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Master's Thesis

Improvement of the Finite-Difference Frequency-Domain Method in Fortran

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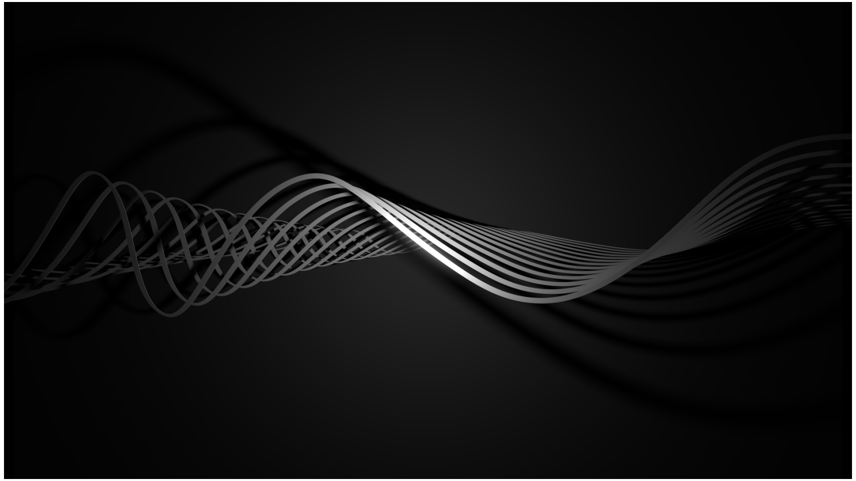
Computational Electromagnetics

- Computational electromagnetics (CEM) research offers an energizing way to initialize improved performance and minimize spatial constraints in practical realizations such as *cell phones, cellular networks, sensor systems, radar,* and *stealth systems* by introducing the potential for innovation in critical components and the ability to perform analysis of modern electrical structures.

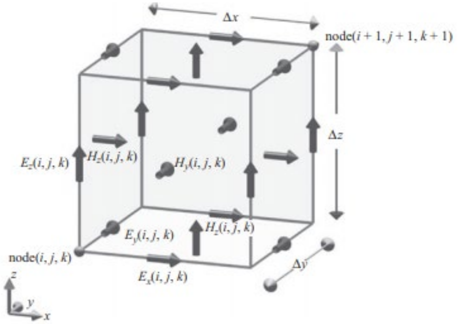


Time-Domain Vs. Frequency Domain

- All computational electromagnetic algorithms use numerical methods to solve the Maxwell equations in either the time or frequency domain.
- Depending on the temporal domain, a field propagator can be chosen (IE or DE) where harmonic time variation is assumed as the basis mode of operation.
- The most suitable selection of domain is dependent upon the type of problem space being modelled.
- The frequency domain technique tends to be more useful for problems with narrow bandwidth and high Q-factors.
- Historically, the frequency domain has been favored for canonical problems that are useful for verifying numerical results in real-world applications.
- Time domain techniques are more often useful for broadband structures with complex and large geometries.

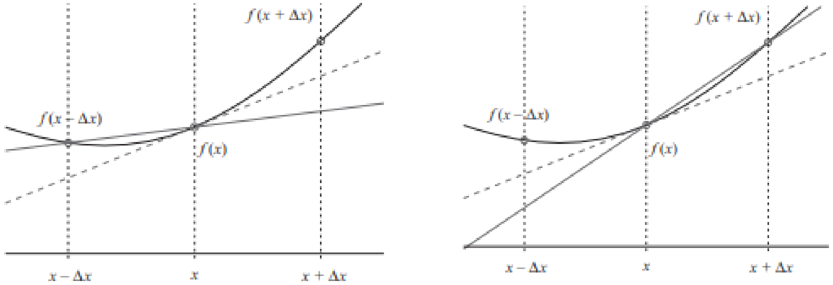
What is the Finite-Difference Method?

- The FD technique is based on discretizing Maxwell curl equations in both the time and frequency domains and implementing linear field equations upon a special lattice known as a Yee grid that defines the computational space.
- The Yee lattice can be a single lattice or superimposed double lattice that allows for field values to occupy every edge and vertex of the Yee cell and allows support of both bi-isotropic and bianisotropic problem spaces.
- The FDFD method is a differential equation method that relies on both of Maxwell differential curl equations that describe Faraday's Law of Induction as well as Ampere's Circuital Law.
- The derivative operators in the finite-difference scheme are based on the forward, backward, and central differences with multiple orders of accuracy derived from their respective Taylor expansions.



Why are Finite-Differences Important?

- The FDFD method has no analytical load since there is no need for structure dependent Green's functions.
- This enables the FDFD method to compute electromagnetic problems with complex geometric structures that would otherwise be extremely difficult to analyze with other methods.
- The FDFD method is also very easy to apply to non-uniform media, enabling material generality to be broad.
- Geometric versatility is very good, allowing FDFD to model complex-shaped and inhomogeneous structures.
- The FDFD method handles dispersive, frequency dependent media very easily while being accurate and robust, being more stable than it's time domain counterparts like FDTD.



Objectives of Research

- Improve computational efficiency and accuracy of established FDFD method.
  - Implement divergence re-enforcing finite-difference frequency-domain equations.
  - Establish faithful comparison between analytic and CEM results.

Core Features of FDFD

- Field Equations
- Perfectly Matched Layers
- Krylov Subspace
- Equivalence Principle

Annihilation

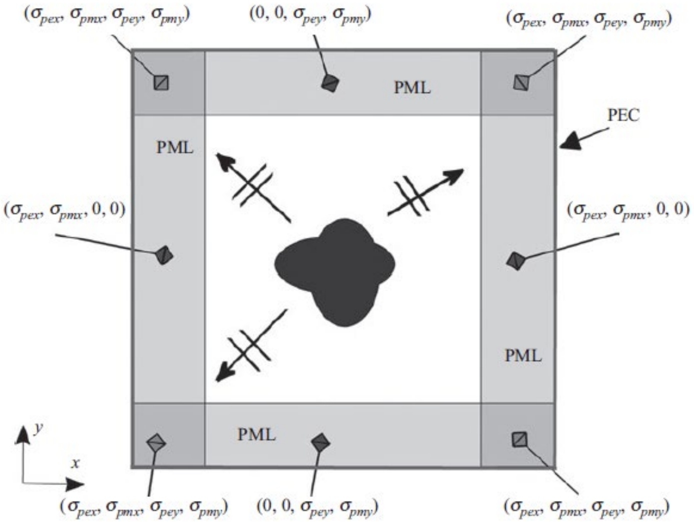


Diagram of PML Region

$$R(\varphi_0) = e^{-2\frac{\sigma\cos(\varphi_0)}{\varepsilon_0 c}\delta}$$

Theoretical Reflection Coefficient

$$\sigma(\rho) = \sigma_{max}\left(\frac{\rho}{\delta}\right)^{n_{pml}}$$

Power Increasing PML Loss Function

$$\sigma_{max} = -\frac{(n_{pml}+1)\varepsilon_0 c \ln(R(0))}{2\Delta s N}$$

Power Increasing PML Loss Function

BiCGSTAB – The Perfect Iterative Method

For each matrix A and vector v, the nested sequence of Krylov subspaces defined by

$$\mathcal{K}_n(A, v) \equiv span(v, Av, \dots, A^{n-1}v), \text{for } n = 1,2, \dots$$

will eventually stop to grow and become invariant under A. If d is the grade of v with respect to A, then

$$A^d v \in \mathcal{K}_d(A, v)$$

and hence,

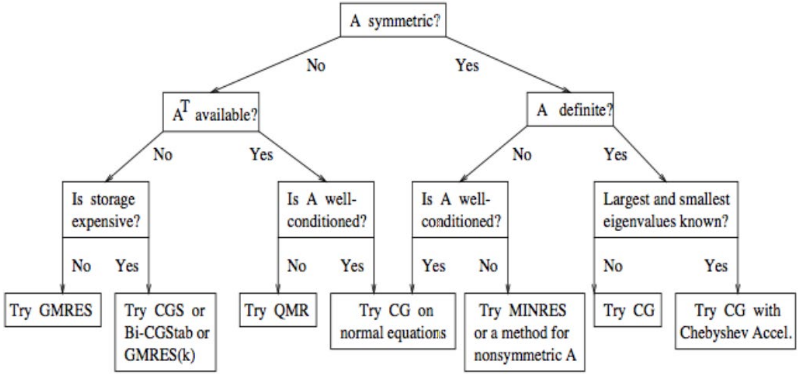
$$A\mathcal{K}_d(A, v) \subseteq \mathcal{K}_d(A, v)$$

