PHYS440 - Exercises from Wong 2022

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November 2023

1.1. How many possible states do (a) four coins have? (b) five coins?

Four coins have $2^4 = 16$ possible states. Five coins have $2^5 = 32$ possible states.

1.2. How many possible states do (a) four dice have? (b) five dice?

Four dice have $6^4 = 1296$ possible states. Five dice have $6^5 = 7776$ possible states.

1.3. Some board games use a twenty-sided die. How many twenty-sided dice does it take to code the seven colors of the rainbow?

Seven colors can be encoded with a single twenty-sided die, because $7 < 20$.

1.4. How many (a) coins and (b) six-sided dice would it take to represent the 26 letters of the English alphabet? Ignore upper and lowercase, spaces, punctuation etc, so there's only 26 letters total.

26 letters can be encoded with 5 coins, because $2^5 = 32 > 26$. The letters can be encoded with two dice, because $6^2 = 36 > 26$.

Another way to compute it is to use the logarithm:

$$\lceil \log_2 26 \rceil = 5$$
$$\lceil \log_6 26 \rceil = 2$$

1.5. Convert the following binary (base 2) numbers to decimal numbers (base 10): (a) 10111_2, (b) 11001010_2.

$$2^4 + 2^2 + 2^1 + 2^0 = 16 + 4 + 2 + 1 = 23$$
$$2^7 + 2^6 + 2^3 + 2^1 = 128 + 64 + 8 + 2 = 202$$

1.6. Convert the following decimal (base 10) numbers to binary numbers (base 2): (a) 42, (b) 495.

1. Divide 42 by 2: the result is 21, with remainder 0.
2. Divide 21 by 2: the result is 10, with remainder 1.
3. Divide 10 by 2: the result is 5, with remainder 0.
4. Divide 5 by 2: the result is 2, with remainder 1.
5. Divide 2 by 2: the result is 1, with remainder 0.
6. Divide 1 by 2: the result is 0, with remainder 1.
7. The binary number is 101010_2.

1. Divide 495 by 2: the result is 247, with remainder 1.
2. Divide 247 by 2: the result is 123, with remainder 1.
3. Divide 123 by 2: the result is 61, with remainder 1.
4. Divide 61 by 2: the result is 30, with remainder 1.
5. Divide 30 by 2: the result is 15, with remainder 0.
6. Divide 15 by 2: the result is 7, with remainder 1.
7. Divide 7 by 2: the result is 3, with remainder 1.
8. Divide 3 by 2: the result is 1, with remainder 1.
9. Divide 1 by 2: the result is 0, with remainder 1.
10. The binary number is 111101111_2.

1.7. Base 16, commonly called hexadecimal, is another frequently used number system in computing. The sixteen digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. So the letter A is ten in decimal, ... and F is fifteen in decimal.

(a) Convert the hexadecimal number 3B7C to a decimal (base 10) number.

(b) Convert the hexadecimal number FF to a binary (base 2) number. (So two hexadecimal numbers can represent eight bits.)

(c) Convert the hexadecimal numbers FA, 10 and E4 to decimal.

(a) $16^3 \times 3 + 16^2 \times 11 + 16^1 \times 7 + 16^0 \times 12 = 15228$.

(b) $FF_{16} = 11111111_2$.

(c) $16 \times 15 + 10 = 250$, $16 + 0 = 16$, $16 \times 14 + 4 = 228$.

1.8. Negative numbers can be encoded in binary using *two's complement*, where the most significant bit is negative, while the remaining bits are positive. Convert each of these two's complement numbers to decimal: 000, 001, 010, 011, 100, 101, 110, 111.

Binary	Decimal
000	0
001	1
010	2
011	3
100	-4
101	-3
110	-2
111	-1

1.9. Write your name as an ASCII bit string.

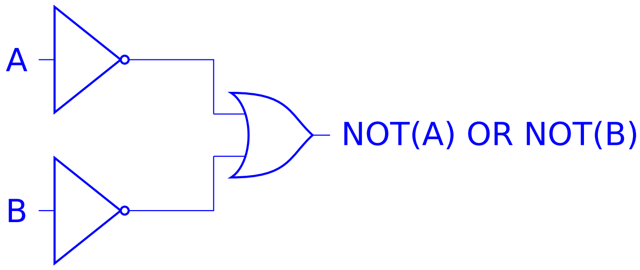
`"John Hurst"` is encoded as

Character	ASCII	Binary
J	74	01001010
o	111	01101111
h	104	01101000
n	110	01101110
(space)	32	00100000
H	72	01001000
u	117	01110101
r	114	01110010
s	115	01110011
t	116	01110100

