# MATH355 2017-10-11

Last time we saw the sacred formula

$$\dim V = \dim \ker \phi + \dim \operatorname{Im} \phi$$

where $\phi\colon V \to W$ is a linear map.

And we mentioned that $V$ could be "chopped up" into two pieces: $\ker \phi$ and some other subspace $U$, where $\phi$ was injective. Let's make this chopping thing rigorous.

Example 1. Let $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbf{R}^2$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbf{R}^2$. They form a basis, this means that given any $\vec{v} \in \mathbf{R}^2$, there are $a, b$ such that

$$\vec{v} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let us rephrase this a little. Call $U := \operatorname{Span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$, $W := \operatorname{Span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$. Then

$$\vec{v} = \vec{u} + \vec{w}$$

where $\vec{u} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{w} = b\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Notice $\vec{u} \in U$ and $\vec{w} \in W$.

In the example above, we saw that any vector $\vec{v}$ may be "broken up" into a sum $\vec{u} + \vec{w}$ where $\vec{u} \in U, \vec{w} \in W$.

Let's make this definition.

Definition 2. Let $V$ be a vector space. Let $U, W \subset V$ be subspaces. We define their *sum* to be

$$\begin{aligned} U + W &:= \{\vec{v} \in V \mid \exists \vec{u} \in U, \vec{w} \in W, \vec{v} = \vec{u} + \vec{w}\} \\ &:= \{\vec{u} + \vec{w} \mid \vec{u} \in U, \vec{w} \in W\} \end{aligned}$$

When $U \cap W = \{\vec{0}\}$, we say $U$ and $W$ are in *direct sum* and write $U \oplus W$.

Remark 3. If $U, W < V$ are subspaces, then $U \cap W$ is also a subspace.

Proposition 4. $U + W$ is a subspace.

Proposition 5. If $U = \operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_r\}$, $W = \operatorname{Span}\{\vec{w}_1, \ldots, \vec{w}_s\}$ then

$$U + W = \operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_r, \vec{w}_1, \ldots, \vec{w}_s\}$$

Let's see an example.

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Example 6. Take $U = \text{Span}\{\begin{pmatrix}1\\1\\0\end{pmatrix}\}$, $W = \text{Span}\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\}$. What is $U + W$? Notice that $U + W$ is the $xy$-plane.

What is $U \cap W$? Notice that $U \cap W = \{\vec{0}\}$. So $U \oplus W$ (they are in direct sum), however $U \oplus W \neq \mathbf{R}^3$.

Example 7. Take $U := \text{Span}\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\}$, $W := \{\begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}\}$. What is $U + W$? Well, certainly

$$\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\}, \{\begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix} \in U + W$$

and $\text{Span}\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\}, \{\begin{pmatrix}0\\-1\\-1\end{pmatrix}\} = \mathbf{R}^3$. Hence, $U + W = \mathbf{R}^3$.

What is $U \cap W$? Well, $\begin{pmatrix}-1\\0\\0\end{pmatrix} \in W$, so $\begin{pmatrix}1\\0\\0\end{pmatrix} \in W$. Therefore $\begin{pmatrix}1\\0\\0\end{pmatrix} \in U \cap W$. Hence, $\text{Span}\begin{pmatrix}1\\0\\0\end{pmatrix} \subset U \cap W$. That is, the $x$-axis is contained in $U \cap W$. However, a picture suggests $U \cap W$ *is* the $x$-axis. More on this later.

One last example.

Example 8. Consider

$$\begin{aligned} U &:= \text{Span}\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\} \\ &= \{\begin{pmatrix}x\\y\\0\\0\end{pmatrix} \mid x, y \in \mathbf{R}\} \\ W &:= \text{Span}\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\} \\ &= \{\begin{pmatrix}0\\0\\s\\t\end{pmatrix} \mid s, t \in \mathbf{R}\} \end{aligned}$$

Notice that $U + W = \mathbf{R}^4$ and $U \cap W = \{\vec{}\}$. So, in four dimensions there is enough space to find two planes intersecting *only* at the origin.

Here is another awesome formula.

Theorem 9. Let $V$ be a vector space. Let $U, W < V$ be subspaces. Then

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

*Proof.* Let's give an idea. First, pick a basis $\vec{v}_1, \ldots, \vec{v}_a$ of $U \cap W$. Extend it to a basis $\vec{v}_1, \ldots, \vec{v}_a, \vec{u}_1, \ldots, \vec{u}_r$ of $U$. Extend it to a basis $\vec{v}_1, \ldots, \vec{v}_a, \vec{w}_1, \ldots, \vec{w}_s$ of $W$. Notice that,

$$\dim U \cap W = a, \dim U = a + r, \dim W = a + s.$$

We want to show that

$$\dim U + W = \dim U + \dim W - \dim U \cap W = (a + r) + (a + s) - a = a + r + s.$$

How to show this? We already have a candidate basis for $U+W$, namely $\vec{v}_1, \ldots, \vec{v}_a, \vec{u}_1, \ldots, \vec{u}_r, \vec{w}_1, \ldots, \vec{w}_s$. So, it suffices to show that:

$$U + W = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_a, \vec{u}_1, \ldots, \vec{u}_r, \vec{w}_1, \ldots, \vec{w}_s\}$$

and $\vec{v}_1, \ldots, \vec{v}_a, \vec{u}_1, \ldots, \vec{u}_r, \vec{w}_1, \ldots, \vec{w}_s$ are linearly independent.

The first is easy (try it!), the second is not hard but a little tricky (try it!). □