2017 Fall - Math 355 - Homework 10

Due: Friday, November 17 *in class.*[1]
Unless specified otherwise, you must always show your work and justify your answers.

(1) Let V be a vector space and let $f\colon V \to V$ be a linear. Suppose $\vec{v}, \vec{w} \in V$ are two *nonzero* vectors. Suppose there exist scalars λ, μ such that $f(\vec{v}) = \lambda\vec{v}$, $f(\vec{w}) = \mu\vec{w}$. Show that if $\lambda \neq \mu$ than $\vec{v}$ and $\vec{w}$ are linearly independent.

(2) Let $V = M_{2\times 2}(\mathbb{C})$ be the complex vector space of two-by-two matrices with complex entries. If $A \in V$ is a matrix, its transpose A^t is defined as follows.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Consider the linear map $\tau\colon V \to V$ defined by $\tau(A) = A + A^t$.[2]

(a) What are $\dim_{\mathbb{C}} \ker\tau$, $\dim_{\mathbb{C}} \mathrm{Im}\tau$?

(b) Is $V = \ker\tau \oplus \mathrm{Im}\tau$? Why, why not?[3]

(3) Consider the matrix

$$A = \begin{pmatrix} -2 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{pmatrix}$$

which we view as representing a linear map f in the standard basis.[4]

(a) Is A diagonalizable over $\mathbb{R}$?

(b) Is A diagonalizable over $\mathbb{C}$?

(c) If A is diagonalizable, find a basis $\mathbb{B}$ (of $\mathbb{R}^3$ or $\mathbb{C}^3$) consisting of eigenvectors for A.

(d) Compute $\mathrm{Rep}_{\mathbb{B}}\, f$.

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[1] This file was last updated at 15:05 on Friday 10th November, 2017.

[2] Notice that $(A+B)^t = A^t + B^t$ for any $A, B \in V$. Also, $(\alpha B)^t = \alpha B^t$ for $B \in V$, $\alpha \in \mathbb{R}$.

[3] A matrix in $\ker\tau$ is called *antisymmetric*.

[4] To simply the computations we point out that 1 is a root of the characteristic polynomial of A.