MATH355 2017-10-25

CHANGE THAT BASIS

Here is a reasonable question: suppose $\vec{v} \in V$ and $\mathbb{B}, \hat{\mathbb{B}}$ are two bases of V. What is the relation between $\operatorname{Rep}_{\mathbb{B}} \vec{v}$ and $\operatorname{Rep}_{\hat{\mathbb{B}}} \vec{v}$? I.e.

How do we pass from the coordinates of $\vec{v}$ wrt $\mathbb{B}$ and wrt to $\hat{\mathbb{B}}$?

Answer: a change of basis matrix.

Let $\mathrm{id}\colon V \to V$ be the *identity*. This is the map defined by $\mathrm{id}(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. The matrix $P := \operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}$ is called a *change of basis matrix*. Let $\vec{x} = \operatorname{Rep}_{\mathbb{B}} \vec{v}$, $\vec{y} = \operatorname{Rep}_{\hat{\mathbb{B}}} \vec{v}$. From the previous lecture, we know that[1]

$$\vec{y} = P\vec{x}.$$

The formula to remember is:

$$\boxed{\operatorname{Rep}_{\hat{\mathbb{B}}} \vec{v} = \left(\operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}\right) \operatorname{Rep}_{\mathbb{B}} \vec{v}} \tag{1}$$

Let's see an example. Consider $V = P_{\leq 2} = \{a_0 + a_1 x + a_2 x^2\}$ the vector space of polynomials of degree at most 2. We have an obvious basis $\mathbb{B} = (1, x, x^2)$, but also $\hat{\mathbb{B}} = (1 + x, 1 - x + x^2, 1)$ is a basis. What is the change of basis matrix $P := \operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}$? Well, let's see:

$$\begin{aligned} 1 &= 0(1+x) + 0(1-x+x^2) + 1 \\ x &= (1+x) + 0(1-x+x^2) - 1 \\ x^2 &= (1+x) + (1-x+x^2) - 2 \cdot 1 \end{aligned}$$

i.e.

$$\operatorname{Rep}_{\hat{\mathbb{B}}} 1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\operatorname{Rep}_{\hat{\mathbb{B}}} x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\operatorname{Rep}_{\hat{\mathbb{B}}} x^2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

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[1] Why is that? Well, call $f = \mathrm{id}$. The previous lecture told us that $\operatorname{Rep}_{\hat{\mathbb{B}}} f(\vec{v}) = (\operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} f)(\operatorname{Rep}_{\mathbb{B}} \vec{v})$. But $f(\vec{v}) = \vec{v}$, as $f = \mathrm{id}$.

thus

$$P := \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

Take for example $\vec{v} := 3 - x + 2x^2 \in V$. We have

$$\vec{x} := \operatorname{Rep}_{\mathbb{B}} \vec{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

how do we compute $\vec{y} = \operatorname{Rep}_{\hat{\mathbb{B}}} \vec{v}$? Using the formula above we have

$$\vec{y} = P\vec{x} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 - 1 + 2 \\ 0 + 0 + 2 \\ 3 + 1 - 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Indeed,

$$(1 + x) + 2(1 - x + x^2) + 0 \cdot 1 = 3 - x + 2x^2 = \vec{v}.$$

1. Change of basis for maps

OK, here's an even better question. Say we have V, W vector spaces, $f\colon V \to W$ a linear map and $\mathbb{B}, \hat{\mathbb{B}}, \mathbb{D}, \hat{\mathbb{D}}$ bases for V and W.

Is there a relation between $\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f$ and $\operatorname{Rep}_{\hat{\mathbb{B}},\hat{\mathbb{D}}} f$?

