(1)

(a) $\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1+3 \\ 1+2 \\ 2-1+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 3 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1-3 & 2-2 & 2-3+9 \\ 1-2 & 2 & 3+6 \\ -2-1 & 4-4 & 4-6+3 \end{pmatrix}$

$= \begin{pmatrix} -4 & 0 & 8 \\ -1 & 2 & 9 \\ -3 & 0 & 1 \end{pmatrix}$

(c) $U_1 := \mathrm{Span}\{\vec{e}_1\}$ $\quad U_2 := \mathrm{Span}\{\vec{e}_2\}$ $\quad U_3 = \mathrm{Span}\{\vec{e}_3, \vec{e}_4\}$

$\mathbb{R}^4 = U_1 + U_2 + U_3$.

~~A~~ ~~A~~ ~~B~~ ~~C~~ ~~D~~ ~~2~~ ~~3~~ ~~4~~

(2)

(a) $f: \mathbb{R}^3 \to \mathbb{R}^3$, ~~f iso~~ $A = \begin{pmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

f iso if and only if $\text{rk} A = 3$.

Use row reduction:

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rk} A = 2$$

~~A~~ f is <u>not</u> an iso.

[Alternatively: notice that column 3 is = column 1 + column 2
so f can't be an iso]

(b) To compute $\text{Rep}_{std,B}$ we only need to express $f(\vec{e}_i)$ in terms of $\mathbb{B}$.

$$f(\vec{e}_1) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} a=0 \\ b=1 \\ c=0 \end{array}$$

$$f(\vec{e}_2) = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} a=2 \\ b=-2 \\ 2=-b+c \end{array}$$

$$\Rightarrow c = 2+b = 2-2 = 0$$

(2 b continued)

$$f(\vec{e}_3) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad 2 = a, \quad -1 = b, \quad 1 = -b + c$

$\Rightarrow \quad c = 1 + b = 1 - 1 = 0$

so $\operatorname{Rep}_{std, B} f = \begin{bmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

~~4~~ 3

a) TRUE $\dim \operatorname{Im} f \leq 2$

and $4 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \ker f \geq 2 \Rightarrow \ker f \neq \{\vec{0}\}$

b) FALSE Take $U = \mathbb{R}^5$, $W = \mathbb{R}^5$.

c) TRUE (was proved in class)

d) false take $B = \text{std}$ $\hat{B} = (\vec{e}_2, \vec{e}_1, \vec{e}_3)$

then $\operatorname{Rep}_{B,\hat{B}} \operatorname{id} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I$.

e) FALSE Take $f \equiv \vec{0}$ the zero map.

f) TRUE. Say $B = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ if $\operatorname{Rep}_{B,B} f = I$

then $f(\vec{v}_1) = \vec{v}_1$, $f(\vec{v}_2) = \vec{v}_2$ $f(\vec{v}_3) = \vec{v}_3$

$\Rightarrow$ if $\vec{v} \in \mathbb{R}^3$ $\vec{v} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ so

$f(\vec{v}) = af(\vec{v}_1) + bf(\vec{v}_2) + cf(\vec{v}_3) =$

$= a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{v}$

so $f = \operatorname{id}$ ■

~~3~~ 4

(a) $g : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. ...

notice $B = \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ is a basis for $\mathbb{R}^3$

to define g, it's enough to declare $\vec{u}_1 = g\begin{pmatrix}1\\0\\1\end{pmatrix}$ ~~g~~ $\vec{u}_2 = g\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\vec{u}_3 = g\begin{pmatrix}1\\1\\1\end{pmatrix}$

set $\vec{u}_1 := \begin{pmatrix}1\\0\\1\end{pmatrix}$ $\vec{u}_2 := \begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\vec{u}_3 := \begin{pmatrix}0\\0\\0\end{pmatrix}$

notice $\mathrm{Rep}_{B,B}\, g = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$

so $\dim \mathrm{Im}\, g = 2$

also $g(W) \subseteq W$ and $g\begin{pmatrix}1\\1\\1\end{pmatrix} = \vec{0}$.

5 $\boxed{\text{no}}$

take $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

i.e.

$f(\vec{e_1}) = \vec{e_2}$ $\quad f(\vec{e_2}) = \vec{0}$

$\ker f = \text{Span}\{\vec{e_2}\}$ $\quad \text{Im} f = \text{Span}\{\vec{e_2}\}$

So $(\text{Im} f) + (\ker f) = \text{Span}\{\vec{e_2}\} \neq \mathbb{R}^2$.