# MATH 355 HOMEWORK 3

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## Problem 1

$W$ is not a subspace of $M_{2\times 2}$. It in fact satisfies none of the three axioms for a subspace. The zero vector in $M_{2\times 2}$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which does not belong to $W$. Moreover, $W$ is not closed under addition:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

but the matrix on the right hand side does not belong to $W$ even though both matrices on the left hand side of the equals sign do. Finally $W$ is not closed under scalar multiplication:

$$2 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

*Remark.* To prove something is *not* a subspace, it suffices to show that it does not satisfy only one of the axioms for a subspace. We include all the ways that $W$ fails to be a subspace for completeness.

## Problem 2

$U$ is not a subspace of $M_{2\times 2}$. It is closed under addition and contains the zero vector, but it is not closed under scalar multiplication. The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

belongs to $U$ as $1 + 1 = 2 \ \geq 0$, but multiplying by $-1$ we obtain

$$-1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is not in $U$ as $(-1) + (-1) = -2 < 0$.

## Problem 3

**Part a.**

$$\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}^t = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}.$$

**Part b.** $V$ is a subspace of $M_{2\times 2}$. The zero vector belongs to $V$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^t + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next note that $V$ is closed under scalar multiplication. Indeed for $\alpha \in \mathbf{R}$ and $A \in M_{2\times 2}$, we have $(\alpha \cdot A)^t = \alpha \cdot A^t$. To see this let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\left(\alpha \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^t = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}^t = \begin{pmatrix} \alpha a & \alpha c \\ \alpha b & \alpha d \end{pmatrix} = \alpha \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \alpha \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \alpha \cdot A^t.$$

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Now we can easily prove that $V$ is closed under scalar multiplication. Let $A \in V$ and $\alpha \in \mathbf{R}$. Then $\alpha \cdot A + (\alpha \cdot A)^t = \alpha \cdot A + \alpha \cdot A^t = \alpha \cdot (A + A^t) = \alpha \cdot 0 = 0$. Thus if $A \in V$, then $\alpha \cdot A \in V$. Finally, we prove a proposition that will let us prove that $V$ is closed under addition.

**Proposition.** *For $A, B \in M_{2\times 2}$, we have $(A+B)^t = A^t + B^t$.*

*Proof.* We write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\begin{aligned}(A+B)^t &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)^t \\ &= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}^t \\ &= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix} \\ &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix} \\ &= A^t + B^t.\end{aligned}$$

□

Now let $A, B \in V$. Then $(A+B) + (A+B)^t = A + B + A^t + B^t = (A + A^t) + (B + B^t) = 0 + 0 = 0$. Thus $(A+B) \in V$, and we've proved that it is a subspace.

## Problem 4

We first observe that

$$\operatorname{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\} = \left\{a \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + b \cdot \begin{pmatrix}-1\\1\\1\end{pmatrix} \,\middle|\, a, b \in \mathbf{R}\right\} = \left\{\begin{pmatrix}a\\a\\a\end{pmatrix} + \begin{pmatrix}-b\\b\\b\end{pmatrix} \,\middle|\, a, b \in \mathbf{R}\right\} = \left\{\begin{pmatrix}a-b\\a+b\\a+b\end{pmatrix} \,\middle|\, a, b \in \mathbf{R}\right\}.$$

Now suppose $\alpha \in \mathbf{R}$ is such that

$$\begin{pmatrix}2+\alpha\\3+\alpha\\4+2\alpha\end{pmatrix} \in \left\{\begin{pmatrix}a-b\\a+b\\a+b\end{pmatrix} \,\middle|\, a, b \in \mathbf{R}\right\}.$$

Then we must have $a, b \in \mathbf{R}$ such that

$$\begin{pmatrix}2+\alpha\\3+\alpha\\4+2\alpha\end{pmatrix} = \begin{pmatrix}a-b\\a+b\\a+b\end{pmatrix}.$$

Hence we must have $3 + \alpha = 4 + 2\alpha$, so $\alpha = -1$ is the only possible value. However it remains to show that when we plug in $\alpha = -1$ we actually get something in the span. In particular we need to find $a, b \in \mathbf{R}$ such that

$$\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}a-b\\a+b\\a+b\end{pmatrix}.$$

So we need

$$\begin{aligned} a - b &= 1 \\ a + b &= 2. \end{aligned}$$

Solving this linear system, we obtain $a = 3/2$ and $b = 1/2$. So we conclude that only $\alpha = -1$ works.

