MATH 355 HOMEWORK 12

NICHOLAS ROUSE

Problem 1

It is not true that if two vectors $\vec{v}, \vec{w}$ are both eigenvectors for some linear map $T : V \to V$ with the same eigenvalue λ that they are necessarily linearly independent. As a counterexample, let $V = \mathbf{R}^2$, $T = Id$, $\vec{v} = (1, 0)$, and $\vec{w} = (0, 1)$. Then both $\vec{v}$ and $\vec{w}$ are eigenvectors for T with eigenvalue 1, but they are linearly independent.

The other direction, that if $\vec{v}$ and $\vec{w}$ are eigenvectors for some linear map with different eigenvalues then $\vec{v}$ and $\vec{w}$ are linearly independent, is true and was problem 1 on last week's homework.

Problem 2

Part a. It is true that if A is similar to B, then A^2 is similar to B^2. To see this let $A = P^{-1}BP$ for some invertible matrix P. Then

$$A^2 = \left(P^{-1}BP\right)\left(P^{-1}BP\right) = P^{-1}B^2P.$$

Part b. This is not true. As a counterexample, let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note first that A and B are *not* similar: A has rank 1 and B has rank 0. However

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = B^2,$$

so A^2 and B^2 are certainly similar.

Problem 3

First Bullet Point. There are two keys fact to know about geometric multiplicities. The first is that they are always less than the corresponding algebraic multiplicities. Second, if λ is a root of the characteristic polynomial (i.e. λ has algebraic multiplicity at least 1) then its geometric multiplicity is at least one. So once we observe that the only roots of the characteristic polynomial are 5 and -1, with algebraic multiplicities 3 and 1 respectively, we can easily solve the problem. In particular, the geometric multiplicity for 5 can be 1, 2, or 3, and the geometric multiplicity for -1 is necessarily 1.

Second Bullet Point. In each case -1 has geometric multiplicity equal to 1, so always have one Jordan block of size one for that eigenvalue. Then, we only need to vary the geometric multiplicity for 5. If 5 has geometric multiplicity 3, then we have three Jordan blocks of size one for the eigenvalue 5, so we have

$$J_{3,1} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

Date: December 1, 2017.

where $J_{3,1}$ denote the matrix A in Jordan form assuming that 5 has geometric multiplicity 3 and -1 has geometric multiplicity 1. Similar if 5 has geometric multiplicity 2, we have two total Jordan block for 5, so we must have one of size two and one of size one. Then

$$J_{2,1} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally in the case that 5 has geometric multiplicity 1 we only have one Jordan block for 5, so we have

$$J_{1,1} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Remark. The Jordan form doesn't prescribe a particular order for the Jordan blocks, so an answer like

$$J_{2,1} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

for the case where 5 has geometric multiplicity 2 would also be acceptable.

Problem 4

Some preliminaries: we'll be working from the algorithm Dr. Calabrese provided here: `http://math.rice.edu/~friedl/math355_fall04/Jordan.pdf`. In particular we'll follow the algorithm on page 3 of those notes. The algorithm produces the Jordan form with 0 entries below the diagonal and any entries equal to 1 on the first superdiagonal (i.e. the fist diagonal above the main diagonal). Jim Hefferon's book and (I think) Dr. Calabrese's class has them below the diagonal. We'll follow the algorithm and then make the necessary modifications to get the 1 entries below the diagonal. Also I leave out some calculations; if you are unsure how to fill in the gaps send me an email or ask in one of the office hours or review sessions this week. Anyway for the matrix A, the only eigenvalue is 2 (I used software to compute this), so the characteristic polynomial is necessarily $p_A(x) = (x-2)^4$, so 2 has algebraic multiplicity 4. Then we need to know the geometric multiplicity of 2. The way to do this is to find the dimension of the solution space of the linear system

$$\begin{aligned} x - y - z &= 2x \\ x + 2y + z + w &= 2y \\ y + 2z - w &= 2z \\ x + y + 3w &= 2w. \end{aligned}$$

