)^{1/2}$$

$$\vec{D} = \sum_{i=1}^{N} \ell_i$$

$$\vec{D}^2 = \vec{D} \cdot \vec{D}$$

the Maths….

The dot product generates cross-terms, so this is taken into account by running over two indices, i and j.

$$\vec{D} = \sum_{i=1}^{N} \ell_i$$

$$\vec{D}^2 = \vec{D} \cdot \vec{D}$$

$$= \sum_{i=1}^{N} \sum_{i=j}^{N} (\vec{\ell}_i \cdot \vec{\ell}_j)$$

Expanding the sum

Expanding the sum can be a bit tricky. The key is to isolate the like terms and cross-terms.

You can see the (i,i) terms are all single, while there are duplicate cross (i,j) terms.

$$\vec{D} = \sum_{i=1}^{N} \ell_i$$

$$\vec{D}^2 = \vec{D} \cdot \vec{D}$$

$$= \sum_{i=1}^{N} \sum_{i=j}^{N} (\vec{\ell}_i \cdot \vec{\ell}_j)$$

$$\begin{aligned}\vec{D} \cdot \vec{D} =& (\ell_1 \cdot \ell_1) + (\ell_1 \cdot \ell_2) + \cdots + (\ell_1 \cdot \ell_N) \\ &+ (\ell_2 \cdot \ell_1) + (\ell_2 \cdot \ell_2) + \cdots + (\ell_2 \cdot \ell_N) \\ &+ \cdots + (\ell_N \cdot \ell_N)\end{aligned}$$

the Maths....

The (i,i) terms are all the same value, so that sum simplifies easily. The second term however simplifies to a sum over the cosine of the angle between steps i and j.

$$= \sum_{i=1}^{N} \sum_{i=j}^{N} (\vec{\ell}_i \cdot \vec{\ell}_j)$$

$$= \sum_{i=1}^{N} \vec{\ell}_i \cdot \vec{\ell}_i + 2 \sum_{\substack{ij=1 \\ i \neq j}}^{N} \vec{\ell}_i \cdot \vec{\ell}_j$$

$$= N\ell^2 + 2\ell^2 \sum_{\substack{ij=1 \\ i \neq j}}^{N} \cos\theta_{ij}$$

the Clever Maths....

The (i,i) terms are all the same value, so that sum simplifies easily. The second term however simplifies to a sum over the cosine of the angle between steps i and j.

On average, if the angle is rotating through all 360º, then the cosine is alternating from between -1 and 1, so it's average is 0.

$$= \sum_{i=1}^{N} \sum_{i=j}^{N} (\vec{\ell}_i \cdot \vec{\ell}_j)$$

$$= \sum_{i=1}^{N} \vec{\ell}_i \cdot \vec{\ell}_i + 2 \sum_{\substack{ij=1 \\ i \neq j}}^{N} \vec{\ell}_i \cdot \vec{\ell}_j$$

$$= N\ell^2 \boxed{+ 2\ell^2 \sum_{\substack{ij=1 \\ i \neq j}}^{N} \cos\theta_{ij}}$$

the Clever Maths....

The distance you travel is related to the square root of the number of steps.

... Pretty cool!!

$$(\Delta r)^2 = \ell^2 N$$

$$N = \left(\frac{\Delta r}{\ell}\right)^2$$

Applications of Radiative Diffusion

Diffusion Velocity

Of course, photons are not skipping around instantaneously, popping *thither and yon*, they have a finite velocity, namely, the speed of light - c.

What is the diffusion velocity?

$$v_{diff}$$

Diffusion Velocity

What is the diffusion velocity?

Photons are traveling at the speed of light, so for a single step

$$c = \frac{\ell}{\Delta t}$$

Diffusion Velocity

What is the diffusion velocity?

The diffusion velocity is the total path traveled divided by the total time it takes to travel. We can assume that interactions are instantaneous.

$$v_{diff} = \frac{\Delta r}{N \Delta t}$$

Diffusion Velocity

What is the diffusion velocity?

