MATH 355 HOMEWORK 4

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Problem 1

Part (a). We show that $\{\vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$ is linearly independent. To see that this suffices observe that this means that $\{\vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$ form a basis for $\mathbf{R}^4$, so throwing in another vector won't change the span. Suppose we have $a, b, c, d \in \mathbf{R}$ such that $a\vec{v_2} + b\vec{v_3} + c\vec{v_4} + d\vec{v_5} = \vec{0}$. Then

$$a\begin{pmatrix}-1\\1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + d\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

So we have a linear system

$$\begin{aligned}
-a + b + c + d &= 0\\
a - b + c + d &= 0\\
a + b - c + d &= 0\\
a + b + c - d &= 0.
\end{aligned}$$

We could do a row reduction, but this one is simple enough to do "by hand." Adding the third and fourth equations gives $2a + 2b = 0$, so $a + b = 0$. Substituting $a + b = 0$ back into the third equation gives $c = d$. Adding the first and second equations gives $2c + 2d = 0$. Using our previous relation $c = d$ we obtain $4c = 0$, hence $c = 0$ and $d = 0$. Then $a = b$ and $a = -b$, so $b = -b$, so $b = 0$ and $a = 0$. We conclude that the set is linearly independent.

Part (b). As we showed in part (a), we have a basis $\{\vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$. Another basis is $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$. To prove this, it suffices to show that this set is linearly independent because our space is four-dimensional. We can solve a similar system or we can observe that $\vec{v_1} = \frac{1}{2}(\vec{v_2} + \vec{v_3} + \vec{v_4} + \vec{v_5})$, so the exchange lemma (p. 122 in Hefferon) gives us that we have another basis (note that the coefficient of v_5—the one we replaced—in our expression for v_1 is nonzero so the result applies).

Part (c). In this section I found all the expression by solving linear systems. I won't include the row reductions at this point. If you really want to see them, send me an email or talk to me in office hours or recitation. Anyway, call $\mathbf{B}_1 = \{\vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$ and $\mathbf{B}_2 = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$. Then as we remarked in part (b), we have

$$\mathrm{Rep}_{\mathbf{B}_1}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}.$$

Doing a calculation, I found

$$\mathrm{Rep}_{\mathbf{B}_1}\begin{pmatrix}2\\3\\5\\1\end{pmatrix} = \begin{pmatrix}7/4\\5/4\\1/4\\9/4\end{pmatrix}.$$

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The next one is easy since our second basis actually has the first vector in it. That is

$$\mathrm{Rep}_{\mathbf{B}_2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Another calculation for the other vector yields

$$\mathrm{Rep}_{\mathbf{B}_2}\begin{pmatrix}2\\3\\5\\1\end{pmatrix}=\begin{pmatrix}9/2\\-1/2\\-1\\-2\end{pmatrix}.$$

Problem 2

Suppose $W \neq \mathbf{R}^5$, so there exists $\vec{v} \in \mathbf{R}^5$ with $\vec{v} \notin W$. First observe that $\vec{0} \in W$, so $\vec{v} \neq \vec{0}$. Now, since $\dim W = 5$, we can find a set of five linearly independent vectors in W. Call them $\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}$. We claim that the set $\{\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}, \vec{v}\}$ is linearly independent. Indeed suppose that the set is not linearly independent. Then there exist $a_i \in \mathbf{R}$ for $1 \leq i \leq 6$ not all zero such that

$$a_1\vec{w_1} + a_2\vec{w_2} + a_3\vec{w_3} + a_4\vec{w_4} + a_5\vec{w_5} + a_6\vec{v} = \vec{0}.$$

Rearranging we obtain

$$a_1\vec{w_1} + a_2\vec{w_2} + a_3\vec{w_3} + a_4\vec{w_4} + a_5\vec{w_5} = -a_6\vec{v}. \tag{1}$$

Suppose $a_6 = 0$, then

$$a_1\vec{w_1} + a_2\vec{w_2} + a_3\vec{w_3} + a_4\vec{w_4} + a_5\vec{w_5} = \vec{0},$$

but since then at least one of a_i for $1 \leq i \leq 5$ is nonzero, this contradicts the linearly independence of the set $\{\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}\}$. Then we must have that $a_6 \neq 0$. But then 1 implies that

$$\vec{v} = \frac{-1}{a_6}\left(a_1\vec{w_1} + a_2\vec{w_2} + a_3\vec{w_3} + a_4\vec{w_4} + a_5\vec{w_5}\right).$$

This however implies that $\vec{v} \in W$ a contradiction. Then we have that $\{\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}, \vec{v}\}$ is linearly independent, this contradicts the fact that $\dim \mathbf{R}^5 = 5$ by Corollary 2.11 in Hefferon (see p.123 for details) which states that no linearly independent set can have a size greater than the dimension of the enclosing space.

