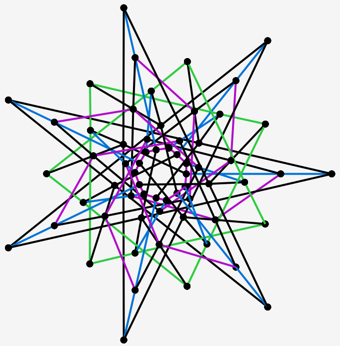
Finite generalised polygons: the final frontier in the classification of spherical buildings



Mathematics: for all

John Bamberg

2025-12-11

The University of Western Australia

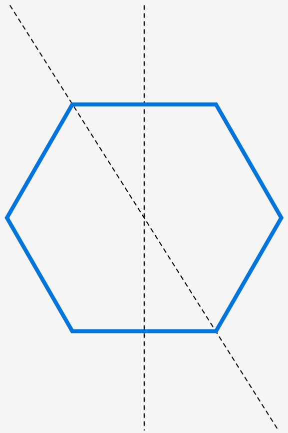
Group

Set G equipped with a binary operation $*$ such that

1. $*$ is associative $\qquad (a * b) * c = a * (b * c)$
2. exists identity e $\qquad g * e = g = e * g$
3. exists inverses $\qquad g * g^{-1} = e = g^{-1} * g$

♣

symmetries of a 6-gon



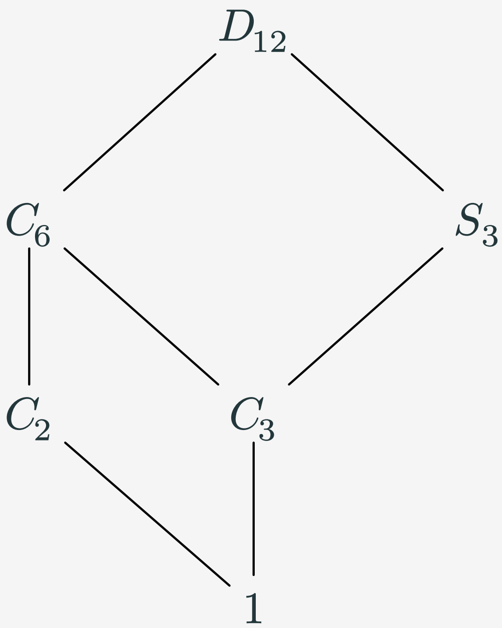
$n \times n$ invertible matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$$

groups can be broken down into indivisible pieces

- normal subgroup N of $G \to$ quotient group G/N
- composition factors … intervals of **simple** quotients

symmetries of a 6-gon



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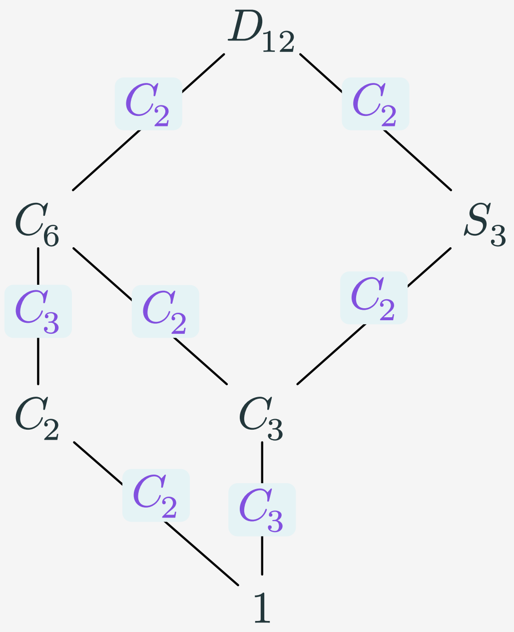

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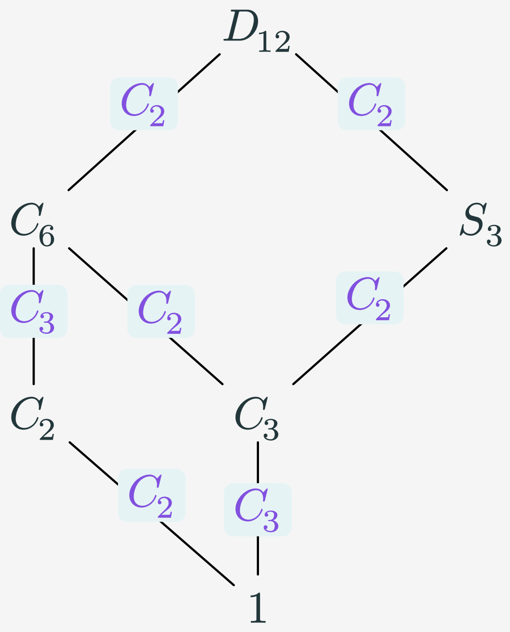
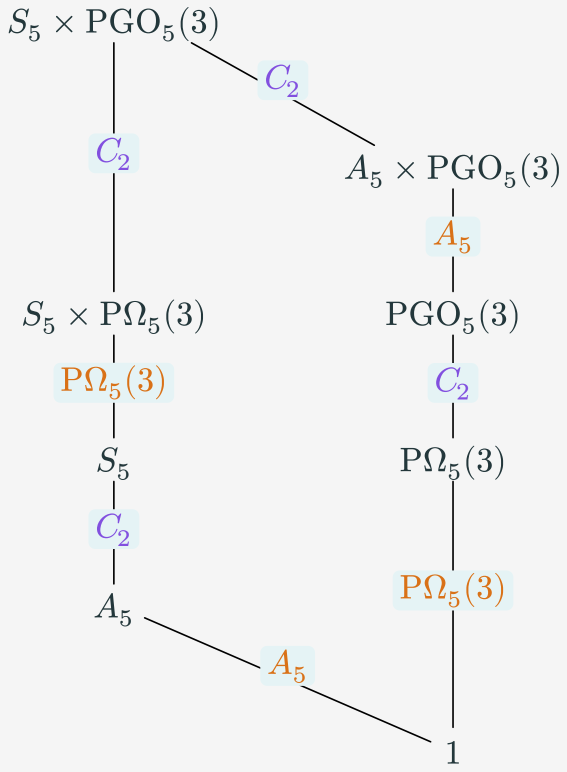


more elaborate



Camille Jordan (1870), Otto Hölder (1889)

The composition factors of a finite group are unique up to reordering.

$$\begin{tikzcd}
S_5 \times \mathrm{PGO}_5(3) \arrow[dd, dash, "C_2"] \arrow[dr, dash, "C_2"] & \\
 & A_5 \times \mathrm{PGO}_5(3) \arrow[d, dash, "A_5"] \\
S_5 \times \mathrm{P\Omega}_5(3) \arrow[d, dash, "\mathrm{P\Omega}_5(3)"] & \mathrm{PGO}_5(3) \arrow[d, dash, "C_2"] \\
S_5 \arrow[d, dash, "C_2"] & \mathrm{P\Omega}_5(3) \arrow[dd, dash, "\mathrm{P\Omega}_5(3)"] \\
A_5 \arrow[dr, dash, "A_5"] & \\
 & 1
\end{tikzcd}$$

$$\{C_2, C_2, A_5, \mathrm{P\Omega}_5(3)\}$$

Classification of Finite Simple Groups

The finite simple groups fall into the following classes:

- cyclic of prime order
- alternating groups
- Lie type $\mathrm{PSL}_n(q), \mathrm{P}\Omega_n(q), \ldots, G_2(q), E_6(q), \ldots, {}^2F_4(q), {}^3D_4(q)$
- 26 sporadic groups $M_{11}, J_3, \mathrm{Co}_2, \ldots, \mathrm{Suz}, \mathrm{Ru}, \mathbb{M}$

Classification of Finite Simple Groups

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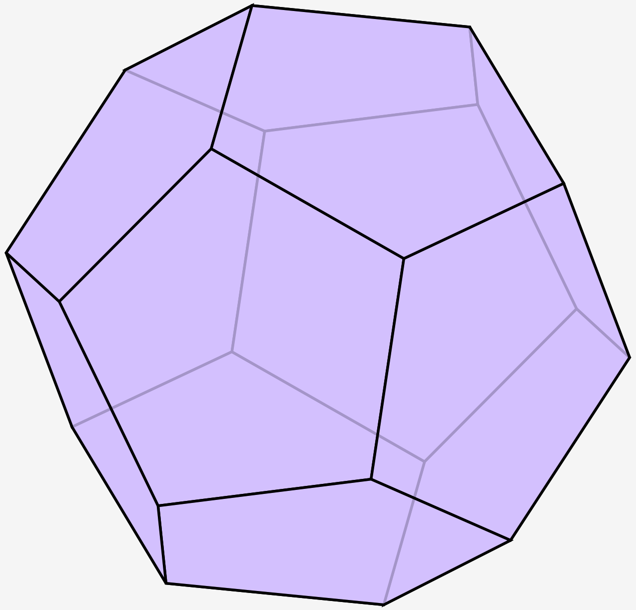
- 26 sporadic groups $M_{11}, J_3, \mathrm{Co}_2, \ldots, \mathrm{Suz}, \mathrm{Ru}, \mathbb{M}$

