1.

(a) $\frac{1}{3}\vec{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$ $\quad -2\vec{w} = \begin{pmatrix} 4 \\ 4 \\ -6 \\ 0 \end{pmatrix}$

$$\frac{1}{3}\vec{v} - 2\vec{w} = \begin{pmatrix} 6 \\ 3 \\ -6 \\ 3 \end{pmatrix}$$

(b) No, any vector in span$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is of the form $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$

but $2 \neq 1$.

(c) Yes

$$\begin{pmatrix} 99 \\ 66 \\ 99 \\ -66 \end{pmatrix} = 99\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 66\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

which is a linear combination of

$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$.

2. (a)

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & \alpha \\ 1 & -1 & 0 & -3 \\ 3 & -1 & -2 & -6 \\ 0 & 2 & -2 & 3 \end{array}\right]$$

$R_2 \rightsquigarrow R_2 - R_1$

$R_3 \rightsquigarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & \alpha \\ 0 & -2 & 2 & -3-\alpha \\ 0 & -4 & 4 & -6-3\alpha \\ 0 & 2 & -2 & 3 \end{array}\right]$$

$R_3 \rightsquigarrow R_3 - 2R_2$

$R_4 \rightsquigarrow R_4 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & \alpha \\ 0 & -2 & 2 & -3-\alpha \\ 0 & 0 & 0 & (-6-3\alpha)-2(-3-\alpha) \\ 0 & 0 & 0 & -\alpha \end{array}\right]$$

It's now in echelon form.

2. (b)

For $\alpha \neq 0$ the system ~~is~~ ~~not~~ ~~does~~ does not have any solutions.

For $\alpha = 0$ the system is equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

~~As~~ This system has infinitely many solutions as it is consistent and z is a free variable.

For no value of $\alpha \in \mathbb{R}$ does the system have a unique solution.

3. (a)

No. $\begin{pmatrix}1\\0\\0\end{pmatrix} \in Z$ as $1-0 \geq 0$

but $-\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} \notin Z$ as $-1-0 \ngeq 0$

(b) No.

$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \in U$ as $1 \cdot 0 = 0$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \in U$ as $1 \cdot 0 = 0$

but $\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \notin U$ as $1 \cdot 1 \neq 0$.

4. a TRUE

b FALSE

c TRUE

d FALSE

5. a

$$\begin{bmatrix} 1 & -2 & -3 & -1 & 1 \\ 1 & 0 & 3 & -1 & -3 \end{bmatrix}$$

$R_2 \rightsquigarrow R_2 - R_1$
$$\begin{bmatrix} 1 & -2 & -3 & -1 & 1 \\ 0 & 2 & 6 & 0 & -4 \end{bmatrix}$$

x, y leading, z, s, t are free.

Now, $2y - 6z - 4t = 0$ so $y = 3z + 2t$

$x - 2y - 3z - 1s + t = 0$ so

$x = 6z + 4t + 3z + s - t = 9z + s + 3t$

So $Sol = \left\{ \begin{pmatrix} 9z+s+3t \\ 3z+2t \\ z \\ s \\ t \end{pmatrix} \middle| z, s, t \in \mathbb{R} \right\} =$

$= \operatorname{Span}\left\{ \begin{pmatrix} 9 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

6. b

We already have a basis, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

to complete consider $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Since $\dim M_{2\times 2} = 4$, it suffices to check they are linearly independent

Suppose

$$a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+d & b \\ c & d-a \end{pmatrix}$ so $b=0,\ c=0$

and $\begin{cases} a+d=0 \\ d-a=0 \end{cases} \Rightarrow 2d=0 \Rightarrow d=0$

so $a=0$. Hence they are lin. indep.

7.

Since $\dim \mathbb{R}^3 = 3$ it suffices to show

$\mathrm{Span}\{\vec{v}_1, \vec{w}, \vec{v}_3\} = \mathbb{R}^3$

We know $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$

~~Since $\vec{v}$~~ so it suffices to show $\vec{v}_2 \in \mathrm{Span}\{\vec{v}_1, \vec{w}, \vec{v}_3\}$

but $\vec{v}_2 = \vec{v}_1 + \vec{w} + 5\vec{v}_3$ hence $\vec{v}_2 \in \mathrm{Span}\{\vec{v}_1, \vec{w}, \vec{v}_3\}$.

