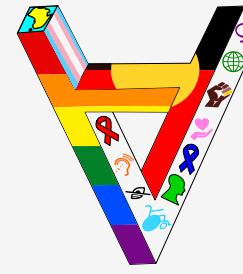
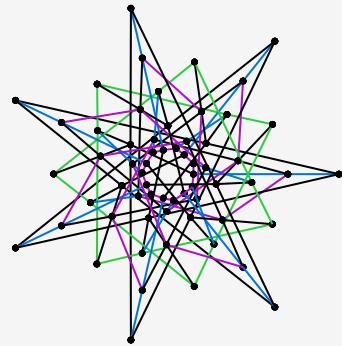


Finite generalised polygons: the final frontier in the classification of spherical buildings



John Bamberg

2025-12-11

The University of Western Australia

Group

Set G equipped with a binary operation $*$ such that

1. $*$ is associative
2. exists identity e
3. exists inverses

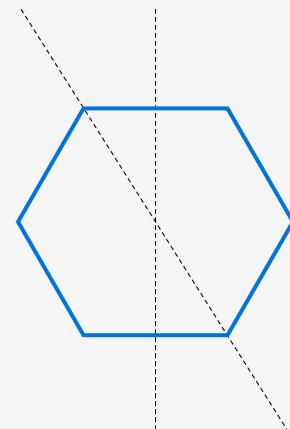
$$(a * b) * c = a * (b * c)$$

$$g * e = g = e * g$$

$$g * g^{-1} = e = g^{-1} * g$$



symmetries of a 6-gon



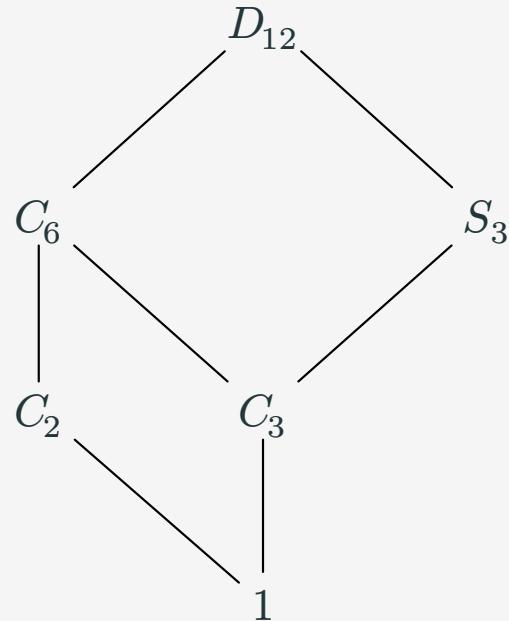
$n \times n$ invertible matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$$

groups can be broken down into indivisible pieces

- normal subgroup N of $G \rightarrow$ quotient group G/N
- composition factors ... intervals of **simple** quotients

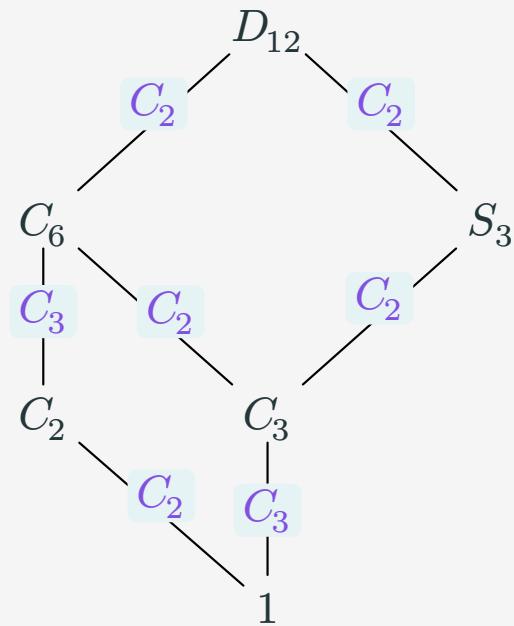
symmetries of a 6-gon



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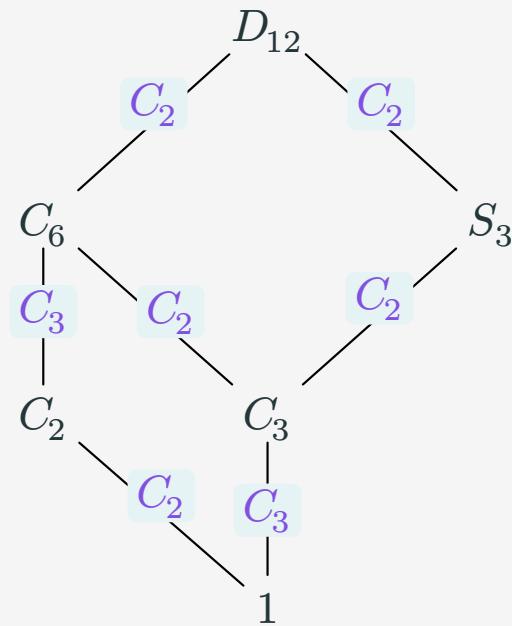
symmetries of a 6-gon



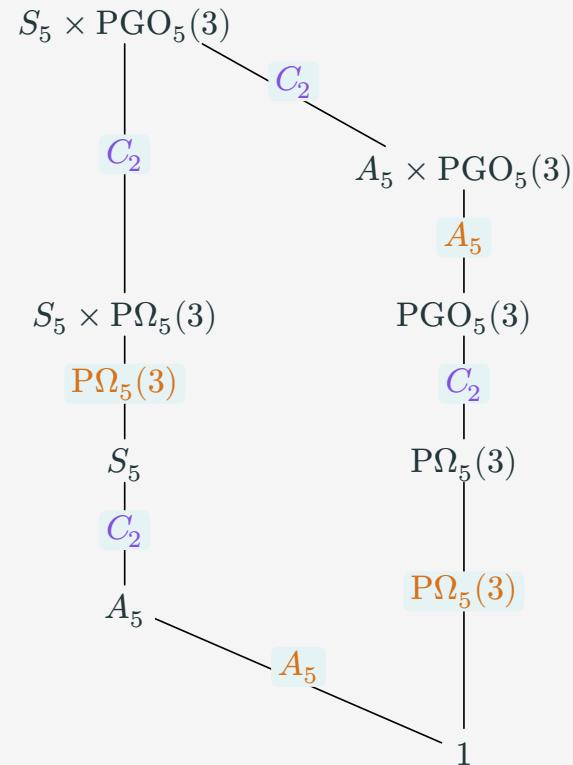
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symmetries of a 6-gon



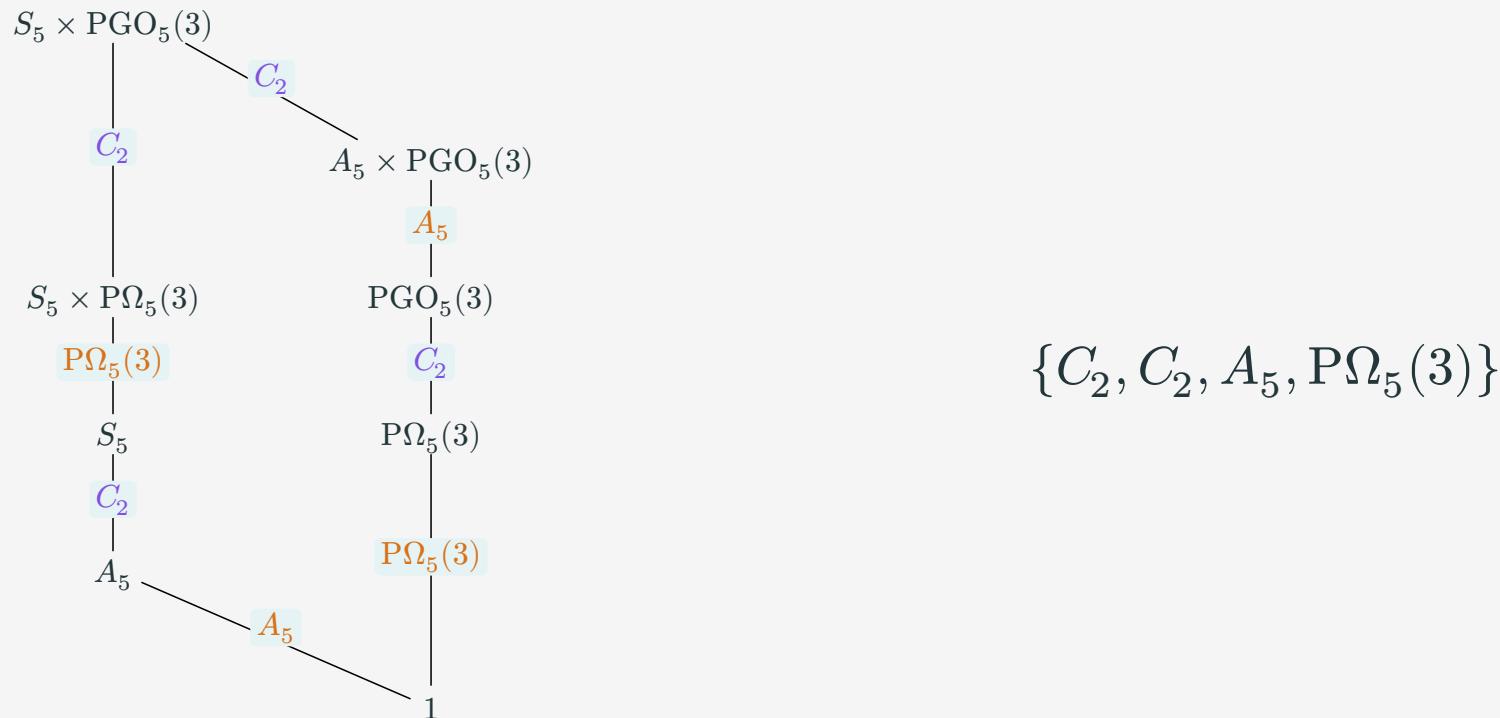
more elaborate



Jordan-Hölder

Camille Jordan (1870), Otto Hölder (1889)

The composition factors of a finite group are unique up to reordering.



