A REMARK ON GENERATORS OF $\mathtt{D}(\mathrm{X})$ AND FLAGS

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Abstract. — We give a simple proof of the following fact. Let X be an n-dimensional, smooth, projective variety with ample or anti-ample canonical bundle, over an algebraically closed base field. Let $\mathrm{Y}_0 \subset \mathrm{Y}_1 \subset \cdots \subset \mathrm{Y}_n = \mathrm{X}$ be a complete flag of closed smooth subvarieties. Then $\mathrm{G} = \bigoplus_{j=0}^{n} \mathcal{O}_{\mathrm{Y}_j}$ is a generator of the (bounded coherent) derived category $\mathtt{D}(\mathrm{X})$. Moreover, from the endomorphism dg-algebra $\mathrm{REnd}_{\mathrm{X}}(\mathrm{G})$ one can recover not only X but also the flag $\mathrm{Y}_0 \subset \mathrm{Y}_1 \subset \cdots \subset \mathrm{Y}_n$.

1. Introduction

There is a remarkable series of papers exploring the connection between marked curves and A_∞-algebras [**Pol11**, **Fis11**, **LP12**, **FP14**, **Pol13**, **LP14**, **Pol15**, **Pol16b**, **Pol16a**]. A very coarse summary goes as follows. Let C be a (smooth projective) curve of genus g and $p_1, \ldots, p_n$ a collection of points. Let $\mathrm{G} = \mathcal{O}_{\mathrm{C}} \oplus \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_n}$, viewed as an object of the derived category $\mathtt{D}(\mathrm{C})$. With G, one can associate two objects: a graded algebra and a differential graded algebra (dg-algebra for short). The former is the Ext-algebra $\mathrm{E} = \mathrm{Ext}^*_{\mathrm{X}}(\mathrm{G}, \mathrm{G})$. The latter is the dg-endomorphism algebra $\mathrm{A} = \mathrm{REnd}_{\mathrm{X}}(\mathrm{G})$, defined using a dg-model of $\mathtt{D}(\mathrm{C})$. By taking cohomology, we have $\mathrm{H}^*(\mathrm{A}) = \mathrm{E}$. The rough idea is that, as the marked curve $(\mathrm{C}, p_1, \cdots, p_n)$ varies in $\mathrm{M}_{g,n}$, E stays constant while A changes.

Using homological perturbation, one can replace the dg-algebra A by a quasi-isomorphic minimal A_∞-algebra. This means that instead of having a cochain complex whose cohomology is E, we equip E itself with higher multiplications m_i. Minimal means $m_1 = 0$.

One then considers M_{E}, the moduli of minimal A_∞-structures over E (up to equivalence). It turns out that the map $\mathrm{M}_{g,n} \to \mathrm{M}_{\mathrm{E}}$ is very interesting. In some cases it provides a *modular* (in the geometric sense) compactification.

Let us go back to the fixed marked curve $(\mathrm{C}, p_1, \ldots, p_n)$ and the dg-algebra A. Let $\mathtt{D}(\mathrm{A})$ be the derived category of dg-A-modules. One can show that G is a generator of $\mathtt{D}(\mathrm{C})$, hence $\mathtt{D}(\mathrm{A})$ is equivalent to $\mathtt{D}(\mathrm{C})$. One can then appeal to, for example [**Ber07**], and recover C from A. However, the results discussed above imply that more is true: from the dg-algebra A one should recover also the configuration of points $p_1, \ldots, p_n$.

It is not obvious how the dg-algebra intrinsically recovers the marked curve $(\mathrm{C}, p_1, \dots, p_n)$. The papers cited above are quite lengthy and technical and there does not appear to be a conceptual explanation for why such a thing should be true. This short note aims to fill precisely this gap.

Conventions. — We work over a fixed field $\mathtt{k}$. All algebras and schemes are assumed to be over $\mathtt{k}$. If X is a scheme, we write $\mathtt{D}(\mathrm{X})$ for its bounded derived category of coherent sheaves. If A is a dg-algebra, we write $\mathtt{D}(\mathrm{A})$ for the bounded derived category of finitely generated right dg-A-modules. All functors, with the exception of global Homs, will be implicitly derived. More precisely, we write $\mathrm{Hom}_{\mathrm{X}}$ for morphisms in the derived category $\mathtt{D}(\mathrm{X})$, while $\mathrm{RHom}_{\mathrm{X}}$ denotes the whole chain complex. In other words $\mathrm{H}^0(\mathrm{RHom}_{\mathrm{X}}) = \mathrm{Hom}_{\mathrm{X}}$.

2. Generators

Let X denote a smooth proper scheme over $\mathtt{k}$ of dimension n. Let $\mathrm{Y}_0 \subset \mathrm{Y}_1 \subset \dots \subset \mathrm{Y}_n = \mathrm{X}$ be a nested sequence of smooth subvarieties. Assume the difference $\mathrm{Y}_j \smallsetminus \mathrm{Y}_{j-1}$ is a dense affine open subset of Y_j (which forces Y_{j-1} to be of pure codimension one inside inside Y_j).

