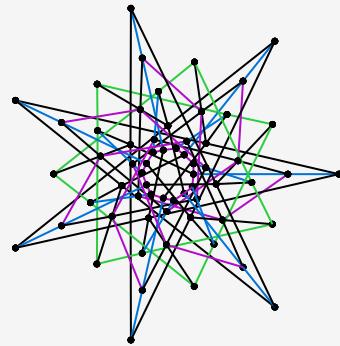


# Finite generalised polygons: the final frontier in the classification of spherical buildings



Mathematics: for all

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John Bamberg

2025-12-11

The University of Western Australia

## Group

Set  $G$  equipped with a binary operation  $*$  such that

1.  $*$  is associative
2. exists identity  $e$
3. exists inverses

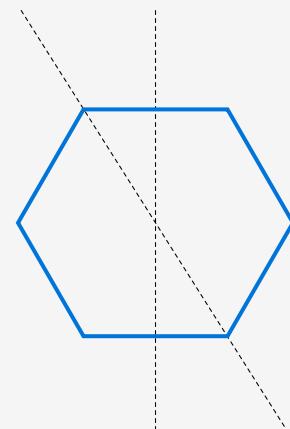
$$(a * b) * c = a * (b * c)$$

$$g * e = g = e * g$$

$$g * g^{-1} = e = g^{-1} * g$$



symmetries of a 6-gon



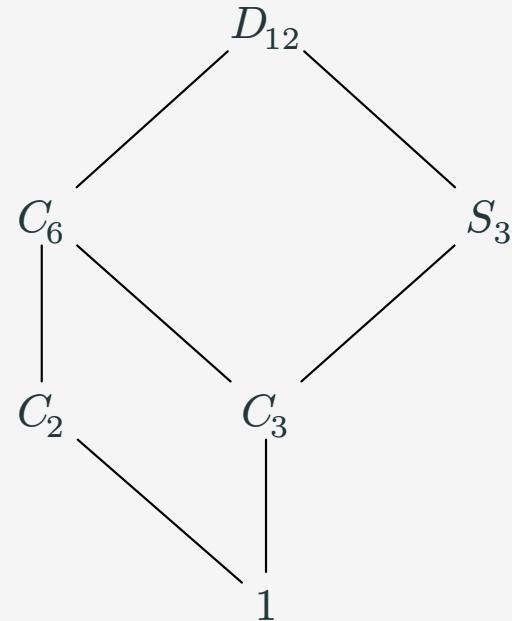
$n \times n$  invertible matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$$

groups can be broken down into indivisible pieces

- normal subgroup  $N$  of  $G \rightarrow$  quotient group  $G/N$
- composition factors ... intervals of **simple** quotients

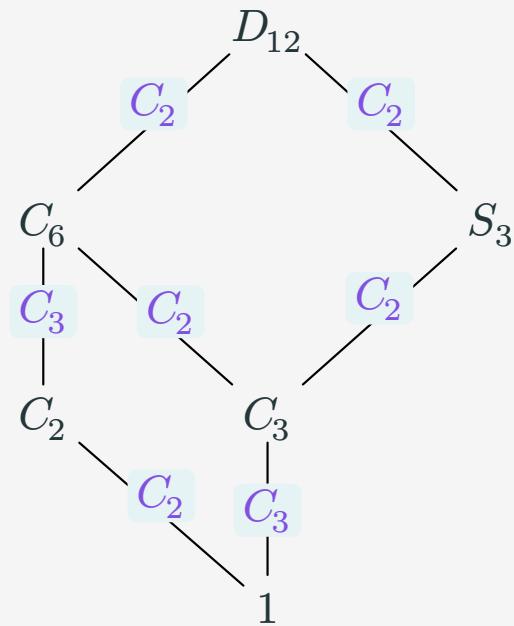
### symmetries of a 6-gon



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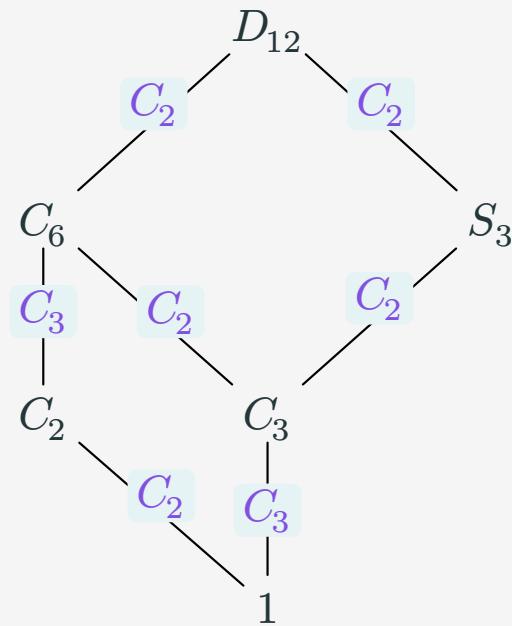
### symmetries of a 6-gon



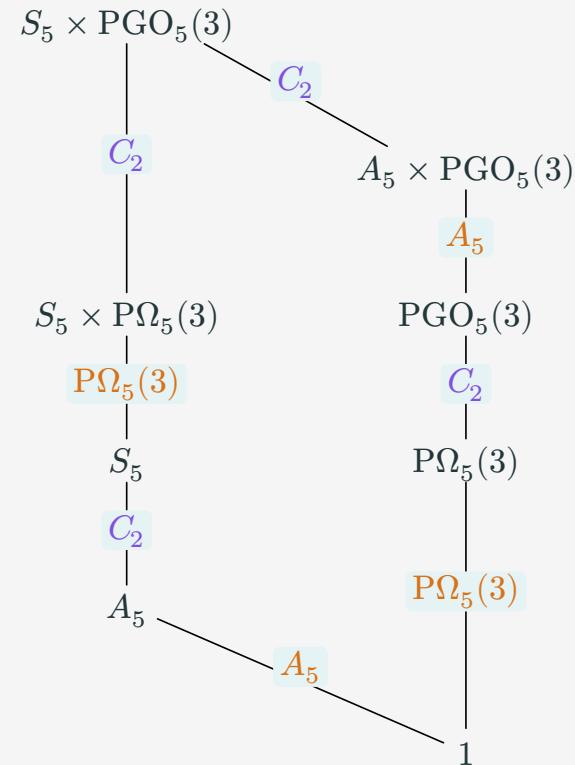
groups can be broken down into indivisible pieces

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### symmetries of a 6-gon



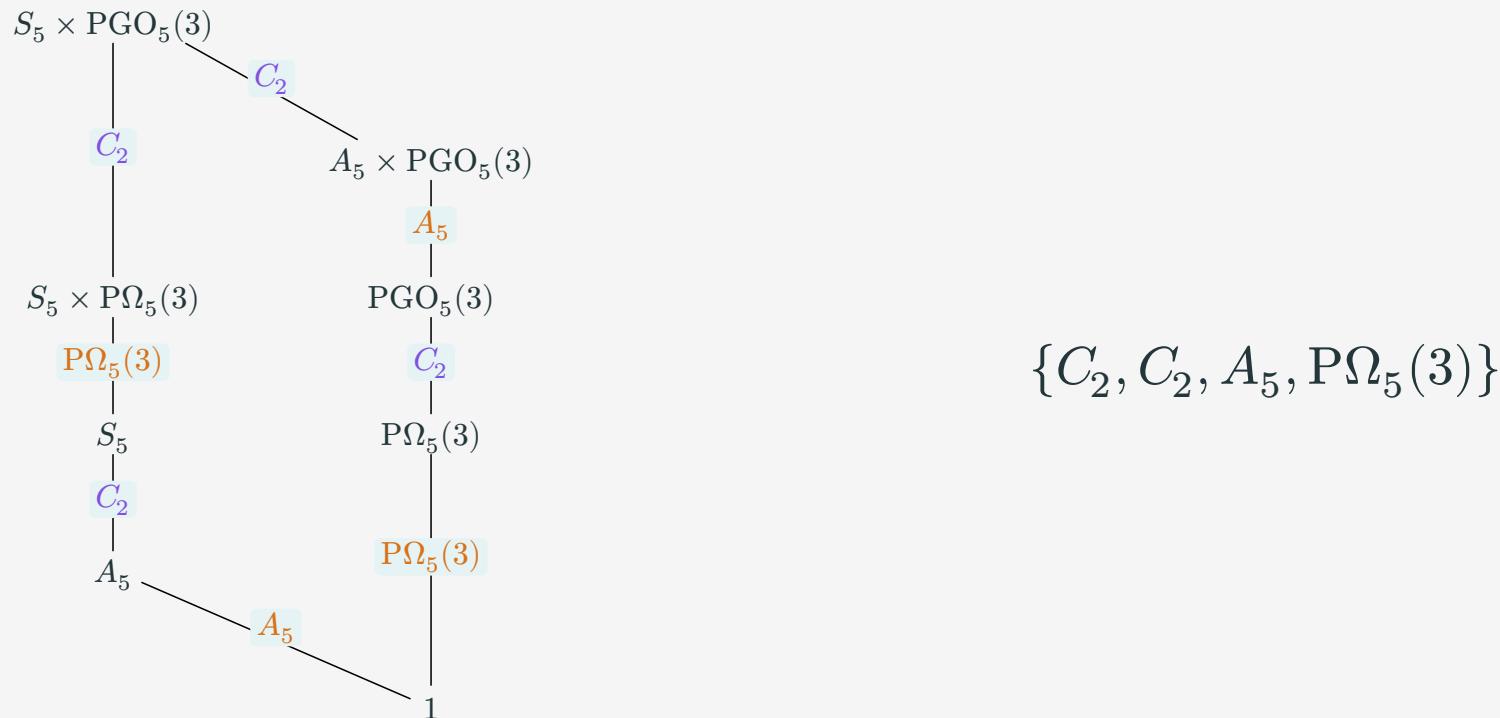
### more elaborate



# Jordan-Hölder

Camille Jordan (1870), Otto Hölder (1889)

The composition factors of a finite group are unique up to reordering.



