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Last Time

$\dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$ for $\varphi : V \to W$ linear.

We also analyzed maps $f : \mathbb{R}^3 \to \mathbb{R}^2$

How to produce such f?

If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ is a matrix, define

$f_A : \mathbb{R}^3 \to \mathbb{R}^2$ by

$f_A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 z \\ a_2 x + b_2 y + c_2 z \end{pmatrix}$ (it's linear)

Notice: $f_A(\vec{e}_1) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $f_A(\vec{e}_2) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ $f_A(\vec{e}_3) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Example $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$ $f_A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $f_A \equiv \vec{0}$

$B := \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \end{pmatrix}$ $f_B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 3y + 3z \end{pmatrix}$

What is $\operatorname{Ker} f_B$?

$$\operatorname{Ker} f_B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| f_B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \right\} = \left\{ \vec{v} \in \mathbb{R}^3 \middle| f_B(\vec{v}) = \vec{0} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \begin{pmatrix} x + 2y + 3z \\ 3y + 3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

i.e. $\operatorname{Ker} f_B$ is the solution space of the homog. lin system

$$\begin{cases} x + 2y + 3z = 0 \\ 3y + 3z = 0 \end{cases}$$

So, ~~Sol~~ to solve the system, write the corresponding matrix

$$\cancel{\begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 \end{bmatrix}} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$ which is in echelon form

1 free variable $\Longrightarrow \dim Sol = \dim \operatorname{Ker} f_B = 1$

So, using the sacred formula,

$$\dim \operatorname{Im} f_B = \dim \mathbb{R}^3 - \dim \operatorname{Ker} f_B = 3 - 1 = 2$$

So f_B is surjective.

[btw, $\dim \operatorname{Im} f_B = rk\, B$]

Ex) $C = \begin{pmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \end{pmatrix}$ $\quad f_C \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + 0 \\ 4x - 2y + 0 \end{pmatrix}$

do same as before,

$$\operatorname{Ker} f_C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{pmatrix} 2x - y \\ 4x - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

to solve the linear system, write the corresponding matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \end{bmatrix}$$ which is not in echelon form

row op $R_2 \rightsquigarrow R_2 - 2R_1$

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

free vars $= 2 \implies \dim \mathrm{Sol} = \dim \mathrm{Ker} f_C = 2$

so $\dim \mathrm{Im} f_C = 3 - 2 = 1$

How to describe $\mathrm{Im} f_C$?

Well, $\mathrm{Im} f_C =$ "span of columns of C" (we will see this later)

so, $\dim \mathrm{Im} f_C = 1$

notice, $f_C(\vec{e}_2) = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \neq \vec{0}$

so $\begin{pmatrix} -1 \\ -2 \end{pmatrix} \in \mathrm{Im} f_C$ and $\dim \mathrm{Im} f_C = 1$

$\rightarrow \mathrm{Im} f_C = \mathrm{Span}\left\{\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right\} = \mathrm{Span}\{f(\vec{e}_2)\}$

<u>notice</u> $\mathrm{Span}\left\{\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right\}$

$= \mathrm{Span}\{f(\vec{e}_1)\}$

Exercise do opposite: $f: \mathbb{R}^2 \to \mathbb{R}^3$

Ok, say $\varphi: V \to W$ linear.

We know $\dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$.

Pick basis $\vec{k}_1, \dots, \vec{k}_s$ of $\operatorname{Ker}\varphi$

extend to basis $\vec{k}_1, \dots, \vec{k}_s, \vec{u}_1, \dots, \vec{u}_r$ for V

call $U := \operatorname{Span}\{\vec{u}_1, \dots, \vec{u}_r\}$

notice: $\dim V = s + r$ and $\dim \operatorname{Ker}\varphi = s$, $\dim U = r$.

notice: $V = \operatorname{Span}\{\vec{k}_1, \dots, \vec{k}_s, \vec{u}_1, \dots, \vec{u}_r\}$

so $\operatorname{Im}\varphi = \varphi(V) = \operatorname{Span}\{\varphi(\vec{k}_1), \dots, \varphi(\vec{k}_s), \varphi(\vec{u}_1), \dots, \varphi(\vec{u}_r)\}$

$= \operatorname{Span}\{\varphi(\vec{u}_1), \dots, \varphi(\vec{u}_r)\}$

~~so~~ $\dim \operatorname{Im}\varphi = r$ so $\varphi(\vec{u}_1), \dots, \varphi(\vec{u}_r)$ is <u>basis</u> for $\operatorname{Im}\varphi$.

Define new linear map $\psi : U \longrightarrow W$ by

$$\psi(\vec{u}) := \varphi(\vec{u})$$

ψ is called the "restriction" of φ to U

we write $\psi = \varphi|U$

Notice ψ is injective $\left(\ker\psi = (\ker\varphi) \cap U = \{\vec{0}\}\right)$

Why interesting? We have split up V in two pieces.

on one piece φ is identically zero $(\ker\varphi)$, on the other φ is injective (U)

Warning there are many choices for U!

Say $\vec{v} \in V$ then

$$\vec{v} = \underbrace{\alpha_1\vec{k}_1 + \dots + \alpha_s\vec{k}_s}_{\in \ \ker\varphi} + \underbrace{\beta_1\vec{u}_1 + \dots + \beta_r\vec{u}_r}_{\in \ U}$$

so $\vec{v} = \vec{k} + \vec{u}$ for $\vec{k} \in \ker\varphi$, $\vec{u} \in U$

it is truly a "splitting" of V.