Derivation of Symmetric Species Divergence Re-Enforcing FDFD Equations

Explanation of derivative operators:

Backward difference on arbitrary function.

$$f(x - n\Delta x) \approx \sum_{i=0}^{\mathrm{N}} \frac{f^{(i)}(x)(-\mathrm{n}\Delta \mathrm{x})^i}{i!}$$

I claim that at every node in the lattice a backward derivative can be applied to each polarization direction of a waveform.

$$f(x - n\Delta x) = \frac{n\Delta x f^0(x)}{0!} - \frac{n^1\Delta x^1 f^1(x)}{1!} + \frac{n^2\Delta x^2 f^2(x)}{2!} - \frac{n^3\Delta x^3 f^3(x)}{3!} + \cdots$$

We can vectorize the Taylor expansion to form a three-dimensional representation.

$$f(\mathbf{r} - s\mathbf{d}) = f(\mathbf{r}) - s\mathbf{d} \cdot \nabla f(\mathbf{r}) + s\mathbf{d} \cdot \nabla\nabla f(\mathbf{r}) \cdot \mathbf{d} - \cdots$$
$$s \in (\infty, 0) \cup (0, \infty)$$

Where

$$f(\mathbf{r}) = \hat{\mathbf{p}} \cdot \mathbf{r} = \hat{\mathbf{p}} \cdot \left(\ell_x(x,y,z)\hat{\mathbf{x}} + \ell_y(x,y,z)\hat{\mathbf{y}} + \ell_z(x,y,z)\hat{\mathbf{z}}\right)$$

$$s\mathbf{d} \; = \; s\big((x - x_0)\hat{\mathbf{x}} + (y - y_0)\hat{\mathbf{y}} + (z - z_0)\hat{\mathbf{z}}\big) \; = \; s(\Delta x\hat{\mathbf{x}} + \Delta y\hat{\mathbf{y}} + \Delta z\hat{\mathbf{z}})$$

$$\Delta x = \Delta y = \Delta z$$

$$\hat{\mathbf{p}}_x \cdot \hat{\mathbf{x}} = 1,\; \hat{\mathbf{p}}_y \cdot \hat{\mathbf{y}} = 1,\; \hat{\mathbf{p}}_z \cdot \hat{\mathbf{z}} = 1$$

$\ell(x,y,z)$ is a function of the metric and $\hat{\mathbf{p}}$ is the unit polarization direction of a waveform.

Also, we have the standard gradient.

$$\nabla = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}} \right)$$

We can project our gradient in any direction.

$$\nabla_x = \frac{f(x) - f(x - \Delta x)}{2\Delta x}\hat{\boldsymbol{x}}$$

$$\nabla_y = \frac{f(y) - f(y - \Delta y)}{2\Delta y}\hat{\boldsymbol{y}}$$

$$\nabla_z = \frac{f(z) - f(z - \Delta z)}{2\Delta z}\hat{\boldsymbol{z}}$$

Since there are two nodes to consider at each step we can average the displacements across each unit wedge or cube specified by the eigenvalue centered on (*i, j, k*) for each projection.

$$\frac{2\Delta x}{2} = \Delta d$$

$$\frac{2\Delta y}{2} = \Delta d$$

$$\frac{2\Delta z}{2} = \Delta d$$

The projections become

$$\nabla_x = \frac{f(x) - f(x - \Delta x)}{\Delta x}\hat{\boldsymbol{x}}$$

$$\nabla_y = \frac{f(y) - f(y - \Delta y)}{\Delta y}\hat{\boldsymbol{y}}$$

$$\nabla_z = \frac{f(z) - f(z - \Delta z)}{\Delta z}\hat{\boldsymbol{z}}$$

We can find the Taylor expansion for any polarization direction we choose. Along the x direction we have:

$$f(\mathbf{r}-s\mathbf{d}) = f(\mathbf{r}) - s\mathbf{d}\cdot\nabla f(\mathbf{r}) + s\mathbf{d}\cdot\nabla\nabla f(\mathbf{r})\cdot\mathbf{d} - \cdots$$