# Classification of Finite Simple Groups

The finite simple groups fall into the following classes:

- cyclic of prime order
- alternating groups
- Lie type
- 26 sporadic groups

$\text{PSL}_n(q), \text{P}\Omega_n(q), \dots, G_2(q), E_6(q), \dots, {}^2F_4(q), {}^3D_4(q)$   
 $M_{11}, J_3, \text{Co}_2, \dots, \text{Suz}, \text{Ru}, \mathbb{M}$



# Classification of Finite Simple Groups

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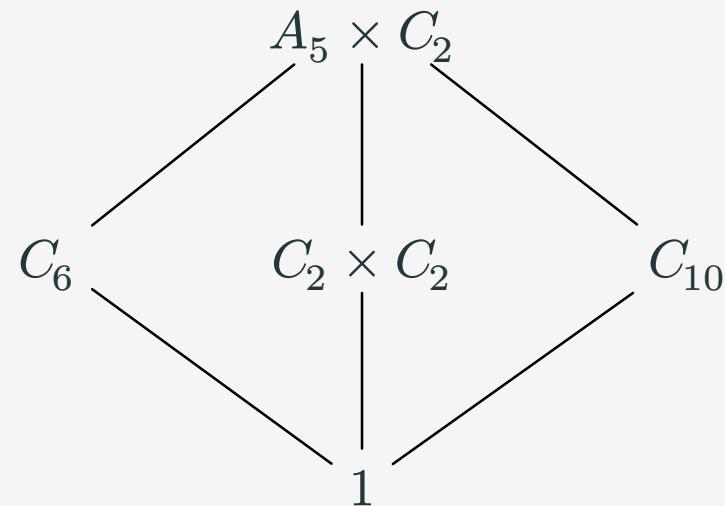
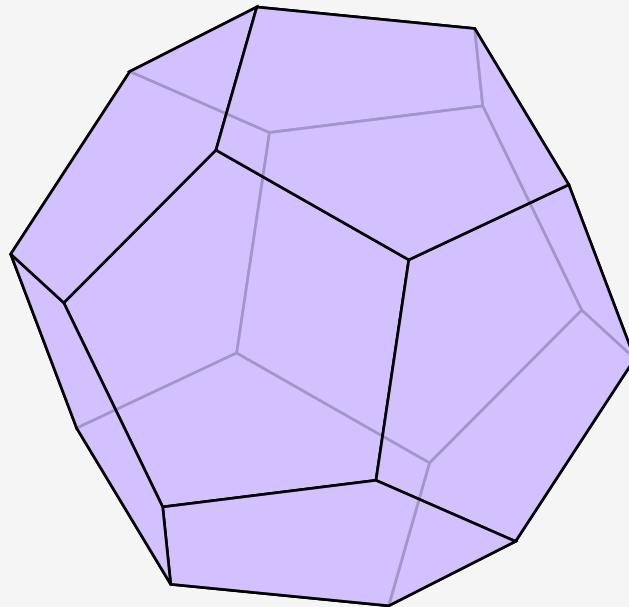
- announced in 1983
- *quasithin* case resolved in 2004 (Aschbacher & Smith)
- proof  $> 10^4$  pages

The classification of finite simple groups (CFSG),  $\dots$ , is one of the monumental achievements of twentieth century mathematics.

— Terence Tao (2013)

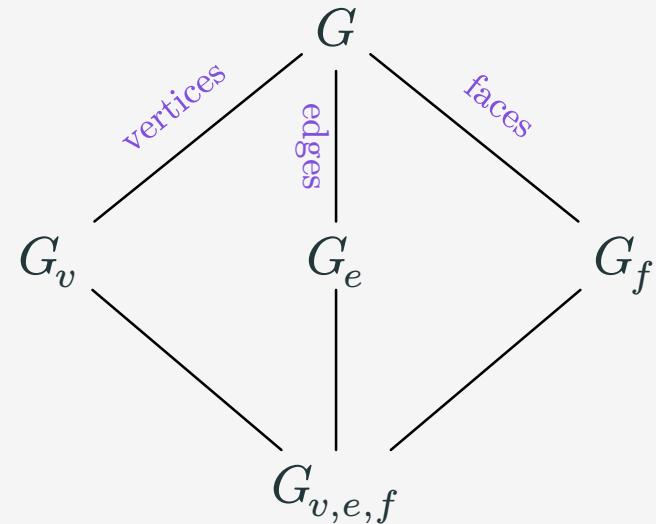
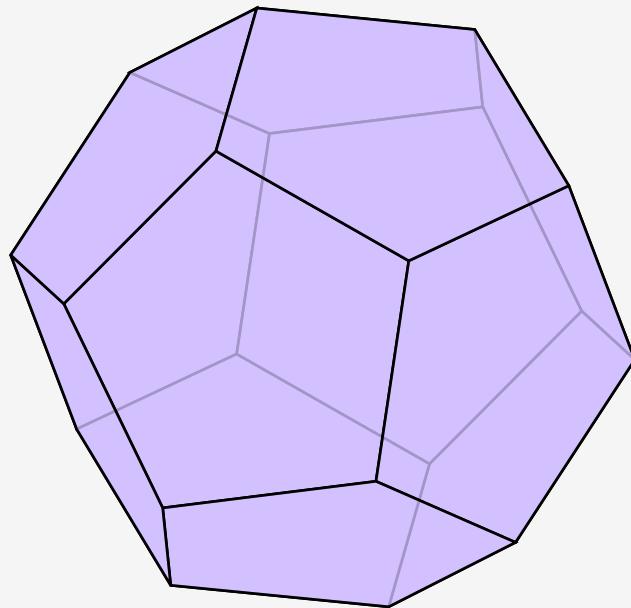
# Erlangen programme

*symmetries of a regular dodecahedron.*



# Erlangen programme

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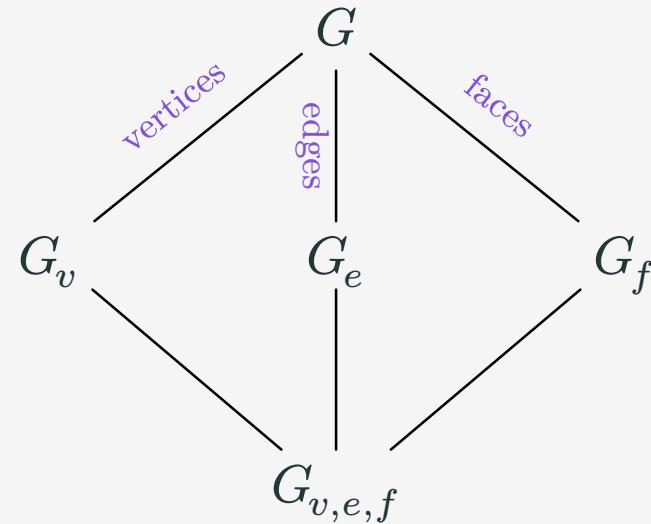
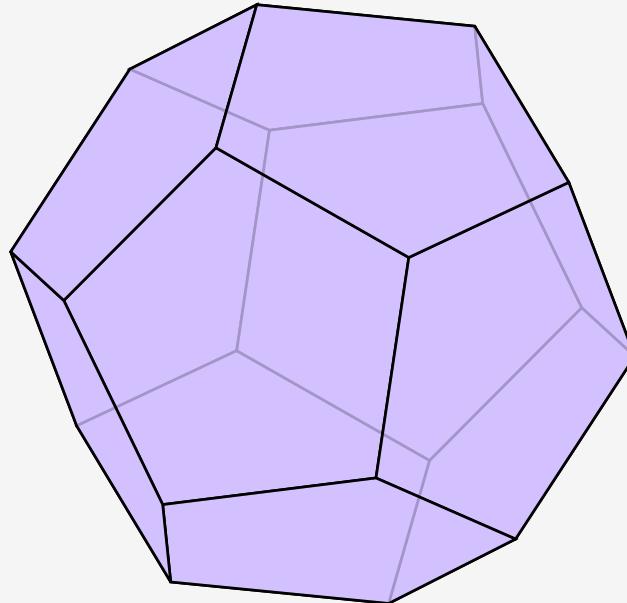