Putting it all together...

$$c = \frac{\ell}{\Delta t}$$

$$v_{diff} = \frac{\Delta r}{N \Delta t}$$

$$v_{diff} = \frac{c}{\sqrt{N}}$$

Diffusion Velocity

It would be useful, for the future perhaps to substitute N in for something with more physical meaning

$$v_{diff} = \frac{c}{\sqrt{N}}$$

Diffusion Velocity

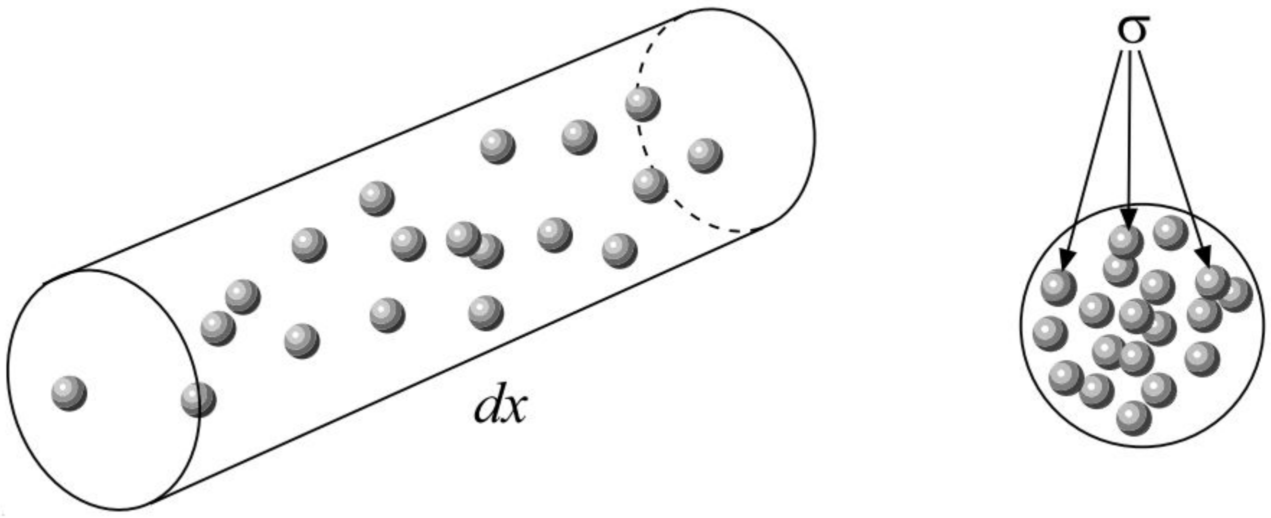
It would be useful, for the future perhaps to substitute N in for something with more physical meaning

$$v_{diff} = \frac{c}{\sqrt{N}}$$

$$v_{diff} = \frac{c\ell}{\Delta r}$$

Mean free path

The mean free path, l, is the characteristic distance between collisions. Consider a photon moving through a cloud of electrons with a number density n, Each electron presents an effective cross-section, σ. Give an analytic expression for the "mean free path" relating these parameters.



Mean free path

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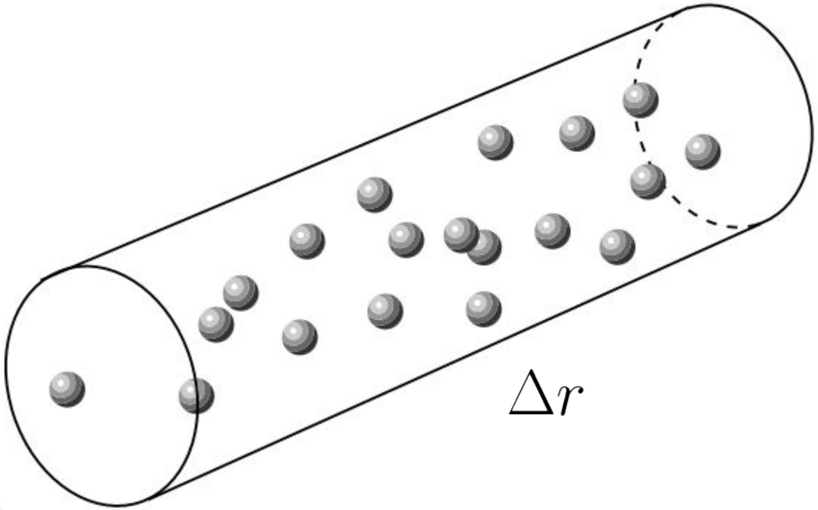
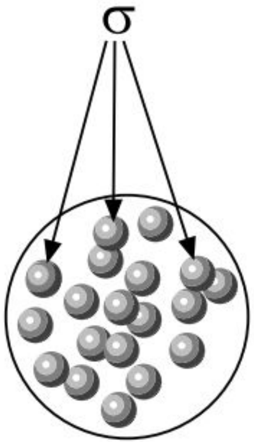




