MATH 355 HOMEWORK 11

Problem 1

Assume for contradiction that $\vec{v}$ and $\vec{w}$ are linearly dependent, so $a\vec{v} + b\vec{w} = 0$ for some $a, b \in \mathbb{R}$, not both zero. Without loss of generality, we may assume $a \neq 0$ so $\vec{v} = c\vec{w}$ (where $c = \frac{b}{a}$). Since $\vec{v}, w \neq \vec{0}$ by assumption, we have $c \neq 0$. Applying f gives

$$\begin{aligned} f(\vec{v}) &= f(c\vec{w}) \\ f(\vec{v}) &= cf(\vec{w}) \\ \lambda\vec{v} &= c\mu\vec{w} \\ \lambda c\vec{w} &= c\mu\vec{w}. \end{aligned}$$

Since $c \neq 0$ and $\vec{w} \neq \vec{0}$, this implies $\lambda = \mu$, a contradiction. Thus, $\vec{v}$ and $\vec{w}$ are linearly independent.

Problem 2

(a). Suppose $\tau(A) = 0$. Then

$$\begin{gathered} A + A^t = 0 \\ \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \iff \\ a = d = 0 \quad b = -c. \end{gathered}$$

Thus,

$$\ker\tau = \left\{ \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\} = \operatorname{Span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

so $\dim_{\mathbb{C}} \ker\tau = 1$. By the dimension formula, we have

$$\dim_{\mathbb{C}} \operatorname{Im}\tau = \dim_{\mathbb{C}} V - \dim_{\mathbb{C}} \ker\tau = 4 - 1 = 3.$$

For the sake of completeness, we can verify this directly by computing

$$\operatorname{Im}\tau = \left\{ \begin{pmatrix} u & v \\ v & w \end{pmatrix} \mid u, v, w \in \mathbb{C} \right\}.$$

To see this, note that

$$\tau\begin{pmatrix} \frac{u}{2} & \frac{v}{2} \\ \frac{v}{2} & \frac{w}{2} \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix},$$

which proves

$$\operatorname{Im}\tau \supseteq \left\{ \begin{pmatrix} u & v \\ v & w \end{pmatrix} \mid u, v, w \in \mathbb{C} \right\}.$$

For the other inclusion, if

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \in \operatorname{Im}\tau,$$

set $u = 2a, v = b + c, w = 2d$. Hence,

$$\operatorname{Im}\tau = \left\{ \begin{pmatrix} u & v \\ v & w \end{pmatrix} \mid u, v, w \in \mathbb{C} \right\} = \operatorname{Span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

so $\dim_C \operatorname{Im}\tau = 3$.

(b). Since $\dim_{\mathbb{C}} \ker \tau = 1$ and $\dim_{\mathbb{C}} \operatorname{Im} \tau = 3$, it is enough to show that $\ker \tau \cap \operatorname{Im} \tau = \{\vec{0}\}$. But this follows directly from the explicit computation of ker and Im above. To be precise, if $A \in \ker \tau$, then $A = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$ for some $z \in \mathbb{C}$. On the other hand, if $A \in \operatorname{Im} \tau$, then $A = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$ for some $u, v, w \in \mathbb{C}$. Hence, if $A \in \ker \tau \cap \operatorname{Im} \tau$, then

$$\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$$

shows that $u = w = 0$ and $v = z = -z \implies v = 0$, so $A = 0$.

Problem 3

The characteristic polynomial of A is

$$\begin{aligned}
\det(\lambda I - A) &= \det \begin{pmatrix} \lambda + 2 & 2 & 9 \\ 1 & \lambda - 1 & 3 \\ -1 & -1 & \lambda - 4 \end{pmatrix} \\
&= (\lambda + 2) \det \begin{pmatrix} \lambda - 1 & 3 \\ -1 & \lambda - 4 \end{pmatrix} - \det \begin{pmatrix} 2 & 9 \\ -1 & \lambda - 4 \end{pmatrix} + (-1) \det \begin{pmatrix} 2 & 9 \\ \lambda - 1 & 3 \end{pmatrix} \\
&= (\lambda + 2)((\lambda - 1)(\lambda - 4) - (-1)(3)) - (2(\lambda - 4) - (-1)(9)) + (-1)((2)(3) - (\lambda - 1)(9)) \\
&= \lambda^3 + 3\lambda^2 - 4\lambda + 2.
\end{aligned}$$