1.10. Decode the following ASCII characters: 1010001, 1110101, 1100001, 1101110, 1110100, 1110101, 1101101.

Binary	ASCII	Character
01010001	81	Q
01110101	117	u
01100001	97	a
01101110	110	n
01110100	116	t
01110101	117	u
01101101	109	m

1.11. Consider the following gate that inverts the inputs before passing them into an OR gate, sometimes called a *negative-OR gate*:



(a) Write the truth table for this circuit.

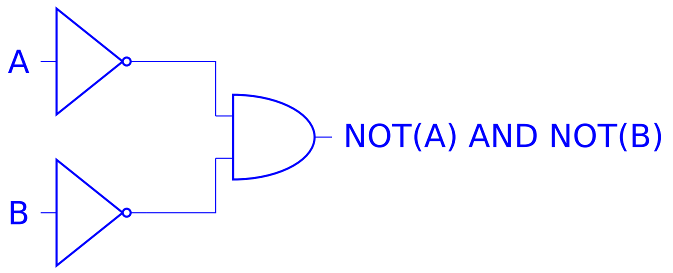
(b) What logic gate is this equivalent to?

(a) The truth table is:

A	B	NOT(A) OR NOT(B)
0	0	1
0	1	1
1	0	1
1	1	0

(b) This is equivalent to a NAND gate.

1.12. Consider the following gate that inverts the inputs before passing them into an AND gate, sometimes called a *negative-AND gate*:



(a) Write the truth table for this circuit.

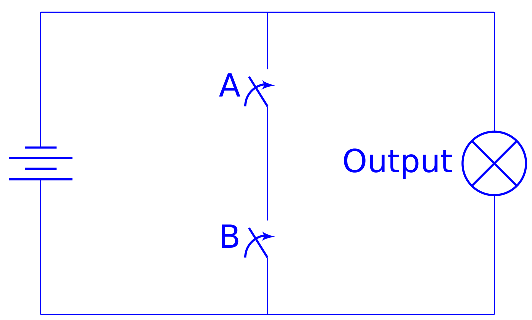
(b) What logic gate is this equivalent to?

(a) The truth table is:

A	B	NOT(A) AND NOT(B)
0	0	1
0	1	0
1	0	0
1	1	0

(b) This is equivalent to a NOR gate.

1.13. Consider the following electrical circuit:



Answer the following questions:

(a) Say $A = 0$ and $B = 0$. Is the light bulb off or on?

(b) Say $A = 0$ and $B = 1$. Is the light bulb off or on?

(c) Say $A = 1$ and $B = 0$. Is the light bulb off or on?

(d) Say $A = 1$ and $B = 1$. Is the light bulb off or on?

(e) What logic gate does this circuit correspond to?

(a) When both switches are off, the current flows through the light bulb, so it is on.

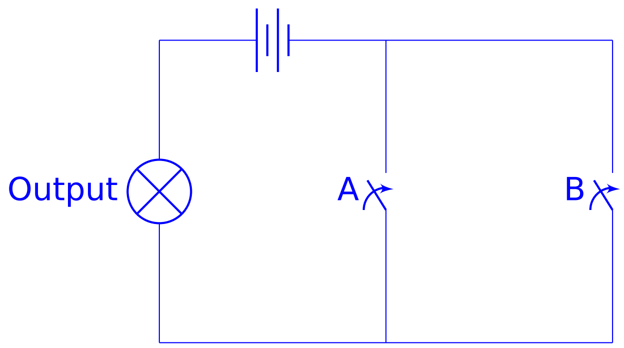
(b) When switch A is off, the current flows through the light bulb, so it is on.

(c) When switch B is off, the current flows through the light bulb, so it is on.

(d) When both switches are on, the current flows through the switches, so the light bulb is off.

(e) This circuit corresponds to a NAND gate.

1.14. Consider the following electrical circuit:



Answer the following questions:

(a) Say $A = 0$ and $B = 0$. Is the light bulb off or on?