- elements are cosets

$$G_v x = \{g x : g \in G_v\}$$

# Erlangen programme

*symmetries of a regular dodecahedron.*

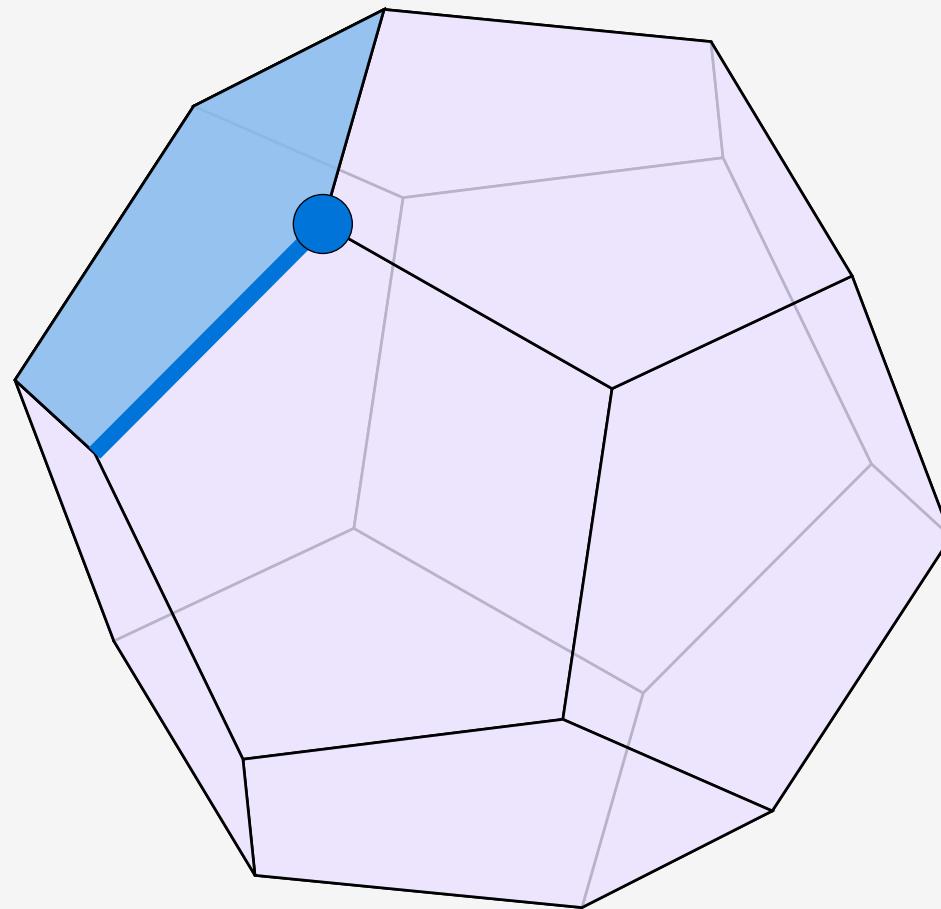


- elements are cosets
- incidence can be recaptured

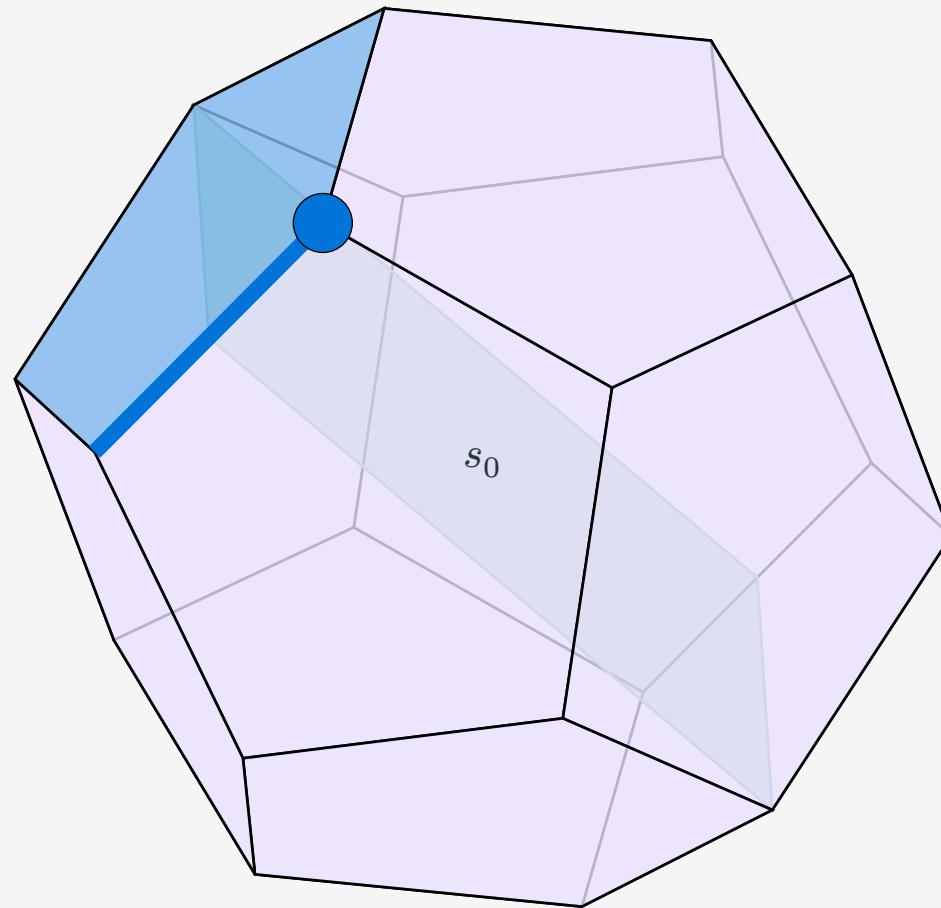
$$G_v x = \{g x : g \in G_v\}$$