The answer is, once again, matrices (two of them, two changes of bases). Let $P := \operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}}$ and $Q := \operatorname{Rep}_{\mathbb{D},\hat{\mathbb{D}}}$ be the change of basis matrices. Let $A := \operatorname{Rep}_{\mathbb{B},\mathbb{D}} f$ and let $\hat{A} = \operatorname{Rep}_{\hat{B},\hat{D}}$ be matrices representing f. Then, the previous discussion combined with the last lecture gives us

$$\hat{A}P = QA$$

Notice that

$$\begin{aligned} &\hat{A}P = QA \\ \Rightarrow &(\hat{A}P)P^{-1} = (QA)P^{-1} \\ \Rightarrow &\hat{A}(PP^{-1}) = QAP^{-1} \\ \Rightarrow &\hat{A}I = QAP^{-1} \\ \Rightarrow &\hat{A} = QAP^{-1} \end{aligned}$$

where I is the identity matrix. In other words, the formula to remember is

$$\hat{A} = QAP^{-1} \tag{2}$$

i.e. the new matrix is equal to the old matrix multiplied left and right by changes of bases.

Can we say something more about P^{-1}? Like, what if we don't like the fact that it is an inverse of something?

Proposition 1. Let V be a vector space, let $\mathbb{B}, \hat{\mathbb{B}}$ be bases. Then

$$\operatorname{Rep}_{\hat{\mathbb{B}},\mathbb{B}} \mathrm{id} = \left(\operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}\right)^{-1}. \tag{3}$$

Proof. Well, remember that $\operatorname{Rep}_{\mathbb{D},\mathbb{E}} g \operatorname{Rep}_{\mathbb{B},\mathbb{D}} f = \operatorname{Rep}_{\mathbb{B},\mathbb{E}} g \circ f$. Take $\mathbb{D} = \hat{\mathbb{B}}$, $\mathbb{E} = \mathbb{B}$, $f = \mathrm{id} = g$. Then

$$\operatorname{Rep}_{\hat{B},B} \mathrm{id} \operatorname{Rep}_{\mathbb{B},\hat{B}} \mathrm{id} = \operatorname{Rep}_{\mathbb{B},\mathbb{B}} \mathrm{id} \circ \mathrm{id} = \operatorname{Rep}_{\mathbb{B},\mathbb{B}} \mathrm{id} = I$$

Let $P = \operatorname{Rep}_{\mathbb{B},\hat{B}} \mathrm{id}$, let $R = \operatorname{Rep}_{\hat{\mathbb{B}},\mathbb{B}}$. The equation above says $RP = I$.

$$\begin{aligned}
&RP = I \\
\Rightarrow &R^{-1}(RP) = R^{-1} I \\
\Rightarrow &(R^{-1}R)P = R^{-1} \\
\Rightarrow &IP = R^{-1} \\
\Rightarrow &P = R^{-1} \\
\Rightarrow &P^{-1} = \left(R^{-1}\right)^{-1} \\
\Rightarrow &P^{-1} = R.
\end{aligned}$$

□

Let's summarize the discussion above.

$$\left(\operatorname{Rep}_{\hat{\mathbb{B}},\hat{\mathbb{D}}} f\right)\left(\operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}\right) = \left(\operatorname{Rep}_{\mathbb{D},\hat{\mathbb{D}}} \mathrm{id}\right)\left(\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f\right) \tag{4}$$

which implies

$$\left(\operatorname{Rep}_{\hat{\mathbb{B}},\hat{\mathbb{D}}} f\right) = \left(\operatorname{Rep}_{\mathbb{D},\hat{\mathbb{D}}} \mathrm{id}\right)\left(\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f\right)\left(\operatorname{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}\right)^{-1} \tag{5}$$

which may also be read as

$$\left(\operatorname{Rep}_{\hat{\mathbb{B}},\hat{\mathbb{D}}} f\right) = \left(\operatorname{Rep}_{\mathbb{D},\hat{\mathbb{D}}} \mathrm{id}\right)\left(\operatorname{Rep}_{\mathbb{B},\mathbb{D}} f\right)\left(\operatorname{Rep}_{\hat{\mathbb{B}},\mathbb{B}} \mathrm{id}\right) \tag{6}$$

