MATH355 2017-10-23

LAST WEEK/THIS WEEK

Before we begin: I highly recommend you read the section Three.Topic.Geometry of Linear Maps from Hefferon.

Fix two vector spaces V, W. We saw a while ago that a linear map $f\colon V \to W$ is uniquely determined by its value on a basis $f(\vec{b}_1), \cdots, f(\vec{b}_n)$. Conversely, given a choice of vectors $\vec{w}_1, \ldots, \vec{w}_m \in W$ there exists a unique linear map $f\colon V \to W$ such that $f(\vec{b}_i) = \vec{w}_i$.

OK, this was great. But what was special about this? Well, the key point is that we fixed a basis for V.

Question: what happens if we also fix a basis on W?

Answer: *matrices*.

1. Matrices

Let us fix for now the following data: V, W vector spaces, $\vec{b}_1, \ldots, \vec{b}_n$ an (ordered) basis for V, $\vec{d}_1, \ldots, \vec{d}_m$ an (ordered) basis for W. We call $\mathbb{B}$ the basis $(\vec{b}_1, \ldots, \vec{b}_n)$ and $\mathbb{D}$ the basis $(\vec{d}_1, \ldots, \vec{d}_m)$.

Theorem 1. Any linear map $f\colon V \to W$ may be represented as a matrix A with respect to the bases $\mathbb{B}, \mathbb{D}$. We write

$$A = \operatorname{Rep}_{\mathbb{B},\mathbb{D}} f.$$

Conversely, given an $m \times n$ matrix A, there *exists a unique* linear map $f\colon V \to W$ such that $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = A$.

In other words the theorem is saying that the *choice* of the bases $\mathbb{B}, \mathbb{D}$ gives a bijection between the set of linear maps $\operatorname{Hom}(V, W)$ and the set of matices $M_{m\times n}$.[1]

Let's recall how this works. Let $f\colon V \to W$ be linear. The vector $f(\vec{b}_j) \in W$ is a linear combination of the $\vec{d}_i$

$$f(\vec{b}_j) = \sum \alpha_{ij}\vec{d}_i$$

We record this data in the $m \times n$ matrix

$$A := \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

Why is this helpful? Because now we have an algorithm to compute what a matrix does to a vector. Indeed, say $\vec{v} \in V$. Then $\vec{v}$ can be expressed in terms of $\mathbb{B}$: $\vec{v} = \sum_j x_j \vec{b}_j$. Then $f(\vec{v}) =$

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[1]Different bases will give rise to different bijections.

$\sum_j x_j f(\vec{b}_j)$ by linearity. But we know what $f(\vec{b}_j)$ is. Thus,

$$f(\vec{v}) = \sum_{ij} \alpha_{ij} x_j \vec{d}_i. \tag{1}$$

We can go further and make this even nicer. Since we are using bases everywhere, we should really be thinking of the coordinates of $\vec{v}$ with respect to $\mathbb{B}$.

$$\operatorname{Rep}_{\mathbb{B}} \vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{2}$$

So the question we ask is: what are the coordinates of the vector $f(\vec{v})$ with respect to the basis $\mathbb{D}$? We already know the answer, it's given by (1)!

$$\operatorname{Rep}_{\mathbb{D}} f(\vec{v}) = \begin{pmatrix} \sum_j \alpha_{1j} x_j \\ \sum_j \alpha_{2j} x_j \\ \vdots \\ \sum_j \alpha_{mj} x_j \end{pmatrix}$$

Notice how $\operatorname{Rep}_{\mathbb{B}} \vec{v}$ has n components, while $\operatorname{Rep}_{\mathbb{D}} f(\vec{v})$ has m components. To say it in a different way: $\operatorname{Rep}_{\mathbb{D}} f(\vec{v})$ is obtained by applying the matrix $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f$ to the column vector $\operatorname{Rep}_{\mathbb{B}} v$.

$$\operatorname{Rep}_{\mathbb{D}} f(\vec{v}) = \left(\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f\right) \operatorname{Rep}_{\mathbb{B}} \vec{v} \tag{3}$$