$$\lambda_{mfp} = \frac{1}{n\sigma}$$

Time for a photon to escape the Sun

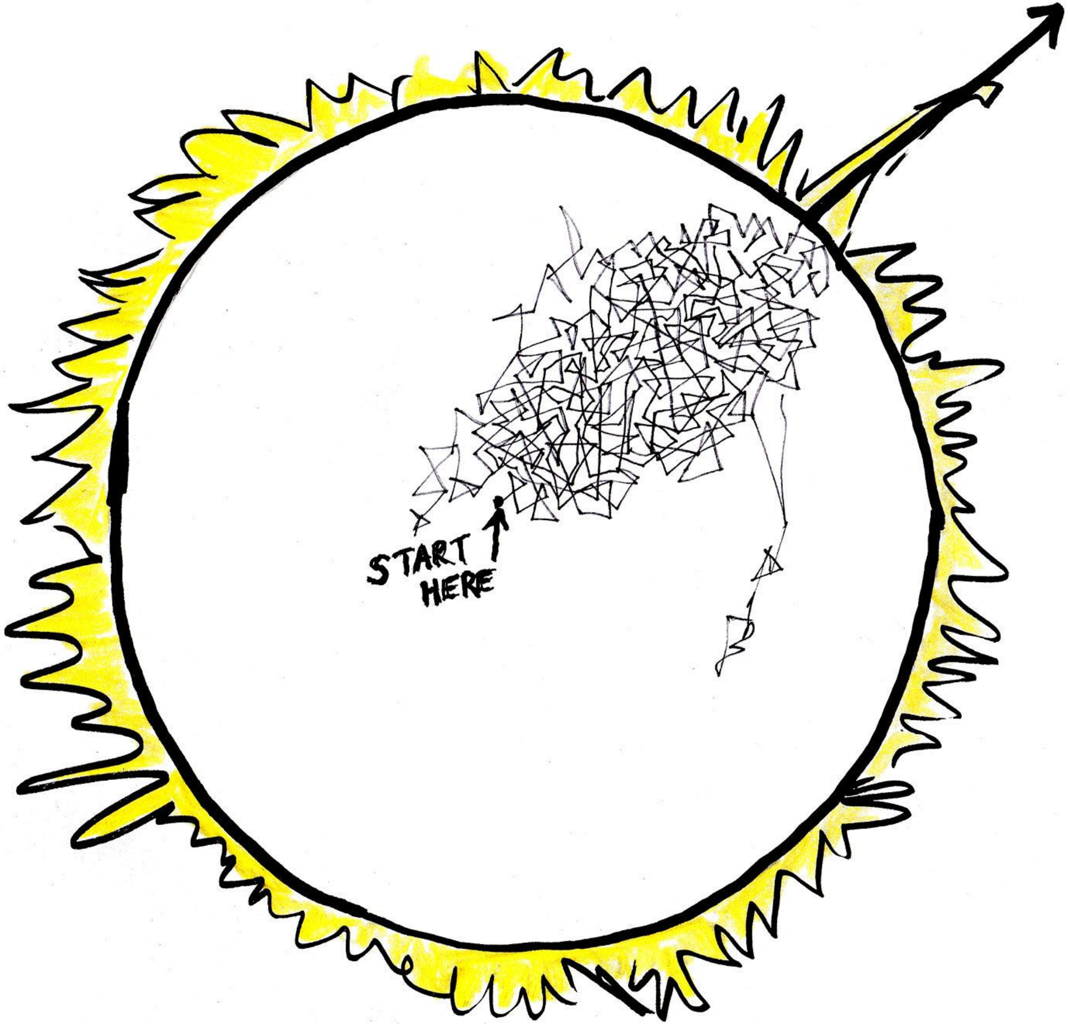
Using what we know, determine how long it takes a photon to leave the Sun if it only scatters off electrons. The scattering cross-section of electrons is the Thomson scattering cross-section