2. Once more

Here is a slightly more highbrow way to look at things. First off, any linear map $\phi\colon \mathbf{R}^n \to \mathbf{R}^m$ is given by a matrix B. Explicitly, given ϕ there exists a unique matrix $B \in M_{m\times n}$ such that $\phi(\vec{x}) = B\vec{x}$ for any column vector $\vec{x} \in \mathbf{R}^n$.[2]

[2] How do we see this? Well, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n x_i \vec{e}_i$. By linearity, $\phi(\vec{x}) = \sum_{i=1}^n x_i \phi(\vec{e}_i)$. For each j, $\phi(\vec{e}_j) \in \mathbf{R}^m$, so it's also a column vector (but of size m). Let B be the matrix whose j-th column is $\phi(\vec{e}_j)$. Using the definition of matrix multiplication, we see that $B\vec{x} = \phi(\vec{x})$.

Let V be a vector space and let $\mathbb{B} = (\vec{b}_1, \dots, \vec{b}_n)$ be a basis.[3] Let $n = \dim V$. We have an isomorphism $\mathrm{Rep}_{\mathbb{B}}\colon V \to \mathbf{R}^n$, which takes $\vec{v}$ to its column vector of coordinates $\mathrm{Rep}_{\mathbb{B}}\,\vec{v}$. Recall that its inverse is given by sending a vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ to $\sum_{i=1}^{n} x_i \vec{b}_i$.

If $f\colon V \to W$ is a linear map, we can also fix a basis $\mathbb{D}$ for W. We summarize our current situation with a diagram.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\mathrm{Rep}_{\mathbb{B}}"'] & W \arrow[d, "\mathrm{Rep}_{\mathbb{D}}"] \\ \mathbf{R}^n & \mathbf{R}^m \end{tikzcd}$$

> Question: is there a linear map $\mathbf{R}^n \to \mathbf{R}^m$ making the diagram above into a commutative square?[4]

Linear maps $\mathbf{R}^n \to \mathbf{R}^m$ are the same thing as matrices, so we are looking for a matrix to close up the square. Of course, the answer is given by $\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\mathrm{Rep}_{\mathbb{B}}"'] & W \arrow[d, "\mathrm{Rep}_{\mathbb{D}}"] \\ \mathbf{R}^n \arrow[r, "\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f"] & \mathbf{R}^m \end{tikzcd}$$

Indeed, the characterizing property of $\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f$ was

$$\boxed{\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f\, \mathrm{Rep}_{\mathbb{B}}\, \vec{v} = \mathrm{Rep}_{\mathbb{D}}\, f(\vec{v})} \tag{7}$$

which is precisely saying: start in the upper left corner with $\vec{v}$: doing down-right is the same as doing right-down.

However, the awesomeness of this Rep construction does not stop here! It has at least three supernatural powers.

First,

$$\mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, 0 = 0 \tag{8}$$

where the first 0 indicates the zero linear map and the second zero indicates the zero matrix.

Second,

$$\mathrm{Rep}_{\mathbb{B},\mathbb{B}}\, \mathrm{id} = I$$

where id is the identity map and I is the identity matrix.

But the real superpower of Rep is the compatibility with compositions.[5] What does this mean? If V, W, U are vector spaces, $f\colon V \to W$ $g\colon W \to U$ are linear maps, and $\mathbb{B}, \mathbb{D}, \mathbb{E}$ are bases, then we can write two different diagrams. The first is

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[3] Don't forget: the order of the basis vectors matters!

[4] A square diagram is *commutative* if we start in the top left corner and end up in the bottom right corner it didn't matter if we did "down followed by right" or "right followed by down".

[5] This "compatibility" is sometimes called *functoriality*.

