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Erratum to

Donaldson–Thomas invariants and flops

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This is an erratum for [6] (which essentially coincides with [4]). The main result [6, Corollary 3.27] is correct. Indeed, if one assumes throughout to be working in [6, Situation 3.24] (namely, if only cares about flops) then the whole paper is fine as is. However, the key [6, Theorem 3.23] is *incorrect* as stated in its full generality. To reflect this, the arXiv version of the paper has been updated to [5].

This erratum is divided in three sections. The first provides an explanation as to why some statements in [6] need to be modified. The second goes through the differences between [6] and [5]. The third is an appendix to [6], which is parallel to [5, Section 4].

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1. Explanation

The paper [6] is concerned with proving a comparison formula for Donaldson–Thomas (DT) invariants. Let us recall the main strategy. One starts with a flopping contraction $Y \to X$ and a flop $Y^+ \to X$. The main goal is to prove the formula

$$(\star) \qquad \mathrm{DT}^{\vee}_{\mathrm{exc}}(Y) \cdot \mathrm{DT}(Y) = \mathrm{DT}^{\vee}_{\mathrm{exc}}(Y^+) \cdot \mathrm{DT}(Y^+).$$

The proof goes via the fact that Y and Y^+ are derived equivalent. The respective categories of perverse coherent sheaves are mapped to each other under Bridgeland's equivalence, hence the perverse Hilbert schemes are isomorphic. Up to some additional details, this formally implies that the perverse DT numbers of Y and Y^+ are the same [6, Theorem 3.26].

The main focus of the paper actually lies in proving [6, Theorem 3.23], which compares the perverse DT invariants with the ordinary ones. In this sense, all the action happens on just one side of the flop.

Let us go back to our flopping contraction $Y \to X$. Readers solely interested in flops can assume that the singular locus of X is zero-dimensional. In this case, the statements and proofs of [6] can be left untouched and one can safely ignore the present text.

Nonetheless, the actual perverse/ordinary comparison formula is of interest beyond the context of flops (for example in the setting of the crepant resolution conjecture for Donaldson–Thomas invariants [2, 3]).

If we allow the singular locus of X to be a curve then [6, Theorem 3.23] needs to be modified into [5, Theorem 3.23]. The latter statement, however, contains the symbol DT^{∂} which we now explain.

The proof of [6, Proposition 3.5] contains a mistake, the root of which can be traced back to the following fact. Inside the category of perverse coherent sheaves ${}^p\mathcal{A}$ we find ${}^p\mathcal{A}_{\leq 1}$, consisting of those complexes having support of dimension at most one (recall that the support of a complex is the union of the supports of the cohomology sheaves).

The subcategory ${}^p\mathcal{A}_{\leq 1} \subset {}^p\mathcal{A}$ *is not closed under subobjects nor quotients.*

Let us illustrate this with an example, for which we thank the referee. For simplicity, let us assume $p = -1$. Suppose $f\colon Y \to X$ is contracting a divisor $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ to a $\mathbb{P}^1$, via projection to the second factor. Let C be a horizontal curve, i.e. $C \simeq \{l\} \times \mathbb{P}^1$ for some $l \in \mathbb{P}^1$, so that we have $f(C) = \mathbb{P}^1$. Take the following short exact sequence:

$$0 \to \mathcal{O}_D(-C) \to \mathcal{O}_D \to \mathcal{O}_C \to 0$$

As $f_*\mathcal{O}_D(-C) = 0$ we see that $\mathcal{O}_D(-C) \in {}^p\mathcal{F}$. As $\mathcal{O}_D$ and $\mathcal{O}_C$ are quotients of $\mathcal{O}_Y$ they both lie in ${}^p\mathcal{T}$. Therefore,

$$0 \to \mathcal{O}_D \to \mathcal{O}_C \to \mathcal{O}_D(-C)[1] \to 0$$

is a short exact sequence in ${}^p\mathcal{A}$ where the middle term lies in ${}^p\mathcal{A}_{\leq 1}$ while the outer terms do not.

