Apparently a Bourbaki talk on semiorthogonal decompositions

John Calabrese, August 10, 2014.

Abstract

All mistakes appearing on this pdf are mine – you cannot have them.

Derived categories are useful. Trust $\mathrm{D}(\mathrm{X})$.

Usually the first time one meets a derived category it's there so that one has a good place where to speak of derived functors. In this note we take the point of view that $\mathrm{D}(\mathrm{X})$ is an interesting object on its own sake, which contains information about X (I think it's even a conjecture of Orlov that $\mathrm{D}(\mathrm{X})$ should know about some flavour of the *motive* of X itself – so it's meant to contain a lot of information!). More precisely, we'll try and have a look at *semiorthogonal decompositions* (ØDs), which are a useful tool to study derived categories.

> *We'll only talk about smooth and projective varieties over* $\mathbb{C}$. *When I write* $\mathrm{D}(\mathrm{X})$ *I mean* $\mathrm{D}^b(\mathtt{Coh}(\mathrm{X}))$ *– the bounded derived category of coherent sheaves on* X.

First some moral philosophy. You might want to study X by studying vector bundles on it. Or might take it to the next level and try to understand the *category* of vector bundles on it. From a categorical point of view, vector bundles is not good. So one enlarges it and considers $\mathtt{Coh}(\mathrm{X})$.

> We might think of this process as an attempt to attach a (very rich and complicated) invariant to a variety. We might construe it as some form of "linearization" or an attempt to turn geometry into algebra.

Unfortunately, $\mathtt{Coh}(\mathrm{X})$ is in some sense too rigid to be useful.

Theorem (Gabriel, ...) – If $\mathtt{Coh}(\mathrm{X}) \simeq \mathtt{Coh}(\mathrm{Y})$ then $\mathrm{X} \simeq \mathrm{Y}$. $\ast$

So, OK, let's do something crazier and attach $\mathrm{D}(\mathrm{X})$ to X. Gabriel's theorem breaks down. Although,

Theorem (Bondal-Orlov) – If ω_{X} is ample or anti-ample, then $\mathrm{D}(\mathrm{X}) \simeq \mathrm{D}(\mathrm{Y})$ implies $\mathrm{X} \simeq \mathrm{Y}$. $\ast$

In spite of this, we'll see that derived categories are flexible enough to be useful, even in this (anti-)Fano context.

Let's recall a few basic properties of $\mathrm{D}(\mathrm{X})$ – remember that X is smooth and projective.

— it's additive (actually, $\mathbb{C}$-linear)

— there is a shift functor $[1]$

— it's a triangulated category, so we can't talk about short exact sequences but we have exact triangles

— it's Ext-finite, in the sense that for any $E, F \in D(X)$, then $\bigoplus_i \mathrm{Ext}^i(E, F) = \bigoplus_i \mathrm{Hom}_{D(X)}(E, F[i])$ is a finite dimensional $\mathbb{C}$-vector space

— there is Serre duality, ie $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, E \otimes \omega_X[\dim X])^\vee$.

We want to abstract this last notion, we call $S_X(-) = (-) \otimes \omega_X[\dim X]$ the *Serre functor* of $D(X)$. Whenever on a triangulated category D there is an auto-equivalence S_D, such that $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, S_D(E))^\vee$, we call S_D the Serre functor of D.

If you are russian enough, you might think of any Ext-finite triangulated category with a Serre functor as coming from a smooth and proper "non-commutative" variety. We won't take this point of view *too* seriously, but well have a stab at it.

What information does $D(X)$ *contain?*

Pinning down exactly what can be deduced about X from $D(X)$ is a hard an important questions (words like Hochschild pop up). We won't go into a list of what is known, a good reference for that is Huybrechts's book on Fourier-Mukai transforms.

What we'll do here is focus instead on how one might study $D(X)$. Here is where semiorthogonal decompositions come in. First, a remark.

Remark. Let's break our convention and take $X = Y \sqcup Z$ any disconnected scheme. If E is some sheaf on X, then there is a natural morphism $E \to j_* j^* E$ which is (split-)surjective with kernel $i_* i^* E$, where i, j are the inclusions of respectively Y, Z. For E a complex it's exactly the same, only we can just say there is a split exact triangle

$$i_* i^* E \to E \to j_* j^* E.$$

So, in some sense to be made precise below, $D(X)$ splits as a sum of $D(Y)$ and $D(Z)$. *

Definition – Let $\mathtt{A}, \mathtt{B}$ be two triangulated subcategories of $D(X)$. We say that $\mathtt{A}, \mathtt{B}$ form an *orthogonal decomposition* (OD) of $D(X)$ if

— $\mathrm{Hom}(\mathtt{B}, \mathtt{A}) = 0 = \mathrm{Hom}(\mathtt{A}, \mathtt{B})$ ("orthogonality")

— for any E there is a unique exact triangle

$$A \to E \to B$$

with $A \in \mathtt{A}$, $B \in \mathtt{B}$ ("fullness").

