2017 Fall - Math 355 - Homework 5

Due: Friday, October 6 *in class.*[1]
Unless specified otherwise, you must always show your work and justify your answer.

(1) Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by
$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \end{pmatrix}.$$
 (a) Show that f is linear.
 (b) Is f injective?
 (c) Is f surjective?
 (d) Describe $K := \ker f$.
 (e) What is $\dim K$? If it is greater than zero, find a basis for K.

(2) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}$ be a two-by-two matrix. We define a linear map $f_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows:
$$f_A\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$
 (a) Find an $A \in M_{2\times 2}$ such that f_A is injective.
 (b) Find a $B \in M_{2\times 2}$ such that f_B is not injective.
 (c) Find a $C \in M_{2\times 2}$ such that f_C is surjective.
 (d) Find a $D \in M_{2\times 2}$ such that f_D is not surjective.
 (e) Find an $E \in M_{2\times 2}$ such that f_E is an isomorphism.
 (f) Find an $F \in M_{2\times 2}$ such that f_F is not an isomorphism.

(3) Let $f \in \mathrm{Hom}(V,W)$. Suppose $\vec{u},\vec{v} \in V$ are linearly independent. Do $f(\vec{u}), f(\vec{v}) \in W$ also have to be linearly independent?

(4) Let $f \in \mathrm{Hom}(V,W)$. Suppose $\vec{u},\vec{v} \in V$ are linearly dependent. Do $f(\vec{u}), f(\vec{v}) \in W$ also have to be linearly dependent?

(5) Let $B = (\vec{v},\vec{w}) \in V$ be a basis. Show that the map $\mathrm{Rep}_B\colon V \to \mathbb{R}^2$ is linear.

(6) Let $\vec{v}_1,\dots,\vec{v}_k \in V$. Consider the map
$$\psi\colon \mathbb{R}^k \to V$$
$$\psi\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \mapsto \sum_{i=1}^{k} x_i\vec{v}_i$$
which is linear.
 (a) Show that ψ is injective if and only if $\vec{v}_1,\dots,\vec{v}_k$ are linearly independent.
 (b) Show that ψ is surjective if and only if $\mathrm{Span}\{\vec{v}_1,\dots,\vec{v}_k\} = V$.
 (c) Show that ψ is an isomorphism if and only if $\vec{v}_1,\dots,\vec{v}_k$ is a basis of V.
 (d) Assume ψ is an isomorphism. Call $B = (\vec{v}_1,\dots,\vec{v}_k)$ the given basis of V. Show that $\psi^{-1} = \mathrm{Rep}_B$.

[1] This file was last updated at 21:47 on Friday 29th September, 2017.