- announced in 1983
- *quasithin* case resolved in 2004 (Aschbacher & Smith)
- proof $> 10^4$ pages

> The classification of finite simple groups (CFSG), … , is one of the monumental achievements of twentieth century mathematics.
>
> — Terence Tao (2013)

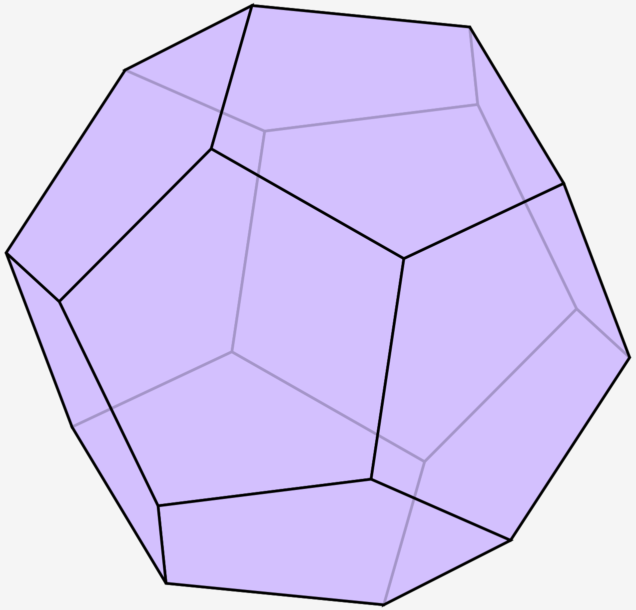
Erlangen programme

symmetries of a regular dodecahedron.



$$\begin{tikzcd}[column sep=small]
 & A_5 \times C_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
C_6 \arrow[dr, dash] & C_2 \times C_2 \arrow[d, dash] & C_{10} \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}$$

symmetries of a regular dodecahedron.

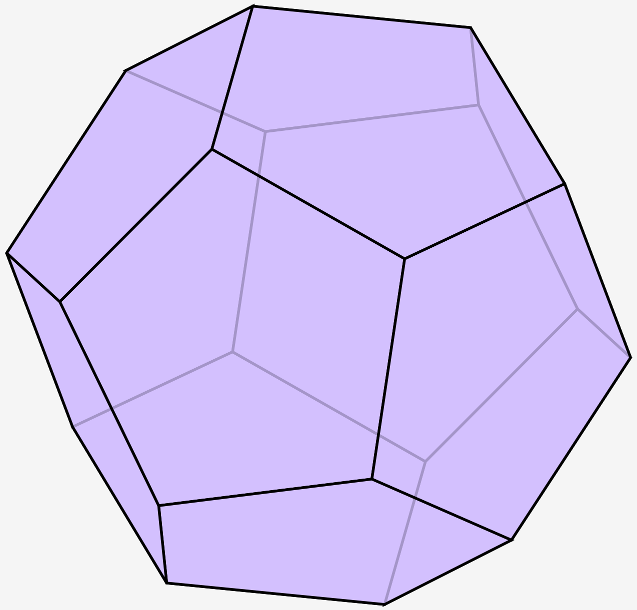


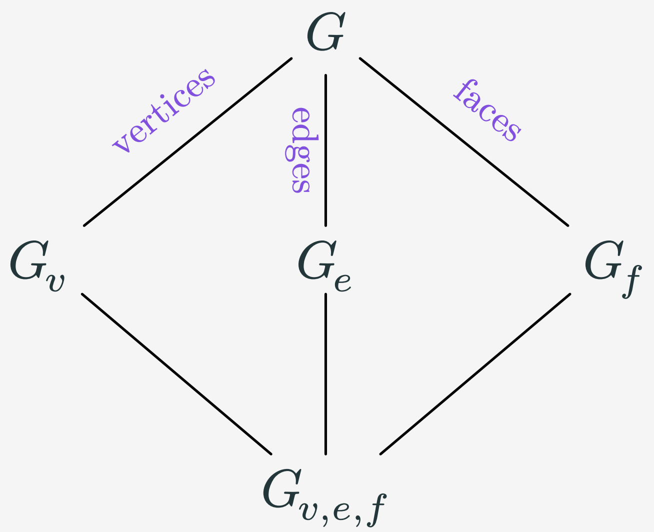
$$\begin{tikzcd}
& G \arrow[dl, dash, "\text{vertices}"'] \arrow[d, dash, "\text{edges}"'] \arrow[dr, dash, "\text{faces}"] & \\
G_v \arrow[dr, dash] & G_e \arrow[d, dash] & G_f \arrow[dl, dash] \\
& G_{v,e,f} &
\end{tikzcd}$$

- elements are cosets

$$G_v x = \{g\,x : g \in G_v\}$$

symmetries of a regular dodecahedron.



