MATH355 2017-09-25

WHAT'S A LINEAR MAP?

Definition 1. Say V, W are vector spaces. A function $f\colon V \to W$ is *linear* if

(1) $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$
(2) $f(\alpha\vec{v}) = \alpha f(\vec{v})$

for all $\vec{v}_1, \vec{v}_2, \vec{v} \in V$ and $\alpha \in \mathbf{R}$.

Slogan:

Linear maps are precisely those which preserve the vector space structure.

Homomorphisms, Linear maps, linear functions, linear operators, linear applications, linear mappings,... all mean the same thing.

Example 2. What does a linear map $f\colon \mathbf{R} \to \mathbf{R}$ look like? Well, 1 forms a basis for $\mathbf{R}$, so let's see where that is sent to: define $a := f(1)$. If $x \in \mathbf{R}$, then

$$f(x) = f(x \cdot 1) = xf(1) = ax.$$

So linear maps correspond to a choice of a scaling factor $a \in \mathbf{R}$.

We see here that linear maps $\mathbf{R} \to \mathbf{R}$ are in 1-1 correspondence with real numbers. In turn, real numbers correspond to 1-by-1 matrices. This, while being silly, is no accident.

Fact 3. If $f\colon V \to W$ is a linear map, then $f(\vec{0}) = \vec{0}$.

Proof. Indeed,

$$f(\vec{0}) = f(\vec{0} + \vec{0}) = f(\vec{0}) + f(\vec{0})$$

by adding $-f(\vec{0})$ to both sides we get

$$f(\vec{0}) = \vec{0}.$$

$\square$

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