MATH355 2017-11-06

SIMILARITY

We are now entering Chapter Five of Hefferon. Section I is about complex vector spaces, which were covered last Friday. The reason we need complex numbers and complex vector spaces is ultimately due to the fact that the polynomial $x^2 + 1$ has no real solutions. In any case, today we begin with Section II and the concept of *similarity*.

Our goal now is to study linear maps $f\colon V \to V$. These are called *endomorphisms* of V. Recall a result from a while ago.

Theorem 1. Let $f\colon V \to V$ be linear. There exist bases $\mathbb{B}, \mathbb{D}$ of V such that $\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f$ is made up of four blocks: upper left is I_r the $r \times r$ identity matrix (where r is the rank of f) and the other three blocks are all zero.

This is a great result, but it destroys a lot of information.

Example 2. For example, consider $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$, i.e. a rotation by $\frac{\pi}{2}$. We have

$$\mathrm{Rep}_{\mathrm{std}} f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let $\mathbb{B} = (\vec{e}_2, -\vec{e}_1)$. Then

$$\mathrm{Rep}_{\mathrm{std},\mathbb{B}} f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but f is *not* the identity.

Example 3. Let $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ be given by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ 3x \end{pmatrix}$. Then

$$B := \mathrm{Rep}_{\mathrm{std}}\, g = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

and $\det B = -6$ (the standard basis vectors are both stretched and swapped).

Let $\mathbb{D} = (\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix})$. Then

$$\mathrm{Rep}_{\mathrm{std},\mathbb{D}}\, g = I$$

is the identity matrix. But $\det I = 1 \neq -6$.

Date: John Calabrese, November 6, 2017.

So we want a different way of viewing linear maps, which remembers some of the geometry. For this reason, we will focus on $\mathrm{Rep}_{\mathbb{B},\mathbb{B}} f$ for a choice of basis $\mathbb{B}$. [same basis on departure and arrival] We will write

$$\mathrm{Rep}_{\mathbb{B}} f = \mathrm{Rep}_{\mathbb{B},\mathbb{B}} f.$$

Proposition 4. Let $\mathbb{B}, \hat{\mathbb{B}}$ be two bases for V. Let $f\colon V \to V$ be a linear map. Let $A = \mathrm{Rep}_{\mathbb{B}} f$, $\hat{A} = \mathrm{Rep}_{\hat{\mathbb{B}}} f$. Then

$$\det \hat{A} = \det A.$$

Proof. We already know this! Let $P := \mathrm{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}$. Then recall that $\hat{A} = PAP^{-1}$.

$$\begin{aligned}
\det \hat{A} &= \det(PAP^{-1}) \\
&= \det(P)\det(A)\det(P^{-1}) \\
&= \det(P)\det(P^{-1})\det(A) \\
&= \det(P)\det(P)^{-1}\det(A) \\
&= \det A.
\end{aligned}$$ □

OK, great. So this means that switching from $\mathbb{B}$ to $\hat{\mathbb{B}}$ must remember something about the geometry of our linear map.

We need a little piece of notation. Call $GL_n \subset M_{n\times n}$ the subset of invertible matrices. This is called the *general linear group*.

Definition 5. We say A is *similar* to B if there exists $P \in GL_n$ such that $B = PAP^{-1}$. We write $A \sim B$.

There are two ways to view similarity.

(1) View both A and B as defining linear maps with respect to the standard basis, similarity is telling you how to express one transformation in terms of the other.
(2) View A as defining a linear map with respect to the standard basis and view P as a change of basis matrix! This way, B is representing the *same* linear map but with respect to a different basis (i.e. a different set of coordinates).

Proposition 6. Being similar is an equivalence relation on $M_{n\times n}$. I.e.

(1) $A \sim A$
(2) if $A \sim B$ then $B \sim A$.
(3) if $A \sim B$, $B \sim C$ then $A \sim C$.