This is the sort of calculation we did early in the course, so I'll skip the calculation and point out that the solution space has dimension 2. This completes step 1 of the algorithm in the link. Next up we have to compute the dimension of the so-called E-spaces (I say "so-called" because I've never heard this term before, and when I Google "E-spaces" I get results for an office rental company in Nashville; the more standard term is "generalized eigenspaces") Anyway the kth E-space for eigenvalue λ is denoted E_λ^k and is defined as

$$E_\lambda^k := \{\vec{v} \in V \mid (A - \lambda I)^k \vec{v} = \vec{0}\} = \ker(A - \lambda I)^k.$$

Note that the nonzero elements of E_λ^1 are just the eigenvectors with eigenvalue λ, so one can calculate that

$$E_2^1 = \operatorname{span}\left\{(-1,1,0,1), (-1,0,1,0)\right\}.$$

Next we want to calculate the kernel of $(A - \lambda I)^2$, but note that $(A - 2I)^2 = 0$, where 0 here denotes the 4×4 matrix with all entries equal to 0. Then we have that $E_2^2 = \mathbf{R}^4$. This completes step 2 of the algorithm. Now we have

$$\begin{aligned} d_1 &= \dim E_2^1 = 2, \\ d_2 &= \dim E_2^2 - \dim E_2^1 = 4 - 2 = 2. \end{aligned}$$

Now we make a diagram as follows. We have $d_1 = 2$ boxes in the first row and $d_2 = 2$ boxes in the second row, so we get

$$\begin{array}{l}\square\square\\ \square\square.\end{array}$$

This completes step 3. We now fill in the boxes in the second row with linearly independent vectors that are in E_2^2 that are not in E_2^1, so we just need linearly independent vectors that are not eigenvectors. One can check that $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ work. Then we fill in the first row with $(A-2I)\vec{e_3} = (-1,1,0,1)$ and $(A-2I)\vec{e_4} = (0,1,-1,1)$. Then we can read off the matrix Q by taking the first column to be the top left entry of our diagram and filling in the columns of the matrix by reading down the columns of the diagram and moving to the top of the next column to the right once we reach the bottom. So we have

$$Q = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

One can verify (but I wouldn't recommend it) that

$$Q^{-1}AQ = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The above matrix is in Jordan form, and we can read off the basis $\mathbb{B}$ such that $\operatorname{Rep}_{\mathbb{B}} f = Q^{-1}AQ$ where $f : \mathbf{R}^4 \to \mathbf{R}^4$ is the abstract linear transformation in the background (i.e. A is the representation of f with respect to some basis; one would typically assume the standard basis absent other information, but the problem is agnostic about the original basis in the sense that A is could be the representation of f with respect to any basis and we would get the same answer via the same procedure). Recall that for changing bases from $\mathbb{D}$ to $\mathbb{B}$ we produce Q such that $Q^{-1}AQ = B$ and $Q = \operatorname{Rep}_{\mathbb{B},\mathbb{D}} I$ where $I : \mathbf{R}^4 \to \mathbf{R}^4$ is the identity transformation. So in our case, we're regarding $A = \operatorname{Rep}_{\mathbb{D}} f$. Write $\mathbb{D} = \{\vec{d_1}, \vec{d_2}, \vec{d_3}, \vec{d_4}\}$ and $\mathbb{B} = \{\vec{b_1}, \vec{b_2}, \vec{b_3}, \vec{b_4}\}$. Then recall that $Q\vec{b_1} = \operatorname{Rep}_{\mathbb{D}} I\vec{b_1} = \operatorname{Rep}_{\mathbb{D}} \vec{b_1}$, but $Q\vec{b_1}$ is just the first column of Q. Similar arguments for the other $\vec{b_i}$ show that $\mathbb{B} = \{(-1,1,0,1), (0,0,1,0), (0,1,-1,1), (0,0,0,1)\}$, with the only caveat that the four vectors are understood to be expressed with respect to whatever basis A was originally expressed with respect to. I expect that A should be understood as expressed with respect to the standard basis though, so one can think of this basis just as four elements of $\mathbf{R}^4$ expressed with respect to the standard basis. The only problem we have now is that $J = Q^{-1}AQ$ upper-triangular, and we might consider the Jordan form as a lower-triangular matrix form. Fortunately, we can just read the basis backwards for each Jordan block to fix this. That is if we set $\vec{b_1'} = \vec{b_2}, \vec{b_2'} = \vec{b_1}, \vec{b_3'} = \vec{b_4}, \vec{b_3'} = \vec{b_4}$, we find that