So, if $A = \operatorname{Rep}_{\mathbb{B},\mathbb{D}}$ and $\operatorname{Rep}_{\mathbb{B}} \vec{v} = (x_1, \ldots, x_n)$ we write[2]

$$\operatorname{Rep}_{\mathbb{D}} f(\vec{v}) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Let's see an example. Let $V = \mathbf{R}^3$ and $W = \mathbf{R}^3$ and let $\mathbb{B} = \mathbb{D}$ be the standard basis.[3] Since we fixed a basis for V and a basis for W, matrices now give rise to linear maps. For example, consider

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

What is the corresponding linear map $f_A \colon \mathbf{R}^3 \to \mathbf{R}^3$? Well, we know that

$$f_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0x + 0y - z \\ 0x + y + 0z \\ x + 0y + 0z \end{pmatrix} = \begin{pmatrix} -z \\ y \\ x \end{pmatrix}.$$

Notice that the y-axis is fixed under f_A: i.e. $f_A \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$. You should convince yourselves that f_A is a rotation of ninety degrees about the y-axis (clockwise or counterclockwise, depending on how you look at it).

[2] Consult section Three.IV.3 for how this works mechanically.

[3] It is standard to use the standard basis for $\mathbf{R}^n$. *However*, this may not always be the case.

2. Rank

Fix now a linear map $f\colon V \to W$. Writing f as a matrix depends on the choice of bases for V and W. Let's see why. Take for example $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ which sends $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}-1\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}0\\3\end{pmatrix}$. In the standard bases, this is represented by

$$\operatorname{Rep} f = \begin{pmatrix}-1 & 0\\ -1 & 3\end{pmatrix}$$

Take now $\mathbb{B} = (\vec{e}_1, \vec{e}_2)$ and $\mathbb{D} = (\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix})$. Then

$$\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

So, if f is a linear map and $A = (\alpha_{ij})_{ij}$ is a matrix representing it,

So the actual numbers α_{ij} are not *intrinsic* to f.

> Question: what can we read off a matrix for f, independently of the choice of bases?

Answer: the *rank*.

Theorem 2. Let $f\colon V \to W$ be linear. Let $\mathbb{B}, \mathbb{B}', \mathbb{D}, \mathbb{D}'$ be bases. Then

$$\operatorname{rk}\left(\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f\right) = \dim \operatorname{Im} f = \operatorname{rk}\left(\operatorname{Rep}_{\mathbb{B}',\mathbb{D}'} f\right)$$

Corollary 3. If $A \in M_{m\times n}$, the *nullity* of A is $n - \operatorname{rk} A$. The nullity is also independent of the choice of bases.

OK, suppose $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} = A$. Recall that $\operatorname{rk} A$ was defined as the dimension of the column space of A. But what are the columns of A? The j-th column is nothing but $\operatorname{Rep}_{\mathbb{D}} f(\vec{b}_j)$ where $\vec{b}_j$ is the j-th basis vector in $\mathbb{B}$. Therefore, $\operatorname{rk} A = \dim \operatorname{Im} f$, which does not depend on the bases $\mathbb{B}, \mathbb{D}$.

Similarly, the nullity of A is $n - \operatorname{rk} A = \dim \ker f$, by the sacred formula: $\dim V = \dim f - \dim \ker f$. Boom goes the dynamite.

3. Composition

Here's another natural question. Let $f\colon V \to W$ and let $g\colon W \to Z$ be two linear maps. We know the composition $g\circ f\colon V \to Z$ is also linear. Pick bases $\mathbb{B}$ for V, $\mathbb{D}$ for W and $\mathbb{E}$ for Z. What's the relationship between $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f$, $\operatorname{Rep}_{\mathbb{D},\mathbb{E}} g$ and $\operatorname{Rep}_{\mathbb{B},\mathbb{E}} g \circ f$?

> Answer: composition is given by *matrix multiplication*.

