JOHN CALABRESE — RESEARCH STATEMENT

Overview. I completed my DPhil in 2013 at the University of Oxford, where my advisor was Tom Bridgeland (now at Sheffield). For the remainder of the year, I was an **EPSRC Doctoral Prize Fellow** at Imperial College, mentored by Richard Thomas. In January 2014 I moved to Rice as an RTG Lovett Instructor, mentored by Brendan Hassett (now at Brown). Later that year I was awarded an **AMS-Simons travel grant**. In 2015 I was awarded an **NSF Mathematical Sciences Postdoctoral Research Fellowship**, and I am now also a G.C. Evans Instructor. In 2017 I was awarded an **NSF conference grant** to organize the Texas Algebraic Geometry Symposium. I work in *algebraic geometry*, with a focus on *categories of sheaves* and applications to *Donaldson-Thomas* theory.

Background. The classical objects of study in algebraic geometry are (smooth) projective varieties, which are subsets $X \subset \mathbf{CP}^N$ defined by the vanishing of homogeneous polynomials. On the one hand, a given such X is a *complex manifold* and can therefore be studied borrowing techniques from *geometry*. On the other hand, since X is defined by polynomials, one can also employ techniques coming from *algebra*. The overarching theme of my research is to understand the geometry of a variety by means of algebraic *invariants*.

The seminal 1955 paper of Serre [Ser55] introduced the notion of *coherent sheaf*, which is now ubiquitous in the subject. Roughly, a coherent sheaf V on a variety X is a collection of vector spaces V_x, one for each point $x \in X$. Unlike vector bundles, the dimension of the fibre V_x needn't be constant: it is allowed to jump along closed subsets. The collection of all coherent sheaves forms a category, $\mathtt{Coh}(X)$. In contrast to vector bundles, $\mathtt{Coh}(X)$ is a particularly pleasant category, as it is closed under all the operations one wishes to perform: kernels, images, direct sums, etc.

With the advent of *homological algebra*, it was later realized one should enlarge $\mathtt{Coh}(X) \subset \mathtt{D}(X)$, passing to its *derived category*. The latter consists of chain complexes of coherent sheaves, providing a more flexible framework for derived functors. For example, derived categories are indispensable to even state the dualities of Grothendieck-Verdier, far-reaching generalizations of *Poincaré duality* (in topology) and *Serre duality* (in algebraic geometry). But it wasn't until Mukai [Muk81], that derived categories became objects of independent study, and were later essential in Kontsevich's *homological mirror symmetry* and Bridgeland's *stability conditions*.

Part of my work is devoted to the study the categories $\mathtt{Coh}(X)$ and $\mathtt{D}(X)$, while another part seeks for applications, specifically to *Donaldson–Thomas* invariants. In his thesis, Thomas introduced virtual counts of sheaves for Calabi-Yau 3-folds (a special class of varieties), marking the birth of Donaldson-Thomas theory. These invariants are closely related to the *curve-counting* invariants of *Gromov-Witten* and to *BPS counts* in mathematical physics.

Past. Two varieties X, Y are called *birational* if there are closed subvarieties $W \subset X$, $Z \subset Y$ such that $X \setminus W$ and $Y \setminus Z$ are isomorphic. While this notion is somewhat useless in classical topology, it has always played a central role in algebraic geometry (due to the more rigid nature of the Zariski topology). In my thesis, I studied the behavior of Donaldson-Thomas invariants under birational maps [Cal16a].

At its very core, [Cal16a] may be seen as an application of an equivalence of categories of Bridgeland [Bri02], together with the theory of *motivic Hall algebras* of Joyce-Song and Kontsevich-Soibelman [JS12, KS10]. The answer was a simple and explicit formula in the case of *flops* (which are the building blocks of birational maps between Calabi-Yau 3-folds). The formula and methods of [Cal16a] were applied by Maulik in [Mau16], settling conjectures of Oblomkov-Shende and Diaconescu-Hua-Soibelman on the *HOMFLY polynomial* of an algebraic link.