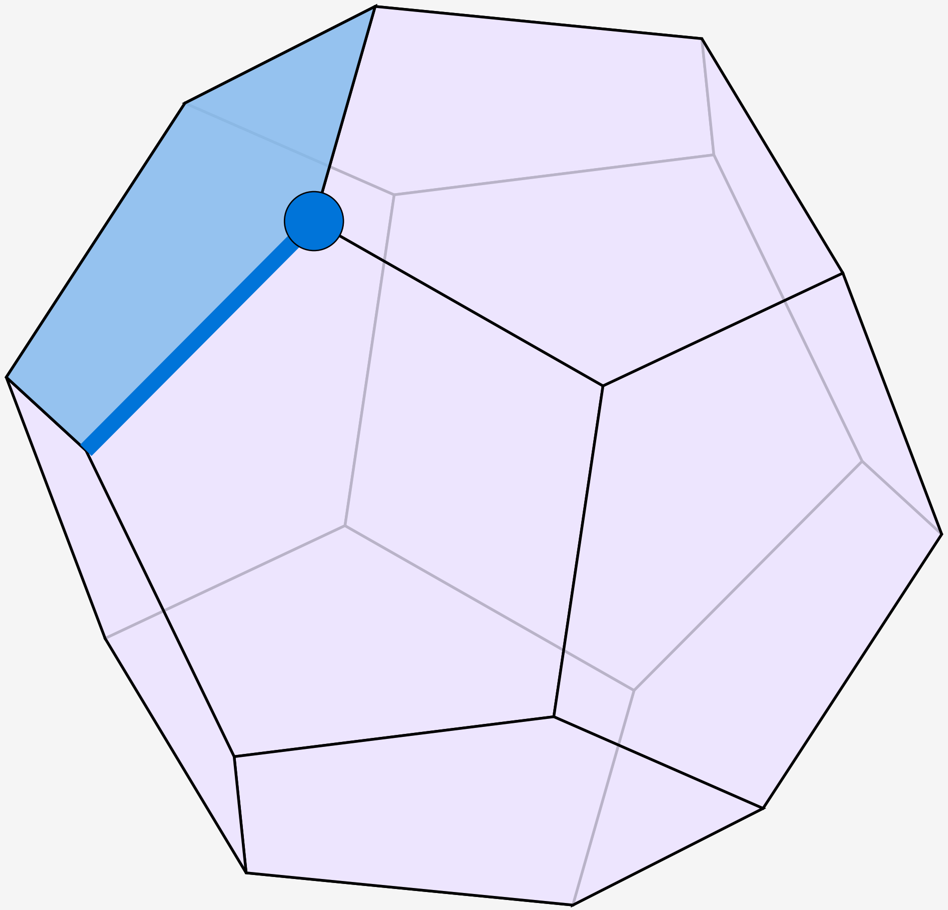


- elements are cosets
- incidence can be recaptured

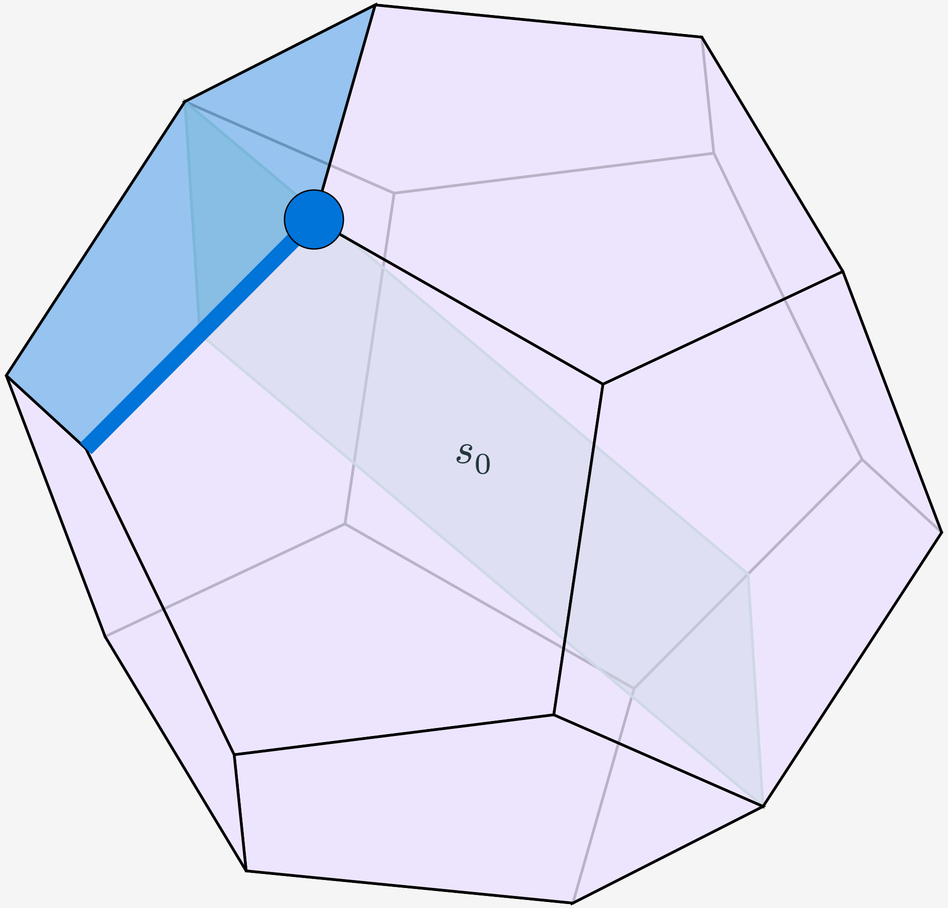
$$G_v x = \{g\,x : g \in G_v\}$$

$$G_v x \,\mathrm{I}\, G_f y \quad \Leftrightarrow \quad G_v x \cap G_f y \neq \varnothing$$

