MATH 355 HOMEWORK 8

Problem 1

(a). $\begin{pmatrix} 1 & 0 & -1 \\ 2 & -2 & 3 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3-4 \\ 6+6+12 \\ -9-3+4 \end{pmatrix} = \begin{pmatrix} -1 \\ 24 \\ -8 \end{pmatrix}$

(b). $\begin{pmatrix} 1 & 0 & -1 \\ 2 & -2 & 3 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & -3 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 1 \\ 7 & 4 & -5 \\ -14 & 0 & 0 \end{pmatrix}$

Problem 2

As a linear map, A is determined by what it does to a basis of $\mathbb{R}^3$. The matrix representation

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

tells us exactly that A maps $e_1 \mapsto e_1$, $e_2 \mapsto e_3$, $e_3 \mapsto e_2$. Thus AA maps $e_i \mapsto e_i$, $i = 1, 2, 3$, so $A = A^{-1}$. We can check directly that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Problem 3

Yes. Consider the linear map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$. The matrix representation in the standard basis is

$$\operatorname{Rep}_{\text{std}} f = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

However, choosing $\mathbb{B} = \text{std}_{\mathbb{R}} = \{1\}$ as the basis for $\mathbb{R}$ and $\mathbb{D} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ as the basis for $\mathbb{R}^2$, we have

$$\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

since $f(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Problem 4

(a). Let

$$b_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, b_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

We compute (by solving linear systems)

$$\begin{aligned} f(e_1) &= b_1 - 3b_2 - 2b_3 \\ f(e_2) &= b_1 - 2b_2 - 2b_3 \\ f(e_3) &= -2b_1 + 6b_2 + 5b_3. \end{aligned}$$

Thus by definition,

$$\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = \begin{pmatrix} 1 & 1 & -2 \\ -3 & -2 & 6 \\ -2 & -2 & 5 \end{pmatrix}.$$

The matrix $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f$ by construction tells us exactly how to write vectors in $\mathbb{R}^3$ as linear combinations of elements of $\mathbb{D}$:

$$\begin{pmatrix} 1 & 1 & -2 \\ -3 & -2 & 6 \\ -2 & -2 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \implies \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = b_2$$

$$\begin{pmatrix} 1 & 1 & -2 \\ -3 & -2 & 6 \\ -2 & -2 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \implies \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = -b_2 + b_3$$

$$\begin{pmatrix} 1 & 1 & -2 \\ -3 & -2 & 6 \\ -2 & -2 & 5 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ -8 \end{pmatrix} \implies \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = 5b_1 - 12b_2 - 8b_3.$$

Problem 5

(a). If f is an isomorphism, then in particular f is injective, so $\ker f = 0$. On the other hand, if $\ker f = 0$, then f is injective, and the dimension formula gives

$$\dim W = \dim V = \dim \ker f + \dim \operatorname{Im} f = \dim \operatorname{Im} f,$$

so f is surjective. Since f is injective and surjective, it is an isomorphism.

(b). This is false. A counterexample is $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$.

(c). If f is an isomorphism, then in particular f is surjective, so $\operatorname{Im} f = W$. On the other hand, if $\operatorname{Im} f = W$, then $\dim \operatorname{Im} f = \dim W$, so the dimension formula gives

$$\begin{aligned} \dim V &= \dim \ker f + \dim \operatorname{Im} f \\ \dim W &= \dim \ker f + \dim \operatorname{Im} f \\ \dim \operatorname{Im} f &= \dim \ker f + \dim \operatorname{Im} f \\ 0 &= \dim \ker f, \end{aligned}$$

so f is injective, hence an isomorphism.

(d). This is false. A counterexample is $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x$.