Computing dyadic monomial moments from the Hermite moments

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November 15, 2023

The first several Hermite polynomials are

$$p_0(v) = He_0\left(\frac{v}{v_0}\right) = 1 \qquad p_2(v) = He_2\left(\frac{v}{v_0}\right) = \frac{(v/v_0)^2 - 1}{\sqrt{2}}$$
$$p_1(v) = He_1\left(\frac{v}{v_0}\right) = v/v_0 \qquad p_3(v) = He_3\left(\frac{v}{v_0}\right) = \frac{(v/v_0)^3 - v/v_0}{\sqrt{6}}.$$

These can be easily inverted:

$$1 = He_0\left(\frac{v}{v_0}\right) \qquad v^2 = v_0^2\left[\sqrt{2}He_2\left(\frac{v}{v_0}\right) + He_0\left(\frac{v}{v_0}\right)\right]$$
$$v = v_0 He_1\left(\frac{v}{v_0}\right) \qquad v^3 = v_0^3\left[\sqrt{6}He_3\left(\frac{v}{v_0}\right) + He_1\left(\frac{v}{v_0}\right)\right].$$

Denote the moment with respect to the dyadic velocity monomial $v_x^a v_y^b v_z^c$ by M_{abc}:

$$M_{abc} = \int_{\mathbb{R}^3} f v_x^a v_y^b v_z^c \, \mathrm{d}v_x v_y v_z,$$

and the moment with respect to the tensor product Hermite polynomial $He_n(v_x/v_0)He_m(v_y/v_0)He_p(v_z/v_0)$ by H_{nmp}:

$$H_{nmp} = \int_{\mathbb{R}^3} f He_n(v_x/v_0)He_m(v_y/v_0)He_p(v_z/v_0) \, \mathrm{d}v_x v_y v_z.$$

The first step to making sense of the Hermite moments is expanding the dyadic velocity monomial moments in terms of the Hermite moment tensors. By solving for the dyadic monomials $v_x^a v_y^b v_z^c$ in terms of the Hermite polynomials up to the same degree, we get the following relationships.

Zeroth-order moment

$$1 = He_0(v_x/v_0) \implies M_{000} = H_{000}$$

First-order moments

$$v_x = v_0 He_1(v_x/v_0) \implies \begin{aligned} M_{100} &= v_0 H_{100} \\ M_{010} &= v_0 H_{010} \\ M_{001} &= v_0 H_{001} \end{aligned}$$

Second-order moments

$$\begin{aligned} v_x v_y = v_0^2 He_1(v_x/v_0) He_2(v_y/v_0) &\implies M_{110} = v_0^2 H_{110} \\ &\quad\;\; M_{101} = v_0^2 H_{101} \\ &\quad\;\; M_{011} = v_0^2 H_{011} \\ v_x^2 = v_0^2(\sqrt{2} He_2(v_x/v_0) + He_0(v_x/v_0)) &\implies M_{200} = v_0^2(\sqrt{2} H_{200} + H_{000}) \\ &\quad\;\; M_{020} = v_0^2(\sqrt{2} H_{020} + H_{000}) \\ &\quad\;\; M_{002} = v_0^2(\sqrt{2} H_{002} + H_{000}) \end{aligned}$$

Third-order moments:

$$\begin{aligned} v_x v_y v_z = v_0^3 He_1\left(\frac{v_x}{v_0}\right) He_1\left(\frac{v_y}{v_0}\right) He_1\left(\frac{v_z}{v_0}\right) &\implies M_{111} = v_0^3 H_{111} \\ v_x^2 v_y = v_0^2\left(\sqrt{2} He_2\left(\frac{v_x}{v_0}\right) + He_0\left(\frac{v_x}{v_0}\right)\right) v_0 He_1\left(\frac{v_y}{v_0}\right) &\implies M_{210} = v_0^3(\sqrt{2} H_{210} + H_{010}) \\ &\quad\;\; M_{201} = v_0^3(\sqrt{2} H_{201} + H_{001}) \\ &\quad\;\; \vdots \\ v_x^3 = v_0^3\left(\sqrt{6} He_3\left(\frac{v_x}{v_0}\right) + He_1\left(\frac{v_x}{v_0}\right)\right) &\implies M_{300} = v_0^3(\sqrt{6} H_{300} + H_{100}). \end{aligned}$$

Fourth and higher total degree moments will start to require expanding the products like $v_x^2 v_y^2$ in terms of the sums of Hermite moments.

