Math 355 – Midterm 2 – Fall 2017

This exam has 5 problems worth 40 points distributed over 6 pages, including this one.

Instructions: This is a 2 hour exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem and carefully justify all answers. Points will be deducted for irrelevant, incoherent or incorrect statements, and no points will be awarded for illegible work. If you you run out of room, you may work answers on the back of pages or on attached scratch paper. Be sure to clearly indicate when work is continued on another page.

Name:

Honor Pledge:

On my honor, I have neither given nor received any unauthorized aid on this exam.

Signature:

Question	Points	Score
1	6	
2	8	
3	10	
4	8	
5	8	
Total:	40	

1. (a) (2 points) Perform the following operation.

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

(b) (2 points) Multiply out the following matrices.

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 3 \\ -1 & 0 & 3 \end{pmatrix}$$

(c) (2 points) Write $\mathbb{R}^4$ as the sum of three non-trivial subspaces.[1]

[1] We say a subspace $U < \mathbb{R}^4$ is *non-trivial* if $\{\vec{0}\} \neq U \neq \mathbb{R}^4$.

2. (a) (4 points) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by the matrix

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Is f an isomorphism? Justify your answer.

(b) (4 points) Consider the basis

$$\mathbb{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Compute $\mathrm{Rep}_{\mathrm{std},\mathbb{B}} f$, where std denotes the standard basis.

3. For the following TRUE/FALSE questions please write clearly either TRUE or FALSE. You do not need to justify your answer.

   (a) (1 point) If $f\colon \mathbb{R}^4 \to \mathbb{R}^2$ is linear, then $\ker f \neq \{\vec{0}\}$.

   (b) (2 points) Let U, W be two subspaces of $\mathbb{R}^5$. If $U + W = \mathbb{R}^5$ then $U \cap W = \{\vec{0}\}$.

   (c) (1 point) Let $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map. The rank of the matrix $\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f$ does not depend on the choice of bases $\mathbb{B}, \mathbb{D}$.

   (d) (2 points) Let $\mathrm{id}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the identity. Then $\mathrm{Rep}_{\mathbb{B},\hat{\mathbb{B}}}\mathrm{id} = I$, regardless of the choice of bases $\mathbb{B}, \hat{\mathbb{B}}$.

   (e) (2 points) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. There always exist bases $\mathbb{B}, \hat{\mathbb{B}}$ such that $\mathrm{Rep}_{\mathbb{B},\hat{\mathbb{B}}} f = I$.

   (f) (2 points) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Let $\mathbb{B}$ be a basis of $\mathbb{R}^3$. If $\mathrm{Rep}_{\mathbb{B},\mathbb{B}} f = I$ then $f = \mathrm{id}$.

4. (a) (8 points) Exhibit a linear map $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the following conditions are *all* met.

   - $g(W) \subseteq W$ where $W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.
   - $g\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$.
   - $\dim \operatorname{Im} g = 2$.

   Justify your answer.

5. (a) (8 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear. Must we have $\ker f + \operatorname{Im} f = \mathbb{R}^2$? Justify your answer.

   [Hint: $\ker f + \operatorname{Im} f = \mathbb{R}^2$ if and only if $\dim(\ker f + \operatorname{Im} f) = 2$. What is the formula to calculate $\dim(\ker f + \operatorname{Im} f)$?]