Classification of Finite Simple Groups

The finite simple groups fall into the following classes:

- cyclic of prime order
- alternating groups
- Lie type
- 26 sporadic groups

$\text{PSL}_n(q), \text{P}\Omega_n(q), \dots, G_2(q), E_6(q), \dots, {}^2F_4(q), {}^3D_4(q)$
 $M_{11}, J_3, \text{Co}_2, \dots, \text{Suz}, \text{Ru}, \mathbb{M}$



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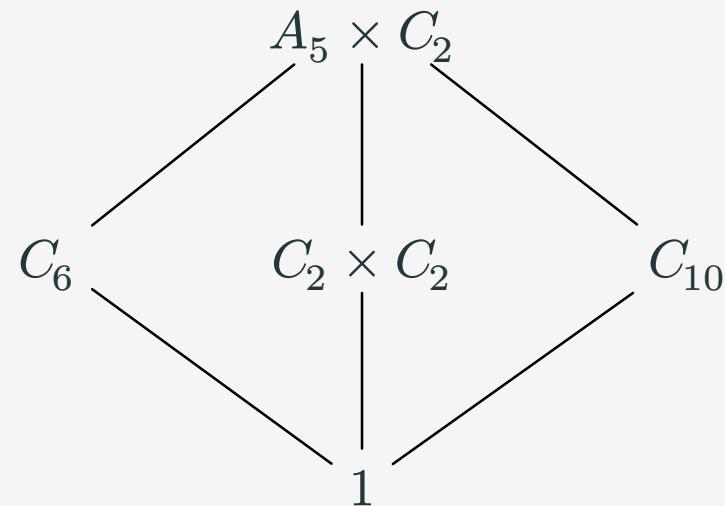
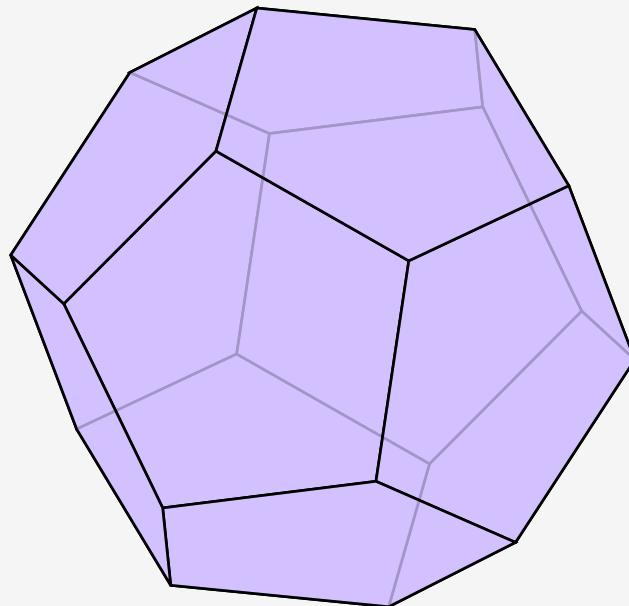
- announced in 1983
- *quasithin* case resolved in 2004 (Aschbacher & Smith)
- proof $> 10^4$ pages

The classification of finite simple groups (CFSG), \dots , is one of the monumental achievements of twentieth century mathematics.

— Terence Tao (2013)

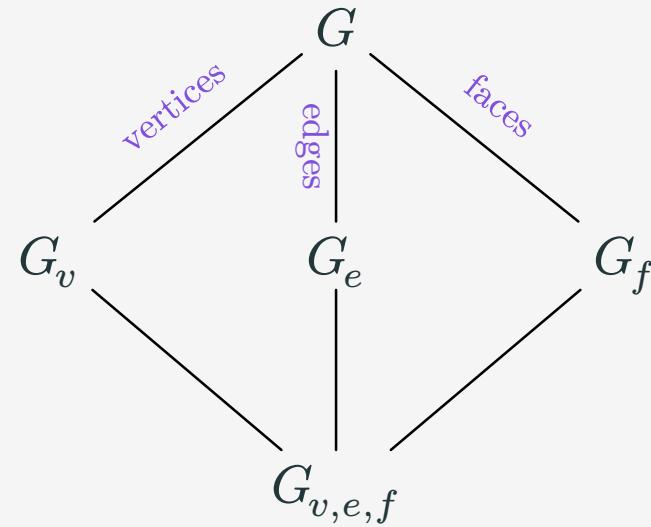
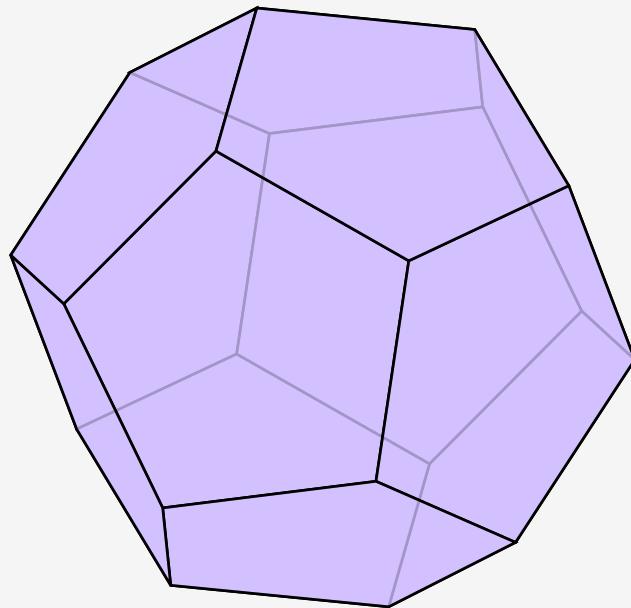
Erlangen programme

symmetries of a regular dodecahedron.



Erlangen programme

symmetries of a regular dodecahedron.

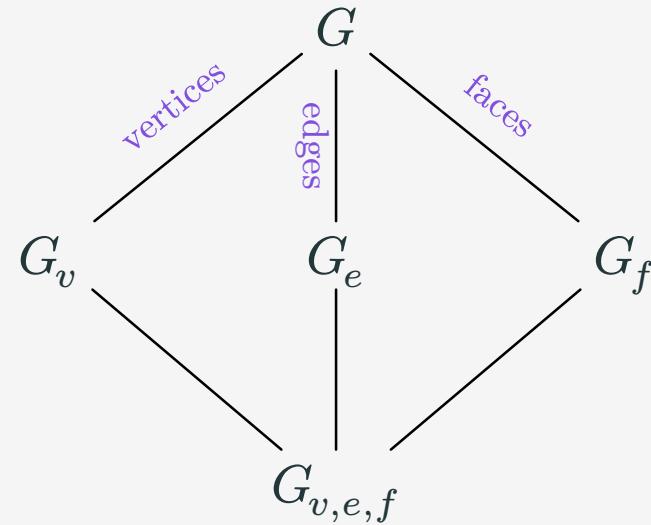
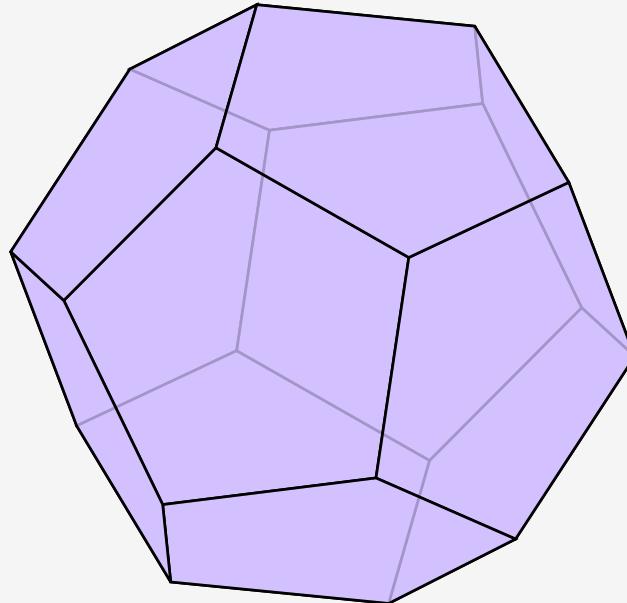


- elements are cosets

$$G_v x = \{g x : g \in G_v\}$$

Erlangen programme

symmetries of a regular dodecahedron.

