MATH355 2017-11-10

EIGENSTUFF

Just to be clear, let us spell out once again the difference between an $\mathbf{R}$-linear and a $\mathbf{C}$-linear map. Recall, if V, W are vector spaces, then $f: V \to W$ is *linear* if

- $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$
- $f(\alpha\vec{v}) = \alpha f(\vec{v})$ for all $\vec{v} \in V$ and $\alpha \in \mathbf{R}$.

Since we are using that our scalars α live in $\mathbf{R}$, we should really be calling this $\mathbf{R}$-*linear*.

Analogously, say V, W are complex vector spaces. Then $g: V \to W$ is $\mathbf{C}$-*linear* if

- $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$
- $f(\alpha\vec{v}) = \alpha f(\vec{v})$ for all $\vec{v} \in V$ and $\alpha \in \mathbf{C}$.

So the only difference is whether I allow to pull out i or not.

Example 1. Let $\phi: \mathbf{C}^2 \to \mathbf{C}^2$ defined by $\phi\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \Re z + \Im w \\ z + w \end{pmatrix}$. Check that ϕ is $\mathbf{R}$-linear but not $\mathbf{C}$-linear.

OK, let's go back to similarity. Recall that two matrices A, B are similar if there exists an invertible matrix P such that $PAP^{-1} = B$. We say A is diagonalizable, if it is similar to a diagonal matrix. When we need to be precise, we will distinguish between being $\mathbf{R}$-*diagonalizable* and $\mathbf{C}$-*diagonalizable*, i.e. whether I can find the matrix P in $M_{n\times n}(\mathbf{R})$ or in $M_{n\times n}(\mathbf{C})$.

We like to rephrase the existence of the matrix P, by viewing it as a change of basis matrix.

Remark 2. A is diagonalizable if and only if there exist n linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_n$ and n scalars $\lambda_1, \ldots, \lambda_n$ such that

$$A\vec{v}_j = \lambda_j \vec{v}_j$$

for all $j = 1, \ldots, n$.

Of course, depending on whether we want diagonalizability over $\mathbf{R}$ or $\mathbf{C}$ we will require $\vec{v}_j \in \mathbf{R}^n$ and $\lambda_j \in \mathbf{R}$ or $\vec{v}_j \in \mathbf{C}^n$ and $\lambda_j \in \mathbf{C}$.

We've already seen examples:

- $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$ is $\mathbf{R}$-diagonalizable (and hence $\mathbf{C}$-diagonalizable);
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $\mathbf{C}$-diagonalizable but *not* $\mathbf{R}$-diagonalizable.

Let's see another example.

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Example 3. Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ We will now show that N is neither diagonalizable over **R** or over **C**.

OK, say we can find two linearly independent vectors $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \vec{w} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$, two scalars λ, μ such that

$$N\vec{v} = \lambda\vec{v}, N\vec{w} = \mu\vec{w}.$$

Let us try and solve these two equations.

$$\begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix} = \lambda\vec{v} = N\vec{v} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$$

so

$$\begin{cases} \lambda\alpha = \beta \\ \lambda\beta = 0 \end{cases}$$

If $\lambda = 0$, then $\beta = 0$, so that $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \operatorname{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If $\lambda \neq 0$, then I can divide by it in the second equation to obtain $\beta = 0$. But then I can divide again in the first equation, obtaining $\alpha = 0$.

So what have we learned? If $\lambda \neq 0$ there is no nonzero vector that satisfies that equation (and same goes if I had written μ instead of λ and $\vec{w}$ instead of $\vec{v}$).

This means that if N is similar to a matrix $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ we would have $\lambda = 0 = \mu$. But $\operatorname{rk} N = 1$, so that's impossible.

More explicitly, a vector satisfying $N\vec{v} = 0\vec{v} = \vec{0}$ must live in $\operatorname{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is one-dimensional. So I cannot find two linearly independent vectors in it!

Notice that in the example just now, it didn't matter whether we used real or complex numbers: that matrix is simply never diagonalizable!

Let us introduce some terminology which will be useful later.

Definition 4. Let $f\colon V \to V$ be a linear map. An *eigenvalue* is a scalar λ such that there exists a nonzero vector $\vec{v}$ such that

$$f(\vec{v}) = \lambda\vec{v}.$$

Such a $\vec{v}$ is called a λ-*eigenvector* for f.

We emphasize once more: λ is allowed to be whatever it wants, but $\vec{v}$ *cannot* be the zero vector. Why? Well, the equation $f(\vec{0}) = \lambda\vec{0}$ is always true, no matter what λ is. So that's not very helpful.

Remark 5. f is diagonalizable if and only if there exists a basis of eigenvectors for f

OK, say we have a matrix A, which we view as representing some linear map f. How do we find eigenvectors and eigenvalues for A? Let us start with eigenvalues. We are trying to find λ and $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

Let us rewrite this as $A\vec{v} - \lambda\vec{v} = \vec{0}$. But this is the same as

$$(A - \lambda I)\vec{v} = \vec{0}$$

In other words,

> $\vec{v}$ is a λ-eigenvector if and only if $\vec{v}$ belongs to the kernel of the matrix $A - \lambda I$.

So, λ is an eigenvalue if and only if the matrix $A - \lambda I$ has a non-zero kernel. We can rephrase this by saying λ is an eigenvalue if and only if the matrix $A - \lambda I$ is *not* invertible. But a square matrix is non-invertible if and only if it has zero determinant! To rephrase this yet again

> λ is an eigenvalue for A if and only if $\det(A - \lambda I) = 0$.

The next few lectures will be devoted to studying precisely the quation $\det(A - \lambda I) = 0$.

To conclude, let x be an indeterminate. It can be shown that $\det(A - xI)$ is a polynomial, called $p_A(x)$ the *characteristic polynomial* of A. To rephrase once more:

> The *eigenvalues* of A correspond precisely to the *roots* of the characteristic polynomial of A.