CATEGORIES OF COHERENT SHEAVES AND APPLICATIONS TO DONALDSON-THOMAS THEORY.

JOHN CALABRESE — PROJECT DESCRIPTION

The overarching theme of this proposal is the study of categories of coherent sheaves. The focus is twofold. On the one hand, the PI proposes to investigate *Torelli-type* theorems, in the context of both *abelian* and *derived* categories, and their connection to *birational* and *non-commutative* geometry. On the other hand, the PI aims to settle a conjecture in *Donaldson-Thomas* theory, using the machinery of *Hall algebras*, *stability conditions* and *wall-crossing*, the three of which rely fundamentally on categories of coherent sheaves.

The proposal discusses three main research objectives:

(1) Study the quotient categories $\mathtt{Coh}(X)/\mathtt{Coh}_{\leq k}(X)$ and their relation to birational geometry. [Main Objective 1, discussed in Section 3].
(2) Discover new Torelli-type theorems for (semi-orthogonal) components of the derived category. [Main Objective 2, discussed in Section 4]
(3) Prove the Crepant Resolution Conjecture for Donaldson-Thomas invariants. [Main Objective 3, discussed in Section 5]

The first two objectives grew out of the PI's program investigating *point-like objects*, initiated in the joint work with Groechenig [CG15] and pursued further in [Cal17a, Calb, Cal17b]. The second objective is strongly influenced by the PI's work with Thomas [CT16], in the context of Kuznetsov's *homological projective duality*. The third is a continuation of the PI's work [Cal16a, Cal16b] on Donaldson-Thomas theory.

Throughout the proposal, the PI discusses several research objectives, which vary in scope. Among these, Research Objectives 1 and 8 would be particularly well-suited for a *PhD student*. Details are provided in the relevant sections.

1. Prior Support

The PI is supported by NSF MSPRF DMS-1502916 ($150,000.00), August 2015 – July 2018 "Derived Categories for Homological Projective Duality and Donaldson-Thomas Invariants".

Intellectual Merit. The PI has produced three published research articles: [Cal17a, Cal17b, Calb], and a survey [Cala], to be included in the proceedings of the Workshop on New Connections in Algebraic Geometry for recent PhDs, organized by Lieblich and Olsson. The PI has matured a research program on the study of point-like objects in abelian and derived categories, stemming from [CG15] and pursued further in [Cal17a, Cal17b, Calb].

Broader Impacts. The PI has been very active in participating and organizing learning seminars (as detailed below). The PI mentored closely a graduate student of Várilly-Alvarado, in preparation for their advanced exam. The PI is currently mentoring an undergraduate student (Patrick Girardet) by providing a reading course in algebraic geometry.

The PI was supported by NSF DMS–1723842 ($15,000), May 2017 – April 2018 "Texas Algebraic Geometry Symposium - 2017". TAGS is a regional conference hosted yearly by one of Rice, Texas A&M and UT Austin.

Intellectual Merit. Research seminars were delivered by Frank Sottile (TAMU), Melody Chan (Brown), Angela Gibney (UGA), Donu Arapura (Purdue), Inna Zakharevich (Cornell), Nicolas Addington (U Oregon), Raphael Rouquier (UCLA), Burt Totaro (UCLA). One special lecture for graduate students was delivered by Anthony Várilly-Alvarado (Rice).

Broader Impacts. In addition to the organizer and speakers, 62 registered participants attended the event. Of these 62: 24 were Asian, 4 were Hispanic or Latino and 4 registered as Two or more races. Of these 62: 17 were female. A website was created hosting reading material (suggested by the speakers) and notes (taken by local graduate students).

2. Broader Impacts

The PI has always been very active in participating and organizing learning seminars and other related activities. In 2012, the PI co-organized a year-long reading seminar on stacks and derived algebraic geometry. In 2013, the PI co-organized a reading seminar on Hilbert schemes and a seminar on extended topological field theories. The PI spoke several times at the Junior Geometry seminar at Imperial College and at Oxford's Junior Geometry and Topology seminar.

In Spring 2014, the PI redesigned the format of MATH 390, a course aimed at giving undergraduate students lecturing experience. The class met on a weekly basis. The lectures were delivered by students, covering basic material on the subject of matrix groups. Students coordinated with the PI before presenting in class, and received detailed feedback after each lecture.

At Rice, the PI has been involved in several learning seminars: Spring 2014 (Intersection Theory), Fall 2014 (Representation Theory), Spring 2015 (Algebraic Surfaces), Fall 2015 (Algebraic Groups), Spring 2016 (Étale Fundamental Groups), Fall 2017 (Rational Points).

Throughout the academic year 2016-2017, the PI held weekly hour-long meetings with Doctoral Candidate Stephen Wolff, who at the time was preparing for his advanced exam (he is now a student of Várilly-Alvarado).

In Spring 2017, the PI organized the Texas Algebraic Geometry Symposium. This is a regional conference aimed at attracting researchers, from the graduate level upwards, coming from Texas and neighboring states.

The PI offered a graduate course on derived categories in Spring 2017, a graduate course on Lie theory in Fall 2017 and an undergraduate linear algebra course in Fall 2017. For these three courses, the PI has written detailed lecture notes, available on the PI's website.

The PI looks forward to continue fostering a welcoming and productive research environment. The PI will continue to mentor students, both at the graduate and undergraduate level, by organizing learning seminars, contributing to undergraduate colloquia and increasing the mathematical resources already available on the PI's website. The PI looks forward to taking PhD students in the near future. Finally, the PI will continue to organize regional conferences, workshops and events.

3. Categories of Coherent Sheaves

The main research objective of this section is the study of the quotient categories $\mathtt{Coh}(X)/\mathtt{Coh}_{\leq k}(X)$, and their relation to birational and non-commutative geometry. This project grew out of the program [CG15, Calb, Cal17b, Cal17a] aimed at investigating point-like objects and reconstruction theorems (also known as Torelli-type theorems).

Background. The seminal 1955 paper of Serre [Ser55] introduced the notion of *coherent sheaf*, which is now ubiquitous in algebraic geometry. We can think of coherent sheaves as possibly singular vector bundles. Roughly speaking, a coherent sheaf V on a variety X is a collection of vector spaces V_x, one for each point $x \in X$. Unlike vector bundles, the dimension of the fibre V_x needn't be constant: it is allowed to jump along closed subsets. The collection of all coherent sheaves forms a category, $\mathtt{Coh}(X)$. Precisely because the rank is allowed to vary, this category is particularly pleasant to work with: it is closed under all the operations one wishes to perform (kernels, images, cokernels, et cetera), in contrast with the category of vector bundles. We say $\mathtt{Coh}(X)$ is an *abelian category*. The passage from vector bundles to coherent sheaves becomes especially natural in the affine case $X = \operatorname{Spec} R$. Indeed, vector bundles on X correspond to projective modules over R, while $\mathtt{Coh}(X)$ corresponds to all (finitely generated) modules.

Gabriel's theorem. Let us recall the following foundational result.

Theorem (Gabriel [Gab62]) – Let X and Y be two varieties,[1] then $\mathtt{Coh}(X) \simeq \mathtt{Coh}(Y)$ if and only if $X \simeq Y$. $*$

We interpret this result as saying that $\mathtt{Coh}(X)$ as a *complete invariant* of X. The key fact making this theorem possible is a characterization of the *minimal objects* of $\mathtt{Coh}(X)$. We say $F \in \mathtt{Coh}(X)$ is minimal if it has no non-trivial sub-objects. It is easy to show that the minimal objects of $\mathtt{Coh}(X)$ are precisely the *skyscraper sheaves* $\kappa(x)$, with $x \in X$. It is then immediate that an equivalence $\mathtt{Coh}(X) \simeq \mathtt{Coh}(Y)$ induces a bijection between points of X and points of Y.

