MATH 355 HOMEWORK 2

Problem 2

Part a. After the row operation $R_2 \mapsto R_2 - 2R_1$ the system is in row-echelon form:

$$\begin{aligned} x + \quad y + 2z \qquad &= 0 \\ -2y - 3z + 6w &= 1 \end{aligned}$$

From this, we see that the leading variables are x and y, and the free variables are z and w.

Part b. Since the system is consistent and there are free variables, the system has infinitely many solutions.

Part c. We're looking to write the general solution to the system in the form $\vec{p} + \vec{v}_h$ where $\vec{p}$ is a *single* particular solution to the inhomogeneous system, and $\vec{v}_h$ is the *general* solution to the associated homogeneous system. By inspection, we can see a particular solution is given by $x = y = z = 0, w = 1/6$, so $\vec{p} = (0, 0, 0, 1/6)$. To find solutions to the associated homogeneous system, we solve for the leading variables x, y in terms of the free variables z, w. From the row-echelon form (of the homogeneous system), we see

$$\begin{aligned} y &= -\frac{3}{2}z + 3w \\ x &= -y - 2z = -\frac{1}{2}z - 3w. \end{aligned}$$

Hence, every solution $\vec{v}_h = (x, y, z, w)$ to the associated homogeneous system is of the form

$$\vec{v}_h = \begin{pmatrix} -\frac{1}{2}z - 3w \\ -\frac{3}{2}z + 3w \\ z \\ w \end{pmatrix} = -\frac{1}{2}z \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix} + w \begin{pmatrix} -3 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

Since z and w are free variables, we can set $c_1 = -\frac{1}{2}z$, $c_2 = w$ to write the solution set as

$$\text{Sol} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/6 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 3 \\ 0 \\ 1 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

Part d. The associated homogeneous system is

$$\begin{aligned} x + y + 2z &= 0 \\ 2x + z + 6w &= 0. \end{aligned}$$

Problem 3

We will use this definition from page 78 of the text for the next two problems.

Definition. A *vector space* (over $\mathbb{R}$) consists of a set $\mathbf{V}$ along with two operations '+' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in \mathbf{V}$ and all *scalars* $r, s \in \mathbb{R}$:

(1) $\vec{v} + \vec{w} \in \mathbf{V}$ ($\mathbf{V}$ is closed under $+$)
(2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ ($+$ is commutative)
(3) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$ ($+$ is associative)
(4) there is a *zero vector* $\vec{0} \in \mathbf{V}$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in \mathbf{V}$ ($+$ has an identity element)
(5) each $\vec{v} \in \mathbf{V}$ has an *additive inverse* $\vec{w} \in \mathbf{V}$ such that $\vec{w} + \vec{v} = \vec{0}$ ($+$ has inverses)
(6) $r \cdot \vec{v} \in \mathbf{V}$ ($\mathbf{V}$ is closed under $\cdot$)
(7) $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$ ($\cdot$ distributes over scalar addition)

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(8) $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$ ($\cdot$ distributes over vector addition)
(9) $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$ (multiplication of scalars associates with $\cdot$)
(10) $1 \cdot \vec{v} = \vec{v}$ (1 is the identity element for $\cdot$).

Now, we want to show that if $\vec{v}_0 + \vec{w} = \vec{w}$ for all $\vec{w} \in \mathbf{V}$, then $\vec{v}_0 = \vec{0}$. By axiom (5), there exists an element $\vec{u} \in \mathbf{V}$ such that $\vec{u} + \vec{w} = \vec{0}$. Then we have

$$\begin{aligned}
\vec{v}_0 + \vec{w} &= \vec{w} && \\
\vec{u} + (\vec{v}_0 + \vec{w}) &= \vec{u} + \vec{w} && \\
\vec{u} + (\vec{v}_0 + \vec{w}) &= \vec{0} && \text{by (5)} \\
(\vec{v}_0 + \vec{w}) + \vec{u} &= \vec{0} && \text{by (2)} \\
\vec{v}_0 + (\vec{w} + \vec{u}) &= \vec{0} && \text{by (3)} \\
\vec{v}_0 + (\vec{u} + \vec{w}) &= \vec{0} && \text{by (2)} \\
\vec{v}_0 + \vec{0} &= \vec{0} && \text{by (5)} \\
\vec{v}_0 &= \vec{0} && \text{by (4).}
\end{aligned}$$

Remark. This argument shows that the additive identity in a vector space is unique.

Problem 4

We have

$$\begin{aligned}
\vec{u} + \vec{v} &= \vec{0} && \text{first equation from the problem} \\
(\vec{u} + \vec{v}) + \vec{w} &= \vec{0} + \vec{w} && \\
(\vec{u} + \vec{v}) + \vec{w} &= \vec{w} + \vec{0} && \text{by (2)} \\
(\vec{u} + \vec{v}) + \vec{w} &= \vec{w} && \text{by (4)} \\
\vec{u} + (\vec{v} + \vec{w}) &= \vec{w} && \text{by (3)} \\
\vec{u} + \vec{0} &= \vec{w} && \text{second equation from the problem} \\
\vec{u} &= \vec{w} && \text{by (4).}
\end{aligned}$$

Remark. This argument shows that additive inverses in a vector space are unique.