MATH355 2017-09-18

DIMENSION

Recall that if $\vec{u}, \vec{v}, \vec{w} \in \mathbf{R}^5$, then their *span* is the subspace of all possible linear combinations in $\vec{u}, \vec{v}, \vec{w}$.

Example 1. For example, $W := \operatorname{Span}\left\{\begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$ is

$$W = \left\{ a\begin{pmatrix}2\\2\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\-1\end{pmatrix} \middle| a, b \in \mathbf{R} \right\}$$
$$= \left\{ \begin{pmatrix}2a\\2a+b\\-b\end{pmatrix} \middle| a, b \in \mathbf{R} \right\}$$

This description is useful, as we can instantly say that (for example)

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} \in W, \text{ corresponding to } a = 1, b = 1$$
$$\begin{pmatrix}2\\19\\5\end{pmatrix} \notin W, \text{ because } 19 - 2 \neq -5.$$

Last time we were talking about bases.

The vectors $\vec{v}_1, \dots, \vec{v}_n \in V$ form a *basis* if they are linearly independent and $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$.

Example 2. Check that $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ is a basis for $\mathbf{R}^3$.

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Definition 3. Recall that if $B = (\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V then any vector $\vec{v} \in V$ may be written *uniquely* as a linear combination of the $\vec{v}_i$. In other words,

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

where the $c_i \in \mathbf{R}$ are uniquely determined by $\vec{v}$. We define

$$\operatorname{Rep}_B \vec{v} := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

the *representation* of $\vec{v}$ in the basis B. The coefficients c_i are called the *coordinates* of $\vec{v}$ with respect to B.

Example 4. Let $B = \left(\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$ be the basis above. Let $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. What is $\operatorname{Rep}_B \vec{v}$?

Think about it...A little bit more... Come on, seriously now... I don't want to just give you the answer... All right... Here it comes... No, not yet... Did you guess it? ... Not yet? ... All right, I'll tell you later.

The book has a really good explanation of why one would like to pick a basis different from the standard one. It's about crystals, salt and pencils and other cool stuff.

The real MVP of linear algebra is the concept of *dimension*.

Definition 5. The *dimension* of a vector space V is the number of vectors $\vec{v}_1, \dots, \vec{v}_n$ in *any* given basis for it. We write

$$\dim V = n.$$

But wait, if we give two bases for V, how do we know that they will have the same number of vectors?

Theorem 6. Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis for V. Let $\vec{w}_1, \dots, \vec{w}_m$ be another one. Then $n = m$.

The slogan is

Any two bases have the same size.

Before we prove this, here are two useful facts. The first follows directly from the definition.

Fact 7. If $\vec{v}_1, \dots, \vec{v}_n$ is a basis for V and $\vec{w} \in V$ is any vector, then $\vec{v}_1, \dots, \vec{v}_n, \vec{w}$ are linearly dependent. In particular, $\vec{v}_1, \dots, \vec{v}_n, \vec{w}$ is not a basis.

For the second, we will give a proof.

Fact 8. If $\vec{v}_1, \dots, \vec{v}_n$ is a basis and $\vec{w} \in V$, write $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. If $a_i \neq 0$, then we may replace $\vec{v}_i$ with $\vec{w}$ and still have a basis. In other words,

$$\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{w}, \vec{v}_{i+1}, \dots, \vec{v}_n$$

is a basis for V.

Proof. We need to show two things: linear independence + span. We know that $\vec{w} = \sum_{j=1}^{n} a_j \vec{v}_j$ with $a_i \neq 0$. Up to relabeling the $\vec{v}_i$, we can assume $i = 1$. So $a_1 \neq 0$ and we wish to show that $\vec{w}, \vec{v}_2, \dots, \vec{v}_n$ is a basis. Suppose we write $\vec{0}$ as a linear combination of the new basis. By substituting for $\vec{w}$, we have

$$\begin{aligned}\vec{0} &= c_1\vec{w} + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \\ &= c_1a_1\vec{v}_1 + (c_2 + c_1a_2)\vec{v}_2 + \cdots + (c_n + c_1a_n)\vec{v}_n\end{aligned}$$

and since $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent, we have

$$c_1a_1 = 0, \quad (c_2 + c_1a_2) = 0, \dots, \quad (c_n + c_1a_n) = 0.$$

But $a_1 \neq 0$, so $c_1 = 0$. This implies $c_2 = 0, \dots, c_n = 0$ as desired.