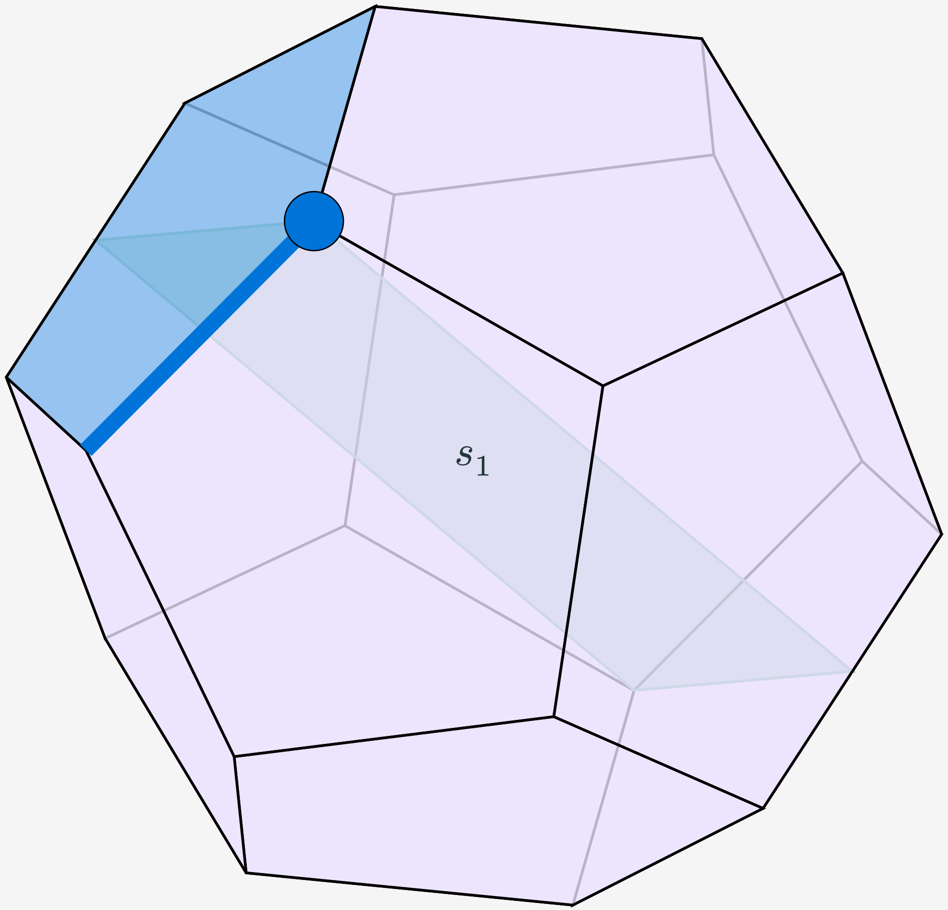
Erlangen programme



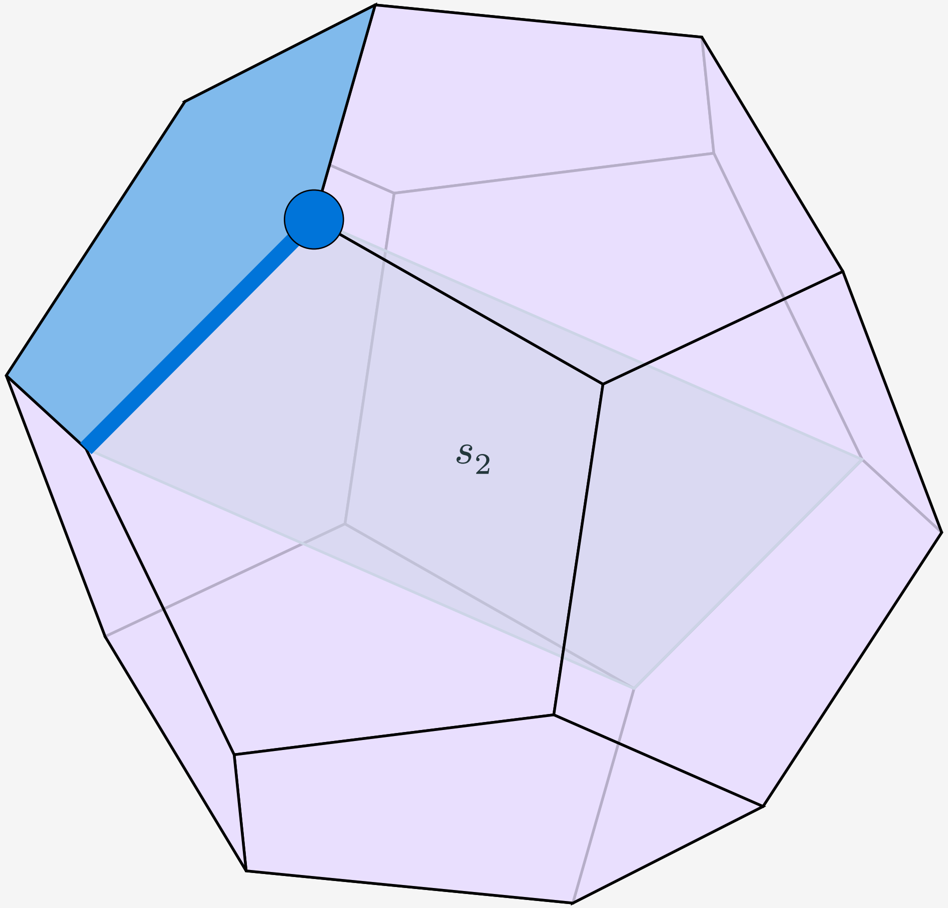
Erlangen programme



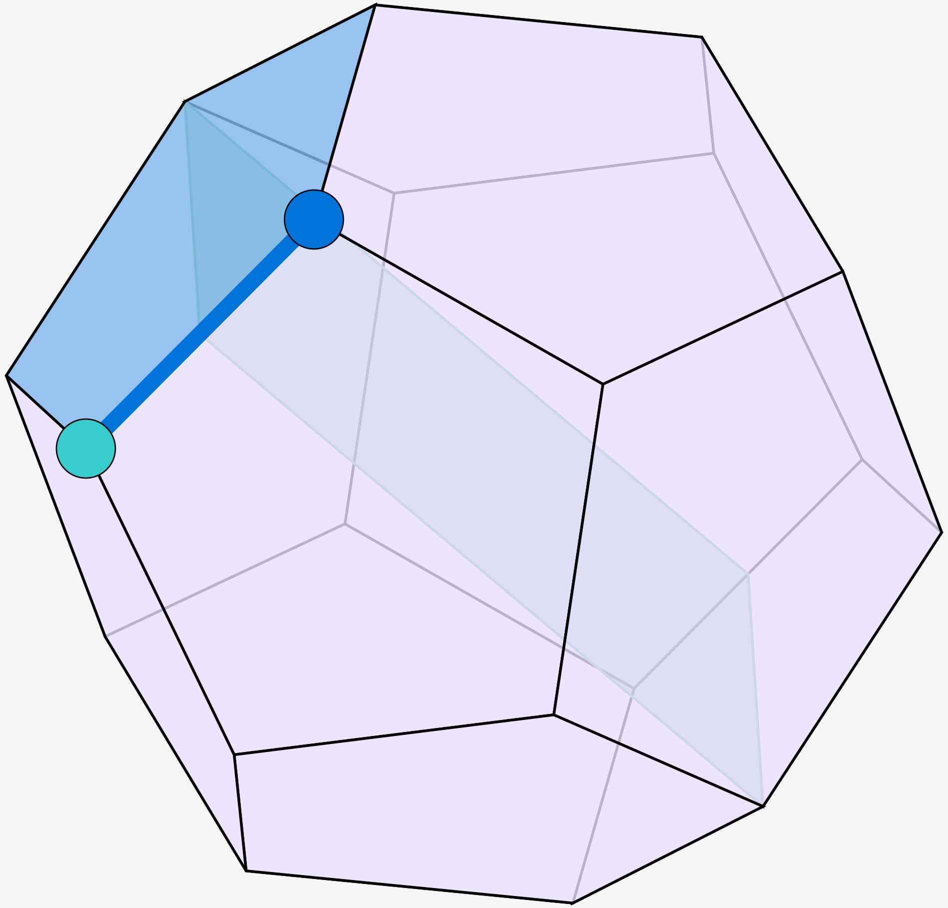
Erlangen programme



Erlangen programme

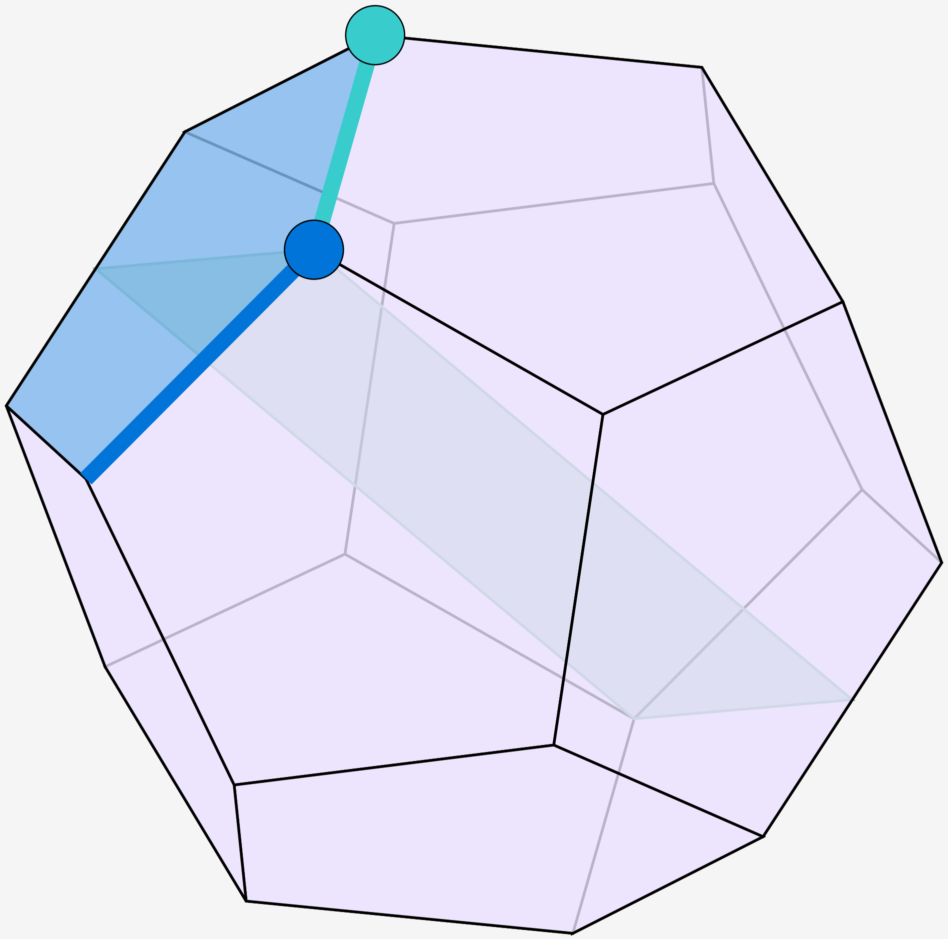


s_0



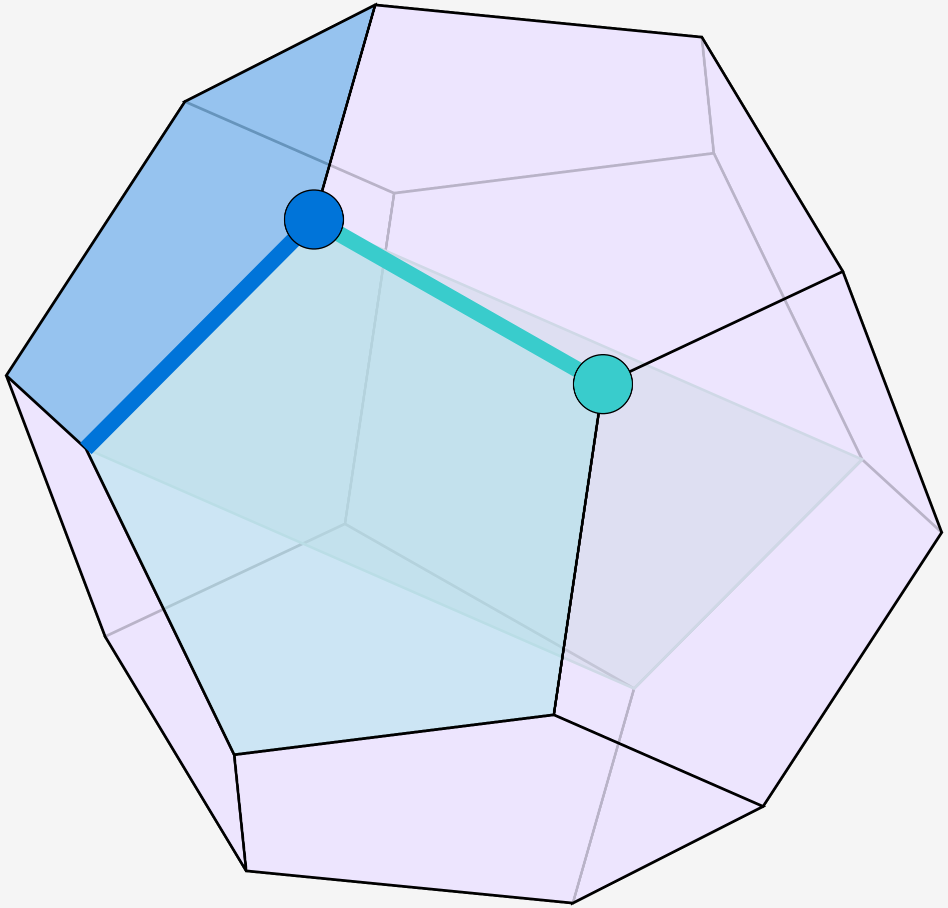
Erlangen programme

$s_0 s_1$



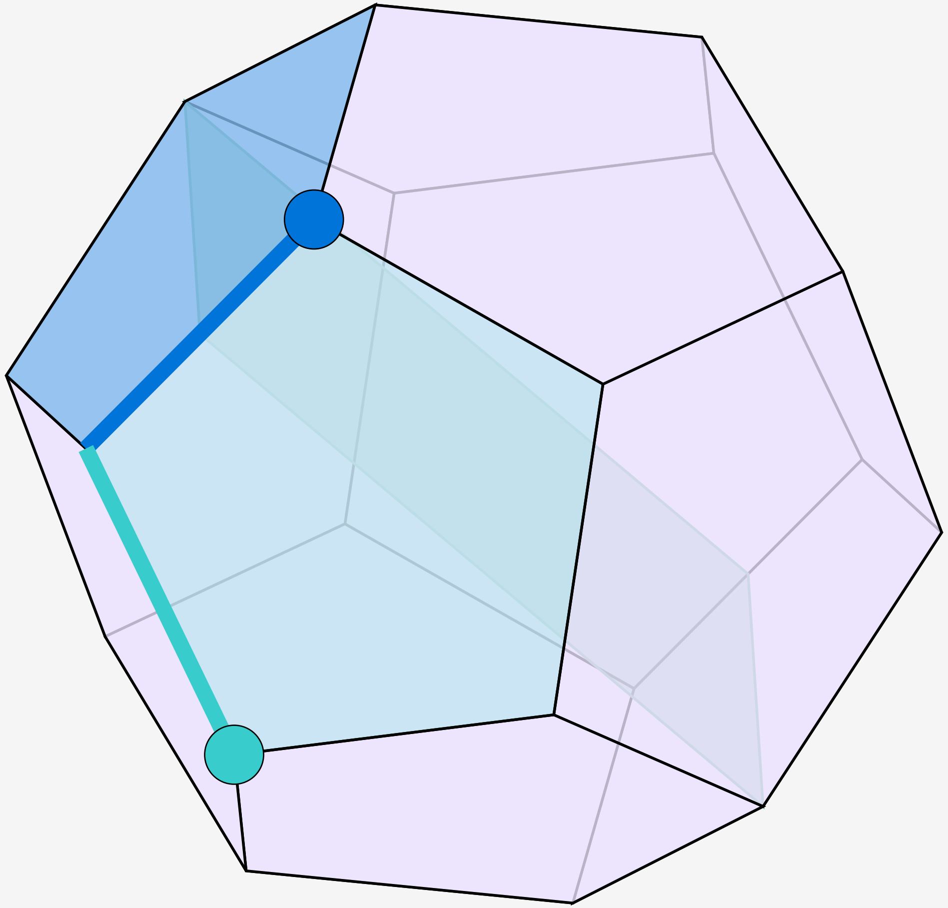
Erlangen programme

$s_0 s_1 s_2$



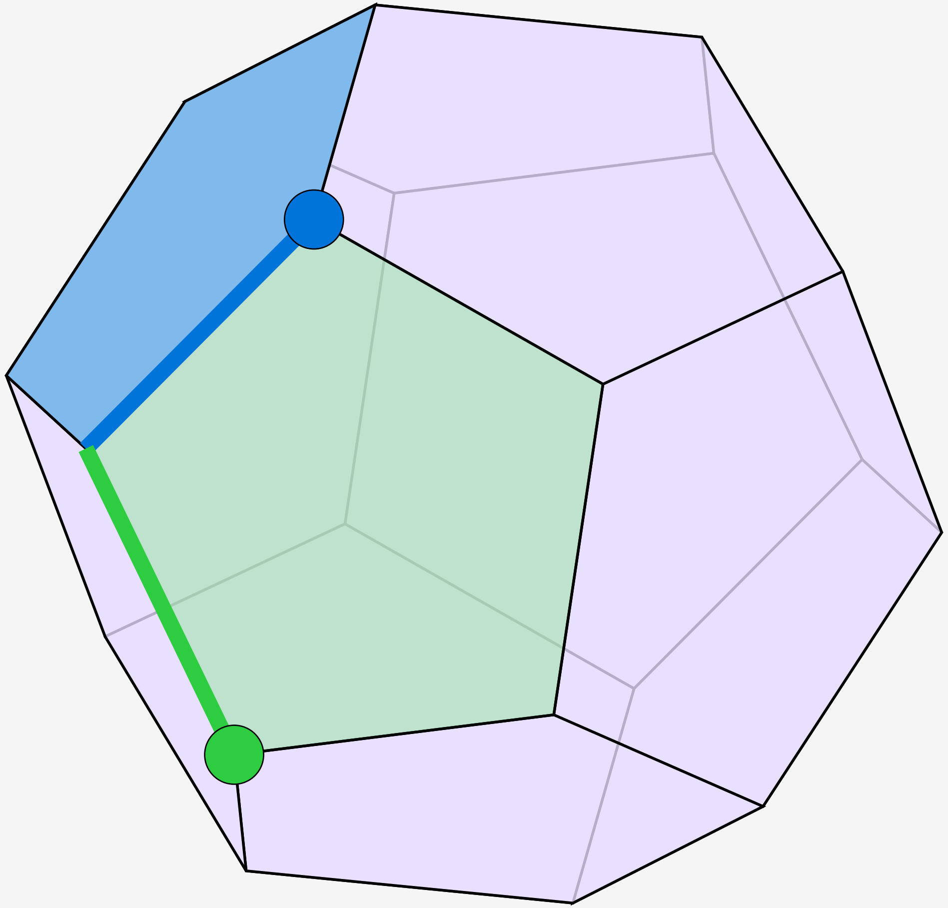
Erlangen programme

$s_0 s_1 s_2 s_0$



Erlangen programme

$$C \xrightarrow{s_0 s_1 s_2 s_0} C'$$



- **chamber (maximal flag)** vertex $\subset$ edge $\subset$ face
- **Weyl group** W = group generated by reflections S
- **W-distance on chambers**

$$(v, e, f) \xrightarrow{w} (v', e', f')$$

1. $C \xrightarrow{1} C' \Leftrightarrow C = C'$
2. $C \xrightarrow{w} C'$ and $C' \xrightarrow{s} C'' \Rightarrow C \xrightarrow{ws \text{ or } w} C''$
3. $C \xrightarrow{w} C'$ and $s \in S \Rightarrow (\exists C'') \quad C \xrightarrow{ws} C''$

introduced by Jacques Tits in 1959

Jacques Tits (1930 - 2021)

buildings

an example from projective geometry

- vector space $\mathbb{F}^4$ over $\mathbb{F}$
- projective special linear group $\mathrm{PSL}_4(\mathbb{F})$
- $W = S_4$

analogy	**thing**
vertices	1-spaces $\langle v \rangle$
edges	2-spaces $\langle u, v \rangle$
faces	3-spaces $\langle u, v, w \rangle$

buildings

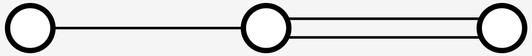
an example from finite polar geometry

- $\mathbb{F}_q^6$ equipped with bilinear form