$$f(\mathbf{r}-s\mathbf{d}) = \widehat{\mathbf{p}}_x \cdot \left(\ell_x(x,y,z)\hat{\mathbf{x}} + \ell_y(x,y,z)\hat{\mathbf{y}} + \ell_z(x,y,z)\hat{\mathbf{z}}\right) -$$
$$s(\Delta x\hat{\mathbf{x}} + \Delta y\hat{\mathbf{y}} + \Delta z\hat{\mathbf{z}})\cdot\left(\frac{\partial}{\partial x}\widehat{\boldsymbol{x}}\right)\left(\widehat{\mathbf{p}}_x \cdot \left(\ell_x(x,y,z)\hat{\mathbf{x}} + \ell_y(x,y,z)\hat{\mathbf{y}} + \ell_z(x,y,z)\hat{\mathbf{z}}\right)\right) + \cdots$$

$$\ell_x(x-\Delta x,y,z) = \widehat{\mathbf{p}}_x \cdot \left(\ell_x(x,y,z)\hat{\mathbf{x}} + \ell_y(x,y,z)\hat{\mathbf{y}} + \ell_z(x,y,z)\hat{\mathbf{z}}\right) -$$
$$s(\Delta x\hat{\mathbf{x}} + \Delta y\hat{\mathbf{y}} + \Delta z\hat{\mathbf{z}})\cdot\left(\frac{\partial}{\partial x}\widehat{\boldsymbol{x}}\right)\left(\widehat{\mathbf{p}}_x \cdot \left(\ell_x(x,y,z)\hat{\mathbf{x}} + \ell_y(x,y,z)\hat{\mathbf{y}} + \ell_z(x,y,z)\hat{\mathbf{z}}\right)\right) + \cdots$$

$$\ell_x(x-\Delta x,y,z) = \ell_x(x,y,z) - s\left(\Delta x\frac{\partial}{\partial x}\ell_x(x,y,z)\right) + \cdots$$

$$\ell_x(x-\Delta x,y,z) = \ell_x(x,y,z) - s\Delta d\frac{\partial}{\partial x}\ell_x(x,y,z) + \cdots$$

$$\ell_x(x-\Delta x,y,z) = \ell_x(x,y,z) - s\Delta d\ell_x'\,(x,y,z) + \cdots$$

$$\ell_x'\,(x,y,z) = \frac{\ell_x(x,y,z) - \ell_x(x-\Delta x,y,z)}{s\Delta d}$$

$$\ell_x'(i,j,k) = \frac{\ell_x(i,j,k) - \ell_x(i-1,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

Similarly, along the remaining directions we have

$$\ell_y'(i,j,k) = \frac{\ell_y(i,j,k) - \ell_y(i,j-1,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_z'(i,j,k) = \frac{\ell_z(i,j,k) - \ell_z(i,j,k-1)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

These are first order second order accurate double backward derivatives as a function of the entire Yee lattice. The eigenvalues $\boldsymbol{s}$ may affect convergence rate. For higher order normal and mixed derivatives, we apply the derivative operation in succession, or find systems of Taylor expansions and optimize error convergence.

For this research, the finite-difference schemes used are as follows:

First order backward first order accurate derivatives.

$$\ell_x'(i,j,k) = \frac{\ell_x(i,j,k) - \ell_x(i-1,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_y'(i,j,k) = \frac{\ell_y(i,j,k) - \ell_y(i,j-1,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_z'(i,j,k) = \frac{\ell_z(i,j,k) - \ell_z(i,j,k-1)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Second order backward first order accurate derivatives.

$$\ell_x''(i,j,k) = \frac{\ell_x(i,j,k) - \ell_x(i-1,j,k) + \ell_x(i-2,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_y''(i,j,k) = \frac{\ell_y(i,j,k) - \ell_y(i,j-1,k) + \ell_y(i-2,j,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_z''(i,j,k) = \frac{\ell_z(i,j,k) - \ell_z(i,j,k-1) + \ell_z(i-2,j,k)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

First order forward first order accurate derivatives.