To show that $\mathrm{Span}\{\vec{w}, \vec{v}_2, \dots, \vec{v}_n\} = V$, it suffices to show that $\vec{v}_1 \in \mathrm{Span}\{\vec{w}, \vec{v}_2, \dots, \vec{v}_n\}$ (why?). But we know since $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ hence

$$\vec{v}_1 = \frac{1}{a_1}\vec{w} + \frac{a_2}{a_1}\vec{v}_2 + \cdots + \frac{a_n}{a_1}\vec{v}_n$$

which concludes the proof. □

Proof of the Theorem. First off, if $m > n$ we relabel the $\vec{v}_i$ as $\vec{w}_i$ and viceversa. So, we can assume $n \leq m$. Since $\vec{v}_1, \dots, \vec{v}_n$ is a basis, $\vec{w}_1 = \sum_i a_i\vec{v}_i$. Since $\vec{w}_1 \neq 0$ (because it's part of a basis!), we must have $a_i \neq 0$ for some i. Up to reindexing the $\vec{w}_i$, we can assume $i = 1$. By the Fact above, $\vec{w}_1, \vec{v}_2, \dots, \vec{v}_n$ is a new basis of V.

Write now $\vec{w}_2 = b_1\vec{w}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$. If $b_2 = 0, b_3 = 0, \dots, b_n = 0$, then $\vec{w}_1$ and $\vec{w}_2$ would be linearly dependent, which is impossible as they are part of a basis. So there must be at least one $i \geq 2$ such that $b_i \neq 0$. Again, we may reindex the $\vec{w}_i$, and assume $b_2 \neq 0$. By the Fact above, $\vec{w}_1, \vec{w}_2, \vec{v}_3, \dots, \vec{v}_n$ is a new basis for V.

Write now $\vec{w}_3 = c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{v}_3 + \cdots + c_n\vec{v}_n$. If $c_3 = 0, c_4 = 0, \dots, c_n = 0$, then $\vec{w}_1, \vec{w}_2, \vec{w}_3$ would be linearly dependent. This would be a contradiction (since they are part of a basis), so there is $i \geq 3$ such that $c_i \neq 0$. Up to reindexing the $\vec{w}_i$, we can assume $i = 3$. By the Fact above, $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{v}_4, \dots, \vec{v}_n$ is a new basis of V.

Write now $\vec{w}_4$...

We keep doing this until we've converted the $\vec{v}_i$ basis into a $\vec{w}_i$ basis. But what have we learned? We learned that $\vec{w}_1, \dots, \vec{w}_n$ is a basis for V. Notice that I wrote $\vec{w}_1, \dots, \vec{w}_n$ and NOT $\vec{w}_1, \dots, \vec{w}_m$. Gasp! So we only needed the first n vectors to form a basis. By the first Fact above, if $\vec{w}_1, \dots, \vec{w}_n$ is a basis then $\vec{w}_1, \dots, \vec{w}_m, \vec{w}_{m+1}$ cannot be linearly independent. Hence m is forced to be equal to n. □

Example 9. Go back to Example 4. If you haven't figured it out yet, you just need to write $\vec{v}$ as a linear combination of the $\vec{v}_i$. You can either eyeball it, or use row reduction, in any case the unique solution is $\vec{v} = 2\vec{v}_2 - \vec{v}_3$. Hence, $\mathrm{Rep}_B\, \vec{v} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.