MATH355 2017-11-01

DETERMINANT: FRIEND OR FOE?

Friend.

The determinant is a function det: $M_{n\times n} \to \mathbf{R}$ satisfying a ton of awesome properties.

- $\det A \neq 0$ if and only if A is invertible.
- $\det I = 1$.
- $\det(\lambda A) = \lambda^n \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- If A is invertible, $\det(A^{-1}) = (\det A)^{-1}$.
- $\det(A^t) = \det(A)$.
- If B is obtained from A by swapping two columns (or two rows) then $\det B = -\det A$.
- If any subset of the columns (or rows) of A is linearly dependent, then $\det A = 0$.

OK, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\det A := ad - bc.$$

Why? Well, $\vec{v}_1 = A\vec{e}_1, \vec{v}_2 = A\vec{e}_2$ are the images of the standard basis vectors of $\mathbf{R}^2$. A geometric argument shows that $\det A$ is the (signed!) area of the parallelogram spanned by $\vec{v}_1, \vec{v}_2$.

For instance, if $\vec{v}_1, \vec{v}_2$ are lie on a line (i.e. they are linearly dependent, i.e. $\operatorname{rk} A \leq 1$) then they span a degenerate parallelogram, which has area zero. And viceversa, if $\det A = 0$, it means that the parallelogram was degenerate, therefore $\vec{v}_1, \vec{v}_2$ are linearly dependent (so A can't be invertible).

How to compute determinants in higher dimensions? Laplace expansion.

Date: John Calabrese, November 3, 2017.