

Set Comprehensions

In mathematics, the comprehension notation can be used to construct new sets from old sets.

$$\{x^2 \mid x \in \{1\dots 5\}\}$$

The set $\{1,4,9,16,25\}$ of all numbers x^2 such that x is an element of the set $\{1\dots 5\}$.

Lists Comprehensions

In Haskell, a similar comprehension notation can be used to construct new lists from old lists.

```
[x^2 | x ← [1..5]]
```

The list `[1,4,9,16,25]` of all numbers x^2 such that x is an element of the list `[1..5]`.

Note:

- The expression $x \leftarrow [1..5]$ is called a generator, as it states how to generate values for x .
- Comprehensions can have multiple generators, separated by commas. For example:

```
> [(x,y) | x <- [1,2,3], y <- [4,5]]  
[(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)]
```

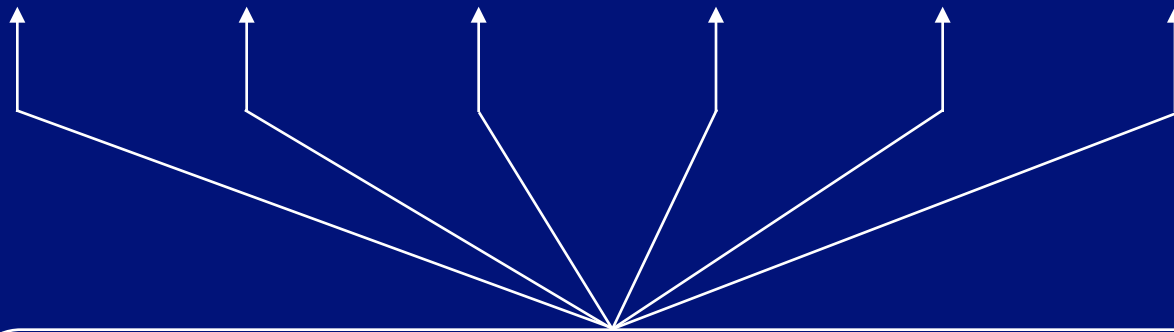
- Changing the order of the generators changes the order of the elements in the final list:

```
> [(x,y) | y ← [4,5], x ← [1,2,3]]  
[(1,4),(2,4),(3,4),(1,5),(2,5),(3,5)]
```

- Multiple generators are like nested loops, with later generators as more deeply nested loops whose variables change value more frequently.

■ For example:

```
> [(x,y) | y ← [4,5], x ← [1,2,3]]  
[(1,4), (2,4), (3,4), (1,5), (2,5), (3,5)]
```



$x \leftarrow [1,2,3]$ is the last generator, so the value of the x component of each pair changes most frequently.

Dependant Generators

Later generators can depend on the variables that are introduced by earlier generators.

$[(x,y) \mid x \leftarrow [1..3], y \leftarrow [x..3]]$

The list $[(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)]$ of all pairs of numbers (x,y) such that x,y are elements of the list $[1..3]$ and $y \geq x$.

Using a dependant generator we can define the library function that concatenates a list of lists:

```
concat    :: [[a]] → [a]
concat xss = [x | xs ← xss, x ← xs]
```

For example:

```
> concat [[1,2,3],[4,5],[6]]
[1,2,3,4,5,6]
```

Guards

List comprehensions can use guards to restrict the values produced by earlier generators.

```
[x | x ← [1..10], even x]
```

The list [2,4,6,8,10] of all numbers x such that x is an element of the list [1..10] and x is even.

Using a guard we can define a function that maps a positive integer to its list of factors:

```
factors  :: Int → [Int]
factors n =
    [x | x ← [1..n], n `mod` x == 0]
```

For example:

```
> factors 15
[1,3,5,15]
```

A positive integer is prime if its only factors are 1 and itself. Hence, using factors we can define a function that decides if a number is prime:

```
prime  :: Int → Bool  
prime n = factors n == [1,n]
```

For example:

```
> prime 15  
False  
  
> prime 7  
True
```

Using a guard we can now define a function that returns the list of all primes up to a given limit:

```
primes  :: Int → [Int]
primes n = [x | x ← [2..n], prime x]
```