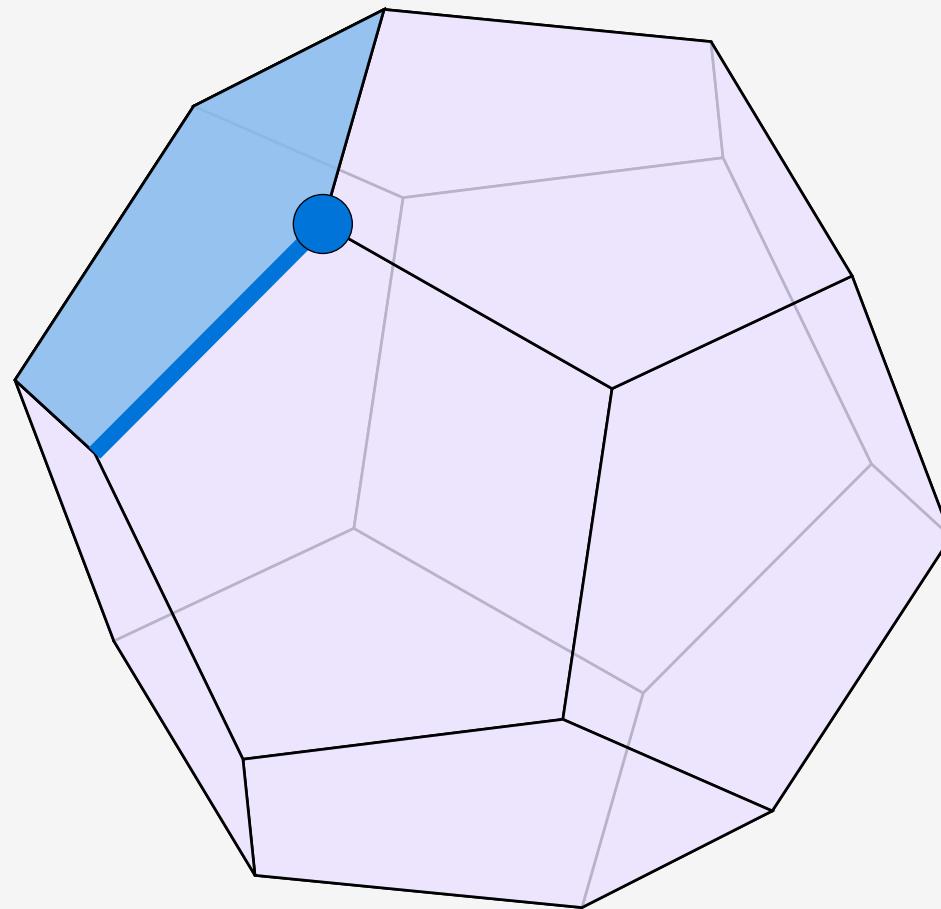


- elements are cosets
- incidence can be recaptured

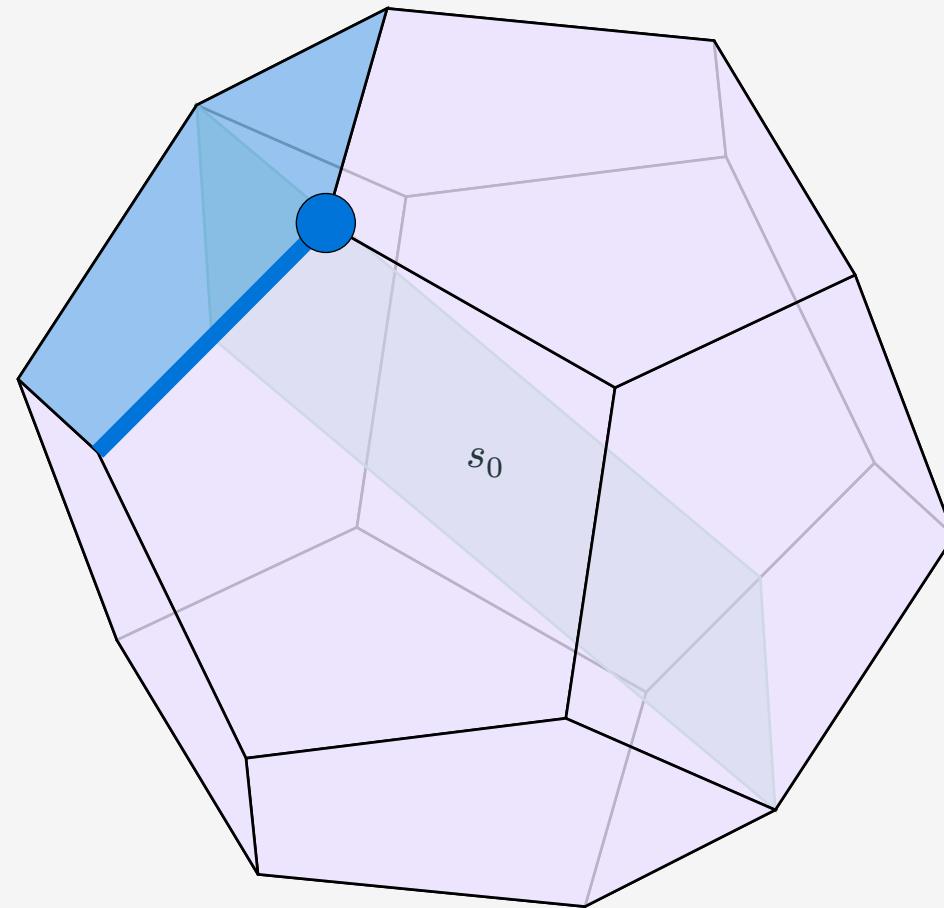
$$G_v x = \{g x : g \in G_v\}$$

$$G_v x \perp G_f y \iff G_v x \cap G_f y \neq \emptyset$$

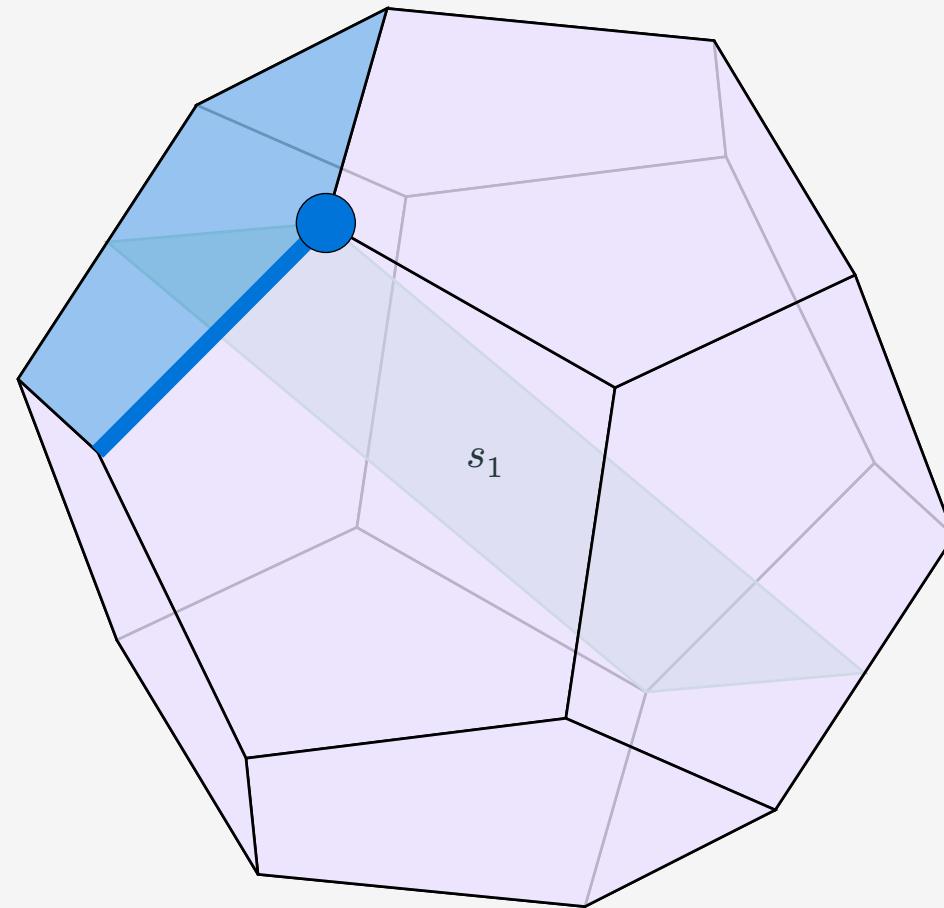
Erlangen programme



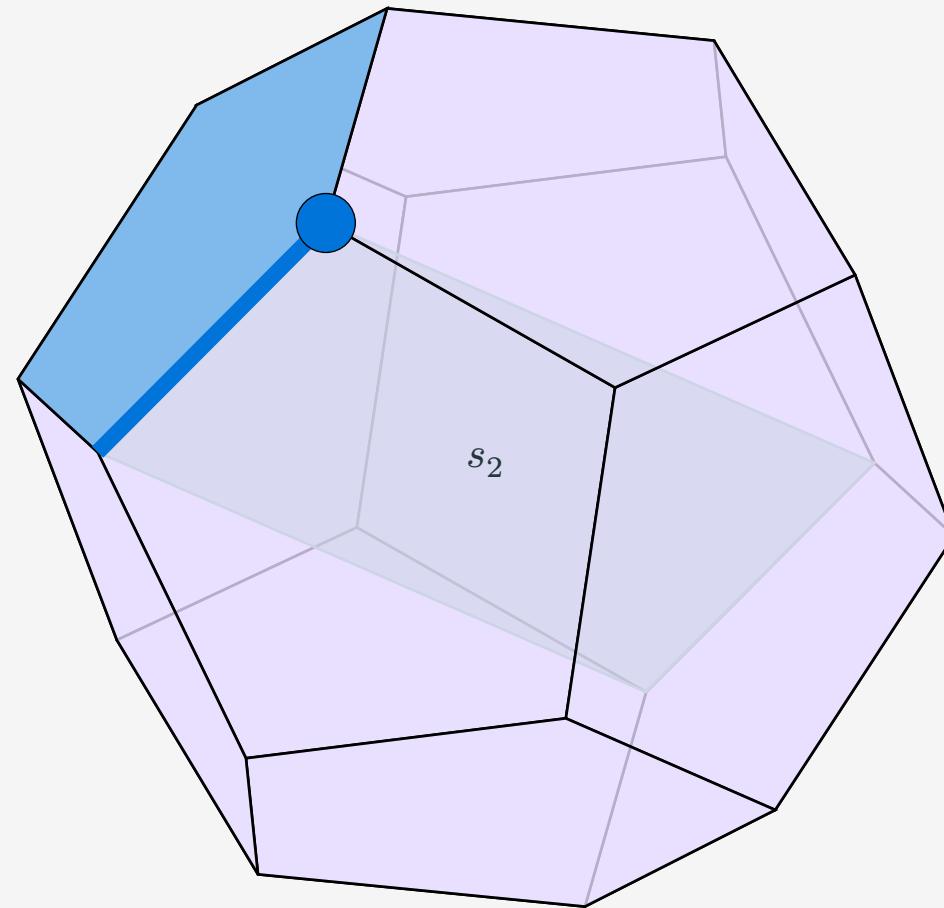
Erlangen programme



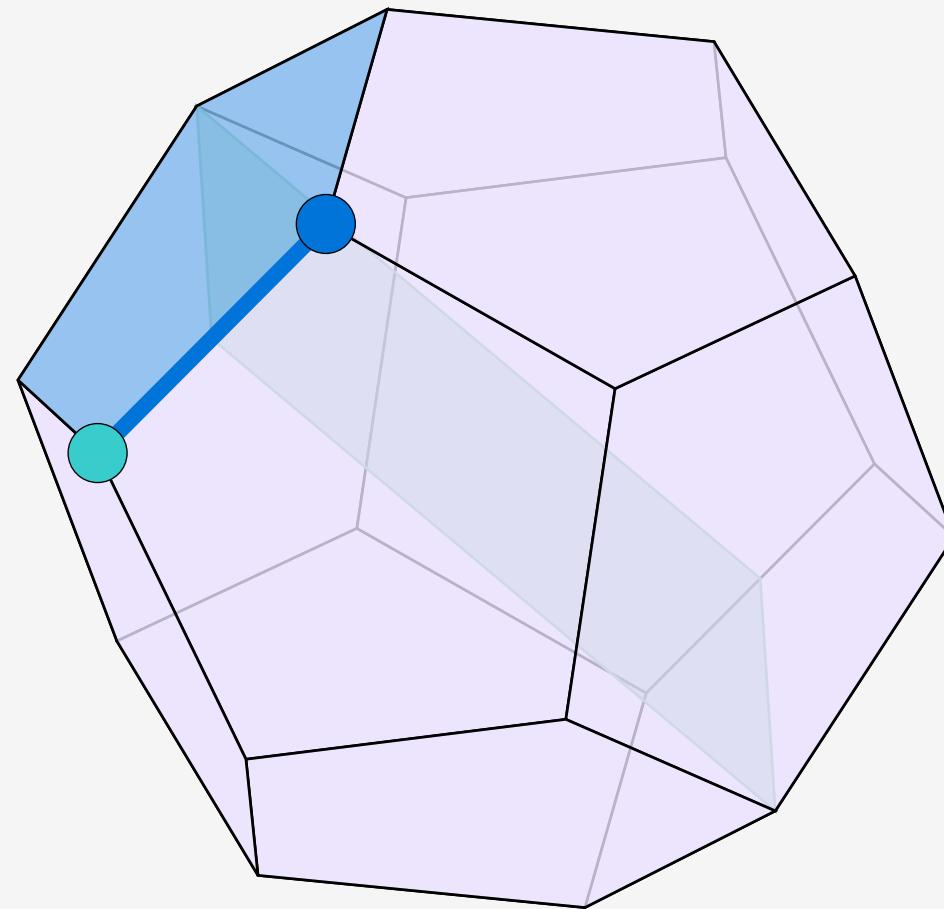
Erlangen programme



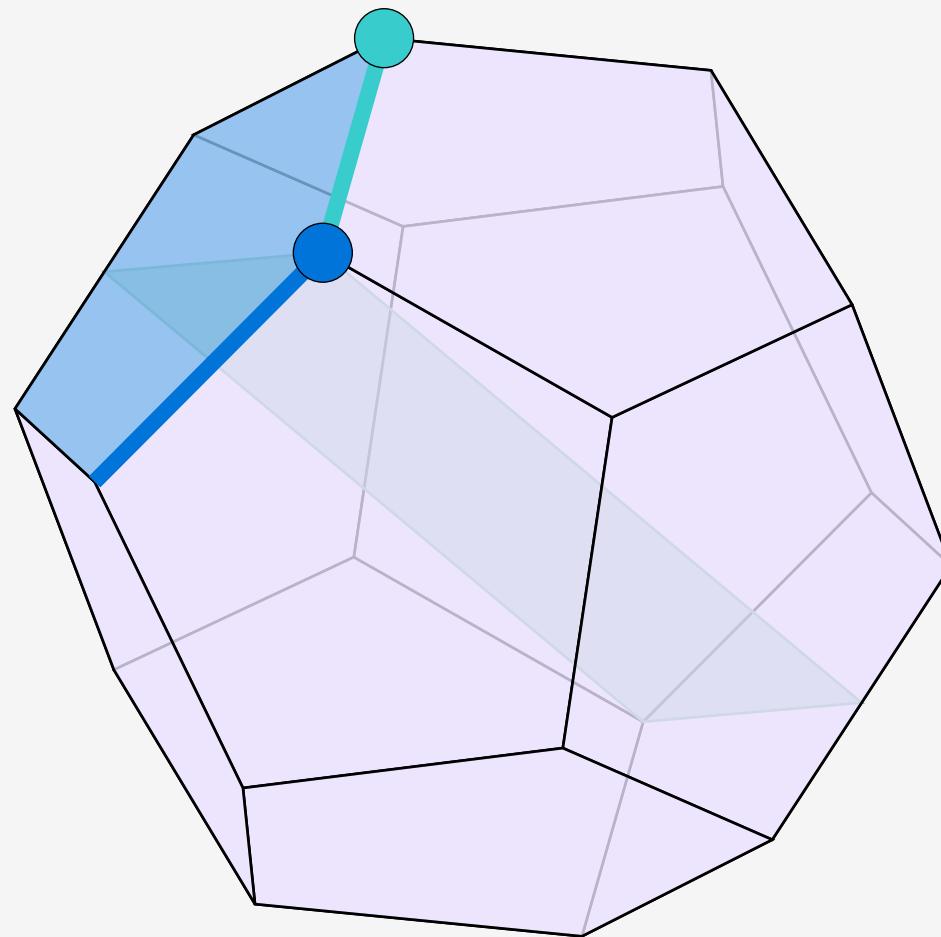
Erlangen programme



Erlangen programme

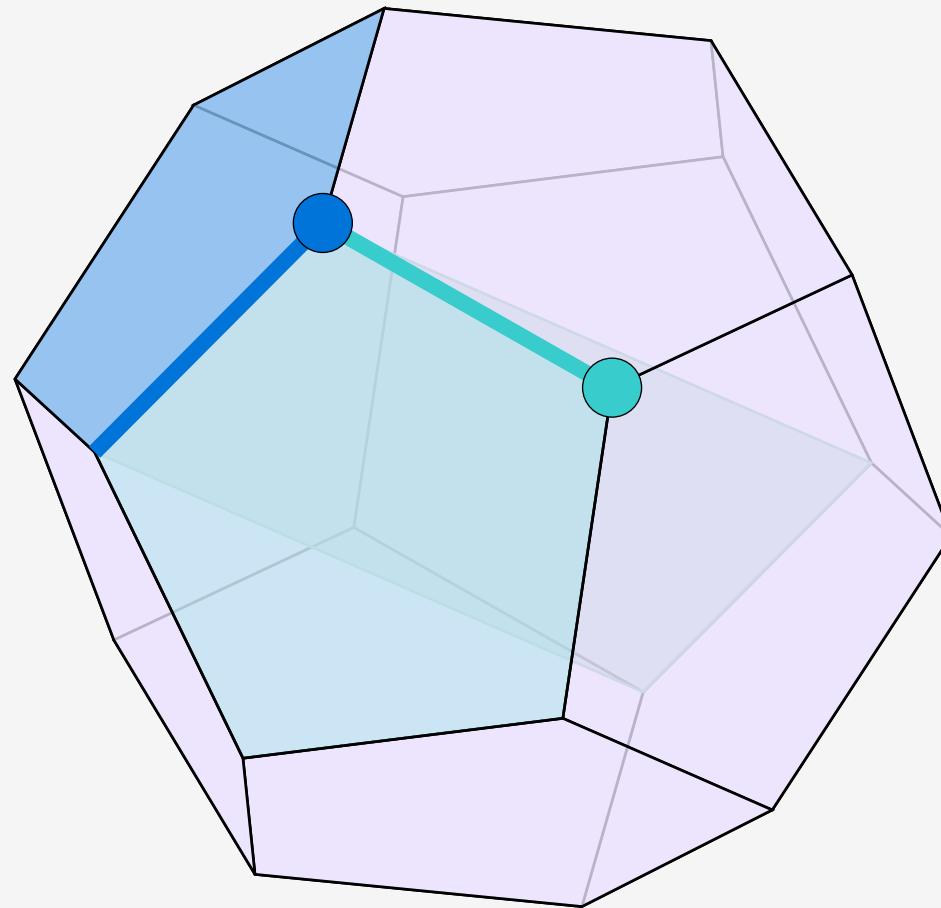


Erlangen programme



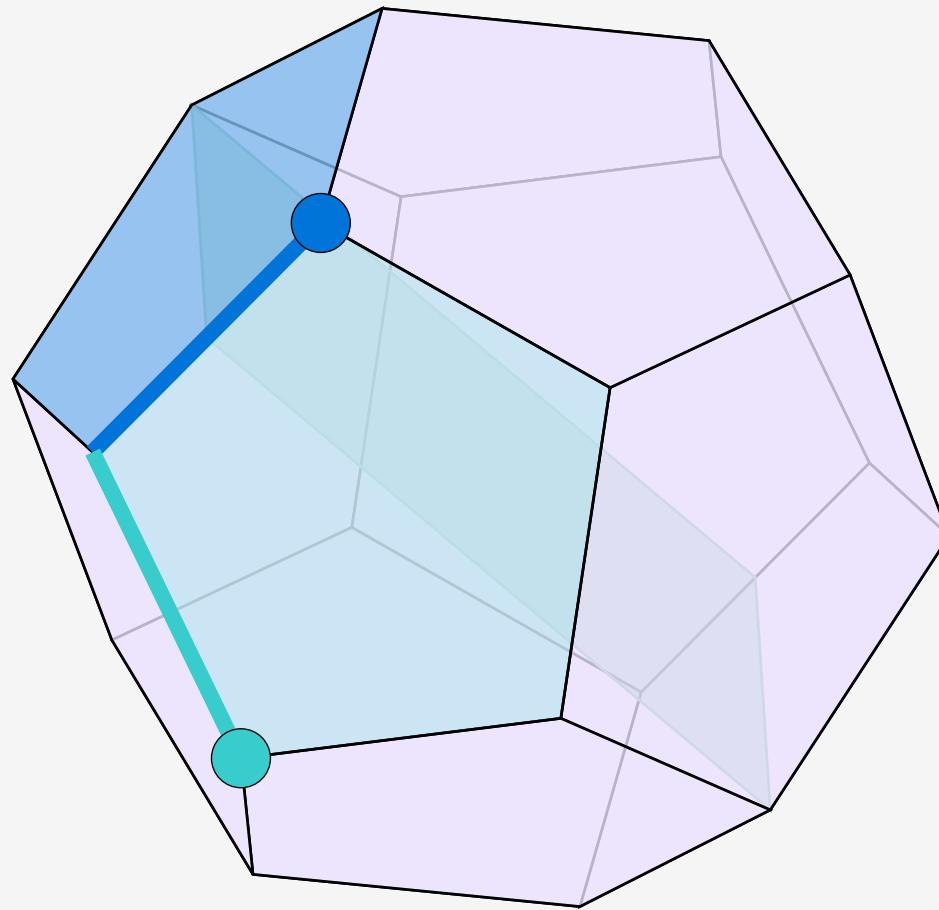
Erlangen programme

$s_0 s_1 s_2$



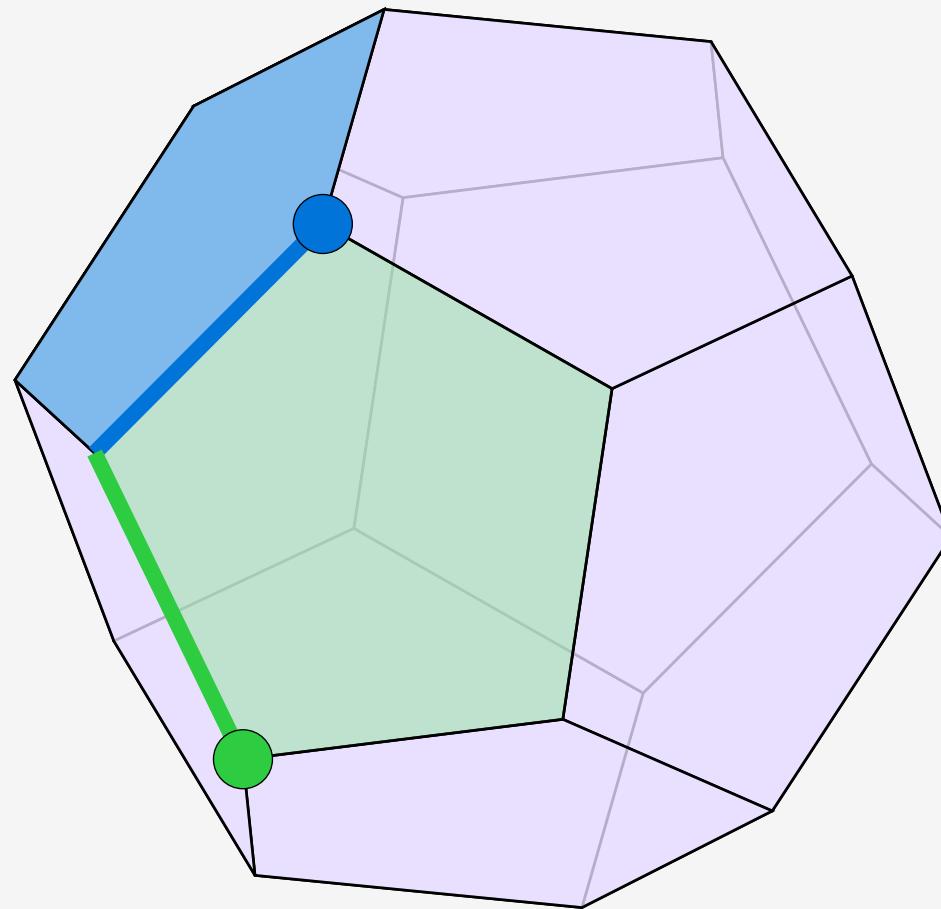
Erlangen programme

$s_0 s_1 s_2 s_0$



Erlangen programme

$$C \xrightarrow{s_0 s_1 s_2 s_0} C'$$



- **chamber (maximal flag)** $\text{vertex} \subset \text{edge} \subset \text{face}$
- **Weyl group** $W = \text{group generated by reflections } S$
- **W -distance on chambers**

$$(v, e, f) \xrightarrow{w} (v', e', f')$$

1. $C \xrightarrow{1} C' \Leftrightarrow C = C'$
2. $C \xrightarrow{w} C'$ and $C' \xrightarrow{s} C'' \Rightarrow C \xrightarrow{ws \text{ or } w} C''$