For technical reasons we define

$$\mathcal{K}_0(A, v) = 0$$

- This mathematical foundation is the basis for the most used iterative methods used in CEM today such as, Bi-CG, Bi-CGSTAB, and GMRES among others. Depending on the solving algorithm, as well as the architecture of the system simulations are being performed other methods may be preferable. In this research, the preferred method is the Bi-CGSTAB method is utilized due its efficiency and ability to perform computations without transposition, inversion, and the expensive nature of memory and compared to GMRES.

BiCGSTAB – The Perfect Iterative Method



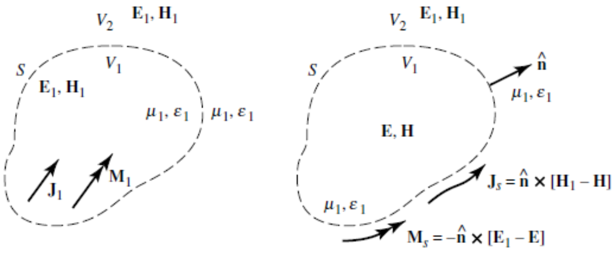
Krylov Subspace Decision Tree

- For most linear systems of equations arising from realistic electromagnetic problems, the Bi-CGSTAB algorithm to solve these equations is most attractive due to irregular convergence behavior found in other methods such as rounding errors resulting in severe cancellation effects in the solution.
- The Bi-CGSTAB(2) method does not suffer from these negative effects and is much more efficient for electromagnetic problems.

Field Equivalence Principle and RCS

$$E_r \simeq 0$$
$$E_\theta \simeq -\frac{jke^{-jkr}}{4\pi r}(L_\phi + \eta N_\theta)$$
$$E_\phi \simeq +\frac{jke^{-jkr}}{4\pi r}(L_\theta - \eta N_\phi)$$
$$H_r \simeq 0$$
$$H_\theta \simeq \frac{jke^{-jkr}}{4\pi r}\left(N_\phi - \frac{L_\theta}{\eta}\right)$$
$$H_\phi \simeq -\frac{jke^{-jkr}}{4\pi r}\left(N_\theta + \frac{L_\phi}{\eta}\right)$$

Spherical Field Equivalence Principle Fields



(a) Actual problem (b) Equivalent problem

$$N_\theta = \iint_S [J_x \cos\theta \cos\phi + J_y \cos\theta \sin\phi - J_z \sin\theta] e^{+jkr'\cos\psi}\, ds'$$
$$N_\phi = \iint_S [-J_x \sin\phi + J_y \cos\phi] e^{+jkr'\cos\psi}\, ds'$$
$$L_\theta = \iint_S [M_x \cos\theta \cos\phi + M_y \cos\theta \sin\phi - M_z \sin\theta] e^{+jkr'\cos\psi}\, ds'$$
$$L_\phi = \iint_S [-M_x \sin\phi + M_y \cos\phi] e^{+jkr'\cos\psi}\, ds'$$

Spherical Field Equivalence Principle Surface Integrals

- Electric and magnetic fields are impressed on the object in the problem space and are afflicted by surface currents found from the scattered fields that we can use to calculate the electric and magnetic far fields for calculation of the radar cross sections.

Maxwell Equations with Divergence in Frequency Domain

$$\vec{\mathbf{H}}_{\mathbf{T}} = \vec{\mathbf{H}}_{\mathbf{scat}} + \vec{\mathbf{H}}_{\mathbf{inc}}$$

Total Field Formulation for Magnetism

$$\vec{\mathbf{E}}_{\mathbf{T}} = \vec{\mathbf{E}}_{\mathbf{scat}} + \vec{\mathbf{E}}_{\mathbf{inc}}$$

Total Field Formulation for Electricity

$$\vec{\nabla} \cdot \mu\vec{\mathbf{H}}_{\mathbf{T}} = 0$$

Gauss's Law for Magnetism

$$\vec{\nabla} \cdot \varepsilon\vec{\mathbf{E}}_{\mathbf{T}} = 0$$

Gauss's Law for the Electric-Displacement with No Free Charges

$$\vec{\mathbf{D}}_{\mathbf{T}} = \varepsilon\vec{\mathbf{E}}_{\mathbf{T}}$$

Electric Linear Isotropic Constitutive Relation

$$\vec{\mathbf{B}}_{\mathbf{T}} = \mu\vec{\mathbf{H}}_{\mathbf{T}}$$

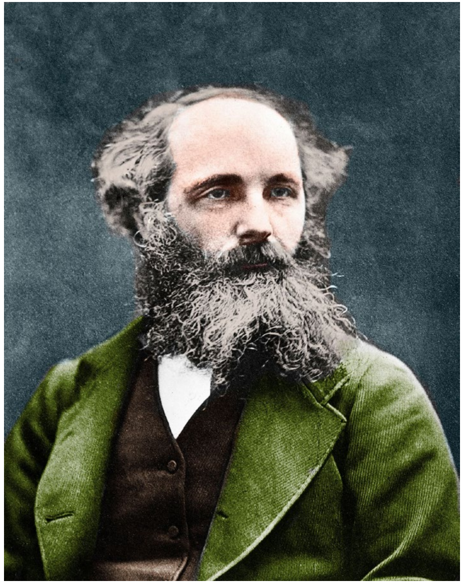
Magnetic Linear Isotropic Constitutive Relation

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{T}} + \vec{\nabla}(\vec{\nabla} \cdot \varepsilon\vec{\mathbf{E}}_{\mathbf{T}}) = -j\omega\mu\vec{\mathbf{H}}_{\mathbf{T}}$$

Faraday's Law of Induction with Divergence Re-Enforced

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{T}} + \vec{\nabla}(\vec{\nabla} \cdot \mu\vec{\mathbf{H}}_{\mathbf{T}}) = j\omega\varepsilon\vec{\mathbf{E}}_{\mathbf{T}}$$

Ampere's Circuital Law with Divergence Re-Enforced



James Clerk Maxwell (1831-1879)

Derivation of Symmetric Species Divergence Re-Enforcing FDFD Equations

For this research, the finite-difference schemes used are as follows.