(b) Say $A = 0$ and $B = 1$. Is the light bulb off or on?

(c) Say $A = 1$ and $B = 0$. Is the light bulb off or on?

(d) Say $A = 1$ and $B = 1$. Is the light bulb off or on?

(e) What logic gate does this circuit correspond to?

(a) When both switches are off, the current does not flow, so the light bulb is off.

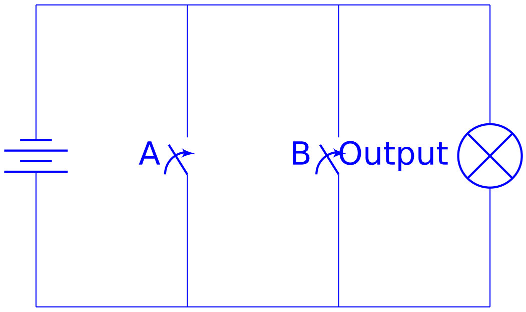
(b) When switch A is off, and switch B is on, the current flows through switch B, so the light bulb is on.

(c) When switch A is on, and switch B is off, the current flows through switch A, so the light bulb is on.

(d) When both switches are on, the current flows through both switches, so the light bulb is on.

(e) This circuit corresponds to an OR gate.

1.15. Consider the following electrical circuit:



Answer the following questions:

(a) Say $A = 0$ and $B = 0$. Is the light bulb off or on?

(b) Say $A = 0$ and $B = 1$. Is the light bulb off or on?

(c) Say $A = 1$ and $B = 0$. Is the light bulb off or on?

(d) Say $A = 1$ and $B = 1$. Is the light bulb off or on?

(e) What logic gate does this circuit correspond to?

(a) When both switches are off, the current flows through the light bulb, so it is on.

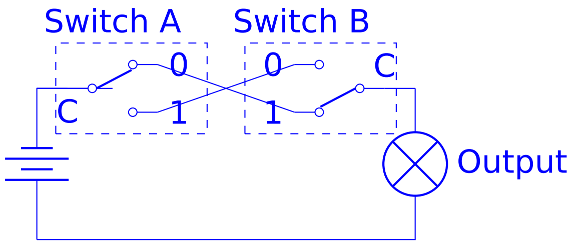
(b) When switch A is off and switch B is on, the current flows through switch B, so the light bulb is off.

(c) When switch A is on and switch B is off, the current flows through switch A, so the light bulb is off.

(d) When both switches are on, the current flows through both switches, so the light bulb is off.

(e) This circuit corresponds to a NOR gate.

1.16. In some homes in the United States, special switches are used so that two switches control a single light. Often, these switches are located at opposite ends of a stairway or a hallway, and either switch can be used to turn the light on or off. A traditional switch enables or disables the flow of electricity through a single wire. In contrast, these special switches, called *three pole switches*, choose between two different wires. The following electrical circuit gives an example:



Each switch has three poles, labeled C, 0 and 1. Switch A is currently flipped up, which connects C and 0. Switch B is currently flipped down, which connects C and 1. In this configuration, there is a complete path for the electricty to flow. It comes out of the positive end of the battery, through Switch A along $A=0$, then down to $B=1$, then through Switch B, then down through the light bulb, left through the bottom wire, and up to the negative end of the battery. So, the light bulb is on when $A=0$ and $B=1$.

(a) Say $A=0$ and $B=0$. Is the light bulb off or on?

(b) Say $A=1$ and $B=0$. Is the light bulb off or on?

(c) Say $A=1$ and $B=1$. Is the light bulb off or on?

(d) What logic gate does this circuit correspond to?

(a) When switch A is up (0) and switch B is up (0), the current does not flow, so the light bulb is off.

(b) When switch A is down (1) and switch B is up (0), the current flows through switch A and then switch B, so the light bulb is on.

(c) When switch A is down (1) and switch B is down (1), the current does not flow, so the light bulb is off.

(d) This circuit corresponds to an XOR gate.

1.17. Visit the Wikipedia article "Transistor count"[1].

(a) Pick an older computer processor. Which one did you pick, what year was it introduced, and how many transistors did it have?

(b) Pick a newer computer processor. Which one did you pick, what year was it introduced, and how many transistors did it have?