  $B(x,y) = x_1y_6 - x_6y_1 + x_2y_5 - x_5y_2 + x_3y_4 - x_4y_3$
- totally isotropic subspace U:

$$B(x,y) = 0 \quad \forall x, y \in U$$

- symplectic group $\mathrm{PSp}_6(q)$
- $W = C_2 \wr S_3$



analogy	thing
vertices	totally isotropic 1-spaces
edges	totally isotropic 2-spaces
faces	totally isotropic 3-spaces

- spherical = finite Weyl group W
- rank = number of things

Jacques Tits 1974

All spherical buildings of rank at least 3 are known and are associated with simple groups of Lie type.

Antecedents

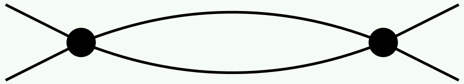
- All projective spaces of rank at least 3 arise from vector spaces. (Hilbert 1899, Vahlen, Hessenberg 1905)
- All polar spaces of rank at least 3 arise from vector spaces equipped with sesquilinear or quadratic forms. (Buekenhout & Shult 1974)

generalised polygons

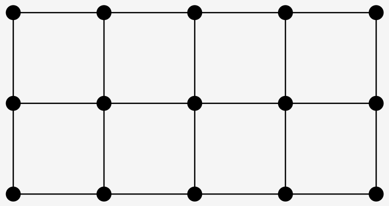
generalised n-gon

Incidence geometry of points and lines such that

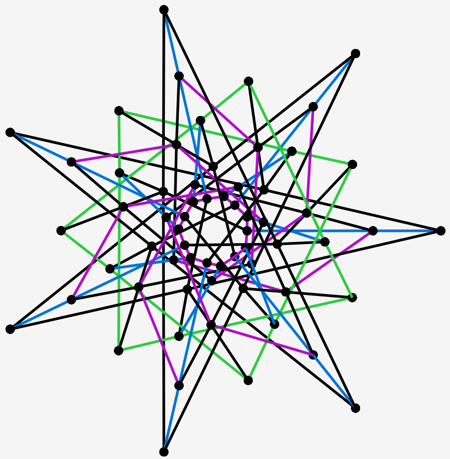
- every 2 elements lie in some ordinary n-gon
- there are no ordinary k-gons for $2 \leqslant k < n$



ordinary 2-gon



a generalised 4-gon



a generalised 6-gon

generalised n-gon

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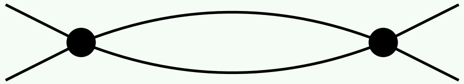
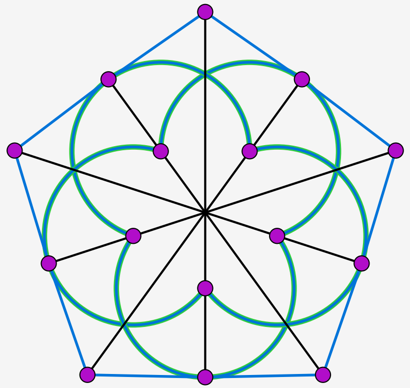
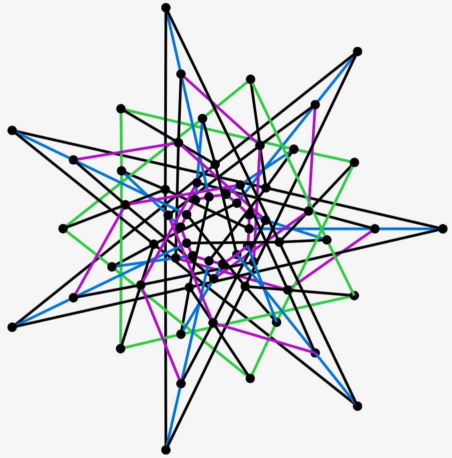


ordinary 2-gon



a generalised 4-gon



a generalised 6-gon

Warning

We will assume our geometries are **finite** from now on.