The hard and technical part is upgrading this bijection to an actual isomorphism of varieties. Gabriel originally dealt with this issue by developing a theory of localizations for abelian categories, by relying on appropriate *quotients* $\mathtt{Coh}(X)/\mathtt{S}$. Quotient categories are the subject of the proposed Main Objective 1.

Generalizing Gabriel. To be more precise, Gabriel proved his theorem in the context of locally noetherian schemes. To any (reasonable) abelian category $\mathtt{A}$, he attached a locally ringed space $Z(\mathtt{A})$ called its center. When $\mathtt{A} = \mathtt{Coh}(X)$, one has $Z(\mathtt{Coh}(X)) = X$.[2] This perspective opens an avenue for *non-commutative geometry*, as we may view any (reasonable) abelian category as a *non-commutative space*.

Work of Rosenberg [Ros04], with contributions by Gabber and later Brandenburg [Bra13], generalized the theorem to encompass any quasi-separated scheme. A different approach to Gabriel's theorem was given by Rouquier [Rou10].

[1] The term *variety* is used here rather loosely. We will see below the theorem holds much more generally for noetherian schemes and beyond.

[2] This is slightly imprecise: one should really be working with the category $\mathtt{QCoh}(X)$ of *quasi-coherent* sheaves. This is only a technical point, which we will ignore for simplicity. Indeed, when X is noetherian $\mathtt{QCoh}(\mathtt{X})$ is merely the Ind-completion of $\mathtt{Coh}(X)$. Conversely, $\mathtt{Coh}(X)$ may be recovered as the subcategory of compact objects.

Generalizing away from schemes, we see that the theorem fails even for the simplest algebraic stacks. For example, if $Y = \mathrm{B}\mathbf{Z}/\mathbf{2Z}$ and X is the disjoint union of two points, then $\mathtt{Coh}(Y) \simeq \mathtt{Coh}(X)$, as both categories simply consist of pairs of vector spaces.

Living in between schemes and stacks are *algebraic spaces*. Even if one is solely interested in varieties, algebraic spaces present themselves in the following contexts: quotients by free group actions, resolutions of singularities, and moduli problems. The most famous example probably being Hironaka's complex (non-projective!) variety whose Hilbert scheme is actually *not* a scheme. In [CG15], we proved the following.

Theorem 3.1 (C.-Groechenig [CG15]) – Let X and Y be two (quasi-compact and separated) algebraic spaces over a base ring R. Then X and Y are isomorphic over R if and only if $\mathtt{Coh}(X)$ is equivalent to $\mathtt{Coh}(Y)$ as R-linear categories. ∗

The result and methods of [CG15] were used by Cantadore [Can17] to prove an interesting result on moduli of point-like objects for the Kuznetsov component of a cubic fourfold. We discuss ideas from Kuznetsov's theory below, in Section 4

The modular approach. The proof of Gabriel's theorem in [CG15] necessarily took a different route from the previous approaches, as algebraic spaces *cannot* be described as locally ringed spaces. Indeed, our strategy was to construct a *moduli functor* $\mathrm{Pt}_{\mathtt{A}}$ parameterizing minimal (or *point-like*) objects of $\mathtt{A}$. We then showed that, for X an algebraic space, $\mathrm{Pt}_{\mathtt{Coh}X}$ is X itself. This was especially challenging in the non-noetherian setting. Nevertheless we succeeded, thanks to a clever and unexpected use of the derived category of $\mathtt{A}$.

Căldăraru's conjecture. Categories of *twisted coherent sheaves* have gained much attention recently, especially in the study of K3 surfaces (see below for a manifestation of these within Kuznetsov's theory of homological projective duality). In 2002, Căldăraru conjectured a version of Gabriel's theorem for $\mathtt{Coh}(X, \alpha)$, the category of α-twisted coherent sheaves [Cal02]. Progress towards this conjecture followed in [Per09, CS07] under some restrictive assumptions. The conjecture was settled for schemes in [Ant13] and in complete generality in [CG15].[3]

Theorem 3.2 (C.-Groechenig [CG15]) – Let X and Y be (quasi-compact and separated) algebraic spaces over a ring R. Let $\alpha \in H^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$, $\beta \in H^2_{\text{ét}}(Y, \mathbf{G}_{\mathrm{m}})$. Then the categories of twisted sheaves $\mathtt{Coh}(X, \alpha)$, $\mathtt{Coh}(Y, \beta)$ are equivalent as R-linear categories if and only if X and Y are isomorphic as R-spaces, via an isomorphism which pulls β back to α. ∗

This theorem came as a consequence of the construction of $\mathrm{Pt}_{\mathtt{A}}$ from [CG15]. Indeed, we were slightly imprecise earlier: $\mathrm{Pt}_{\mathtt{Coh}(X)}$ is not X, but rather $X \times \mathrm{B}\mathbf{G}_{\mathrm{m}}$, the trivial $\mathbf{G}_{\mathrm{m}}$-gerbe over X. More generally, one sees that $\mathrm{Pt}_{\mathtt{Coh}(X,\alpha)}$ is the $\mathbf{G}_{\mathrm{m}}$-gerbe corresponding to α.

Perverse coherent sheaves. One is naturally led to ask whether the construction of $\mathrm{Pt}_{\mathtt{A}}$ can be fruitful for other abelian categories of geometric nature. The PI proposes here to study a category appearing in the context of *threefold flops*.

When discussing the flop formula for Donaldson-Thomas invariants below in Section 5, the category $\mathtt{Per}(X/W)$ will play a central role. Objects of $\mathtt{Per}(X/W)$ are called *perverse coherent sheaves*. Here, $f\colon X \to W$ is a flopping contraction between varieties of dimension three (over a field) and $\mathtt{Per}(X/W)$ is a specific heart of a t-structure sitting inside the derived

[3] It should be noted that [Ant13] and [CG15] use vastly different methods: the former utilizes the derived Azumaya algebras of Toën, while the latter proof is elementary using moduli spaces.

category $\mathtt{D}(X)$. The category $\mathtt{Per}(X/W)$ was defined by Bridgeland to settle (in dimension 3) a deep conjecture of Bondal-Orlov [Bri02]. Its definition is

$$\begin{aligned}\mathtt{Per}(X/W) := \{E \in \mathtt{D}(X) \mid & H^i(E) = 0, \text{ for } i \neq -1, 0; \\ & f_* H^{-1}(E) = 0; \\ & R^1 f_* H^0(E) = 0\}\end{aligned}$$

where we omitted a small technical assumption for simplicity. The second condition implies that $H^{-1}(E)$ is supported on the contracted locus of f. Hence, if we restrict away from this locus, we see that the complex E is just an ordinary coherent sheaf.

Under nice homological assumptions (such as X smooth and W Gorenstein), Bridgeland constructed a moduli space Y of point-like objects in $\mathtt{Per}(X/W)$. He proved (among other things) the following striking results:

- $Y \to W$ is the *flop* of $X \to W$.
- The derived categories $\mathtt{D}(X)$ and $\mathtt{D}(Y)$ are equivalent.