(a) The Zilog Z80 processor was introduced in 1976 and had 8500 transistors.

(b) The 11th gen Intel Core processor was introduced in 2021 and has more than 6 billion transistors.

2.8. A qubit is in the state

$$\frac{e^{i\pi/8}}{\sqrt{5}}\,|0\rangle + \beta\,|1\rangle\,.$$

What is a possible value of β?

$$\begin{aligned}
\left|\frac{e^{i\pi/8}}{\sqrt{5}}\right|^2 + |\beta|^2 &= 1\\
\frac{1}{5} + |\beta|^2 &= 1\\
|\beta|^2 &= \frac{4}{5}\\
|\beta| &= \frac{2}{\sqrt{5}}\\
\beta &= \frac{2e^{i\theta}}{\sqrt{5}}
\end{aligned}$$

2.9. A qubit is in the state

$$A\left(2e^{i\pi/6}\,|0\rangle - 3\,|1\rangle\right).$$

(a) Normalize the state (i.e. find A).

(b) If you measure the qubit, what is the probability that you get $|0\rangle$?

(c) If you measure the qubit, what is the probability that you get $|1\rangle$?

(a)

$$\left|2Ae^{i\pi/6}\right|^2 + |3A|^2 = 1$$
$$4|A|^2 + 9|A|^2 = 1$$
$$13|A|^2 = 1$$
$$|A|^2 = \frac{1}{13}$$
$$|A| = \frac{1}{\sqrt{13}}$$
$$A = \frac{e^{i\theta}}{\sqrt{13}}$$

(b) The probability of measuring $|0\rangle$ is $|2A|^2 = \frac{4}{13}$.

(c) The probability of measuring $|1\rangle$ is $|3A|^2 = \frac{9}{13}$.

2.10. A qubit is in the state

$$\frac{1}{2}|0\rangle - \frac{\sqrt(3)}{2}|1\rangle.$$

(a) If you measure it in the Z-basis $\{|0\rangle, |1\rangle\}$, what states can you get and with what probabilities?

(b) Write the qubit's state in terms of the X-basis $\{|+\rangle, |-\rangle\}$.

(c) If you measure it in the X-basis, what states can you get and with what probabilities?

(a) The probability of measuring $|0\rangle$ is $\left|\frac{1}{2}\right|^2 = \frac{1}{4}$. The probability of measuring $|1\rangle$ is $\left|-\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$.

(b) The X-basis states are $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Thus we have:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$|+\rangle + |-1\rangle = \sqrt{2}\,|0\rangle \qquad |0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$|+\rangle - |-1\rangle = \sqrt{2}\,|1\rangle \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

From these we calculate the qubit's state in the X-basis:

$$\begin{aligned}\frac{1}{2}|0\rangle-\frac{\sqrt{3}}{2}|1\rangle &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)\right)-\frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}(|+\rangle-|-\rangle)\right)\\ &= \frac{1}{2\sqrt{2}}|+\rangle+\frac{1}{2\sqrt{2}}|-\rangle-\frac{\sqrt{3}}{2\sqrt{2}}|+\rangle+\frac{\sqrt{3}}{2\sqrt{2}}|-\rangle\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}|+\rangle+\frac{1+\sqrt{3}}{2\sqrt{2}}|-\rangle\end{aligned}$$

(c) The probability of measuring $|+\rangle$ is $\left|\frac{1-\sqrt{3}}{2\sqrt{2}}\right|^2=\frac{4-2\sqrt{3}}{8}=\frac{2-\sqrt{3}}{4}$. The probability of measuring $|-\rangle$ is $\left|\frac{1+\sqrt{3}}{2\sqrt{2}}\right|^2=\frac{4+2\sqrt{3}}{8}=\frac{2+\sqrt{3}}{4}$.

2.11. The following states are opposite points on the Bloch sphere:

$$|a\rangle=\frac{\sqrt{3}}{2}|0\rangle+\frac{i}{2}|1\rangle$$
$$|b\rangle=\frac{i}{2}|0\rangle+\frac{\sqrt{3}}{2}|1\rangle$$

So, we can measure relative to them. Now consider a qubit in the state

$$\frac{1}{2}|0\rangle-\frac{\sqrt{3}}{2}|1\rangle.$$

(a) Write the qubit's state in terms of $|a\rangle$ and $|b\rangle$.

