If $f\colon V \to W$ is a linear map, we define

$$\begin{aligned}\operatorname{Im} f &:= \{\vec{w} \in W \mid \exists \vec{v} \in V, f(\vec{v}) = \vec{w}\}\\ &= \{f(\vec{v}) \mid \vec{v} \in V\}\end{aligned}$$

the *image* of f.

Remark 1. f is surjective if and only if $\operatorname{Im} f = W$.

Remark 2. f is an isomorphism if and only if $\ker f = \{\vec{0}\}$ *and* $\operatorname{Im} f = W$.

Proposition 3. $\ker f < V$ is a subspace and $\operatorname{Im} f < W$ is a subspace.

Proof. We already proved that $\ker f$ is a subspace of V in a previous lecture. Let us show that $\operatorname{Im} f < W$ is a subspace. We need to check our usual three things.

- Does $\vec{0} \in \operatorname{Im} f$?

Well, $f(\vec{0}) = \vec{0}$ because f is a linear map, so $\vec{0} \in \operatorname{Im} f$.

- If $\vec{w}_1, \vec{w}_2 \in \operatorname{Im} f$, does $\vec{w}_1 + \vec{w}_2 \in \operatorname{Im} f$?

By definition, there are $\vec{v}_1, \vec{v}_2 \in V$ with $f(\vec{v}_1) = \vec{w}_1, f(\vec{v}_2) = \vec{w}_2$. Since f is linear, $\vec{w}_1 + \vec{w}_2 = f(\vec{v}_1) + f(\vec{v}_2) = f(\vec{v}_1 + \vec{v}_2)$. So $\vec{w}_1 + \vec{w}_2 \in \operatorname{Im} f$.

- If $\vec{w} \in \operatorname{Im} f$ and $\alpha \in \mathbf{R}$, does $\alpha\vec{w} \in \operatorname{Im} f$?

By definition, there exists $\vec{v} \in V$ with $f(\vec{v}) = \vec{w}$. By linearity, $\alpha\vec{w} = \alpha f(\vec{v}) = f(\alpha\vec{v})$. So, $\alpha\vec{w} \in \operatorname{Im} f$. ☐

Example 4. Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix}$. Then, $\ker f$ is the y-axis while $\operatorname{Im} f$ is the x-axis.

Proof. By definition, $\ker f = \{\vec{v} \in \mathbf{R}^2 \mid f(\vec{v}) = \vec{0}\}$. So

$$\begin{aligned}\ker f &= \{\begin{pmatrix}x\\y\end{pmatrix} \in \mathbf{R}^2 \mid \begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\}\\ &= \{\begin{pmatrix}0\\y\end{pmatrix} \mid y \in \mathbf{R}\}\end{aligned}$$

which is the y-axis.

By defintion, $\operatorname{Im} f = \{f(\vec{v}) \mid \vec{v} \in \mathbf{R}^2\}$. So,

$$\operatorname{Im} f = \{\begin{pmatrix}x\\0\end{pmatrix} \mid x \in \mathbf{R}\}$$

which is the x-axis. ☐

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More generally, we may define the image of anything in V.

Definition 5. Let $f\colon V \to W$ be linear, let $U < V$ be a subspace. We define

$$\begin{aligned} f(U) &:= \{\vec{w} \in W \mid \exists \vec{u} \in U, f(\vec{u}) = \vec{w}\} \\ &= \{f(\vec{u}) \mid \vec{u} \in U\} \end{aligned}$$

and call it the *image of* U *under* f.

Proposition 6. If $U < V$ is a subspace, then $f(U) < W$ is also a subspace.

The proof is similar to the proof from the previous page.

Fair Question: if $f\colon V \to W$ is an isomorphism, then is $\dim V = \dim W$?

Yes!

Proposition 7. Suppose $f\colon V \to W$ is an isomorphism. Then $\dim V = \dim W$.

Proof. Pick a basis $\vec{v}_1, \dots, \vec{v}_n$ of V. Here $n = \dim V$. By a Lemma we saw in class, $f(\vec{v}_1), \dots, f(\vec{v}_n)$ are linearly independent. Hence, $\dim V = n \le \dim W$.

Consider, $g := f^{-1}\colon W \to V$. We know that g is also an isomorphism. We repeat the same argument above, with f replaced by g. This implies that $\dim W \le \dim V$.

Combining both inequalities, we have $\dim V = \dim W$. □

In class we also saw a different proof of this fact. Can you remember what it was? If not, can you think of a different way to prove this without using the inverse map f^{-1}?

What about the converse to the previous question?

Proposition 8. Let V, W be vector spaces. Suppose $\dim V = \dim W$, then $V \simeq W$.

Recall that $V \simeq W$ means there exists some isomorphism between V and W. This is pretty amazing result, if you think about it. $\dim V$ is just a mere number, and yet it controls completely which vector space you are working with (up to isomorphisms).

Proof. Let V be a vector space. Pick a basis $B = (\vec{v}_1, \dots, \vec{v}_n)$ for it. In the homework, you have shown that

$$\mathrm{Rep}_B\colon V \to \mathbf{R}^n$$

is linear and also an isomorphism. You have also shown that the inverse is given by the linear map

$$\mathrm{Rep}_B^{-1} := \psi_B\colon \mathbf{R}^n \to V$$

$$\psi_B \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

Suppose now $\dim V = \dim W$. Pick bases $B = (\vec{v}_1, \dots, \vec{v}_n)$ of V and $D = (\vec{w}_1, \dots, \vec{w}_n)$ of W. Notice that $\dim V = n = \dim W$. We have the isomorphisms $\mathrm{Rep}_B\colon V \to \mathbf{R}^n$ and $\mathrm{Rep}_D\colon W \to \mathbf{R}^n$. Hence, the composition $\mathrm{Rep}_D^{-1} \circ \mathrm{Rep}_B = \psi_D \circ \mathrm{Rep}_B\colon V \to W$ is the desired isomorphism between V and W. □

In the proof above we assumed the following two facts.

Proposition 9. Say $f\colon V \to W$ and $g\colon W \to Z$ are both linear maps of vector spaces. Then their composition $g \circ f\colon V \to Z$ is also linear.

Proposition 10. Say $f\colon V \to W$ and $g\colon W \to Z$ are both isomorphisms of vector spaces. Then their composition $g \circ f\colon V \to Z$ is also an isomorphism of vector spaces.

Both propositions follow directly from the definitions. I encourage you to give it a shot and try to prove them on your own.