MATH 355 HOMEWORK 5

Problem 1

(a). By definition, f is linear if $f(a\vec{v}+b\vec{w}) = af(\vec{v}) + bf(\vec{w})$ for all $v, w \in \mathbf{R}^3$, $a, b \in \mathbf{R}$. So we check

$$\begin{aligned}
f\left(a\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} + b\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}\right) &= f\left(\begin{pmatrix}ax_1+bx_2\\ay_1+by_2\\az_1+bz_2\end{pmatrix}\right)\\
&= \begin{pmatrix}ax_1+bx_2+ay_1+by_2\\ax_1+bx_2+az_1+bz_2\end{pmatrix}\\
&= \begin{pmatrix}ax_1+ay_1\\ax_1+az_1\end{pmatrix} + \begin{pmatrix}bx_2+by_2\\bx_2+bz_2\end{pmatrix}\\
&= a\begin{pmatrix}x_1+y_1\\x_1+z_1\end{pmatrix} + b\begin{pmatrix}x_2+y_2\\x_2+z_2\end{pmatrix}\\
&= af\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} + bf\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix},
\end{aligned}$$

which shows that f is linear.

(b). No, f is not injective. For example,

$$f\begin{pmatrix}1\\-1\\-1\end{pmatrix} = f\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

c. Yes, f is surjective. Let $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbf{R}^2$ be arbitrary. Then

$$f\begin{pmatrix}0\\a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}.$$

d and e. By definition,

$$K = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbf{R}^3 : f\begin{pmatrix}x\\y\\z\end{pmatrix} = \vec{0}\right\}.$$

But

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\x+z\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \iff y = z = -x.$$

Thus,

$$K = \left\{\begin{pmatrix}\lambda\\-\lambda\\-\lambda\end{pmatrix} : \lambda \in \mathbf{R}\right\} = \operatorname{Span}\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\},$$

so $\dim K = 1$ and $\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}$ is a basis for K.

Problem 2

a,c,e. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $f_A = \mathrm{Id}_A$, so it is injective, surjective, and an isomorphism. Another popular choice is $A = \begin{pmatrix} 1 + \frac{1}{1+\frac{1}{1+\dots}} & 3 \uparrow\uparrow\uparrow\uparrow 3 \\ i^i & \int_0^\infty \frac{e^{-x}-1}{x^{3/2}} dx \end{pmatrix}$.

b. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Then $f_A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f_A \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

d. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $f_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin \mathrm{image}(f_A)$.

f. The matrix A from part b or part d works. So does $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Problem 3

No. The zero map is an easy counterexample. For a different counterexample, let $f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Then f sends both of the linearly independent vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Problem 4

Yes. Let $\vec{u}, \vec{v} \in V$ be linearly dependent. By definition there exist $a, b \in \mathbf{R}$ not both zero such that $a\vec{u} + b\vec{v} = 0$. Then by linearity,

$$af(\vec{u}) + bf(\vec{v}) = f(a\vec{u} + b\vec{v}) = f(\vec{0}) = \vec{0},$$

so $f(\vec{u}), f(\vec{v})$ are linearly dependent.

Problem 5

Let's recall the definition of the map $\mathrm{Rep}_B : V \to \mathbf{R}^2$, where $B = (\vec{v}, \vec{w})$ is a basis for V. Let $\vec{x} \in V$. Since B is a basis, there exist unique real numbers a, b such that $\vec{x} = a\vec{v} + b\vec{w}$. Then we define

$$\mathrm{Rep}_B(\vec{x}) = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now, given $\vec{x}_1, \vec{x}_2 \in V$, $r_1, r_2 \in \mathbf{R}$, we want to show

$$\mathrm{Rep}_B(r_1\vec{x}_1 + r_2\vec{x}_2) = r_1\mathrm{Rep}_B(\vec{x}_1) + r_2\mathrm{Rep}_B(\vec{x}_2).$$

Since B is a basis, let $a_1, b_1, a_2, b_2 \in \mathbf{R}$ be the unique real numbers such that $\vec{x}_1 = a_1\vec{v} + b_1\vec{w}$ and $\vec{x}_2 = a_2\vec{v} + b_2\vec{w}$. Applying the definition of Rep_B, we get

$$\begin{aligned}
\mathrm{Rep}_B(r_1\vec{x}_1 + r_2\vec{x}_2) &= \mathrm{Rep}_B(r_1(a_1\vec{v} + b_1\vec{w}) + r_2(a_2\vec{v} + b_2\vec{w})) \\
&= \mathrm{Rep}_B((r_1a_1 + r_2a_2)\vec{v} + (r_1b_1 + r_2b_2)\vec{w}) \\
&= \begin{pmatrix} r_1a_1 + r_2a_2 \\ r_1b_1 + r_2b_2 \end{pmatrix} \\
&= r_1\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\
&= r_1\mathrm{Rep}_B(\vec{x}_1) + r_2\mathrm{Rep}_B(\vec{x}_2).
\end{aligned}$$

PROBLEM 6

a. First assume ψ is injective and suppose

$$x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = 0.$$

To show that $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, we must show $x_1 = \cdots = x_k = 0$. Notice

$$x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \psi\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix},$$

so the injectivity of ψ implies $\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \vec{0}$, as desired.

On the other hand, assume that $v_1, \dots, v_k$ are linearly independent. To show that ψ is injective, it suffices to show that its kernel is trivial. Let $\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \ker\psi$. Then

$$\vec{0} = \psi\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k,$$

so the linear independence of $\vec{v}_1, \dots, \vec{v}_k$ implies $x_1 = \cdots = x_k = 0$, so $\ker\psi$ is trivial.

b. First assume ψ is surjective, and let $v \in V$. By surjectivity, there exists $\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \in \mathbf{R}^k$ such that

$$\vec{v} = \psi\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} = y_1\vec{v}_1 + \cdots + y_k\vec{v}_k.$$

Hence $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

On the other hand, if $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then there exist $y_1, \dots, y_k \in \mathbf{R}$ such that $\vec{v} = y_1\vec{v}_1 + \cdots + y_k\vec{v}_k$. But

$$\psi\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} = y_1\vec{v}_1 + \cdots + y_k\vec{v}_k = \vec{v},$$

so $v \in \operatorname{image}(\psi)$. Since $\vec{v} \in V$ was arbitrary, this shows that ψ is surjective.

c. By definition, ψ is an isomorphism if and only if ψ is both injective and surjective. By parts a and b, ψ is both injective and surjective if and only if $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent and span V. By definition, $\vec{v}_1, \dots, \vec{v}_k$ form a basis for V if and only if they are linearly independent and span V.

d. We need to show $\operatorname{Rep}_B \circ \psi = \operatorname{Id}_{\mathbf{R}^k}$ and $\psi \circ \operatorname{Rep}_B = \operatorname{Id}_V$. For the first equality, we see

$$\operatorname{Rep}_B \circ \psi\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \operatorname{Rep}_B(x_1\vec{v}_1 + \cdots + x_k\vec{v}_k) = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}.$$

For the second, let $\vec{v} = y_1\vec{v}_1 + \cdots + y_k\vec{v}_k \in V$. Then

$$\psi(\mathrm{Rep}_B(\vec{v})) = \psi\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} = y_1\vec{v}_1 + \cdots + y_k\vec{v}_k = \vec{v}.$$

Note that since we already showed that ψ is an isomorphism, it's actually enough to show that either one of these equations holds.