First order backward first order accurate derivatives:

$$\ell'_x(i,j,k) = \frac{\ell_x(i,j,k) - \ell_x(i-1,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell'_y(i,j,k) = \frac{\ell_y(i,j,k) - \ell_y(i,j-1,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell'_z(i,j,k) = \frac{\ell_z(i,j,k) - \ell_z(i,j,k-1)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Second order backward first order accurate derivatives:

$$\ell''_x(i,j,k) = \frac{\ell_x(i,j,k) - \ell_x(i-1,j,k) + \ell_x(i-2,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell''_y(i,j,k) = \frac{\ell_y(i,j,k) - \ell_y(i,j-1,k) + \ell_y(i-2,j,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell''_z(i,j,k) = \frac{\ell_z(i,j,k) - \ell_z(i,j,k-1) + \ell_z(i-2,j,k)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Derivation of Symmetric Species Divergence Re-Enforcing FDFD Equations

First order forward first order accurate derivatives.

$$\ell'_x(i,j,k) = \frac{\ell_x(i+1,j,k) - \ell_x(i,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell'_y(i,j,k) = \frac{\ell_y(i,j+1,k) - \ell_y(i,j,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell'_z(i,j,k) = \frac{\ell_z(i,j,k+1) - \ell_z(i,j,k)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Second order forward first order accurate derivatives.

$$\ell''_x(i,j,k) = \frac{\ell_x(i+2,j,k) - \ell_x(i+1,j,k) + \ell_x(i,j,k)}{\Delta x}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell''_y(i,j,k) = \frac{\ell_y(i,j+2,k) - \ell_y(i,j+1,k) + \ell_y(i,j,k)}{\Delta y}, \boldsymbol{s} = 1, O(\Delta d)$$

$$\ell''_z(i,j,k) = \frac{\ell_z(i,j,k+2) - \ell_z(i,j,k+1) + \ell_z(i,j,k)}{\Delta z}, \boldsymbol{s} = 1, O(\Delta d)$$

Derivation of Symmetric Species Divergence Re-Enforcing FDFD Equations

From here we need to think about symmetry and causality. Totally symmetric application of the total fields with divergence by means of decoupling the fields ensures divergence is enforced on the scattered and incident fields alone at each node. We can find coefficients with the Maxwell electric curl equation. For instance, in the x-direction we have:

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{T}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon \vec{\mathbf{E}}_{\mathbf{T}})\right) = -j\omega\mu\vec{\mathbf{H}}_{\mathbf{T}}$$

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{inc}} + \nabla \times \vec{\mathbf{E}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}})\right) = -j\omega\mu\vec{\mathbf{H}}_{\mathbf{inc}} - j\omega\mu\vec{\mathbf{H}}_{\mathbf{scat}}$$

$$-j\omega\mu_0\vec{\mathbf{H}}_{\mathbf{inc}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}})\right) + \nabla \times \vec{\mathbf{E}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}})\right) - j\omega\mu_{xi}\vec{\mathbf{H}}_{\mathbf{inc}} - j\omega\mu_{xi}\vec{\mathbf{H}}_{\mathbf{scat}}$$

$$\nabla \times \vec{\mathbf{E}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}})\right) + j\omega\mu_{xi}\vec{\mathbf{H}}_{\mathbf{scat}} = j\omega\mu_0\vec{\mathbf{H}}_{\mathbf{inc}} - j\omega\mu_{xi}\vec{\mathbf{H}}_{\mathbf{inc}}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}\varepsilon_{xi}E_{x,scat} + \frac{\partial}{\partial y}E_{z,scat} - \frac{\partial}{\partial z}E_{y,scat} + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\varepsilon_{xi}E_{x,inc} + j\omega\mu_{xi}H_{x,scat} = j\omega(\mu_0 - \mu_{xi})H_{x,inc}$$

The diagonal terms in the permeability dyadic become off-diagonal terms in the derivative.

$$\frac{\varepsilon_{xi}}{j\omega\mu_{xi}}\frac{\partial}{\partial x}\frac{\partial}{\partial x}E_{x,scat}+\frac{1}{j\omega\mu_{xy}}\frac{\partial}{\partial y}E_{z,scat}-\frac{1}{j\omega\mu_{xz}}\frac{\partial}{\partial z}E_{y,scat}+\frac{\varepsilon_{xi}}{j\omega\mu_{xi}}\frac{\partial}{\partial x}\frac{\partial}{\partial x}E_{x,inc}+H_{x,scat}=\frac{\mu_0-\mu_{xi}}{\mu_{xi}}H_{x,inc}$$

Discretizing, we have:

X-Direction:

$$\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\Delta x}\cdot\Big(E_{x,scat}(i+2,j,k)-E_{x,scat}(i+1,j,k)+E_{x,scat}(i,j,k)\Big)+$$

$$\frac{1}{j\omega\mu_{xy}(i,j,k)\Delta y}\cdot\Big(E_{z,scat}(i,j+1,k)-E_{z,scat}(i,j,k)\Big)-$$

$$\frac{1}{j\omega\mu_{xz}(i,j,k)\Delta z}\cdot\Big(E_{y,scat}(i,j,k+1)-E_{y,scat}(i,j,k)\Big)+$$

$$\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\Delta x}\cdot\Big(E_{x,inc}(i+2,j,k)-E_{x,inc}(i+1,j,k)+E_{x,inc}(i,j,k)\Big)+$$

$$H_{x,scat}(i,j,k)=\frac{\mu_0(i,j,k)-\mu_{xi}(i,j,k)}{\mu_{xi}(i,j,k)}H_{x,inc}(i,j,k)$$

Our coefficients for magnetic incidence in the x-direction become:

$$\mathbf{C_{hxhx}}=\frac{\mu_0(i,j,k)-\mu_{xi}(i,j,k)}{\mu_{xi}(i,j,k)}$$

$$\mathbf{C_{hxei}}=\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\Delta x}$$

$$\mathbf{C_{hxex}}=\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\Delta x}$$

$$\mathbf{C_{hxey}}=\frac{1}{j\omega\mu_{xz}(i,j,k)\Delta z}$$

$$\mathbf{C_{hxez}}=\frac{1}{j\omega\mu_{xy}(i,j,k)\Delta y}$$

Total Field Symmetry with Divergence Re-Enforced

Maxwell Electric Curl Equations

$$\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\Delta x}\cdot\left(E_{x,scat}(i+2,j,k)-E_{x,scat}(i+1,j,k)+E_{x,scat}(i,j,k)\right)+\frac{1}{j\omega\mu_{xy}(i,j,k)\Delta y}\cdot\left(E_{z,scat}(i,j+1,k)-E_{z,scat}(i,j,k)\right)-$$

$$\frac{1}{j\omega\mu_{xz}(i,j,k)\Delta z}\cdot\left(E_{y,scat}(i,j,k+1)-E_{y,scat}(i,j,k)\right)+\frac{\varepsilon_{xi}(i,j,k)}{j\omega\mu_{xi}(i,j,k)\Delta x}\cdot\left(E_{x,inc}(i+2,j,k)-E_{x,inc}(i+1,j,k)+E_{x,inc}(i,j,k)\right)+H_{x,scat}(i,j,k)=\frac{\mu_0(i,j,k)-\mu_{xi}(i,j,k)}{\mu_{xi}(i,j,k)}H_{x,inc}(i,j,k)$$

Equation 1: Faraday's Law of Induction in the X-Direction with Divergence Re-Enforced

$$\frac{\varepsilon_{yi}(i,j,k)}{j\omega\mu_{yi}(i,j,k)\Delta y}\cdot\left(E_{y,scat}(i,j+2,k)-E_{y,scat}(i,j+1,k)+E_{y,scat}(i,j,k)\right)+\frac{1}{j\omega\mu_{yx}(i,j,k)\Delta x}\cdot\left(E_{z,scat}(i+1,j,k)-E_{z,scat}(i,j,k)\right)-$$

$$\frac{1}{j\omega\mu_{yz}(i,j,k)\Delta z}\cdot\left(E_{x,scat}(i,j,k+1)-E_{x,scat}(i,j,k)\right)+\frac{\varepsilon_{yi}(i,j,k)}{j\omega\mu_{yi}(i,j,k)\Delta y}\cdot\left(E_{y,inc}(i,j+2,k)-E_{y,inc}(i,j+1,k)+E_{y,inc}(i,j,k)\right)+H_{y,scat}(i,j,k)=\frac{\mu_0(i,j,k)-\mu_{yi}(i,j,k)}{\mu_{yi}(i,j,k)}H_{y,inc}(i,j,k)$$