Related to the flop formula, Bryan-Cadman-Young [BCY12] formulated a *Crepant Resolution Conjecture* for Donaldson-Thomas invariants. This conjecture predicts a transformation rule for the invariants, in the setting of the *McKay Correspondence* [BKR01]. This conjecture consists of two statements: I proved the first and made progress towards the second in the paper [Cal16b].

A celebrated theorem in algebraic geometry is *Gabriel's theorem*, which says two varieties X and Y are isomorphic if and only if the categories $\mathtt{Coh}(X)$ and $\mathtt{Coh}(Y)$ are equivalent. In other words, $\mathtt{Coh}(X)$ is a *complete invariant* of the variety X. While in Oxford, I started collaborating with Michael Groechenig (now at Freie Universität Berlin), which resulted in the paper [CG15]. We provided a modern approach to Gabriel's theorem, leading to a considerable generalization. The approach of [CG15], later led to [Cal17b, Cal17c, Cal17a] which explore related questions.

As a fellow at Imperial College, I worked with Richard Thomas on applying Kuznetsov's recent theory of *homological projective duality* [Kuz07]. While $\mathtt{Coh}(X)$ is a complete invariant of X, $\mathtt{D}(X)$ is not. An active area of research aims to understand which varieties X, Y are *Fourier-Mukai partners*, meaning $\mathtt{D}(X)$ is equivalent to $\mathtt{D}(Y)$. This led to the work [CT16], where we constructed new derived equivalences of Calabi-Yau 3-folds arising from pencils of cubic 4-folds. The survey [Cala] provides an overview of the ideas involved. Work in [CT16] was later shown to be part of a broader picture in [BL16].

Present. I am currently collaborating with Roberto Pirisi (UBC), on an exciting variant of Gabriel's reconstruction theorem [CP17]. Using work of Meinhardt-Partsch as a springboard [MP14], we have studied the quotients categories $\mathtt{Coh}(X)/\mathtt{Coh}_{\leq k}(X)$. We show that the k-th quotient controls the isomorphism type of X, *up to codimension* $\dim X - k$. In other words, these quotients interpolate between Gabriel's theorem and *birational geometry*. For example, from the quotient $\mathtt{Coh}(X)/\mathtt{Coh}_{\leq \dim X-1}(X)$ one extracts the function field of X, and hence the birational type of X. This work is a continuation of the program initiated in [CG15].

A conjecture of Kawamata [Kaw02], stated that a variety has at most finitely many Fourier-Mukai partners. In [AT09], it was shown that a variety admits at most countably many partners. Kawamata's conjecture was recently disproved by Lesieutre [Les15]. However, variations on the theme of Kawamata's conjecture can be explored for (semiorthogonal) *components* of $\mathtt{D}(X)$. The study of these components has received renewed attention due to Kuznetsov's theory [Kuz07], mentioned above. Borrowing ideas from my previous work [CT16], I have been working on (infinitesimal) Torelli-type theorems for these components [Calb], akin to [HR16].

Future. Continuing from [CP17], I hope to make contact with *non-commutative geometry*, in the sense of Artin, Smith, Van den Bergh et al. Indeed, there are many constructions of *non-commutative projective varieties*, and the work [PVdB16] produces examples of *birational* non-commutative surfaces. The works [CP17, PVdB16] are certainly related, and a close investigation will perhaps also lead to higher dimensional constructions, such as *non-commutative flops*.

Together with Jørgen Rennemo (Oxford) and Sjoerd Beentjes (Edinburgh), we have been working on completing the proof of the crepant resolution conjecture. The Donaldson-Thomas numbers of a variety are typically organized in a generating series, which a posteriori is the expansion of a rational function. What seems missing from [Cal16b] is a clever use of the Hall algebra, to ensure two different generating series are actually the expansions of the same rational function, after a change of variables. We have recently made a breakthrough, and are able to complete the proof in a special case.

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