2017 Fall - Math 355 - Homework 9

Due: Friday, November 3 *in class.*[1]
Unless specified otherwise, you must always show your work and justify your answers.
This homework consists of two pages.

(1) Compute the inverse B of the following matrix by using row and column operations.

$$A := \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$$

Verify that $AB = I = BA$.

(2) Exhibit a linear map $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that the following conditions are *all* met.

- Let $W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$. Then $f(W) \subset W$.
- $f\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}$.
- $\dim \ker f = 1$.

(a) For the f you found, compute $\operatorname{Rep}_{\text{std}} f$, where std stands for standard basis.

(b) Can you find a bases $\mathbb{B}$ such that $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$?

If not, why? If so, find such $\mathbb{B}$ and $\mathbb{D}$.

(3) Let $f\colon V \to V$ be a linear map. Let $g = f \circ f\colon V \to V$ be f composed with itself.

(a) Show that $\ker f \subseteq \ker g$.
(b) Show that $\mathrm{Im} g \subseteq \mathrm{Im} f$.
(c) Let $A \in M_{n\times n}$. Show that $\operatorname{rk}(A^2) \leq \operatorname{rk} A$.
(d) Show that $\mathrm{Im} g = f(\mathrm{Im} f)$.

(4) Consider the linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ 3x - 2y - z \end{pmatrix}.$$

(a) Compute $A := \operatorname{Rep}_{\text{std}} f$.
(b) What is $\operatorname{rk} A$?

[1]This file was last updated at 13:10 on Friday 27$^{\text{th}}$ October, 2017.

(c) Write down a basis for $\ker f$.

(d) Find bases $\mathbb{B}, \mathbb{D}$, such that

$$\hat{A} := \operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(e) Compute the change of basis matrix P from std to $\mathbb{B}$.

(f) Compute the change of basis matrix Q from std to $\mathbb{D}$.

(g) Verify that $\hat{A} = QAP^{-1}$.