3. $C \xrightarrow{w} C'$ and $s \in S \Rightarrow (\exists C'') \quad C \xrightarrow{ws} C''$



introduced by Jacques Tits in 1959



Jacques Tits (1930 - 2021)

Ω an example from projective geometry

- vector space \mathbb{F}^4 over \mathbb{F}
- projective special linear group $\mathrm{PSL}_4(\mathbb{F})$
- $W = S_4$



| analogy | thing |
|----------|------------------------------------|
| vertices | 1-spaces $\langle v \rangle$ |
| edges | 2-spaces $\langle u, v \rangle$ |
| faces | 3-spaces $\langle u, v, w \rangle$ |

Ω an example from finite polar geometry

- \mathbb{F}_q^6 equipped with bilinear form

$$B(x, y) = x_1y_6 - x_6y_1 + x_2y_5 - x_5y_2 + x_3y_4 - x_4y_3$$

- totally isotropic subspace U :

$$B(x, y) = 0 \quad \forall x, y \in U$$

- symplectic group $\mathrm{PSp}_6(q)$
- $W = C_2 \wr S_3$



| analogy | thing |
|----------|-------------------------------|
| vertices | totally isotropic 1-spaces |
| edges | totally isotropic 2-spaces |
| faces | totally isotropic 3-spaces |

buildings

- spherical = finite Weyl group W
- rank = number of things

Jacques Tits 1974