$$\sigma_T = 7 \times 10^{-25}\,\mathrm{cm}^2$$

Relevant equations

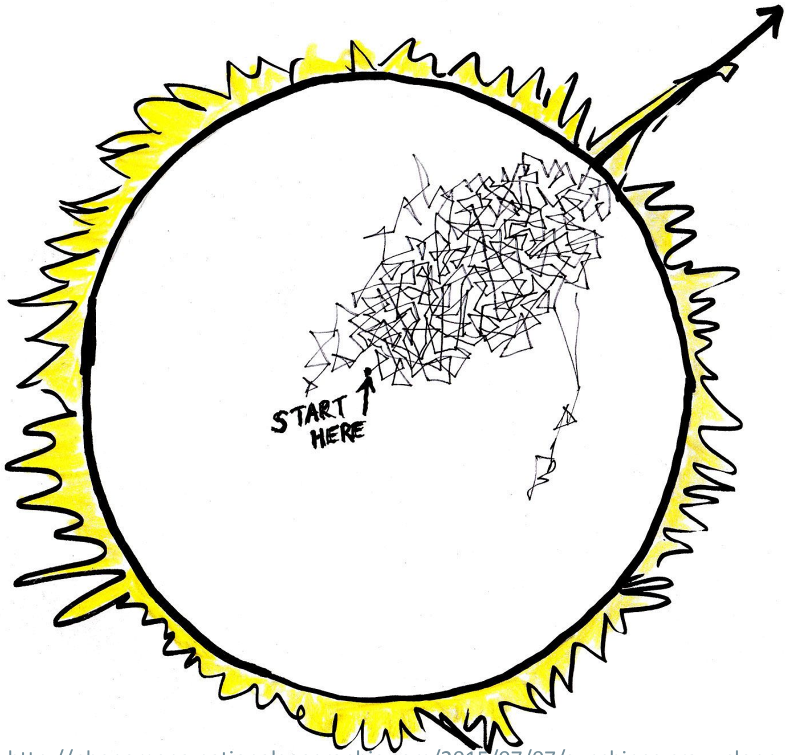
$$v_{diff} = \frac{\Delta r}{\tau_{diff}}$$

$$\lambda_{mfp} = \frac{1}{n\sigma}$$



http://phenomena.nationalgeographic.com/2015/07/07/sunshines-crazy-sloppy-path-to-you/

Time for a photon to escape the Sun



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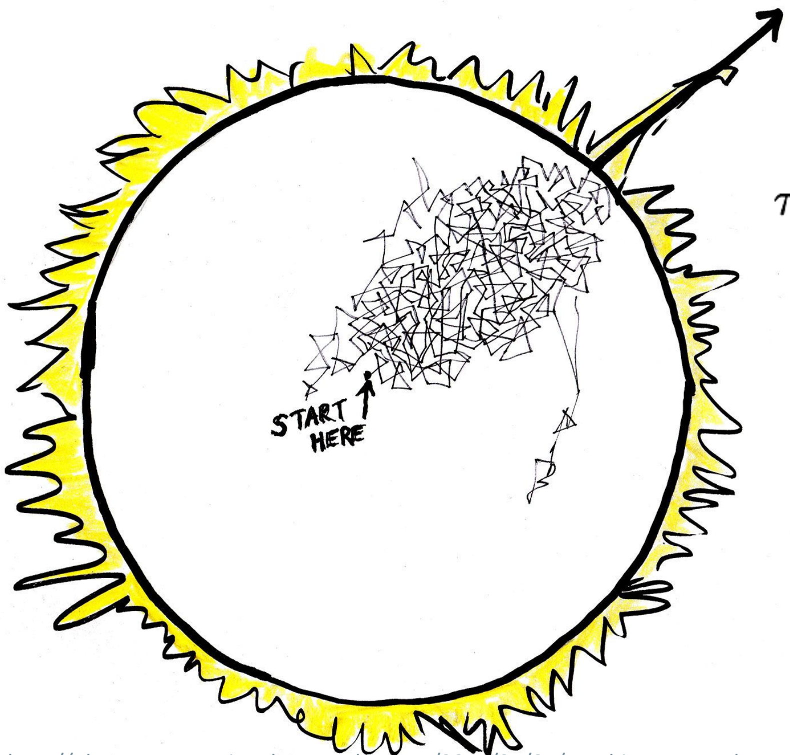
What is Δr in this case?

