Last name: __________________ First name: __________________

QUIZ #2

No books, notes, or calculators are allowed. **20 Points, 20 Minutes.**

1. (6 points) Prove that the real vector space of all continuous real-valued function on the interval $(0,1)$ is infinite-dimensional.

(Solution) Let V denote the vector space of continous real-valued functions on the interval. We show it is infinite dimensional by showing there is an infinite sequence of vectors, $v_1, \dots,$ such that $v_1, \dots v_m$ is linearly independant for all m. This shows that the vector space is infinite, since the dimension is always larger then the size of any linearly indepedant list, which follows from (2.38).

We select $v_i = x^i$, the i-th monomial. We will show that for fixed m, the set $x^1, \dots x^m$ is linearly independant. Take any vanishing linear combination, $0 = \sum_{i=1}^m a_i v_i = \sum_{i=1}^m a_i x^i$. We wish to show this implies that $a_i = 0$. First note, that if we take the m-th derivative of this function, we obtain $0 = m! a_m$, so that $a_m = 0$, and $\sum_{i=1}^m a_i x^i = \sum_{i=1}^{m-1} a_i x^i = 0$. We may now take the $(m-1)$-th derivative, so obtain $(m-1)! a_{m-1} = 0$, so that $a_{m-1} = 0$, and repeat this process by downward induction until $m = 0$, to obtain $a_m = a_{m-1} = \dots a_0 = 0$, so that there are no-nonzero linear depedances, so that the set is linearly independant, as desired.

2. (6 points) Suppose that v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 - v_2, v_2 + v_3, v_3 - v_4, v_4$$

is also a basis of V.

(Solution) First note that, if we show that the vectors are a spanning list, then since they have the same size as our given basis, v_1, v_2, v_3, v_4, they are also a basis by (2.42) in the book.

Now note that

$$\begin{aligned} v_1 &= (v_1 - v_2) + (v_2 + v3) - (v_3 - v_4) - v_4 \\ v_2 &= (v_2 + v_3) - (v_3 - v_4) + v_4 \\ v_3 &= (v_3 - v_4) + v_4 \\ v_4 &= v_4 \end{aligned}$$

Thus we see that $\{v_1, v_2, v_3, v_4\} \subset$Span$(v_1 - v_2, v_2 + v_3, v_3 - v_4, v_4)$.

Thus we have that V=Span$(v_1, v_2, v_3, v_4) \subset$Span$(v_1 - v_2, v_2 + v_3, v_3 - v_4, v_4)$.

Which shows that $v_1 - v_2, v_2 + v_3, v_3 - v_4, v_4$ is a spanning set.

3. (8 points) Suppose that $V_1, V_2, ..., V_m$ are finite-dimensional subspaces of V. Prove that

$$\dim(V_1 + V_2 + ... + V_m) \leq \dim V_1 + \dim V_2 + ... + \dim V_m.$$

(Solution) I will first clear up a point of confusion. We have

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

We do not have

$$\dim(V_1 + \dots V_n) = \dim(V_1) + \dots \dim(V_n) - \dim(V_1 \cap V_2 \cap \dots \cap V_n)$$

Or

$$\dim(V_1 + \dots V_n) = \dim(V_1) + \dots \dim(V_n) - \dim(V_1 \cap V_2) - \dots \dim(V_{n-1} \cap V_n) + \dots - \dim(V_1 \cap V_2 \cap \dots \cap V_n)$$

Or any other formula of this type. It is not possible to express the dimension of $\dim(V_1 + \dots V_n)$ in terms of the dimensions of the various intersections. Here is a conterexample. Take $V_1 = span((1,0))$, $V_2 = span((0,1))$, $V_3 = span((1,1))$. There all are lines in F^2, and intersect trivially. We also that $\dim(V_1 + V_2 + V_3) = \dim(F^2) = 2$, but we have that $\dim(V_1) + \dim(V_2) + \dim(V_3) = 3$, and the dimensions of the intersections are all zero, so we cannot possibly have any formula of this type.

Remeber though, we do have the formula $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$, so that $\dim(V + W) \leq \dim(V) + \dim(W)$.

Now for the problem solution. We proceed by induction. For $m = 1$, we simply have $\dim(V_1) = \dim(V_1)$. Assuming the $(m-1)$-case, we may write $\dim(V_1 + \dots V_m) = \dim((V_1 + \dots V_{m-1}) + V_m) \leq \dim(V_1 + \dots V_{m-1}) + \dim(V_m)$, applying the above inequality. Now we may use the inductive set $\dim(V_1 + \dots V_{m-1}) + \dim(V_m) \leq \dim(V_1) + \dots \dim(V_{m-1}) + \dim(V_m)$, which completes the induction.