MATH355 - 2017-09-11 BONUS

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Exercise. Let $V = \mathbb{R}^3$ and let $W \subset V$ be defined as

$$W = \left\{ \begin{pmatrix} x \\ x+z \\ z \end{pmatrix} \middle| x, z \in \mathbb{R} \right\}$$

Show that W is a subspace of $\mathbb{R}^3$.

Solution. We have to show that the three defining conditions of being a subspace are met.

- $\vec{0} \in W$

Indeed, $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and since $0 = 0 + 0$ we have $\vec{0} \in W$.

- If $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$.

Let us write out explicitly the components of $\vec{u}$ and $\vec{v}$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

We know $\vec{u} \in W$ if and only if $u_2 = u_1 + u_3$, similarly $v_2 = v_1 + v_3$. The vector $\vec{u} + \vec{v}$ has components $(u_1 + v_1, u_2 + v_2, u_3 + v_3)$. Since

$$u_2 + v_2 = (u_1 + u_3) + (v_1 + v_3) = u_1 + v_1 + u_3 + v_3$$

we have $\vec{u} + \vec{v} \in W$.

- If $a \in \mathbb{R}$, $\vec{w} \in W$, then $a\vec{w} \in W$

Suppose $\vec{w} = (r, l, s)$. Since $\vec{w} \in W$, we know $l = r + s$. Now, $a\vec{w} = (ar, al, as)$ and

$$al = a(r + s) = ar + as$$

hence $a\vec{w} \in W$.

We have just proved that W is a subspace of $\mathbb{R}^3$.