Last time we saw the ancient formula

$$\dim U + W = \dim U + \dim W - \dim U \cap W.$$

handed down to us by the heroes of humankind.

Proposition 1. Two planes (through the origin) in $\mathbf{R}^3$ are either equal or they meet in a line.

Proof. Call U, W the two planes. The formula tells us

$$\dim U + W = \dim U + \dim W - \dim U \cap W = 2 + 2 - \dim U \cap W = 4 - \dim U \cap W.$$

Since $U+W < \mathbf{R}^3$, we must have $\dim U+W \le 3$. Hence, $\dim U\cap W \ge 1$ (otherwise $\dim U+W = 4$). Since $U\cap W < U$, we have $\dim U\cap W \le \dim U = 2$. So, two possibilities: either $\dim U\cap W = 1$ or $\dim U \cap W = 2$. $\square$

Example 2. Take $U := \operatorname{Span}\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$ and $W := \operatorname{Span}\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$. Notice, $U + W = \mathbf{R}^3$, as it contains a basis for $\mathbf{R}^3$. By the sacred formula, $\dim U \cap W = 1$, so it's a line. Which line is it? How to describe it? Well, since $\dim U \cap W = 1$ we know $U \cap W = \operatorname{Span}\{\vec{v}\}$ for some $\vec{v} \ne 0$. Can we find such a $\vec{v}$? Say $\vec{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. Since $\vec{v} \in U$, we must have $z = 0$. Since $\vec{v} \in W$, we must have

$$\vec{v} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix}$$

thus $x = a = y$. Can we find such a vector? Sure, take $\vec{v} = \begin{pmatrix}-\pi\\-\pi\\0\end{pmatrix}$. So, $U \cap W$ is the line $\operatorname{Span}\{\begin{pmatrix}-\pi\\-\pi\\0\end{pmatrix}\}$.

Recall $U, W \subset V$ are in *direct sum* if $U \cap W = \{\vec{0}\}$. In that case we write $U \oplus W$ instead of $U + W$, to remind ourselves of this awesome feature.

Proposition 3. If $V = U \oplus W$, then any $\vec{v} \in V$ may be written *uniquely* as

$$\vec{v} = \vec{u} + \vec{w}$$

with $\vec{u} \in U, \vec{w} \in W$.

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Proof. OK, $V = U \oplus W$ means $V = U + W$ and $U \cap W = \{\vec{0}\}$. Since $V = U + W$, any $\vec{v}$ is the sum $\vec{u} + \vec{w}$. Let's prove uniqueness. Suppose, $\vec{v} = \vec{u} + \vec{w}$ but also $\vec{v} = \vec{u}_1 + \vec{w}_1$, with $\vec{u}, \vec{u}_1 \in U$, $\vec{w}, \vec{w}_1 \in W$. Then

$$\vec{0} = \vec{v} - \vec{v} = (\vec{u} + \vec{w}) - (\vec{u}_1 + \vec{w}_1) = (\vec{u} - \vec{u}_1) + (\vec{w} - \vec{w}_1)$$

therefore $\vec{u} - \vec{u}_1 = \vec{w} - \vec{w}_1$. So, $\vec{u} - \vec{u}_1 \in U \cap W = \{\vec{0}\}$, hence $\vec{u} = \vec{u}_1$. Same for $\vec{w}$ and $\vec{w}_1$. □

Remark 4. If $U \cap W \neq \{\vec{0}\}$ te theorem is not true. Indeed, take $V = \mathbf{R}^3$, $U := \mathrm{Span}\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$, $W := \mathrm{Span}\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$. Take $\vec{v} := \begin{pmatrix}1\\1\\2\end{pmatrix}$. Then

$$\begin{pmatrix}1\\1\\2\end{pmatrix} = \left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right) + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

but also

$$\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \left(\begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

and these are two different ways to write $\vec{v}$ as a sum of a vector in U and a vector in W.

OK, here's a question: if $V = W + W_1$ and $V = W + W_2$ then is $W_1 = W_2$?

Example 5. Let $V = \mathbf{R}^2$, let $W := \mathrm{Span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$, $W_1 := \mathrm{Span}\{\begin{pmatrix}0\\1\end{pmatrix}\}$, $W_2 := \mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}\}$. But $\mathbf{R}^2 = W \oplus W_1$ and also $\mathbf{R}^2 = W \oplus W_2$. So, yeah, no. There is no *cancellation* in sums of vector spaces.

1. And now, for something completely different

Suppose $f\colon V \to W$ is a linear map. Take $\vec{b}_1, \dots, \vec{b}_n \in V$ a basis. Define $\vec{w}_i := f(\vec{b}_i)$. Let $\vec{v} \in V$ be any vector. We know it can be written *uniquely* as

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{b}_i.$$

Therefore

$$f(\vec{v}) = f\Big(\sum_{i=1}^{n} \alpha_i \vec{b}_i\Big) = \sum_{i=1}^{n} \alpha_i f(\vec{b}_i) = \sum_{i=1}^{n} \alpha_i \vec{w}_i.$$

What have we learned?

f is *uniquely determined* by its values on a basis!

We can actually do a converse of sorts.

Theorem 6. Let V, W be vector spaces. Let $\vec{b}_1, \dots, \vec{b}_n$ be a basis of V. Let's pick vectors $\vec{w}_i \in W$ for all i. Then there *exists* a *unique* $\phi\colon V \to W$ linear such that $\phi(\vec{b}_i) == \vec{w}_i$ for all i.

Proof. How? Well, for $\vec{v} \in V$ write it as $\vec{v} = \sum_i \alpha_i \vec{b}_i$.Declare

this can only be done in a unique way, so there's no ambiguity!

$$\phi(\vec{v}) := \sum_i \alpha_i \vec{w}_i.$$

Exercise: show that it is well-defined (i.e. we did not mess up anything when we defined this), and show that ϕ is linear. [you might want to do only the case where $\dim V = 2$, it's easier on the notation] $\square$