$$\ell_x'(i,j,k) = \frac{\ell_x(i+1,j,k) - \ell_x(i,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_y'(i,j,k) = \frac{\ell_y(i,j+1,k) - \ell_y(i,j,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_z'(i,j,k) = \frac{\ell_z(i,j,k+1) - \ell_z(i,j,k)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Second order forward first order accurate derivatives.

$$\ell_x''(i,j,k) = \frac{\ell_x(i+2,j,k) - \ell_x(i+1,j,k) + \ell_x(i,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_y''(i,j,k) = \frac{\ell_y(i,j+2,k) - \ell_y(i,j+1,k) + \ell_y(i,j,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell_z''(i,j,k) = \frac{\ell_z(i,j,k+2) - \ell_z(i,j,k+1) + \ell_z(i,j,k)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Next, we use these derivatives in the derivation of the electric and magnetic field equations with the understanding that the origination of the of the electric and magnetic fields exist everywhere on the Yee lattice, and propagation direction depends on the eigenvalue of the incidence field.

Total field formulation:

$$\vec{\mathbf{H}}_{\mathbf{T}} = \vec{\mathbf{H}}_{\mathbf{scat}} + \vec{\mathbf{H}}_{\mathbf{inc}}$$

Total Field Formulation for Magnetism

$$\vec{\mathbf{E}}_{\mathbf{T}} = \vec{\mathbf{E}}_{\mathbf{scat}} + \vec{\mathbf{E}}_{\mathbf{inc}}$$

Total Field Formulation for Electricity

Isotropic constitutive relations:

$$\vec{\mathbf{D}}_{\mathbf{T}} = \boldsymbol{\varepsilon}\vec{\mathbf{E}}_{\mathbf{T}}$$

Electric Linear Isotropic Constitutive Relation

$$\vec{\mathbf{B}}_{\mathbf{T}} = \mu\vec{\mathbf{H}}_{\mathbf{T}}$$

Magnetic Linear Isotropic Constitutive Relation

Maxwell Equations in frequency domain:

$$\vec{\nabla} \cdot \mu \vec{\mathbf{H}}_{\mathbf{T}} = 0$$

Gauss's Law for Magnetism

$$\vec{\nabla} \cdot \varepsilon \vec{\mathbf{E}}_{\mathbf{T}} = 0$$

Gauss's Law for Electric Field for Zero Free Charges

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{T}} + \vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon \vec{\mathbf{E}}_{\mathbf{T}}\right) = -j\omega\mu \vec{\mathbf{H}}_{\mathbf{T}}$$

Faraday's Law of Induction with Divergence Enforced

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{T}} + \vec{\nabla}\left(\vec{\nabla} \cdot \mu \vec{\mathbf{H}}_{\mathbf{T}}\right) = j\omega\varepsilon \vec{\mathbf{E}}_{\mathbf{T}}$$

Ampere's Circuital Law with Divergence Enforced

From here we need to think about symmetry and causality. Totally symmetric application of the total fields with divergence ensures divergence is enforced on the scattered and incident fields at each node. We can find coefficients with the Maxwell electric curl equation. For instance, in the x-direction we have:

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{T}} + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon \vec{\mathbf{E}}_{\mathbf{T}}\right)\right) = -j\omega\mu \vec{\mathbf{H}}_{\mathbf{T}}$$

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{inc}} + \nabla \times \vec{\mathbf{E}}_{\mathbf{scat}} + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}}\right)\right) + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}}\right)\right) = -j\omega\mu \vec{\mathbf{H}}_{\mathbf{inc}} - j\omega\mu \vec{\mathbf{H}}_{\mathbf{scat}}$$

$$-j\omega\mu_0 \vec{\mathbf{H}}_{\mathbf{inc}} + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}}\right)\right) + \nabla \times \vec{\mathbf{E}}_{\mathbf{scat}} + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}}\right)\right) = -j\omega\mu_{xi} \vec{\mathbf{H}}_{\mathbf{inc}} - j\omega\mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}}$$

