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If $f\colon V \to W$ is a linear map, we call V the *domain* of f and W the *codomain* of f. In the definition of surjective, the codomain matters.

Example 1. Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Then f is NOT surjective, for example $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can never by $f\begin{pmatrix} x \\ y \end{pmatrix}$, for any values x, y.

Example 2. Let $g\colon \mathbf{R}^2 \to \mathbf{R}$ given by $f\begin{pmatrix} x \\ y \end{pmatrix} = x$. The map g *is* surjective. Indeed, if $a \in \mathbf{R}$ is arbitrary, then $a = f\begin{pmatrix} a \\ 129 \end{pmatrix}$.

In the two examples above, although f and g look like they are defining the same function, they are *NOT*! Their codomains are different.

What is an *isomorphism*? It's the correct way to identify one vector space with another.

Example 3. Let $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbf{R}\}$ the vector space of polynomials of degree at most two. We have a map $P_2 \to \mathbf{R}^3$ given by

$$a + bx + cx^2 \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This is an example of an isomorphism.

Example 4. Let V be an abstract vector space and let $B = (\vec{v}_1, \ldots, \vec{v}_n)$ be an ordered basis. Then

$$\begin{aligned} \mathrm{Rep}_B\colon V &\to \mathbf{R}^n \\ \vec{v} &\mapsto \mathrm{Rep}_B\,\vec{v} \end{aligned}$$

is another example of an isomorphism.

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Example 5. Another example is

$$\begin{aligned} M_{1\times n} &\to M_{n\times 1} \\ (a_1, \ldots, a_n) &\mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{aligned}$$

This is taking a row vector to its transpose, which is a column vector.

OK, let's give the precise definition.

Definition 6. A map $f\colon V \to W$ is a (linear) *isomorphism* if

- f is linear
- f is bijective

We write $V \simeq W$.

More examples! (page 170 of the book)

Example 7. Dilations. Pick $s \in \mathbf{R}$. Define

$$\begin{aligned} d_s\colon \mathbf{R}^2 &\to \mathbf{R}^2 \\ \vec{v} &\mapsto d_s\vec{v} := s\vec{v} \end{aligned}$$

the dilation of factor s. Check that, for $s \neq 0$, d_s is an isomorphism.

Example 8. Same example as before, but for an abstract vector space. Let $s \in \mathbf{R}$, V a vector space. Define $d_s\colon V \to V$ by $d_s(\vec{v}) = s\vec{v}$. If $s \neq 0$, then d_s is an isomorphism.

Example 9. A rotation of the plane by an angle θ, that's also an isomorphism.

Example 10. A reflection of the plane about some axis (through the origin), that's also an isomorphism.

OK, take a break from linear algebra to review some terminology.

Definition 11. If $f\colon X \to Y$ is a bijection, we may define $g\colon Y \to X$ by

$$g(y) = x$$

where

$x \in X$ is the unique element of X such that $f(x) = y$.

One usually calls this map the *inverse of* f and denotes it f^{-1}.

Example 12. Consider $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$. Then f is bijective. Its inverse is given by $f^{-1}(y) = \sqrt[3]{y}$.

Example 13. Consider $g\colon \mathbf{R} \to \mathbf{R}$ given by $g(x) = x^2$. Then g is neither injective nor surjective.

Example 14. Consider $h\colon \mathbf{R} \to \mathbf{R}_{\geq 0}$, given by $h(x) = x^2$. Here $\mathbf{R}_{\geq 0}$ means all $x \in \mathbf{R}$ satisfying $x \geq 0$. Then h is surjective but not injective.

Example 15. Consider $\sigma\colon \mathbf{R}_{\geq 0} \to R_{\geq 0}$ given by $\sigma(x) = x^2$. Then σ is a bijection.

Finally, we recall a useful criterion to know when a function is bijective.

Theorem 16. Let $f\colon X \to Y$ be a function. Then f is bijective if and only if there exists $g\colon Y \to X$ such that

$$g \circ f = \mathrm{id}_X$$
$$f \circ g = \mathrm{id}_Y$$

OK, let's go back to linear algebra.

> Fair question: if $f\colon V \to W$ is an isomorphism, then it has an inverse map $f^{-1}\colon W \to V$: is f^{-1} linear?

Yes.

Proposition 17. Let $f\colon V \to W$ be an isomorphism. Let $f^{-1}\colon W \to V$ be the inverse. Then f^{-1} is also an isomorphism.

Proof. For ease of notation, call $g := f^{-1}$. We need to check that

- $g(\vec{w}_1 + \vec{w}_2) = g(\vec{w}_1) + g(\vec{w}_2)$
- $g(\alpha\vec{w}) = \alpha g(\vec{w})$

for all $\vec{w}_1, \vec{w}_2, \vec{w} \in W$ and $\alpha \in \mathbf{R}$.

OK, let's do the first. call $\vec{v} := g(\vec{w}_1 + \vec{w}_2)$, $\vec{v}_1 := g(\vec{w}_1)$, $\vec{v}_2 := g(\vec{w}_2)$. By definition, $\vec{v}$ is the *unique* vector such that $f(\vec{v}) = \vec{w}_1 + \vec{w}_2$. Similarly for $\vec{v}_1, \vec{v}_2$. But f is *linear*, so

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) = \vec{w}_1 + \vec{w}_2 = f(\vec{v})$$

But f is injective, so $\vec{v}_1 + \vec{v}_2 = \vec{v}$. Hence,

$$g(\vec{w}_1 + \vec{w}_2) = \vec{v} = \vec{v}_1 + \vec{v}_2 = g(\vec{w}_1) + g(\vec{w}_2).$$

OK, let's do the second. Call $\vec{v} = g(\vec{w})$, $\vec{v}' = g(\alpha\vec{w})$. Since f is linear we have

$$f(\alpha\vec{v}) = \alpha f(\vec{v}) = \alpha\vec{w} = f(\vec{v}')$$

and since f is injective, we have $\alpha\vec{v} = \vec{v}'$. Hence,

$$g(\alpha\vec{w}) = \vec{v}' = \alpha\vec{v} = \alpha g(\vec{w}).$$

And we are done. □