*

The really simple proposition one deduces is the following.

Proposition – X is disconnected if and only if it admits a (non-trivial) orthogonal decomposition. *

So we learned that $D(X)$ knows when X is connected. Now, while there isn't an intermediate between being connected and being disconnected, on the algebraic side one can weaken the definition.

Definition – A pair of triangulated subcategories $\mathtt{A}, \mathtt{B} \subset D(X)$ form a *semiorthogonal decomposition* (ØD) of $D(X)$ if

— $\mathrm{Hom}(\mathtt{B}, \mathtt{A}) = 0$ ("semi-orthogonality")

— for any E there is a unique exact triangle

$$A \to E \to B$$

with $A \in \mathtt{A}$, $B \in \mathtt{B}$ ("fullness").

When this occurs we write $D(X) = \langle \mathtt{A}, \mathtt{B} \rangle$. Of course, there is no reason why we should restrict to ØD s consisting of two components (for example one might be able to decompose $\mathtt{A}$ or $\mathtt{B}$ further). In general a semiorthogonal decomposition consists of a finite number of pieces and one writes $D(X) = \langle \mathtt{A}_1, \ldots, \mathtt{A}_n \rangle$. ⋆

Being able to do things like this is one of the reasons we love algebra. Before giving the first example let's make a silly observation.

Let $E \in D(X)$. There is a functor $\phi_E : D(\mathtt{Vect}) \to D(X)$ which takes a complex of vector spaces V to $V \otimes_{\mathbb{C}} E$. Notice that $D(\mathtt{Vect}) = D(\mathrm{pt})$, the derived category of a point. In some sense we would like to think of the image of ϕ_E as being the minimal triangulated category containing E. However, ϕ_E must be fully faithful for this to be equal to $D(\mathrm{pt})$.

Definition – An object E is called *exceptional* if ϕ_E is fully faithful. More concretely, E is exceptional if and only if $\mathcal{H}om(E, E) = \mathbb{C}$ and $\mathrm{Hom}(E, E[k]) = 0$ for $k \neq 0$. ⋆

By the way, a semiorthogonal decomposition consisting purely of exceptional objects is called a *full exceptional collection*. So, great.

Theorem (Beilinson) –

$$D(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle$$

where to be pedantic one should have written

$$\langle \phi_{\mathcal{O}}(D(\mathtt{Vect})), \ldots, \phi_{\mathcal{O}(n)}(D(\mathtt{Vect})) \rangle.$$

⋆

This is one of those theorems which starts a whole industry of maths. There has been a lot of work in exceptional collections for various varieties.

Remark. A(n obvious but important) warning is in order though. The notation is quite misleading, in the sense that it's lacking vital information. For example, $D(\mathrm{pt} \sqcup \mathrm{pt}) = \langle D(\mathrm{pt}), D(\mathrm{pt}) \rangle$, but also $D(\mathbb{P}^1) = \langle D(\mathrm{pt}), D(\mathrm{pt}) \rangle$. What makes the two categories non equivalent is that in the first case the two categories are completely orthogonal, while in the latter there are maps going in the opposite direction, which can be thought as some *gluing data* to obtain $D(\mathbb{P}^1)$. *

Kapranov exhibited an ØD for grassmannians, but it gets more involved. OK, so here's a non-example.

Lemma – Let $\omega_X = \mathcal{O}_X$ (we loosely say X is *Calabi-Yau*), then $D(X)$ admits only the trivial ØD . Notice that when this holds, the Serre functor of $D(X)$ is just the shift $[\dim X]$. ⋆

OK, given a ØD , how do you get new ones?

Theorem (Orlov) – Let E be a vector bundle of rank $n+1$ on X and let $p : \mathbb{P}_X(E) \to X$ the projectivization. Then

$$D(\mathbb{P}(E)) = \langle p^*D(X), p^*D(X) \otimes \mathcal{O}_p(1), \ldots, p^*D(X) \otimes \mathcal{O}_p(n) \rangle.$$

⋆

A twisted version.