Equation 2: Faraday's Law of Induction in the Y-Direction with Divergence Re-Enforced

$$\frac{\varepsilon_{zi}(i,j,k)}{j\omega\mu_{zi}(i,j,k)\Delta z}\cdot\left(E_{z,scat}(i+2,j,k)-E_{z,scat}(i+1,j,k)+E_{z,scat}(i,j,k)\right)+\frac{1}{j\omega\mu_{zx}(i,j,k)\Delta x}\cdot\left(E_{y,scat}(i+1,j,k)-E_{y,scat}(i,j,k)\right)-$$

$$\frac{1}{j\omega\mu_{zy}(i,j,k)\Delta y}\cdot\left(E_{x,scat}(i,j+1,k)-E_{x,scat}(i,j,k)\right)+\frac{\varepsilon_{zi}(i,j,k)}{j\omega\mu_{zi}(i,j,k)\Delta z}\cdot\left(E_{z,inc}(i+2,j,k)-E_{z,inc}(i+1,j,k)+E_{z,inc}(i,j,k)\right)+H_{z,scat}(i,j,k)=\frac{\mu_0(i,j,k)-\mu_{zi}(i,j,k)}{\mu_{zi}(i,j,k)}H_{z,inc}(i,j,k)$$

Equation 3: Faraday's Law of Induction in the Z-Direction with Divergence Re-Enforced

Total Field Symmetry with Divergence Re-Enforced

We use the Maxwell magnetic curl equation using the same method to find the other coefficients. For instance, in the x-direction we have:

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{T}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu \vec{\mathbf{H}}_{\mathbf{T}})\right) = j\omega\varepsilon \vec{\mathbf{E}}_{\mathbf{T}}$$

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{inc}} + \nabla \times \vec{\mathbf{H}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_0 \vec{\mathbf{H}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}})\right) = j\omega\varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}} + j\omega\varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}}$$

$$j\omega\varepsilon_0 \vec{\mathbf{E}}_{\mathbf{inc}} + \nabla \times \vec{\mathbf{H}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_0 \vec{\mathbf{H}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}})\right) = j\omega\varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}} + j\omega\varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}}$$

$$\nabla \times \vec{\mathbf{H}}_{\mathbf{scat}} + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_0 \vec{\mathbf{H}}_{\mathbf{inc}})\right) + diag\left(\vec{\nabla}(\vec{\nabla} \cdot \mu_{xi} \vec{\mathbf{H}}_{\mathbf{scat}})\right) + j\omega\varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{scat}} = j\omega\varepsilon_{xi} \vec{\mathbf{E}}_{\mathbf{inc}} - j\omega\varepsilon_0 \vec{\mathbf{E}}_{\mathbf{inc}}$$

$$\mu_{xi} \frac{\partial}{\partial x}\frac{\partial}{\partial x} H_{x,scat} + \frac{\partial}{\partial y} H_{z,scat} - \frac{\partial}{\partial z} H_{y,scat} + \mu_{xi} \frac{\partial}{\partial x}\frac{\partial}{\partial x} H_{x,inc} - j\omega\varepsilon_{xi} E_{x,scat} = j\omega(\varepsilon_{xi} - \varepsilon_0) E_{x,inc}$$

Total Field Symmetry with Divergence Re-Enforced

The diagonal terms in the permeability dyadic become off-diagonal terms in the derivative.

$$\frac{\mu_{xi}}{j\omega\varepsilon_{xi}}\cdot\frac{\partial}{\partial x}\frac{\partial}{\partial x}H_{x,scat}+\frac{1}{j\omega\varepsilon_{xy}}\cdot\frac{\partial}{\partial y}H_{z,scat}-\frac{1}{j\omega\varepsilon_{xz}}\cdot\frac{\partial}{\partial z}H_{y,scat}+\frac{\mu_{xi}}{j\omega\varepsilon_{xi}}\cdot\frac{\partial}{\partial x}\frac{\partial}{\partial x}H_{x,inc}-E_{x,scat}=\frac{(\varepsilon_{xi}-\varepsilon_0)}{\varepsilon_{xi}}E_{x,inc}$$

Discretizing, we have:

<u>X-Direction</u>:

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\Delta x}\cdot\left(H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)\right)+$$

$$\frac{1}{j\omega\varepsilon_{xy}(i,j,k)\Delta y}\cdot\left(H_{z,scat}(i,j,k)-H_{z,scat}(i,j-1,k)\right)-$$

$$\frac{1}{j\omega\varepsilon_{xz}(i,j,k)\Delta z}\cdot\left(H_{y,scat}(i,j,k)-H_{y,scat}(i,j,k-1)\right)+$$

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\Delta x}\cdot\left(H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)\right)+$$

$$E_{x,scat}(i,j,k)=\frac{\varepsilon_{xi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{xi}(i,j,k)}E_{x,inc}(i,j,k)$$

Our coefficients for magnetic incidence in the x-direction become:

$$\mathbf{C_{exex}}=\frac{\varepsilon_{xi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{xi}(i,j,k)}$$

$$\mathbf{C_{exhi}}=\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\Delta x}$$

$$\mathbf{C_{exhx}}=\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\Delta x}$$

$$\mathbf{C_{exhy}}=\frac{1}{j\omega\varepsilon_{xy}(i,j,k)\Delta y}$$

$$\mathbf{C_{exhz}}=\frac{1}{j\omega\varepsilon_{xz}(i,j,k)\Delta z}$$

Total Field Symmetry with Divergence Re-Enforced

Maxwell Magnetic Curl Equations

$$\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\Delta x}\cdot\Big(H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)\Big)+\frac{1}{j\omega\varepsilon_{xy}(i,j,k)\Delta y}\cdot\Big(H_{z,scat}(i,j,k)-H_{z,scat}(i,j-1,k)\Big)-\frac{1}{j\omega\varepsilon_{xz}(i,j,k)\boldsymbol{s}\Delta z}\cdot\Big(H_{y,scat}(i,j,k)-H_{y,scat}(i,j,k-1)\Big)$$
$$+\frac{\mu_{xi}(i,j,k)}{j\omega\varepsilon_{xi}(i,j,k)\Delta x}\cdot\Big(H_{x,scat}(i,j,k)-H_{x,scat}(i-1,j,k)+H_{x,scat}(i-2,j,k)\Big)+E_{x,scat}(i,j,k)=\frac{\varepsilon_{xi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{xi}(i,j,k)}E_{x,inc}(i,j,k)$$

