MATH355 2017-11-08

COMPLEX MATRICES ARE SOMETIMES EASIER

For $A, B \in M_{2\times 2}$, recall we say $A \sim B$ (A is similar to B) if there exists an invertible matrix $P \in GL_n$ such that $B = PAP^{-1}$.

> We view similarity as follows. A and B are both representing the same linear map but in different bases. The relevant change of basis matrix is given precisely by P.

We call a matrix D *diagonal* if the only nonzero entries are on the diagonal. I.e., if $D = (d_{ij})$ then $d_i j = 0$ for $i \neq j$.

We call a matrix A *diagonalizable* of it is similar to a diagonal matrix. Explicitly, there exists P such that PAP^{-1} is a diagonal matrix. We say that P *diagonalizes* A.

Example 1. Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\det A = 1$ but A is not similar to I, the identity matrix. Why? Well, if $PAP^{-1} = I$ then $A = P^{-1}IP = P^{-1}P = I$, but $A \neq I$.

Example 2. Consider $B = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$. Take $P = \begin{pmatrix} \frac{-\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} \end{pmatrix}$. Verify that $P^{-1} = \begin{pmatrix} -\sqrt{\frac{3}{2}} & \sqrt{\frac{2}{3}} \\ 1 & 1 \end{pmatrix}$. A quick calculation reveals that

$$D := PBP^{-1} = \begin{pmatrix} -\sqrt{6} & 0 \\ 0 & \sqrt{6} \end{pmatrix}$$

which is a diagonal matrix!

Why is this so great? Well, remember we interpret similarity as a change of basis. Let us do it explicitly here. Write $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ for the linear map defined as

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ 3x \end{pmatrix}$$

so that $\mathrm{Rep}_{\mathrm{std}}\, g = B$. Now, consider the basis $\mathbb{B} = (\vec{v}_1, \vec{v}_2)$ where

$$\vec{v}_1 = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 1 \end{pmatrix}$$

so that $\mathrm{Rep}_{\mathbb{B},\mathrm{std}}\, \mathrm{id} = P^{-1}$ and $\mathrm{Rep}_{\mathbb{B}}\, g = D$.

Date: John Calabrese, November 13, 2017.

Wait, how did we figure out this last part (in the previous example) without having to compute anything? Well, first off, P^{-1} is an invertible matrix so its columns must form a basis (why?). The vectors $\vec{v}_1, \vec{v}_2$ are indeed the columns of P^{-1}.

What is $\mathrm{Rep}_{\mathbb{B},\mathrm{std}}\,\mathrm{id}$? Well, the first column is $\mathrm{Rep}_{\mathrm{std}}\,\vec{v}_1$ and the second column is $\mathrm{Rep}_{\mathrm{std}}\,\vec{v}_2$. So, $\mathrm{Rep}_{\mathbb{B},\mathrm{std}}\,\mathrm{id} = P^{-1}$. Also, we showed a long time ago that we always have

$$P = (P^{-1})^{-1} = \mathrm{Rep}_{\mathrm{std},\mathbb{B}}\,\mathrm{id}\,.$$

OK, now we nailed down our change of basis matrix. How do we compute $\mathrm{Rep}_{\mathbb{B}}\,g$? Once again, we know how to do this already (we've seen this formula many times!):

$$\mathrm{Rep}_{\mathbb{B}}\,g = \mathrm{Rep}_{\mathbb{B},\mathbb{B}}\,g = \mathrm{Rep}_{\mathrm{std},\mathbb{B}}\,\mathrm{Rep}_{\mathrm{std}}\,g\,\mathrm{Rep}_{\mathbb{B},\mathrm{std}} = PBP^{-1}.$$

So the basis $\mathbb{B}$ is the best possible basis for g, as $\mathrm{Rep}_{\mathbb{B}}\,g$ is diagonal! Indeed, you can easily calculate that

$$g(\vec{v}_1) = -\sqrt{6}\vec{v}_1$$
$$g(\vec{v}_2) = \sqrt{6}\vec{v}_2$$

The vectors $\vec{v}_1$ and $\vec{v}_2$ are called *eigenvectors* (for g). The scalars $-\sqrt{6}$, $\sqrt{6}$ are called *eigenvalues*.

Example 3. What about our $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$? Can A be diagonalized? In other words, can we find P such that PAP^{-1} is diagonal? More explicitly, does there exist P and scalars $\lambda, \mu \in \mathbf{R}$ such that

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}?$$

Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the linear map $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. So that $A = \mathrm{Rep}_{\mathrm{std}}\,f$. If A were diagonalizable, then (by proceeding in the same way as the previous example) there would exist a basis $\vec{v}_1, \vec{v}_2$ such that $f(\vec{v}_1) = \lambda\vec{v}_1$, $f(\vec{v}_2) = \mu\vec{v}_2$.