Walter Feit & Graham Higman 1964

If every line has at least 3 points and every point lies on at least 3 lines, then

$$n \in \{3, 4, 6, 8\}.$$

⚠ Warning

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Walter Feit & Graham Higman 1964

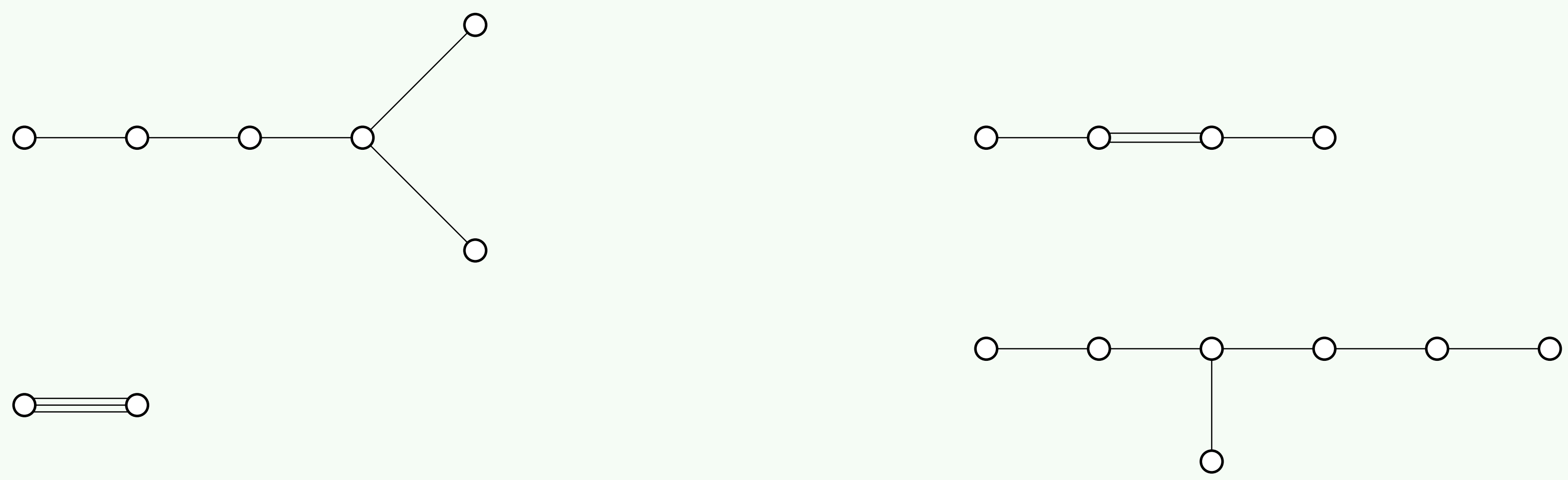
If every line has at least 3 points and every point lies on at least 3 lines, then

$$n \in \{3, 4, 6, 8\}.$$

Related to integers k such that $\sin^2\left(\frac{\pi}{k}\right) \in \mathbb{Q}$.

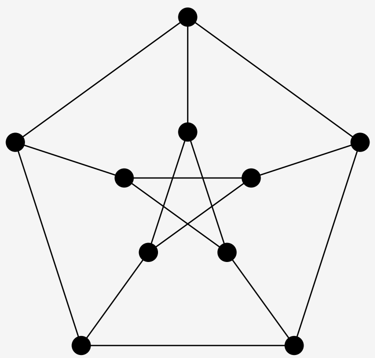
applications

- generalised n-gons are rank 2 buildings
- buildings can be broken down into generalised n-gons (residues)



The cage problem

Given $d > 1$, $g > 2$, find the smallest number of possible vertices that a regular graph of **degree** d and **girth** g can have.



Petersen graph: cage(3,5)

A cage(d, g) has at least

$$1 + d + d(d-1) + \cdots + d(d-1)^{\frac{g-3}{2}}$$

vertices

A bipartite graph meeting the bound (*Moore graph*) **is** a generalised n-gon.

Extremal combinatorics

- optimal clique-free pseudorandom graphs (Mubayi & Verstraëte[1])
- new asymptotic bounds on cycle-complete Ramsey numbers
- smallest degree of Ramsey graphs that are minimal for cliques (JB, Bishnoi, & Lesgourgues[2])
- $r(4,t) = \Omega\left(\frac{t^3}{\log^4 t}\right)$ (Mattheus & Verstraëte[3])

♣

[1] "A note on pseudorandom Ramsey graphs", doi:10.4171/JEMS/1359

[2] "The minimum degree of minimal Ramsey graphs for cliques", doi:10.1112/blms.12658″

[3] "The asymptotics of $r(4,t)$ ", doi:10.4007/annals.2024.199.2.8

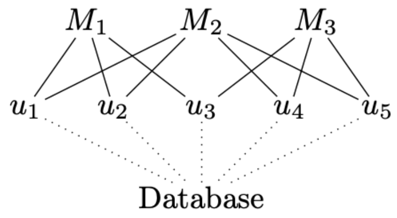
User-Private Information Retrieval schemes

In such a scheme, a set of users collaborate to retrieve files from a database without revealing to observers which participant in the scheme requested the file.

- generalised quadrangles were used[1] to create a method that is substantially more secure against eavesdropping than was previously known.

They have also appeared in

- the construction of codes
- quantum error-correction
- quasisymmetric designs
- association schemes



	u_1	u_2	u_3	u_4	u_5
M_1	1	1	1	0	0
M_2	1	1	0	1	1
M_3	0	0	1	1	1

Figure 1: A visualisation of a UPIR system.

[1] Gnilke, Greferath, Hollanti, Nuñez Ponasso, Ó Catháin, Swartz, “Improved user-private information retrieval via finite geometry”, doi:10.1007/s10623-018-00591-9

dual generalised n-gon: swapping the roles of points and lines.

3-gons (projective planes)

- Desarguesian ("classical") $\rightarrow$ $\mathrm{PSL}_3(q)$
- lots of non-Desarguesian planes known

4-gons (generalised quadrangles)

- arising from formed vector spaces ("classical") $\rightarrow$ $\mathrm{PSp}_4(q), \mathrm{PSU}_4(q), \mathrm{PSU}_5(q)$
- lots not arising from formed vector spaces

6-gons (generalised hexagons)

- classical $\rightarrow$ $G_2(q), {}^3D_4(q)$

8-gons (generalised octagons)

- classical $\rightarrow$ ${}^2F_4(q)$

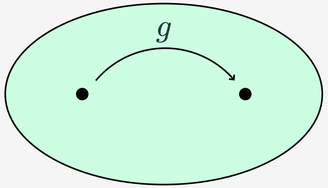
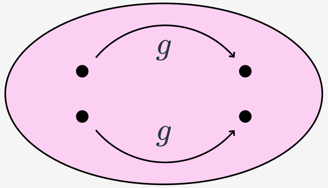
known examples

Bose-Nair, Cameron, Cohen-Tits, Dixmier-Zara, Hall, Hall-Swift-Walker, Lam-Kolesova-Thiel, Lam-Thiel-Swiercz, MacInnes, Payne, Payne-Thas, Tarry, Thas

All generalised n-gons with at most 124 points have been classified.

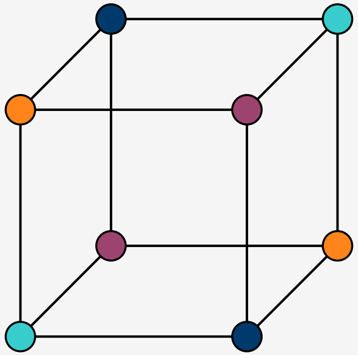
n	# points	# symmetries	n	# points	# symmetries
3	7	168	3	13	5616
3	21	120960	3	31	372000
3	57	5630688	3	73	49448448
3	91	84913920	3	91	311040
3	91	33696	4	15	720
4	27	51840	4	40	51840
4	64	138240	4	85	1958400
4	112	26127360	6	63	12096

G acts on Ω

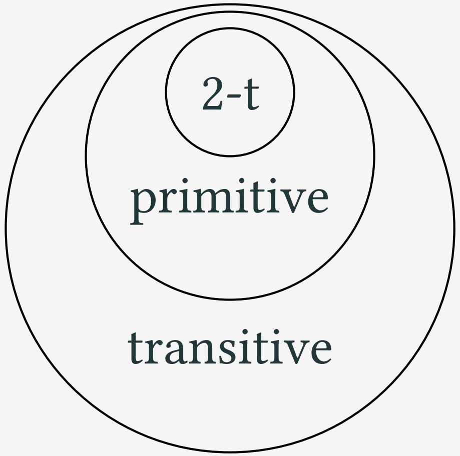
transitive		$(\forall \omega, \omega' \in \Omega)(\exists g \in G) \quad \omega' = \omega^g$
2-transitive		$(\forall \omega_1, \omega_2, \omega_1', \omega_2' \in \Omega)(\exists g \in G)$ $\omega_1' = \omega_1^g$ and $\omega_2' = \omega_2^g$

- **intransitive** G preserves a subset
- **imprimitive** G is transitive and preserves a partition

symmetries of a cube (imprimitive)



Venn diagram



primitive groups are atoms for **transitive groups**

convention

We will associate an adjective for the symmetry group with the object.