By our keen powers of perception, we see that 1 is a root of this polynomial, so the characteristic polynomial factors as $(\lambda - 1)(\lambda^2 - 2\lambda + 2)$. Thus, the eigenvalues of A are $1, 1 + i, 1 - i$.

(a). Since A has non-real eigenvalues, it is not diagonalizable over $\mathbb{R}$.

(b). Since A has three distinct eigenvalues, it is diagonalizable over $\mathbb{C}$.

(c). Let $\vec{v}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be an eigenvector for $\lambda = 1$. Then $A\vec{v}_1 = \vec{v}_1$, so we get a system of equations

$$\begin{aligned}
-2a - 2b - 9c &= a \\
-a + b - 3c &= b \\
a + b + 4c &= c.
\end{aligned}$$

Since the eigenspace for $\lambda = 1$ is one-dimensional, this equation will have infinitely many solutions. To simplify computations, let's set $c = 1$ and hope that there is still a solution:

$$\begin{aligned}
3a + 2b &= -9 \\
a &= -3 \\
a + b &= -3
\end{aligned}$$

We have a solution $a = -3, b = 0, c = 1$, so we set $\vec{v}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

Let $\vec{v}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be an eigenvector for $\lambda = 1 + i$. Then $A\vec{v}_2 = (1 + i)\vec{v}_2$ yields the system

$$\begin{aligned}
-2a - 2b - 9c &= a + ai \\
-a + b - 3c &= b + bi \\
a + b + 4c &= c + ci
\end{aligned}$$

Let's again set $c = 1$, so the system becomes

$$\begin{aligned} 3a + 2b &= -9 - ai \\ a &= -3 - bi \\ a + b &= -3 + i \end{aligned}$$

The second two equations yield $b - bi = i$. Splitting b into its real and imaginary parts $b = b_0 + b_1 i$, we have

$$\begin{gathered} b_0 + b_1 i - b_0 i + b_1 = i \\ (b_0 + b_1) + (b_1 - b_0)i = i, \end{gathered}$$

which yields the real system

$$\begin{aligned} b_0 + b_1 &= 0 \\ -b_0 + b_1 &= 1, \end{aligned}$$

so $b = b_0 + b_1 i = \frac{-1+i}{2}$. From the original system, we have $a = -3 - bi = \frac{-5+i}{2}$. We could take $\vec{v}_2$ with these entries, but multiplying by 2 yields another eigenvector with the same eigenvalue, so we can take $\vec{v}_2$ to be the slightly more pleasing vector

$$\vec{v}_2 = \begin{pmatrix} -5+i \\ -1+i \\ 2 \end{pmatrix}.$$

For the last eigenvector, we're saved from further computation. Since $1 + i$ and $1 - i$ are conjugate, an eigenvector for $1 - i$ is just the conjugate of an eigenvector for $1 + i$. Hence, we can take

$$\vec{v}_3 = \overline{\vec{v}_2} = \begin{pmatrix} -5-i \\ -1-i \\ 2 \end{pmatrix}.$$

Thus,

$$\mathbb{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -5+i \\ -1+i \\ 2 \end{pmatrix}, \begin{pmatrix} -5-i \\ -1-i \\ 2 \end{pmatrix} \right\}$$

is a basis of eigenvectors for $\mathbb{C}^3$.

(d). Since $\mathbb{B}$ is a basis of eigenvectors, $\mathrm{Rep}_{\mathbb{B}} f$ is diagonal with entries corresponding to the eigenvalues of the basis elements. Thus,

$$\mathrm{Rep}_{\mathbb{B}} f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix}.$$