Hi Neil!

Recall: $f: V \to V$ linear, λ scalar

$$V_\lambda = \{\vec{v} \in V \mid f(\vec{v}) = \lambda \vec{v}\}$$ "λ-eigenspace"

Note: λ is eigenvalue for $f \iff \dim V_\lambda > 0$.

Ex) Say $f: \mathbb{R}^3 \to \mathbb{R}^3$ is the zero map

$(\mathbb{R}^3)_0 = \mathbb{R}^3$ $\quad (\mathbb{R}^3)_\lambda = \{\vec{0}\}$ for any $\lambda \neq 0$.

Ex) $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ $\quad f\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a+b \\ 3b \end{pmatrix}$ $\quad \mathrm{Rep}_{std} f = A$

what are eigenvalues of f?

$$\det(A - xI) = \det\begin{pmatrix} 3-x & 1 \\ 0 & 3-x \end{pmatrix} = (3-x)^2$$

so only eigenvalue is 3

what is $(\mathbb{R}^2)_3 = \{\vec{v} \in \mathbb{R}^2 \mid A\vec{v} = 3\vec{v}\} = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

well $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ so

$$(\mathbb{R}^2)_3 = \left\{\begin{pmatrix} a \\ b \end{pmatrix} \middle| \, b = 0\right\} = \mathrm{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ which is 1-dim

$\Rightarrow f$ not diagonalizable.

Useful Fact:

Prop Let $f: V \to V$ be linear. Let $\lambda_1, \dots, \lambda_k$ be ~~eigen~~ distinct eigenvals

let $\vec{v}_1, \dots, \vec{v}_k \in V$

$\vec{v}_j$ a λ_j-eigenvector then $\vec{v}_1, \dots, \vec{v}_k$ are linearly indep.

Ex) $B = \begin{pmatrix} 2 & 0 & 0 \\ 7 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is B diag'z?

expand along first row

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 7 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\Big((-1-\lambda)(2-\lambda)\Big)$$

$$= -(\lambda+1)(2-\lambda)^2$$

"$\lambda=1$ has multiplicity 1

$\lambda=2$ has multiplicity 2"

Let's find eigenvectors:

$\boxed{\lambda=-1}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 7 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3a \\ 7a+c \\ 3c \end{bmatrix} = \vec{0} \text{ iff } a=0=c$$

So $V_{-1} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \, a=0=c \right\} = \mathrm{Span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

[here $V = \mathbb{R}^3$]

$\boxed{\lambda = 2}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 7 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 7a - 3b + c \\ 0 \end{bmatrix}$$

$= \vec{0}$ iff $7 - 3b + c = 0$

So $V_2 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; 7a - 3b + c = 0 \right\} = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$

pick $\mathbb{B} = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right)$ basis

~~Rep~~ let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be s.t. $g\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ 7a - b + c \\ 2c \end{pmatrix}$

i.e. $\mathrm{Rep}_{std}\, g = B$

~~to~~ what is $\mathrm{Rep}_{\mathbb{B}}\, g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ · by construction

<u>Compute</u> $B\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $B\begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$ $B\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ to make sure. (sanity check)

do this in class if you have time, o/w exercise for them.

what is P s.t. $PBP^{-1} = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 2 \end{pmatrix}$?

well we know $P^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 7 & 3 \end{pmatrix}$

and P we can compute.

Ex) if A is matrix 3×3 and char pol of A has three distinct real roots. Is A diagz? (Yes! why? use ~~that vectors~~ prev. prop)

Ex) $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ is A diagz?

what about $B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$?

(discuss these if you have time?)