Suppose now we can find a vector $\vec{v} \in \mathbf{R}^2$ such that $f(\vec{v}) = \lambda\vec{v}$ (for whatever value of $\lambda \in \mathbf{R}$). Say $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, then

$$f(\vec{v}) = \begin{pmatrix} -b \\ a \end{pmatrix}$$
$$\lambda\vec{v} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$

so

$$\begin{cases} -b = \lambda a \\ a = \lambda b \end{cases}$$

so that $-b = \lambda a = \lambda^2 b$ which means $(\lambda^2 + 1)b = 0$. Since $(\lambda^2 + 1)$ is never zero (for *real* values of λ), we must have $b = 0$. But then the second equation above implies $a = 0$. So $\vec{v} = \vec{0}$.

This means we can *never* find the basis we want. So A cannot be diagonalized!

Nevertheless, we persist. Indeed, check this out. Consider the matrix $Q = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, where $i \in \mathbf{C}$ is that funny number such that $i^2 = -1$. Consider also $R = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$. A quick calculation will show that $QR = I$ is the identity! So $Q = R^{-1}$. Also, compute the following amazing fact

$$RAR^{-1} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

WAIT A MINUTE! The matrix on the RHS is diagonal! I thought we couldn't diagonalize A??? The point is, we lacked imagination.

In the example above we used that $(\lambda^2 + 1)$ is never zero. But if λ is allowed to be complex number, then that quantity *can* be zero: $(\pm i)^2 + 1 = 0$! So, we should have enlarged the domain of definition of A (and viewed it as a linear map of $\mathbf{C}^2$, rather than just $\mathbf{R}^2$).

The point being:

> sometimes we make things bigger to understand them.

For example, say we have a polynomial ax^2+bx+c and we wish to find the roots. Luckily, we have a formula that tells us the answer! However, even if $a, b, c \in \mathbf{R}$ the formula (implicitly) goes via the complex numbers. It's a good thing that we have a formula for the roots, even when the roots aren't real. The point being: if we didn't know about the complex numbers we couldn't have a homogeneous formula to calculate the roots of a quadratic polynomial. The same philosophy applies to diagonalizing matrices.

OK, what is a complex vector space anyway? Well, it's a set of vectors V with addition and scalar multiplication, only now the scalars are allowed to be complex numbers. For instance, if $\vec{v} \in V$ the quantity $(3 + 2i)\vec{v} = 3\vec{v} + i(2\vec{v}) \in V$ makes sense.

Example 4. $\mathbf{R}^3$ is *not* a complex vector space: the only meaning we give to the quantity $i\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ is the vector $\begin{pmatrix} 3i \\ i \\ 2i \end{pmatrix}$ which lives in $\mathbf{C}^3$ but not $\mathbf{R}^3$.

On the other hand, $\mathbf{C}^3$ is a complex vector space.

Now, if $\vec{v}_1, \ldots, \vec{v}_k \in \mathbf{C}^n$ are a bunch of vectors in $\mathbf{C}^n$ (or any complex vector space V), we should be careful to distinguish between two things.

A $\mathbf{C}$-*linear combination* is $\sum_j \lambda_j \vec{v}_j$ with $\lambda_j \in \mathbf{C}$

An $\mathbf{R}$-*linear combination* is $\sum_j \alpha_j \vec{v}_j$ with $\alpha_j \in \mathbf{R}$.

> This is crux the matter: whether we allow arbitrary complex numbers as scalars, or just real numbers.

Using this, we have different notions of span, linear independence, linear dependence, subspace, dimension etc.

Example 5. Take $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ i \end{pmatrix}$ in $\mathbf{C}^2$. Then

$$\begin{aligned} \mathrm{Span}_{\mathbf{C}}\{\vec{v}, \vec{w}\} &= \mathbf{C}^2 \\ \mathrm{Span}_{\mathbf{R}}\{\vec{v}, \vec{w}\} &\neq \mathbf{C}^2 \end{aligned}$$

(exercise!) indeed,

$$\mathrm{Span}_{\mathbf{R}}\{\vec{v}, \vec{w}\} = \{\begin{pmatrix} a \\ ib \end{pmatrix} \mid a, b \in \mathbf{R}\}$$

and this is only a *real* subspace of $\mathbf{C}^2$ but not a complex one (why? what does real/complex subspace mean? what's the difference?)

Example 6. The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ are a $\mathbf{C}$-basis of $\mathbf{C}^2$ (why?). While $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ are *not* an $\mathbf{R}$-basis of $\mathbf{C}^2$. However, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2i \end{pmatrix}$ *is* an $\mathbf{R}$-basis of $\mathbf{C}^2$ (why?).

So we see that $\dim_{\mathbf{C}} \mathbf{C}^2 = 2$ while $\dim_{\mathbf{R}} \mathbf{C}^2 = 4$.

In any case, don't stress too much about vector spaces over the complex numbers. I promise you that things will become obvious in a few days (you'll realize complex vector spaces aren't at all that complex).

We will follow the following convention:

> if we don't say anything, vector spaces, matrices, linear maps, bases, dimension etc. will be over the real numbers.