MATH 355 HOMEWORK 10

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Problem 1

I picked the third row and column because they both have two zeros, so the Laplace expansions will only involve computing two 3×3 determinants each. Let

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 3 & 1 & 1 \\ -1 & 0 & 0 & 2 \\ 1 & -1 & 3 & 1 \end{pmatrix}.$$

Then

$$\det A = (-1)\begin{vmatrix} 1 & -2 & 4 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -2 & 4 \\ 0 & 3 & 1 \\ -1 & 0 & 2 \end{vmatrix}.$$

The first determinant on the right-hand side of the above equation is zero because (for example) the third column is -2 times time the first column minus 3 times the second column. We compute the other determinant by expanding along the first column.

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 4 \\ 0 & 3 & 1 \\ -1 & 0 & 2 \end{vmatrix} &= 1\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} \\ &= 6 - (-2 - 12) \\ &= 20. \end{aligned}$$

We conclude that $\det A = -60$. Now we expand along the third column in one heroic calculation

$$\begin{aligned} \det A &= (-1)\begin{vmatrix} -2 & 0 & 4 \\ 3 & 1 & 1 \\ -1 & 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 3 \end{vmatrix} \\ &= (-1)\left((-2)\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}\right) + (-2)\left((1)\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}\right) \\ &= (-1)(-2(1-3) + 4(9+1)) + (-2)(9 + 1 + 2(-1)) \\ &= (-1)(44) + (-2)(8) \\ &= -44 - 16 \\ &= -60. \end{aligned}$$

Problem 2

Part a. This map is $\mathbf{R}$-linear. To see that γ is additive let $z = a + bi$ and $w = c + di$ for $a, b, c, d \in \mathbf{R}$. Then

$$\gamma(z + w) = \gamma(a + bi + c + di) = \gamma((a + c) + (b + d)i) = a + c - (b + d)i = a - bi + c - di = \gamma(z) + \gamma(w).$$

Now let $\alpha \in \mathbf{R}$ and consider

$$\gamma(\alpha z) = \gamma(\alpha(a + bi)) = \gamma(\alpha a + \alpha bi) = \alpha a - \alpha bi = \alpha(a - bi) = \alpha\gamma(z).$$

We conclude that the map is $\mathbf{R}$-linear.

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Part b. The map is not $\mathbf{C}$-linear. For a counterexample, observe that

$$\gamma(i \cdot i) = \gamma(-1) = -1 \neq 1 = i \cdot (-i) = i\gamma(i).$$

Problem 3

Part a. $\mathbf{B} = \{1, i\}$ forms an $\mathbf{R}$-basis for $\mathbf{C}$. Indeed we know that any complex number (i.e. an element of $\mathbf{C}$) can be written as $a+bi$ for $a, b \in \mathbf{R}$. This shows that $\mathbf{B}$ is spanning. To see that $\mathbf{B}$ is linearly independent over $\mathbf{R}$ is suffices to observe that there is no real number α such that $\alpha \cdot 1 = \alpha = i$. That is, it suffices to show that i is not a real number, and we know that it is not.

Remark. To actually show that i is not a real number rather than merely saying it requires some argument that there is no square root of -1 in the real numbers. Such an argument is not hard to give but it's not enlightening in this context, so we omit this detail.

Part b. Two vector spaces over $\mathbf{R}$ are isomorphic if and only if they have they have the same dimension. The previous part shows that $\mathbf{C}$ has dimension two as a real vector space, and we know that $\mathbf{R}^2$ has dimension two as a real vector space, so $\mathbf{C} \cong \mathbf{R}^2$ as real vector spaces.

Problem 4

Part a. $\mathbf{B} = \{1\}$ forms a basis for $\mathbf{C}$ over $\mathbf{C}$. This set is spanning since for any $z \in \mathbf{C}$, we can write $z = z \cdot 1$. Moreover the set is linearly independent. Suppose that $z \cdot 1 = 0$ for some $z \in \mathbf{C}$, then we immediately see that $z = 0$, so the set is linearly independent.

Part b. The set $\{1, i\}$ forms an $\mathbf{R}$-basis for $\mathbf{C}$ as we showed in Part a of Problem 3.

Remark. The terms "extract" may have been a bit confusing. The question is just asking how to find an $\mathbf{R}$-basis given that you have a $\mathbf{C}$-basis in hand.

Problem 5

Part a. This is a special case ($V = \mathbf{C}^2$) of Part a of Problem 6. We give a fuller argument there.

Part b. We can view $\mathbf{C}^2$ as the set of ordered pairs of complex numbers. That is $\mathbf{C}^2 = \{(z, w) \mid z, w \in \mathbf{C}\}$. We claim that $\mathbf{B} = \{(1, 0), (0, 1)\}$ is a $\mathbf{C}$-basis for $\mathbf{C}^2$. This is proved *mutatis mutandis* as proving $\mathbf{R}^2$ is a two-dimensional vector space over the real numbers. In particular suppose that there are complex numbers $\alpha, \beta \in \mathbf{C}$ such that $\alpha(1, 0) + \beta(9, 1) = (0, 0)$. Then we have $(\alpha, \beta) = (0, 0)$, so $\alpha = 0$ and $\beta = 0$. We then find that $\mathbf{B}$ is linearly independent. Moreover it is clearly spanning as for any element $(\alpha, \beta) \in \mathbf{C}^2$, we have $\alpha(1, 0) + \beta(0, 1) = (\alpha, \beta)$.