$$\begin{aligned} J\vec{b_1'} &= J\vec{b_2} = \vec{b_1} + 2\vec{b_2} = 2\vec{b_1'} + \vec{b_2'} \\ J\vec{b_2'} &= J\vec{b_1} = 2\vec{b_1} = 2\vec{b_2'}, \end{aligned}$$

and similarly for the other Jordan block. So with respect to $\mathbb{B}' = \{(0,0,1,0), (-1,1,0,1), (0,0,0,1), (0,1,-1,1)\}$, A has the form

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

For the second matrix B, we're going to omit most of the discussion we did for A and just compute stuff. Mathematica tells me that B has eigenvalues 2 and 3. One computes that the geometric multiplicity of each eigenvalue is 1, so there is only one Jordan block for each eigenvalue, so we know the Jordan form must be (using the lower-triangular form)

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

We sort of strip down the algorithm to compute the basis. We can do this one eigenvalue at a time. Let's start with 2. We need an eigenvector for B with eigenvalue 2. One can compute that $(1, 0, 0, 1)$ works. Then we need a vector $\vec{b_1}$ such that $(B-2I)^2\vec{b_1} = \vec{0}$, but $\vec{b_1}$ is not an eigenvector. We compute that $\vec{b_1} = (0, 0, 1, 0)$ suffices. Then we compute that $(B - 2I)\vec{b_1} = (1, 0, 0, 1)$.Then we have our first two basis entries, $(0, 0, 1, 0)$ and $(1, 0, 0, 1)$. Following the same procedure for the eigenvalue 3 we get our last two basis entries: $(1, 1, 0, 0)$ and $(-1, 0, 1, 0)$. In summary our basis is $\mathbb{B} = \{(0, 0, 1, 0), (1, 0, 0, 1), (1, 1, 0, 0), (-1, 0, 1, 0)\}$.

Problem 5

This is not enough to deduce the Jordan form. In particular one can have either two Jordan blocks of size two or two Jordan blocks, one of size three and one of size one. So the possibilities are

$$J_1 = \begin{pmatrix} 64 & 0 & 0 & 0 \\ 1 & 64 & 0 & 0 \\ 0 & 0 & 64 & 0 \\ 0 & 0 & 1 & 64 \end{pmatrix},$$

or

$$J_2 = \begin{pmatrix} 64 & 0 & 0 & 0 \\ 1 & 64 & 0 & 0 \\ 0 & 1 & 64 & 0 \\ 0 & 0 & 0 & 64 \end{pmatrix}.$$

This matrices are not similar. It is a general fact that two matrices are similar if and only if they have the same Jordan form (except for permuting the Jordan blocks), and these matrices are both in Jordan form yet differ. This is perhaps not a fact covered in class, however, so we instead check the dimensions of the E-spaces. One should convince oneself that the dimensions of the E-spaces do not depend on the basis with respect to which the original matrix is calculated. Anyway, $\dim\ker\left(J_1 - 64I\right)^2 = 4$, but $\dim\ker\left(J_2 - 64I\right)^2 = 3$, as one can check, so we conclude that the matrices J_1 and J_2 are not similar even though they both have only one eigenvalue, 64, with algebraic multiplicity 4 and geometric multiplicity 2.