Theorem (Bernardara) – Let $p : Y \to X$ a Brauer-Severi variety (ie an étale $\mathbb{P}^n$-bundle[1]) and let β be the corresponding Brauer class.

$$D(Y) = \langle p^*D(X), p^*D(X, \beta) \otimes \mathcal{O}_p(1), \ldots, p^*D(X, \beta^n) \otimes \mathcal{O}_p(n) \rangle.$$

[1] There is an étale (or analytic) open cover of X, such that on each patch $U \subset X$, one has $p^{-1}(U) = U \times \mathbb{P}^n$. These things are PGL_{n+1}-principal bundles (aka torsors) and are classified by $H^1_{\text{ét}}(X, PGL_{n+1})$. There is a short exact sequence $0 \to \mathbb{G}_m \to GL_{n+1} \to PGL_{n+1} \to 1$. So from a Brauer-Severi variety we get a class $\beta \in H^2(X, \mathbb{G}_m)$, which is called the *Brauer class* of p.

[one should point out that $\mathcal{O}_p(i)$ is well-defined as an $f^*\beta^i$-twisted sheaf] *

Theorem (Orlov) – Let $Y \subset X$ a smooth subvariety of codimension c. Blow it up.

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "p"'] & \mathrm{Bl}_Y X \arrow[d, "\pi"] \\ Y \arrow[r] & X \end{tikzcd}$$

$$D(\mathrm{Bl}_Y X) = \langle \pi^* D(X), i_* p^* D(Y), i_* p^* D(Y) \otimes \mathcal{O}_p(1), \dots, i_* p^* D(Y) \otimes \mathcal{O}_p(c-2) \rangle$$

*

OK, what if one has a semiorthogonal collection of subcategories, how do we know they form a decomposition? Put differently, how do we know it's full? How can we check it spans the whole derived category?

Remark. Whenever you have a semiorthogonal decomposition the Grothendieck group splits (in fact, as far as I understand there is a unifying and more general statement about motives). Precisely, if $D(X) = \langle \mathtt{A}_1, \dots, \mathtt{A}_n \rangle$, then $K_0(X) = \bigoplus_i K_0(\mathtt{A}_i)$. *

It was conjectured for a while that if you have a semiorthogonal collection, $\mathtt{A}_1, \dots, \mathtt{A}_n$, such that $\bigoplus_i K_0(\mathtt{A}_i) = K_0(X)$ then the collection was in fact full. Any collection can be completed to a decomposition, just by considering the subcategory orthogonal to the rest. In other words, given a semiorthogonal collection $\mathtt{A}_1, \dots, \mathtt{A}_r$, we always have

$$D(X) = \langle \mathtt{A}_1, \dots, \mathtt{A}_r, \mathtt{B} \rangle$$

where $\mathtt{B}$ consists of all objects B such that $\mathrm{Hom}(B, A)$ for all $A \in \mathtt{A}_i$ and for all i. So the conjecture boiled down to:

> can one find an example of $\mathtt{B}$ such that $K_0(\mathtt{B}) = 0$?

Such a category is called a *phantom*. Just to introduce other terminology found in the literature, $\mathtt{B}$ is a *quasi-phantom* if $K_0(\mathtt{B})$ is torsion. Unfortunately, it turned out that the conjecture was too optimistic. An example of X was found with a $\mathtt{B}$ with trivial K-motive. This implies that all invariants one cook up for $\mathtt{B}$ vanish: K_0, higher K-theory, Hochschild homology...

> :(

HPD

OK, now let's have a look at a newer thing, called *homological projective duality*. I'll discuss only a dumbed down version, you can read up on the real deal in Kuznetsov's ICM notes. The goal is to understand the derived categories of hyperplane sections and how they vary in families. Let's consider $\mathbb{P}^n$ and $X = \{s = 0\} \subset \mathbb{P}^n$ for s of degree $d \leq n$. Then

$$D(X) = \langle \mathtt{A}_X, \mathcal{O}_X, \dots, \mathcal{O}_X(n-d) \rangle$$

and, amazingly, we do *not* need to assume X to be smooth! We like to think the bundles $\mathcal{O}_X(i)$ in the decomposition as coming from the embedding, while $\mathtt{A}_X$ is the interesting part of $D(X)$. In fact, we might even invent a notation $D_{\mathrm{int}}(X) := \mathtt{A}_X$. We can also keep intersecting, if Y is a hypersurface of degree e, such that $\dim X = n - d - e$, then

$$D(X) = \langle \mathtt{A}_{X\cap Y}, \mathcal{O}, \dots, \mathcal{O}(n-d-e) \rangle.$$

Similarly for the intersection of $X_1, \dots, X_r$, as long as $\sum_i d_i \le n$, where d_i is the degree of X_i, and as long as the intersection is complete (if you want to weaken this, then the decomposition still holds but you have to interpret $X_1 \cap \cdots \cap X_r$ as a derived intersection).