Part c. We claim that $\mathbf{B} = \{(1, 0), (i, 0), (0, 1), (0, i)\}$ is an $\mathbf{R}$-basis for $\mathbf{C}^2$. To see linear independence, suppose that $a, b, c, d \in \mathbf{R}$ and $a(1, 0) + b(0, i) + c(1, 0) + d(0, i) = (0, 0)$. Then $(a + bi, c + di) = (0, 0)$. Then $a + bi = 0$ and $c + di = 0$. A complex number is zero if and only if its real and imaginary parts are equal to zero, so we conclude that $a, b, c, d = 0$ and that $\mathbf{B}$ is linearly independent. To see spanning we observe that any pair of complex numbers (z, w) can be represented by $(a+bi, c+di)$ for $a, b, c, d \in \mathbf{R}$. Thus $(z, w) = a(1, 0) + b(0, i) + c(1, 0) + d(0, i)$, and $\mathbf{B}$ is spanning.

Part d. This is the same as Part b of Problem 3: Both $\mathbf{C}^2$ and $\mathbf{R}^4$ have a basis consisting of four elements, so both of them are dimension four and hence isomorphic.

Problem 6

Part a. Recall the axioms for a vector space. All of the axioms that don't make reference to the underlying scalars (in our case either $\mathbf{R}$ or $\mathbf{C}$) are immediate. For example if one can add two vectors in a $\mathbf{C}$-vector space then one can add them in exactly the same way in the space viewed as an $\mathbf{R}$-vector space. For the axioms involving the scalars, we satisfy the axioms by restricting the multiplication. For example one of the axioms specifies that for any $\alpha \in \mathbf{C}$ and $\vec{v} \in V$, we have $\alpha\vec{v} \in V$. Since $\mathbf{R}$ is a subset of $\mathbf{C}$ we in particular know that for any $\alpha \in \mathbf{R}$ and $\vec{v} \in V$, we have $\alpha\vec{v} \in V$, so the analogous axiom for real vector spaces is satisfied. The other axioms are demonstrated in the same way.

Part b. Suppose that $\mathbf{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a $\mathbf{C}$-basis for V. We claim that $\mathbf{D} = \{\vec{v_1}, i\vec{v_1}, \ldots, \vec{v_n}, i\vec{v_n}\}$ is an $\mathbf{R}$-basis for V. To see linear independence, suppose that there exist real numbers $a_1, a_1', \ldots a_n, a_n'$ such that $a_1\vec{v_1} + a_1' \cdot i\vec{v_1} + \ldots + a_n\vec{v_n} + a_n' \cdot i\vec{v_n} = 0$. Then we have that $(a_1 + ia_1')\vec{v_1} + \ldots + (a_n + ia_n')\vec{v_n} = 0$. This is a $\mathbf{C}$-linear combination of vectors from $\mathbf{B}$, which is linearly independent. Then we find that $a_j + ia_j' = 0$ for all $1 \leq j \leq n$. Hence $a_j = a_j' = 0$ for all $1 \leq j \leq n$, so $\mathbf{D}$ is linearly independent. To see that $\mathbf{D}$ is spanning, write an arbitrary element $\vec{v} \in V$ as a $\mathbf{C}$-linear combination of elements of $\mathbf{B}$: $\vec{v} = b_1\vec{v_1} + \ldots + b_n\vec{v_n}$. We can now write each b_j as $b_j = a_j + a_j'$ for $a_j, a_j' \in \mathbf{R}$. Then we have $\vec{v} = (a_1 + ia_1')\vec{v_1} + \ldots + (a_n + ia_n')\vec{v_n} = a_1\vec{v_1} + a_1' \cdot i\vec{v_1} + \ldots + a_n\vec{v_n} + a_n' \cdot i\vec{v_n}$, which is an $\mathbf{R}$-linear combination of elements of $\mathbf{D}$, so we see that $\mathbf{D}$ is spanning. We conclude that $\mathbf{D}$ is a basis.

Remark. Part b of Problem 4 and Part c of Problem 5 are special cases of the above exercise.

Part c. Let $n = \dim_{\mathbf{C}} V$, so we can write down an $\mathbf{C}$-basis consisting of n elements: $\mathbf{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$. By the previous part, we know that $\mathbf{D} = \{\vec{v_1}, i\vec{v_1}, \ldots, \vec{v_n}, i\vec{v_n}\}$ is an $\mathbf{R}$-basis for V. Observe that $\mathbf{D}$ has twice as many elements as $\mathbf{B}$, so $\dim_{\mathbf{R}} V = 2n = 2\dim_{\mathbf{C}} V$.