To fix the mistake, one needs (among other things) to change the main identity (3.4) of [6]. The element $\mathcal{H}_{\leq 1}$ needs to be replaced by $\mathcal{K}_{\leq 1}$, which corresponds to the subscheme $\mathrm{Hilb}^{\partial}_{\leq 1}(Y) \subset \mathrm{Hilb}_{\leq 1}(Y)$ parameterizing those surjections $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_Z$ in $\mathrm{Coh}(Y)$ whose perverse cokernel lies in ${}^p\mathcal{F}_{\leq 1}[1]$. This is explained in [5, Remark 3.5]. We remark that the perverse cokernel of any surjection $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_Z$ lies in ${}^p\mathcal{F}[1]$.

When X happens to have singular locus of dimension zero, ${}^p\mathcal{F}_{\leq 1} = {}^p\mathcal{F}$ and thus $\mathrm{Hilb}^{\partial}_{\leq 1}(Y) = \mathrm{Hilb}_{\leq 1}(Y)$. However, in general the two are different. Let us illustrate this by means of an example. Suppose $p = -1$ and assume again $f\colon Y \to X$ is contracting a divisor $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ to a $\mathbb{P}^1$. Let as before C be a horizontal curve and let I_C be its ideal sheaf in Y, so that we have a short exact sequence

$$0 \to I_C \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$$

in $\mathrm{Coh}(Y)$. Let now

$$0 \to T \to I_C \to F \to 0$$

be the unique exact sequence with respect to the torsion pair $({}^p\mathcal{T}, {}^p\mathcal{F})$. It follows that $F[1]$ is the perverse cokernel of $\mathcal{O}_Y \to \mathcal{O}_C$.

We also have a short exact sequence

$$0 \to \mathcal{O}_Y(-D) \to I_C \to \mathcal{O}_D(-C) \to 0.$$

Since we have $\mathcal{O}_D(-C) \in {}^p\mathcal{F}$, the surjection $I_C \to \mathcal{O}_D(-C)$ factors through a surjection $F \to \mathcal{O}_D(-C)$. Hence the support of F is two-dimensional.

Similarly, we have two perverse Hilbert schemes: ${}^p\mathrm{Hilb}_{\mathrm{ch}\le 1}$ and ${}^p\mathrm{Hilb}_{\le 1}$. The first parameterizes perverse quotients $\mathcal{O}_Y \twoheadrightarrow E$ with $\mathrm{ch}_0(E) = 0 = \mathrm{ch}_1(E)$, while the second parameterizes quotients $\mathcal{O}_Y \twoheadrightarrow E$ where the dimension of the support of E is at most one. As a consequence, we introduce *partial* invariants DT^∂ by integrating $\mathcal{K}_{\le 1}$ or, equivalently, by taking the weighted Euler characteristic of $\mathrm{Hilb}^\partial_{\le 1}(Y)$.

2. Differences

We proceed here by detailing how [6] was modified to obtain [5] (we will omit a few minor changes which have no impact on the mathematical content).

- For organization purposes, we moved Lemma 3.2 to [5, Lemma 1.4] and added [5, Lemma 1.5]. The latter makes it clear that the subcategory ${}^p\mathcal{F}_{\le 1}[1] \subset {}^p\mathcal{A}_{\le 1}$ is closed under quotients and extensions.
- Section 2 can be left untouched.
- In [5], we added Remark 3.5 to explain why the element $\mathcal{K}_{\le 1}$ is necessary.
- In the identity (3.4), $\mathcal{H}_{\le 1}$ should be replaced by $\mathcal{K}_{\le 1}$, cf. [5, (3.6)].
- The definitions of the stacks $\mathfrak{M}_R$ and $\mathfrak{N}$ need to be changed: one must require in both cases the morphism $\mathcal{O}_Y \to T$ in the displayed diagram to have perverse cokernel in ${}^p\mathcal{F}_{\le 1}[1]$.
- The proof of Proposition 3.5 needs to be modified slightly (by including an application of the snake lemma, cf. [5, Proposition 3.10]).
- In the identity prior to Section 3.6, one should replace $\mathcal{H}_{\le 1}$ by $\mathcal{K}_{\le 1}$.
- In the statement of Proposition 3.15, $\mathcal{H}_{\le 1}$ should be replaced by $\mathcal{K}_{\le 1}$. Similarly, in its proof Hilb should be replaced by Hilb^∂, cf. [5, Proposition 3.21].
- Similarly, in the first identity of Section 3.7, $\mathcal{H}_{\le 1}$ should be replaced by $\mathcal{K}_{\le 1}$.
- Remark 3.21 should be extended to point out that it holds ditto for $\mathrm{DT}^\partial(Y)$.