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{scat}} + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}}\right)\right) + diag\left(\vec{\nabla}\left(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}}\right)\right) + j\omega\mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}} = j\omega\mu_0 \vec{\mathbf{H}}_{\mathbf{inc}} - j\omega\mu_{xi} \vec{\mathbf{H}}_{\mathbf{inc}}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}\varepsilon_{xi} E_{x,scat} + \frac{\partial}{\partial y} E_{z,scat} - \frac{\partial}{\partial z} E_{y,scat} + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\varepsilon_{xi} E_{x,inc} + j\omega\mu_{xi} H_{x,scat} = j\omega(\mu_0 - \mu_{xi}) H_{x,inc}$$

The diagonal terms in the permeability dyadic become off-diagonal terms in the derivative.

$$\frac{\varepsilon_{xi}}{j\omega\mu_{xi}}\frac{\partial}{\partial x}\frac{\partial}{\partial x}E_{x,scat}+\frac{1}{j\omega\mu_{xy}}\frac{\partial}{\partial y}E_{z,scat}-\frac{1}{j\omega\mu_{xz}}\frac{\partial}{\partial z}E_{y,scat}+\frac{\varepsilon_{xi}}{j\omega\mu_{xi}}\frac{\partial}{\partial x}\frac{\partial}{\partial x}E_{x,inc}+$$
$$H_{x,scat}=\frac{\mu_0-\mu_{xi}}{\mu_{xi}}H_{x,inc}$$

Discretizing, we have:

X-Direction:

$$\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(E_{x,scat}(i+2,j,k)-E_{x,scat}(i+1,j,k)+E_{x,scat}(i,j,k)\Big)+$$
$$\frac{1}{j\omega\mu_{xy}(i,j,k)\boldsymbol{s}\Delta y}\cdot\Big(E_{z,scat}(i,j+1,k)-E_{z,scat}(i,j,k)\Big)-$$
$$\frac{1}{j\omega\mu_{xz}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(E_{y,scat}(i,j,k+1)-E_{y,scat}(i,j,k)\Big)+$$
$$\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(E_{x,inc}(i+2,j,k)-E_{x,inc}(i+1,j,k)+E_{x,inc}(i,j,k)\Big)+$$
$$H_{x,scat}(i,j,k)=\frac{\mu_0(i,j,k)-\mu_{xi}(i,j,k)}{\mu_{xi}(i,j,k)}H_{x,inc}(i,j,k)$$

Our coefficients for magnetic incidence in the x-direction become:

$$\mathbf{C_{hxhx}}=\frac{\mu_0(i,j,k)-\mu_{xi}(i,j,k)}{\mu_{xi}(i,j,k)}$$

$$\mathbf{C_{hxei}}=\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{hxex}}=\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{hxey}}=\frac{1}{j\omega\mu_{xz}(i,j,k)\boldsymbol{s}\Delta z}$$

$$\mathbf{C_{hxez}} = \frac{1}{j\omega\mu_{xy}(i,j,k)\boldsymbol{s}\Delta y}$$

Similarly, we can find the coefficient in the remaining incidence directions using the same method in the y-direction and z-direction, respectively.

Y-Direction:

$$\frac{\varepsilon_{yi}(i,j,k)}{j\omega\mu_{yi}(i,j,k)\boldsymbol{s}\Delta y}\cdot\left(E_{y,scat}(i,j+2,k) - E_{y,scat}(i,j+1,k) + E_{y,scat}(i,j,k)\right) +$$
$$\frac{1}{j\omega\mu_{yx}(i,j,k)\boldsymbol{s}\Delta x}\cdot\left(E_{z,scat}(i+1,j,k) - E_{z,scat}(i,j,k)\right) -$$
$$\frac{1}{j\omega\mu_{yz}(i,j,k)\boldsymbol{s}\Delta z}\cdot\left(E_{x,scat}(i,j,k+1) - E_{x,scat}(i,j,k)\right) +$$
$$\frac{\varepsilon_{yi}(i,j,k)}{j\omega\mu_{yi}(i,j,k)\boldsymbol{s}\Delta y}\cdot\left(E_{y,inc}(i,j+2,k) - E_{y,inc}(i,j+1,k) + E_{y,inc}(i,j,k)\right) +$$
$$H_{y,scat}(i,j,k) = \frac{\mu_0(i,j,k) - \mu_{yi}(i,j,k)}{\mu_{yi}(i,j,k)}H_{y,inc}(i,j,k)$$