The result was extended to allow some singularities for X in [Che02, AC05].

Research Objective 1 – Study the relation between Y and the space $\mathrm{Pt}_{\mathtt{Per}(X/Y)}$. $*$

One would reasonably expect $\mathrm{Pt}_{\mathtt{Per}(X/W)}$ to coincide with the flop Y. However, the definition of point-like adopted by Bridgeland is not intrinsic to the category $\mathtt{Per}(X/W)$ and differs from the one appearing in [CG15]. The correct approach is to view $\mathtt{Per}(X/W)$ as a *sheaf of categories* over W, as showed in [VdB04]. In [CG15] a sheafified version of Theorem 3.1 already appears, and we expect the *relative* moduli space $\mathrm{Pt}_{\mathtt{Per}(X/W)/W}$ to yield Y. Pursuing this avenue will lead to a more general version of Bridgeland's construction, encompassing a broader class of singularities than those allowed in [Che02, AC05].

The PI believes Research Objective 1 to be well suited for a PhD student. To understand $\mathrm{Pt}_{\mathtt{Per}(X/W)/W}$ one first needs a good knowledge of algebraic geometry (moduli theory, derived categories, and some birational geometry). The category $\mathtt{Per}(X/W)$ was crucial in settling the conjecture of Bondal-Orlov. Nevertheless, the conjecture is still *open* in higher dimensions (but important progress has been made by Namikawa, Cautis and Halpern-Leistner) and research in derived categories is currently very active. Finally, generalizing this construction from flops to *flips* has provided elusive: a new approach is needed.

Birational Gabriel. In [MP14] Meinhardt and Partsch studied stability conditions on the *quotient categories* $\mathtt{C}_k(X) := \mathtt{Coh}(X)/\mathtt{Coh}_{\leq k}(X)$. Here $\mathtt{Coh}_{\leq k}(X)$ denotes the subcategory of sheaves whose support is of dimension at most k. The quotient $\mathtt{C}_k(X)$ is obtained from $\mathtt{Coh}(X)$ by formally setting all objects in $\mathtt{Coh}_{\leq k}(X)$ equal to zero.

A connection with birational geometry was already apparent in [MP14]. Let $n = \dim X$. They showed that, for X smooth and projective over a field, the category $\mathtt{C}_{n-1}(X)$ is equivalent to the category of vector spaces over $K(X)$, the function field of X. By Morita theory, if $\mathtt{C}_{n-1}(X) \simeq \mathtt{C}_{n-1}(Y)$ then $K(X) \simeq K(Y)$. In turn, X and Y must be *birational*!

Recall that X and Y are birational if they share a common Zariski-open (and thus dense) subset. More precisely, there exist closed subvarieties $Z \hookrightarrow X$, $V \hookrightarrow Y$, and an isomorphism $X \setminus Z \simeq Y \setminus W$. For a general birational equivalence $X \sim Y$, there is no control over the codimensions $\operatorname{codim}_X Z$, $\operatorname{codim}_Y V$. We say X and Y are *isomorphic in codimension c* if in the birational equivalence above $\operatorname{codim}_X Z \geq c, \operatorname{codim}_Y V \geq c$. For example, if X is a blowup of Y, then X and Y are isomorphic in codimension zero.

Inspired by [MP14], the PI conjectures the following variant of Gabriel's theorem.

CONJECTURE 3.3 – The varieties X and Y are isomorphic in codimension c if and only if the categories $\mathtt{C}_{\dim X-c-1}(X)$, $\mathtt{C}_{\dim Y-c-1}(Y)$ are equivalent. $*$

As pointed out above, the extreme cases of this conjecture are Gabriel's theorem and the fact that two varieties are birational if and only if they have isomorphic function fields. Thus, the quotient categories $\mathtt{C}_k(X)$ are *interpolating* between these two classic theorems. In [MP14] the case $c = 1$ was settled for X, Y smooth and projective over a field.

MAIN OBJECTIVE 1 – Settle Conjecture 3.3 in the affirmative. $*$

The PI has been working on the following strategy to settle this conjecture with Roberto Pirisi (UBC).

RESEARCH OBJECTIVE 2 – Characterize the minimal objects of $\mathtt{C}_k(X)$. $*$

Just as in Gabriel's original theorem, the minimal (or point-like) objects seem to be key. We believe that the minimal objects of $\mathtt{C}_k(X)$ correspond to closed, irreducible subvarieties $Z \hookrightarrow X$ with $\dim Z = k + 1$.

If this were true, then given an equivalence $\mathtt{C}_k(X) \simeq \mathtt{C}_k(Y)$ one would be able to identify the generic points of the subvarieties of dimension $k + 1$. If one had also access to the corresponding *local rings*, then by localizing further one would obtain an isomorphism of the generic points of X and Y and hence a birational map.

RESEARCH OBJECTIVE 3 – Construct a locally ringed space ${}^kX$ from the category $\mathtt{C}_k(X)$. $*$

The space ${}^kX$ should contain all the geometry of X, up to codimension $\dim X - k$. The PI expects the construction of ${}^kX$ to be similar to the Gabriel spectrum Z, discussed at the beginning of this section. Hence, given an equivalence $\mathtt{C}_k(X) \simeq \mathtt{C}_k(Y)$ we would obtain an isomorphism of locally ringed spaces ${}^kX \simeq {}^kY$ which in turn would yield the desired birational equivalence.

Non-commutative Geometry. There are many flavors of non-commutative geometry. Using Gabriel's theorem as a starting point, one may view an abstract abelian category as (the category of coherent sheaves of) a *non-commutative space*. This point of view (and its derived counterpart using triangulated categories) has received much attention from algebraic geometers. Settling Conjecture 3.3 would open the door for many new questions in non-commutative birational geometry.

Indeed, upon completing the objectives outlined above, the outcome should be an *intrinsic* characterization of the categories $\mathtt{Coh}_{\leq k}(X)$ inside $\mathtt{Coh}(X)$.

RESEARCH OBJECTIVE 4 – Characterize the subcategories $\mathtt{Coh}_{\leq k}(X) \subset \mathtt{Coh}(X)$ intrinsically with respect to the category $\mathtt{Coh}(X)$. $*$

We expect the following should be true. Let $\mathtt{A} = \mathtt{Coh}(X)$. Then $\mathtt{S}_0 = \mathtt{Coh}_{\leq 0}(X)$ is the smallest *Serre subcategory* containing all the minimal objects of $\mathtt{A}$. Consider the quotient $\mathtt{A}_1 = \mathtt{A}/\mathtt{A}_0$. By Research Objective 2, the minimal objects of $\mathtt{A}_1$ will correspond to curves in X. Hence, we expect that $\mathtt{S}_1 = \mathtt{Coh}_{\leq 1}(X)$ is the smallest Serre subcategory containing $\mathtt{S}_0$ and the (pre-images of the) minimal objects of $\mathtt{A}_1$. Once again, we may quotient to form another category $\mathtt{A}_2 = \mathtt{A}/\mathtt{S}_1$. Iterating this process, we arrive at the final quotient $\mathtt{C}_{\dim X-1}(X)$, which in turn is the same as $\mathrm{Mod}(K(X))$.

Notice that no geometry was used in the construction above, hence the process makes sense for any non-commutative space. Although this process will not always converge, we believe that for nice non-commutative algebras these quotients will be interesting to study.

For example, in [PVdB16] many non-commutative versions of birational maps of surfaces were constructed.

