# PHYS440 - Exercises from Nielsen and Chuang (2016)

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**2.1.** (Linear dependence: example) Show that $(1, -1)$, $(1, 2)$ and $(2, 1)$ are linearly dependent.

$$(1, -1) + (1, 2) = (2, 1)$$

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**2.2.** (Matrix representations: example) Suppose $V$ is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and $A$ is a linear operator from $V$ to $V$ such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Find the matrix representation for $A$, with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of $A$.

With respect to the basis $|0\rangle, |1\rangle$, the vector representation of $|0\rangle$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector representation of $|1\rangle$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then

$$A|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$
$$A|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Now consider the basis $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Then $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$.

So with respect to the basis $|+\rangle, |-\rangle$, the vector representation of $|0\rangle$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and the vector representation of $|1\rangle$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Let's denote the matrix representation of the operator $A$ with respect to the basis $|+\rangle, |-\rangle$ as $A'$.

Then

$$\begin{aligned} A' \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ A' &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

**2.3.** (Matrix representation for operator products) Suppose $A$ is a linear operator from vector space $V$ to vector space $W$, and $B$ is a linear operator from vector space $W$ to vector space $X$. Let $|v_i\rangle$, $|w_j\rangle$ and $|x_k\rangle$ be bases for the vector spaces $V$, $W$ and $X$, respectively. Show that the matrix representation for the linear transformation $BA$ is the matrix product of the matrix representations for $B$ and $A$, with respect to the appropriate bases.

Let $l = \dim V$, $m = \dim W$ and $n = \dim X$.

If we take a vector $\vec{v} \in V$, we can write it as a linear combination of the basis vectors $|v_i\rangle$:

$$\vec{v} = \sum_{i=1}^{l} v_i \, |v_i\rangle$$

where we interpret $|v_i\rangle$ as a basis vector, and $v_i$ by itself as the coefficient of that basis vector.

Then the matrix representation of the linear operator $A$ is

$$A\vec{v} = \sum_{j=1}^{m} \left[ \sum_{i=1}^{l} A_{ji} v_i \right] \left| w_j \right\rangle .$$

Similarly, for the linear operator $B$,

$$B\vec{w} = \sum_{k=1}^{n} \left[ \sum_{j=1}^{m} B_{kj} w_j \right] \left| x_k \right\rangle .$$

Therefore, the matrix representation of the linear operator $BA$ is

$$\begin{aligned} B(A\vec{v}) &= \sum_{k=1}^{n} \left[ \sum_{j=1}^{m} B_{kj} \left( \sum_{i=1}^{l} A_{ji} v_i \right) \right] |x_k\rangle \\ &= \sum_{k=1}^{n} \left[ \sum_{j=1}^{m} \sum_{i=1}^{l} B_{kj} A_{ji} v_i \right] |x_k\rangle \\ &= \sum_{k=1}^{n} \left[ \sum_{i=1}^{l} \left( \sum_{j=1}^{m} B_{kj} A_{ji} \right) v_i \right] |x_k\rangle \\ &= \sum_{k=1}^{n} \left[ \sum_{i=1}^{l} (BA)_{ki} v_i \right] |x_k\rangle \end{aligned}$$

That is, the matrix representation of the linear operator $BA$ is the matrix product of the matrix representations for $B$ and $A$.

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**2.11.** (Eigendecomposition of the Pauli matrices) Find the eigenvectors, eigenvalues and diagonal representations of the Pauli matrices $X$, $Y$ and $Z$.

$X$ matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$y = \lambda x$$

$$x = \lambda y$$

This gives two solutions for $X$: eigenvalue $\lambda = 1$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda = -1$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The diagonal form of $X$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$Y$ matrix:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda x = -iy$$

$$ix = \lambda y$$

$$y = \lambda^2 y$$

This gives two solutions for $Y$: eigenvalue $\lambda = 1$ with eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$, and eigenvalue $\lambda = -1$ with eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

A diagonal form of $Y$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$Z$ matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = \lambda x$$

$$y = -\lambda y$$

This gives two solutions for $Z$: eigenvalue $\lambda = 1$ with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and eigenvalue $\lambda = -1$ with eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The $Z$ matrix is already in diagonal form.