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Let us fix an abstract vector space V. Today we discussed when a subset contained in V, $W \subset V$ behaves well with respect to the vector space structure.

Definition. A subset $W \subset V$ is a (vector) *subspace* if the following three conditions are met:

(1) $\vec{0} \in W$
(2) if $\vec{v}, \vec{w} \in W$ then $\vec{v} + \vec{w} \in W$
(3) if $a \in \mathbb{R}$, $\vec{w} \in W$ then $a\vec{w} \in W$.

The best way to understand what this means, is to see a lot of examples (the book has MORE!).

- $\mathbb{R}^7$ is a subspace if $\mathbb{R}^7$ (why?)
- Recall that $\vec{0} \in \mathbb{R}^7$ means $\vec{0} = (0,0,0,0,0,0,0)$. The subset $\{\vec{0}\}$ is a subspace of $\mathbb{R}^7$ (why?)
- In any vector space V, $\{\vec{0}\}$ is a subspace of V (why?).
- The subset $\{(3,5,7)\}$ is *not* a subspace of $\mathbb{R}^3$ (why? it does not contain zero!)
- The subset $\{(x,y,0,t) \mid x,y,t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$ (why?)
- The subset $\{(x,y,17,t) \mid x,y,t \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^4$ (why?)
- The subset $\{(x,y,2,t) \mid x,y,t \in \mathbb{R}\} \cup \{(x,y,-2,t) \mid x,y,t \in \mathbb{R}\}$ is *not* a subspace.

Here is the last example we saw in class. Consider $W = \{(x,y) \mid x \in \mathbb{R}, y \geq 0\}$. Is W a subspace of $\mathbb{R}^2$? Before reading the answer, draw a picture of W.

- $\vec{0} \in W$

indeed, $\vec{0} = (0,0)$ and $0 \geq 0$.

- if $\vec{v}, \vec{w} \in W$ then $\vec{v} + \vec{w} \in W$

indeed, $\vec{v} = (v_1, v_2) \in W$ means $v_2 \geq 0$. Similarly, $\vec{w} = (w_1, w_2) \in W$ means $w_2 \geq 0$. In turn, $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2)$ and $v_2 + w_2 \geq 0$. So $\vec{v} + \vec{w} \in W$.

- if $a \in \mathbb{R}$ and $\vec{w} \in W$, does $a\vec{v} \in W$?

Well sometimes yes, but NOT ALWAYS!

To disprove something, you need to exhibit a COUNTEREXAMPLE!

Let us exhibit a counterexample. Take $\vec{w} = (3,1)$. Since $1 \geq 0$, $\vec{w} \in W$. Let $a = -1$, then $-\vec{w} = (-3,-1)$ but $-1 < 0$ so $-\vec{w}$ does not belong to W. Hence, W is not a subspace of $\mathbb{R}^2$.

Here is another weird example. Consider $W = \{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}$. This means W consists of vectors which are *either* of the form $(x,0)$ or of the form $(0,y)$. Is W a subspace of $\mathbb{R}^2$? Before reading the answer, try to draw a picture of W.

- $\vec{0} \in W$

indeed, $(0,0)$ is of the form $(x,0)$.

- if $a \in \mathbb{R}$ and $\vec{v} \in W$, then $a\vec{v} \in W$

indeed, fix $a \in \mathbb{R}$. If $\vec{v}$ is of the form $(x,0)$ then $a\vec{v} = (ax,0)$ which belongs to W. If $\vec{v}$ is of the form $(0,y)$ then $a\vec{v} = (0,ay)$ which belongs to W. Thus, W is closed under multiplication by scalars?

- Is W closed under addition? I.e. if $\vec{v}, \vec{w} \in W$ does $\vec{v} + \vec{w} \in W$?

No. Let us exhibit a counterexample. Consider $\vec{v} = (1,0) \in W$ and $\vec{w} = (0,1) \in W$. Then $\vec{v} + \vec{w} = (1,1)$. But $(1,1) \notin W$ as it is neither of the form $(x,0)$ nor $(0,y)$. Hence, W is not a subspace of $\mathbb{R}^2$.

Perhaps now that we have developed this vocabulary, it is useful to revisit an example we saw friday. Let $V = P_{\leq 3}$ be the vector space of polynomials of degree *at most* 3. Consider $W = \{0\} \cup P_{=3}$ where

$$P_{=3} = \{p(x) \in P_{\leq 3} \mid \deg p(x) = 3\}$$

Is W a subspace of $P_{\leq 3}$?

Well, $0 \in W$ by fiat (the very definition of W is $P_{=3}$ together with 0). However, W is not closed under addition. Indeed, consider $\vec{v} = x^2 - x^3$ and $\vec{w} = x^3$. Then $\vec{v}, \vec{w} \in W$ as $\deg \vec{v} = 3$ and $\deg w = 3$. However, $\vec{v} + \vec{w} = x^2 - x^3 + x^3 = x^2 \notin W$ as $\deg x^2 = 2 \neq 3$ and $x^2 \neq 0$.