Computing dyadic monomial moments from the Hermite moments

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The first several Hermite polynomials are

$$p_0(v) = He_0\left(\frac{v}{v_0}\right) = 1 \qquad p_2(v) = He_2\left(\frac{v}{v_0}\right) = \frac{(v/v_0)^2 - 1}{\sqrt{2}}$$
$$p_1(v) = He_1\left(\frac{v}{v_0}\right) = v/v_0 \qquad p_3(v) = He_3\left(\frac{v}{v_0}\right) = \frac{(v/v_0)^3 - v/v_0}{\sqrt{6}}.$$

These can be easily inverted:

$$1 = He_0\left(\frac{v}{v_0}\right) \qquad v^2 = v_0^2\left[\sqrt{2}He_2\left(\frac{v}{v_0}\right) + He_0\left(\frac{v}{v_0}\right)\right]$$
$$v = v_0 He_1\left(\frac{v}{v_0}\right) \qquad v^3 = v_0^3\left[\sqrt{6}He_3\left(\frac{v}{v_0}\right) + He_1\left(\frac{v}{v_0}\right)\right].$$

Denote the moment with respect to the dyadic velocity monomial $v_x^a v_y^b v_z^c$ by M_{abc}, and the moment with respect to the tensor product Hermite polynomial $He_n(v_x/v_0)He_m(v_y/v_0)He_p(v_z/v_0)$ by H_{nmp}.

Zeroth-order moment

$$M_{000} = H_{000}$$

First-order moments

$$\begin{aligned} M_{100} &= v_0 H_{100} \\ M_{010} &= v_0 H_{010} \\ M_{001} &= v_0 H_{001} \end{aligned}$$

Second-order moments

$$\begin{aligned} M_{110} &= v_0^2 H_{110} \\ M_{101} &= v_0^2 H_{101} \\ M_{011} &= v_0^2 H_{011} \\ M_{200} &= v_0^2(\sqrt{2}H_{200} + H_{000}) \\ M_{020} &= v_0^2(\sqrt{2}H_{020} + H_{000}) \\ M_{002} &= v_0^2(\sqrt{2}H_{002} + H_{000}) \end{aligned}$$

Third-order moments:

$$\begin{aligned} M_{111} &= v_0^3 H_{111} \\ M_{210} &= v_0^3(\sqrt{2}H_{210} + H_{010}) \\ M_{201} &= v_0^3(\sqrt{2}H_{201} + H_{001}) \\ &\vdots \\ M_{300} &= v_0^3(\sqrt{6}H_{300} + H_{100}). \end{aligned}$$

Fourth and higher total degree moments will start to require expanding the products like $v_x^2 v_y^2$ in terms of the sums of Hermite moments.

1 Observables

Now consider the dyadic centered moments, where we subtract off the velocity to get the random velocity part:

$$\rho = M_{000}, \quad \rho \boldsymbol{u} = \begin{pmatrix} M_{100} \\ M_{010} \\ M_{001} \end{pmatrix}.$$

$$\begin{aligned}\int f(\boldsymbol{v}-\boldsymbol{u}) \otimes (\boldsymbol{v}-\boldsymbol{u})\, d\boldsymbol{v} &= \int f \boldsymbol{v} \otimes \boldsymbol{v}\, \mathrm{d}\boldsymbol{v} - \boldsymbol{u} \otimes \int f \boldsymbol{v}\, \mathrm{d}\boldsymbol{v} - \left(\int f \boldsymbol{v}\, \mathrm{d}\boldsymbol{v}\right) \otimes \boldsymbol{u} + \rho(\boldsymbol{u} \otimes \boldsymbol{u}) \\ &= \underbrace{\begin{pmatrix} M_{200} & M_{110} & M_{101} \\ M_{110} & M_{020} & M_{011} \\ M_{101} & M_{011} & M_{002} \end{pmatrix}}_{\mathbb{M}^2} - \rho \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}\end{aligned}$$

$$|\boldsymbol{v}-\boldsymbol{u}|^2 = (\boldsymbol{v}-\boldsymbol{u}) \cdot (\boldsymbol{v}-\boldsymbol{u}) = |\boldsymbol{v}|^2 - 2\boldsymbol{v} \cdot \boldsymbol{u} + |\boldsymbol{u}|^2.$$

$$\begin{aligned}\int f|\boldsymbol{v}-\boldsymbol{u}|^2\, \mathrm{d}\boldsymbol{v} &= \int f|\boldsymbol{v}|^2\, \mathrm{d}\boldsymbol{v} - \rho|\boldsymbol{u}|^2 \\ &= M_{200} + M_{020} + M_{002} - \rho|\boldsymbol{u}|^2 \\ &= \mathrm{Tr}(\mathbb{M}^2) - \rho|\boldsymbol{u}|^2\end{aligned}$$

$$\begin{aligned}\int f(\boldsymbol{v}-\boldsymbol{u})|\boldsymbol{v}-\boldsymbol{u}|^2\, \mathrm{d}\boldsymbol{v} &= \int f(\boldsymbol{v}|\boldsymbol{v}|^2 - 2\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{u}) + \boldsymbol{v}|\boldsymbol{u}|^2 - \boldsymbol{u}|\boldsymbol{v}|^2 + 2\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{u}) - \boldsymbol{u}|\boldsymbol{u}|^2)\, \mathrm{d}\boldsymbol{v} \\ &= \underbrace{\begin{pmatrix} M_{300} + M_{120} + M_{102} \\ M_{210} + M_{030} + M_{012} \\ M_{201} + M_{021} + M_{003} \end{pmatrix}}_{\mathbb{Q}^3} - 2\mathbb{M}^2 \cdot \boldsymbol{u} + \rho\boldsymbol{u}|\boldsymbol{u}|^2 - \boldsymbol{u}\mathrm{Tr}(\mathbb{M}^2) + 2\rho\boldsymbol{u}|\boldsymbol{u}|^2 - \rho\boldsymbol{u}|\boldsymbol{u}|^2 \\ &= \mathbb{Q}^3 - 2\mathbb{M}^2 \cdot \boldsymbol{u} - \boldsymbol{u}\mathrm{Tr}(\mathbb{M}^2) + 2\rho\boldsymbol{u}|\boldsymbol{u}|^2\end{aligned}$$