MATH355 2017-09-22

DIMENSION IS SERIOUSLY USEFUL

Recall that the dimension of V is the number of vectors in any basis.

Example 1. $\dim \mathbf{R}^2 = 2$. Indeed $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ is a basis, and it is made up of two vectors.

Example 2. $\dim \mathbf{R}^n = n$. Indeed, $\vec{e}_1, \ldots, \vec{e}_n$ is a basis (this is called the *standard* basis or the *canonical* bais).

Example 3. $\dim P_{\leq 3} = 4$. Recall $P_{\leq 3} = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbf{R}\}$. A basis consists of $\{1, x, x^2, x^3\}$ (why?).

Example 4. $\dim P_{\leq n} = n + 1$. Indeed a basis is given by $\{1, x, x^2, \ldots, x^n\}$.

Example 5. What is $\dim M_{3\times 2}$? Recall that

$$M_{3\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \middle| a, b, c, d, e, f \in \mathbf{R} \right\}.$$

A basis is given by $\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\\0 & 1\end{pmatrix}$

Why?

Example 6. In general, $\dim M_{m\times n} = mn$.

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It is a good exercise to try and prove the following facts (see also the textbook).

Fact 7. Any set of linearly independent vectors can be extended to a basis.

Example 8. Suppose $\vec{u}, \vec{v}, \vec{w} \in \mathbf{R}^5$ are linearly independent. Then we can always find $\vec{s}, \vec{t} \in \mathbf{R}^5$ such that $\vec{u}, \vec{v}, \vec{w}, \vec{s}, \vec{t}$.

Fact 9. Any spanning set can be shrunk to a basis.

Example 10. Take $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. Check that $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \mathbf{R}^3$. But $\vec{v}_1, \ldots, \vec{v}_4$ is *not* a basis. (why?) However $\vec{v}_1, \vec{v}_3, \vec{v}_4$ is a basis, and so is $\vec{v}_2, \vec{v}_3, \vec{v}_4$ (why?).

Fact 11. If $W < V$ is a subspace, then $\dim W \leq \dim V$.

Write $k = \dim W$, and $n = \dim V$. The Fact above says that $0 \leq k \leq n$. It's useful to see the extreme cases.

$$\dim W = 0 \text{ if and only if } W = \{\vec{0}\}$$
$$\dim W = n \text{ if and only if } W = V$$

The following result is extremely important.

Lemma 12. Say $\dim V = n$, say $\vec{v}_1, \ldots, \vec{v}_n \in V$. Then

$$\vec{v}_1, \ldots, \vec{v}_n \text{ are linearly independent}$$

if and only if

$$\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = V.$$

Concretely, this means that (*once you know the dimension!*) to check a set of vectors is a basis you only need to check one of the two conditions.

Corollary 13. Say $\dim V = n$, say $\vec{v}_1, \ldots, \vec{v}_n \in V$.
If $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, then they form a basis.
If $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = V$, then they form a basis.

Example 14. Are $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $1\vec{2}1$, $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ a basis for $\mathbf{R}^3$? Well, $\dim \mathbf{R}^3 = 3$ so it suffices to check whether they are linearly independent.

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\2\\1\end{pmatrix} + z\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}x+y+z\\x+2y\\x+y-z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Write the matrix corresponding to the system of equations

$$\begin{pmatrix}1&1&1\\1&2&0\\1&1&-1\end{pmatrix}$$

whose echelon form is

$$\begin{pmatrix}1&1&1\\0&1&-1\\0&0&-2\end{pmatrix}$$

so the only solution is the trivial solution, hence the three vectors are linear independent. By the Lemma, we deduce they form a basis of $\mathbf{R}^3$.

Once again: we had to know beforehand that $\dim \mathbf{R}^3 = 3$!

Example 15. Are $\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}-3\\4\end{pmatrix}$ linearly independent? Of course not! Why? If they were, this would mean that $\dim \mathbf{R}^2 \geq 3$, but $\dim \mathbf{R}^2 = 2$!

Let us generalize the previous example and state it as a fact.

Fact 16. Say $\dim V = n$, say $\vec{w}_1, \ldots, \vec{w}_k$ are linearly independent vectors. Then $k \leq n$.

We now want to relate what we've learned so far, to systems of linear equations. Say

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

is a *homogeneous* system of equations. First off,

Sol $< \mathbf{R}^n$ is a subspace.

Notice that n is the number of variables $x_1, \ldots, x_n$. But what is the dimension of Sol?

$$\boxed{\dim \text{Sol} = \text{number of free variables}}$$

Example 17. Consider the system

$$\begin{cases} -3x + 6y + z + s = 0 \\ 5z - 5s + 3t = 0 \\ 6s = 0 \end{cases}$$

With corresponding matrix

$$\begin{pmatrix} -3 & 6 & 1 & 1 & 0 \\ 0 & 0 & 5 & -5 & 3 \\ 0 & 0 & 0 & 6 & 0 \end{pmatrix}$$

which is conveniently already in echelon form. Observe, x, y, s are leading variables, therefore y, t are free. Hence, $\dim \mathrm{Sol} = 2$.