Research Objective 5 – Interpret the birational equivalences of [PVdB16] in the framework of quotient categories. $*$

For example, in [PVdB16] we find a notion of *non-commutative function field*. If $\mathtt{A}$ is the abelian category corresponding to one of these surfaces, we may start taking the quotients $\mathtt{A} \twoheadrightarrow \mathtt{A}_1 \twoheadrightarrow \mathtt{A}_2 \twoheadrightarrow \cdots$ from above. Assuming the process terminates (which we expect), then the last non-zero quotient will be some category $\mathtt{A}_d$.

Research Objective 6 – Show that $\mathtt{A}_d$ is Morita-equivalent to the function field defined in [PVdB16]. $*$

Finally, one can also see further directions stemming from this research project, such as the construction of higher dimensional examples of birational non-commutative spaces. It would be very interesting to construct examples of non-commutative spaces being *isomorphic in codimension one*, rather than merely birational. For example, a concrete goal would be to construct an example of a *non-commutative flop*.

4. Derived Categories

The main research objective of this section is the construction of new Torelli-type theorems for components of the derived category, especially in relation to *phantoms* and Kuznetsov's *homological projective duality*.

Background. Derived categories were born as a tool to perform homological algebra. With work of Mukai [Muk81], derived categories became objects of independent study, and were later essential in Kontsevich's *homological mirror symmetry* and Bridgeland's *stability conditions*.

If X is a variety, we will write $\mathtt{D}(X)$ for the bounded derived category of $\mathtt{Coh}(X)$. The objects of $\mathtt{D}(X)$ are chain complexes of coherent sheaves and the morphisms are obtained by formally inverting all quasi-isomorphisms. The category $\mathtt{D}(X)$ is no longer abelian, but it possesses a rich structure: it is a *triangulated category*.

If X and Y are two varieties such that $\mathtt{D}(X) \simeq \mathtt{D}(Y)$ are equivalent, we say X and Y are *derived equivalent* or, equivalently, X and Y are (Fourier-Mukai) *partners*. In light of the previous section, it is only natural to ask whether a Gabriel theorem exists for the derived category. In other words: must derived equivalent varieties be isomorphic? The answer is negative. The work [Muk81] cited above showed that any abelian variety and its dual are derived equivalent (but certainly not isomorphic in dimension ≥ 2).

Nevertheless, we have the following foundational result.

Theorem 4.1 (Bondal-Orlov [BO01]) – Let X, Y be smooth and projective varieties. Assume ω_X is ample (or anti-ample). Then $X \simeq Y$ if and only if $\mathtt{D}(X) \simeq \mathtt{D}(Y)$. $*$

In [Bal11], Ballard extended the Bondal-Orlov theorem to the case of Gorenstein varieties. In [MN10] the theorem was generalized to the derived category of twisted sheaves (similarly to Caldararu's conjecture). Finally, in [SdSSdS12] a version of the theorem working relatively over a base scheme is shown to hold.

As an extension of the program investigating point-like objects of abelian categories, the PI proved a general version of the Bondal-Orlov theorem which simultaneously unifies and generalizes the above.

THEOREM 4.2 (C. [Cal17a]) – Let S be a noetherian algebraic stack. Let X, Y be stacks separated and of finite type over S. For each $s \in S$, assume the fibres X_s, Y_s to be Gorenstein. Assume moreover X_s to have ample (or anti-ample) canonical bundle. Let α, β be two Brauer classes on X and Y. Then, $\mathtt{D}(X, \alpha) \simeq \mathtt{D}(Y, \beta)$ as S-linear categories if and only if $X \simeq Y$ as S-stacks and via an isomorphism which pulls β back to α. $*$

Decompositions. In the beginning of the subject, much research was devoted to searching for *exceptional collections*, which are the simplest kind of *decompositions* of the derived category. Kuznetsov's theory of *homological projective duality* has shed light on the remarkable features of more general decompositions. For what follows, X is a smooth and projective variety over the complex numbers.

DEFINITION – An *orthogonal decomposition* of $\mathtt{D}(X)$ consists of a pair of triangulated subcategories $\mathtt{A}, \mathtt{B} \subset \mathtt{D}(X)$ such that:

(1) $\mathrm{Hom}(A, B) = 0$, for all $A \in \mathtt{A}$, $B \in \mathtt{B}$;
(2) $\mathrm{Hom}(B, A) = 0$, for all $A \in \mathtt{A}$, $B \in \mathtt{B}$;
(3) if $E \in \mathtt{D}(X)$, there is an exact triangle $A \to E \to B \to A[1]$, with $A \in \mathtt{A}, B \in \mathtt{B}$. $*$

If $\mathtt{D}(X)$ admits no non-trivial orthogonal decompositions we say it is *indecomposable* (by non-trivial we mean that $\mathtt{A}$, $\mathtt{B}$ both contain non-zero objects). The following is a basic result.

PROPOSITION – A variety X is connected if and only if $\mathtt{D}(X)$ is indecomposable. $*$

While on X there is no middle ground between being connected or disconnected, on $\mathtt{D}(X)$ one can find a compromise: we just omit condition 1. A pair $\mathtt{A}, \mathtt{B}$ satisfying (2), (3) above is then called a *semiorthogonal decomposition* and we write $\mathtt{D}(X) = \langle \mathtt{A}, \mathtt{B} \rangle$. Of course, $\mathtt{A}$ or $\mathtt{B}$ might in turn admit a semiorthogonal decomposition of their own, so we may decompose $\mathtt{D}(X)$ further.

Probably the most famous semiorthogonal decomposition was constructed by Beilinson. For projective space we have $\mathtt{D}(\mathbf{P}^N) = \langle \mathtt{D}(\mathrm{Mod}(\mathbf{C})), \ldots, \mathtt{D}(\mathrm{Mod}(\mathbf{C})) \rangle$. To be more precise, we should view each $\mathtt{D}(\mathrm{Mod}(\mathbf{C}))$ as being the category generated by the line bundle $\mathcal{O}(k)$. One usually writes $\mathtt{D}(\mathbf{P}^N) = \langle \mathcal{O}, \ldots, \mathcal{O}(N) \rangle$.

Generalizing this, Orlov provided semiorthogonal decompositions for projective bundles and for blowups. Antithetically, by Serre duality a variety with trivial canonical bundle cannot admit any non-trivial semiorthogonal decompositions.

If X is a hypersurface in $\mathbf{P}^N$ then there is always a semiorthogonal decomposition $\mathtt{D}(X) = \langle \mathtt{A}_X, \mathcal{O}, \ldots, \mathcal{O}(N-d) \rangle$, where $d \leq N+1$ is the degree of X. We think of $\langle \mathcal{O}, \ldots, \mathcal{O}(N-d) \rangle$ as coming from the embedding $X \hookrightarrow \mathbf{P}^N$, while $\mathtt{A}_X$ is the "interesting" part of the derived category of X. A natural question arises.

Question: is $\mathtt{A}_X$ related to some other variety Y?

Wanting to be overly optimistic, we might ask for an equivalence $\mathtt{A}_X \simeq \mathtt{D}(Y)$. If such a Y exists we informally call it the *homological projective dual* (HP-dual) variety of X. Let us have a look at examples.

When $d = 1$ we have $X = \mathbf{P}^{N-1}$ and $\mathtt{A}_X = 0$, so nothing interesting occurs: the HP-dual of $\mathbf{P}^{N-1}$ is the empty variety.

