Ex) $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$\pi(\vec{e}_1) = \vec{e}_1 + 0\vec{e}_2 \qquad \pi(\vec{e}_2) = 0\vec{e}_1 + 0\vec{e}_2$$

$$\mathrm{Rep}_{std}\,\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Say $\mathbb{B} := \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = (\vec{b}_1, \vec{b}_2)$

$\mathbb{D} := \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right) = (\vec{d}_1, \vec{d}_2)$

What is $\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\,\pi$ ?

It's columns are given by coordinates of $\pi(\vec{b}_i)$ wrt $\mathbb{D}$

i.e. $j$-th column $= \mathrm{Rep}_{\mathbb{D}}\,\pi(\vec{b}_j)$

So

$$\pi(\vec{b}_1) = \pi\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -\vec{d}_1 + \frac{1}{2}\vec{d}_2$$

$$\pi(\vec{b}_2) = \pi\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \vec{d}_1 - \frac{1}{2}\vec{d}_2$$

So $\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\,\pi = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix}$

Say now $\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \in \mathbb{R}^2$

$\pi(\vec{v}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ but what is $\mathrm{Rep}_{\mathbb{D}} \pi(\vec{v})$ ?

well, $\mathrm{Rep}_{\mathbb{B}} \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as $\vec{v} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

so $\mathrm{Rep}_{\mathbb{D}} \pi(\vec{v}) = \mathrm{Rep}_{\mathbb{B},\mathbb{D}} \pi \ \mathrm{Rep}_{\mathbb{B}} \vec{v} = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2+3 \\ 1-\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

indeed $\pi(\vec{v}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ so

$\mathrm{Rep}_{\mathbb{D}} \pi(\vec{v}) = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$.

<u>Key Fact</u> if $f: V \to W$ linear, $\vec{v} \in V$, $\mathbb{B}, \mathbb{D}$ bases

$$\mathrm{Rep}_{\mathbb{D}} f(\vec{v}) = \left(\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f\right)\left(\mathrm{Rep}_{\mathbb{B}} \vec{v}\right)$$

<u>wow!</u>

$\text{rk Rep}_{B,D} f = \dim \text{Im} f$ (does not depend on choice of $B, D$)

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Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 2a-b \\ 2a-b \end{pmatrix}$$

$$\text{Im} f = \left\{ \begin{pmatrix} 2a-b \\ 2a-b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$\dim \text{Im} f = 1.$

- $\text{Rep}_{std} f = ?$ well $f(\vec{e}_1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\vec{e}_1 + 2\vec{e}_2$

$f(\vec{e}_2) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -\vec{e}_1 - \vec{e}_2$

so $\text{Rep}_{std} f = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} =: A$

rk $A$ can be computed via row reduction

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{rk} A = 1.$$

Say now

$$\mathbb{B} = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) \qquad \mathbb{D} = \left( \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \right)$$

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

$$f\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = -6\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} - 9\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

so $$\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f = \cancel{\begin{pmatrix} 2 \\ -6 \end{pmatrix}} \begin{pmatrix} 2 & -6 \\ 3 & -9 \end{pmatrix} =: B$$

$\mathrm{rk}\, B = ?$

$$\begin{pmatrix} 2 & -6 \\ 3 & -9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad \mathrm{rk}\, B = 1.$

Important fact: matrix multiplication is composition of linear maps.

Say $V, W, Z$ v.sp. $\dim V = n$ $\dim W = m$ $\dim Z = r$

$\mathbb{B}, \mathbb{D}, \mathbb{E}$ bases for $V, W, Z$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "g\circ f"'] & W \arrow[d, "g"] \\ & Z \end{tikzcd}$$

becomes

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f"] \arrow[dr, "\mathrm{Rep}_{\mathbb{B},\mathbb{E}}\, g\circ f"'] & \mathbb{R}^m \arrow[d, "\mathrm{Rep}_{\mathbb{D},\mathbb{E}}\, g"] \\ & \mathbb{R}^r \end{tikzcd}$$

$$\mathrm{Rep}_{\mathbb{B},\mathbb{E}}\, g\circ f = \left(\mathrm{Rep}_{\mathbb{D},\mathbb{E}}\, g\right) \cdot \left(\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f\right)$$

whoa!

Example

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} a-b \\ a+b \\ b \end{pmatrix}$$

$$g: \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} z \\ x+y+z \\ -y+2x \\ x \end{pmatrix}$$

$$g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto g\left(f\begin{pmatrix} a \\ b \end{pmatrix}\right) = g\begin{pmatrix} a-b \\ a+b \\ b \end{pmatrix} = \begin{pmatrix} b \\ a-b+a+b+b \\ -a-b+2a-2b \\ a-b \end{pmatrix} = \begin{pmatrix} b \\ 2a+b \\ a-3b \\ a-b \end{pmatrix}$$

$$\operatorname{Rep}_{\text{std}, \cancel{B,D}} f = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} =: A$$

$$\operatorname{Rep}_{\cancel{D}, \text{std}}\, g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =: B$$

$$\operatorname{Rep}_{\text{std}}\, g \circ f = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 1 & -3 \\ 1 & -1 \end{pmatrix} =: C$$

wha

$$BA = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0+0+0 & -0+0+1 \\ 1+1+0 & -1+1+1 \\ 2-1+0 & -2-1+0 \\ 1+0+0 & -1+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 1 & -3 \\ 1 & -1 \end{pmatrix} = C$$

yes!