All spherical buildings of rank at least 3 are known and are associated with simple groups of Lie type.

Antecedents

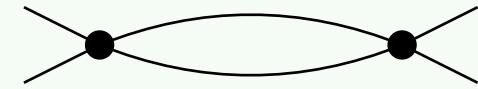
- All projective spaces of rank at least 3 arise from vector spaces. (Hilbert 1899, Vahlen, Hessenberg 1905)
- All polar spaces of rank at least 3 arise from vector spaces equipped with sesquilinear or quadratic forms. (Buekenhout & Shult 1974)

generalised polygons

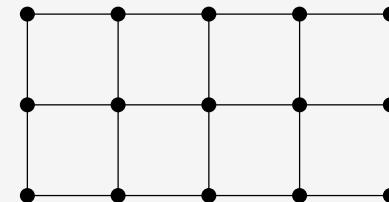
generalised n -gon

Incidence geometry of points and lines such that

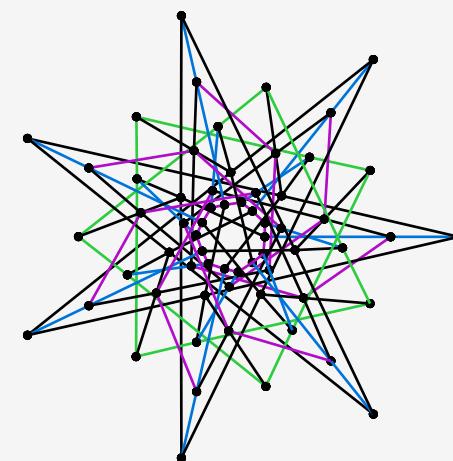
- every 2 elements lie in some ordinary n -gon
- there are no ordinary k -gons for $2 \leq k < n$



ordinary 2-gon



a generalised 4-gon



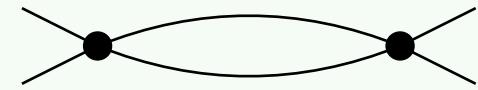
a generalised 6-gon

generalised polygons

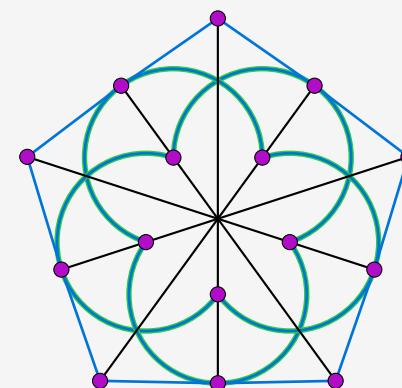
generalised n -gon

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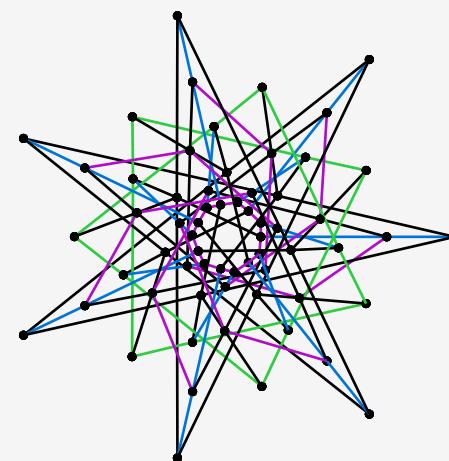
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ordinary 2-gon



a generalised 4-gon



a generalised 6-gon

⚠ Warning

We will assume our geometries are **finite** from now on.