When $d = 2$, X is a quadric. For simplicity, let's assume it to be smooth and even-dimensional. Kuznetsov [Kuz08] (generalizing Kapranov [Kap86]), showed that $\mathtt{A}_X = \mathtt{D}(\mathscr{C}_0\text{-mod})$ is the derived category of modules over the even part of the Clifford algebra corresponding to the quadric X. It is a standard fact that $\mathscr{C}_0$ is isomorphic as an algebra to $\mathbf{C} \times \mathbf{C}$ (here we are using smoothness and even-dimensionality of X). Thus we see that the

HP-dual of a quadric consists of two points. The modules $\mathbf{C} \times \{0\}$ and $\{0\} \times \mathbf{C}$ correspond to the so-called *spinor bundles* S_+, S_- on X.

When $d \geq 3$ there is no uniform geometric answer. Although there exists an interpretation of $\mathtt{A}_X$ in terms of A_∞-algebras [BDF$^+$], one would like to find a geometric description for $\mathtt{A}_X$. When $d = 3$ and $N = 5$, i.e. X is a cubic fourfold, one can show that $\mathtt{A}_X$ bears an uncanny resemblance with the derived category of a K3 surface, in the sense that it possesses the correct categorical properties (such as Serre duality).

As a counterpart to the Hodge theoretic work of Hassett [Has00], Kuznetsov [Kuz04] conjectured that the cubic X is *rational* (i.e. birational to $\mathbf{P}^4$) if and only if $\mathtt{A}_X = \mathtt{D}(K3)$, for some K3 surface.

There are indeed examples of cubic fourfolds for which an interesting K3 surface can be constructed geometrically. These are explored in [Kuz04], which is the starting point of the PI's work [CT16], discussed below.

Cubic Fourfolds. Let $X \subset \mathbf{P}^5$ be a degree three hypersurface. From the general theory we obtain a semiorthogonal decomposition

$$\mathtt{D}(X) = \langle \mathtt{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$$

and in our notation $\mathtt{D}_{\text{int}}(X) = \mathtt{A}_X$. One can compute the Serre functor of $\mathtt{A}_X$ and it turns to be [2], the shift by two. Put differently, $\mathtt{A}_X$ is a CY_2 (or $K3$) category. Kuznetsov proved that in fact $\mathtt{A}_X = \mathtt{D}(K3)$ for an actual $K3$ surface in the following cases [Kuz04].[4]

- X contains a plane P (here one allows $K3$ surfaces twisted by a Brauer class)
- X is Pfaffian
- X has a single node

In all three cases there is a geometric construction producing a K3 surface. We will focus on the first, which ties in with our discussion on quadrics.

Let $\mathbf{P}^5 = \mathbf{P}(V \oplus W)$, where V, W are two copies of $\mathbf{C}^3$. Assume $\mathbf{P}(V) \subset X \subset \mathbf{P}^5$ is a general cubic fourfold containing a plane. Projecting away onto $\mathbf{P}(W)$ we have the map $Z = \mathrm{Bl}_{\mathbf{P}(V)} X \to \mathbf{P}(W)$. A quick computation shows that the fibres of $Z \to \mathbf{P}(W)$ are quadric surfaces. Just as a quadratic form of a vector space gives rise to a Clifford algebra, a family of quadric surfaces gives rise to a *sheaf* of Clifford algebras. A relative version of HPD for quadrics [Kuz08, ABB14] may be performed, and one obtains a semiorthogonal decomposition of Z. The first component is $\mathtt{D}(\mathbf{P}(W), \mathscr{C}_0)$, where $\mathscr{C}_0$ is once again the (even part of) the corresponding sheaf of Clifford algebras. By taking the centre of this algebra, we obtain a double cover $K3 \to \mathbf{P}(W)$, branched over the locus of singular quadrics, together with a Brauer class $\alpha \in H^2(K3, \mathcal{O}^\times)$. Roughly speaking, α is encoding the monodromy of maximally isotropic subspaces. One may also decompose the derived category Z using Orlov's theorem. Using a series of mutations, Kuznetsov showed that $\mathtt{A}_X$ is indeed equivalent to $\mathtt{D}(K3, \alpha)$. The latter category of the derived category of *α-twisted sheaves* on the derived category of the given K3 surface. We mentioned twisted sheaves above, in connection to Căldăraru's conjecture.

Inspired by HPD, one might wonder what happens when intersecting two cubic fourfolds X_0, X_1. Attached to X_0, X_1 is the *pencil* of cubic fourfolds spanned by the corresponding cubic forms. Let $X = X_0 \cap X_1$ be the baselocus of the pencil. This is a Calabi-Yau complete intersection of dimension three. Let $\mathcal{H} \subset \mathbf{P}^5 \times \mathbf{P}^1$ be the total family of such a pencil. One has a decomposition $\mathtt{D}(\mathcal{H}) = \langle \mathtt{A}_{\mathcal{H}}, \mathcal{O}_{\mathcal{H}}(3,0), \mathcal{O}_{\mathcal{H}}(3,1), \mathcal{O}_{\mathcal{H}}(4,0), \mathcal{O}_{\mathcal{H}}(4,1), \mathcal{O}_{\mathcal{H}}(5,0), \mathcal{O}_{\mathcal{H}}(5,1) \rangle$.

[4] There is also more recent work of Kuznetsov [Kuz17] devoted to studying the case of del Pezzo fibrations, linked to the new examples of *rational* cubic fourfolds of [AHTV16].

In our notation we would write $\mathtt{A}_{\mathcal{H}} = \mathtt{D}_{\mathrm{int}}(\mathcal{H})$. From the general theory of HPD we have that $\mathtt{A}_{\mathcal{H}} = \mathtt{D}(X)$. Recall that Calabi-Yau varieties do not admit semiorthogonal decompositions and thus $D_{\mathrm{int}}(X) = \mathtt{D}(X)$. In [CT16] we give a proof of the equivalence without appealing to the general theory and using only Orlov's theorem and mutations.

To produce a geometric incarnation for $\mathtt{A}_{\mathcal{H}}$, one should consider special pencils. Consider two cubics $X_0, X_1 \supset \mathbf{P}(V)$ containing the same plane and the pencil they generate. Let $X_0 \cap X_1 = X \supset \mathbf{P}(V)$ be the base locus and let $\mathcal{H} \subset \mathbf{P}^5 \times \mathbf{P}^1$ be the total family.

HPD suggests that Kuznetsov's strategy over $\mathbf{P}^1$ should produce an equivalence $\mathtt{D}(X) = \mathtt{D}(Y, \alpha)$, where Y is the Calabi-Yau threefold doubly covering $\mathbf{P}(W) \times \mathbf{P}^1$. Unfortunately, one cannot apply the theory directly as X is not smooth. Any pencil of cubic fourfolds containing a plane (even if the said plane is allowed to vary) will have singular base locus.