Remark. The structure of the proof includes a number of smaller proofs by contradiction to establish for example the linear independence of the set or that $a_6 \neq 0$. This can be a little confusing because one has to keep track of what is being supposed for contradiction and what has actually been proved once one arrives at a particular contradiction. This pattern of argument is however typical of proofs in linear algebra.

Problem 3

Part (a). We're blessed in that the augmented matrix we get from this system of equations is already in echelon form:

$$\left(\begin{array}{ccccc|c}1 & -1 & 1 & 4 & -6 & 0\\0 & 0 & 1 & 0 & 1 & 0\end{array}\right).$$

Then our leading variables are x_1 and x_3 and our free variables are x_2, x_4, and x_5. Since we have three free variables, we conclude that $\dim \mathrm{Sol} = 3$.

Part (b). Let's first describe the solution space for Sol. Our augmented matrix tells us that

$$\begin{aligned}x_1 &= x_2 - x_3 - 4x_4 + 6x_5\\x_3 &= -x_5.\end{aligned}$$

Expressing our leading variables entirely in terms of our free variables yields

$$\begin{aligned}x_1 &= x_2 - 4x_4 + 7x_5\\x_3 &= -x_5.\end{aligned}$$

Then we can write

$$\begin{aligned}
\mathrm{Sol} &= \left\{ \begin{pmatrix} x_2 - 4x_4 + 7x_5 \\ x_2 \\ -x_5 \\ x_4 \\ x_5 \end{pmatrix} \middle| x_2, x_4, x_5 \in \mathbf{R} \right\} \\
&= \left\{ \begin{pmatrix} x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 7x_5 \\ 0 \\ -x_5 \\ 0 \\ x_5 \end{pmatrix} \middle| x_2, x_4, x_5 \in \mathbf{R} \right\} \\
&= \left\{ x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| x_2, x_4, x_5 \in \mathbf{R} \right\}
\end{aligned}.$$

This suggests a basis of

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Our above calculation shows that the set we wrote down is spanning, and because it is spanning set with the same size as the dimension of Sol we know it's a basis.

Part (c). We claim that

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbf{R}^5$. Since the set has five elements it suffices to show that it is spanning. We will show spanning but showing that all of the standard basis vectors for $\mathbf{R}^5$, $\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}, \vec{e_5}$ can be obtained by linear combinations of the vectors in our set. Note that $\vec{e_4}$ and $\vec{e_5}$ are elements of our set, so we already have them. Now

$$\left(\frac{-1}{4}\right) \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \left(\frac{1}{4}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we are free to use $\vec{e_1}$ in our linear combinations (because a linear combination of linear combinations is another linear combination). Then we have

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is $\vec{e_2}$. Finally,

$$(-1) \begin{pmatrix} 7 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

which is $\vec{e_3}$. Another basis is

$$\left\{ \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-4\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}7\\0\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\}.$$

That this set is a basis is easy to see because we can just scale our fourth vector by $\frac{1}{2}$ and get our first basis back.

Remark 0.2. There are (infinitely) many ways both to give a basis for Sol and to complete that basis to one for $\mathbf{R}^5$. Cf. Lemma 2.4 of Hefferon (p.122).

Part(c). Call our first basis $\mathbf{B}_1$. We wish to find $a, b, c, d, e \in \mathbf{R}$ such that

$$a\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}-4\\0\\0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\-1\\0\\0\\1\end{pmatrix} + d\begin{pmatrix}0\\0\\0\\2\\0\end{pmatrix} + e\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\2\\2\\1\end{pmatrix}.$$

Note that our five vectors' forming a basis guarantees a unique solution (a, b, c, d, e) to this equation. Anyway our equations are

$$\begin{aligned} a - 4b + 7c &= 0 \\ a &= -1 \\ c &= 2 \\ b + d &= 2 \\ c + e &= 1. \end{aligned}$$

A little calculation gives $a = -1, b = \frac{-13}{4}, c = 2, d = \frac{21}{4}, e = -1$. So we have

$$\mathrm{Rep}_{\mathbf{B}_1}\begin{pmatrix}0\\-1\\2\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-\frac{13}{4}\\2\\\frac{21}{4}\\-1\end{pmatrix}.$$

For the second basis $\mathbf{B}_2$ the only change is $c + e = 1$ is now $c + 2e = 1$, so we need $e = -\frac{1}{2}$. That is

$$\mathrm{Rep}_{\mathbf{B}_1}\begin{pmatrix}0\\-1\\2\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-\frac{13}{4}\\2\\\frac{21}{4}\\-\frac{1}{2}\end{pmatrix}.$$