Before we talk about families, can anything be said about $\mathtt{A}_X$? Well, it's a triangulated category and since it appears as a component in a semiorthogonal it's Ext-finite and has a Serre functor. There is a formula to compute the Serre functor $S_{\mathtt{A}}$ but it's hard to pin down exactly. However, one can always understand a power of it.

Lemma – Assume X is smooth hypersurface of degree d in $\mathbb{P}^n$. Then

$$S_{\mathtt{A}}^{d/\gcd(n+1,d)} = \left[\frac{(d-2)(n+1)}{\gcd(n+1,d)}\right].$$

*

Let's say this again: a power of the Serre functor is just a shift. Since a Calabi-Yau variety has Serre functor just $[\dim]$, we think of $\mathtt{A}_X$ as being a *fractional Calabi-Yau* category. So, you might think that $\mathtt{A}_X$ is the derived category of a non-commutative Calabi-Yau of fractional dimension.

But let's see a more seriously instance of this. If one picks the numerology correctly, one can find actual Calabi-Yau categories popping up. A case which has sparked a lot of interest is $n = 5$, $d = 3$: cubic fourfolds. So, for a cubic fourfold X, we have

$$D(X) = \langle \mathtt{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$$

and $S_{\mathtt{A}} = [2]$: a K3-category!

Q: Is $\mathtt{A} = D(K3)$ for an actual K3 surface?

There is a conjecture by Kuznetsov (based also on some Hochschild homology computations and following work on the geometry of cubic fourfolds by Hassett) which says that X is rational if and only if $\mathtt{A} = D(K3)$ for a geometric K3.

This would have the following nice consequence. Generically one shows that $\operatorname{rk} K_0^{\mathrm{num}}(\mathtt{A}) = 2$ (the rank of the numerical Grothendieck group), however $\operatorname{rk} K_0^{\mathrm{num}}(K3) \ge 3$. In particular, the generic cubic fourfolds is irrational. So far no one has been able to write down a non-rational cubic fourfold.[2]

For some cubic fourfolds the K3 surface can be seen geometrically. I'll talk about the case of cubics containing a plane at the end.

OK, so let's try and move a hypersurface in a family. For each $X = \{s = 0\} \subset \mathbb{P}^n$ of degree d we have a category $\mathtt{A}_s$. Consider the universal family

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d] & \mathbb{P}^n \times \mathbb{P}(V^*) \\ \mathbb{P}(V^*) & \end{tikzcd}$$

where $V^* = H^0(\mathbb{P}^n, \mathcal{O}(d))$ and $H = \{(p,s) | s(p) = 0\}$. Similarly to the case of hypersurfaces, there is a ØD on H.

$$D(H) = \langle \mathtt{A}, D(\mathbb{P}(V^*)) \boxtimes \mathcal{O}(d), D(\mathbb{P}(V^*)) \boxtimes \mathcal{O}(d+1), D(\mathbb{P}(V^*)) \boxtimes \mathcal{O}(n)\rangle.$$

Moreover, for $s \in V^*$, we can base change the decomposition above and obtain the decomposition for $H_s = \{s = 0\}$ which we talked about earlier. In other words $\mathtt{A} \otimes_{D(\mathbb{P}(V^*))} D(\mathrm{pt}) = \mathtt{A}_s$. Because of this, we think of $\mathtt{A}$ as being the total family of the categories $\mathtt{A}_s$.

More generally, consider a linear system $L \subset V^*$. We have a diagram

[2] Although a conditional proof of the fact that the generic cubic is irrational has been given by Galkin-Shinder.

$$
\begin{tikzcd}
H_L \arrow[r] \arrow[d] & H \arrow[d] \\
\mathbb{P}(L) \arrow[r, hook] & \mathbb{P}(V^*)
\end{tikzcd}
$$

and a ØD

$$D(H_L) = \langle \mathtt{A}_L, D(\mathbb{P}(L)) \boxtimes \mathcal{O}(d), D(\mathbb{P}(L)) \boxtimes \mathcal{O}(d+1), D(\mathbb{P}(L)) \boxtimes \mathcal{O}(n) \rangle.$$

The nice thing now is that these families of categories relate to the corresponding base loci. Let $L^\perp = \ker(V \to L^*) \subset V$. Consider the fibre product

$$
\begin{tikzcd}
X_L \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\
\mathbb{P}(L^\perp) \arrow[r] & \mathbb{P}(V)
\end{tikzcd}
$$

here I'm thinking $\mathbb{P}^n \to \mathbb{P}(V)$ embedded by the degree d Veronese embedding. If you think about it, X_L is the base locus of L. Now, here you either need to assume X_L to have the expected dimension or have to be willing to take the derived fibre product. In any case, the non trivial part of $D(X_L)$, let's call it $\mathtt{A}_L$ is equal to the $\mathtt{A}_L$ in the decomposition of $\mathtt{A}_L$.