To overcome this, in [CT16] we take a small resolution $\widetilde{X}$ of X, by blowing up the plane $\mathbf{P}(V)$ inside it. This resolution is the base locus of a pencil on $\mathrm{Bl}_{\mathbf{P}(V)} \mathbf{P}^5$ and recall that by projecting away we map $\mathrm{Bl}_{\mathbf{P}(V)} \mathbf{P}^5 \to \mathbf{P}(W)$. Applying relative HPD (with base $\mathbf{P}(W)$) we deduce that $\mathtt{D}(\widetilde{X}) = \mathtt{D}(\mathbf{P}(W) \times \mathbf{P}^1, \mathscr{C}_0)$, where once more $\mathscr{C}_0$ is the even part of the relevant sheaf of Clifford algebras. To make the latter category more geometric, this time it does not suffice to take the double cover of $Y_0 \to \mathbf{P}(W) \times \mathbf{P}^1$. One has to furthermore take a small resolution $Y \to Y_0$, which inherits a Brauer class α from $\mathscr{C}_0$. The upshot is the following.

Theorem 4.3 (C.-Thomas [CT16]) – $\mathtt{D}(\widetilde{X}) = \mathtt{D}(Y, \alpha)$. $*$

In [CT16] it is also proved that Y *cannot* be Kähler. This example was later shown to be part of a more general construction in [BL16]. In [CT16] pencils of cubic fourfolds with a single node were also studied. The strategy is once more to blow up the common intersection (in this case the point where the cubics have a node) and then to work with the obtained resolution. Two Calabi-Yau 'duals' varieties $\tilde{X}, Y$ are thus produced.

Theorem 4.4 (C.-Thomas [CT16]) – We have an equivalence $\mathtt{D}(\widetilde{X}) \simeq \mathtt{D}(Y)$. Moreover, $\tilde{X}, Y$ are birational. However, the equivalence is not induced by a birational map. $*$

The fact that the equivalence is not induced by a rational map may be checked, for example, by showing that the class of a skyscraper is sent to a complex of high rank.

Research Objective 7 – Understand explicitly the equivalence $\mathtt{D}(\tilde{X}) \simeq \mathtt{D}(Y)$. $*$

Groups of auto-equivalences of derived categories of Calabi-Yau threefolds are highly mysterious. For elliptic curves there is a complete classification. For K3 surfaces, there is a conjectural description due to Bridgeland, in terms of spherical twists and stability conditions. This was settled only in the Picard rank 1 case by Bayer-Bridgeland. In dimension three, very little is known. We expect the equivalence above to exotic, i.e. not in the subgroup generated by standard equivalences and spherical twists.

Research Objective 8 – Produce a new equivalence of Calabi-Yau threefolds, using pencils of Pfaffian cubic fourfolds. $*$

The Pfaffian case is the one missing from [CT16]. A priori it would seem that the techniques of [CT16] do not apply to this context. However, [Bea00] Beauville showed that a cubic fourfold is Pfaffian if and only if it containins a degree five del Pezzo surface. Thus we will be able to proceed by performing a (tweaked) family version of [Kuz04] for the Pfaffian case. In other words, if $X_0, X_1 \supset S$ are two cubic fourfolds containing a common del Pezzo surface, we aim to produce a dual variety Y by passing to $\mathrm{Bl}_S \mathbf{P}^5$ and mimicking the constructions of [CT16, Kuz04].

We believe the Objectives 8 to be ideal for a PhD student: the goal is concrete, a path to a result is clear to the PI, and the background needed will provide indispensable knowledge to tackle future problems in derived categories.

Torelli-type theorems. Let us go back to semi-orthogonal decompositions. If X a cubic *threefold*, we have a decomposition

$$\mathtt{D}(X) = \langle \mathtt{A}_X, \mathcal{O}, \mathcal{O}(1) \rangle$$

In this case, one easily shows that $\mathtt{A}_X$ can never be the derived category of a variety. On the other hand, [BMMS12] shows that $\mathtt{A}_X$ is a complete invariant of the cubic threefold. More precisely, if X, Y are cubic threefolds, then $\mathtt{A}_X \simeq \mathtt{A}_Y$ if and only if $X \simeq Y$. Several other examples of *derived Torelli theorems* can be found in the recent work [HR16].

Main Objective 2 – Prove new derived Torelli theorems for different families of varieties (for example hypersurfaces or varieties with phantom categories). ∗

Let us present a concrete method to produce such results. Our strategy will be broken down into the Research Objectives below.

A conjecture of Kawamata stated that a (smooth, complex, projective) variety has at most finitely many Fourier-Mukai partners. In [AT09], it was shown that a variety admits at most *countably* many partners. In [Les15], an example of a threefold with countably many distinct partners was unearthed, hence disproving Kawamata's original conjecture.

The approach of [AT09] can be summarized as follows. There is a stack $\mathcal{V}$ parameterizing *all* smooth, proper, connected varieties over $\mathbf{C}$. There is also a (pre-)stack $\mathcal{D}$, parameterizing all smooth, proper, connected $\mathbf{C}$-linear dg-categories. There is a map, which Anel-Toën call the *period map*, $\mathcal{V} \to \mathcal{D}$ sending X to $\mathtt{D}(X)$. They then study the derivative of the period map, which turns out to be injective.

We want to consider variants of this period map. For example, if we were interested in cubic threefolds, we would replace $\mathcal{V}$ with the moduli space $\mathcal{M}$ of cubic threefolds. A new period map $\mathcal{M} \to \mathcal{D}$ would then be defined by sending a cubic threefold X to the category $\mathtt{A}_X$. This procedure works as long as the given semi-orthogonal decomposition behaves well in families.

Research Objective 9 – Develop a theory of moduli spaces of varieties with decompositions well-behaved in families. ∗

We expect this task to be straightforward, thanks to the technical results of Kuznetsov [Kuz11], which ensure robust compatibilities between decompositions and base change.

Obviously, there will be trivial cases of this notion. For example, we could consider the period map which sends X to the zero category. This will obviously never satisfy any Torelli property. Thus, we need a condition ensuring the derivative of the period map to be injective.

Research Objective 10 – Find sufficient (and possibly necessary) conditions for the period map to be unramified. ∗

Although there isn't a purely cohomological description of deformations of derived categories (unlike deformations of varieties), we expect Hochschild cohomology to play a key role here. Indeed, there is a map from first-order deformations of $\mathtt{D}(X)$ to $HH^2(X)$, the second Hochschild cohomology group. Given appropriate non-vanishing conditions, unramifiedness of the period map will follow.

Research Objective 11 – Find families of varieties satisfying the prescribed vanishing on Hochschild cohomology. *

Thankfully, there are many computations already carried out by Kuznetsov [Kuz15]. Strikingly, even in the case of *phantoms* (i.e. components undetectable by invariants such as K-theory) the correct vanishing seems to be satisfied.

5. Donaldson-Thomas invariants

The Main Objective in this section is the proof of the *crepant resolution conjecture* for Donaldson-Thomas (DT) invariants. DT theory originated with Thomas's thesis, as a holomorphic analogue of the Casson invariant for real 3-manifolds. A theory to *virtually* enumerate stable coherent sheaves on Calabi-Yau threefolds was developed. Nowadays the theory has seen many generalizations, variants, and categorifications, most notably work of Kontsevich-Soibelman and Joyce-Song.

The invariants. The flavor of DT theory we are interested in is that of *curve counting*. Instead of following the original approach (using virtual cycles), we will define the invariants by taking a shortcut, provided to us by the groundbreaking work of Behrend [Beh09].