Equation 4: Ampere’s Circuital in the X-Direction with Divergence Re-Enforced

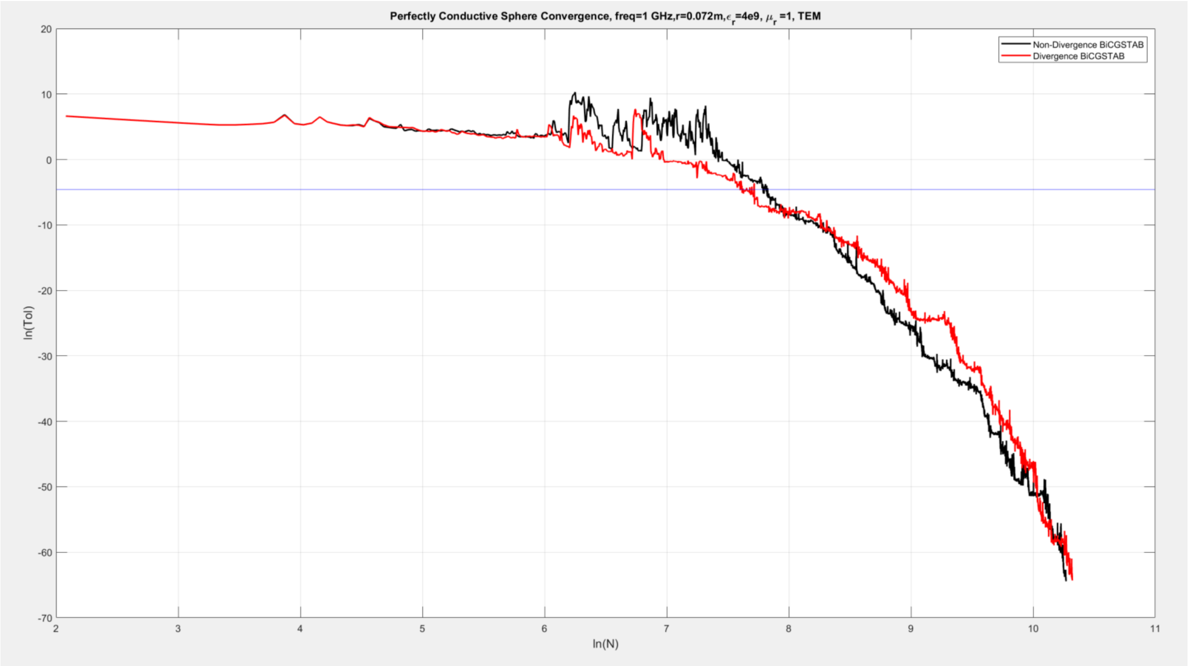
$$\frac{\mu_{yi}(i,j,k)}{j\omega\varepsilon_{yi}(i,j,k)\Delta y}\cdot\Big(H_{y,scat}(i,j,k)-H_{y,scat}(i-1,j,k)+H_{y,scat}(i-2,j,k)\Big)+\frac{1}{j\omega\varepsilon_{yx}(i,j,k)\Delta x}\cdot\Big(H_{z,scat}(i,j,k)-H_{z,scat}(i-1,j,k)\Big)-\frac{1}{j\omega\varepsilon_{yz}(i,j,k)\Delta z}\cdot\Big(H_{x,scat}(i,j,k)-H_{x,scat}(i,j,k-1)\Big)$$
$$+\frac{\mu_{yi}(i,j,k)}{j\omega\varepsilon_{yi}(i,j,k)\Delta y}\cdot\Big(H_{y,scat}(i,j,k)-H_{y,scat}(i-1,j,k)+H_{y,scat}(i-2,j,k)\Big)+E_{y,scat}(i,j,k)=\frac{\varepsilon_{yi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{yi}(i,j,k)}E_{y,inc}(i,j,k)$$

Equation 5: Ampere’s Circuital Law in the Y-Direction with Divergence Re-Enforced

$$\frac{\mu_{zi}(i,j,k)}{j\omega\varepsilon_{zi}(i,j,k)\Delta z}\cdot\Big(H_{z,scat}(i,j,k)-H_{z,scat}(i-1,j,k)+H_{z,scat}(i-2,j,k)\Big)+\frac{1}{j\omega\varepsilon_{zx}(i,j,k)\Delta x}\cdot\Big(H_{y,scat}(i,j,k)-H_{y,scat}(i-1,j,k)\Big)-\frac{1}{j\omega\varepsilon_{zy}(i,j,k)\Delta y}\cdot\Big(H_{x,scat}(i,j,k)-H_{x,scat}(i,j-1,k)\Big)+$$
$$\frac{\mu_{zi}(i,j,k)}{j\omega\varepsilon_{zi}(i,j,k)\Delta z}\cdot\Big(H_{z,scat}(i,j,k)-H_{z,scat}(i-1,j,k)+H_{z,scat}(i-2,j,k)\Big)+E_{z,scat}(i,j,k)=\frac{\varepsilon_{zi}(i,j,k)-\varepsilon_0(i,j,k)}{\varepsilon_{zi}(i,j,k)}E_{z,inc}(i,j,k)$$

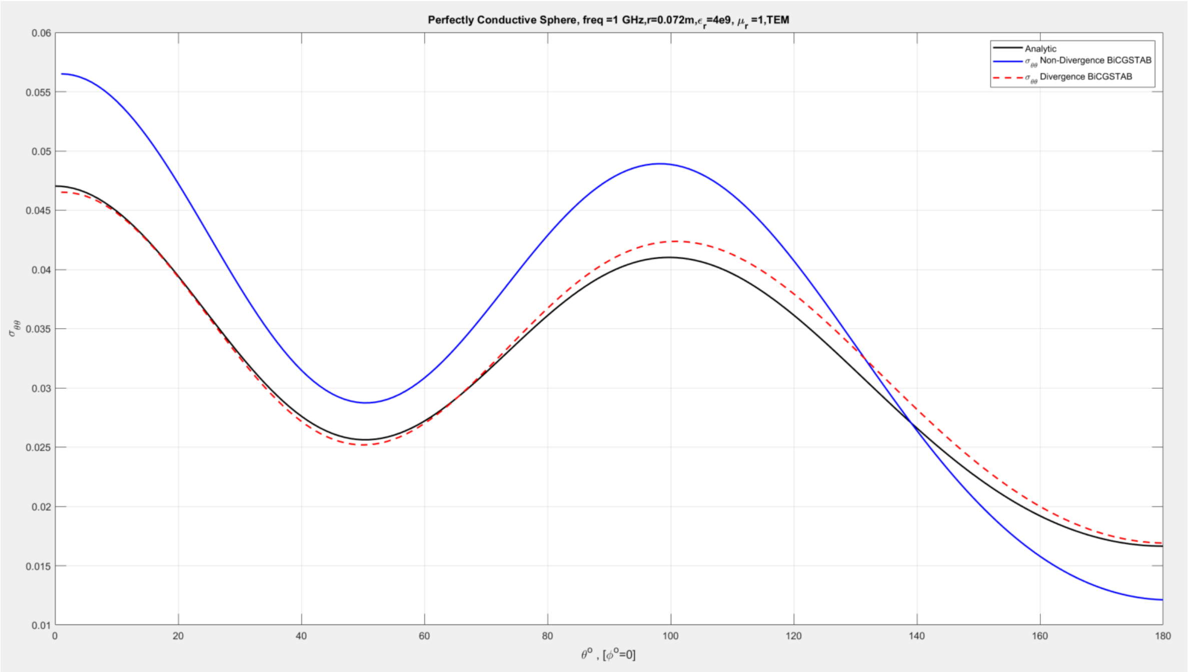
Equation 6: Ampere’s Circuital Law in the Z-Direction with Divergence Re-Enforced

