

# **Categories for the Curious**

**A Light Intro to Category Theory**

John Cavanaugh

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# Preface

This site contains the course notes for an informal one-semester seminar on category theory at CSU East Bay.

# 1 Introduction

## 1.1 What is a category?

We will learn the [abstract definition of a category](#) later. However, it would be counterproductive to state the precise definition right now.

## 1.2 Why would we want to learn this?

Throughout our mathematical education, we have likely encountered many types of mathematical objects (e.g., groups, rings, vector spaces, fields, topological spaces, sets, functions, manifolds). We might have learned that a group is “a *set* equipped with an associative binary operation having an identity element, and for which every element has an inverse,” or that a “vector space is a *set* equipped with the operations of vector addition and scalar multiplication satisfying the usual axioms.” It may have been a while since we’ve dealt with these definitions, or we may not have learned them at all. That is okay. What is important to focus on now is that both of these objects are defined as a *set* equipped with some additional structure. We have defined them in terms of “sets.”

So what is a “set,” exactly? We might see it defined as “a collection of objects, which are called its elements,” which then begs the question: what is a “collection”? How is it that we can give such precise definitions of all of the structures mentioned above (vector spaces, groups, rings, etc.), yet we seem to have difficulty making precise the concept of a set, on which the definitions of all of these structures rely?

The uncomfortable truth is that modern mathematics, with its rigorous definitions and axioms, cannot exist *ex nihilo*. To formulate a rigorous definition, we must first precisely define each term within that definition, and so on for the definitions of each of those terms. This process cannot end without either defining two objects “circularly,” such that some object is defined in terms of another and vice versa, or declaring once and for all that certain objects are atomic, and they cannot be defined in terms of any other object.

Categories constitute an alternative foundation for mathematics; all of the structures mentioned above could instead be defined in terms of categories. While

## 1.3 Our Goal

## 2 Sets

## 3 Partially Ordered Sets

### 3.1 The “Divides” Relation

**Definition 3.1.** Let  $m$  and  $n$  be integers. We say  $m$  *divides*  $n$ , and we write  $m \mid n$  if there exists an integer  $q$  such that  $mq = n$ .

For example, 3 divides 6 since  $3 \cdot 2 = 6$ , and  $-5$  divides 15 since  $-5 \cdot 3 = 15$ .

**Exercise 3.1.** Show that for any integer  $a$  we have  $a \mid a$ .

Solution

We have  $a \cdot 1 = a$ , so  $a$  satisfies Definition 3.1 with 1 in place of  $q$ .

This is called *reflexivity*, and we say that  $\mid$  is *reflexive* because it satisfies this property.

Another important note about the “divides” relation is that to say “ $a$  divides  $b$ ” is not the same as saying “ $b/a$  is an integer.” The latter is a statement about the result of applying the division binary *operation* to  $a$  and  $b$ .<sup>1</sup> The number 0 is what primarily causes issues here: “0 divides 0” is true, since  $0 \cdot q = 0$  for any number  $q$ . Meanwhile,  $0/0$  is undefined.

**Exercise 3.2.** Show that if  $a$ ,  $b$ , and  $c$  are integers, and  $a \mid b$  and  $b \mid c$ , then  $a \mid c$ .

Solution

If  $a \mid b$ , then there exists an integer  $k_1$  such that  $ak_1 = b$ . If  $b \mid c$ , then there exists an integer  $k_2$  such that  $bk_2 = c$ . Hence,  $a(k_1k_2) = (ak_1)k_2 = bk_2 = c$ . So  $a$  divides  $c$  with  $k_1k_2$  in place of  $q$  in Definition 3.1.

This property is called *transitivity*, and we say that  $\mid$  is a *transitive* relation on the integers.

For all that follows, we denote by  $\mathbb{N}$  the *natural numbers*, consisting of the non-negative integers  $0, 1, 2, \dots$ . Take special care that we are including zero in this set.<sup>2</sup> It will soon be apparent why we do this.

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<sup>1</sup>A binary operation on a set  $X$  is a mapping  $X \times X \rightarrow X$ . A relation is not a mapping; it is simply a statement (something that is either true or false).

<sup>2</sup>This differs from another convention that excludes zero, which we may be used to.