Proposition 2.1. — The object $\mathrm{G} = \bigoplus_{j=0}^{n} \mathcal{O}_{\mathrm{Y}_j}$ generates $\mathtt{D}(\mathrm{X})$.

To be precise, by generating $\mathtt{D}(\mathrm{X})$ we mean the following. Let A be the endomorphism dg-algebra $\mathrm{REnd}_{\mathrm{X}}(\mathrm{G})$. There is a functor $\Phi\colon \mathtt{D}(\mathrm{X}) \to \mathtt{D}(\mathrm{A})$ given by sending E to $\mathrm{RHom}_{\mathrm{X}}(\mathrm{G}, \mathrm{E})$, with left adjoint Ψ given by sending M to $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{G}$. We say G generates if Φ is an equivalence. It is well know that the composition $\Phi\Psi$ is the identity as

$$\mathrm{RHom}_{\mathrm{X}}(\mathrm{G}, \mathrm{M} \otimes_{\mathrm{A}} \mathrm{G}) = (\mathrm{M} \otimes_{\mathrm{A}} \mathrm{G}) \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{G}^{\vee} = \mathrm{M} \otimes_{\mathrm{A}} \mathrm{RHom}_{\mathrm{X}}(\mathrm{G}, \mathrm{G}) = \mathrm{M} \otimes_{\mathrm{A}} \mathrm{A} = \mathrm{M}$$

hence Ψ is fully faithful.

Proof. — It suffices to show that, for any $\mathrm{E} \in \mathtt{D}(\mathrm{X})$, if $\mathrm{RHom}_{\mathrm{X}}(\mathrm{G}, \mathrm{E}) = 0$ then $\mathrm{E} = 0$. We shall prove this by induction on the dimension of X.

When $\dim \mathrm{X} = 0$ this is obvious as X is affine: $0 = \mathrm{RHom}_{\mathrm{X}}(\mathrm{G}, \mathrm{E}) = \mathrm{RHom}_{\mathrm{X}}(\mathcal{O}_{\mathrm{X}}, \mathrm{E}) = \mathrm{E}$. Suppose the theorem is true in dimension $n-1$ and assume $\dim \mathrm{X} = n$. Let $\mathrm{Y} = \mathrm{Y}_{n-1}$ and write $i\colon \mathrm{Y} \to \mathrm{X}$ for the inclusion. Notice that $\mathrm{G} = \mathcal{O}_{\mathrm{X}} \oplus i_* \mathrm{F}$ where F is a generator of Y (by the inductive assumption). Let $\mathrm{E} \in \mathtt{D}(\mathrm{X})$ and suppose $\mathrm{RHom}_{\mathrm{X}}(\mathrm{G}, \mathrm{E}) = 0$. Then $0 = \mathrm{RHom}_{\mathrm{X}}(i_* \mathrm{F}, \mathrm{E}) = \mathrm{RHom}_{\mathrm{X}}(\mathrm{F}, i^! \mathrm{E})$ which implies $i^! \mathrm{E} = 0$. It follows that $\mathrm{supp}\, \mathrm{E} \subset \mathrm{X} \smallsetminus \mathrm{Y} = \mathrm{U}$. Let $\mathrm{Z} = \mathrm{supp}\, \mathrm{E}$, which is closed in X. Write $\mathrm{V} = \mathrm{X} \smallsetminus \mathrm{Z}$ and write $j\colon \mathrm{U} \to \mathrm{X}$, $h\colon \mathrm{V} \to \mathrm{X}$, $k\colon \mathrm{U} \cap \mathrm{V} \to \mathrm{X}$ for the inclusions. We have the following Mayer–Vietoris triangle

$$\mathrm{E} \to j_* j^* \mathrm{E} \oplus h_* h^* \mathrm{E} \to k_* k^* \mathrm{E} \to \mathrm{E}[1].$$

Since $\mathrm{supp}\, \mathrm{E} \subset \mathrm{U}$ we have $j^* \mathrm{E} = 0 = k^* \mathrm{E}$, hence $\mathrm{E} \simeq j_* j^* \mathrm{E}$. But now we may use $0 = \mathrm{RHom}_{\mathrm{X}}(\mathcal{O}_{\mathrm{X}}, \mathrm{E}) = \mathrm{RHom}_{\mathrm{U}}(\mathcal{O}_{\mathrm{U}}, j^* \mathrm{E})$ which implies (as U is affine) $j^* \mathrm{E} = 0$. Hence the claim follows. $\square$

As an immediate corollary we have that, if I_j denotes the ideal sheaf of Y_j, then $\bigoplus_{j=0}^{n} \mathrm{I}_j$ is a generator of $\mathtt{D}(\mathrm{X})$.

3. Algebras

We will now start with an abstract algebra and define a space, together with a chain of subsets of its $\mathtt{k}$-points. Let A be a smooth and proper dg-algebra and let R be an ordinary $\mathtt{k}$-algebra. Recall [**BO01**, **Cal16**] that an object $P \in \mathtt{D}(A \otimes_{\mathtt{k}} R)$ is a *Bondal–Orlov* point if the following are true.

