# 2017 Fall - Math 355 - Homework 6

Due: Friday, October 13 *in class.*[1]
Unless specified otherwise, you must always show your work and justify your answers.

(1) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map.
  (a) Discuss all possible values of $\dim\ker f, \dim \mathrm{Im} f$.
  (b) For each value you found, exhibit a concrete example.
  (c) For each example, describe $\mathrm{Im} f$ and find a basis for it.
  (d) For each example, describe $\ker f$ and find a basis for it.

(2) If $A$ is a matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

we define the corresponding linear map

$$f_A\colon \mathbb{R}^3 \to \mathbb{R}^3$$
$$f_A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xa_1 + yb_1 + zc_1 \\ xa_2 + yb_2 + zc_2 \\ xa_3 + yb_3 + zc_3 \end{pmatrix}$$

  (a) Find $A$ such that $\ker f_A = \mathrm{Span}\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$.
  (b) Find $B$ such that $\mathrm{Im} f_B = \mathrm{Span}\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\}$.
  (c) Find $C$ such that $f_C(\vec{e}_1), f_C(\vec{e}_3)$ are linearly dependent.

(3) Let $\phi\colon V \to V$ be a linear map.
  (a) Is it always true that $(\ker\phi) \cap (\mathrm{Im}\phi) = \{\vec{0}\}$? Why?
  (b) We know that $\dim V = \dim\ker\phi + \dim\mathrm{Im}\phi$. Is it also always true that $V = (\ker\phi) + (\mathrm{Im}\phi)$? Why?

For this last part, recall that if $U, W < V$ are subspaces, we define $U + W$ to be their *sum*, this means

$$\begin{aligned} U + W &= \{\vec{v} \in V \mid \exists \vec{u} \in U, \vec{w} \in W, \vec{v} = \vec{u} + \vec{w}\} \\ &= \{\vec{u} + \vec{w} \mid \vec{u} \in U, \vec{w} \in W\} \end{aligned}$$

[1] This file was last updated at 21:29 on Friday 6th October, 2017.