$$G_v x \perp G_f y \iff G_v x \cap G_f y \neq \emptyset$$

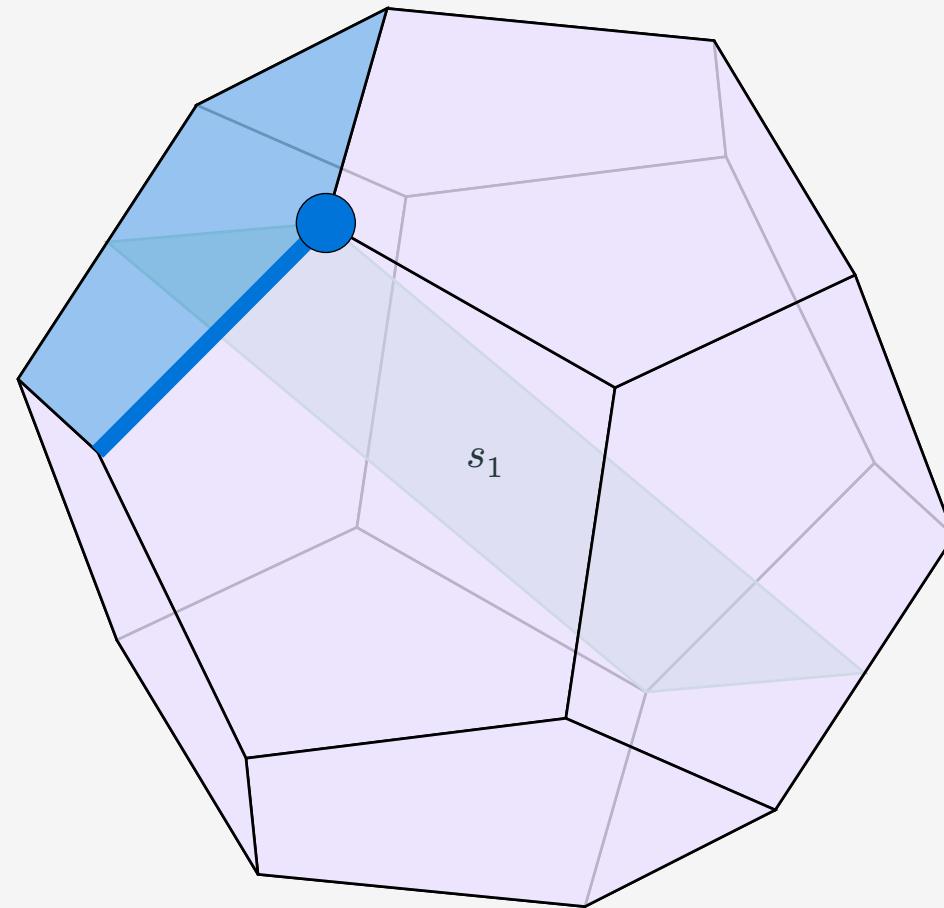
# Erlangen programme



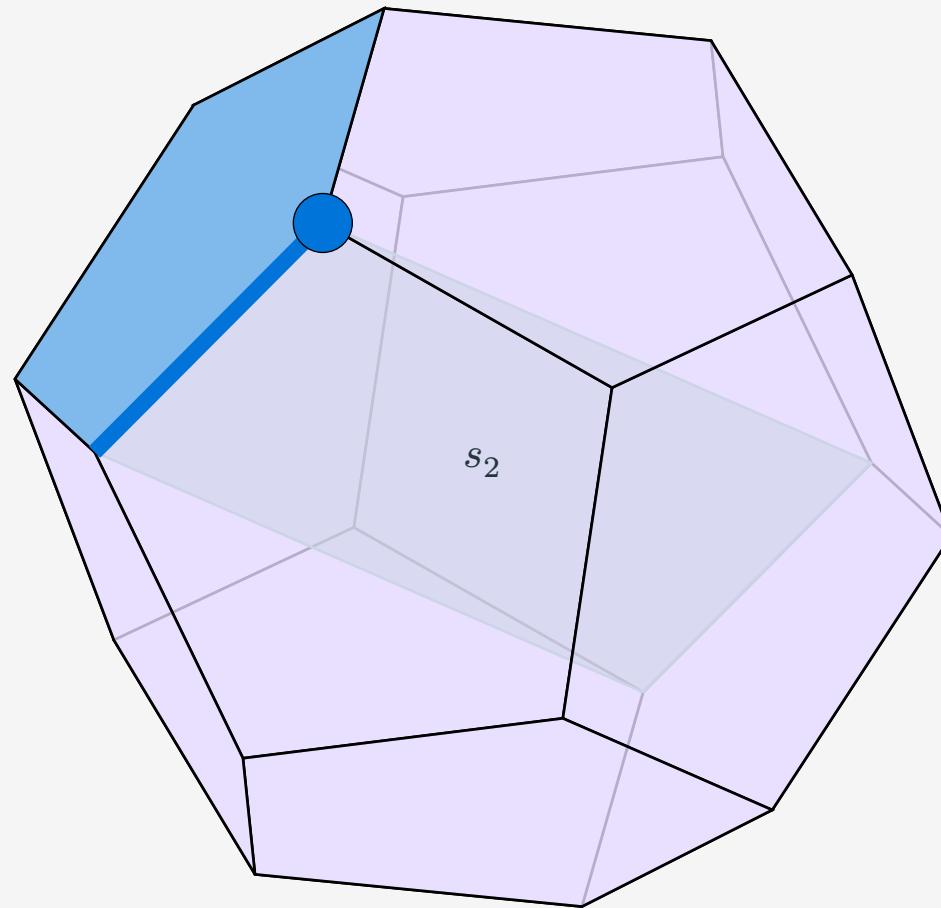
# Erlangen programme



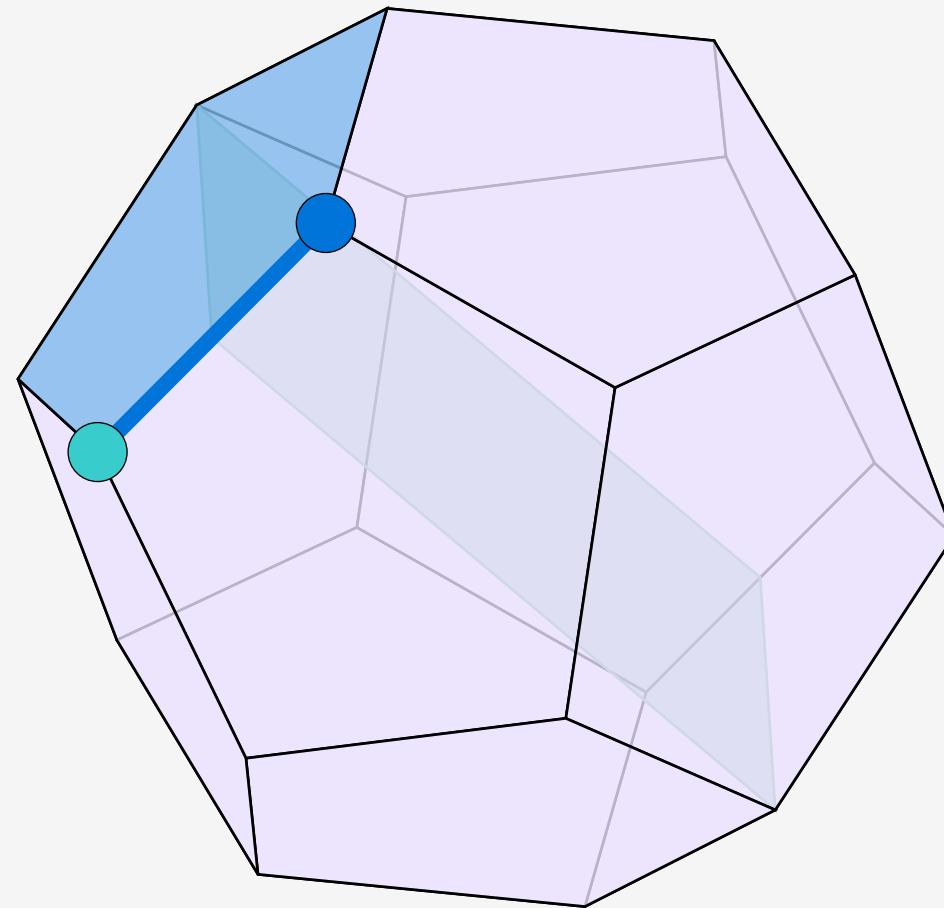
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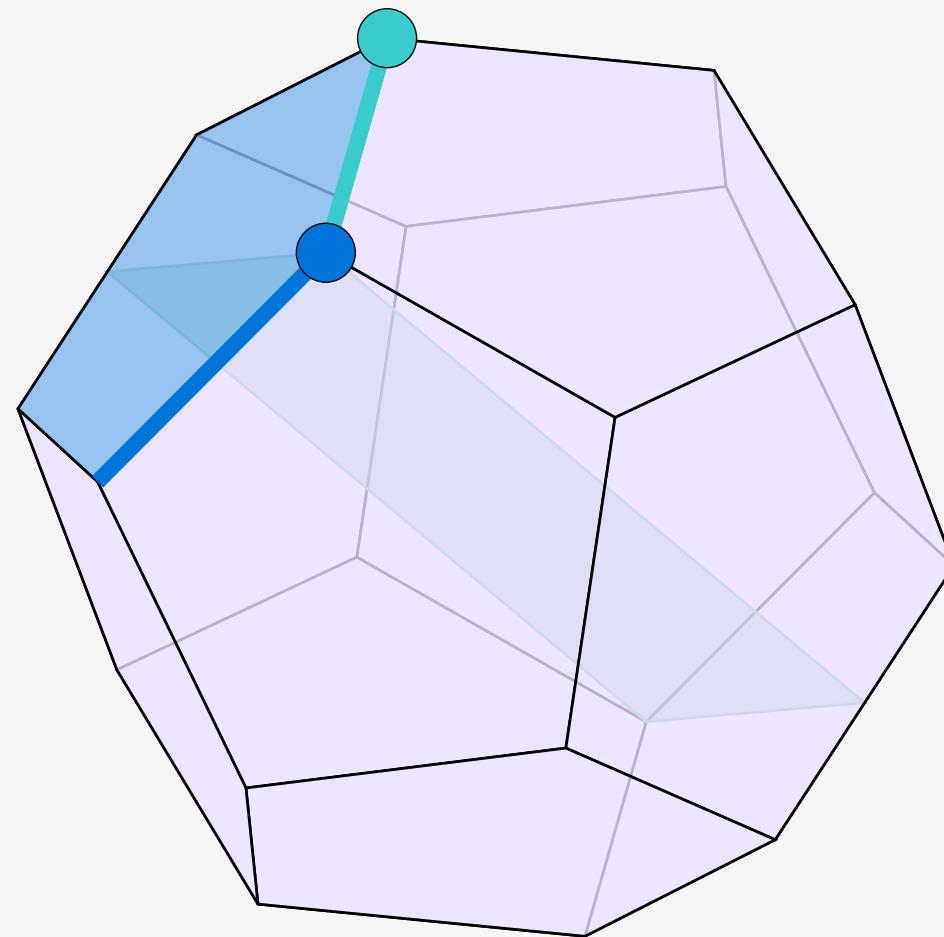
# Erlangen programme