Our coefficients for magnetic incidence in the y-direction become:

$$\mathbf{C_{hyhy}} = \frac{\mu_0(i,j,k) - \mu_{yi}(i,j,k)}{\mu_{yi}(i,j,k)}$$

$$\mathbf{C_{hyex}} = \frac{1}{j\omega\mu_{yz}(i,j,k)\boldsymbol{s}\Delta z}$$

$$\mathbf{C_{hyei}} = \frac{\varepsilon_{yi}(i,j,k)}{j\omega\mu_{yi}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{hyey}} = \frac{\varepsilon_{yi}(i,j,k)}{j\omega\mu_{yi}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{hyez}} = \frac{1}{j\omega\mu_{yx}(i,j,k)\boldsymbol{s}\Delta x}$$

Z-Direction:

$$\frac{\varepsilon_{zi}(i,j,k)}{j\omega\mu_{zi}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(E_{z,scat}(i+2,j,k)-E_{z,scat}(i+1,j,k)+E_{z,scat}(i,j,k)\Big)+$$

$$\frac{1}{j\omega\mu_{zx}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(E_{y,scat}(i+1,j,k)-E_{y,scat}(i,j,k)\Big)-$$

$$\frac{1}{j\omega\mu_{zy}(i,j,k)\boldsymbol{s}\Delta y}\cdot\Big(E_{x,scat}(i,j+1,k)-E_{x,scat}(i,j,k)\Big)+$$

$$\frac{\varepsilon_{zi}(i,j,k)}{j\omega\mu_{zi}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(E_{z,inc}(i+2,j,k)-E_{z,inc}(i+1,j,k)+E_{z,inc}(i,j,k)\Big)+$$

$$H_{z,scat}(i,j,k)=\frac{\mu_0(i,j,k)-\mu_{zi}(i,j,k)}{\mu_{zi}(i,j,k)}H_{z,inc}(i,j,k)$$

Our coefficients for magnetic incidence in the z-direction become:

$$\mathbf{C_{hzhz}}=\frac{\mu_0(i,j,k)-\mu_{zi}(i,j,k)}{\mu_{zi}(i,j,k)}$$

$$\mathbf{C_{hzex}}=\frac{1}{j\omega\mu_{zy}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{hzey}}=\frac{1}{j\omega\mu_{zx}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{hzei}}=\frac{\varepsilon_{zi}(i,j,k)}{j\omega\mu_{zi}(i,j,k)\boldsymbol{s}\Delta z}$$

$$\mathbf{C_{hzez}}=\frac{\varepsilon_{zi}(i,j,k)}{j\omega\mu_{zi}(i,j,k)\boldsymbol{s}\Delta z}$$

We use the Maxwell magnetic curl equation using the same method to find the other coefficients. For instance, in the x-direction we have:

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{T}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu \vec{\mathbf{H}}_{\mathbf{T}})\right) = j\omega\varepsilon \vec{\mathbf{E}}_{\mathbf{T}}$$

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{inc}} + \nabla \times \vec{\mathbf{H}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_0 \vec{\mathbf{H}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}})\right) = j\omega\varepsilon_{xi}\vec{\mathbf{E}}_{\mathbf{inc}} + j\omega\varepsilon_{xi}\vec{\mathbf{E}}_{\mathbf{scat}}$$

$$j\omega\varepsilon_0 \vec{\mathbf{E}}_{\mathbf{inc}} + \nabla \times \vec{\mathbf{H}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_0 \vec{\mathbf{H}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}})\right) = j\omega\varepsilon_{xi}\vec{\mathbf{E}}_{\mathbf{inc}} + j\omega\varepsilon_{xi}\vec{\mathbf{E}}_{\mathbf{scat}}$$

