Categories for the Curious

A Light Intro to Category Theory

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Preface

This site contains the course notes for an informal one-semester seminar on category theory at CSU East Bay.

1 Introduction

1.1 What is a category?

The definition is simple, but the level of abstraction in its statement may make the concept seem alien (even though it should be familiar).

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

1. A class of *objects* denoted $\mathrm{Obj}(\mathcal{C})$.
2. For each pair of objects x and y, there is a set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ containing the *morphisms* with *source* x and *target* y (morphisms are also called *arrows*).
3. For any pair of morphisms f and g such that the target of f is equal to the source of g there is a morphism $g \circ f$ (or simply gf) called the *composition* of f with g. This morphism shares its source with that of f and its target with that of g.

Additionally, the following properties must be satisfied:

1. Each object x has an identity morphism 1_x such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ we have $f \circ 1_x = f = 1_y \circ f$.
2. Composition is associative, so that whenever $f \in \mathrm{Hom}_{\mathcal{C}}(w, x)$, $g \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(y, z)$, we have $(hg)f = h(gf)$.

We will learn how to draw conclusions from the abstract definition of a category later. It's useful to have the definition on hand, but if one is intimidated by this, then one can informally think of a category as something where we have objects, a way of mapping between those objects (along with an identity map on each object), and a way of composing those maps (where composition is associative). Almost everything in mathematics satisfies these criteria. We'll see that what can be considered a "map" is fairly flexible, and the ways we might compose those maps can also vary quite a bit.

1.2 Categories as a foundation for mathematics

Throughout our mathematical education, we have likely encountered many types of mathematical objects (e.g., groups, rings, vector spaces, fields, topological spaces, sets, functions, manifolds). We might have learned that a group is "a *set* equipped with an associative binary operation having an identity element, and for which every element has an inverse," or that a "vector space is a *set* equipped with the operations of vector addition and scalar multiplication satisfying the usual axioms." It may have been a while since we've dealt with these definitions, or we may not have learned them at all. That is okay. What is important to focus on now is that both of these objects are defined as a *set* equipped with some additional structure. We have defined them in terms of "sets."

So what is a "set," exactly? We might see it defined as "a collection of objects, which are called its elements," which then begs the question: what is a "collection"? How is it that we can give such precise definitions of all of the structures mentioned above (vector spaces, groups, rings, etc.), yet we seem to have difficulty making precise the concept of a set, on which the definitions of all of these structures rely?

The uncomfortable truth is that modern mathematics, with its rigorous definitions and axioms, cannot exist *ex nihilo*. To formulate a rigorous definition, we must first precisely define each term within that definition, and so on for the definitions of each of those terms. This process cannot end without either defining two objects "circularly," such that some object is defined in terms of another and vice versa, or declaring once and for all that certain objects are atomic, and they cannot be defined in terms of any other object. This is where sets lie: we do not bother defining what a set is, and instead we focus on the conclusions drawn from a set of axioms we assume to hold about sets.

We could do away with set theory, and instead formally define all the mathematical structures above in terms of categories. Thus, categories constitute an alternative foundation for mathematics. But category theory does not bypass the problem of definability. There are still terms which we take as "atomic," and we do not bother precisely defining them (e.g., the objects and morphisms).

1.3 Categories reveal structure

1.4 Our Goal

This is meant to be a very approachable introduction to category theory. All that is needed to progress through these notes is a persisting curiosity. Ideally, one is comfortable with sets and functions between them, but beyond that not much else is required. There are more sections of the notes which may assume more background, but these are clearly marked and can be skipped without disrupting the flow of the material. Every such section is indicated like so:

> **Warning**
>
> This section assumes prior knowledge. It may be safely skipped.

These sections will typically be found near the end of each chapter.

2 Sets

3 Partially Ordered Sets

3.1 The "Divides" Relation

Definition 3.1. Let m and n be integers. We say m *divides* n, and we write $m \mid n$ if there exists an integer q such that $mq = n$.

For example, 3 divides 6 since $3 \cdot 2 = 6$, and -5 divides 15 since $-5 \cdot 3 = 15$.

Exercise 3.1. Show that for any integer a we have $a \mid a$.

Solution

We have $a \cdot 1 = a$, so a satisfies Definition 3.1 with 1 in place of q.

This is called *reflexivity*, and we say that $\mid$ is *reflexive* because it satisfies this property.

Another important note about the "divides" relation is that to say "a divides b" is not the same as saying "b/a is an integer." The latter is a statement about the result of applying the division binary *operation* to a and b.[1] The number 0 is what primarily causes issues here: "0 divides 0" is true, since $0 \cdot q = 0$ for any number q. Meanwhile, $0/0$ is undefined.

Exercise 3.2. Show that if a, b, and c are integers, and $a \mid b$ and $b \mid c$, then $a \mid c$.

Solution

If $a \mid b$, then there exists an integer k_1 such that $ak_1 = b$. If $b \mid c$, then there exists an integer k_2 such that $bk_2 = c$. Hence, $a(k_1k_2) = (ak_1)k_2 = bk_2 = c$. So a divides c with k_1k_2 in place of q in Definition 3.1.

This property is called *transitivity*, and we say that $\mid$ is a *transitive* relation on the integers.

For all that follows, we denote by $\mathbb{N}$ the *natural numbers*, consisting of the non-negative integers $0, 1, 2, \ldots$. Take special care that we are including zero in this set.[2] It will soon be apparent why we do this.

[1] A binary operation on a set X is a mapping $X \times X \to X$. A relation is not a mapping; it is simply a statement (something that is either true or false).

[2] This differs from another convention that excludes zero, which we may be used to.