Let us spell things out. Let $\mathbb{B} = (\vec{b}_1, \dots, \vec{b}_n)$, $\mathbb{D} = (\vec{d}_1, \dots, \vec{d}_m)$, $\mathbb{E} = (\vec{e}_1, \dots, \vec{e}_l)$. We write $f(\vec{b}_j) = \sum_i \alpha_{ij}\vec{d}_i$, also $g(\vec{d}_i) = \sum_k \beta_{ki}\vec{e}_k$ and finally $g \circ f(\vec{b}_j) = \sum_k \gamma_{kj}\vec{e}_k$. We have $A = (\alpha_{ij}) = \operatorname{Rep}_{\mathbb{B},\mathbb{D}} f$, $B = (\beta_{ki}) = \operatorname{Rep}_{\mathbb{D},\mathbb{E}} g$, $C = (\gamma_{kj}) = \operatorname{Rep}_{\mathbb{B},\mathbb{E}} g \circ f$.

On the other hand,

$$g \circ f(\vec{b}_j) = g(\sum_i \alpha_{ij}\vec{d}_i) = \sum_i \alpha_{ij}\, g(\vec{d}_i) = \sum_i \alpha_{ij} \sum_k \beta_{ki}\vec{e}_k = \sum_k (\alpha_{ij}\beta_{ki})\,\vec{e}_k$$

Hence,

$$\gamma_{kj} = \sum_i \alpha_{ij}\beta_{ki} \tag{4}$$

And this is precisely how we define matrix multiplication. If $A = \mathrm{Rep}_{\mathbb{B},\mathbb{D}} f$, $B = \mathrm{Rep}_{\mathbb{D},\mathbb{E}} g$, $C = \mathrm{Rep}_{\mathbb{B},\mathbb{E}} g \circ f$ then

$$C = \mathrm{Rep}_{\mathbb{B},\mathbb{E}} g \circ f = BA = \mathrm{Rep}_{\mathbb{D},\mathbb{E}} g \, \mathrm{Rep}_{\mathbb{B},\mathbb{D}} f.$$

In other words, the kj entry of BA is obtained by running through the j-th row of A and the k-th column of B.

However, this discussion with indices is incredibly unhelpful. The best way to understand this is to work through a bunch of examples. See Chapter Three.IV.3.

4. The vector space of matrices

Fix once again vector spaces V, W. Recall that by $\mathrm{Hom}(V, W)$ we mean the set of all linear maps from V to W. We will now show that $\mathrm{Hom}(V, W)$ is in fact a vector space. Fixing bases $\mathbb{B}, \mathbb{D}$ will then give a vector space isomorphism between $\mathrm{Hom}(V, W)$ and the vector space of matrices $M_{m \times n}$.

Indeed, define $\vec{0}: V \to W$ to be the map $\vec{0}\vec{v} = \vec{0}$ for all $\vec{v} \in V$. The zero map.

If $f, g \in \mathrm{Hom}(V, W)$, define $f + g: V \to W$ by

$$f + g(\vec{v}) = f(\vec{v}) + g(\vec{v}).$$

Show that $f + g$ is linear.

If $f \in \mathrm{Hom}(V, W)$ and $\alpha \in \mathbf{R}$ define αf by

$$(\alpha f)(\vec{v}) = \alpha f(\vec{v}).$$

Show that αf is linear. Show that indeed the two operations just defined on $\mathrm{Hom}(V, W)$ do make up a vector space.

So, how does this translate in turn of matrices? Recall that $M_{m \times n}$ is also a vector space. How? We have the zero matrix $0 \in M_{m \times n}$, the matrix whose all entries are zero.

If $A, B \in M_{m \times n}$ are matrices, we have $A + B$ the matrix given by adding entry by entry.

If $A \in M_{m \times n}$ and $\alpha \in \mathbf{R}$, then αA is the matrix obtained by multiplying each entry of A by α.

Now, fix bases $\mathbb{B}, \mathbb{D}$ for V and W. Then $f \in \mathrm{Hom}(V, W)$ turns into a matrix $\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f$.

Proposition 4. We have the following:

$$\begin{aligned} \mathrm{Rep}_{\mathbb{B},\mathbb{D}} \vec{0} &= 0 \\ \mathrm{Rep}_{\mathbb{B},\mathbb{D}} f + g &= \mathrm{Rep}_{\mathbb{B},\mathbb{D}} f + \mathrm{Rep}_{\mathbb{B},\mathbb{D}} g \\ \mathrm{Rep}_{\mathbb{B},\mathbb{D}} \alpha f &= \alpha \, \mathrm{Rep}_{\mathbb{B},\mathbb{D}} f. \end{aligned}$$