Efficiency Improvement



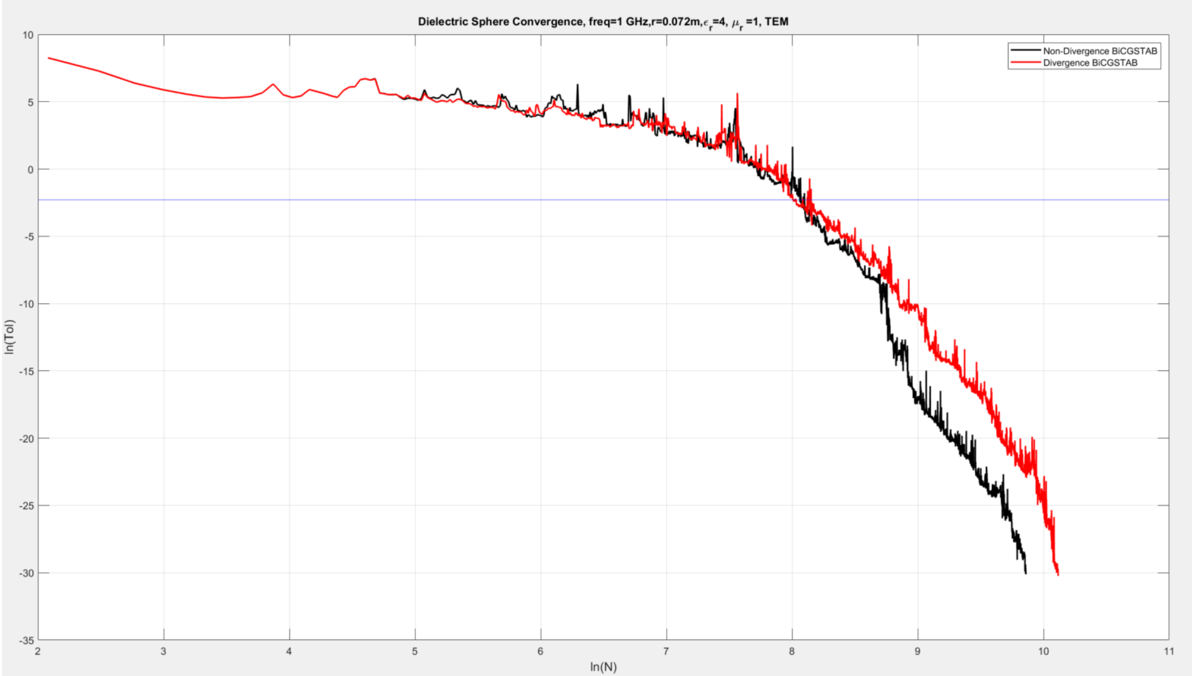
Perfectly Conductive Sphere Convergence

Accuracy Improvement



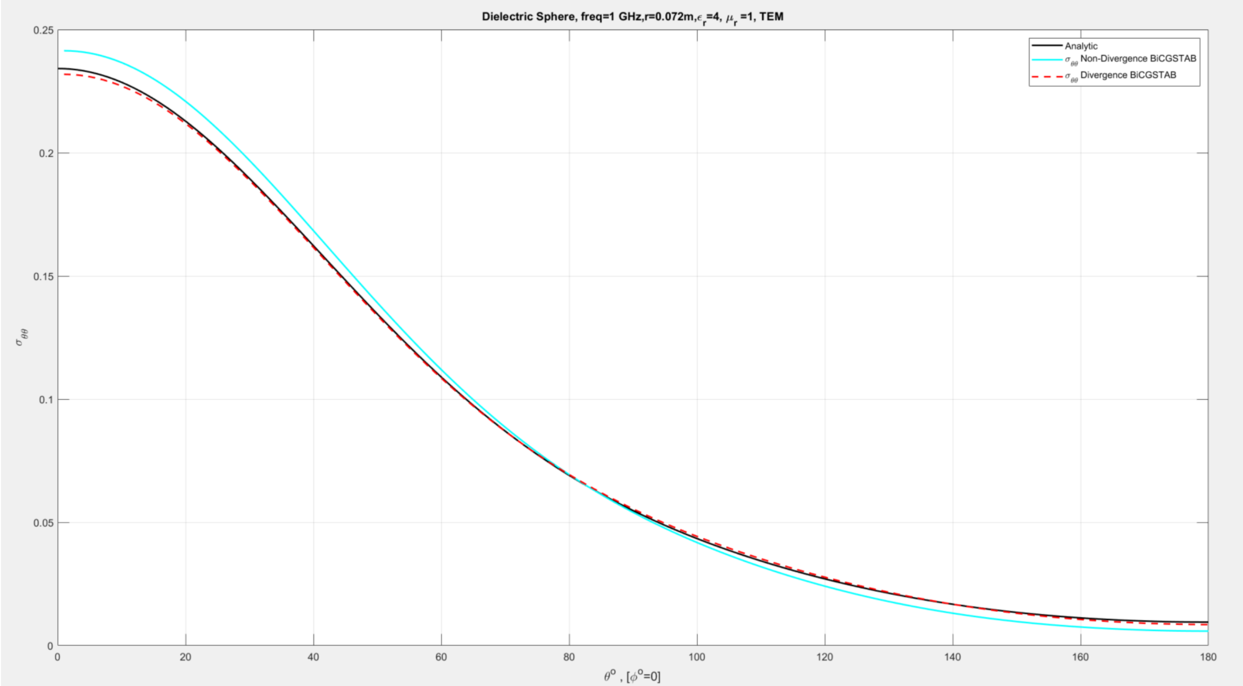
Perfectly Conductive Sphere RCS

Efficiency Improvement



Dielectric Sphere Convergence

Accuracy Improvement



Dielectric Sphere RCS

Results

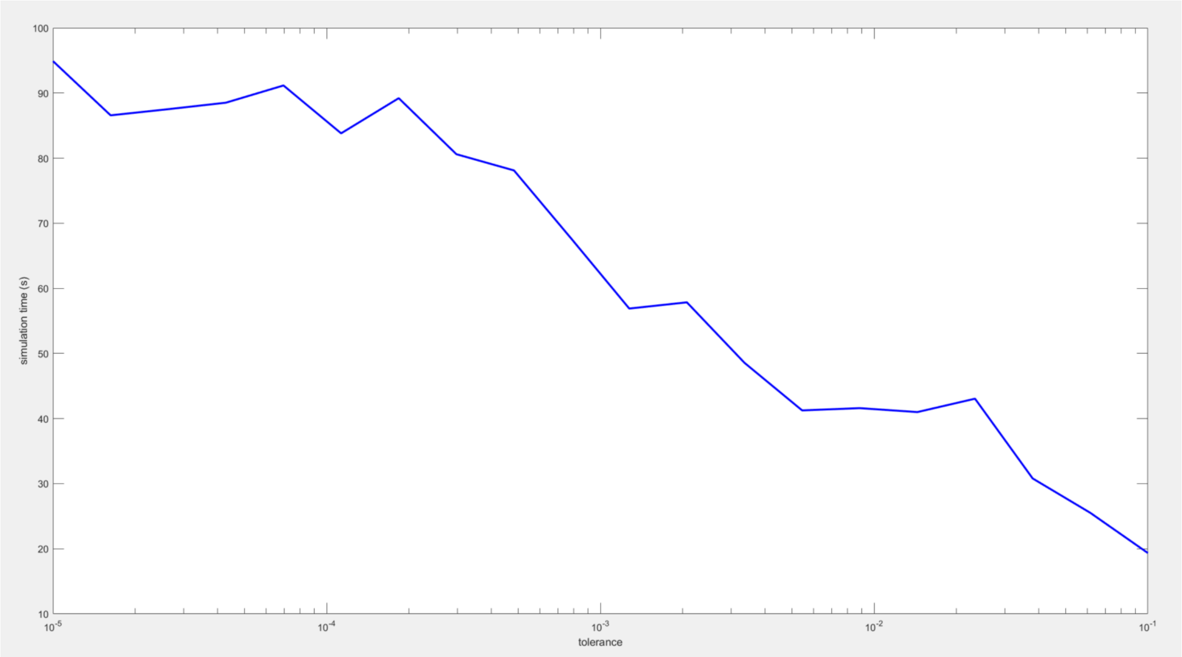
FDFD	NMATVEC	RCS RMS Error	Tolerance
Vanilla	1638	0.00500	0.01
Divergence	1345	0.00083	0.01

Table 1: Perfectly Conducting Sphere Comparison Table

FDFD	NMATVEC	RCS RMS Error	Tolerance
Vanilla	1284	0.0034	0.1
Divergence	1389	0.0015	0.1