5. (a)

$$\begin{bmatrix} 1 & -2 & -3 & -1 & 1 \\ 1 & 0 & 3 & -1 & -3 \end{bmatrix}$$

$R_2 \rightsquigarrow R_2 - R_1$
$$\begin{bmatrix} 1 & -2 & -3 & -1 & 1 \\ 0 & 2 & 6 & 0 & -4 \end{bmatrix}$$

x, y leading, z, s, t are free.

Now, $2y - 6z - 4t = 0$ so $y = 3z + 2t$

$x - 2y - 3z - 1s + t = 0$ so

$x = 6z + 4t + 3z + s - t = 9z + s + 3t$

So $Sol = \left\{ \begin{pmatrix} 9z+s+3t \\ 3z+2t \\ z \\ s \\ t \end{pmatrix} \middle| \ z, s, t \in \mathbb{R} \right\} =$

$= \cancel{\{}$ $\mathrm{Span} \left\{ \begin{pmatrix} 9 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

5. (b)

dim $W = 3$ as its dimension equals the number of free variables

5. (c)

A basis is given by

$$\begin{pmatrix} 9 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

because they ~~span W and~~ 3 vectors spanning a space of dimension 3.

6. (a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A \in W$ iff $d = -a$

So $W = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

Thus,

$W = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

We check they are linearly independent:

If $a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $b = 0$, $c = 0$, $a = 0$ so they form a basis for W.

$\dim W = 3$.

6. b

We already have a basis, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

to complete consider $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Since $\dim M_{2\times 2} = 4$, it suffices to check they are linearly independent

Suppose

$$a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+d & b \\ c & d-a \end{pmatrix}$ so $b=0, c=0$

and $\begin{cases} a+d=0 \\ d-a=0 \end{cases} \Rightarrow 2d=0 \Rightarrow d=0$

so $a=0$. Hence they are lin. indep.

7.

Since $\dim \mathbb{R}^3 = 3$ it suffices to show

$\mathrm{Span}\{\vec{v}_1, \vec{w}, \vec{v}_3\} = \mathbb{R}^3$

We know $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$

~~Since $\vec{v}$~~ so it suffices to show $\vec{v}_2 \in \mathrm{Span}\{\vec{v}_1, \vec{w}, \vec{v}_3\}$

but $\vec{v}_2 = \vec{v}_1 + \vec{w} + 5\vec{v}_3$ hence $\vec{v}_2 \in \mathrm{Span}\{\vec{v}_1, \vec{w}, \vec{v}_3\}$.

3. (a)

Suppose $a\vec{u}+b\vec{w}=\vec{0}$

then $a\vec{u}=-b\vec{w}$

if $a\neq 0$ $\vec{u}=-\frac{b}{a}\vec{w} \Rightarrow \vec{u}\in \text{Span}\{\vec{w}\}\subseteq W$

hence, $\vec{u}\in U\cap W=\{\vec{0}\}$ hence $\vec{u}=\vec{0}$. A contradiction.

Assume then $a=0$

then $b\vec{w}=0$.

If $b\neq 0$ then $\vec{w}=\frac{1}{b}\vec{0}=\vec{0}$ A contradiction.

Hence we must have $a=b=0$. I.e. $\vec{u},\vec{w}$ are lin. indep.

8. b

~~Call~~ $\cancel{\vec{u}}$

Suppose $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k \vec{u}_k + b_1\vec{w}_1 + \cdots + b_\ell \vec{w}_\ell = \vec{0}$

Call $\vec{u} := a_1\vec{u}_1 + \cdots + a_k\vec{u}_k$

$\vec{w} := -b_1\vec{w}_1 + \cdots + (-b_\ell)\vec{w}_\ell$

then $\vec{u} = \vec{w}$

but W is closed under linear combinations

so $\vec{w} \in W$, so $\vec{u} \in U \cap W = \{\vec{0}\}$

hence $\vec{u} = 0$.

Similarly, $\vec{w} \in U \cap W = \{\vec{0}\}$ thus $\vec{w} = \vec{0}$.

8. (c)

~~If $U \cap W = \{0$~~

Pick a basis $\vec{u}_1, \ldots, \vec{u}_{100}$ of U

$\vec{w}_1, \ldots, \vec{w}_{100}$ of W

by part (b) $\vec{u}_1, \ldots, \vec{u}_{100}, \vec{w}_1, \ldots, \vec{w}_{100}$

are linearly independent,

hence $\dim V \geq 100 + 100 = 200$

but $200 \not\leq 137$

hence $U \cap W \neq \{\vec{0}\}$.