For example:

```
> primes 40
[2,3,5,7,11,13,17,19,23,29,31,37]
```

The Zip Function

A useful library function is zip, which maps two lists to a list of pairs of their corresponding elements.

```
zip :: [a] → [b] → [(a,b)]
```

For example:

```
> zip ['a', 'b', 'c'] [1,2,3,4]  
[('a',1),('b',2),('c',3)]
```

Using `zip` we can define a function returns the list of all pairs of adjacent elements from a list:

```
pairs    :: [a] → [(a,a)]  
pairs xs = zip xs (tail xs)
```

For example:

```
> pairs [1,2,3,4]  
[(1,2), (2,3), (3,4)]
```

Using pairs we can define a function that decides if the elements in a list are sorted:

```
sorted    :: Ord a => [a] -> Bool
sorted xs =
    and [x ≤ y | (x,y) ← pairs xs]
```

For example:

```
> sorted [1,2,3,4]
True

> sorted [1,3,2,4]
False
```

Using `zip` we can define a function that returns the list of all positions of a value in a list:

```
positions :: Eq a => a -> [a] -> [Int]
positions x xs =
    [i | (x', i) <- zip xs [0..n], x == x' ]
    where n = length xs - 1
```

For example:

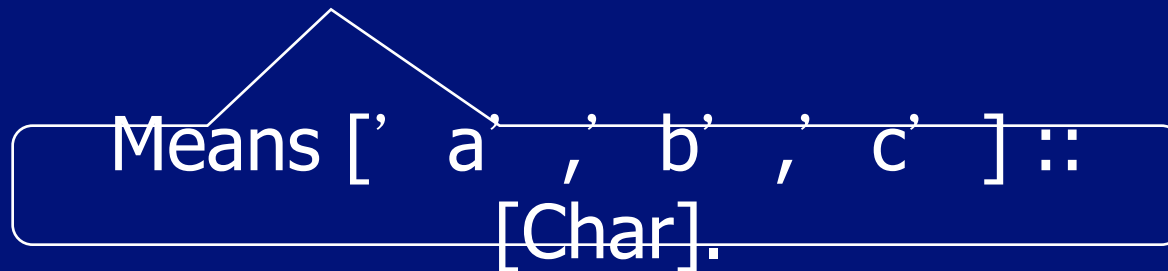
```
> positions 0 [1,0,0,1,0,1,1,0]
[1,2,4,7]
```

String Comprehensions

A string is a sequence of characters enclosed in double quotes. Internally, however, strings are represented as lists of characters.

```
"abc" :: String
```

Means ['a', 'b', 'c'] ::
[Char].



Because strings are just special kinds of lists, any polymorphic function that operates on lists can also be applied to strings. For example:

```
> length "abcde"
```

```
5
```

```
> take 3 "abcde"
```

```
"abc"
```

```
> zip "abc" [1,2,3,4]
```

```
[('a',1),('b',2),('c',3)]
```

Similarly, list comprehensions can also be used to define functions on strings, such as a function that counts the lower-case letters in a string:

```
lowers    :: String → Int
lowers xs =
    length [x | x ← xs, isLower x]
```

For example:

```
> lowers "Haske11"
6
```

Exercises

- (1) A triple (x,y,z) of positive integers is called pythagorean if $x^2 + y^2 = z^2$. Using a list comprehension, define a function

```
pyths :: Int → [(Int,Int,Int)]
```

that maps an integer n to all such triples with components in $[1..n]$. For example:

```
> pyths 5  
[(3,4,5), (4,3,5)]
```

(2) A positive integer is perfect if it equals the sum of all of its factors, excluding the number itself. Using a list comprehension, define a function

```
perfects :: Int → [Int]
```

that returns the list of all perfect numbers up to a given limit. For example:

```
> perfects 500
```

```
[6, 28, 496]
```

- (3) The scalar product of two lists of integers `xs` and `ys` of length `n` is give by the sum of the products of the corresponding integers:

$$\sum_{i=0}^{n-1} (xs_i * ys_i)$$

Using a list comprehension, define a function that returns the scalar product of two lists.