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_0 \vec{\mathbf{H}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}})\right) + j\omega\varepsilon_{xi}\vec{\mathbf{E}}_{\mathbf{scat}} = j\omega\varepsilon_{xi}\vec{\mathbf{E}}_{\mathbf{inc}} - j\omega\varepsilon_0\vec{\mathbf{E}}_{\mathbf{inc}}$$

$$\mu_{xi}\frac{\partial}{\partial x}\frac{\partial}{\partial x}H_{x,scat} + \frac{\partial}{\partial y}H_{z,scat} - \frac{\partial}{\partial z}H_{y,scat} + \mu_{xi}\frac{\partial}{\partial x}\frac{\partial}{\partial x}H_{x,inc} - j\omega\varepsilon_{xi}E_{x,scat} = j\omega(\varepsilon_{xi} - \varepsilon_0)E_{x,inc}$$

The diagonal terms in the permeability dyadic become off-diagonal terms in the derivative.

$$\frac{\mu_{xi}}{j\omega\varepsilon_{xi}} \cdot \frac{\partial}{\partial x}\frac{\partial}{\partial x}H_{x,scat} + \frac{1}{j\omega\varepsilon_{xy}} \cdot \frac{\partial}{\partial y}H_{z,scat} - \frac{1}{j\omega\varepsilon_{xz}} \cdot \frac{\partial}{\partial z}H_{y,scat} + \frac{\mu_{xi}}{j\omega\varepsilon_{xi}} \cdot \frac{\partial}{\partial x}\frac{\partial}{\partial x}H_{x,inc} - E_{x,scat} = \frac{(\varepsilon_{xi} - \varepsilon_0)}{\varepsilon_{xi}}E_{x,inc}$$

Discretizing, we have:

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)}\cdot\frac{H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)}{\boldsymbol{s}\Delta x}+$$

$$\frac{1}{j\omega\varepsilon_{xy}(i,j,k)}\cdot\frac{H_{z,scat}(i,j,k)-H_{z,scat}(i,j-1,k)}{\boldsymbol{s}\Delta y}-$$

$$\frac{1}{j\omega\varepsilon_{xz}(i,j,k)}\cdot\frac{H_{y,scat}(i,j,k)-H_{y,scat}(i,j,k-1)}{\boldsymbol{s}\Delta z}+$$

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)}\cdot\frac{H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)}{\boldsymbol{s}\Delta x}+$$

$$E_{x,scat}(i,j,k)=\frac{\varepsilon_{xi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{xi}(i,j,k)}E_{x,inc}(i,j,k)$$

<u>**X-Direction**</u>:

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)\Big)+$$

$$\frac{1}{j\omega\varepsilon_{xy}(i,j,k)\boldsymbol{s}\Delta y}\cdot\Big(H_{z,scat}(i,j,k)-H_{z,scat}(i,j-1,k)\Big)-$$

$$\frac{1}{j\omega\varepsilon_{xz}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(H_{y,scat}(i,j,k)-H_{y,scat}(i,j,k-1)\Big)+$$

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)\Big)+$$

$$E_{x,scat}(i,j,k)=\frac{\varepsilon_{xi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{xi}(i,j,k)}E_{x,inc}(i,j,k)$$

Our coefficients for electric incidence in the x-direction become:

$$\mathbf{C_{exex}}=\frac{\varepsilon_{xi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{xi}(i,j,k)}$$

$$\mathbf{C_{exhi}}=\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{exhx}} = \frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{exhy}} = \frac{1}{j\omega\varepsilon_{xy}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{exhz}} = \frac{1}{j\omega\varepsilon_{xz}(i,j,k)\boldsymbol{s}\Delta z}$$

Similarly, we can find the coefficient in the remaining incidence directions using the same method in the y-direction and z-direction, respectively.

