

# Introduction

As we have seen, many functions can naturally be defined in terms of other functions.

```
factorial  :: Int → Int  
factorial n = product [1..n]
```

factorial maps any integer  $n$  to the product of the integers between 1 and  $n$ .

Expressions are evaluated by a stepwise process of applying functions to their arguments.

For example:

```
factorial 4
=
product [1..4]
=
product [1,2,3,4]
=
1*2*3*4
=
24
```

# Recursive Functions

In Haskell, functions can also be defined in terms of themselves. Such functions are called recursive.

```
factorial 0      = 1
factorial (n+1) = (n+1) * factorial n
```

factorial maps 0 to 1, and any other positive integer to the product of itself and the factorial of its predecessor.

For example:

`factorial 3`  
=  
`3 * factorial 2`  
=  
`3 * (2 * factorial 1)`  
=  
`3 * (2 * (1 * factorial 0))`  
=  
`3 * (2 * (1 * 1))`  
=  
`3 * (2 * 1)`  
=  
`3 * 2`  
=  
`6`

## Note:

- factorial 0 = 1 is appropriate because 1 is the identity for multiplication:  $1 * x = x = x * 1$ .
- The recursive definition diverges on integers  $< 0$  because the base case is never reached:

```
> factorial (-1)
```

```
Error: Control stack overflow
```

# Why is Recursion Useful?

- Some functions, such as factorial, are simpler to define in terms of other functions.
- As we shall see, however, many functions can naturally be defined in terms of themselves.
- Properties of functions defined using recursion can be proved using the simple but powerful mathematical technique of induction.

# Recursion on Lists

Recursion is not restricted to numbers, but can also be used to define functions on lists.

```
product      :: [Int] → Int
product []    = 1
product (n:ns) = n * product ns
```

product maps the empty list to 1,  
and any non-empty list to its head  
multiplied by the product of its tail.

For example:

```
= product [2,3,4]
= 2 * product [3,4]
= 2 * (3 * product [4])
= 2 * (3 * (4 * product []))
= 2 * (3 * (4 * 1))
= 24
```



Using the same pattern of recursion as in product we can define the length function on lists.

```
length      :: [a] → Int
length []   = 0
length (_:xs) = 1 + length xs
```

length maps the empty list to 0,  
and any non-empty list to the  
successor of the length of its tail.

For example:

```
length [1,2,3]
=
1 + length [2,3]
=
1 + (1 + length [3])
=
1 + (1 + (1 + length []))
=
1 + (1 + (1 + 0))
=
3
```

Using a similar pattern of recursion we can define the reverse function on lists.

```
reverse      :: [a] → [a]
reverse []    = []
reverse (x:xs) = reverse xs ++ [x]
```

reverse maps the empty list to the empty list, and any non-empty list to the reverse of its tail appended to its head.

For example:

```
reverse [1,2,3]
=
reverse [2,3] ++ [1]
=
(reverse [3] ++ [2]) ++ [1]
=
((reverse [] ++ [3]) ++ [2]) ++ [1]
=
(([] ++ [3]) ++ [2]) ++ [1]
=
[3,2,1]
```

# Multiple Arguments

Functions with more than one argument can also be defined using recursion. For example:

- Zipping the elements of two lists:

```
zip      :: [a] → [b] → [(a,b)]
zip []   _      = []
zip _    []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

## ■ Remove the first n elements from a list:

```
drop :: Int → [a] → [a]
drop 0 xs = xs
drop (n+1) [] = []
drop (n+1) (_:xs) = drop n xs
```

## ■ Appending two lists:

```
(++) :: [a] → [a] → [a]
[] ++ ys = ys
(x:xs) ++ ys = x : (xs ++ ys)
```

# Quicksort

The quicksort algorithm for sorting a list of integers can be specified by the following two rules:

- The empty list is already sorted;
- Non-empty lists can be sorted by sorting the tail values  $\leq$  the head, sorting the tail values  $>$  the head, and then appending the resulting lists on either side of the head value.

Using recursion, this specification can be translated directly into an implementation:

```
qsort      :: [Int] → [Int]
qsort []   = []
qsort (x:xs) =
    qsort smaller ++ [x] ++ qsort larger
  where
    smaller = [a | a ← xs, a ≤ x]
    larger  = [b | b ← xs, b > x]
```

Note:

- This is probably the simplest implementation of quicksort in any programming language!



For example (abbreviating qsort as q):

q [3, 2, 4, 1, 5]



q [2, 1] ++ [3] ++ q [4, 5]



q [1] ++ [2] ++ q []

q [] ++ [4] ++ q [5]



[1]

[]

[]

[5]

# Exercises

(1) Without looking at the standard prelude, define the following library functions using recursion:

- Decide if all logical values in a list are true:

```
and :: [Bool] → Bool
```

- Concatenate a list of lists:

```
concat :: [[a]] → [a]
```

- Produce a list with n identical elements:

```
replicate :: Int → a → [a]
```

- Select the nth element of a list:

```
(!!) :: [a] → Int → a
```

- Decide if a value is an element of a list:

```
elem :: Eq a ⇒ a → [a] → Bool
```

## (2) Define a recursive function

```
merge :: [Int] → [Int] → [Int]
```

that merges two sorted lists of integers to give a single sorted list. For example:

```
> merge [2,5,6] [1,3,4]  
[1,2,3,4,5,6]
```

### (3) Define a recursive function

```
msort :: [Int] → [Int]
```

that implements merge sort, which can be specified by the following two rules:

- Lists of length  $\leq 1$  are already sorted;
- Other lists can be sorted by sorting the two halves and merging the resulting lists.