# Erlangen programme

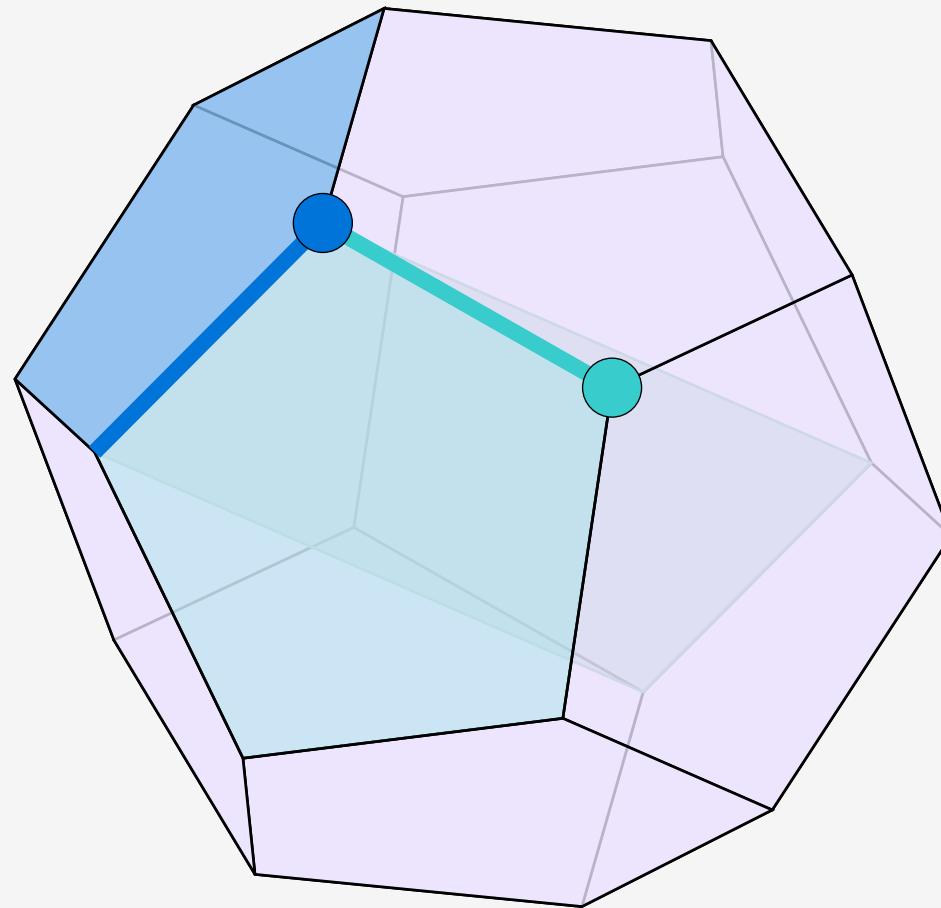


# Erlangen programme



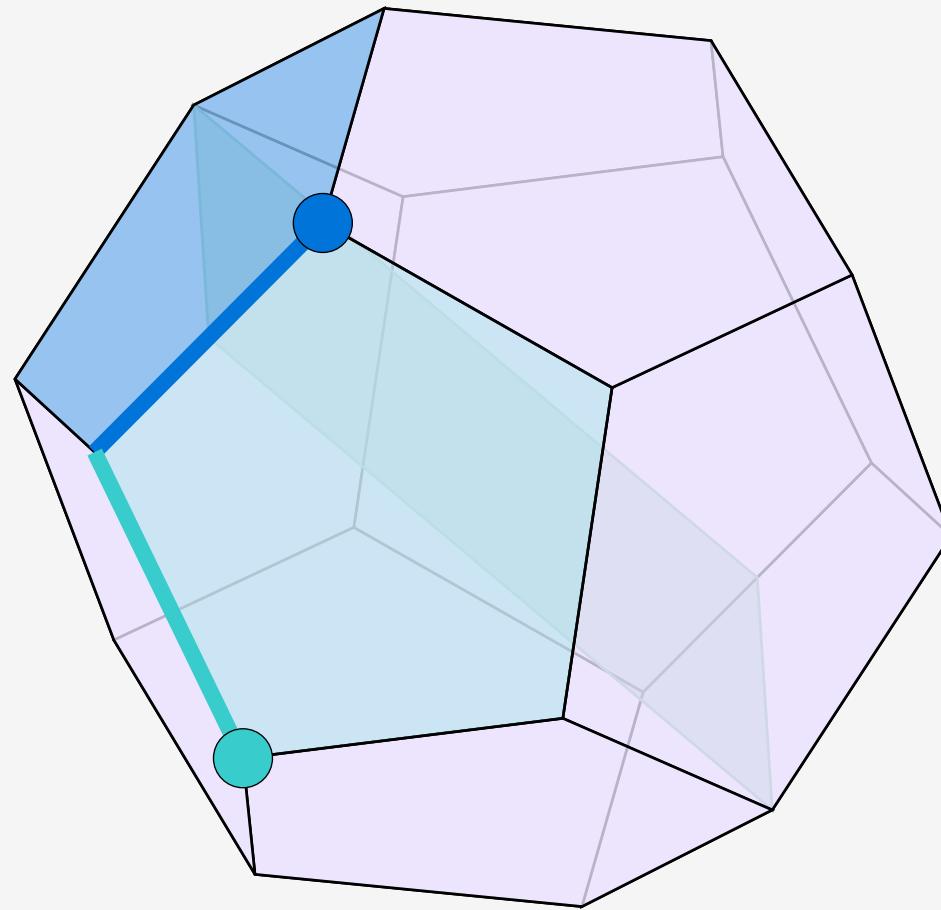
# Erlangen programme

$s_0 s_1 s_2$



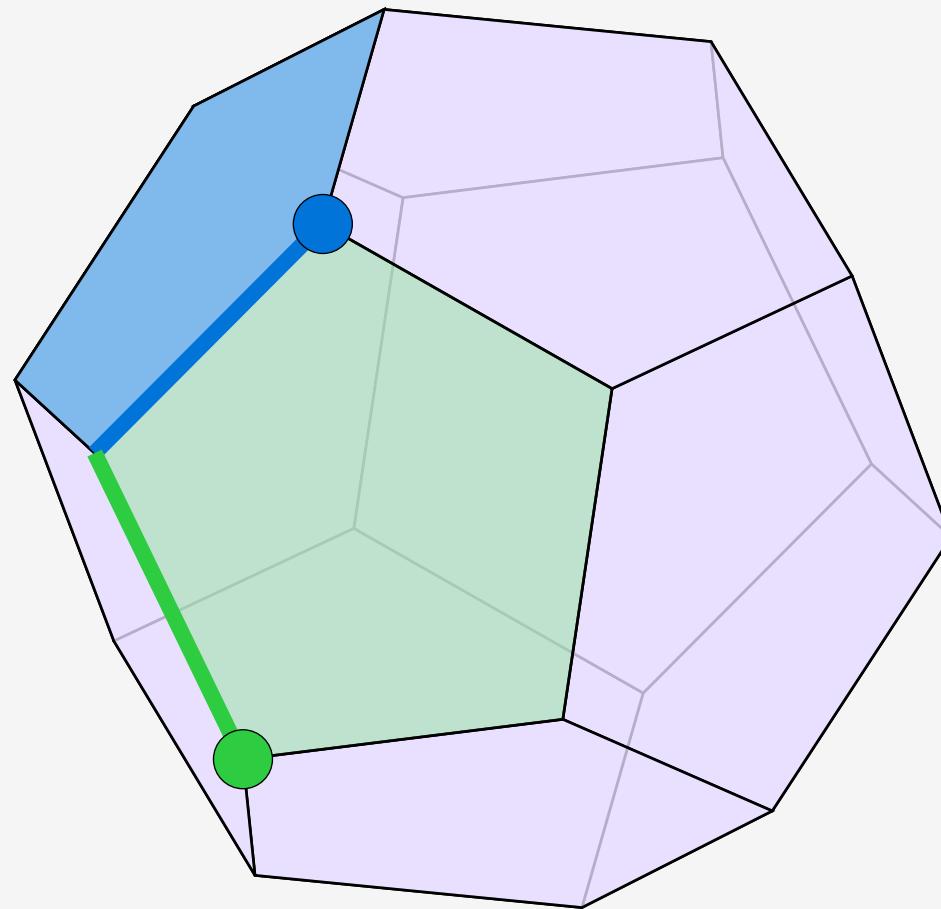
# Erlangen programme

$s_0 s_1 s_2 s_0$



# Erlangen programme

$$C \xrightarrow{s_0 s_1 s_2 s_0} C'$$



- **chamber (maximal flag)**  $\text{vertex} \subset \text{edge} \subset \text{face}$
- **Weyl group**  $W = \text{group generated by reflections } S$
- **$W$ -distance on chambers**

$$(v, e, f) \xrightarrow{w} (v', e', f')$$

1.  $C \xrightarrow{1} C' \Leftrightarrow C = C'$
2.  $C \xrightarrow{w} C'$  and  $C' \xrightarrow{s} C'' \Rightarrow C \xrightarrow{ws \text{ or } w} C''$
3.  $C \xrightarrow{w} C'$  and  $s \in S \Rightarrow (\exists C'') \quad C \xrightarrow{ws} C''$



introduced by Jacques Tits in 1959



Jacques Tits (1930 - 2021)

## Ω an example from projective geometry

- vector space  $\mathbb{F}^4$  over  $\mathbb{F}$
- projective special linear group  $\mathrm{PSL}_4(\mathbb{F})$
- $W = S_4$



analogy	thing
vertices	1-spaces $\langle v \rangle$
edges	2-spaces $\langle u, v \rangle$
faces	3-spaces $\langle u, v, w \rangle$

## $\Omega$ an example from finite polar geometry

- $\mathbb{F}_q^6$  equipped with bilinear form

$$B(x, y) = x_1y_6 - x_6y_1 + x_2y_5 - x_5y_2 + x_3y_4 - x_4y_3$$

- totally isotropic subspace  $U$ :

$$B(x, y) = 0 \quad \forall x, y \in U$$

- symplectic group  $\mathrm{PSp}_6(q)$
- $W = C_2 \wr S_3$



analogy	thing
vertices	totally isotropic 1-spaces
edges	totally isotropic 2-spaces
faces	totally isotropic 3-spaces

# buildings

- spherical = finite Weyl group  $W$
- rank = number of things

Jacques Tits 1974

All spherical buildings of rank at least 3 are known and are associated with simple groups of Lie type.

Antecedents

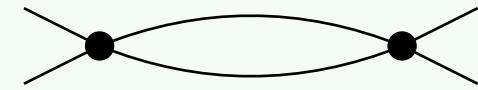
- All projective spaces of rank at least 3 arise from vector spaces. (Hilbert 1899, Vahlen, Hessenberg 1905)
- All polar spaces of rank at least 3 arise from vector spaces equipped with sesquilinear or quadratic forms. (Buekenhout & Shult 1974)

# generalised polygons

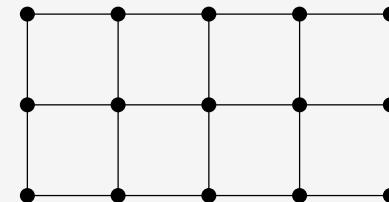
## generalised $n$ -gon

Incidence geometry of points and lines such that

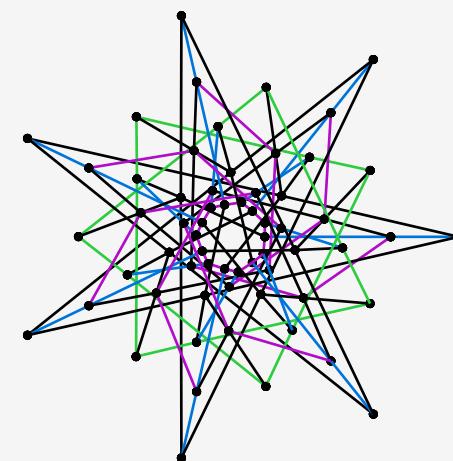
- every 2 elements lie in some ordinary  $n$ -gon
- there are no ordinary  $k$ -gons for  $2 \leq k < n$



ordinary 2-gon



a generalised 4-gon



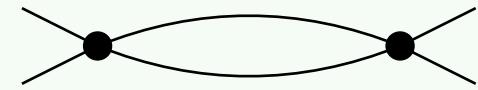
a generalised 6-gon

# generalised polygons

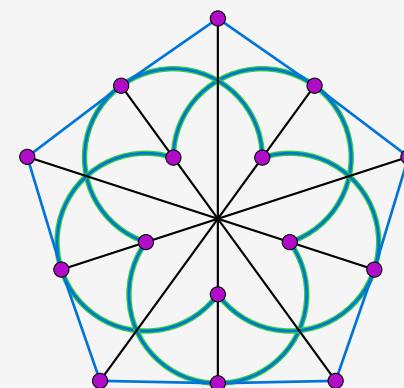
## generalised $n$ -gon

Incidence geometry of points and lines such that

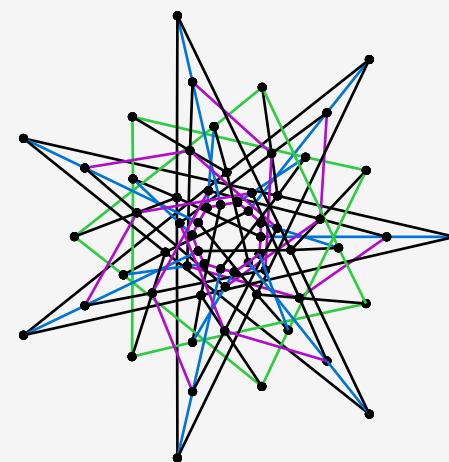
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ordinary 2-gon



a generalised 4-gon



a generalised 6-gon

## ⚠ Warning

We will assume our geometries are **finite** from now on.

