MATH355 2017-09-15

LINEARLY DEPENDENT AND INDEPENDENT VECTORS: WHAT DO THEY KNOW? DO THEY KNOW THINGS? LET'S FIND OUT!

Example 1. Let

$$W := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x, y, z \in \mathbf{R}, x + y + z = 0 \right\}$$

we can rewrite

$$W = \left\{ \begin{pmatrix} y - z \\ y \\ z \end{pmatrix} \middle| y, z \in \mathbf{R} \right\} = \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| y, z \in \mathbf{R} \right\}$$

Call $\vec{v}_1 := \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, then

$$W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}.$$

Example 2. The xy-plane $P \subset \mathbf{R}^3$ is also a span. Indeed,

$$P := \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x, y \in \mathbf{R} \right\} = \operatorname{Span}\{\vec{e}_1, \vec{e}_2\}.$$

Example 3. Recall $P_{\leq 2} = \{ax^2 + bx + c \mid a, b, c \in \mathbf{R}\}$ is the space of polynomials of degree at most two. We have

$$P_{\leq 2} = \operatorname{Span}\{1, x, x^2\} \text{ (why?)}$$

Inside $P_{\leq 2}$, we can define

$$W := \{ax^2 + bx + c \mid a, b, c \in \mathbf{R}, a + b + c = 0\}$$

Question: is W a subspace? The answer is... (think about it first)... (ready?) ... (ok, here it comes) ... yes. In fact, we will write it as a span. Indeed,

$$W = \{(-b - c)x^2 + bx + c \mid b, c \in \mathbf{R}\} = \{b(-x^2 + x) + c(-x^2 + 1) \mid b, c \in \mathbf{R}\}$$

so

$$W = \operatorname{Span}\{-x^2 + x, -x^2 + 1\}$$

Actually, if you think about it, this is the same as example 1. The link between the two is matching $\vec{e}_1$ with x^2, $\vec{e}_2$ with x and $\vec{e}_3$ with the polynomial 1.

The slogan to keep in mind is:

Subspaces are spans.

Date: John Calabrese, September 18, 2017.

Example 4. Consider $\mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\} \subset \mathbf{R}^2$. Since

$$\begin{pmatrix}2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix}$$

we have that $\begin{pmatrix}2\\2\end{pmatrix} \in \mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}\}$, therefore

$$\mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\} = \mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}\}$$

Example 5. We can also reverse the example from before. Since $\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\2\end{pmatrix}$, we have

$$\mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\} = \mathrm{Span}\{\begin{pmatrix}2\\2\end{pmatrix}\}$$

Exercise 6. Show that

$$\mathrm{Span}\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\} = \mathrm{Span}\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\}$$

What we don't like about the previous three examples is the *redundancy* of vectors. Having *minimal* spanning sets would be much better.

******* Take a break. *******

Definition 7. The vectors $\vec{v}_1, \dots, \vec{v}_k \in V$ are *linearly independent* if whenever

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$$

then we must have

$$a_1 = 0, a_2 = 0, \dots, a_k = 0.$$

In other words,

> $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent is if the only way to write $\vec{0}$ is with the stupid linear combination.

Definition 8. Conversely, $\vec{v}_1, \dots, \vec{v}_k$ are *linearly dependent* if they are not independent.

Concretely, if $\vec{v}_1, \dots, \vec{v}_k$ are linearly *dependent*, then we may find $a_1, \dots, a_k \in \mathbf{R}$ such that

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0}$$

with at least one j such that $a_j \neq 0$.

Example 9. Suppose $\vec{v}_1 - \vec{v}_2 + 5\vec{v}_3 = \vec{0}$. This means that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent. But what is really cool about this is the ability to express some vectors in terms of the others. For example

$$\vec{v}_2 = \vec{v}_1 + 5\vec{v}_3$$

or

$$\vec{v}_1 = -5\vec{v}_3 + \vec{v}_2$$

or

$$\vec{v}_3 = \frac{1}{5}\vec{v}_2 - \frac{1}{5}\vec{v}_1.$$

Example 10. The vectors $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent. How do we show this? Well, suppose

$$a_1\begin{pmatrix}1\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix} = \vec{0}$$

this means

$$\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

but then $a_1 = 0 = a_2$. Therefore, $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent.

Example 11. More generally, the vectors $\vec{e}_1, \ldots, \vec{e}_n \in \mathbf{R}^n$ are linearly independent, where

$$\vec{e}_1 = \begin{pmatrix}1\\0\\0\\0\\\vdots\\0\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}0\\1\\0\\0\\\vdots\\0\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}0\\0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix}0\\0\\0\\\vdots\\0\\1\end{pmatrix}$$

Example 12. Are the vectors

$$\vec{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 0 \\ 1 \\ -3 \\ \frac{1}{2} \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

linearly dependent or independent? I don't know, let's find out! Suppose we have a linear combination

$$a\vec{u} + b\vec{v} + c\vec{w}\vec{0}$$

this means

$$\begin{pmatrix} 2a + b0 + c0 \\ a0 + b + c0 \\ a - 3b + c0 \\ -a + \frac{b}{2} + 5c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

therefore

$$2a = 0, b = 0, a - 3b = 0, -a + \frac{b}{2} + 5c = 0$$

hence we must have

$$a = 0 = b = c$$

thus $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

Example 13. The vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are linearly dependent. Indeed,

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is a non-trivial linear combination.

In general, if a set of vectors contains $\vec{0}$, that set will be linearly dependent.

Example 14. Consider $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \vec{w} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$. Does $\{\vec{u}, \vec{v}\}$ have the same span as $\{\vec{u}, \vec{v}, \vec{w}\}$? Well, if $\vec{w}$ is in the span of $\vec{u}, \vec{v}$ the answer is yes. Otherwise, it's no. Since

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

we have $\vec{w} \in \text{Span}\{\vec{u}, \vec{v}\}$. Thus $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent and the two spans coincide.

This fact was mentioned in class.

Fact 15. Any subset of linearly independent vectors is linearly independent.

Concretely, suppose $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent vectors. Let $\vec{v}_1, \ldots, \vec{v}_r$ with $r \leq k$. Then $\vec{v}_1, \ldots, \vec{v}_r$ are also linearly independent.

Proof. Let us show $\vec{v}_1, \dots, \vec{v}_r$ are linearly independent. We write

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_r\vec{v}_r = \vec{0}$$

and we would like to show that $a_1 = 0 = a_2 = \cdots = a_r$. We extend that linear combination to

$$\vec{0} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_r\vec{v}_r + 0\vec{v}_{r+1} + 0\vec{v}_{r+2} + \cdots + 0\vec{v}_k$$

which is a linear combination of $\vec{v}_1, \dots, \vec{v}_k$. Since $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, we must have $a_1 = 0 = a_2 = \cdots = a_r$, as wanted. □

Remark 16. Notice that the order doesn't really matter. For example, if $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent, then so will $\vec{v}_2, \vec{v}_4$ but also $\vec{v}_1, \vec{v}_4, \vec{v}_3$ and all other possible combinations.

Finally, we introduced the notion of basis.

Definition 17. A *basis* of a vector space V is a set S of linearly independent vectors which span V.

The slogan is:

basis = linearly independent + span.

Concretely, $\vec{v}_1, \dots, \vec{v}_n$ are a basis of V if

- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$
- $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

Example 18. The vectors $\vec{e}_1, \dots, \vec{e}_n$ are a basis of $\mathbf{R}^n$.

Exercise 19. The vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are a basis of $\mathbf{R}^2$.