What is *n*?

$$v_{diff} = \frac{\Delta r}{\tau_{diff}} = \frac{c\,\lambda_{mfp}}{\Delta r}$$

$$\lambda_{mfp} = \frac{1}{n\sigma}$$

Time for a photon to escape the Sun

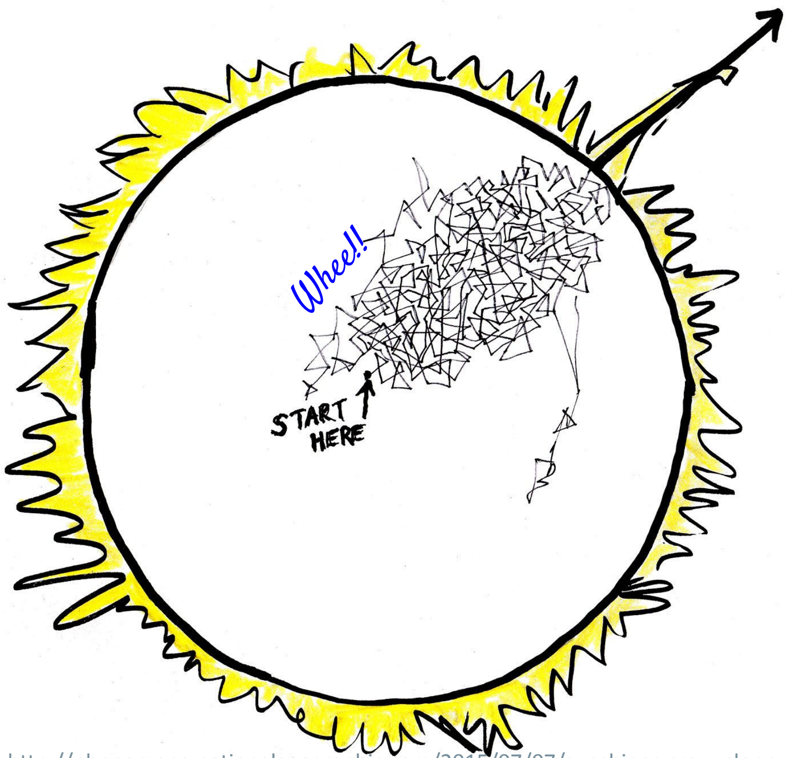


$$\tau_{diff} = \frac{r_\odot^2 \rho_\odot \sigma_T}{c\, m_H}$$

$$= \frac{(7\times 10^{10}\mathrm{cm})^2\,(1.4\ \mathrm{g/cm^3})\,(7\times 10^{-25}\ \mathrm{cm^2})}{(3\times 10^{10}\ \mathrm{cm/s})\,(1.7\times 10^{-24}\ \mathrm{g})}$$

http://phenomena.nationalgeographic.com/2015/07/07/sunshines-crazy-sloppy-path-to-you/

Time for a photon to escape the Sun



$$\tau_{diff} \approx 10 \times 10^{10} \text{ s}$$
$$\approx 3000 \text{ yr}$$

http://phenomena.nationalgeographic.com/2015/07/07/sunshines-crazy-sloppy-path-to-you/