Homework 7 Solutions

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#1: Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$ be the standard basis vectors for $\mathbb{R}^5$.
Let $U = \text{span}\{\vec{e}_1, \vec{e}_2\}$ and $V = \text{span}\{\vec{e}_3, \vec{e}_4, \vec{e}_5\}$. We claim $\mathbb{R}^5 = U \oplus V$.

Indeed, we can write every element $(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$ as $a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5$. This shows that $\mathbb{R}^5 = U + V$. Moreover since $\{\vec{e}_1, \dots, \vec{e}_5\}$ is a basis, the representation $a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5$ is unique (i.e. there is no other linear combination of the $\vec{e}_i$ giving the same element of $\mathbb{R}^5$). Thus we can uniquely represent $\vec{x} \in \mathbb{R}^5$ as $\vec{u} + \vec{v}$ for $\vec{u} \in U$ and $\vec{v} \in V$. Then the sum is direct.

#2: Let $\{\vec{e}_1, \dots, \vec{e}_5\}$ be the standard basis vectors for $\mathbb{R}^5$. Put $U = \text{span}\{\vec{e}_1\}$, $V = \text{span}\{\vec{e}_2\}$, $W = \text{span}\{\vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$. Then since $\{\vec{e}_1, \dots, \vec{e}_5\}$ forms a basis, we can write $\vec{x} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5$. $[\vec{x} \in \mathbb{R}^5]$
Set $\vec{u} = a_1\vec{e}_1 \in U$, $\vec{v} = a_2\vec{e}_2 \in V$, $\vec{w} = a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5 \in W$. Then $\vec{x} \in \mathbb{R}^5$ is the sum $U+V+W$. This sum is <u>not</u> direct. One can choose subspaces to write $\mathbb{R}^5$ as a direct sum of three subspaces, however.

#3: We have the relation $\dim(U+W) = \dim U + \dim W - \dim(U \cap V)$. In our case we then have $\dim(U+W) + \dim U \cap W = 6$. We also see immediately (since $U \subseteq U+W$) that $3 \leq \dim(U+W) \leq 5 = \dim \mathbb{R}^5$. In fact we can obtain each: $\dim(U+W)$ can be 3, 4, or 5. Then $\dim U \cap W$ can be 3, 2, or 1. We make a table of examples.

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$ be the standard basis for $\mathbb{R}^5$.

	U	W	$U \cap W$	$U + W$
1	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$
2	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_4\}$	$\text{span}\{\vec{e}_1, \vec{e}_2\}$	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$
3	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	$\text{span}\{\vec{e}_1, \vec{e}_4, \vec{e}_5\}$	$\text{span}\{\vec{e}_1\}$	$\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, e_4, \vec{e}_5\}$

In case 1 $\dim(U \cap W) = 3$ and $\dim(U+W) = 3$. In case 2 $\dim(U \cap W) = 2$ and $\dim(U+W) = 4$. In case 3 $\dim(U \cap W) = 1$ and $\dim(U+W) = 5$. For completeness, we show explicitly why $U \cap W$ and $U+W$ are what we claimed in the table for case 3. The other cases are analogous. Suppose $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \cap \text{span}\{\vec{e}_1, \vec{e}_4, \vec{e}_5\}$. Write $\vec{x}$ in terms of the standard bases: $\vec{x} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5$. Since $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, we have $a_4 = a_5 = 0$. Similar $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_4, \vec{e}_5\}$ implies $a_2 = a_3 = 0$. Hence $\vec{x} \in a_1\vec{e}_1$. This shows $U \cap W \subseteq \text{span}\{\vec{e}_1\}$. Suppose $\vec{y} \in \text{span}\{\vec{e}_1\}$, so $\vec{y} = \alpha\vec{e}_1$. Clearly we have $\alpha\vec{e}_1 \in \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\alpha\vec{e}_1 \in \text{span}\{\vec{e}_1, \vec{e}_4, \vec{e}_5\}$. Hence $U \cap W \subseteq \text{span}\{\vec{e}_1\}$ as claimed. For $U+V$, we note that we can write any $\vec{x} \in \mathbb{R}^5$ as $\vec{x} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5$. Then $\vec{u} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 \in \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\vec{w} = 0 \cdot \vec{e}_1 + a_4\vec{e}_4 + a_5\vec{e}_5 \in \text{span}\{\vec{e}_1, \vec{e}_4, \vec{e}_5\}$. Then $\vec{x} = \vec{u} + \vec{w}$. This shows $\mathbb{R}^5 \subseteq U+W$, but V and W are subspaces of $\mathbb{R}^5$. Hence $U+W = \mathbb{R}^5 = \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$.