From here on out, we should introduce the notation $\mathrm{DT}^\partial(Y/X)$ for consistency with $\mathrm{DT}^\partial(Y)$ and $\mathrm{DT}(Y)$, cf. [5, Remark 3.28]. Namely, $\mathrm{DT}(Y/X)$ comes from integrating ${}^p\mathrm{Hilb}_{\mathrm{ch}\le 1}(Y/X)$ while $\mathrm{DT}^\partial(Y/X)$ comes from ${}^p\mathrm{Hilb}_{\le 1}(Y/X)$. From [6, Remark 3.21] onwards, one should replace $\mathrm{DT}(Y/X)$ with $\mathrm{DT}^\partial(Y/X)$. Similarly, $\mathcal{K}_{\le 1}$ should replace $\mathcal{H}_{\le 1}$ in all Hall algebra identities.

Finally, [5] contains a new section where we investigate what happens when we restrict to curve classes contracted by f.

3. One-dimensional singular locus

Recall our general setup.

Situation 3.1. Fix a smooth and projective variety Y of dimension three, over $\mathbb{C}$, with trivial canonical bundle $\omega_Y \cong \mathcal{O}_Y$ and satisfying $H^1(Y, \mathcal{O}_Y) = 0$. Fix a map $f : Y \to X$ satisfying the following properties:

- f is birational and its fibres are at most one-dimensional;
- X is projective and Gorenstein;
- $Rf_*\mathcal{O}_Y = \mathcal{O}_X$.

As we have remarked earlier the category ${}^p\mathcal{A}_{\leq 1}$, consisting of those $E \in {}^p\mathcal{A}$ with $\dim \operatorname{supp} E \leq 1$, is not closed under quotients. This fact is the core reason why we had to replace $\mathcal{H}$ with $\mathcal{K}$ and is what causes the appearance of the "partial" DT numbers.

It is not clear whether there is a compact formula in the Hall algebra relating ${}^p\mathcal{H}$ with $\mathcal{H}$. Any approach seems to involve dealing with surface classes contracted by f. To complicate matters further, the basic identity (in H_∞) $1^{\mathcal{O}}_{{}^p\mathcal{A}} = 1^{\mathcal{O}}_{{}^p\mathcal{F}[1]} * 1^{\mathcal{O}}_{{}^p\mathcal{T}}$ does not even hold in the full Hall algebra: given $E \in {}^p\mathcal{A}$, together with its torsion and torsion-free parts $F[1]$ and T, there is an exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}_Y, F[1]) \to \operatorname{Hom}(\mathcal{O}_Y, E) \to \operatorname{Hom}(\mathcal{O}_Y, T) \to \operatorname{Hom}(\mathcal{O}_Y, F[2])$$

where the last group is equal to $H^2(Y, F) = H^1(X, R^1 f_* F)$. This group vanishes when $F \in {}^p\mathcal{F}_{\leq 1}$ (this assumption, in conjunction with $f_* F = 0$, forces the support of F to be a union of curves contracted to points, hence the support of $R^1 f_*$ is zero-dimensional), but in general it may not be the case.

This being said, let us come to the good news. If we restrict to the subcategory ${}^p\mathcal{A}_{\mathrm{exc}}$ then we can work around these obstacles.

Lemma 3.2. *Let $E \in {}^p\mathcal{A}$ be such that* $\operatorname{ch}_0(E) = \operatorname{ch}_1(E) = 0$ *and* $\dim \operatorname{supp} Rf_* E = 0$. *Then* $\dim \operatorname{supp} E \leq 1$, *in other words $E \in {}^p\mathcal{A}_{\leq 1}$.*

In other words, given $E \in {}^p\mathcal{A}$ with $Rf_* E$ a skyscraper, $E \in {}^p\mathcal{A}_{\leq 1}$ if and only if $\operatorname{ch}_0(E) = \operatorname{ch}_1(E) = 0$ *(i.e. for these complexes, being supported in dimension at most one is a condition on their Chern characters).*

Before we prove this lemma, we recall the key technical result of Van den Bergh [1, Lemmas 3.1.3, 3.1.5].

