MATH355 2017-09-13

Throughout, V will denote an abstract vector space. If $W \subset V$ is a subspace, we will sometimes write $W < V$. Also, if $W < V$ is a subspace, then W itself is a vector space, with the operations inherited by V.

Definition 1. Let $\vec{v}_1, \dots, \vec{v}_k \in V$ be vectors. A *linear combination* of $\vec{v}_1, \dots, \vec{v}_k$ is a sum

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k \in V$$

where $a_i \in \mathbf{R}$ can be any real number and $k \in \mathbf{N}$ can be any natural number.

Notice that $\vec{0} = 0\vec{v}_1$ is always a linear combination.

Example 2. Any vector in $\mathbf{R}^2$ is a linear combination of $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Indeed, if $\vec{v} \in \mathbf{R}^2$, then $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ for $a, b \in \mathbf{R}$. Hence, $\vec{v} = a\vec{e}_1 + b\vec{e}_2$, which is a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

Lemma 3. Let $W < V$ be a subspace, then W is closed under linear combinations.

Concretely, this means that if $\vec{w}_1, \cdots, \vec{w}_r \in W$ then $a_1\vec{w}_1 + \cdots + a_r\vec{w}_r \in W$ for any $a_1, \dots, a_r \in \mathbf{R}$.

Proof. By induction. ☐

Definition 4. Let $S \subset V$ be any subset. The *span* of S is

$$\begin{aligned} \operatorname{Span} S &= \{\text{all possible linear combinations of vectors in } S\} \\ &= \{a_1\vec{s}_1 + \cdots + a_k\vec{s}_k \mid a_1, \dots, a_k \in \mathbf{R}, \vec{s}_1, \dots, \vec{s}_k \in S, \forall k \in \mathbf{N}\} \end{aligned}$$

By convention, the "empty linear combination" is just $\vec{0}$ and $\operatorname{Span} \varnothing = \{\vec{0}\}$, the span of the empty set is the zero vector space.

Lemma 5. Let $S \subset V$. Then $\operatorname{Span} S$ is a subspace.

Proof. Let $W := \operatorname{Span} S$. We must show the usual three things: $\vec{0} \in W$; if $\vec{u}, \vec{v} \in W$ then $\vec{u} + \vec{v} \in W$; if $a \in \mathbf{R}$, $\vec{u} \in W$ then $a\vec{v} \in W$.

If $S = \varnothing$, then $\operatorname{Span} S = \{\vec{0}\}$, which is always a subspace. If $S \neq \varnothing$, then there exists some $\vec{s} \in S$. Since $\vec{0} = 0\vec{s}$ is a linear combination of elements of S, we have $\vec{0} \in W$.

Suppose $\vec{w} = a_1\vec{s}_1 + \cdots + a_k\vec{s}_k$ is a linear combination of vectors in S. Thus $\vec{w} \in W$. Let $a \in \mathbf{R}$, then $a\vec{w} = (aa_1)\vec{s}_1 + \cdots + (aa_k)\vec{s}_k$ is also a linear combination of vectors in S. Hence $a\vec{w} \in W$. In other words W is closed under scalar multiplication. Closure under addition is similar (exercise). ☐

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Example 6. Let $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in \mathbf{R}^3$. What is Span $\vec{v}$? Well

$$\begin{aligned} \operatorname{Span} \vec{v} &= \{a_1\vec{v} + \cdots + a_k\vec{v} \mid a_i \in \mathbf{R}, \forall i\} \\ &= \{(a_1 + \cdots + a_k)\vec{v} \mid a_i \in \mathbf{R}, \forall i\} \\ &= \{a\vec{v} \mid a \in \mathbf{R}\} \\ &= \{\begin{pmatrix} a \\ 0 \\ 2a \end{pmatrix} \mid a \in \mathbf{R}\} \end{aligned}$$

Example 7. Let $\vec{0} \in \mathbf{R}^4$, i.e. $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then

$$\operatorname{Span}\{\vec{0}\} = \{a_1\vec{0} + \cdots + a_k\vec{0} \mid a_i \in \mathbf{R}, k \in \mathbf{N}\} = \{\vec{0}\}.$$

Example 8. What is Span $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$? Call $W :=$ Span $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$. Notice that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

which is a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \in W$. Also, $\frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \in W$ as it is a linear combination of elements in W. Hence, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W$. Consider instead

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \in W$$

The upshot now is that both $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in W. By W is closed under linear combinations and we know that any vector in $\mathbf{R}^2$ is a linear combination of $\vec{e}_1$ and $\vec{e}_2$. Thus, $\mathbf{R}^2 \subset W$. But since $W \subset \mathbf{R}^2$, we have

$$\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = W = \mathbf{R}^2.$$