Type Declarations

In Haskell, a new name for an existing type can be defined using a type declaration.

```
type String = [Char]
```

String is a synonym for the type [Char].

Type declarations can be used to make other types easier to read. For example, given

```
type Pos = (Int,Int)
```

we can define:

```
origin     :: Pos
origin      = (0,0)

left       :: Pos → Pos
left (x,y) = (x-1,y)
```

Like function definitions, type declarations can also have <u>parameters</u>. For example, given

```
type Pair a = (a,a)
```

we can define:

```
mult       :: Pair Int → Int
mult (m,n) = m*n

copy       :: a → Pair a
copy x      = (x,x)
```

Type declarations can be nested:

```
type Pos   = (Int,Int)

type Trans = Pos → Pos
```

However, they cannot be recursive:

```
type Tree = (Int,[Tree])
```

Data Declarations

A completely new type can be defined by specifying its values using a data declaration.

```
data Bool = False | True
```

Bool is a new type, with two new values False and True.

Note:

- The two values False and True are called the <u>constructors</u> for the type Bool.

- Type and constructor names must begin with an upper-case letter.

- Data declarations are similar to context free grammars. The former specifies the values of a type, the latter the sentences of a language.

Values of new types can be used in the same ways as those of built in types. For example, given

```
data Answer = Yes | No | Unknown
```

we can define:

```
answers      :: [Answer]
answers       = [Yes,No,Unknown]

flip         :: Answer → Answer
flip Yes      = No
flip No       = Yes
flip Unknown  = Unknown
```

The constructors in a data declaration can also have parameters. For example, given

```
data Shape = Circle Float
           | Rect Float Float
```

we can define:

```
square          :: Float → Shape
square n         = Rect n n

area            :: Shape → Float
area (Circle r) = pi * r^2
area (Rect x y) = x * y
```

Note:

- Shape has values of the form Circle r where r is a float, and Rect x y where x and y are floats.

- Circle and Rect can be viewed as <u>functions</u> that construct values of type Shape:

```
Circle :: Float → Shape

Rect   :: Float → Float → Shape
```

Not surprisingly, data declarations themselves can also have parameters. For example, given

```
data Maybe a = Nothing | Just a
```

we can define:

```
safediv     :: Int → Int → Maybe Int
safediv _ 0 = Nothing
safediv m n = Just (m `div` n)

safehead    :: [a] → Maybe a
safehead [] = Nothing
safehead xs = Just (head xs)
```

Recursive Types

In Haskell, new types can be declared in terms of themselves. That is, types can be recursive.

```
data Nat = Zero | Succ Nat
```

Nat is a new type, with constructors Zero :: Nat and Succ :: Nat → Nat.

Note:

- A value of type Nat is either Zero, or of the form Succ n where n :: Nat. That is, Nat contains the following infinite sequence of values:

```
Zero

Succ Zero

Succ (Succ Zero)
```

⋮

- We can think of values of type Nat as natural numbers, where Zero represents 0, and Succ represents the successor function 1+.

- For example, the value

```
Succ (Succ (Succ Zero))
```

represents the natural number

```
1 + (1 + (1 + 0))  =  3
```

Using recursion, it is easy to define functions that convert between values of type Nat and Int:

```
nat2int          :: Nat → Int
nat2int Zero      = 0
nat2int (Succ n) = 1 + nat2int n

int2nat          :: Int → Nat
int2nat 0         = Zero
int2nat (n+1)     = Succ (int2nat n)
```

Two naturals can be added by converting them to integers, adding, and then converting back:

```
add     :: Nat → Nat → Nat
add m n = int2nat (nat2int m + nat2int n)
```

However, using recursion the function add can be defined without the need for conversions:

```
add Zero     n = n
add (Succ m) n = Succ (add m n)
```

For example:

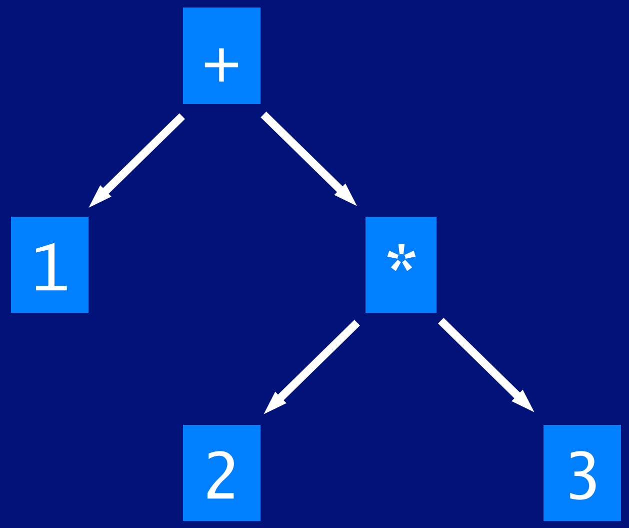
```
  add (Succ (Succ Zero)) (Succ Zero)
=
  Succ (add (Succ Zero) (Succ Zero))
=
  Succ (Succ (add Zero (Succ Zero))
=
  Succ (Succ (Succ Zero))
```

Note:

- The recursive definition for add corresponds to the laws 0+n = n and (1+m)+n = 1+(m+n).