Example 18. Continuing from the example above, let us write a basis for Sol. First, we need to describe it.

$$\begin{aligned} \mathrm{Sol} &= \left\{ \begin{pmatrix} 2y - \frac{1}{5}t \\ y \\ -\frac{3}{5}t \\ t \\ 0 \end{pmatrix} \middle| y, t \in \mathbf{R} \right\} \\ &= \left\{ y\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{5} \\ 0 \\ -\frac{3}{5} \\ 1 \\ 0 \end{pmatrix} \middle| y, t \in \mathbf{R} \right\} \end{aligned}$$

How do you know that a basis for Sol is given by

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} \\ 0 \\ -\frac{3}{5} \\ 1 \\ 0 \end{pmatrix}?$$

Well, it's two vectors which span a two dimensional subspaces, hence they must be linearly independent, by Fact 12!

Proofs

In class, we did not have time to explain why the Facts listed above are true. I recommend you give it a shot on your own, before reading below.

Before we begin, recall the following fact mentioned a while ago.

Fact 19. Suppose $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent. Let $\vec{v}_{k+1} \in V$. Then $\vec{v}_1, \ldots, \vec{v}_{k+1}$ are linearly independent if and only if $\vec{v}_{k+1} \notin \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$.

Proof. Suppose $\vec{v}_1, \ldots, \vec{v}_{k+1}$ are linearly independent. If you wrote $\vec{v}_{k+1} = \sum_{i=1}^{k} a_i \vec{v}_i$, then $\vec{0} = \left(\sum_{i=1}^{k} a_i \vec{v}_i\right) - \vec{v}_{k+1}$. By linear independence, we must have that all coefficients are zero: so $a_i = 0$ for all i, but also $-1 = 0$. Which is absurd.

Conversely, assume $\vec{v}_{k+1} \notin \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. Suppose $\sum_{i=1}^{k+1} a_i \vec{v}_i = \vec{0}$. Then $-a_{k+1}\vec{v}_{k+1} = \sum_{i=1}^{k} a_i \vec{v}_i$. If $a_{k+1} = 0$, then $a_i = 0$ for all i, by linear independence. If $a_{k+1} \neq 0$, then $\vec{v}_{k+1} = \sum_{i=1}^{k} -\frac{a_i}{a_{k+1}} \vec{v}_i$. But then $\vec{v}_{k+1} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, which contradicts our assumption. □

OK, let's move on to proving the other stuff. The first two proofs are more "proof-sketches" than rigorous proofs, but I think they convey better the idea of why things work.

Proof of Fact 7. We will give an algorithm to extended bases.

Step 1. Suppose $\vec{v}_1, \ldots, \vec{v}_k \in V$ are linearly independent. Let $W := \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$.

Step 2. If $W = V$, then they were a basis to begin with.

If not, $W \subsetneq V$, and there must be $\vec{v}_{k+1} \notin W$. Since $\vec{v}_{k+1}$ does not belong to the span of the first k vectors, $\vec{v}_1, \ldots, \vec{v}_{k+1}$ are all linearly independent (by Fact 19 above).

Start over from Step 1, but now with one more vector.

The process terminates since V is assumed to be finite-dimensional to begin with. □

Proof of Fact 9. Let $S \subset V$ with $\operatorname{Span} S = V$. If all vectors $\vec{v} \in S$ are zero, then $V = \{\vec{0}\}$ and we are done.

If not, pick $\vec{v}_1 \in S, \vec{v}_1 \neq 0$. If $\operatorname{Span}\{\vec{v}_1\} = V$ we are done.

If not, there must be $\vec{v}_2 \in S$, with $\vec{v}_1, \vec{v}_2$ linearly independent. Indeed, if such $\vec{v}_2$ did not exist, this means all $\vec{v} \in S$ belong to $\operatorname{Span}\{\vec{v}_1\}$. [why?]. But so $\operatorname{Span}\{\vec{v}_1\} \supset \operatorname{Span}\{S\} = V$, which is a contradiction.

If now $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = V$ we are done. If not, there must be $\vec{v}_3 \in S$ with $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent. Indeed, if this weren't the case, then all other $\vec{v}$ would belong to $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. This means $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \supset \operatorname{Span} S = V$. Contradiction.

If now $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = V$ we are done. If not, there must be $\vec{v}_4 \in S$ with...

The process terminates as V is assumed to be finite-dimensional. □

Proof of Fact 11. Let $W < V$ be a subspace. Let $n = \dim V$, $k = \dim W$. By definition of dimension, there is a basis of W made up of $\vec{w}_1, \ldots, \vec{w}_k$. But these vectors are linearly independent, so we may complete them to a basis of V. Since we are adding vectors to the list, this means $\dim V \geq k$. □

Proof of the Box below Fact 11. Recall that if $\vec{v} \in W$, then $\{\vec{v}\}$ is linearly independent if and only if $\vec{v} \neq 0$. If $\dim W = 0$, it means that no vector is linearly independent (otherwise $\dim W \geq 1$). Hence, all vectors are zero.

Suppose $\dim W = \dim V$. If $W \subsetneq V$, there must be $\vec{v} \notin W$. Pick a basis $\vec{w}_1, \ldots, \vec{w}_n$ of W. Since $\vec{v} \notin W = \operatorname{Span}\{\vec{w}_1, \ldots, \vec{w}_n\}$, this implies $\vec{w}_1, \ldots, \vec{w}_n, \vec{v}$ are linearly independent. This means $\dim V > n = \dim W$. Contradiction. □

I will leave the proof of Lemma 12 and Fact 16 as an exercise. Corollary 13 is obvious, once you know Lemma 12.