(b) If you measure the qubit in the basis $\{|a\rangle,|b\rangle\}$, what states can you get and with what probabilities?

(a) We can write the qubit's state in terms of $|a\rangle$ and $|b\rangle$ using these expressions for $|0\rangle$ and $|1\rangle$:

$$|0\rangle=\frac{\sqrt{3}}{2}|a\rangle-\frac{i}{2}|b\rangle \qquad |1\rangle=-\frac{i}{2}|a\rangle+\frac{\sqrt{3}}{2}|b\rangle$$

Then:

$$\begin{aligned}\frac{1}{2}|0\rangle-\frac{\sqrt{3}}{2}|1\rangle &= \frac{1}{2}\left(\frac{\sqrt{3}}{2}|a\rangle-\frac{i}{2}|b\rangle\right)-\frac{\sqrt{3}}{2}\left(-\frac{i}{2}|a\rangle+\frac{\sqrt{3}}{2}|b\rangle\right)\\ &= \frac{\sqrt{3}}{4}|a\rangle-\frac{i}{4}|b\rangle+\frac{i\sqrt{3}}{4}|a\rangle-\frac{3}{4}|b\rangle\\ &= \frac{\sqrt{3}(1+i)}{4}|a\rangle-\frac{3+i}{4}|b\rangle\end{aligned}$$

(b) The probability of measuring $|a\rangle$ is $\left|\frac{\sqrt{3}(1+i)}{4}\right|^2 = \frac{3\times(1+1)}{16} = \frac{3}{8}$. The probability of measuring $|b\rangle$ is $\left|-\frac{3+i}{4}\right|^2 = \frac{1+9}{16} = \frac{5}{8}$.

2.12. A qubit is in the state $|0\rangle$. If you measure it in the X-basis ($\{|+\rangle, |-\rangle\}$), and then measure it again in the Z-basis ($\{|0\rangle, |1\rangle\}$), what is the probability of getting

(a) $|0\rangle$?

(b) $|1\rangle$?

After measuring in the X-basis, the qubit is in the state $|+\rangle$ or $|-\rangle$. From either of these states, a measurement in the Z-basis will give

(a) $P(|0\rangle) = \frac{1}{2}$

(b) $P(|1\rangle) = \frac{1}{2}$.

2.13. Is there a measurement that can distinguish the following pairs of states? If yes, give a measurement. If no, explain your reasoning.

(a) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $e^{i\pi/8}|+\rangle = \frac{e^{i\pi/8}}{\sqrt{2}}(|0\rangle + |1\rangle)$

(b) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

(c) $|0\rangle$ and $e^{i\pi/4}|0\rangle$

(a) These states cannot be distinguished, because they differ only by a global phase.

(b) These states can be distinguished by measuring in the X-basis ($\{|+\rangle, |-\rangle\}$).

(c) These states cannot be distinguished, because they differ only by a global phase.

2.14. A qubit is in the state

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle).$$

(a) Where on the Bloch sphere is this state? Give your answer in (θ, ϕ) coordinates.

(b) Sketch the point on the Bloch sphere.

(a)

$$
\begin{aligned}
|i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \\
&= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \\
\implies \cos\left(\frac{\theta}{2}\right) &= \frac{1}{\sqrt{2}} \\
\frac{\theta}{2} &= \frac{\pi}{4} \\
\theta &= \frac{\pi}{2} \\
e^{i\phi} &= i \\
\phi &= \frac{\pi}{2}
\end{aligned}
$$

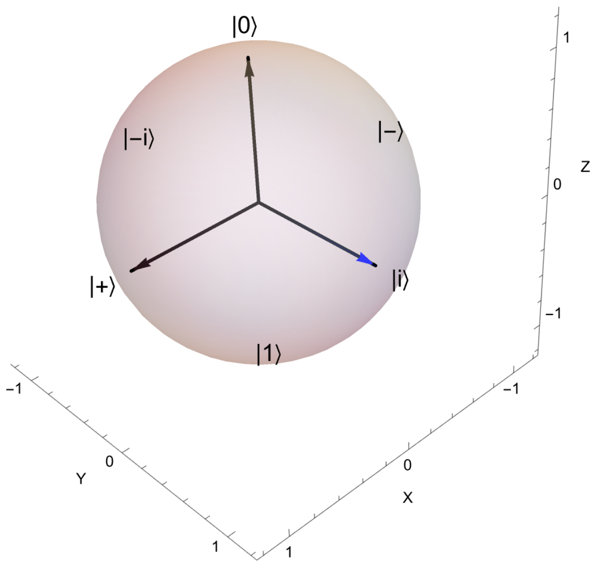


Figure 1: The point on the Bloch sphere.