Walter Feit & Graham Higman 1964

If every line has at least 3 points and every point lies on at least 3 lines, then

$$n \in \{3, 4, 6, 8\}.$$



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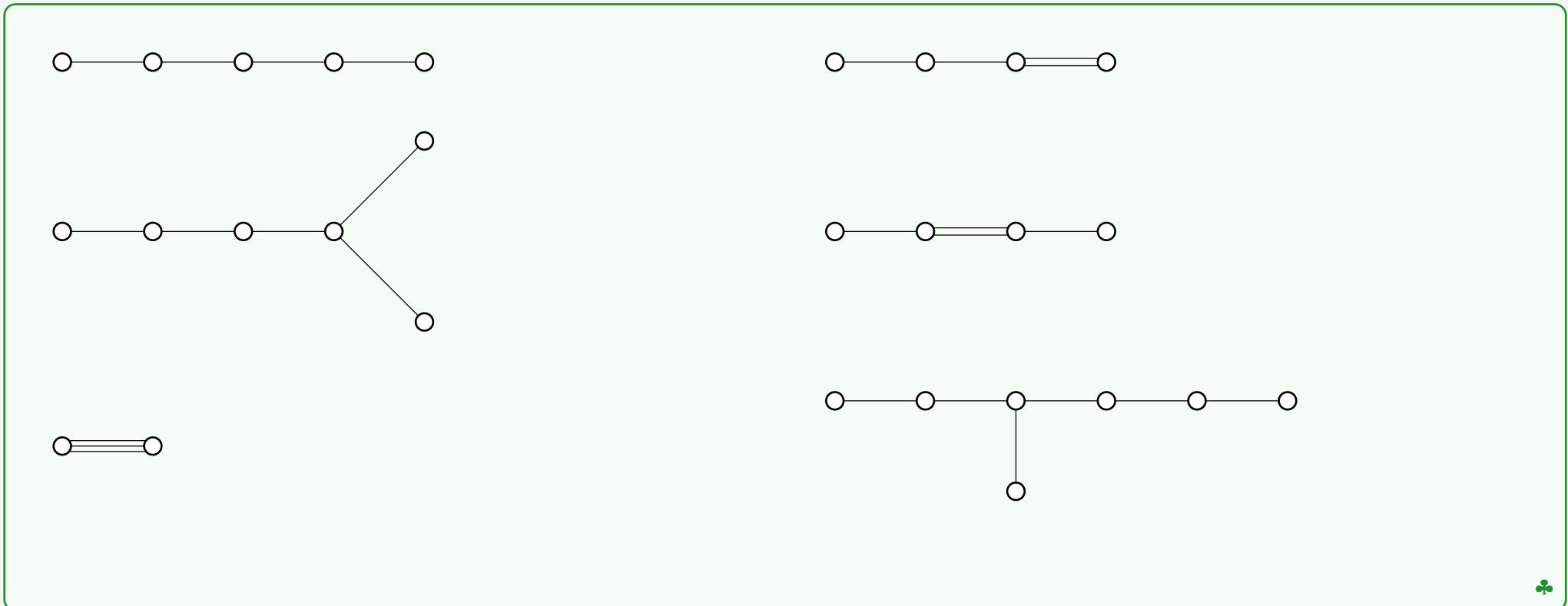
$$n \in \{3, 4, 6, 8\}.$$

Related to integers k such that $\sin^2\left(\frac{\pi}{k}\right) \in \mathbb{Q}$.



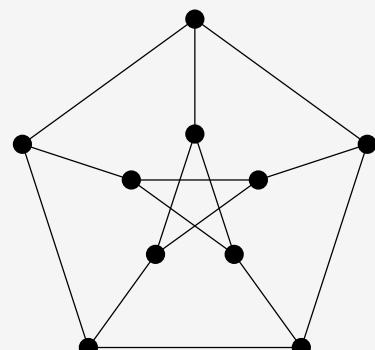
applications

- generalised n -gons are rank 2 buildings
- buildings can be broken down into generalised n -gons (residues)



The cage problem

Given $d > 1, g > 2$, find the smallest number of possible vertices that a regular graph of **degree d** and **girth g** can have.



Petersen graph: cage(3,5)

A cage(d, g) has at least

$$1 + d + d(d - 1) + \cdots + d(d - 1)^{\frac{g-3}{2}}$$

vertices

A bipartite graph meeting the bound (*Moore graph*) **is** a generalised n -gon.



Extremal combinatorics

- optimal clique-free pseudorandom graphs (Mubayi & Verstraëte¹)
- new asymptotic bounds on cycle-complete Ramsey numbers
- smallest degree of Ramsey graphs that are minimal for cliques (JB, Bishnoi, & Lesgourgues²)
- $r(4, t) = \Omega\left(\frac{t^3}{\log^4 t}\right)$ (Mattheus & Verstraëte³)



¹“A note on pseudorandom Ramsey graphs”, doi:10.4171/JEMS/1359

²“The minimum degree of minimal Ramsey graphs for cliques”, doi:10.1112/blms.12658”

³“The asymptotics of $r(4, t)$ “, doi:10.4007/annals.2024.199.2.8

User-Private Information Retrieval schemes

In such a scheme, a set of users collaborate to retrieve files from a database without revealing to observers which participant in the scheme requested the file.

- generalised quadrangles were used¹ to create a method that is substantially more secure against eavesdropping than was previously known.



They have also appeared in

- the construction of codes
- quantum error-correction
- quasisymmetric designs
- association schemes

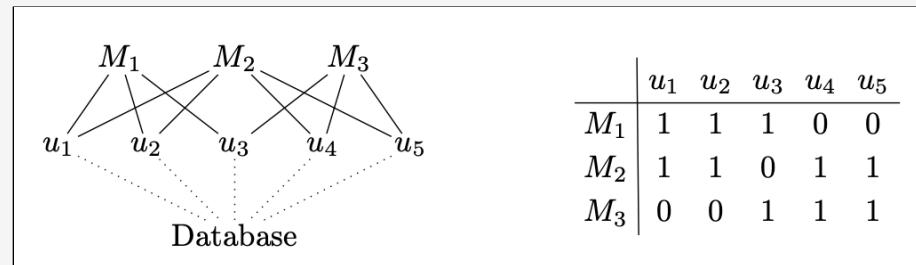


Figure 1: A visualisation of a UPIR system.

¹Gnilke, Greferath, Hollanti, Nuñez Ponasso, Ó Catháin, Swartz, “Improved user-private information retrieval via finite geometry”, doi:10.1007/s10623-018-00591-9

known examples

dual generalised n -gon: swapping the roles of points and lines.

3-gons (projective planes)

- Desarguesian (“classical”) $\rightarrow \text{PSL}_3(q)$
- lots of non-Desarguesian planes known

4-gons (generalised quadrangles)

- arising from formed vector spaces (“classical”) $\rightarrow \text{PSp}_4(q), \text{PSU}_4(q), \text{PSU}_5(q)$
- lots not arising from formed vector spaces

6-gons (generalised hexagons)

- classical $\rightarrow G_2(q), {}^3D_4(q)$

8-gons (generalised octagons)

- classical $\rightarrow {}^2F_4(q)$

known examples

Bose-Nair, Cameron, Cohen-Tits, Dixmier-Zara, Hall, Hall-Swift-Walker, Lam-Kolesova-Thiel, Lam-Thiel-Swiercz, MacInnes, Payne, Payne-Thas, Tarry, Thas

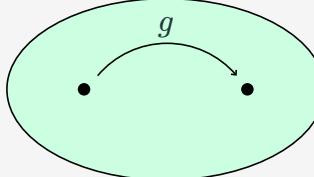
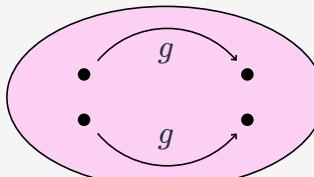
All generalised n -gons with at most 124 points have been classified.



| n | # points | # symmetries | n | # points | # symmetries |
|-----|----------|--------------|-----|----------|--------------|
| 3 | 7 | 168 | 3 | 13 | 5616 |
| 3 | 21 | 120960 | 3 | 31 | 372000 |
| 3 | 57 | 5630688 | 3 | 73 | 49448448 |
| 3 | 91 | 84913920 | 3 | 91 | 311040 |
| 3 | 91 | 33696 | 4 | 15 | 720 |
| 4 | 27 | 51840 | 4 | 40 | 51840 |
| 4 | 64 | 138240 | 4 | 85 | 1958400 |
| 4 | 112 | 26127360 | 6 | 63 | 12096 |

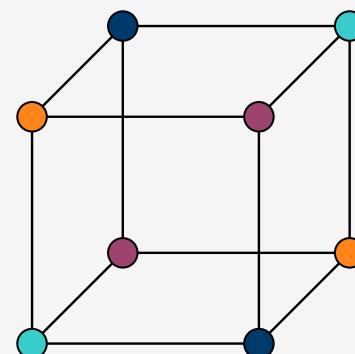
actions

G acts on Ω

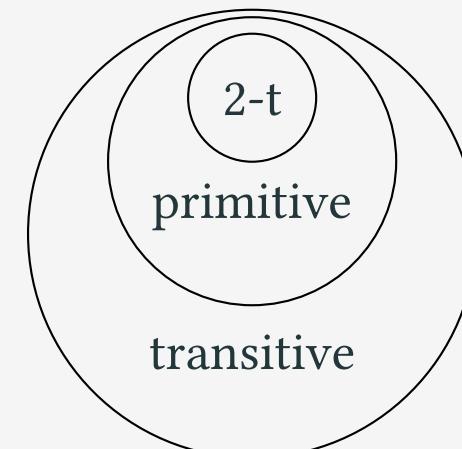
| | | |
|---------------------|--|---|
| transitive |  | $(\forall \omega, \omega' \in \Omega)(\exists g \in G) \quad \omega' = \omega^g$ |
| 2-transitive |  | $(\forall \omega_1, \omega_2, \omega'_1, \omega'_2 \in \Omega)(\exists g \in G)$ $\omega'_1 = \omega_1^g$ and $\omega'_2 = \omega_2^g$ |

- **intransitive** G preserves a subset
- **imprimitive** G is transitive and preserves a partition

symmetries of a cube (imprimitive)



Venn diagram



primitive groups are atoms for **transitive groups**

convention

We will associate an adjective for the symmetry group with the object.

1. “**point-2-transitive** projective plane” \Rightarrow
“the group of symmetries of the projective plane is **2-transitive** on **points**.”
2. “**point-primitive** and **flag-transitive** generalised quadrangle” \Rightarrow
“the group of symmetries of the generalised quadrangle is **primitive** on **points** and
transitive on flags.”

Ostrom & Wagner (1959)

A finite point-2-transitive projective plane is classical.