1. "**point-2-transitive** projective plane" $\Longrightarrow$
   "the group of symmetries of the projective plane is **2-transitive** on **points**."
2. "**point-primitive** and **flag-transitive** generalised quadrangle" $\Longrightarrow$
   "the group of symmetries of the generalised quadrangle is **primitive** on **points** and **transitive** on **flags**."

Ostrom & Wagner (1959)

A finite point-2-transitive projective plane is classical.

> This was the first time 2-transitivity produced a complete classification of finite geometries. Since then the notion of a geometric classification in terms of a group-theoretic hypothesis has become commonplace. That was not the case 35 years ago, and it is a measure of these papers' influence that this type of hypothesis is now regarded as a natural extension of Klein's Erlangen program.
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> — Bill Kantor (1993)

Ostrom & Wagner (1959)

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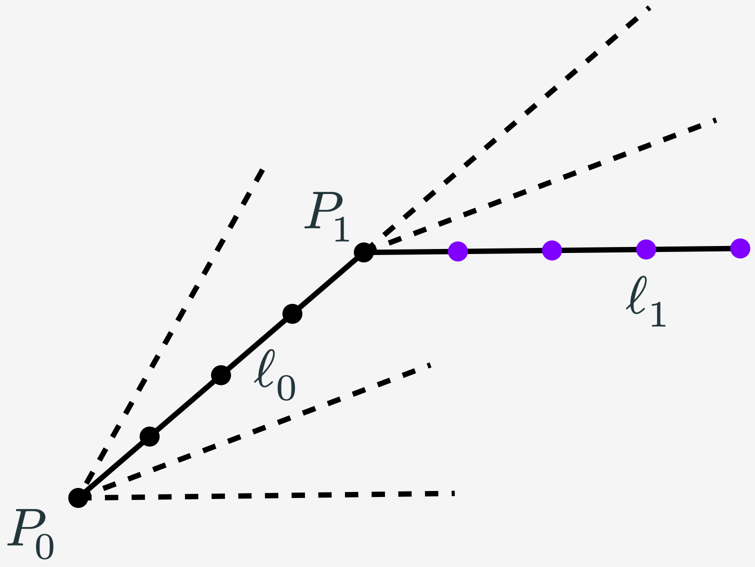
Conjecture (Bruck - early 1950's)

A finite point-transitive projective plane is classical.

2-transitive	Ostrom & Wagner 1959
↓	
flag-transitive	Kantor 1987
↓	
point-primitive	Kantor 1987
↓	
point-transitive	Gill 2007, 2016

K. Thas and Zagier (2008)

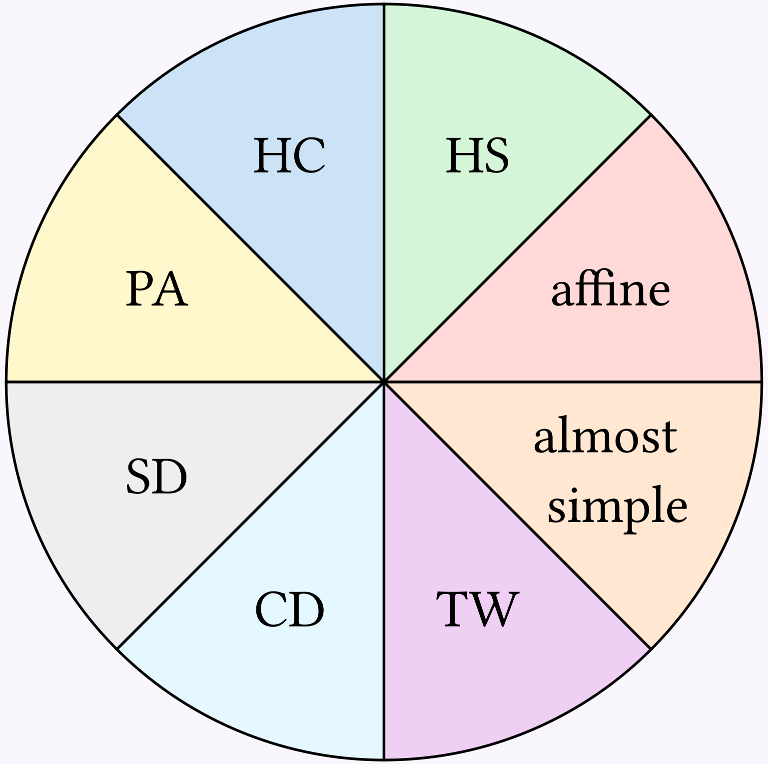
A point-primitive non-classical projective plane has at least 4×10^{22} points.



Fong & Seitz 1973/1974

A finite *Moufang* generalised polygon is classical.

The O'Nan-Scott Theorem for finite primitive groups



generalised hexagons/octagons: characterisations

distance-transitive on points

↓

flag-transitive + point-primitive + line-primitive

↓

point-primitive

Buekenhout & Van Maldeghem (1994)

Schneider & Van Maldeghem (2008)

JB, Glasby, Popiel, Praeger, Schneider (2017)

JB, Glasby, Popiel, Praeger, Schneider (2017)

If G acts primitively on the points of a finite GH or GO, then G is `almost simple`.

Morgan & Popiel (2016)

(i) ${}^2B_2(q), {}^2G_2(q) \not\leq G$. (ii) ${}^2F_4(q) \leq G \Rightarrow$ known example.

generalised quadrangles: a sporadic example

- first constructed by Ronald Ahrens and George Szekeres (UNSW) in 1969
- 64 points, 96 lines
- *link to demo*
- flag-transitive, points-distance transitive, line-distance intransitive, Moufang
- point-primitive, line-imprimitive

ON A COMBINATORIAL GENERALIZATION OF 27 LINES ASSOCIATED WITH A CUBIC SURFACE

R. W. AHRENS and G. SZEKERES

(Received 26 March 1969)