Let X be a variety, which we will always take to be smooth and projective over the complex numbers. Since we are interested in counting curves in X, it is only natural to look at the *Hilbert scheme* $\mathrm{Hilb}_{\leq 1}(X)$, which parameterizes all subschemes $Z \hookrightarrow X$ with $\dim Z \leq 1$. This moduli space splits as a disjoint union according to topological type

$$\mathrm{Hilb}_{\leq 1}(X) = \coprod_{\beta,n} \mathrm{Hilb}_X(\beta,n)$$

where $\mathrm{Hilb}_X(\beta,n)$ is parameterizing subschemes $Z \hookrightarrow X$ with $\dim Z \leq 1$, $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$. Now, $\mathrm{Hilb}_X(\beta,n)$ is a projective scheme over $\mathbf{C}$, hence compact (in the analytic topology). In an ideal situation, $\mathrm{Hilb}_X(\beta,n)$ would moreover be zero-dimensional, so that we could define the curve-count to be $N_X(\beta,n) := |\,\mathrm{Hilb}_X(\beta,n)|$.

Since $\mathrm{Hilb}_X(\beta,n)$ is typically *not* zero-dimensional, we instead define numbers

$$\bar{DT}_X(\beta,n) := \chi_{\mathrm{top}}\left(\mathrm{Hilb}_X(\beta,n)\right)$$

which we call the *naive* DT number of X of class (β,n). In a nutshell, we are viewing the topological Euler characteristic χ_{top} as a higher-dimensional analogue of cardinality. But let us roughly explain how to obtain the genuine DT invariants. Behrend showed that every scheme M (of finite type over $\mathbf{C}$) comes equipped with a constructible function $\nu_M\colon M(\mathbf{C}) \to \mathbf{Z}$. Given such a function, we may define a variant of the Euler characteristic $\chi_\nu(M) := \sum_k \chi_{\mathrm{top}}(\nu_M^{-1}(k))k$. We mention that when M is smooth of dimension d, then $\nu_M \equiv (-1)^d$, so that $\chi_\nu(M) = (-1)^d\chi_{\mathrm{top}}(M)$. Additionally, χ_ν is sensitive to thickenings. For example, if $M = \mathrm{Spec}(\mathbf{C}[\epsilon]/(\epsilon^2))$, we have $\chi_{\mathrm{top}}(M) = 1$ while $\chi_\nu(M) = 2$. Loosely speaking, χ_ν is remembering that M is obtained by letting two points collide together. Thus we define $DT_X(\beta,n) := \chi_\nu(\mathrm{Hilb}_X(\beta,n))$. When X is a *Calabi-Yau threefold* (CY3), which we take to mean X is smooth and projective of dimension 3, has trivial canonical bundle and $H^1(X,\mathcal{O}_X) = 0$, Behrend showed that these numbers coincide with the original invariants.

Structure. Before we move on, let us take a step back and see what we can say about "point-counting". Let us write $\mathrm{Hilb}_n X = \mathrm{Hilb}_X(0,n)$ for the Hilbert scheme of points in X.

We have the remarkable formula [Gö90, Che96].

$$\sum_{n\geq 0} \chi_{\mathrm{top}}(\mathrm{Hilb}_n(X))q^n = \begin{cases} (1-q)^{-\chi_{\mathrm{top}}(X)} & \text{if } \dim X = 1 \\ \prod_{m\geq 1}(1-q^m)^{-\chi_{\mathrm{top}}(X)} & \text{if } \dim X = 2 \\ \prod_{m\geq 1}(1-q^m)^{-m\chi_{\mathrm{top}}(X)} & \text{if } \dim X = 3 \end{cases} \tag{5.1}$$

The next natural step is to consider curves. Inevitably, the problem becomes much more intricate. Nevertheless, we have the following striking result.

Theorem 5.2 (Bridgeland [Bri11], Toda [Tod10]) – Let X be CY3. Fix an integral curve class β. Then the quotient series

$$\frac{\sum_n \chi_{\mathrm{top}}\left(\mathrm{Hilb}_X(\beta,n)\right)q^n}{\sum_n \chi_{\mathrm{top}}\left(\mathrm{Hilb}_n X\right)q^n}$$

is the Laurent expansion of a rational function of q, invariant under $q \mapsto q^{-1}$. ∗

Recall that we defined naive DT numbers precisely as Euler characteristics, so the result above may expressed in terms of those. For genuine DT invariants, define $DT_\beta(X) := \sum_n DT_X(\beta,n)q^n$ and $DT_0(X) = \sum_k DT_X(0,n)q^n$. We have an analogous result.

Theorem 5.3 (Bridgeland [Bri11], Toda [Tod10]) – Suppose again X is CY3 and fix a class β. The quotient series $\frac{DT_\beta(X)}{DT_0(X)}$ is the Laurent expansion of a rational function, invariant under $q \mapsto q^{-1}$. ∗

It is no accident that the two above theorems have identical statements, as they are proved using the same strategy, which involves categories in an essential way.

> From now on, we will work with DT numbers, although the results which follow hold just the same for the plain Euler numbers $\bar{DT}$.

Categories, Hall algebras, numbers. Let us now give an idea of how categories enter the picture in the proof of Theorem 5.2. We do so, as these same methods will be fundamental to tackle the Research Objectives below.

The key tool is the *motivic Hall algebra* $H(\mathtt{Coh}(X))$, of the category of coherent sheaves. Let us describe the salient features of this algebra. Any moduli space (or stack) of sheaves on X is represented by a corresponding element in $H(\mathtt{Coh}(X))$. For example, we write $1_{\mathtt{Coh}(X)} \in H(\mathtt{Coh}(X))$ for the element corresponding to the stack of *all* coherent sheaves on X. We have an element $\mathscr{H}$, corresponding to the Hilbert scheme. Finally, we have an element $1^{\mathcal{O}}_{\mathtt{Coh}(X)}$, corresponding to the moduli stack parameterizing pairs (F,s), where $F \in \mathtt{Coh}(X)$ and $s\colon \mathcal{O}_X \to F$, i.e. sheaves equipped with a global section.

The point of the Hall algebra is to encode *categorical relations* into *algebraic identities*. For example, $\mathtt{Coh}(X)$ has a first isomorphism theorem: any map $\mathcal{O}_X \to F$ may be factored as a surjection $\mathcal{O}_X \twoheadrightarrow G$ (by taking G to be the image) followed by an injection $G \hookrightarrow F$. But recall that the Hilbert scheme $\mathrm{Hilb}(X)$ may be viewed both as the moduli space of closed subschemes of X or as the moduli space of surjections $\mathcal{O}_X \twoheadrightarrow G$: the point being, any such G must be of the form $\mathcal{O}_Z$ for some closed subscheme $Z \hookrightarrow X$. The considerations above are encoded by the elegant identity: $1^{\mathcal{O}}_{\mathtt{Coh}(X)} = \mathscr{H} * 1_{\mathtt{Coh}(X)}$.

In turn, the Hall algebra (or, more precisely, a suitably defined subalgebra) comes equipped with what is called an *integration map*. For simplicity, we restrict to $\mathtt{Coh}_{\leq 1}(X)$, the category of sheaves supported in dimension at most one. With this simplification, the integration map takes values in a ring generated by symbols q^n, for $n \in \mathbf{Z}$, and t^β, for β an effective curve-class.

The crucial point is that applying the integration map to $\mathscr{H}$, yields the generating series for DT invariants:[5] $\mathscr{H} \mapsto DT(X) := \sum_{\beta,n} DT_X(\beta,n)t^\beta q^n$.

To summarize, categorical relations give rise to identities in the Hall algebra, which may be used to prove equalities between the generating series.