This was the first time 2-transitivity produced a complete classification of finite geometries. Since then the notion of a geometric classification in terms of a group-theoretic hypothesis has become commonplace. That was not the case 35 years ago, and it is a measure of these papers' influence that this type of hypothesis is now regarded as a natural extension of Klein's Erlangen program.

— Bill Kantor (1993)

Ostrom & Wagner (1959)

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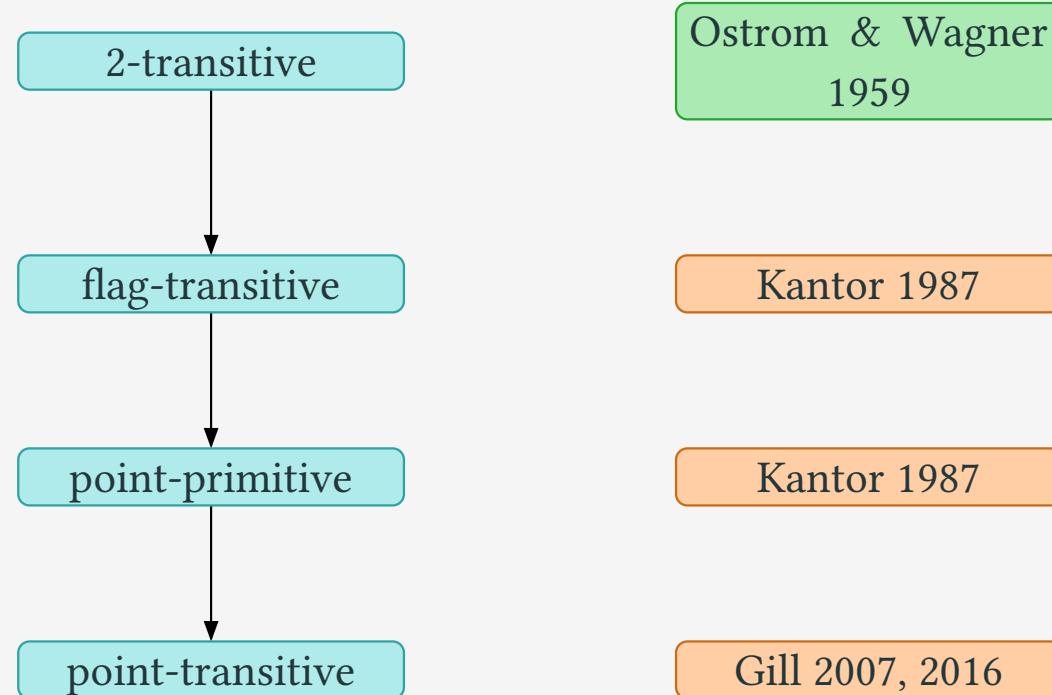
— Bill Kantor (1993)

Conjecture (Bruck - early 1950's)

A finite point-transitive projective plane is classical.



projective planes: characterisations

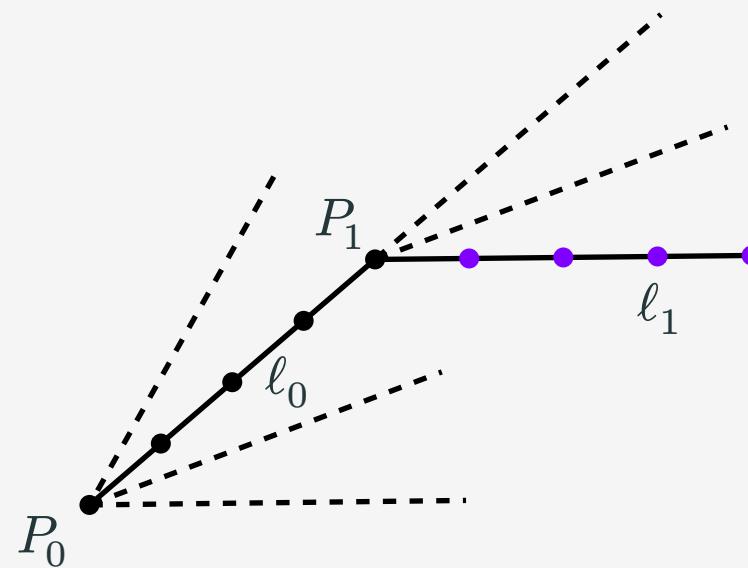


K. Thas and Zagier (2008)

A point-primitive non-classical projective plane has at least 4×10^{22} points.



general results



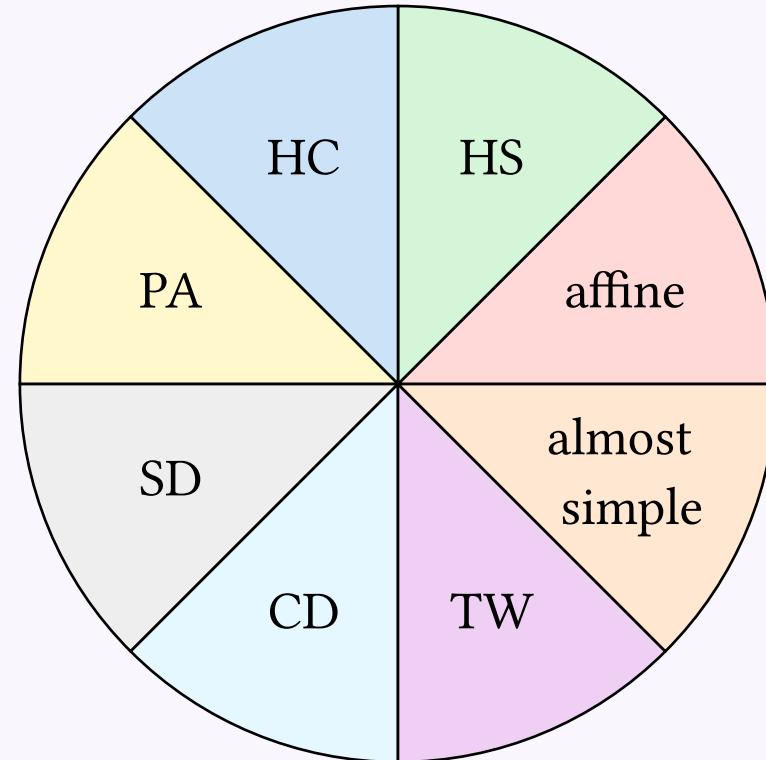
Fong & Seitz 1973/1974

A finite *Moufang* generalised polygon is classical.

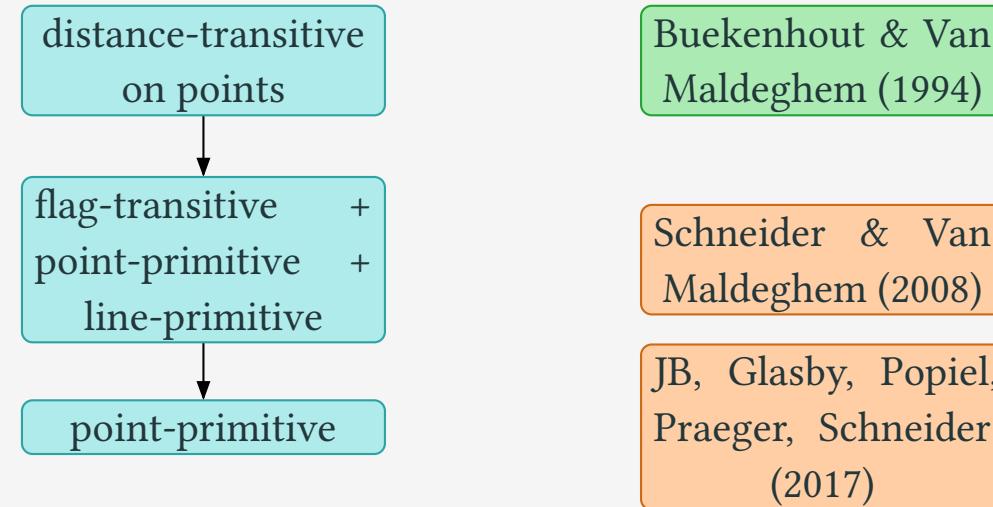


primitive groups (atoms for transitive groups)

The O'Nan-Scott Theorem for finite primitive groups



generalised hexagons/octagons: characterisations



JB, Glasby, Popiel, Praeger, Schneider (2017)

If G acts primitively on the points of a finite GH or GO, then G is almost simple.

Morgan & Popiel (2016)

(i) ${}^2B_2(q), {}^2G_2(q) \not\subset G$. (ii) ${}^2F_4(q) \subset G \Rightarrow$ known example.

generalised quadrangles: a sporadic example

- first constructed by Ronald Ahrens and George Szekeres (UNSW) in 1969
- 64 points, 96 lines
- *link to demo*
- flag-transitive, points-distance transitive, line-distance intransitive, Moufang
- point-primitive, line-imprimitive

ON A COMBINATORIAL GENERALIZATION OF 27 LINES ASSOCIATED WITH A CUBIC SURFACE

R. W. AHRENS and G. SZEKERES

(Received 26 March 1969)

