PHYS440 - Exercise on basis states

John Hurst

March 2024

2.3. Given basis states $\{|0\rangle, |1\rangle\}$ for measurements on the Z axis and $\{|+\rangle, |-\rangle\}$ for measurements on the X axis find suitable basis states for measurements on the Y axis.

We'll follow the outline provided in exercise 2.3 in Susskind and Friedman[1].

We start with the states $|0\rangle$ and $|1\rangle$ for the Z axis.

For the X axis, we have:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Rearranging, we also have:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

And also these facts:

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$
$$\langle 0|1\rangle = \langle 1|0\rangle = 0$$
$$\langle +|+\rangle = \langle -|-\rangle = 1$$
$$\langle +|-\rangle = \langle -|+\rangle = 0$$
$$|\langle 0|+\rangle|^2 = \langle 0|+\rangle\langle +|0\rangle = \frac{1}{2}$$
$$|\langle 0|-\rangle|^2 = \langle 0|-\rangle\langle -|0\rangle = \frac{1}{2}$$
$$|\langle 1|+\rangle|^2 = \langle 1|+\rangle\langle +|1\rangle = \frac{1}{2}$$
$$|\langle 1|-\rangle|^2 = \langle 1|-\rangle\langle -|1\rangle = \frac{1}{2}$$

We will denote the basis states for the Y axis as $|a\rangle$ and $|b\rangle$ for now, and we will define them as a superposition of the Z axis basis states:

$$|a\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$|b\rangle = \gamma|0\rangle + \delta|1\rangle$$

where

$$\alpha = \langle 0|a\rangle \qquad \overline{\alpha} = \langle a|0\rangle$$
$$\beta = \langle 1|a\rangle \qquad \overline{\beta} = \langle a|1\rangle$$
$$\gamma = \langle 0|b\rangle \qquad \overline{\gamma} = \langle b|0\rangle$$
$$\delta = \langle 1|b\rangle \qquad \overline{\delta} = \langle b|1\rangle$$

For symmetry between the Z and Y axes, we have these relationships:

$$|\langle 0|a\rangle|^2 = \langle 0|a\rangle\langle a|0\rangle = \alpha\overline{\alpha} = \frac{1}{2}$$
$$|\langle 1|a\rangle|^2 = \langle 1|a\rangle\langle a|1\rangle = \beta\overline{\beta} = \frac{1}{2}$$
$$|\langle 0|b\rangle|^2 = \langle 0|b\rangle\langle b|0\rangle = \gamma\overline{\gamma} = \frac{1}{2}$$
$$|\langle 1|b\rangle|^2 = \langle 1|b\rangle\langle b|1\rangle = \delta\overline{\delta} = \frac{1}{2}$$

We also have similar relationships between the X and Y axes, in terms of $|+\rangle$

and $|-\rangle$, which we can rewrite in terms of $|0\rangle$ and $|1\rangle$. We can use $\langle +|a\rangle$ as follows:

$$\begin{aligned}
|\langle +|a\rangle|^2 = \langle +|a\rangle \langle a|+\rangle &= \frac{1}{2} \\
\frac{1}{\sqrt{2}}(\langle 0|a\rangle + \langle 1|a\rangle)\frac{1}{\sqrt{2}}(\langle a|0\rangle + \langle a|1\rangle) &= \frac{1}{2} \\
\frac{1}{2}(\langle 0|a\rangle \langle a|0\rangle + \langle 0|a\rangle \langle a|1\rangle + \langle 1|a\rangle \langle a|0\rangle + \langle 1|a\rangle \langle a|1\rangle) &= \frac{1}{2} \\
\frac{1}{2} + \langle 0|a\rangle \langle a|1\rangle + \langle 1|a\rangle \langle a|0\rangle + \frac{1}{2} &= 1 \\
\langle 0|a\rangle \langle a|1\rangle + \langle 1|a\rangle \langle a|0\rangle &= 0 \\
\alpha\overline{\beta} + \beta\overline{\alpha} &= 0 \\
\alpha\overline{\beta} &= -\overline{(\alpha\overline{\beta})}
\end{aligned}$$

For $z = -\overline{z}$, we have z purely imaginary, so $\alpha\overline{\beta}$ is purely imaginary, and α and β cannot both be real.