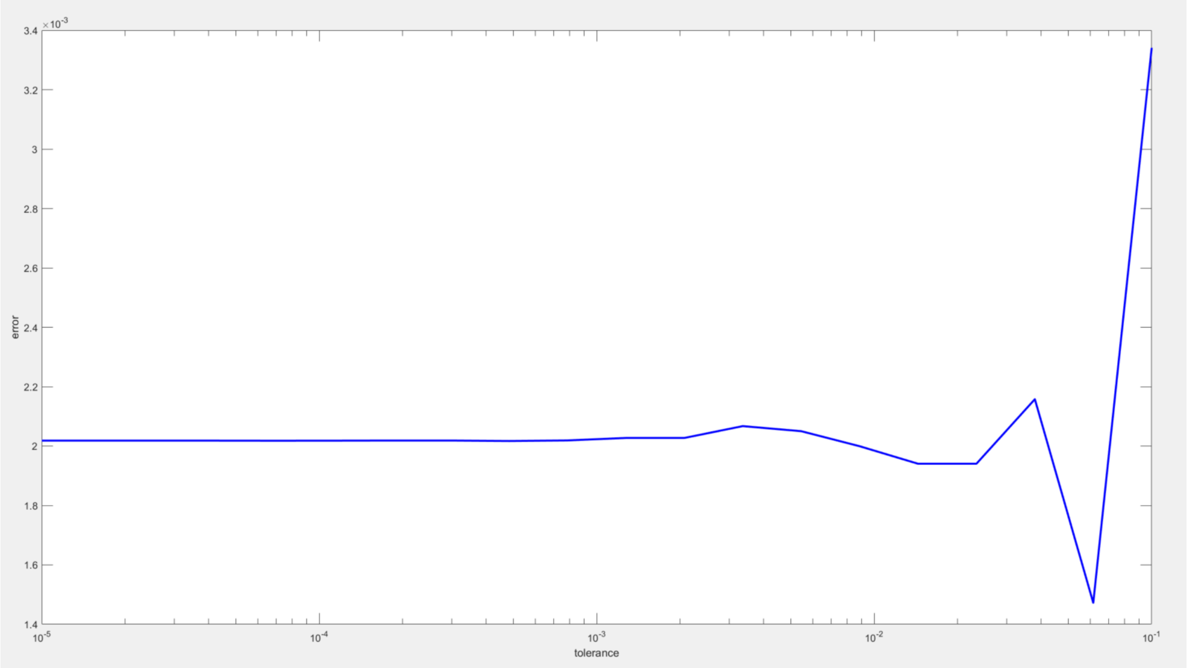
Table 2: Dielectric Sphere Comparison Table

Simulation Time



Simulation Convergence Time Versus Tolerance

Convergence Error



Simulation Error Versus Tolerance

FDFD Fortran Code

```
Cexhy = 1/(j*w*s*dz*eps_xz); Cexhz = 1/(j*w*s*dy*eps_xy); Cexex = -(eps_xi - eps_o) / eps_xi;
Ceyhz = 1/(j*w*s*dx*eps_yx); Ceyhx = 1/(j*w*s*dz*eps_yz); Ceyey = -(eps_yi - eps_o) / eps_yi;
Cezhx = 1/(j*w*s*dy*eps_zy); Cezhy = 1/(j*w*s*dx*eps_zx); Cezez = -(eps_zi - eps_o) / eps_zi;

Chxey = 1/(j*w*s*dz*mu_xz); Chxez = 1/(j*w*s*dy*mu_xy); Chxhx = (mu_xi - mu_o) / mu_xi;
Chyez = 1/(j*w*s*dx*mu_yx); Chyex = 1/(j*w*s*dz*mu_yz); Chyhy = (mu_yi - mu_o) / mu_yi;
Chzex = 1/(j*w*s*dy*mu_zy); Chzey = 1/(j*w*s*dx*mu_zx); Chzhz = (mu_zi - mu_o) / mu_zi;

! divergence re-enforcing coefficients

Cexhx = mu_xi/(j*w*s*dx*eps_xi); Cexhi = mu_xi/(j*w*s*dx*eps_xi);
Ceyhy = mu_yi/(j*w*s*dy*eps_yi); Ceyhi = mu_yi/(j*w*s*dy*eps_yi);
Cezhz = mu_zi/(j*w*s*dz*eps_zi); Cezhi = mu_zi/(j*w*s*dz*eps_zi);

Chxex = eps_xi/(j*w*s*dx*mu_xi); Chxei = eps_xi/(j*w*s*dx*mu_xi);
Chyey = eps_yi/(j*w*s*dy*mu_yi); Chyei = eps_yi/(j*w*s*dy*mu_yi);
Chzez = eps_zi/(j*w*s*dz*mu_zi); Chzei = eps_zi/(j*w*s*dz*mu_zi);

Bex = Cexex * Einc_x; Bey = Ceyey * Einc_y; Bez = Cezez * Einc_z;
Bhx = Chxhx * Hinc_x; Bhy = Chyhy * Hinc_y; Bhz = Chzhz * Hinc_z;
```

FDFD Fortran Code

```fortran
! divergence re-enfocing equations

Bex(3:nx,2:ny,2:nz) = Bex(3:nx,2:ny,2:nz) - &
                (-Cexhz(3:nx,2:ny,2:nz)*Bhz(3:nx,2:ny,2:nz)+Cexhz(3:nx,2:ny,2:nz)*Bhz(3:nx,1:ny-1,2:nz)   &
                 +Cexhy(3:nx,2:ny,2:nz)*Bhy(3:nx,2:ny,2:nz)-Cexhy(3:nx,2:ny,2:nz)*Bhy(3:nx,2:ny,1:nz-1)   &

                 -Cexhi(3:nx,2:ny,2:nz)*Bhxi(3:nx,2:ny,2:nz)+Cexhi(3:nx,2:ny,2:nz)*Bhxi(2:nx-1,2:ny,2:nz)-Cexhi(3:nx,2:ny,2:nz)*Bhxi(1:nx-2,2:ny,2:nz) &
                 +Cexhx(3:nx,2:ny,2:nz)*Bhx(3:nx,2:ny,2:nz)-Cexhx(3:nx,2:ny,2:nz)*Bhx(2:nx-1,2:ny,2:nz)+Cexhx(3:nx,2:ny,2:nz)*Bhx(1:nx-2,2:ny,2:nz));

Bey(2:nx,3:ny,2:nz) = Bey(2:nx,3:ny,2:nz) - &
                (-Ceyhx(2:nx,3:ny,2:nz)*Bhx(2:nx,3:ny,2:nz)+Ceyhx(2:nx,3:ny,2:nz)*Bhx(2:nx,3:ny,1:nz-1)   &
                 +Ceyhz(2:nx,3:ny,2:nz)*Bhz(2:nx,3:ny,2:nz)-Ceyhz(2:nx,3:ny,2:nz)*Bhz(1:nx-1,3:ny,2:nz)   &

                 -Ceyhi(2:nx,3:ny,2:nz)*Bhyi(2:nx,3:ny,2:nz)+Ceyhi(2:nx,3:ny,2:nz)*Bhyi(2:nx,2:ny-1,2:nz)-Ceyhi(2:nx,3:ny,2:nz)*Bhyi(2:nx,1:ny-2,2:nz) &
                 +Ceyhy(2:nx,3:ny,2:nz)*Bhy(2:nx,3:ny,2:nz)-Ceyhy(2:nx,3:ny,2:nz)*Bhy(2:nx,2:ny-1,2:nz)+Ceyhy(2:nx,3:ny,2:nz)*Bhy(2:nx,1:ny-2,2:nz));

Bez(2:nx,2:ny,3:nz) = Bez(2:nx,2:ny,3:nz) - &
                (-Cezhy(2:nx,2:ny,3:nz)*Bhy(2:nx,2:ny,3:nz)+Cezhy(2:nx,2:ny,3:nz)*Bhy(1:nx-1,2:ny,3:nz)   &
                 +Cezhx(2:nx,2:ny,3:nz)*Bhx(2:nx,2:ny,3:nz)-Cezhx(2:nx,2:ny,3:nz)*Bhx(2:nx,1:ny-1,3:nz)   &

                 -Cezhi(2:nx,2:ny,3:nz)*Bhzi(2:nx,2:ny,3:nz)+Cezhi(2:nx,2:ny,3:nz)*Bhzi(2:nx,2:ny,2:nz-1)-Cezhi(2:nx,2:ny,3:nz)*Bhzi(2:nx,2:ny,1:nz-2) &
                 +Cezhz(2:nx,2:ny,3:nz)*Bhz(2:nx,2:ny,3:nz)-Cezhz(2:nx,2:ny,3:nz)*Bhz(2:nx,2:ny,2:nz-1)+Cezhz(2:nx,2:ny,3:nz)*Bhz(2:nx,2:ny,1:nz-2));
```