Walter Feit & Graham Higman 1964

If every line has at least 3 points and every point lies on at least 3 lines, then

$$n \in \{3, 4, 6, 8\}.$$



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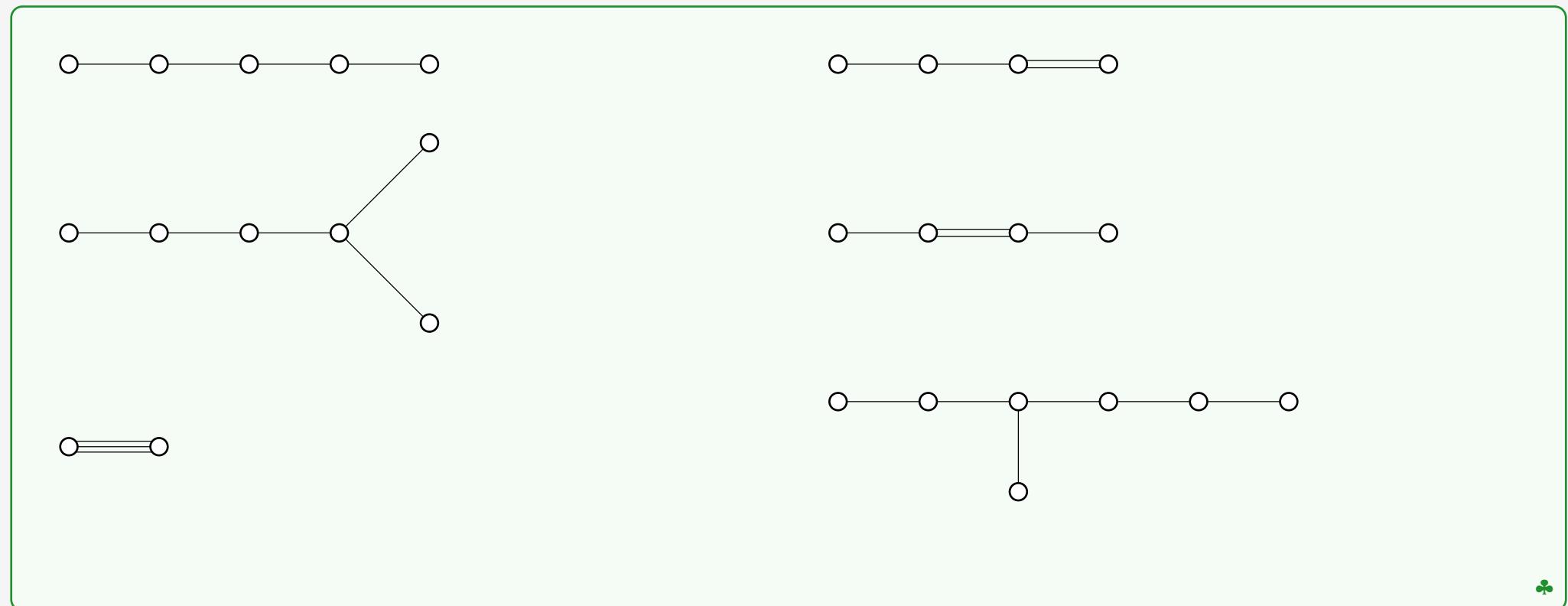
$$n \in \{3, 4, 6, 8\}.$$

Related to integers  $k$  such that  $\sin^2\left(\frac{\pi}{k}\right) \in \mathbb{Q}$ .



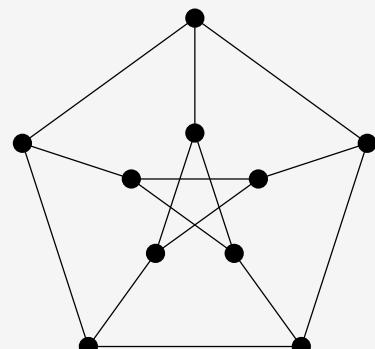
# applications

- generalised  $n$ -gons are rank 2 buildings
- buildings can be broken down into generalised  $n$ -gons (residues)



## The cage problem

Given  $d > 1, g > 2$ , find the smallest number of possible vertices that a regular graph of **degree  $d$**  and **girth  $g$**  can have.



Petersen graph: cage(3,5)

A cage( $d, g$ ) has at least

$$1 + d + d(d - 1) + \cdots + d(d - 1)^{\frac{g-3}{2}}$$

vertices

A bipartite graph meeting the bound (*Moore graph*) **is** a generalised  $n$ -gon.



## Extremal combinatorics

- optimal clique-free pseudorandom graphs (Mubayi & Verstraëte<sup>1</sup>)
- new asymptotic bounds on cycle-complete Ramsey numbers
- smallest degree of Ramsey graphs that are minimal for cliques (JB, Bishnoi, & Lesgourgues<sup>2</sup>)
- $r(4, t) = \Omega\left(\frac{t^3}{\log^4 t}\right)$  (Mattheus & Verstraëte<sup>3</sup>)



<sup>1</sup>“A note on pseudorandom Ramsey graphs”, doi:10.4171/JEMS/1359

<sup>2</sup>“The minimum degree of minimal Ramsey graphs for cliques”, doi:10.1112/blms.12658”

<sup>3</sup>“The asymptotics of  $r(4, t)$  “, doi:10.4007/annals.2024.199.2.8

## User-Private Information Retrieval schemes

In such a scheme, a set of users collaborate to retrieve files from a database without revealing to observers which participant in the scheme requested the file.

- generalised quadrangles were used<sup>1</sup> to create a method that is substantially more secure against eavesdropping than was previously known.



They have also appeared in

- the construction of codes
- quantum error-correction
- quasisymmetric designs
- association schemes

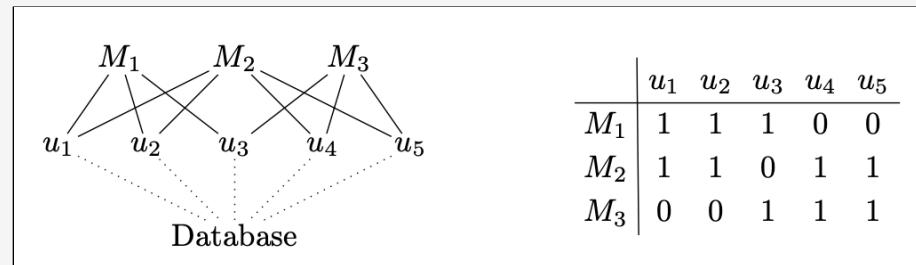


Figure 1: A visualisation of a UPIR system.

<sup>1</sup>Gnilke, Greferath, Hollanti, Nuñez Ponasso, Ó Catháin, Swartz, “Improved user-private information retrieval via finite geometry”, doi:10.1007/s10623-018-00591-9

# known examples

**dual** generalised  $n$ -gon: swapping the roles of points and lines.

## 3-gons (projective planes)

- Desarguesian (“classical”)  $\rightarrow \text{PSL}_3(q)$
- lots of non-Desarguesian planes known

## 4-gons (generalised quadrangles)

- arising from formed vector spaces (“classical”)  $\rightarrow \text{PSp}_4(q), \text{PSU}_4(q), \text{PSU}_5(q)$
- lots not arising from formed vector spaces

## 6-gons (generalised hexagons)

- classical  $\rightarrow G_2(q), {}^3D_4(q)$

## 8-gons (generalised octagons)

- classical  $\rightarrow {}^2F_4(q)$

# known examples

Bose-Nair, Cameron, Cohen-Tits, Dixmier-Zara, Hall, Hall-Swift-Walker, Lam-Kolesova-Thiel, Lam-Thiel-Swiercz, MacInnes, Payne, Payne-Thas, Tarry, Thas

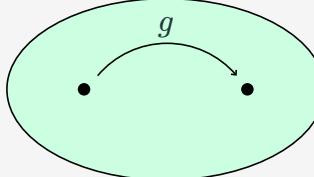
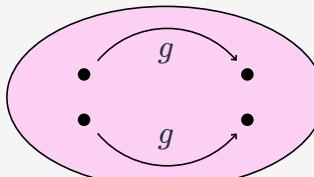
All generalised  $n$ -gons with at most 124 points have been classified.



$n$	# points	# symmetries	$n$	# points	# symmetries
3	7	168	3	13	5616
3	21	120960	3	31	372000
3	57	5630688	3	73	49448448
3	91	84913920	3	91	311040
3	91	33696	4	15	720
4	27	51840	4	40	51840
4	64	138240	4	85	1958400
4	112	26127360	6	63	12096

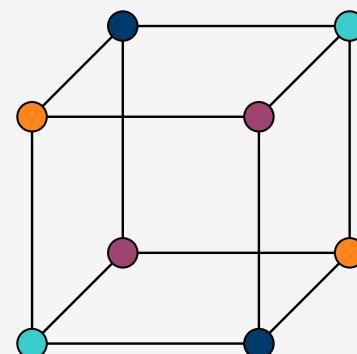
# actions

$G$  acts on  $\Omega$

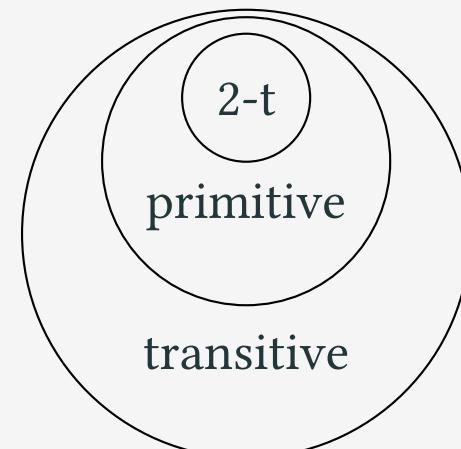
<b>transitive</b>		$(\forall \omega, \omega' \in \Omega)(\exists g \in G) \quad \omega' = \omega^g$
<b>2-transitive</b>		$(\forall \omega_1, \omega_2, \omega'_1, \omega'_2 \in \Omega)(\exists g \in G)$ $\omega'_1 = \omega_1^g$ and $\omega'_2 = \omega_2^g$

- **intransitive**  $G$  preserves a subset
- **imprimitive**  $G$  is transitive and preserves a partition

### symmetries of a cube (imprimitive)



Venn diagram



**primitive groups** are atoms for **transitive groups**

## convention

We will associate an adjective for the symmetry group with the object.

1. “**point-2-transitive** projective plane”  $\Rightarrow$   
“the group of symmetries of the projective plane is **2-transitive** on **points**.”
2. “**point-primitive** and **flag-transitive** generalised quadrangle”  $\Rightarrow$   
“the group of symmetries of the generalised quadrangle is **primitive** on **points** and  
**transitive on flags**.”

### Ostrom & Wagner (1959)

A finite point-2-transitive projective plane is classical.



This was the first time 2-transitivity produced a complete classification of finite geometries. Since then the notion of a geometric classification in terms of a group-theoretic hypothesis has become commonplace. That was not the case 35 years ago, and it is a measure of these papers' influence that this type of hypothesis is now regarded as a natural extension of Klein's Erlangen program.