#4: In terms of $\vec{b}_1, \vec{b}_2, \vec{b}_3$, we have $\begin{pmatrix}1\\3\\1\end{pmatrix} = \vec{b}_1 + 2\vec{b}_2 - 2\vec{b}_3$, hence

$$f\begin{pmatrix}1\\3\\1\end{pmatrix} = f(\vec{b}_1 + 2\vec{b}_2 - 2\vec{b}_3) = f(\vec{b}_1) + 2f(\vec{b}_2) - 2f(\vec{b}_3) = \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-2\\2\end{pmatrix} - \begin{pmatrix}4\\-2\end{pmatrix}$$

$= \begin{pmatrix}-5\\3\end{pmatrix}$. A linear map always takes the zero vector to the zero vector: $f(\vec{0}) = f(0\cdot\vec{0}) = 0f(\vec{0}) = \vec{0}$. Hence $f\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

$\begin{pmatrix}2\\2\\1\end{pmatrix} = \vec{b}_1 + \vec{b}_2$, so $f\begin{pmatrix}2\\2\\1\end{pmatrix} = f(\vec{b}_1 + \vec{b}_2) = f(\vec{b}_1 + \vec{b}_2) = f(\vec{b}_1) + f(\vec{b}_2)$

$= \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Remark: We can write down a matrix expressing this linear map.

$f = \begin{pmatrix}1 & -1 & 2\\-1 & 1 & -1\end{pmatrix}$, where each column is $f(\vec{b}_i)$ for $i=1,2,3$.

To use this matrix, we write $\begin{pmatrix}1\\3\\1\end{pmatrix}$ in terms of the $\vec{b}_i$. This gives the column vector representation $\begin{pmatrix}1\\2\\-2\end{pmatrix}$. For $\begin{pmatrix}2\\2\\1\end{pmatrix}$ we get $\begin{pmatrix}1\\1\\0\end{pmatrix}$. Hence

$$\begin{pmatrix}1 & -1 & 2\\-1 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}1-2-4\\-1+2+2\end{pmatrix} = \begin{pmatrix}-5\\3\end{pmatrix} \text{ and}$$

$$\begin{pmatrix}1 & -1 & 2\\-1 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1-1\\-1+1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

#5: Choose a basis $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ for V. Write $V_i = \mathrm{span}\{\vec{b}_i\}$. We claim $V = V_1 \oplus V_2 \oplus \dots \oplus V_n$. This follows essentially from the definition of basis. Indeed, every $\vec{v} \in V$ can be written as $\vec{v} = a_1\vec{b}_1 + \dots + a_n\vec{b}_n$ for $a_i \in \mathbb{R}$. This shows that $V = V_1 + \dots + V_n$. Since this expression for $\vec{v}$ is unique, the sum is in fact direct: $V = V_1 \oplus \dots \oplus V_n$.

#6: Recall the relation $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$ for U and V linear subspaces of some vector space. In our case we let $U = W_1 + W_2$ and $V = W_3$. Then

$$\dim((W_1+W_2)+W_3) = \dim(W_1+W_2) + \dim W_3 - \dim((W_1+W_2)\cap W_3)$$
$$= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) + \dim W_3 - \dim((W_1+W_2)\cap W_3).$$

By hypothesis, $\dim(W_1+W_2+W_3) = \dim W_1 + \dim W_2 + \dim W_3 = \dim V$, so we have

$0 = -\dim(W_1 \cap W_2) - \dim((W_1+W_2)\cap W_3)$. Multiplying by -1 we get $\dim(W_1 \cap W_2) + \dim((W_1+W_2)\cap W_3) = 0$. Since dimension is nonnegative, we have each term is 0 hence $\dim(W_1 \cap W_2) = 0$. This implies $W_1 \cap W_2 = \{\vec{0}\}$. Hence $(W_1 \cap W_2) \cap W_3 = \{\vec{0}\} \cap W_3 = \{\vec{0}\}$. Hence $\dim(W_1 \cap W_2 \cap W_3) = 0$. A similar calculation (change the labeling) shows $\dim W_2 \cap W_3 = 0$, so $W_2 \cap W_3 = \{\vec{0}\}$.

#7: $W_1 \cap W_2 \cap W_3 = \{\vec{0}\}$. To see this write $W = W_1 \cap W_2 \cap W_3$ and suppose there exists some nonzero $\vec{w} \in W$.

Suppose W_1 has a vector $\vec{x}$ such that $\{\vec{w}, \vec{x}\}$ is linearly independent (at least W_i has such a vector, else $\dim W_1 + \dim W_2 + \dim W_3 = 3$).

Then since $\dim W_1 + \dim W_2 + \dim W_3 = 4$ subject to $\dim W_1 \geq 2$, $\dim W_2 \geq 1$, and $\dim W_3 \geq 1$, we must have $\dim W_1 = 2$, $\dim W_2 = 1$, and $\dim W_3 = 1$. Then $W_1 = \operatorname{span}\{\vec{w}, \vec{x}\}$, $W_2 = \operatorname{span}\{\vec{w}\}$, $W_3 = \operatorname{span}\{\vec{w}\}$. However, $W_1 + W_2 + W_3 = \operatorname{span}\{\vec{w}, \vec{x}\}$ in this case. But $W_1 + W_2 + W_3 = V$ with $\dim V = 3$, so we have a contradiction. We conclude that $W = W_1 \cap W_2 \cap W_3$ contains no nonzero vectors. Hence $W_1 \cap W_2 \cap W_3 = \{\vec{0}\}$.