Exercise 3.3. Show that for natural numbers a and b, if $a \mid b$ and $b \mid a$, then $a = b$.[3]

Solution

If $a \mid b$, then there exists a natural number k_1 such that $ak_1 = b$. Similarly, $b \mid a$ implies there exists a natural number k_2 such that $bk_2 = a$. Hence, $b = ak_1 = bk_1k_2$. We may similarly show that $a = k_1k_2a$. The first equation tells us that $b(1 - k_1k_2) = 0$. Then either $b = 0$ or $k_1 = k_2 = 1$.

If $b = 0$, then because $ak_1 = b$ we have either $a = 0$ (and thus $a = b$), or we have $k_1 = 0$. This would imply that $a = k_1k_2a = 0 \cdot k_2a = 0$. So $a = b = 0$.

If $b \neq 0$ (i.e., $k_1 = k_2 = 1$), then certainly $a = 1 \cdot a = ak_1 = b$. In either case, $a = b$.

This property is known as *antisymmetry*, and we say that $\mid$ is an *antisymmetric* relation on $\mathbb{N}$.

Exercise 3.4. Give an example of a pair of integers for which $a \mid b$ and $b \mid a$, yet $a \neq b$.

3.2 Defining Posets

Given a set X and a binary relation $\preccurlyeq$ defined on X, we say that $(X, \preccurlyeq)$ is a *partially ordered set* and that $\preccurlyeq$ is a *partial order* on X if the relation $\preccurlyeq$ is reflexive, antisymmetric, and transitive.

We will use $\preccurlyeq$ to denote an arbitrary partial order. The set $(\mathbb{N}, \mid)$ is a poset, since we verified that $\mid$ is reflexive, transitive, and antisymmetric on $\mathbb{N}$ in the previous section. On the other hand, $(\mathbb{Z}, \mid)$ is not a poset, since $\mid$ is not antisymmetric if we allow both positive and negative integers (see Exercise 3.4).

Note that for a given poset, it may be the case that for some $x, y \in X$ we have neither $x \preccurlyeq y$ nor $y \preccurlyeq x$. In this case, we say that x and y are *incomparable*.

Exercise 3.5.

1. Show that $(\mathbb{Z}, \leq)$ is a poset, and every pair of elements in $\mathbb{Z}$ are comparable.
2. Show that for an arbitrary set X, we have $(\mathcal{P}(X), \subseteq)$ is a poset, where $\mathcal{P}(X)$ is the *power set* of X, the set of all of its subsets. Show that if $|X| \geq 3$, then there exist incomparable elements of $\mathcal{P}(X)$.
3. Show that $(\mathbb{C}, \preccurlyeq)$ is a poset, where $a + bi \preccurlyeq c + di$ if and only if either $a \leq c$, or $a = c$ while $b \leq d$ (this is called the "lexicographic order" on the complex numbers). Show that every pair of elements of $\mathbb{C}$ are comparable under this order.

[3]If we allow a and b to be negative, then the most general statement we can make is that the magnitudes are equal.

3.3 The Opposite Order

For every poset $(X, \preccurlyeq)$ there is a corresponding poset called the *opposite* poset. The underlying set X remains the same, but we define a new order $\preccurlyeq^{\mathrm{op}}$ on X by $x \preccurlyeq^{\mathrm{op}} y$ if and only if $y \preccurlyeq x$. The opposite poset for $(\mathbb{Z}, \leq)$ from Exercise 3.5 is simply $(\mathbb{Z}, \geq)$.

3.4 Upper and Lower Bounds

Given a subset $A \subseteq X$ of a poset $(X, \preccurlyeq)$, we say that z is an *upper bound* of A if for each $x \in A$ we have $x \preccurlyeq z$. We say z is a *lower bound* of A if for each $x \in A$ we have $z \preccurlyeq x$. If $A = X$, then z is called a *maximum element* if it is an upper bound and a *minimum element* if it is a lower bound.

Exercise 3.6. Show that the poset $(\mathbb{N}, |)$ has a maximum and a minimum element.

Solution

The minimum element is 1, since for all $x \in \mathbb{N}$ we have $1 \mid x$. The maximum element is 0, since for all $x \in \mathbb{N}$ we have $x \mid 0$.

Note that if we excluded zero from our definition of $\mathbb{N}$, then we would only have a minimum element (and no maximum element).

3.5 Advanced Exercises

⚠ Warning

This section assumes prior knowledge. It may be safely skipped.

Exercise 3.7. Show that the following are posets:

1. For an arbitrary group G, the pair $(\mathcal{A}(G), \leq)$ is a partially ordered set, where $\mathcal{A}(G)$ consists of the subgroups of G and $\leq$ is the "subgroup" relation.
2. For a path-connected, locally path-connected, semilocally simply-connected topological space X with a distinguished basepoint $x_0 \in X$, the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a poset under the relation

$$p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0) \preccurlyeq p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$$

if and only if there exists a basepoint-preserving covering map $f : \tilde{X}_2 \to \tilde{X}_1$ such that $p_1 \circ f = p_2$ (this can be rephrased a bit less precisely by saying $\tilde{X}_2$ is a covering of $\tilde{X}_1$) (Hatcher 2002, 68).

3. Given a finite, Galois field extension E/F, there is a partial order on the intermediate fields $F \subseteq K \subseteq E$ under the inclusion relation: $K_1 \preccurlyeq K_2$ if and only if $K_1 \subseteq K_2$.
4. Given a vector space V, the subspaces of V form a partially ordered set under the subspace relation.

Solution

Every group is a subgroup of itself, so the subgroup relation is reflexive.

4 Groups

5 Vector Spaces

6 The Abstract Definition of a Category

7 Functors

8 Natural Transformations

References

Hatcher, Allen. 2002. *Algebraic Topology*. USA: Cambridge University Press. https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.