Lemma 3.3 (Van den Bergh). *Consider the counit morphism $f^* f_* T \to T$. The objects in ${}^{-1}\mathcal{T}$ are precisely those $T \in$* $\operatorname{Coh}(Y)$ *such that $f^* f_* T \to T$ is surjective.*

Given $F \in$ $\operatorname{Coh}(Y)$, *there is a canonical map $\phi_F \colon F \to H^{-1}(f^! R^1 f_* F)$. The objects in ${}^0\mathcal{F}$ are precisely those $F \in$* $\operatorname{Coh}(Y)$ *such that ϕ_F is injective.*

Proof of Lemma 3.2. Let us now prove that a perverse coherent sheaf E satisfying our assumptions is actually supported on a curve. Let $T = H^0(E)$ and $F = H^{-1}(E)$. As $Rf_* E$ is a skyscraper sheaf, it follows that both $f_* T$ and $R^1 f_* F$ are skyscraper sheaves as well.

As $[E] = [T] - [F]$ in $K_0(Y)$ and $\mathrm{ch}_0(E) = \mathrm{ch}_1(E) = 0$, it follows that T is supported in dimension at most one if and only if F is. When the perversity is $p = -1$, we have that $f^{-1}(\operatorname{supp} f_*T) = \operatorname{supp} f^* f_* T \supset \operatorname{supp} T$. But, as f_*T is a skyscraper sheaf and the fibres of f are at most one-dimensional, we have $\dim \operatorname{supp} f^* f_* T \leq 1$ and we can conclude.

When the perversity is $p = 0$, we have

$$\operatorname{supp} F \subset \operatorname{supp} H^{-1}(f^! R^1 f_* F) \subset \operatorname{supp} f^! R^1 f_* F \subset f^{-1}(\operatorname{supp} R^1 f_* F).$$

Again, as $R^1 f_* F$ is a skyscraper sheaf, we are done. □

What truly makes the category ${}^p\mathcal{A}_{\mathrm{exc}}$ robust is the following lemma.

Lemma 3.4. *The category ${}^p\mathcal{A}_{\mathrm{exc}}$ is closed under extensions, quotients and subobjects.*

Proof. Let $A \to B \to C$ be a short exact sequence in ${}^p\mathcal{A}$. First of all, Rf_*B is a skyscraper if and only if both Rf_*A and Rf_*C are skyscraper sheaves. As $[B] = [A] + [C]$ in $K_0({}^p\mathcal{A})$, it follows from Lemma 3.2 that ${}^p\mathcal{A}_{\mathrm{exc}}$ is closed under extensions.

Thus we are left to show that if $B \in {}^p\mathcal{A}_{\mathrm{exc}}$ then $A, C \in {}^p\mathcal{A}_{\mathrm{exc}}$. Consider the long exact sequence of cohomology sheaves.

$$0 \to H^{-1}(A) \to H^{-1}(B) \to H^{-1}(C) \to H^0(A) \to H^0(B) \to H^0(C) \to 0.$$

As $B \in {}^p\mathcal{A}_{\mathrm{exc}}$ it follows that $H^{-1}(A)$ and $H^0(C)$ are both supported in dimension at most one. Thus the only obstruction to concluding is the support of either $H^{-1}(C)$ or $H^0(A)$. However, we know that both $R^1 f_* H^{-1}(C)$ and $f_* H^0(A)$ are skyscraper sheaves. Using Van den Bergh's lemma, we conclude that at least one is (and hence both are) supported in dimension at most one. □

Consequently, Section 3 of [6] can be adapted to the category ${}^p\mathcal{A}_{\mathrm{exc}}$ without needing any "partial" invariants.

Theorem 3.5. *Assume Situation* 3.1. *Then*

$${}^p\underline{DT}_{\mathrm{exc}}(Y/X) = \frac{\underline{DT}^\vee_{\mathrm{exc}} \cdot \underline{DT}_{\mathrm{exc}}(Y)}{DT_0(Y)}$$

holds, where

$${}^p\underline{DT}_{\mathrm{exc}}(Y/X) = \sum_{\substack{\beta,n \\ f_*\beta=0}} \underline{DT}_{Y/X}(\beta,n) q^{(\beta,n)}, \quad \underline{DT}_{Y/X}(\beta,n) = \chi_\mu({}^p\mathrm{Hilb}_{Y/X}(\beta,n)).$$

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