**Exercise 3.3.** Show that for natural numbers  $a$  and  $b$ , if  $a \mid b$  and  $b \mid a$ , then  $a = b$ .<sup>3</sup>

Solution

If  $a \mid b$ , then there exists a natural number  $k_1$  such that  $ak_1 = b$ . Similarly,  $b \mid a$  implies there exists a natural number  $k_2$  such that  $bk_2 = a$ . Hence,  $b = ak_1 = bk_1k_2$ . We may similarly show that  $a = k_1k_2a$ . The first equation tells us that  $b(1 - k_1k_2) = 0$ . Then either  $b = 0$  or  $k_1 = k_2 = 1$ .

If  $b = 0$ , then because  $ak_1 = b$  we have either  $a = 0$  (and thus  $a = b$ ), or we have  $k_1 = 0$ . This would imply that  $a = k_1k_2a = 0 \cdot k_2a = 0$ . So  $a = b = 0$ .

If  $b \neq 0$  (i.e.,  $k_1 = k_2 = 1$ ), then certainly  $a = 1 \cdot a = ak_1 = b$ . In either case,  $a = b$ .

This property is known as *antisymmetry*, and we say that  $\mid$  is an *antisymmetric* relation on  $\mathbb{N}$ .

**Exercise 3.4.** Give an example of a pair of integers for which  $a \mid b$  and  $b \mid a$ , yet  $a \neq b$ .

## 3.2 Defining Posets

Given a set  $X$  and a binary relation  $\preceq$  defined on  $X$ , we say that  $(X, \preceq)$  is a *partially ordered set* and that  $\preceq$  is a *partial order* on  $X$  if the relation  $\preceq$  is reflexive, antisymmetric, and transitive.

We will use  $\preceq$  to denote an arbitrary partial order. The set  $(\mathbb{N}, \mid)$  is a poset, since we verified that  $\mid$  is reflexive, transitive, and antisymmetric on  $\mathbb{N}$  in the previous section. On the other hand,  $(\mathbb{Z}, \mid)$  is not a poset, since  $\mid$  is not antisymmetric if we allow both positive and negative integers (see Exercise 3.4).

**Exercise 3.5.** Show that the following are posets:

1.  $(\mathbb{Z}, \leq)$ .
2. For an arbitrary set  $X$ ,  $(\mathcal{P}(X), \subseteq)$ , where  $\mathcal{P}(X)$  is the *power set* of  $X$ , the set of all of its subsets.
3.  $(\mathbb{C}, \preceq)$ , where  $a + bi \preceq c + di$  if and only if either  $a \leq c$ , or  $a = c$  while  $b \leq d$  (this is called the “lexicographic order” on the complex numbers).
4. Given a vector space  $V$ , the subspaces of  $V$  form a partially ordered set under the subspace relation.

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<sup>3</sup>If we allow  $a$  and  $b$  to be negative, then the most general statement we can make is that the magnitudes are equal.

### 3.3 Advanced Exercises

These assume prior knowledge and may be safely skipped.

**Exercise 3.6.** Show that the following are posets:

1. For an arbitrary group  $G$ , the pair  $(\mathcal{A}(G), \leq)$  is a partially ordered set, where  $\mathcal{A}(G)$  consists of the subgroups of  $G$  and  $\leq$  is the “subgroup” relation.
2. For a path-connected, locally path-connected, semilocally simply-connected topological space  $X$  with a distinguished basepoint  $x_0 \in X$ , the set of basepoint-preserving isomorphism classes of path-connected covering spaces  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  is a poset under the relation:  $(\tilde{X}_1, \tilde{x}_1) \preceq (\tilde{X}_2, \tilde{x}_2)$  if and only if there exists a basepoint-preserving covering map  $f : \tilde{X}_2 \rightarrow \tilde{X}_1$  such that  $p_1 \circ f = p_2$  (this can be rephrased a bit less precisely by saying  $X_2$  is a covering of  $X_1$ ) (Hatcher 2002, 68).
3. Given a finite, Galois field extension  $E/F$ , there is a partial order on the intermediate fields  $F \subseteq K \subseteq E$  under the inclusion relation:  $K_1 \preceq K_2$  if and only if  $K_1 \subseteq K_2$ .

Solution

Every group is a subgroup of itself, so the subgroup relation is reflexive.

## 4 Groups

## 5 Vector Spaces

## **6 The Abstract Definition of a Category**

## 7 Functors

## 8 Natural Transformations

# References

Hatcher, Allen. 2002. *Algebraic Topology*. USA: Cambridge University Press. <https://pi.math.cornell.edu/~hatcher/AT/ATpage.html>.