(b)

2.15. A qubit is in the state

$$
\frac{1-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle .
$$

(a) Where on the Bloch sphere is this state? Give your answer in (θ, ϕ) coordinates.

(b) Sketch the point on the Bloch sphere.

(a)

$$\begin{aligned}\frac{1-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle &= \frac{e^{7\pi i/4}}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \\ &\equiv \frac{1}{2}|0\rangle + \frac{e^{-7i\pi/4}\sqrt{3}}{2}|1\rangle\end{aligned}$$

So:

$$\begin{aligned}\frac{1}{2}|0\rangle + \frac{e^{-7i\pi/4}\sqrt{3}}{2}|1\rangle &= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \\ \implies \cos\left(\frac{\theta}{2}\right) &= \frac{1}{2} \\ \frac{\theta}{2} &= \frac{\pi}{3} \\ \theta &= \frac{2\pi}{3} \\ e^{i\phi} &= e^{-7i\pi/4} \\ &= e^{i\pi/4} \\ \phi &= \frac{\pi}{4}\end{aligned}$$

(b)

2.16. Consider the following two states from Exercise 2.11:

$$\begin{aligned}|a\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle \\ |b\rangle &= \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\end{aligned}$$

Prove that these are opposite points of the Bloch sphere by finding their points in spherical coordinates (θ_a, ϕ_a) and (θ_b, ϕ_b). Verify that $\theta_b = \pi - \theta_a$ and $\phi_b = \phi_a + \pi$, which means they lie on opposite points of the Bloch sphere.

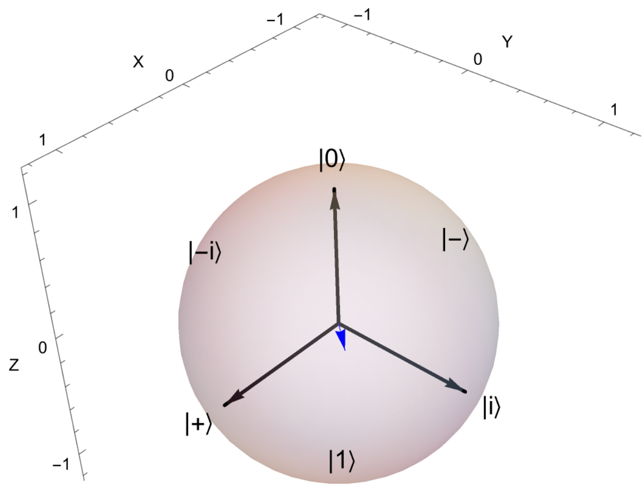


Figure 2: The point on the Bloch sphere.

$$
\begin{aligned}
|a\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle \\
&= \cos\frac{\theta_a}{2}|0\rangle + e^{i\phi_a}\sin\frac{\theta_a}{2}|1\rangle \\
\implies \cos\frac{\theta_a}{2} &= \frac{\sqrt{3}}{2} \\
\frac{\theta_a}{2} &= \frac{\pi}{6} \\
\theta_a &= \frac{\pi}{3} \\
e^{i\phi_a} &= i \\
\phi_a &= \frac{\pi}{2} \\
|b\rangle &= \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \\
&= \frac{e^{i\pi/2}}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \\
&\equiv \frac{1}{2}|0\rangle + \frac{e^{-i\pi/2}\sqrt{3}}{2}|1\rangle \\
&= \frac{1}{2}|0\rangle + \frac{e^{3i\pi/2}\sqrt{3}}{2}|1\rangle \\
&= \cos\frac{\theta_b}{2}|0\rangle + e^{i\phi_b}\sin\frac{\theta_b}{2}|1\rangle \\
\implies \cos\frac{\theta_b}{2} &= \frac{1}{2} \\
\frac{\theta_b}{2} &= \frac{\pi}{3}
\end{aligned}
$$

So:

$$\theta_b = \frac{2\pi}{3} = \pi - \frac{\pi}{3} = \pi - \theta_a$$
$$\phi_b = \frac{3\pi}{2} = \frac{\pi}{2} + \pi = \phi_a + \pi$$

TODO: Check answer; apparently something wrong with ϕ_a, ϕ_b.