In the same way, we can find that $\gamma\overline{\delta}$ is purely imaginary, and so γ and δ cannot both be real.

We can set

$$\begin{aligned}
\alpha &= \frac{1}{\sqrt{2}} \\
\beta &= \frac{i}{\sqrt{2}} \\
\gamma &= \frac{1}{\sqrt{2}} \\
\delta &= -\frac{i}{\sqrt{2}}
\end{aligned}$$

Then we have

$$\begin{aligned}
|a\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \\
|b\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle
\end{aligned}$$

Checking this, we have

$$
\begin{aligned}
\langle a|a\rangle &= \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle + \frac{i}{2}\langle 0|1\rangle - \frac{i}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} + 0 - 0 + \frac{1}{2} \\
&= 1 \\
\langle a|b\rangle &= \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle - \frac{i}{2}\langle 0|1\rangle - \frac{i}{2}\langle 1|0\rangle - \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} - 0 - 0 - \frac{1}{2} \\
&= 0 \\
\langle b|b\rangle &= \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle - \frac{i}{2}\langle 0|1\rangle + \frac{i}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} - 0 + 0 + \frac{1}{2} \\
&= 1 \\
\langle b|a\rangle &= \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle + \frac{i}{2}\langle 0|1\rangle + \frac{i}{2}\langle 1|0\rangle - \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} + 0 + 0 - \frac{1}{2} \\
&= 0
\end{aligned}
$$

We have shown that the basis states for the Y axis require complex numbers. But the particular $|a\rangle$ and $|b\rangle$ we have chosen are not unique. We can introduce an arbitrary phase factor $e^{i\theta}(\theta \in \mathbb{R})$ to $|a\rangle$ and $|b\rangle$, and they will still be valid basis states for the Y axis.

Let

$$
\begin{aligned}
|c\rangle &= e^{i\theta}|a\rangle = \frac{e^{i\theta}}{\sqrt{2}}(|0\rangle + i|1\rangle) \\
|d\rangle &= e^{i\theta}|b\rangle = \frac{e^{i\theta}}{\sqrt{2}}(|0\rangle - i|1\rangle)
\end{aligned}
$$

The magnitudes of the amplitudes are unchanged, so the Born rule requirement is still met.

We can also check the inner products of these states:

$$\begin{aligned}
\langle c|c\rangle &= \frac{e^{-i\theta}}{\sqrt{2}}\left(\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\frac{e^{i\theta}}{\sqrt{2}}\left(|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle + \frac{i}{2}\langle 0|1\rangle - \frac{i}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} + 0 - 0 + \frac{1}{2} \\
&= 1 \\
\langle c|d\rangle &= \frac{e^{-i\theta}}{\sqrt{2}}\left(\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\frac{e^{i\theta}}{\sqrt{2}}\left(|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle - \frac{i}{2}\langle 0|1\rangle - \frac{i}{2}\langle 1|0\rangle - \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} - 0 - 0 - \frac{1}{2} \\
&= 0 \\
\langle d|d\rangle &= \frac{e^{-i\theta}}{\sqrt{2}}\left(\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right)\frac{e^{i\theta}}{\sqrt{2}}\left(|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle - \frac{i}{2}\langle 0|1\rangle + \frac{i}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} - 0 + 0 + \frac{1}{2} \\
&= 1 \\
\langle d|c\rangle &= \frac{e^{-i\theta}}{\sqrt{2}}\left(\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right)\frac{e^{i\theta}}{\sqrt{2}}\left(|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle + \frac{i}{2}\langle 0|1\rangle + \frac{i}{2}\langle 1|0\rangle - \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} + 0 + 0 - \frac{1}{2} \\
&= 0
\end{aligned}$$

The $|a\rangle$ and $|b\rangle$ above are the simplest of infinitely many possible choices. Following the nice convention in Wong[2], let's relabel these:

$$|i\rangle = |a\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$
$$|-i\rangle = |b\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

References

[1] Art Friedman Leonard Susskind. *Quantum Mechanics: The Theoretical Minimum*. 1st ed. Allen Lane, 2014. ISBN: 978-0-241-00344-1.

[2] Thomas G. Wong. *Introduction to Classical and Quantum Computing*. 1st ed. Rooted Grove, 2022. ISBN: 979-8-9855931-0-5.