To Bernhard Hermann Neumann on his 60th birthday

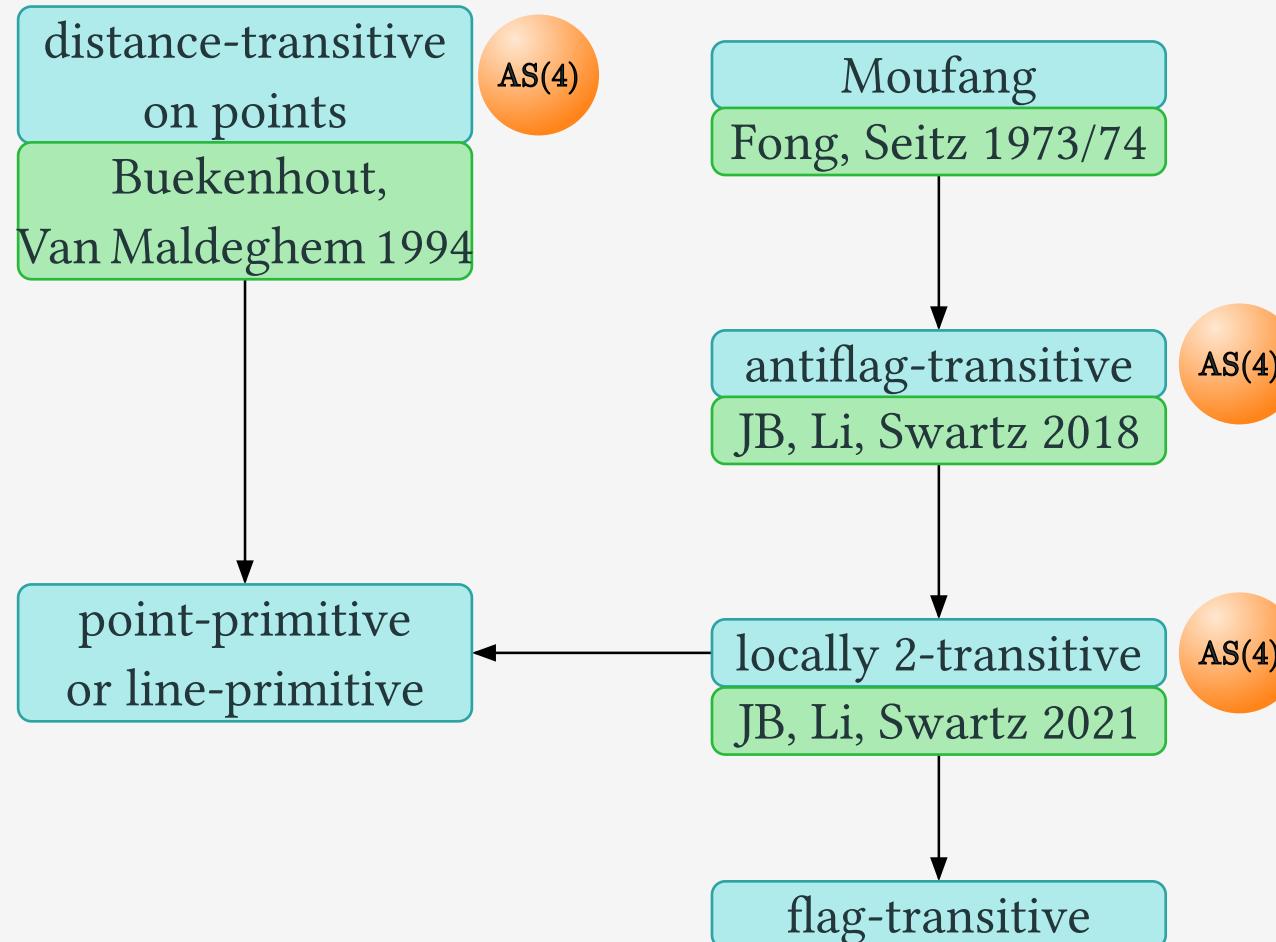
Communicated by G. B. Preston

1.

Given integers $0 < \lambda < k < v$, does there exist a nontrivial graph G with the following properties: G is of order v (i.e. has v vertices), is regular of degree k (i.e. every vertex is adjacent to exactly k other vertices), and every pair of vertices is adjacent to exactly λ others? Two vertices are said to be adjacent if they are connected by an edge. We call a graph with the above properties a symmetric (v, k, λ) graph and refer to the last of the properties as the λ -condition. The complete graph of order v is a trivial example of a symmetric $(v, v-1, v-2)$ graph, but we are of course only interested in non-trivial constructions.

All graphs will be assumed to have no loops or double edges. If the vertices of G are denoted by x_1, \dots, x_v and S_i , $i = 1, \dots, v$ denotes the set of vertices which are adjacent to x_i , then the sets S_i form a symmetric block design with parameters (v, k, λ) i.e. $|S_i| = k$, $|S_i \cap S_j| = \lambda$ for $i \neq j$. Thus the parameters v, k, λ must satisfy the Bruck-Ryser-Chowla conditions ([2], p. 107) in order that such a graph should exist. However these conditions

generalised quadrangles: characterisations



Conjecture (Kantor 1991)

A finite flag-transitive generalised quadrangle is classical, AS(4), or the Lunelli-Sce quadrangle.



generalised quadrangles: characterisations

Conjecture (Kantor 1991)

A finite flag-transitive generalised quadrangle is classical, AS(4), or the Lunelli-Sce quadrangle.



Conjecture

A finite flag-transitive generalised quadrangle is point-primitive or line-primitive.



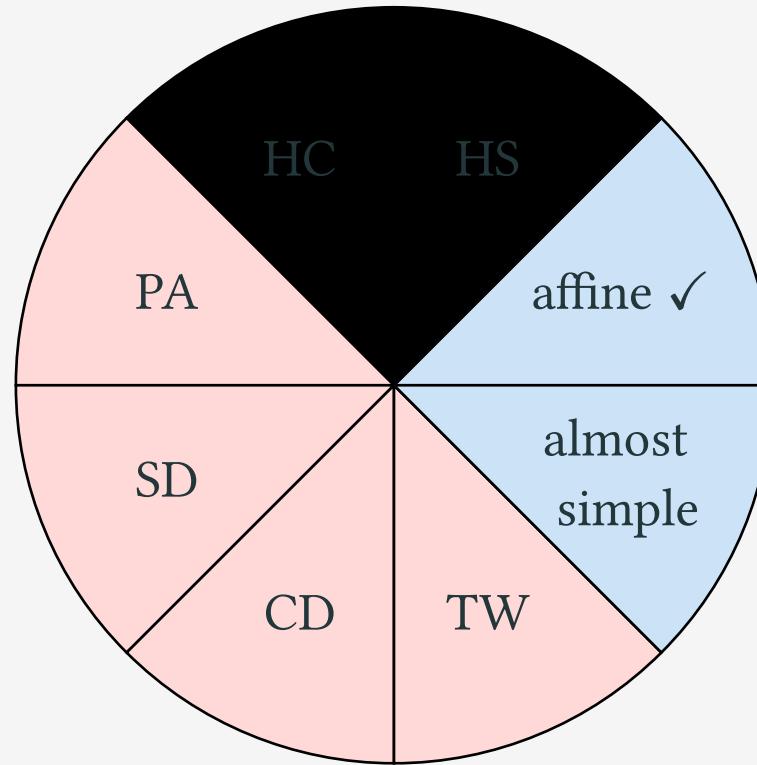
JB, Giudici, Morris, Royle, Spiga (2012)

- G primitive on points and lines of a GQ $\Rightarrow G$ is almost simple.
- If G is also flag-transitive, then G is of Lie type.



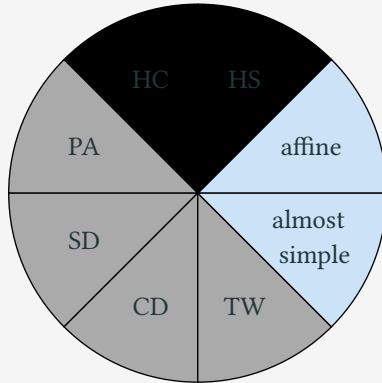
go to conclusion

point-primitive and line-transitive



- JB, Glasby, Popiel, Praeger, *J. Comb. Des.* (2017)
- JB, Popiel, Praeger, *J. Group Theory* (2017)

just point-primitive



Di (2025+)

HS does not arise.



JB, Popiel, Praeger (Nagoya Math. 2019)

- When G is of type PA, SD, CD, TW, the structure of G is heavily restricted.
- Let θ be a nonidentity symmetry of a GQ. Then θ fixes less than $v^{\frac{4}{5}}$ of v points.



exception: unique GQ on 27 points has θ fixing 15.

conclusion

recent results on almost simple type: $S \triangleleft G \leqslant \text{Aut}(S)$

| author (yr) | condition | conclusion |
|--|--|-------------------------|
| JB, Giudici, Morris, Royle, Spiga (2012) | point-primitive, flag-transitive, $S = A_n$ | $n = 6$, known example |
| JB, Evans (2021) | point-primitive, S sporadic | \emptyset |
| Feng, Lu (2023) | point-primitive, line-primitive, $S = \text{PSL}_2(q)$ | $q = 9$, known example |
| Lu, Zhang, Zou (2024) | point-primitive, line-primitive, $S = \text{PSU}_3(q), q \geq 3$ | \emptyset |
| Arumugam, JB, Giudici, (2025) | point-primitive, line-primitive, $S \in \{{}^2B_2(q), {}^2G_2(q)\}$ | \emptyset |
| Arumugam (2026+) | point-primitive, line-primitive, $S = {}^2F_4(q)$ | \emptyset |

conclusion

- local symmetry conditions

Feng and K. Thas (arXiv)

Suppose G acts on a thick generalised quadrangle. If for every point P , the group G_P is not faithful on the points collinear with P , then it is classical.

- geometric way to study symmetry
 - ▶ can we simplify the proof of the CFSG?
- classify geometric objects satisfying a symmetry condition
 - ▶ flag-transitive, point-primitive?
- imposing symmetry allows us to construct new geometric objects

conclusion

“At least two decades have passed since the discovery of a generalised quadrangle with new parameters
... A generalised quadrangle with new parameters would be of much interest.”

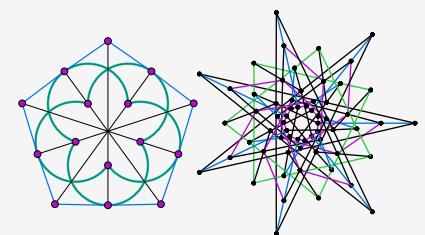
— Simeon Ball (2015)

Open problems

1. Construction of non-classical finite generalised hexagons or octagons.
2. Classification of finite flag-transitive generalised polygons.
3. Classification of finite point-primitive generalised polygons.
4. Are all finite point-transitive projective planes classical?

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conclusion

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