2.22. Consider a map U that transforms the Z-basis states as follows:

$$U|0\rangle = |0\rangle + |1\rangle,$$
$$U|1\rangle = |0\rangle - |1\rangle.$$

Say $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a normalized quantum state, i.e. $|\alpha|^2 + |\beta|^2 = 1$.

(a) Calculate $U|\psi\rangle$.

(b) From your answer to (a), is U a valid quantum gate? Explain your reasoning.

(a) $U|\psi\rangle = (\alpha + \beta)|0\rangle + (\alpha - \beta)|1\rangle$.

(b) U is not a valid quantum gate, because it does not preserve the normalization of the state.

2.23. Consider a map U that transforms the Z-basis states as follows:

$$U|0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{\sqrt{3}+i}{4}|1\rangle,$$
$$U|1\rangle = \frac{\sqrt{3}+i}{4}|0\rangle - \frac{\sqrt{3}+3i}{4}|1\rangle.$$

Say $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a normalized quantum state, i.e. $|\alpha|^2 + |\beta|^2 = 1$.

(a) Calculate $U|\psi\rangle$.

(b) From your answer to (a), is U a valid quantum gate? Explain your reasoning.

(a) $U|\psi\rangle = \left(\frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{3}+i}{4}\beta\right)|0\rangle + \left(\frac{\sqrt{3}+i}{4}\alpha - \frac{\sqrt{3}+3i}{4}\beta\right)|1\rangle$.

(b) U is a valid quantum gate, because $UU^\dagger = I$.

4.3. Calculate the following inner products:

(a) $\langle 10|11\rangle$.

(b) $\langle +-|01\rangle$.

(c) $\langle 1+0|1-0\rangle$.

(a)

$$\begin{aligned}\langle 10|11\rangle &= \langle 1|1\rangle \langle 0|1\rangle \\ &= 1\times 0 \\ &= 0\end{aligned}$$

This is expected because $|10\rangle$ and $|11\rangle$ are orthogonal.

(b)

$$\begin{aligned}\langle +-|01\rangle &= \langle +|0\rangle \langle -|1\rangle \\ &= \left(\frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\langle 1|0\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 0|1\rangle - \frac{1}{\sqrt{2}}\langle 1|1\rangle\right) \\ &= \left(\frac{1}{\sqrt{2}}\times 1 + \frac{1}{\sqrt{2}}\times 0\right)\left(\frac{1}{\sqrt{2}}\times 0 - \frac{1}{\sqrt{2}}\times 1\right) \\ &= -\frac{1}{2}\end{aligned}$$

(c)

$$\begin{aligned}\langle 1+0|1-0\rangle &= \langle 1|1\rangle \langle +|-\rangle \langle 0|0\rangle \\ &= 1\times 0\times 1 \\ &= 0\end{aligned}$$

4.4. Verify that

$$|1\rangle\otimes|1\rangle\otimes|0\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix}.$$

$$
\begin{aligned}
|1\rangle \otimes |1\rangle \otimes |0\rangle &= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \\
&= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix} 0\begin{pmatrix}1\\0\end{pmatrix} \\ 1\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\
&= \begin{pmatrix} 0\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\ 1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix}.
\end{aligned}
$$

4.5. Consider a two-qubit state

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{i}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}+i}{4}|11\rangle\,.$$

(a) What is $|\psi\rangle$ as a (column) vector?

(b) What is $\langle\psi|$ as a (row) vector?

(a)

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{i}{\sqrt{2}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix}.$$

(b)

$$\langle\psi| = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{\sqrt{2}} & \frac{\sqrt{3}-i}{4} \end{pmatrix}.$$

4.6. Show that $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is a complete orthonornmal basis the state of two qubits by showing that it satisfies the completeness relation

$$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| = I.$$

where I is the identity matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned}
&|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

References

[1] Wikipedia. *Transistor count*. URL: https://en.wikipedia.org/wiki/Transistor_count.