# Homework 6 Solutions

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I'm trying a new way of writing solutions this week. Please let me know if you don't like it! Note: I'm not sure if you've covered the rank-nullity theorem, so I won't use it, but you can make some problems easier with it.

**Problem 1:** Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map.

**Part a:** $\dim \operatorname{Im} f \in \{0,1,2\}$. $\dim \operatorname{Ker} f \in \{0,1,2\}$. We give examples of all of these in subsequent sections. For now we show that $\dim \operatorname{Im} f \neq 3$. Suppose $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$. Then $\operatorname{Im} f = \operatorname{span}\{f(e_1), f(e_2)\}$, hence dimension at most 2.

**Part b:** Let $f_0: \mathbb{R}^2 \to \mathbb{R}^3$ be the zero map. I.e. $f_0(x,y) = (0,0,0)$. Then $\dim \operatorname{Im} f = 0$ and $\dim \operatorname{Ker} f = 2$.

Let $f_1: \mathbb{R}^2 \to \mathbb{R}^3$, $f_1(x,y) = (x,0,0)$. Then $\operatorname{Im} f_1 = \operatorname{span}\{(1,0,0)\}$, so $\dim \operatorname{Im} f_1 = 1$. $\operatorname{Ker} f_1 = \{(0,y) \mid y \in \mathbb{R}\} = \operatorname{span}\{(0,1)\}$, so $\dim \operatorname{Ker} f_1 = 1$.

Let $f_2: \mathbb{R}^2 \to \mathbb{R}^3$, $f_2(x,y) = (x,y,0)$. Then $\operatorname{Im} f_2 = \operatorname{span}\{(1,0,0),(0,1,0)\}$, so $\operatorname{Im} f_2$ has dimension 2. The kernel is trivial: $(x,y,0) = (0,0,0)$ if and only if $x=y=0$, so $\dim \operatorname{Ker} f_2 = 0$.

**Part c:** $\operatorname{Im} f_0 = \{\vec{0}\}$, so the image is just the zero vector and the only basis for it is the empty set, $\emptyset$.

We described $\operatorname{Im} f_1$ and $\operatorname{Im} f_2$ in part b. We give bases for them here for completeness. $\{(1,0,0)\}$ is a basis for $\operatorname{Im} f_1$, and $\{(1,0,0),(0,1,0)\}$ is a basis for $\operatorname{Im} f_2$.

**Part d:** $\operatorname{Ker} f_0 = \mathbb{R}^2$, so any basis for $\mathbb{R}^2$ will work, e.g., the standard basis $\{(1,0),(0,1)\}$.

$\operatorname{Ker} f_1$ and $\operatorname{Ker} f_2$ we described in part b. $\{(0,1)\}$ is a basis for $\operatorname{Ker} f_1$, and $\emptyset$ is the unique basis for $\operatorname{Ker} f_2$.

**Problem 2:**

Problem 2:

Part a: Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

So $f_A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+z \\ x+2y+z \\ x+y \end{pmatrix}$

Suppose $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in \operatorname{Ker} f_A$. Then $\begin{pmatrix} x_0+2y_0+z_0 \\ x_0+2y_0+z_0 \\ x_0+y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Solving this linear system we have $x_0 = -y_0$, and $x_0 - 2x_0 + z_0 = 0$, so $z_0 = x_0$. Then the solutions to this linear system are of the form $\begin{pmatrix} x_0 \\ -x_0 \\ x_0 \end{pmatrix}$, so $\operatorname{Ker} f_A = \left\{ x_0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \middle| x_0 \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$, as desired.

Part b: Let

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

Note that $f_B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $f_B\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, and $f_B\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Then $f_B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, so $\operatorname{Im} f_B = \left\{ x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$, so $\operatorname{Im} f_B = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$.

Part c: Let

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $f_C(\vec{e}_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = f_C(\vec{e}_3)$, and any set with the zero vector is linearly dependent.

Problem 3: $\phi: V \to V$ linear

Part a: No. Let $V = \mathbb{R}^2$ and $\phi(x,y) = (y,0)$. Then $\phi(1,0) = (0,0)$, so $(1,0) \in \ker\phi$. However, $\phi(0,1) = (1,0)$, so $(1,0) \in \operatorname{Im}\phi$. Note that $(1,0) \neq \vec{0}$.

Part b: No. Let $V = \mathbb{R}^2$ and $\phi(x,y) = (y,0)$ as before. Then $\phi(0,1) = (1,0) \neq \vec{0}$, so $(0,1) \notin \ker\phi$, but $\operatorname{Im}\phi = \operatorname{span}\{(1,0)\}$, so $(0,1) \notin \operatorname{Im}\phi$ either.