Math 355 – Midterm 1 – Fall 2017

This exam has 8 problems worth 53 points distributed over 9 pages, including this one.

Instructions: This is a 2 hour exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem and carefully justify all answers. Points will be deducted for irrelevant, incoherent or incorrect statements, and no points will be awarded for illegible work. If you you run out of room, you may work answers on the back of pages or on attached scratch paper. Be sure to clearly indicate when work is continued on another page.

Name:

Honor Pledge:

On my honor, I have neither given nor received any unauthorized aid on this exam.

Signature:

Question	Points	Score
1	5	
2	5	
3	4	
4	4	
5	10	
6	7	
7	4	
8	14	
Total:	53	

1. (a) (1 point) Consider the vectors

$$\vec{v} = \begin{pmatrix} 6 \\ -3 \\ 0 \\ 9 \end{pmatrix}, \vec{w} = \begin{pmatrix} -2 \\ -2 \\ 3 \\ 0 \end{pmatrix}$$

What is $\frac{1}{3}\vec{v} - 2\vec{w}$?

(b) (2 points) Does the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ belong to the span of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

(c) (2 points) Consider the subspace

$$W := \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Does $\begin{pmatrix} 99 \\ 66 \\ 99 \\ -66 \end{pmatrix}$ belong to W?

2. Let $\alpha \in \mathbb{R}$ be a fixed real number. Consider the system of linear equations

$$\begin{cases} x + y - 2z &= \alpha \\ x - y &= -3 \\ 3x - y - 2z &= -6 \\ 2y - 2z &= 3 \end{cases}$$

(a) (3 points) Write the corresponding augmented matrix and, using row operations, reduce it to echelon form.

(b) (2 points) For which values of $\alpha \in \mathbb{R}$ does the system have no solutions, a unique solution, infinitely many solutions?

[Note: the answer may also be "this possibility never occurs"]

3. (a) (2 points) Consider the subset $Z \subset \mathbb{R}^3$ given by

$$Z := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x - z \geq 0 \right\}.$$

Is $Z \subset \mathbb{R}^3$ a subspace? Why?

(b) (2 points) Consider the subset $U \subset \mathbb{R}^4$ given by

$$U := \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \middle| yz = 0 \right\}.$$

Is $U \subset \mathbb{R}^4$ a subspace? Why?

4. For the following true/false questions you do NOT need to justify your answer (nor show your work). Please write clearly either TRUE or FALSE.
   (a) (1 point) If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^4$ are vectors, then $\text{Span}\{\vec{u}, \vec{v}, \vec{w}\} \subset \mathbb{R}^4$ is a subspace.

   (b) (1 point) If $U, W < \mathbb{R}^2$ are subspaces, then $U \cup W$ is also a subspace.

   (c) (1 point) If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are linearly independent, then $\vec{u}, \vec{v}$ are also linearly independent.

   (d) (1 point) If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are linearly dependent, then $\vec{u}, \vec{w}$ are also linearly dependent.

5. Consider the following homogenous linear system of equations

$$\begin{cases} x - 2y - 3z - s + t &= 0 \\ x + 3z - s - 3t &= 0 \end{cases}$$

where we view (x, y, z, s, t) as coordinates in $\mathbb{R}^5$.

(a) (3 points) Which variables are leading, which are free? Describe $W := \text{Sol} < \mathbb{R}^5$, the corresponding solution space.

(b) (2 points) What is the dimension of W? (you must justify your answer, either by computation or by referring to the correct fact stated in class)

(c) (2 points) Write down a basis B for W. Why is the collection of vectors you wrote down a basis for W?

(d) (3 points) Let $\vec{v} := \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$. What is $\text{Rep}_B \vec{v}$? Here B is the basis you found above.

6. (a) (3 points) Consider the subspace $W < M_{2\times 2}$ defined as

$$W := \{A \in M_{2\times 2} \mid \mathrm{tr}(A) = 0\}.$$

Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathrm{tr}(A) := a+d$.What is the dimension of W? Why?

(b) (4 points) Write down a basis for W. Complete it to a basis for $M_{2\times 2}$.

7. (4 points) Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form a basis of $\mathbb{R}^3$. Suppose $\vec{w} = -\vec{v}_1 + \vec{v}_2 - 5\vec{v}_3$. Show that $\vec{v}_1, \vec{w}, \vec{v}_3$ also form a basis of $\mathbb{R}^3$.

8. Let $U, W < \mathbb{R}^{137}$ be two subspaces.
   (a) (4 points) Suppose $U \cap W = \{\vec{0}\}$. Let $\vec{0} \neq \vec{u} \in U$, $\vec{0} \neq \vec{w} \in W$. Show that $\vec{u}$ and $\vec{w}$ are linearly independent.

   (b) (4 points) More generally, suppose again $U \cap W = \{\vec{0}\}$. Let $\vec{u}_1, \ldots, \vec{u}_k \in U$ be linearly independent, and let $\vec{w}_1, \ldots, \vec{w}_l \in W$ be linearly independent. Show that $u_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_l$ are all linearly independent.

   (c) (6 points) Assume now $\dim U = \dim W = 100$. Show that $U \cap W \supsetneq \{\vec{0}\}$, i.e. $U \cap W$ is strictly bigger than just $\{\vec{0}\}$.
   [You may use part (b) even if you did not solve it.]