— Bill Kantor (1993)

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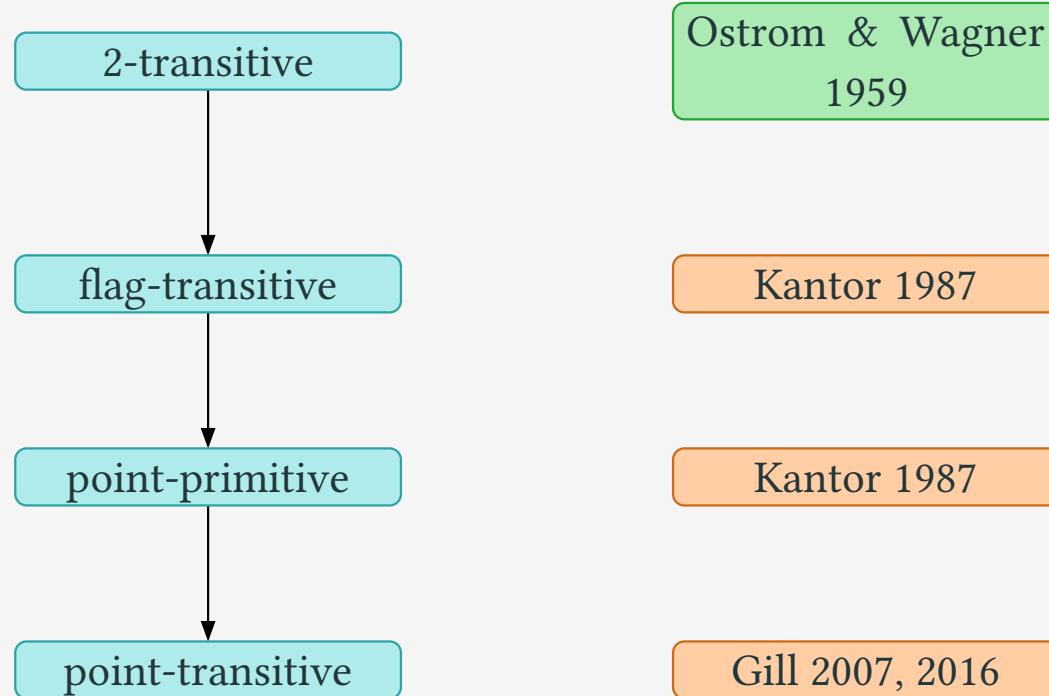
— Bill Kantor (1993)

### Conjecture (Bruck - early 1950's)

A finite point-transitive projective plane is classical.



# projective planes: characterisations

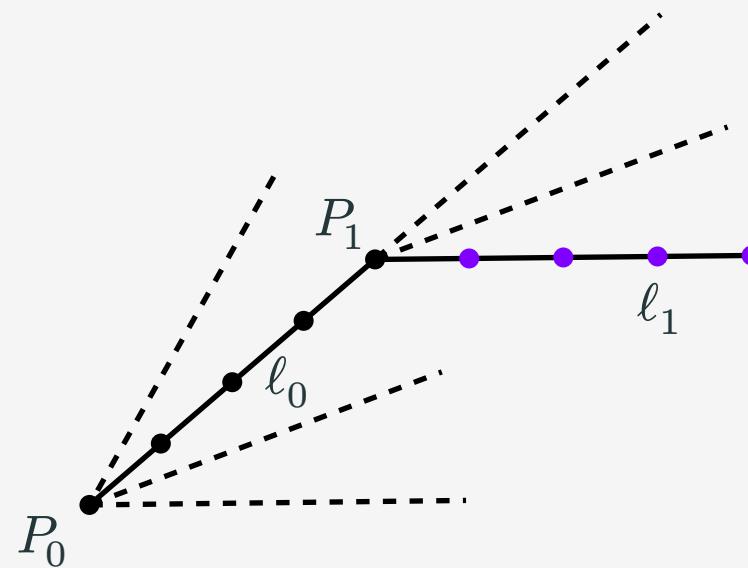


K. Thas and Zagier (2008)

A point-primitive non-classical projective plane has at least  $4 \times 10^{22}$  points.



## general results



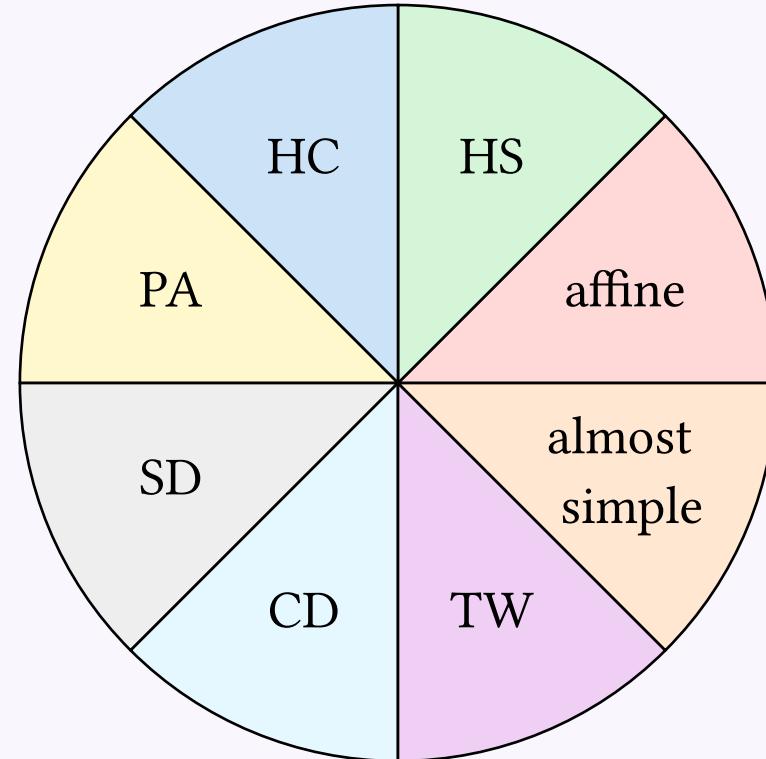
Fong & Seitz 1973/1974

A finite *Moufang* generalised polygon is classical.

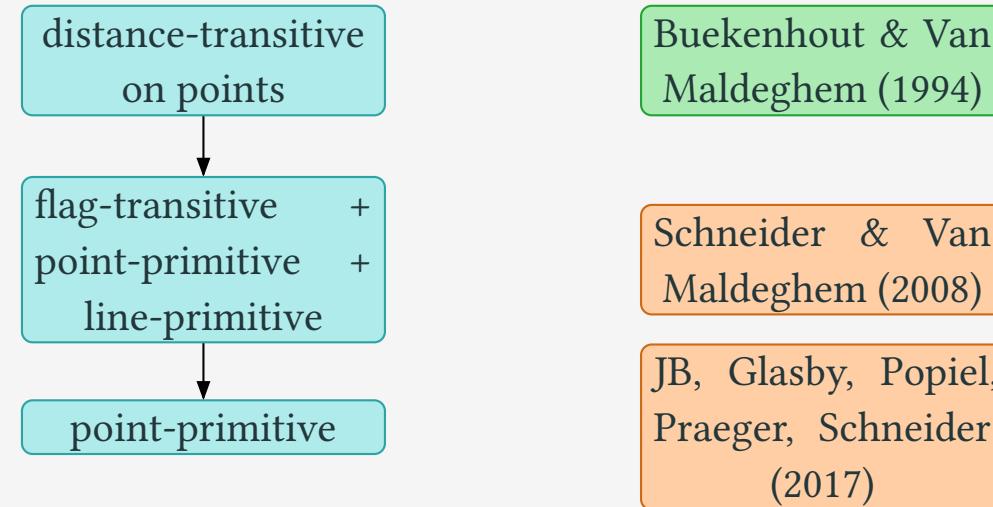


# primitive groups (atoms for transitive groups)

## The O'Nan-Scott Theorem for finite primitive groups



# generalised hexagons/octagons: characterisations



JB, Glasby, Popiel, Praeger, Schneider (2017)

If  $G$  acts primitively on the points of a finite GH or GO, then  $G$  is almost simple.

Morgan & Popiel (2016)

(i)  ${}^2B_2(q), {}^2G_2(q) \not\subset G$ .      (ii)  ${}^2F_4(q) \subset G \Rightarrow$  known example.

# generalised quadrangles: a sporadic example

- first constructed by Ronald Ahrens and George Szekeres (UNSW) in 1969
- 64 points, 96 lines
- *link to demo*
- flag-transitive, points-distance transitive, line-distance intransitive
- point-primitive, line-imprimitive

## ON A COMBINATORIAL GENERALIZATION OF 27 LINES ASSOCIATED WITH A CUBIC SURFACE

R. W. AHRENS and G. SZEKERES

(Received 26 March 1969)

