HW9 Solutions

#1: We wish to compute the inverse of

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$$ by row and column operations.

We follow the method outlined in Lemma 4.7 and subsequent examples. In particular we only use row operations.

$$\left(\begin{array}{ccc|ccc} 7 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 0 & 0 & 1 \end{array}\right).$$

Add 3/7 times row 1 to row 3

$$\left(\begin{array}{ccc|ccc} 7 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 34/7 & -11/7 & 3/7 & 0 & 1 \end{array}\right).$$

Multiply row1 by 1/7

$$\left(\begin{array}{ccc|ccc} 1 & 2/7 & 1/7 & 1/7 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 34/7 & -11/7 & 3/7 & 0 & 1 \end{array}\right)$$

Multiply row 2 by 1/3

$$\left(\begin{array}{ccc|ccc} 1 & 2/7 & 1/7 & 1/7 & 0 & 0 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 34/7 & -11/7 & 3/7 & 0 & 1 \end{array}\right).$$

Add $\frac{-2}{7}$ times row 2 to row1.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5/21 & 1/7 & -2/21 & 0 \end{array}\right.$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5/21 & 1/7 & -2/21 & 0 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 34/7 & -11/7 & 3/7 & 0 & 1 \end{array}\right)$$

Add $\frac{-34}{7}$ times row 2 to row 3

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5/21 & 1/7 & -2/21 & 0 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/21 & 3/7 & -34/21 & 1 \end{array}\right).$$

Add -5 times row 3 to row 1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 8 & -5 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/21 & 3/7 & -34/21 & 1 \end{array}\right).$$

Multiply row 3 by 21

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 8 & 5 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 9 & -34 & 21 \end{array}\right).$$

Add 1/3 times row 3 to row 2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 8 & -5 \\ 0 & 1 & 0 & 3 & -11 & 7 \\ 0 & 0 & 1 & 9 & -34 & 21 \end{array}\right).$$

So we read off the inverse:

$$\begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ \text{[illegible]} & -34 & 21 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 0 & \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}.$$

#2: We want a linear map $f: \mathbb{R}^4 \to \mathbb{R}^4$ s.t.

(1) $f(W) \subseteq W$ where $W = \operatorname{span}\left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \right\}$.

(2) $f\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = f\begin{pmatrix}0\\-1\\2\\0\end{pmatrix}$

(3) $\dim \ker f = 1$.

A linear map is uniquely determined by its values on a basis. Set $f\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $f\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$. This guarantees condition (1).

Denote $b_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $b_2 = \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$. Note that b_1 and b_2 are linearly independent. Set $b_3 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$. Note that $\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \frac{1}{2}b_1 - \frac{1}{2}b_2 + b_3$ and $\begin{pmatrix}0\\-1\\2\\0\end{pmatrix} = b_1 - b_2 - b_3$. Condition (2) now says

$$f\left(\tfrac{1}{2}b_1 - \tfrac{1}{2}b_2 + b_3\right) = f\left(b_1 - b_2 - b_3\right), \text{ so}$$

$$\tfrac{1}{2}f(b_1) - \tfrac{1}{2}f(b_2) + f(b_3) = f(b_1) - f(b_2) - f(b_3), \text{ so}$$

$$2f(b_3) = \tfrac{1}{2}f(b_1) - \tfrac{1}{2}f(b_2), \text{ so}$$

$\begin{pmatrix}0\\ \end{pmatrix}$. So we have condition (2)

$f(b_3) = f(b_1) - f(b_2) = b_1 - b_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$. So we have condition (2) by setting $f(b_3) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$.

Note that $\{b_1, b_2, b_3\}$ is linearly independent. We can complete this to a basis by setting $b_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. To figure out where to send b_4, we have to satisfy condition (3). Note that $\{f(b_1), f(b_2), f(b_3)\}$ is linearly dependent, so the map already has nontrivial kernel, so we want to send b_4 to something not in span$\{f(b_1), f(b_2)\}$. The easiest is

$f(b_4) = b_4$.

Part (a): Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis. We need to compute $f(e_1)$, $f(e_2)$, $f(e_3)$, and $f(e_4)$. Before we had $b_3 = e_2$, so we know

$$f(e_2) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = 2e_3.$$

Also $f(e_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_4$ (since $b_4 = e_4$).

$e_1 = \frac{1}{2} b_1 + \frac{1}{2} b_2$, so $f(e_1) = \frac{1}{2} f(b_1) + \frac{1}{2} f(b_2) = \frac{1}{2} b_1 + \frac{1}{2} b_2 = e_1$.

$e_3 = \frac{1}{2} b_1 - \frac{1}{2} b_2$, so $f(e_3) = e_3$.