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\mathrm{Rep}_{\mathbb{B}}"'] & W \arrow[r, "g"] \arrow[d, "\mathrm{Rep}_{\mathbb{D}}"'] & U \arrow[d, "\mathrm{Rep}_{\mathbb{E}}"'] \\
\mathbf{R}^n \arrow[r, "\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f"'] & \mathbf{R}^m \arrow[r, "\mathrm{Rep}_{\mathbb{D},\mathbb{E}} g"'] & \mathbf{R}^r
\end{tikzcd}$$

and the second is

$$\begin{tikzcd}
V \arrow[r, "g\circ f"] \arrow[d, "\mathrm{Rep}_{\mathbb{B}}"'] & U \arrow[d, "\mathrm{Rep}_{\mathbb{U}}"] \\
\mathbf{R}^n \arrow[r, "\mathrm{Rep}_{\mathbb{B},\mathbb{U}} g\circ f"] & \mathbf{R}^m
\end{tikzcd}$$

But here is the magic: composing the horizontal rows of the first diagram gives the horizontal rows of the second diagram! Indeed, $g \circ f = g \circ f$, but also

$$\mathrm{Rep}_{\mathbb{D},\mathbb{E}}\, g\, \mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f = \mathrm{Rep}_{\mathbb{B},\mathbb{E}}\, g \circ f. \tag{9}$$

The added bonus is that all this change of basis stuff is a mere consequence of what we just said about Rep. Let's see what we mean by this. Consider $W = V$, $\mathbb{D} = \hat{\mathbb{B}}$ and $f = \mathrm{id}$. Then (7) tells me that

$$\mathrm{Rep}_{\mathbb{B},\hat{\mathbb{B}}}\, \mathrm{id}\, \mathrm{Rep}_{\mathbb{B}}\, \vec{v} = \mathrm{Rep}_{\hat{\mathbb{B}}}\, \vec{v}. \tag{10}$$

OK, take now the square

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & W \arrow[d, "\mathrm{id}"] \\
V \arrow[r, "f"] & W
\end{tikzcd}$$

from the vector space point of view, nothing is happening. However, by picking different bases, a lot is going on in terms of coordinates. Indeed, the square above becomes

$$\begin{tikzcd}
\mathbf{R}^n \arrow[r, "\mathrm{Rep}_{\mathbb{B},\mathbb{D}} f"] \arrow[d, "\mathrm{Rep}_{\mathbb{B},\hat{\mathbb{B}}} \mathrm{id}"'] & \mathbf{R}^m \arrow[d, "\mathrm{Rep}_{\mathbb{D},\hat{\mathbb{D}}} \mathrm{id}"] \\
\mathbf{R}^n \arrow[r, "\mathrm{Rep}_{\hat{\mathbb{B}},\hat{\mathbb{D}}} f"'] & \mathbf{R}^m
\end{tikzcd}$$

and since $f \circ \mathrm{id} = \mathrm{id} \circ f$ in the previous square, we must have

$$\mathrm{Rep}_{\hat{\mathbb{B}},\hat{\mathbb{D}}}\, f\, \mathrm{Rep}_{\mathbb{B},\hat{\mathbb{B}}}\, \mathrm{id} = \mathrm{Rep}_{\mathbb{D},\hat{\mathbb{D}}}\, \mathrm{id}\, \mathrm{Rep}_{\mathbb{B},\mathbb{D}}\, f \tag{11}$$

To see how Rep is bridging V, W with $\mathbf{R}^n, \mathbf{R}^m$, it might be helpful to draw the following cubical diagram.

$$\begin{tikzcd}[row sep=small, column sep=small]
V \arrow[rr] \arrow[dr] \arrow[dd] & & W \arrow[dr] \arrow[dd] & \\
& \mathbf{R}^n \arrow[rr] \arrow[dd] & & \mathbf{R}^m \arrow[dd] \\
V \arrow[rr] \arrow[dr] & & W \arrow[dr] & \\
& \mathbf{R}^n \arrow[rr] & & \mathbf{R}^m
\end{tikzcd}$$

where each edge is labelled by the appropriate Rep function and all the faces are commutative squares.