Categories for the Curious

A Light Intro to Category Theory

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Preface

This site contains the course notes for an informal one-semester seminar on category theory at CSU East Bay.

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2.1 The "Divides" Relation

Definition 2.1. Let m and n be integers. We say m *divides* n, and we write $m \mid n$ if there exists an integer q such that $mq = n$.

For example, 3 divides 6 since $3 \cdot 2 = 6$, and -5 divides 15 since $-5 \cdot 3 = 15$.

Exercise 2.1. Show that for any integer a we have $a \mid a$.

Solution

We have $a \cdot 1 = a$, so a satisfies Definition 2.1 with 1 in place of q.

This is called *reflexivity*, and we say that $\mid$ is *reflexive* because it satisfies this property.

Another important note about the "divides" relation is that to say "a divides b" is not the same as saying "b/a is an integer." The latter is a statement about the result of applying the division binary *operation* to a and b.[1] The number 0 is what primarily causes issues here: "0 divides 0" is true, since $0 \cdot q = 0$ for any number q. Meanwhile, $0/0$ is undefined.

This property is known as *antisymmetry*, and we say that $\mid$ is an *antisymmetric* relation on $\mathbb{N}$.

Exercise 2.2. Show that if a, b, and c are integers, and $a \mid b$ and $b \mid c$, then $a \mid c$.

Solution

If $a \mid b$, then there exists an integer k_1 such that $ak_1 = b$. If $b \mid c$, then there exists an integer k_2 such that $bk_2 = c$. Hence, $a(k_1k_2) = (ak_1)k_2 = bk_2 = c$. So a divides c with k_1k_2 in place of q in Definition 2.1.

This property is called *transitivity*, and we say that $\mid$ is a *transitive* relation on the integers.

[1] A binary operation on a set X is a mapping $X \times X \to X$. A relation is not a mapping; it is simply a statement (something that is either true or false).

For all that follows, we denote by $\mathbb{N}$ the *natural numbers*, consisting of the non-negative integers $0, 1, 2, \ldots$. Take special care that we are including zero in this set.[2] It will soon be apparent why we do this.

Exercise 2.3. Show that for natural numbers a and b, if $a \mid b$ and $b \mid a$, then $a = b$.[3]

Solution

If $a \mid b$, then there exists a natural number k_1 such that $ak_1 = b$. Similarly, $b \mid a$ implies there exists a natural number k_2 such that $bk_2 = a$. Hence, $b = ak_1 = bk_1k_2$. We may similarly show that $a = k_1k_2a$. The first equation tells us that $b(1 - k_1k_2) = 0$. Then either $b = 0$ or $k_1 = k_2 = 1$.

If $b = 0$, then because $ak_1 = b$ we have either $a = 0$ (and thus $a = b$), or we have $k_1 = 0$. This would imply that $a = k_1k_2a = 0 \cdot k_2a = 0$. So $a = b = 0$.

If $b \neq 0$ (i.e., $k_1 = k_2 = 1$), then certainly $a = 1 \cdot a = ak_1 = b$. In either case, $a = b$.

Exercise 2.4. Give an example of a pair of integers for which $a \mid b$ and $b \mid a$, yet $a \neq b$.

2.2 Defining Posets

Given a set X and a binary relation $\preccurlyeq$ defined on X, we say that $(X, \preccurlyeq)$ is a *partially ordered set* and that $\preccurlyeq$ is a *partial order* on X if the relation $\preccurlyeq$ is reflexive, antisymmetric, and transitive.

We will use $\preccurlyeq$ to denote an arbitrary partial order. The set $(\mathbb{N}, \mid)$ is a poset, since we verified that $\mid$ is reflexive, transitive, and antisymmetric on $\mathbb{N}$ in the previous section. On the other hand, $(\mathbb{Z}, \mid)$ is not a poset, since $\mid$ is not antisymmetric if we allow both positive and negative integers (see Exercise 2.4).

Exercise 2.5. Show that the following are posets:

1. $(\mathbb{Z}, \leq)$.
2. For an arbitrary set X, $(\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$ is the *power set* of X, the set of all of its subsets.
3. $(\mathbb{C}, \preccurlyeq)$, where $a + bi \preccurlyeq c + di$ if and only if either $a \leq c$, or $a = c$ while $b \leq d$ (this is called the "lexicographic order" on the complex numbers).
4. Given a vector space V, the subspaces of V form a partially ordered set under the subspace relation.

[2] This differs from another convention that excludes zero, which we may be used to.

[3] If we allow a and b to be negative, then the most general statement we can make is that the magnitudes are equal.

2.3 Advanced Exercises

These assume prior knowledge and may be safely skipped.

Exercise 2.6. Show that the following are posets:

1. For an arbitrary group G, the pair $(\mathcal{A}(G), \leq)$ is a partially ordered set, where $\mathcal{A}(G)$ consists of the subgroups of G and $\leq$ is the "subgroup" relation.
2. For a path-connected, locally path-connected, semilocally simply-connected topological space X with a distinguished basepoint $x_0 \in X$, the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0 \to (X, x_0)$ is a poset under the relation: $(\tilde{X}_1, \tilde{x}_1) \preccurlyeq (\tilde{X}_2, \tilde{x}_2)$ if and only if there exists a basepoint-preserving covering map $f : \tilde{X}_2 \to \tilde{X}_1$ such that $p_1 \circ f = p_2$ (this can be rephrased a bit less precisely by saying X_2 is a covering of X_1) (Hatcher 2002, 68).
3. Given a finite, Galois field extension E/F, there is a partial order on the intermediate fields $F \subseteq K \subseteq E$ under the inclusion relation: $K_1 \preccurlyeq K_2$ if and only if $K_1 \subseteq K_2$.

Solution

Every group is a subgroup of itself, so the subgroup relation is reflexive.

References

Hatcher, Allen. 2002. *Algebraic Topology*. USA: Cambridge University Press. https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.