<u>Y-Direction</u>:

$$\frac{\mu_{yi}(i,j,k)}{j\omega\varepsilon_{yi}(i,j,k)\boldsymbol{s}\Delta y}\cdot\Big(H_{y,scat}(i,j,k) - H_{y,scat}(i-1,j,k) + H_{y,scat}(i-2,j,k)\Big) +$$
$$\frac{1}{j\omega\varepsilon_{yx}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(H_{z,scat}(i,j,k) - H_{z,scat}(i-1,j,k)\Big) -$$
$$\frac{1}{j\omega\varepsilon_{yz}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(H_{x,scat}(i,j,k) - H_{x,scat}(i,j,k-1)\Big) +$$
$$\frac{\mu_{yi}(i,j,k)}{j\omega\varepsilon_{yi}(i,j,k)\boldsymbol{s}\Delta y}\cdot\Big(H_{y,scat}(i,j,k) - H_{y,scat}(i-1,j,k) + H_{y,scat}(i-2,j,k)\Big) +$$
$$E_{y,scat}(i,j,k) = \frac{\varepsilon_{yi}(i,j,k) - \varepsilon_0(i,j,k)}{\varepsilon_{yi}(i,j,k)}E_{y,inc}(i,j,k)$$

Our coefficients for electric incidence in the y-direction become:

$$\mathbf{C_{eyey}} = \frac{\varepsilon_{yi}(i,j,k) - \varepsilon_0(i,j,k)}{\varepsilon_{yi}(i,j,k)}$$

$$\mathbf{C_{eyhx}} = \frac{1}{j\omega\varepsilon_{yz}(i,j,k)\boldsymbol{s}\Delta z}$$

$$\mathbf{C_{eyhi}} = \frac{\mu_{yi}(i,j,k)}{j\omega\varepsilon_{yi}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{eyhy}} = \frac{\mu_{yi}(i,j,k)}{j\omega\varepsilon_{yi}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{eyhz}} = \frac{1}{j\omega\varepsilon_{yx}(i,j,k)\boldsymbol{s}\Delta x}$$

<u>Z-Direction</u>:

$$\frac{\mu_{zi}(i,j,k)}{j\omega\varepsilon_{zi}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(H_{z,scat}(i,j,k) - H_{z,scat}(i-1,j,k) + H_{z,scat}(i-2,j,k)\Big) +$$

$$\frac{1}{j\omega\varepsilon_{zx}(i,j,k)\boldsymbol{s}\Delta x}\cdot\Big(H_{y,scat}(i,j,k) - H_{y,scat}(i-1,j,k)\Big) -$$

$$\frac{1}{j\omega\varepsilon_{zy}(i,j,k)\boldsymbol{s}\Delta y}\cdot\Big(H_{x,scat}(i,j,k) - H_{x,scat}(i,j-1,k)\Big) +$$

$$\frac{\mu_{zi}(i,j,k)}{j\omega\varepsilon_{zi}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(H_{z,scat}(i,j,k) - H_{z,scat}(i-1,j,k) + H_{z,scat}(i-2,j,k)\Big) +$$

$$E_{z,scat}(i,j,k) = \frac{\varepsilon_{zi}(i,j,k) - \varepsilon_0(i,j,k)}{\varepsilon_{zi}(i,j,k)}E_{z,inc}(i,j,k)$$

Our coefficients for electric incidence in the z-direction become:

$$\mathbf{C_{ezez}} = \frac{\varepsilon_{zi}(i,j,k) - \varepsilon_0(i,j,k)}{\varepsilon_{zi}(i,j,k)}$$

$$\mathbf{C_{ezhx}} = \frac{1}{j\omega\varepsilon_{zy}(i,j,k)\boldsymbol{s}\Delta y}$$

$$\mathbf{C_{ezhi}} = \frac{1}{j\omega\varepsilon_{zx}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{ezhy}} = \frac{1}{j\omega\varepsilon_{zx}(i,j,k)\boldsymbol{s}\Delta x}$$

$$\mathbf{C_{ezhz}} = \frac{\mu_{zi}(i,j,k)}{j\omega\varepsilon_{zi}(i,j,k)\boldsymbol{s}\Delta z}$$

At this point, it may be necessary to perform analysis and apply either a uniform value for ***s*** in each coefficient expression, or apply different values of ***s*** in each coefficient expression to increase convergence rate as well as accuracy of the divergence re-enforcing field equations of the FDFD method.