- The natural map $R \to \mathrm{Hom}_{A \otimes_{\mathtt{k}} R}(P, P)$ is an isomorphism.
- For all $i < 0$, $\mathrm{Hom}_{A \otimes_{\mathtt{k}} R}(P, P[i]) = 0$.
- If R is a field, there exists an integer m and an isomorphism $\Sigma(P) \cong P[m]$.

Here Σ is the Serre functor, which exists as A was assumed to be smooth and proper. We say P is a *universal* Bondal–Orlov point if, for any $R \to R'$, $P \otimes R'$ is a Bondal–Orlov point.

We define the functor $X'_A \colon \mathtt{Alg}(\mathtt{k}) \to \mathtt{Set}$ from (ordinary) $\mathtt{k}$-algebras to sets as

$$X'_A(R) = \{P \in \mathtt{D}(A \otimes_{\mathtt{k}} R) \mid P \text{ is a universal Bondal–Orlov } \} / \sim$$

where $P \sim P'$ if there exists a line bundle L over R such that $P \otimes_R L \cong P'$ in $\mathtt{D}(A \otimes_{\mathtt{k}} R)$.

Given a BO-point P over R, we may forget the dg-structure and just view it as a complex in $\mathtt{D}(R)$. As such it has cohomology R-modules $H^i_f(P)$. We define $X_A \subset X'_A$ as the subfunctor parameterizing those P such that

$$H^i_f(P) \begin{cases} = 0 & \text{if } i < 0 \\ \neq 0 & \text{if } i = 0. \end{cases}$$

Now we can define our flag. Consider the following subsets of $X_A(\mathtt{k})$.

$$X_{A,i} = \left\{ P \in X_A(\mathtt{k}) \mid \dim_{\mathtt{k}} H^0_f(P) \geq i \right\}.$$

Finally, we can prove our remark.

Theorem 3.1. — Assume X is a smooth and projective variety with ample or anti-ample canonical bundle. Assume $\mathtt{k}$ is algebraically closed. Let $Y_0 \subset \cdots \subset Y_n = X$ be a complete flag as in the previous section. Let G and A be the corresponding generator and dg-algebra. Then $X_A = X$ and $X_{A,n-j+1} = Y_j$.

Proof. — Both assertions are consequences of the Bondal–Orlov theorem (see [**Cal16**] where this moduli theoretic point of view is spelled out). Explicitly, recall that any $P \in X_A(\mathtt{k})$ is of the form $P = \mathcal{O}_p[j]$ for p a closed point of X. As an A-module, P is given by $\mathrm{RHom}_X(G, P)$. In particular, $H^{-j}_f(P) = \mathrm{Hom}_X(G, \mathcal{O}_p)$ (notice that this is very different from the cohomology *sheaf* $H^{-j}(P)$). More generally, we see that $P \in X_A(\mathtt{k})$ if and only if $j = 0$. Hence, the functor X_A is indeed isomorphic to X (again, see [**Cal16**]).

Notice that $\mathrm{Hom}_X(\mathcal{O}_{Y_j}, \mathcal{O}_p) = \mathtt{k}$. It then follows that $X_{A,1} = Y_n(\mathtt{k})$, $X_{A,2} = Y_{n-1}(\mathtt{k})$, $X_{A,3} = Y_{n-2}(\mathtt{k})$ and so on. $\square$

References

[Ber07] Marcello Bernardara. Fourier-Mukai transforms of curves and principal polarizations. *C. R. Math. Acad. Sci. Paris*, 345(4):203–208, 2007.

[BO01] Alexei Bondal and Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Math.*, 125(3):327–344, 2001.

[Cal16] John Calabrese. Relative Singular Twisted Bondal–Orlov. *preprint*, 2016.

[Fis11] R. Fisette. The A-infinity algebra of an Elliptic Curve and the j-invariant. *ArXiv e-prints*, November 2011.

[FP14] Robert Fisette and Alexander Polishchuk. A_∞-algebras associated with curves and rational functions on $M_{g,g}$. I. *Compos. Math.*, 150(4):621–667, 2014.

[LP12] Y. Lekili and T. Perutz. Arithmetic mirror symmetry for the 2-torus. *ArXiv e-prints*, November 2012.

[LP14] Y. Lekili and A. Polishchuk. A modular compactification of $\mathcal{M}\_1, n$ from A_∞-structures. *ArXiv e-prints*, August 2014.

[Pol11] Alexander Polishchuk. A_∞-algebra of an elliptic curve and Eisenstein series. *Comm. Math. Phys.*, 301(3):709–722, 2011.

[Pol13] A. Polishchuk. Moduli of curves as moduli of A-infinity structures. *ArXiv e-prints*, December 2013.

[Pol15] A. Polishchuk. Moduli of curves with nonspecial divisors and relative moduli of A_∞-structures. *ArXiv e-prints*, November 2015.

[Pol16a] A. Polishchuk. A-infinity algebras associated with elliptic curves and Eisenstein-Kronecker series. *ArXiv e-prints*, April 2016.

[Pol16b] A. Polishchuk. Moduli spaces of nonspecial pointed curves of arithmetic genus 1. *ArXiv e-prints*, March 2016.

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