MATH355 2017-09-18

IT'S ALL ABOUT THAT BASIS

Example 1. Are the vectors $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 2 \\ -5 \\ 9 \\ 11 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -3 \end{pmatrix}$ linearly dependent or independent?

I don't know, let's find out! Suppose

$$x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$$

this means

$$\begin{pmatrix} x + 2y \\ -x - 5y + z \\ 3x + 9y - z \\ x + 11y - 3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Uhm, dunno. It's a system of equations though, so we can write down the corresponding augmented matrix to see if there's a solution!

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -5 & 1 & 0 \\ 3 & 9 & -1 & 0 \\ 1 & 11 & -3 & 0 \end{array}\right]$$

... a few row operations later ...

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

as we have one free variable and the system is homogeneous, there are infinitely many solutions. Hence, the three vectors are *not* linearly independent.

Although we don't *need* to find a solution, it might be helpful to do so, as a sanity check. For example, take $z = 3$, so that

$$x + 2y = 0, -3y + 3 = 0$$

hence $y = 1$ and $x = -2$. Indeed,

$$-2\vec{u} + \vec{v} + 3\vec{w} = \begin{pmatrix} -2 + 2 \\ 2 - 5 + 3 \\ -6 + 9 - 3 \\ -6 + 9 - 3 \\ -2 + 11 - 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent.

We also mentioned in class the following fact, which may be useful.

Date: John Calabrese, September 18, 2017.

Lemma 2. A subset $S \subset V$ is linearly independent if and only if for all $\vec{v} \in S$ we have

$$\operatorname{Span}(S \setminus \{\vec{v}\}) \subsetneq \operatorname{Span} S$$

Roughly, the lemma is saying that S is linearly independent if and only if all the vectors in S are essential (no redundancy).

Remark 3. Recall that the symbol $\subsetneq$ means 'contained in but not equal to'. In other words it means $\subset$ and $\neq$ simultaneously. For example, if $B := \{7, 8, 99, -19\}$, $A := \{7, 8, 99\}$, $C := \{-19, 99, 7, 8\}$, then $C \subset B$ and $A \subset B$. But only $A \subsetneq B$.

Recall that

basis = linear independent + span.

Example 4. The vectors $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis of $\mathbf{R}^2$

A vector space has as many bases as your imagination allows.

Example 5. $\vec{v}_1 := \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\vec{v}_2 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis of $\mathbf{R}^2$.

To show this, we need to prove two things: $\vec{v}_1, \vec{v}_2$ are linearly independent, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \mathbf{R}^2$. Let us begin with the former. Suppose

$$a\vec{v}_1 + b\vec{v}_2 = \vec{0}$$

this means

$$\begin{pmatrix} 2a + b \\ 4a + b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and by subtracting the first equation from the first we see that $2a = 0$ so $a = 0$, and the first equation now reads $b = 0$. Hence, the only way to obtain $\vec{0}$ is via the trivial linear combination. In other words, $\vec{v}_1, \vec{v}_2$ are linearly independent.

Let us show they span $\mathbf{R}^2$. Let us fix a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$. Can we represent it as a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Let's find out.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2a + b \\ 4a + b \end{pmatrix}$$

Once again, subtract the second equation from the first, and we obtain $2a = y - x$ so that $a = \frac{1}{2}(y - x)$. The first equation now reads $x = 2a + b = y - x + b$, so $b = 2x - y$. In other words,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(y - x)\vec{v}_1 + (2x - y)\vec{v}_2$$

so $\vec{v}_1, \vec{v}_2$ span $\mathbf{R}^2$.

Notice that when we expressed $\begin{pmatrix} x \\ y \end{pmatrix}$ in terms of our basis, we found a *unique* solution for the coefficients a, b of our linear combination. This is no accident.

Theorem 6. The vectors $\vec{v}_1, \ldots, \vec{v}_n \in V$ form a basis of V if and only if any vector $\vec{v} \in V$ may be expressed in a unique way as a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$.

Proof. We have two statements. The first is

the vectors $\vec{v}_1, \ldots, \vec{v}_n \in V$ form a basis of V

while the second is

any vector $\vec{v} \in V$ may be expressed in a unique way as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

We wish to prove they are 'equivalent', meaning the first implies the second but also the second implies the first.

Let us first show that the first implies the second (which we write as $\Rightarrow$). So, suppose $\vec{v}_1, \dots, \vec{v}_n$ form a basis of V. By definition of basis, any vector $\vec{v}$ is a linear combination of the $\vec{v}_i$. So the only thing left to show is that the coefficients of this linear combination are unique. Suppose then

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

but also

$$\vec{v} = \beta_1 \vec{v}_1 + \dots + \beta_n \vec{v}_n$$

then

$$\vec{0} = \vec{v} - \vec{v} = (\alpha_1 - \beta_1)\vec{v}_1 + \dots + (\alpha_n - \beta_n)\vec{v}_n$$

Since $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent, we must have

We already used the fact that bases span, now we use that bases are linearly independent.

$$\alpha_1 - \beta_1 = 0 = \alpha_2 - \beta_2 = \dots = \alpha_n - \beta_n$$

So, $\alpha_i = \beta_i$ for all i. In other words, the coefficients expressing $\vec{v}$ as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$ are unique.

Now we prove the convers (we write $\Leftarrow$). So, suppose any vector $\vec{v}$ may expressed uniquely as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$. In particular, this means $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$. Hence, we need to show the $\vec{v}_i$ are linearly independent. Suppose

$$\vec{0} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

but we can also write

$$\vec{0} = 0\vec{v}_1 + \dots + 0\vec{v}_n$$

But assumption, the coefficients must be unique, hence $\alpha_i = 0$ for all i. This completes the proof that $\vec{v}_1, \dots, \vec{v}_n$ form a basis of V. □

We now come to the coolest feature of bases.

Definition 7. Assume $B = (\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V. Let $\vec{v} \in V$. Using the previous theorem, we know

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

for *unique* $\alpha_1, \dots, \alpha_n \in \mathbf{R}$. We define

$$\operatorname{Rep}_B \vec{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

the *representation* of $\vec{v}$ with respect to the basis B. The numbers α_i are called the *coordinates* of $\vec{v}$.

This is really amazing: a basis turns an abstract vector $\vec{v}$ into an n-tuple of numbers $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ in $\mathbf{R}^n$!

But, watch out, the *order* in which we write B matters!

Example 8. Let $B = (\vec{e}_1, \vec{e}_2)$ be the standard basis of $\mathbf{R}^2$. Let $\vec{v} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \in \mathbf{R}^2$. Then

$$\vec{v} = 3\vec{e}_1 + (-6)\vec{e}_2$$

hence

$$\operatorname{Rep}_B \vec{v} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

Example 9. Let $B' = (\vec{e}_2, \vec{e}_1)$ be a slightly different basis of $\mathbf{R}^2$. Let $\vec{v} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \in \mathbf{R}^2$. Then

$$\vec{v} = -6\vec{e}_2 + 3\vec{e}_2$$

hence

$$\operatorname{Rep}_{B'} \vec{v} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$$

On page 146 of the book you will find a convincing explanation (based on crystals!) of why one would want to pick different bases.