FDFD Fortran Code

```fortran
! divergence re-enforcing matvec

tmpx(1:Nxm1-1,1:Nym1,1:Nzm1) = Chxez(1:Nxm1-1,1:Nym1,1:Nzm1)*(x(1:Nxm1-1,2:Ny,1:Nzm1,3)-x(1:Nxm1-1,1:Nym1,1:Nzm1,3))        &
+ Chxey(1:Nxm1-1,1:Nym1,1:Nzm1)*(-x(1:Nxm1-1,1:Nym1,2:Nz,2)+x(1:Nxm1-1,1:Nym1,1:Nzm1,2))                        &

+ Chxex(1:Nxm1-1,1:Nym1,1:Nzm1)*(x(3:Nx,1:Nym1,1:Nzm1,1)-x(2:Nxm1,1:Nym1,1:Nzm1,1)+x(1:Nxm1-1,1:Nym1,1:Nzm1,1)) &
- Chxei(1:Nxm1-1,1:Nym1,1:Nzm1)*(-x(3:Nx,1:Nym1,1:Nzm1,1)+x(2:Nxm1,1:Nym1,1:Nzm1,1)-x(1:Nxm1-1,1:Nym1,1:Nzm1,1));

tmpy(1:Nxm1,1:Nym1-1,1:Nzm1) = Chyex(1:Nxm1,1:Nym1-1,1:Nzm1)*(x(1:Nxm1,1:Nym1-1,2:Nz,1)-x(1:Nxm1,1:Nym1-1,1:Nzm1,1))        &
+ Chyez(1:Nxm1,1:Nym1-1,1:Nzm1)*(-x(2:Nx,1:Nym1-1,1:Nzm1,3)+x(1:Nxm1,1:Nym1-1,1:Nzm1,3))                        &

+ Chyey(1:Nxm1,1:Nym1-1,1:Nzm1)*(x(1:Nxm1,3:Ny,1:Nzm1,2)-x(1:Nxm1,2:Nym1,1:Nzm1,2)+x(1:Nxm1,1:Nym1-1,1:Nzm1,2)) &
- Chyei(1:Nxm1,1:Nym1-1,1:Nzm1)*(-x(1:Nxm1,3:Ny,1:Nzm1,2)+x(1:Nxm1,2:Nym1,1:Nzm1,2)-x(1:Nxm1,1:Nym1-1,1:Nzm1,2));

tmpz(1:Nxm1,1:Nym1,1:Nzm1-1) = Chzey(1:Nxm1,1:Nym1,1:Nzm1-1)*(x(2:Nx,1:Nym1,1:Nzm1-1,2)-x(1:Nxm1,1:Nym1,1:Nzm1-1,2))        &
+ Chzex(1:Nxm1,1:Nym1,1:Nzm1-1)*(-x(1:Nxm1,2:Ny,1:Nzm1-1,1)+x(1:Nxm1,1:Nym1,1:Nzm1-1,1))                        &

+ Chzez(1:Nxm1,1:Nym1,1:Nzm1-1)*(x(1:Nxm1,1:Nym1,3:Nz,3)-x(1:Nxm1,1:Nym1,2:Nzm1,3)+x(1:Nxm1,1:Nym1,1:Nzm1-1,3)) &
- Chzei(1:Nxm1,1:Nym1,1:Nzm1-1)*(-x(1:Nxm1,1:Nym1,3:Nz,3)+x(1:Nxm1,1:Nym1,2:Nzm1,3)-x(1:Nxm1,1:Nym1,1:Nzm1-1,3));
```

FDFD Fortran Code

```
y(3:Nx,2:Ny,2:Nz,1) = x(3:Nx,2:Ny,2:Nz,1) &
- (Cexhz(3:Nx,2:Ny,2:Nz)*(-tmpz(3:Nx,2:Ny,2:Nz)+tmpz(3:Nx,1:Nym1,2:Nz))  &
+  Cexhy(3:Nx,2:Ny,2:Nz)*(tmpy(3:Nx,2:Ny,2:Nz)-tmpy(3:Nx,2:Ny,1:Nzm1))   &

+  Cexhx(3:Nx,2:Ny,2:Nz)*(tmpy(3:Nx,2:Ny,2:Nz)-tmpy(2:Nxm1,2:Ny,2:Nz)+tmpy(1:Nxm1-1,2:Ny,2:Nz)) &
-  Cexhi(3:Nx,2:Ny,2:Nz)*(-tmpy(3:Nx,2:Ny,2:Nz)+tmpy(2:Nxm1,2:Ny,2:Nz)-tmpy(1:Nxm1-1,2:Ny,2:Nz)));

y(2:Nx,3:Ny,2:Nz,2) = x(2:Nx,3:Ny,2:Nz,2) &
- (Ceyhx(2:Nx,3:Ny,2:Nz)*(-tmpx(2:Nx,3:Ny,2:Nz)+tmpx(2:Nx,3:Ny,1:Nzm1))  &
+  Ceyhz(2:Nx,3:Ny,2:Nz)*(tmpz(2:Nx,3:Ny,2:Nz)-tmpz(1:Nxm1,3:Ny,2:Nz))   &

+  Ceyhy(2:Nx,3:Ny,2:Nz)*(tmpz(2:Nx,3:Ny,2:Nz)-tmpz(2:Nx,2:Nym1,2:Nz)+tmpz(2:Nx,1:Nym1-1,2:Nz)) &
-  Ceyhi(2:Nx,3:Ny,2:Nz)*(-tmpz(2:Nx,3:Ny,2:Nz)+tmpz(2:Nx,2:Nym1,2:Nz)-tmpz(2:Nx,1:Nym1-1,2:Nz)));

y(2:Nx,2:Ny,3:Nz,3) = x(2:Nx,2:Ny,3:Nz,3) &
- (Cezhy(2:Nx,2:Ny,3:Nz)*(-tmpy(2:Nx,2:Ny,3:Nz)+tmpy(1:Nxm1,2:Ny,3:Nz))  &
+  Cezhx(2:Nx,2:Ny,3:Nz)*(tmpx(2:Nx,2:Ny,3:Nz)-tmpx(2:Nx,1:Nym1,3:Nz))   &

+  Cezhz(2:Nx,2:Ny,3:Nz)*(tmpx(2:Nx,2:Ny,3:Nz)-tmpx(2:Nx,2:Ny,2:Nzm1)+tmpx(2:Nx,2:Ny,1:Nzm1-1)) &
-  Cezhi(2:Nx,2:Ny,3:Nz)*(-tmpx(2:Nx,2:Ny,3:Nz)+tmpx(2:Nx,2:Ny,2:Nzm1)-tmpx(2:Nx,2:Ny,1:Nzm1-1)));
```

Conclusions

- Divergence can be re-enforced in the FDFD method to improve efficiency and accuracy of the FDFD method.
  - Increased convergence rate.
  - Greater accuracy of RCS.
- Periodic Boundary Conditions can be implemented to simulate infinitely periodic structures.

Further Innovation Opportunities

- Advanced Field Formulations
- Periodic Boundary Conditions in FDFD
- Krylov Subspace Optimization
- Pre-Conditioning