Categories $\Rightarrow$ *Identities in Hall algebra* $\Rightarrow$ *Equality of numbers*

For example, the invariance $q \mapsto q^{-1}$ appearing in Theorems 5.2, 5.3 is ultimately a consequence of the symmetry of the Hall algebra induced by the symmetry $F \mapsto \underline{\mathrm{Hom}}(F, \mathcal{O}_X)$.

Birational Modifications. A foundational question one asks is how DT invariants of X change under modifications of X. More concretely, we turn to birational geometry.

> If X and Y are two birational CY3 varieties, is there a relation between their DT invariants?

By [Kol89] we know that, if $X \sim Y$ are birational and CY3, there is a finite chain of *flops* connecting the two. The rough of idea is that we start with a map $X \to W$, contracting a rational curve $C \subset X$. Here, X is a smooth and projective CY3 and W has an isolated Gorenstein singularity (the image of C). The flop Y is obtained by performing surgery on C: this rational curve is replaced by another $C' \subset Y$, which now intersects divisors with opposite sign.

$$\begin{tikzcd} X \arrow[dr, "f"] & & Y \arrow[dl, "g"'] \\ & W & \end{tikzcd}$$

Theorem 5.4 (Toda [Tod13], C. [Cal16a]) – After identifying curve classes on X and Y via the flop, we have $DT^\vee_{\mathrm{exc}}(X)DT(X) = DT^\vee_{\mathrm{exc}}(Y)DT(Y)$. $*$

In the theorem we have the notation

$$DT(X) := \sum_{\beta,n} DT_X(\beta,n)t^\beta q^n, \qquad DT^\vee_{\mathrm{exc}}(X) := \sum_{\substack{\beta,n \\ f_*\beta=0}} DT_X(\beta,n)t^{-\beta}q^n$$

where $f_*\beta = 0$ means the curve corresponding to β is contracted to a point. Since birational CY3s are connected by flops, the theorem completely answers the question posed above. See also [Jia15] for a generalization to stacks.

Central to the proof of the theorem, is the derived equivalence $\mathtt{D}(Y) \simeq \mathtt{D}(X)$ and the equivalence between the abelian categories $\mathtt{Per}(Y/W) \simeq \mathtt{Per}(X/W)$, both due to Bridgeland. These equivalences provide a bridge between X and Y and the motivic Hall algebra (this time of the category $\mathtt{Per}$!) is once again the key.

The formula, but also the methods of [Cal16a], was put to remarkable use by Maulik in [Mau16], settling conjectures of Oblomkov-Shende and Diaconescu-Hua-Soibelman on the HOMFLY polynomial of an algebraic knot.

Crepant resolutions. In 2010, Bryan-Cadman-Young [BCY12] conjectured a comparison formula, in the context of the McKay correspondence: the *Crepant Resolution Conjecture* (CRC) for DT invariants. This is the DT version of Ruan's conjecture for Gromov-Witten invariants [LR01, Rua06]. The CRC predicts the following. Let $\mathscr{X}$ be a smooth, Deligne-Mumford stack of dimension three, with trivial canonical bundle and which is generically a scheme. We require its coarse space X to be projective and we let $Y \to X$ be the crepant

[5] This is slightly imprecise, as we should be looking at $\mathscr{H}_{\leq 1}$, the element corresponding to $\mathrm{Hilb}_{\leq 1}(X)$, rather than the whole Hilbert scheme.

resolution defined as $\mathrm{Hilb}^1(\mathscr{X})$. Assume moreover that $\mathscr{X}$ is a *hard Lefschetz* orbifold, which means that the morphism $Y \to X$ has fibres of dimension at most one.

$$\begin{tikzcd} X \arrow[dr] & & \mathscr{X} \arrow[dl] \\ & Y & \end{tikzcd}$$

In [BKR01], and more generally [CT08], it was shown that there is an equivalence $\mathtt{D}(\mathscr{X}) \simeq \mathtt{D}(Y)$. In this context, the following two identities are conjectured to hold.

$$\frac{DT_{\mathrm{mr}}(\mathscr{X})}{DT_0(\mathscr{X})} = \frac{DT(Y)}{DT_{\mathrm{exc}}(Y)} \tag{5.5}$$

$$DT_0(\mathscr{X}) = \frac{DT_{\mathrm{exc}}(Y)DT^{\vee}_{\mathrm{exc}}(Y)}{DT_0(Y)} \tag{5.6}$$

where the subscript $_{\mathrm{mr}}$ stands for *multiregular*. A substack $Z \hookrightarrow \mathscr{X}$ is multiregular if the sheaf $\mathcal{O}_Z$ is sent to a an object $E \in \mathtt{D}(Y)$, whose support is one-dimensional.

Approaching the CRC. In [Cal16b], the PI showed that the McKay equivalence $\mathtt{D}(\mathscr{X}) \simeq \mathtt{D}(Y)$ restricts to an equivalence $\mathtt{Coh}(\mathscr{X}) \simeq \mathtt{Per}(Y/X)$. The knowledge accumulated in [Cal16a] on the motivic Hall algebra of $\mathtt{Per}(Y/X)$ was crucial in proving the following.

Theorem 5.7 (C. [Cal16b]) – The identity (5.6) of the conjecture holds. $*$

Also in [Cal16b], progress towards (5.5) was made. Later, Ross [Ros17] settled the CRC in the case of toric A-singularities. However, the general case still eludes us.

Proving the CRC. The PI proposes a new strategy to prove the CRC.

Main Objective 3 – Settle the Crepant Resolution Conjecture in the affirmative. $*$

We explain how the approach of the PI is new, and different from [Cal16b, Ros17]. A key observation (missing from previous work) is that the identity (5.5) is not a mere identification of *generating series* but rather of *rational functions*. Implicit in (5.5) is a hidden (and mysterious) change of variables.

However, the true breakthrough comes from the fact that the PI, Beentjes and Rennemo have now a *complete proof* of (5.5) in the explicit case of $\mathscr{X} = [T/(\mathbf{Z}/2\mathbf{Z})]$, where T is the total space of $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$ and $\mathbf{Z}/2\mathbf{Z}$ is acting fiberwise by dilation.

The PI, Beentjes and Rennemo have employed *wall-crossing* methods (as opposed to the combinatorial ones of [Ros17]). Two families of stability conditions, $\sigma_\delta, \sigma'_{\alpha,\omega}$, are defined on $\mathscr{X}$, depending on three real parameters δ, α, ω. These stability conditions interpolate between the *DT* invariants of $\mathscr{X}$ and the *Bryan-Steinberg* invariants [BS16] on Y. The latter invariants represent a different way of expressing the RHS of [BS16]. By a careful use of the wall-crossing formalism, the required change of variables appears and (5.5) is proven.

The current task consists in generalizing the proof above to all hard Lefschetz orbifolds $\mathscr{X}$. For σ_δ the PI sees no obstacle, as the definition will be a variant of slope-stability. However, the definition of $\sigma'_{\alpha,\omega}$ depends instead on the knowledge of the characters of the stabilizer group $\mathbf{Z}/2\mathbf{Z}$. The PI expects the general version of σ' to depend on multiple parameters, one for each extra character of the stabilizer groups of the orbifold $\mathscr{X}$. Thankfully, for a general $\mathscr{X}$, the stabilizer groups of the stacky locus have been classified by Bryan. The wall-crossing occurs as we cross sheaves corresponding to each character passed the regular representation.

This strategy represents the first in the literature where wall-crossing has been applied in a context with *non-trivial* Euler pairing.

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