OK, this was all pretty abstract and somehow tautological: how do we make it more geometric? Well, a *homological projective dual* of (in this case) $\mathbb{P}^n$ with respect to the degree d Veronese is some variety $Y \to \mathbb{P}(V^*)$ mapping to the dual projective space, such that $D(Y) = \mathtt{A}$, where the latte is the one appearing in the ØD of $D(H)$. It follows that for any linear system $L \subset V^*$, and denoting $Y_L = Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$, the interesting part of $D(Y_L)$ is equal to the interesting part of $D(X_L)$.

Let's have a look at an example. For $d = 1$ we only talk about linear spaces. We have $\mathtt{A} = 0$ and the HPdual is in fact the empty scheme $Y = \varnothing$.

Let's do a better example: quadrics. For a single smooth quadric (let's assume even-dimensional) $Q \subset \mathbb{P}^n$, we have

$$D(Q) = \langle D(\mathrm{Cl}_0), \text{whatever's left} \rangle$$

(this was first proved by Kapranov, but the generalization in terms of Clifford algebras was given by Kuznetsov). In turns out that for a smooth quadric, the even part of the Clifford algebra is just a product of matrix algebras, so $D(\mathrm{Cl}_0) = D(\mathrm{pt} \sqcup \mathrm{pt})$. So, quadrics are just pairs of points!

In general, the HPdual is not a genuine variety, but mildly non-commutative thing: it's

$$Y = (\mathbb{P}(V^*), \mathrm{Cl}_0).$$

So, we have that (modulo disregarding non interesting things) we have a description of the derived category of an intersection of qudrics in terms of Clifford algebras. More precisely, if $L \subset \mathbb{P}(V^*)$ parameterizes a linear family of quadrics, then $D(X_L)D(\mathbb{P}(L), \mathrm{Cl}_0)$ (modulo ignoring the boring bits of the derived category and where X_L is the base locus or in other words an intersection of $\dim L - 1$ quadrics).

More geometrically, one can do an intermediate construction and try to analyze $D(X_L)$ by looking at $D(H_L)$. Let's assume $2 | n + 1$ so that we work with even-dimensional quadrics. We said that $D(Q)$ is just $D(\mathrm{pt} \sqcup \mathrm{pt})$. The points in question correspond to some special bundles on Q called the *spinor bundles*.

If $\dim L = 2$ (a pencil), the total family H_L traces around where these two points are moving. So we get a double cover C of $\mathbb{P}(L) = \mathbb{P}^1$, branched where the quadrics in the fibres drop rank. In other words $D_{\mathrm{int}}(Q_0 \cap Q_1) = D_{\mathrm{int}}(H_L) = D(C)$ – where the first two derived categories are .

For $\dim L = 2$, we get a branched double cover of $\mathbb{P}^2$. It turns out to be a K3 surface! However, it turns out that $D(\mathbb{P}^2, \mathrm{Cl}_0)$ does not become trivial by passing to the double cover K3. It actually becomes an Azumaya algebra with Brauer class α and so we have a *twisted variety*. So $D_{\mathrm{int}}(Q_0 \cap Q_1 \cap Q_2) = D(K3, \alpha)$.

For four quadrics this story can be continued and we get a Calabi-Yau threefold, again twisted by a Brauer class. For more quadrics I don't think this story can be extended. Addington's thesis is the best reference for this material.

So what about cubic fourfolds containing a plane? So, let X be a cubic fourfold containing a plane $P \subset X$. Let $\mathbb{P}^2$ be a disjoint plane. Blow up P and project away, one gets a map $X' = \mathrm{Bl}_P X \to \mathbb{P}^2$. One checks that actually $X' \to \mathbb{P}^2$ is a family of quadrics! So (again modulo the boring bits), $D(X') = D(K3, \alpha)$, for the construction performed earlier. Using Orlov's blow up formula (and performing a few mutations), one shows that in fact $D(X) = D(K3, \alpha)$. Since I'm using the "modulo-boring" notation, $D(X)$ really means $\mathcal{A}_X$ from earlier. There are ways to make the Brauer class trivial, for example by taking X to contain two disjoint planes. As cubic fourfolds containing a plane are rational this is mild evidence for the "trivial" case of Kuznetsov's conjecture.