To Bernhard Hermann Neumann on his 60th birthday

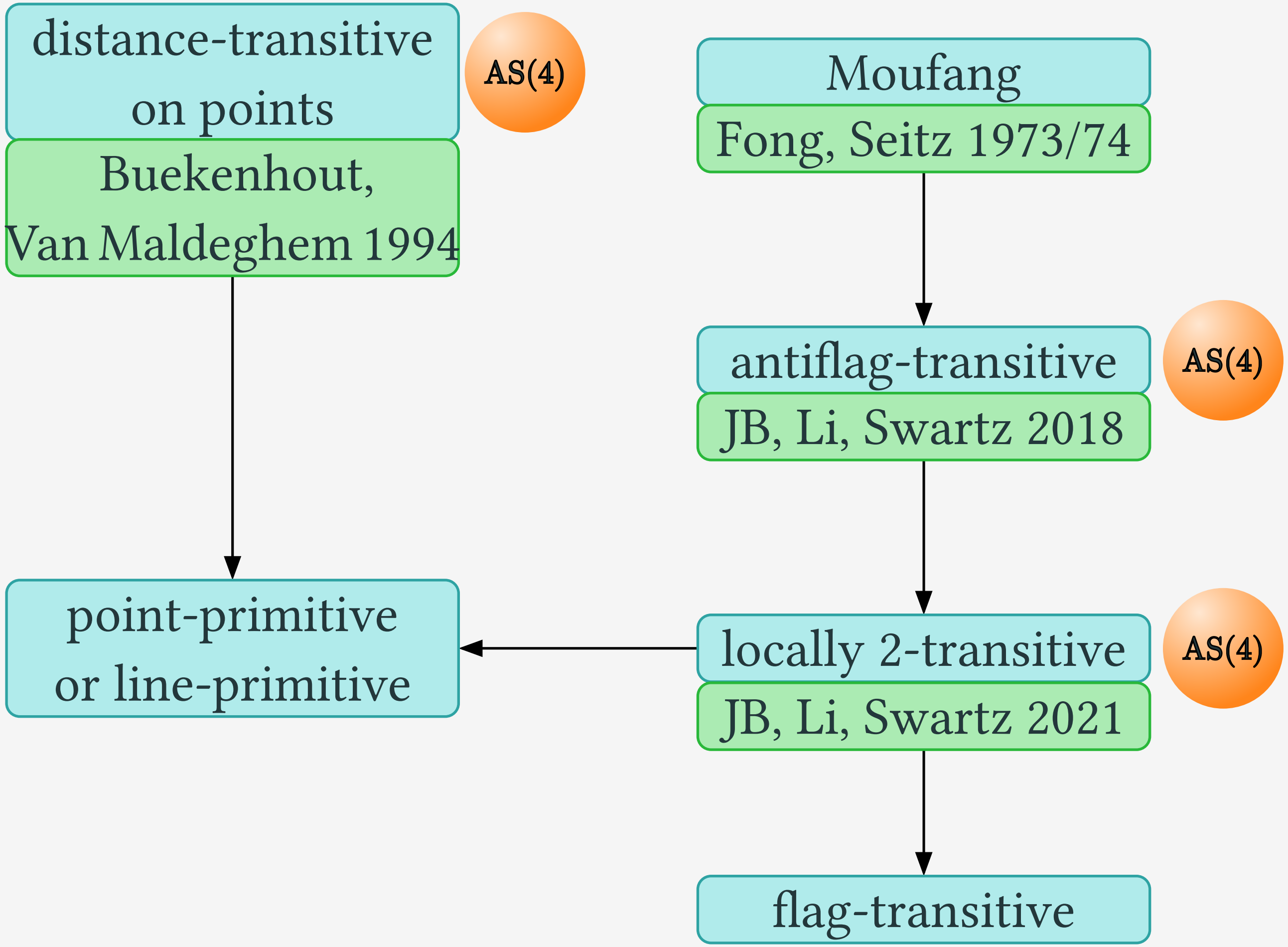
Communicated by G. B. Preston

1.

Given integers $0 < \lambda < k < v$, does there exist a nontrivial graph G with the following properties: G is of order v (i.e. has v vertices), is regular of degree k (i.e. every vertex is adjacent to exactly k other vertices), and every pair of vertices is adjacent to exactly λ others? Two vertices are said to be adjacent if they are connected by an edge. We call a graph with the above properties a symmetric (v, k, λ) graph and refer to the last of the properties as the λ-condition. The complete graph of order v is a trivial example of a symmetric $(v, v-1, v-2)$ graph, but we are of course only interested in non-trivial constructions.

All graphs will be assumed to have no loops or double edges. If the vertices of G are denoted by $x_1, \cdots, x_v$ and S_i, $i = 1, \cdots, v$ denotes the set of vertices which are adjacent to x_i, then the sets S_i form a symmetric block design with parameters (v, k, λ) i.e. $|S_i| = k$, $|S_i \cap S_j| = \lambda$ for $i \neq j$. Thus the parameters v, k, λ must satisfy the Bruck-Ryser-Chowla conditions ([2], p. 107) in order that such a graph should exist. However these conditions

generalised quadrangles: characterisations



Conjecture (Kantor 1991)

A finite flag-transitive generalised quadrangle is classical, $\mathrm{AS}(4)$, or the Lunelli-Sce quadrangle.

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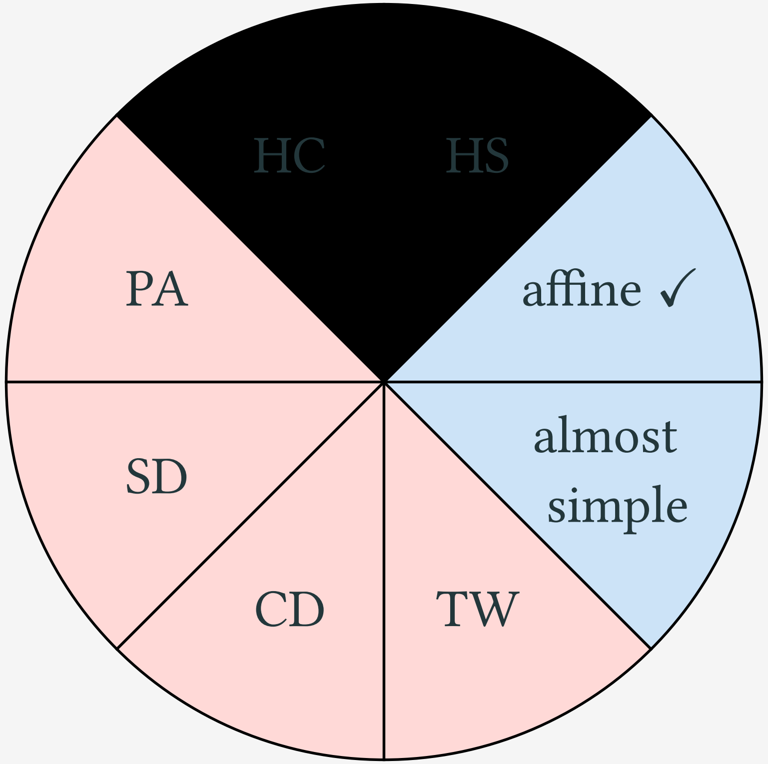
Conjecture

A finite flag-transitive generalised quadrangle is point-primitive or line-primitive.

JB, Giudici, Morris, Royle, Spiga (2012)

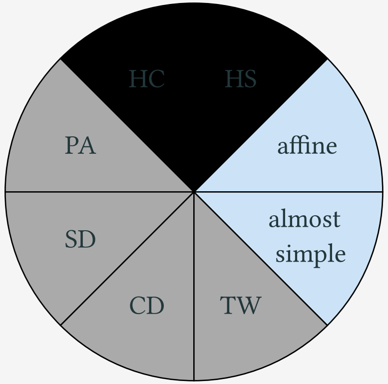
- G primitive on points and lines of a GQ $\Rightarrow$ G is `almost simple`.
- If G is also flag-transitive, then G is of Lie type.

point-primitive and line-transitive



- JB, Glasby, Popiel, Praeger, *J. Comb. Des.* (2017)
- JB, Popiel, Praeger, *J. Group Theory* (2017)

just point-primitive



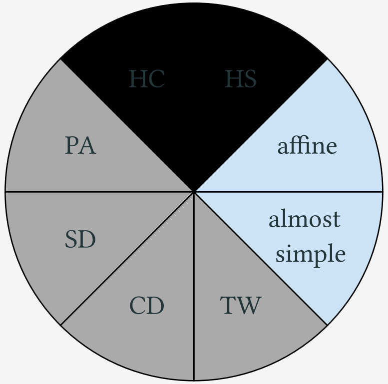
Di (2025+)

HS does not arise.

JB, Popiel, Praeger (Nagoya Math. 2019)

- When G is of type PA, SD, CD, TW, the structure of G is heavily restricted.
- Let θ be a nonidentity symmetry of a GQ. Then θ fixes less than $v^{\frac{4}{5}}$ of v points.

just point-primitive



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exception: unique GQ on on 27 points has θ fixing 15.

recent results on almost simple type: $S \lhd G \leqslant \mathrm{Aut}(S)$

author (yr)	condition	conclusion
JB, Giudici, Morris, Royle, Spiga (2012)	point-primitive, flag-transitive, $S = A_n$	$n = 6$, known example
JB, Evans (2021)	point-primitive, S sporadic	$\varnothing$
Feng, Lu (2023)	point-primitive, line-primitive, $S = \mathrm{PSL}_2(q)$	$q = 9$, known example
Lu, Zhang, Zou (2024)	point-primitive, line-primitive, $S = \mathrm{PSU}_3(q)$, $q \geqslant 3$	$\varnothing$
Arumugam, JB, Giudici, (2025)	point-primitive, line-primitive, $S \in \left\{ {}^2B_2(q), {}^2G_2(q) \right\}$	$\varnothing$
Arumugam (2026+)	point-primitive, line-primitive, $S = {}^2F_4(q)$	$\varnothing$

conclusion

- local symmetry conditions

Feng and K. Thas (arXiv)

Suppose G acts on a thick generalised quadrangle. If for every point P, the group G_P is not faithful on the points collinear with P, then it is classical.

- geometric way to study symmetry
  - can we simplify the proof of the CFSG?
- classify geometric objects satisfying a symmetry condition
  - flag-transitive, point-primitive?
- imposing symmetry allows us to construct new geometric objects

> "At least two decades have passed since the discovery of a generalised quadrangle with new parameters … A generalised quadrangle with new parameters would be of much interest."
>
> — Simeon Ball (2015)

Open problems

1. Construction of non-classical finite generalised hexagons or octagons.
2. Classification of finite flag-transitive generalised polygons.
3. Classification of finite point-primitive generalised polygons.
4. Are all finite point-transitive projective planes classical?

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