1 Random velocity moments

Now consider the dyadic random velocity moments, where we subtract off the velocity to get the random velocity part.

Denote the moment of f with respect to the dyadic random velocity monomial $(v_x - u_x)^a (v_y - u_y)^b (v_z - u_z)^c$ by U_{abc}:

$$U_{abc} = \int_{\mathbb{R}^3} f (v_x - u_x)^a (v_y - u_y)^b (v_z - u_z)^c \,\mathrm{d}\boldsymbol{v}$$

Zeroth order moment

$$U_{000} = M_{000}.$$

First-order moments

$$U_{100} = U_{010} = U_{001} = 0.$$

Second-order moments

$$\begin{aligned} \begin{pmatrix} U_{200} & U_{110} & U_{101} \\ U_{110} & U_{020} & U_{011} \\ U_{101} & U_{011} & U_{002} \end{pmatrix} &= \int f(\boldsymbol{v} - \boldsymbol{u}) \otimes (\boldsymbol{v} - \boldsymbol{u})\, d\boldsymbol{v} \\ &= \int f \boldsymbol{v} \otimes \boldsymbol{v}\, \mathrm{d}\boldsymbol{v} - \boldsymbol{u} \otimes \int f \boldsymbol{v}\, \mathrm{d}\boldsymbol{v} - \left(\int f \boldsymbol{v}\, \mathrm{d}\boldsymbol{v}\right) \otimes \boldsymbol{u} + \rho(\boldsymbol{u} \otimes \boldsymbol{u}) \\ &= \underbrace{\begin{pmatrix} M_{200} & M_{110} & M_{101} \\ M_{110} & M_{020} & M_{011} \\ M_{101} & M_{011} & M_{002} \end{pmatrix}}_{\mathbb{M}^2} - \rho \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix} \end{aligned}$$

Third-order moments corresponding to heat flux

$$
\begin{aligned}
U_{300} &= \int f(v_x - u_x)^3 \, \mathrm{d}\boldsymbol{v} \\
&= \int f(v_x^3 - 3v_x^2 u_x + 3v_x u_x^2 - u_x^3) \, \mathrm{d}\boldsymbol{v} \\
&= M_{300} - 3u_x M_{200} + 3u_x^2 M_{100} - u_x^3 M_{000} \\
&= M_{300} - 3u_x M_{200} + 2u_x^3 M_{000} \\
U_{030} &= M_{030} - 3u_y M_{020} + 2u_y^3 M_{000} \\
U_{003} &= M_{003} - 3u_z M_{002} + 2u_z^3 M_{000} \\
U_{210} &= \int f(v_x - u_x)^2 (v_y - u_y) \, \mathrm{d}\boldsymbol{v} \\
&= \int f \left[v_x^2 - 2v_x u_x + u_x^2\right] (v_y - u_y) \, \mathrm{d}\boldsymbol{v} \\
&= \int f(v_x^2 v_y - 2v_x v_y u_x + v_y u_x^2 - v_x^2 u_y + 2v_x u_x u_y - u_x^2 u_y) \, \mathrm{d}\boldsymbol{v} \\
&= M_{210} - 2u_x M_{110} + u_x^2 M_{010} - u_y M_{200} + 2u_x u_y M_{100} - u_x^2 u_y M_{000} \\
&= M_{210} - 2u_x M_{110} - u_y M_{200} + 2u_x^2 u_y M_{000} \\
U_{201} &= M_{201} - 2u_x M_{101} - u_z M_{200} + 2u_x^2 u_z M_{000} \\
&\vdots \\
U_{012} &= M_{012} - u_y M_{002} - 2u_z M_{011} + 2u_y u_z^2 M_{000}
\end{aligned}
$$

2 Observables

Finally we can discuss the observables corresponding to the second and third moments, namely temperature and heat flux.

$$
\begin{aligned}
T &= \frac{m}{dn}(U_{200} + U_{020} + U_{002}) \\
\boldsymbol{q} &= \frac{m}{2} \begin{pmatrix} U_{300} + U_{120} + U_{102} \\ U_{210} + U_{030} + U_{012} \\ U_{201} + U_{021} + U_{003} \end{pmatrix}.
\end{aligned}
$$

NOTE: What's the meaning of the factor of 1/2 in the heat flux definition?

3 Closure relations

We should be able to recover a relationship of the form

$$\mathbf{q} \approx \kappa \nabla_x T$$

as the collision frequency increases, $\nu_p \to \infty$.