Computing dyadic monomial moments from the Hermite moments

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The first several Hermite polynomials are

$$p_0(v) = He_0\left(\frac{v}{v_0}\right) = 1 \qquad p_2(v) = He_2\left(\frac{v}{v_0}\right) = \frac{(v/v_0)^2 - 1}{\sqrt{2}}$$

$$p_1(v) = He_1\left(\frac{v}{v_0}\right) = v/v_0 \qquad p_3(v) = He_3\left(\frac{v}{v_0}\right) = \frac{(v/v_0)^3 - v/v_0}{\sqrt{6}}.$$

These can be easily inverted:

$$1 = He_0\left(\frac{v}{v_0}\right) \qquad v^2 = v_0^2\left[\sqrt{2}He_2\left(\frac{v}{v_0}\right) + He_0\left(\frac{v}{v_0}\right)\right]$$

$$v = v_0 He_1\left(\frac{v}{v_0}\right) \qquad v^3 = v_0^3\left[\sqrt{6}He_3\left(\frac{v}{v_0}\right) + He_1\left(\frac{v}{v_0}\right)\right].$$

Denote the moment with respect to the dyadic velocity monomial $v_x^a v_y^b v_z^c$ by M_{abc}, and the moment with respect to the tensor product Hermite polynomial $He_n(v_x/v_0)He_m(v_y/v_0)He_p(v_z/v_0)$ by H_{nmp}.

Zeroth-order moment

$$M_{000} = H_{000}$$

First-order moments

$$\begin{aligned}
M_{100} &= v_0 H_{100}\\
M_{010} &= v_0 H_{010}\\
M_{001} &= v_0 H_{001}
\end{aligned}$$

Second-order moments

$$\begin{aligned}
M_{110} &= v_0^2 H_{110}\\
M_{101} &= v_0^2 H_{101}\\
M_{011} &= v_0^2 H_{011}\\
M_{200} &= v_0^2(\sqrt{2}H_{200} + H_{000})\\
M_{020} &= v_0^2(\sqrt{2}H_{020} + H_{000})\\
M_{002} &= v_0^2(\sqrt{2}H_{002} + H_{000})
\end{aligned}$$

Third-order moments:

$$\begin{aligned}
M_{111} &= v_0^3 H_{111}\\
M_{210} &= v_0^3(\sqrt{2}H_{210} + H_{010})\\
M_{201} &= v_0^3(\sqrt{2}H_{201} + H_{001})\\
&\vdots\\
M_{300} &= v_0^3(\sqrt{6}H_{300} + H_{100}).
\end{aligned}$$

Fourth and higher total degree moments will start to require expanding the products like $v_x^2 v_y^2$ in terms of the sums of Hermite moments.

1 Observables

Now consider the dyadic centered moments, where we subtract off the velocity to get the random velocity part:

$$\rho = M_{000}, \quad \rho \boldsymbol{u} = \begin{pmatrix} M_{100} \\ M_{010} \\ M_{001} \end{pmatrix}.$$

$$\int f(\boldsymbol{v} - \boldsymbol{u}) \otimes (\boldsymbol{v} - \boldsymbol{u}) \, d\boldsymbol{v}$$