## Problem 5

The vectors are linearly independent. Suppose we have $\alpha, \beta, \gamma \in \mathbf{R}$ such that

$$\alpha \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we have the linear system

$$\begin{aligned} 2\alpha + \beta - \gamma &= 0 \\ 3\alpha + \beta + \gamma &= 0 \\ 4\alpha + \beta + \gamma &= 0. \end{aligned}$$

Subtracting the second equation from the third we obtain $\alpha = 0$. Then we have

$$\begin{aligned} \beta - \gamma &= 0 \\ \beta + \gamma &= 0. \end{aligned}$$

Adding these two equations we find $2\beta = 0$, hence $\beta = 0$, and then we have $\gamma = 0$. Then the set is linearly independent.

## Problem 6

The vectors are not linearly independent:

$$1 \cdot \begin{pmatrix} -7 \\ -6 \\ -13 \end{pmatrix} + (-2) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

## Problem 7

There are (infinitely) many bases for $\mathbf{R}^3$. Two are $\{\vec{e_1}, \vec{e_2}, \vec{e_2} + \vec{e_3}\}$ and $\{-\vec{e_1}, \vec{e_2}, \vec{e_3}\}$.

## Problem 8

**Part a.** This is true. Suppose $S \subset T \subset V$. Then recall that

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{m} \lambda_i s_i \;\middle|\; m \in \mathbf{N}, \lambda_i \in \mathbf{R}, s_i \in S \right\}.$$

If we have $S \subset T$, then each $s_i \in S$ in fact is an element of $T$, so every linear combination of elements of $S$ (i.e. an element of $\operatorname{span}(S)$) is also a linear combination of elements of $T$.

**Part b.** This is false. Let $S = \{\vec{e_1}\}$ and $T = \{\vec{e_2}\}$ where $\vec{e_1}$ and $\vec{e_2}$ are the standard basis vectors for $\mathbf{R}^2$. Then $S \cup T = \{\vec{e_1}, \vec{e_2}\}$, so $\operatorname{span}(S \cup T) = \mathbf{R}^2$. However, $\operatorname{span}(S) = \{\alpha \cdot \vec{e_1} | \alpha \in \mathbf{R}\}$, and $\operatorname{span}(S) = \{\beta \cdot \vec{e_1} | \beta \in \mathbf{R}\}$. Hence $\operatorname{span}(S) \cup \operatorname{span}(T) = \{\gamma \cdot \vec{e_i} \mid \gamma \in \mathbf{R} \text{ and } i = 1 \text{ or } 2\}$. Then in particular the vector $e_1 + e_2$ is not in $\operatorname{span}(S) \cup \operatorname{span}(T)$.

**Part c.** This is also false. Again let $\vec{e_1}$ and $\vec{e_2}$ be the standard basis vectors for $\mathbf{R}^2$. Now let $S = \{\vec{e_1}, \vec{e_2}\}$ and $T = \{\vec{e_1}, \vec{e_1} + \vec{e_2}\}$. Note that $S$ and $T$ are both bases for $\mathbf{R}^2$, so $\operatorname{span}(S) = \operatorname{span}(T) = \operatorname{span}(S) \cap \operatorname{span}(T) = \mathbf{R}^2$. However, $S \cap T = \{\vec{e_1}\}$, so $\operatorname{span}(S \cap T) = \{\alpha \cdot \vec{e_1} | \alpha \in \mathbf{R}\} \neq \mathbf{R}^2$. For example $\vec{e_2}$ does not belong to $\operatorname{span}(S \cap T)$.

## Problem 9

The thing to keep in mind is that vector subspaces (other than all of $\mathbf{R}^2$ and the zero subspace) of $\mathbf{R}^2$ look exactly like a single line through the origin.

**Part a.** This one is a subspace.

**Part b.** This line doesn't pass through the origin, so it doesn't contain the zero vector of $\mathbf{R}^2$, so it's not a subspace.

**Part c.** This isn't even a line and also doesn't contain the origin. To see the problem with not being a line, try adding two points that happen to lie on the circle and see whether the sum is also on the circle.

**Part d.** This is also not a subspace. It is the union of two subspaces. The problem is that if you add a point on one line to a point from the other line, you (usually) won't land on one of the lines. To be a bit more precise, imagine these lines were the lines $y = x$ and $y = -x$, so then $(1, 1)$ is on the first line and $(-1, 1)$ on the other. However their sum $(0, 2)$ is not on either line. (cf. Problem 8, Part b)

**Part e.** This one is not a subspace. The problem here is that you can scale vectors inside subspaces. Imagine that the corners of this rectangle are $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$, then $2 \cdot (1, 0) = (2, 0)$ is not in the shaded region.