To Bernhard Hermann Neumann on his 60th birthday

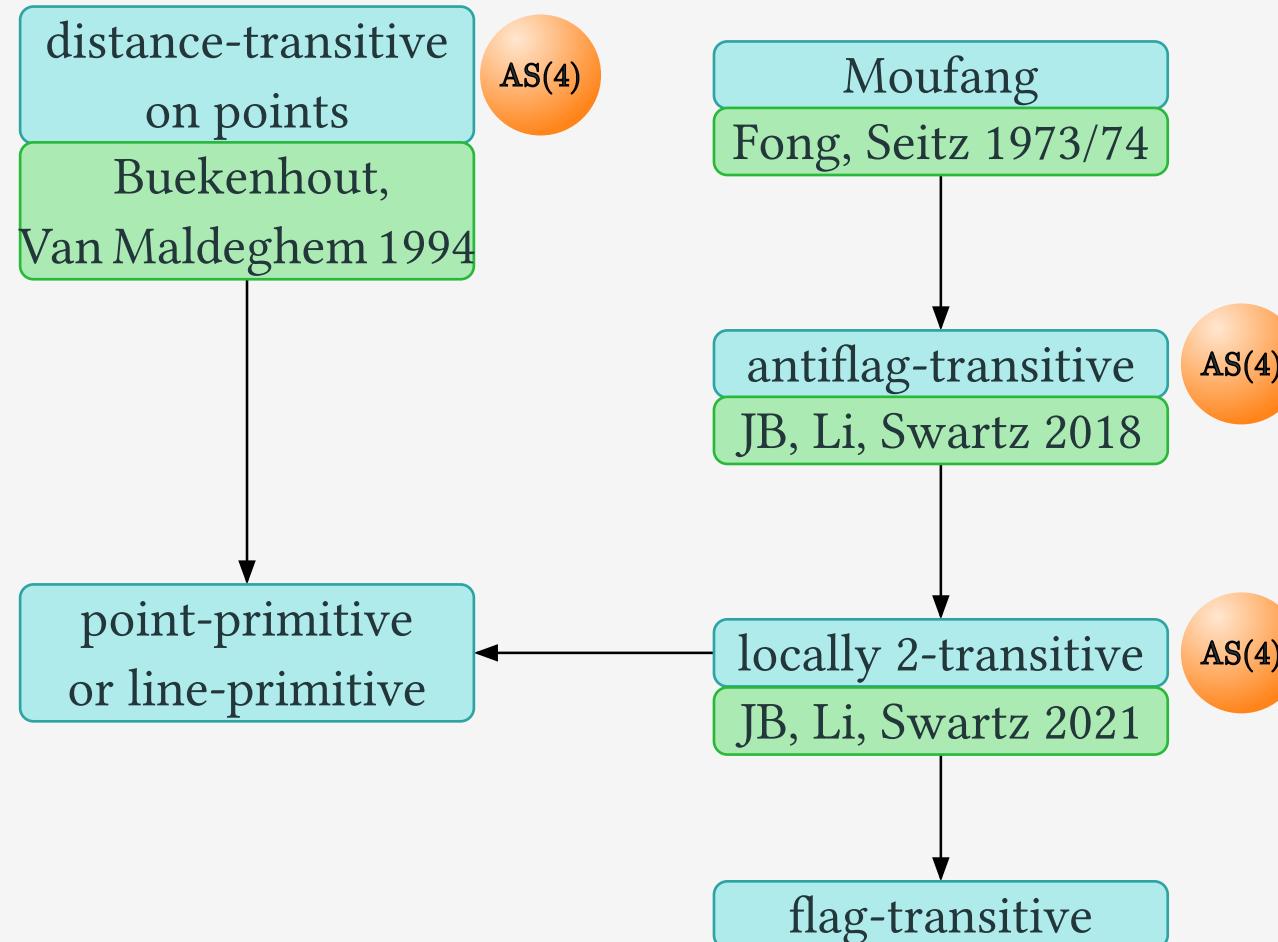
Communicated by G. B. Preston

### 1.

Given integers  $0 < \lambda < k < v$ , does there exist a nontrivial graph  $G$  with the following properties:  $G$  is of order  $v$  (i.e. has  $v$  vertices), is regular of degree  $k$  (i.e. every vertex is adjacent to exactly  $k$  other vertices), and every pair of vertices is adjacent to exactly  $\lambda$  others? Two vertices are said to be adjacent if they are connected by an edge. We call a graph with the above properties a symmetric  $(v, k, \lambda)$  graph and refer to the last of the properties as the  $\lambda$ -condition. The complete graph of order  $v$  is a trivial example of a symmetric  $(v, v-1, v-2)$  graph, but we are of course only interested in non-trivial constructions.

All graphs will be assumed to have no loops or double edges. If the vertices of  $G$  are denoted by  $x_1, \dots, x_v$  and  $S_i$ ,  $i = 1, \dots, v$  denotes the set of vertices which are adjacent to  $x_i$ , then the sets  $S_i$  form a symmetric block design with parameters  $(v, k, \lambda)$  i.e.  $|S_i| = k$ ,  $|S_i \cap S_j| = \lambda$  for  $i \neq j$ . Thus the parameters  $v, k, \lambda$  must satisfy the Bruck-Ryser-Chowla conditions ([2], p. 107) in order that such a graph should exist. However these conditions

# generalised quadrangles: characterisations



## Conjecture (Kantor 1991)

A finite flag-transitive generalised quadrangle is classical, AS(4), or the Lunelli-Sce quadrangle.



# generalised quadrangles: characterisations

## Conjecture (Kantor 1991)

A finite flag-transitive generalised quadrangle is classical, AS(4), or the Lunelli-Sce quadrangle.



## Conjecture

A finite flag-transitive generalised quadrangle is point-primitive or line-primitive.



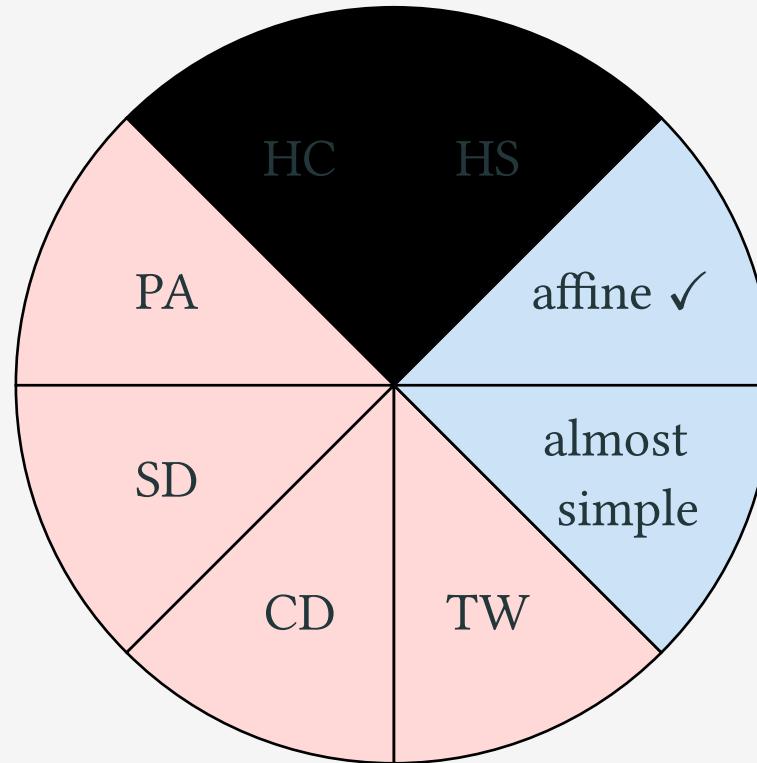
JB, Giudici, Morris, Royle, Spiga (2012)

- $G$  primitive on points and lines of a GQ  $\Rightarrow G$  is almost simple.
- If  $G$  is also flag-transitive, then  $G$  is of Lie type.



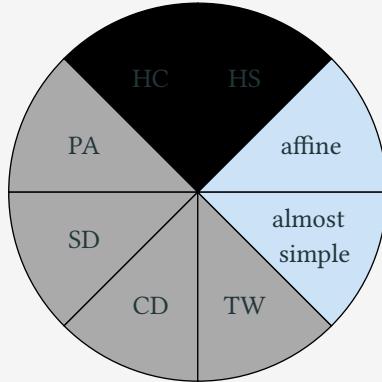
go to conclusion

## point-primitive and line-transitive



- JB, Glasby, Popiel, Praeger, *J. Comb. Des.* (2017)
- JB, Popiel, Praeger, *J. Group Theory* (2017)

## just point-primitive



Di (2025+)

HS does not arise.



JB, Popiel, Praeger (Nagoya Math. 2019)

- When  $G$  is of type PA, SD, CD, TW, the structure of  $G$  is heavily restricted.
- Let  $\theta$  be a nonidentity symmetry of a GQ. Then  $\theta$  fixes less than  $v^{\frac{4}{5}}$  of  $v$  points.



**exception:** unique GQ on 27 points has  $\theta$  fixing 15.

# conclusion

recent results on almost simple type:  $S \triangleleft G \leqslant \text{Aut}(S)$

author (yr)	condition	conclusion
JB, Giudici, Morris, Royle, Spiga (2012)	point-primitive, flag-transitive, $S = A_n$	$n = 6$ , known example
JB, Evans (2021)	point-primitive, $S$ sporadic	$\emptyset$
Feng, Lu (2023)	point-primitive, line-primitive, $S = \text{PSL}_2(q)$	$q = 9$ , known example
Lu, Zhang, Zou (2024)	point-primitive, line-primitive, $S = \text{PSU}_3(q), q \geq 3$	$\emptyset$
Arumugam, JB, Giudici, (2025)	point-primitive, line-primitive, $S \in \{{}^2B_2(q), {}^2G_2(q)\}$	$\emptyset$
Arumugam (2026+)	point-primitive, line-primitive, $S = {}^2F_4(q)$	$\emptyset$

# conclusion

- local symmetry conditions

Feng and K. Thas (arXiv)

Suppose  $G$  acts on a thick generalised quadrangle. If for every point  $P$ , the group  $G_P$  is not faithful on the points collinear with  $P$ , then it is classical.

- geometric way to study symmetry
  - ▶ can we simplify the proof of the CFSG?
- classify geometric objects satisfying a symmetry condition
  - ▶ flag-transitive, point-primitive?
- imposing symmetry allows us to construct new geometric objects

# conclusion

“At least two decades have passed since the discovery of a generalised quadrangle with new parameters  
... A generalised quadrangle with new parameters would be of much interest.”

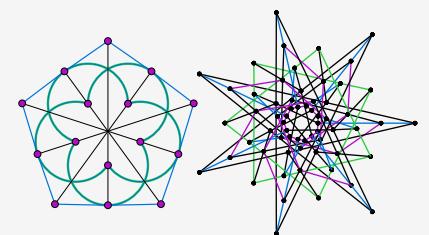
— Simeon Ball (2015)

## Open problems

1. Construction of non-classical finite generalised hexagons or octagons.
2. Classification of finite flag-transitive generalised polygons.
3. Classification of finite point-primitive generalised polygons.
4. Are all finite point-transitive projective planes classical?

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# conclusion

## AustMS2026

- December 1st - 4th, 2026
- The University of Western Australia (Perth)