Then

$$\text{Rep}_{s+d} f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Yes. Pick any nonzero element of the kernel. Call it c_2. Complete $\{c_2\}$ to a basis $\{c_1, c_2, c_3, c_4\}$. Observe that $\{f(c_1), f(c_3), f(c_4)\}$ is linearly independent. (Otherwise the kernel would have dimension greater than 1). Call $d_1 = f(c_1)$, $d_2 = f(c_3)$, $d_4 = f(c_4)$. Then $\{d_1, d_2, d_4\}$ is linearly independent and can be completed to a basis $\{d_1, d_2, d_3, d_4\}$. Then $f(d_2) = 0$ and setting $\mathbb{B} = \{c_1, c_2, c_3, c_4\}$ and $\mathbb{D} = \{d_1, d_2, d_3, d_4\}$ suffices. We can follow this outline to produce the bases $\mathbb{B}$ and $\mathbb{D}$. First we need a nonzero element of the kernel. Note that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ so } \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix} \text{ is in the kernel.}$$

We can complete this to a basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathbb{B}$.

Then $f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $f\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, so we can complete to a basis $\mathbb{D} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Note that the order is relevant.

#3:

#3:

Part a: Let $\vec{v} \in \ker f$, so $f(\vec{v}) = \vec{0}$. Consider $g(\vec{v}) = f(f(\vec{v})) = f(\vec{0}) = \vec{0}$. Hence $\ker f \subseteq \ker g$.

Part b: Let $\vec{v} \in \operatorname{Im} g$, so there exists $\vec{w} \in V$ s.t. $g(\vec{w}) = \vec{v}$, so $f(f(\vec{w})) = \vec{v}$, so set $\vec{u} = f(\vec{w})$, so $f(\vec{u}) = \vec{v}$. Then $\vec{v} \in \operatorname{Im} f$. Hence $\operatorname{Im} g \subseteq \operatorname{Im} f$.

Part c: Let $f: V \to V$ be a linear map that has representation A with respect to some basis. We know $rk(A) = \dim \operatorname{Im} f$ and $rk(A^2) = \dim \operatorname{Im}(f \circ f)$. Then by part b, we have $\dim \operatorname{Im}(f \circ f) \leq \dim \operatorname{Im}(f)$, so $rk(A^2) \leq rk(A)$.

Part d: Let $\vec{v} \in \operatorname{Im} g$, so there exists $\vec{w} \in V$ s.t. $g(\vec{w}) = \vec{v}$, so $f(f(\vec{w})) = \vec{v}$. $f(\vec{w}) \in \operatorname{Im} f$, so $\vec{v} \in f(\operatorname{Im} f)$. Hence $\operatorname{Im} g \subseteq f(\operatorname{Im} f)$.

Conversely let $\vec{v} \in f(\operatorname{Im} f)$, so there exists $\vec{w} \in \operatorname{Im} f$ s.t. $\vec{v} = f(\vec{w})$. Since $\vec{w} \in \operatorname{Im} f$, there exists $\vec{u} \in V$ such that $f(\vec{u}) = \vec{w}$, so $\vec{v} = f(f(\vec{u})) = g(\vec{u})$, so $\vec{v} \in \operatorname{Im} g$. Hence $f(\operatorname{Im} f) \subseteq \operatorname{Im} g$ and we conclude $f(\operatorname{Im} f) = \operatorname{Im} g$.

#4:

Part a: $f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$; $f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$; $f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$. So

$$\operatorname{Rep}_{std} f = \begin{pmatrix}1 & 1 & 1\\ 3 & -2 & -1\end{pmatrix}$$

Part b: $\left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix}\right\}$ is linearly independent, so the image has dimension 2, so $rk(A) = 2$.

Part c: Since the image has dimension 2, the kernel has dimension 1, so any nonzero element of the kernel will form a basis.

Note that $\begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\left\{\begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}\right\}$ is a basis for ker f.

Part d: Let $\mathbb{B} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}\right\}$ and $\mathbb{D} = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$.

Then $f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and $f\begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so $\text{Rep}_{\mathbb{B},\mathbb{D}} f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Part e: We just compute $\text{Rep}_{\mathbb{B}} e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\text{Rep}_{\mathbb{B}} e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and

$\text{Rep}_{\mathbb{B}} e_3 = \begin{pmatrix} 1/5 \\ 4/5 \\ -1/5 \end{pmatrix}$, so $P = \begin{pmatrix} 1 & 0 & 1/5 \\ 0 & 1 & 4/5 \\ 0 & 0 & -1/5 \end{pmatrix}$

Part f: Similarly $\text{Rep}_{\mathbb{D}} e_1 = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$ $\text{Rep}_{\mathbb{D}} e_2 = \begin{pmatrix} 1/5 \\ -1/5 \end{pmatrix}$, so

$Q = \begin{pmatrix} 2/5 & 1/5 \\ 3/5 & -1/5 \end{pmatrix}$.

Part g: $P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -5 \end{pmatrix}$, so

$$\hat{A} = \begin{pmatrix} 2/5 & 1/5 \\ 3/5 & -1/5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1/5 \\ 0 & 1 & 4/5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$, as desired.