Arithmetic Expressions

Consider a simple form of expressions built up from integers using addition and multiplication.



Using recursion, a suitable new type to represent such expressions can be declared by:

```
data Expr = Val Int
          | Add Expr Expr
          | Mul Expr Expr
```

For example, the expression on the previous slide would be represented as follows:

```
Add (Val 1) (Mul (Val 2) (Val 3))
```

Using recursion, it is now easy to define functions that process expressions. For example:

```
size          :: Expr → Int
size (Val n)   = 1
size (Add x y) = size x + size y
size (Mul x y) = size x + size y

eval          :: Expr → Int
eval (Val n)   = n
eval (Add x y) = eval x + eval y
eval (Mul x y) = eval x * eval y
```

Note:

- The three constructors have types:

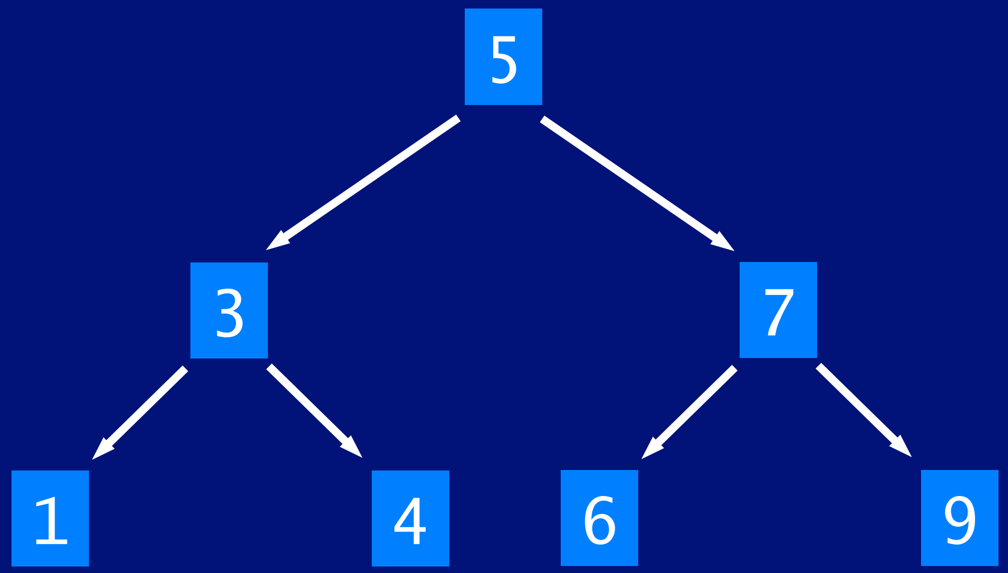
```
Val :: Int → Expr
Add :: Expr → Expr → Expr
Mul :: Expr → Expr → Expr
```

- Many functions on expressions can be defined by replacing the constructors by other functions using a suitable <u>fold</u> function. For example:

```
eval = fold id (+) (*)
```

Binary Trees

In computing, it is often useful to store data in a two-way branching structure or binary tree.



Using recursion, a suitable new type to represent such binary trees can be declared by:

```
data Tree = Leaf Int
          | Node Tree Int Tree
```

For example, the tree on the previous slide would be represented as follows:

```
Node (Node (Leaf 1) 3 (Leaf 4))
     5
     (Node (Leaf 6) 7 (Leaf 9))
```

We can now define a function that decides if a given integer occurs in a binary tree:

```
occurs               :: Int → Tree → Bool
occurs m (Leaf n)     = m==n
occurs m (Node l n r) = m==n
                        || occurs m l
                        || occurs m r
```

But… in the worst case, when the integer does not occur, this function traverses the entire tree.

Now consider the function flatten that returns the list of all the integers contained in a tree:

```
flatten              :: Tree → [Int]
flatten (Leaf n)      = [n]
flatten (Node l n r)  = flatten l
                        ++ [n]
                        ++ flatten r
```

A tree is a search tree if it flattens to a list that is ordered. Our example tree is a search tree, as it flattens to the ordered list [1,3,4,5,6,7,9].

Search trees have the important property that when trying to find a value in a tree we can always decide which of the two sub-trees it may occur in:

```
occurs m (Leaf n)              = m==n
occurs m (Node l n r) | m==n = True
                      | m<n  = occurs m l
                      | m>n  = occurs m r
```

This new definition is more <u>efficient</u>, because it only traverses one path down the tree.

Exercises

(1) Using recursion and the function add, define a function that multiplies two natural numbers.

(2) Define a suitable function fold for expressions, and give a few examples of its use.

(3) A binary tree is complete if the two sub-trees of every node are of equal size. Define a function that decides if a binary tree is complete.