MATH355 2017-09-27

KERNELS

f is linear, f is a homomorphism, f is a linear operator... are all synonyms. When we write

$$f \in \mathrm{Hom}(V, W)$$

we mean

"$f\colon V \to W$ is a linear map going from V to W."

Last time we showed that any linear map f sends $\vec{0}$ to $\vec{0}$. Let's give some examples now.

Example 1. The first is *rotation by* $90°$. Concretely, we define

$$f\colon \mathbf{R}^2 \to \mathbf{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Is f linear? We need to check two conditions. First, if $\vec{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ we must show that $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$. We write

$$f\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} -(y_1 + y_2) \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} -y_2 \\ x_2 \end{pmatrix} = f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Second, if $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$ and $\alpha \in \mathbf{R}$ we need to show that $f(\alpha\vec{v}) = \alpha f(\vec{v})$. We write

$$f(\alpha\vec{v}) = f\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \begin{pmatrix} -\alpha y \\ \alpha x \end{pmatrix} = \alpha\begin{pmatrix} -y \\ x \end{pmatrix} = \alpha f\begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence f is linear.

Example 2. Consider now instead $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ sending $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Is g linear? (yes, exercise!) Geometrically, this is the projection of $\mathbf{R}^2$ onto the x-axis, but viewed as a map $\mathbf{R}^2 \to \mathbf{R}^2$.

Before we move on, we must recall some terminology.

Definition 3. Let $f\colon X \to Y$ a map between any two sets.

We say f is *injective* if

whenever $f(x) = f(x')$ then $x = x'$.

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In other words: "f does not squash anything"

In other words: "f reaches all of Y"

We say f is *surjective* if

$$\text{for all } y \in Y, \text{ there is an } x \in X \text{ with } f(x) = y.$$

Finally, we say f is *bijective* if it is both injective and surjective.

Example 4. We saw last time that if $f\colon \mathbf{R} \to \mathbf{R}$, then $f(x) = ax$ for some $a \in \mathbf{R}$. Show that there is a bijection between the set of linear maps $\mathrm{Hom}(\mathbf{R}, \mathbf{R})$ and $\mathbf{R}$ itself.

Example 5. Let V be any vector space and let $\vec{u}, \vec{v} \in V$ be any two vectors. Consider the map

$$\psi\colon \mathbf{R}^2 \to V$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x\vec{u} + y\vec{v}.$$

Then,

ψ is injective if and only if $\vec{u}, \vec{v}$ are linearly independent
ψ is surjective if and only if $\mathrm{Span}\{\vec{u}, \vec{v}\} = \mathbf{R}^2$.

Exercise for you.

More generally, in the example above we could consider n vectors $\vec{v}_1, \ldots, \vec{v}_n \in V$ and the corresponding $\psi\colon \mathbf{R}^n \to V$. The same two conclusions would hold.

Example 6. Consider $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ from above. Is f injective? Is f surjective? Exercise.

Example 7. Consider $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ from above. Is f injective? No! Why? Well, $f\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = f\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. So f cannot be injective.

Definition 8. Let $f \in \mathrm{Hom}(V, W)$. The *kernel* of f is the subset of V defined by

$$\ker f := \{\vec{v} \in V \mid f(\vec{v}) = \vec{0}\} \subset V$$

Proposition 9. Let $f \in \mathrm{Hom}(V, W)$. Then $\ker f$ is a subspace of V.

Proof. We need to check three things. First, is $\vec{0} \in \ker f$? Well, since $f(\vec{0}) = \vec{0}$, yes.

Second, if $\vec{u}, \vec{v} \in \ker f$, does $\vec{u} + \vec{v} \in \ker f$? Well, $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v}) = \vec{0} + \vec{0} = \vec{0}$, where the second-to-last equality follows from the fact that $\vec{u}, \vec{v} \in \ker f$. Hence, $\vec{u} + \vec{v} \in \ker f$.

Third, if $\alpha \in \mathbf{R}$, $\vec{v} \in \ker f$, does $\alpha\vec{v} \in \ker f$? Well, $f(\alpha\vec{v}) = \alpha f(\vec{v}) = \alpha\vec{0} = \vec{0}$. □

Proposition 10. Let $f \in \mathrm{Hom}(V, W)$. Then f is injective if and only if $\ker f = \{\vec{0}\}$.

Proof. Let us prove one direction. Suppose f is injective. Let $\vec{v} \in \ker f$, i.e. $f(\vec{v}) = \vec{0}$. But $f(\vec{0}) = \vec{0}$, hence by injectivity $\vec{v} = \vec{0}$. Hence, $\ker f = \{\vec{0}\}$.

Let us prove the converse direction. Suppose $\ker f = \{\vec{0}\}$. We want to show f is injective. Suppose $f(\vec{u}) = f(\vec{v})$, we need to show that in fact $\vec{u} = \vec{v}$. Well, $f(\vec{u}) = f(\vec{v})$ so $f(\vec{u}) - f(\vec{v}) = \vec{0}$, but then

$$\vec{0} = f(\vec{u}) - f(\vec{v}) = f(\vec{u}) + f(-\vec{v}) = f(\vec{u} - \vec{v})$$

Hence, $\vec{u} - \vec{v} \in \ker f$. But $\ker f = \{\vec{0}\}$, hence $\vec{u} - \vec{v} = \vec{0}$. This implies $\vec{u} = \vec{v}$. We have shown that f is injective. $\square$

Example 11. Consider $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ from before, $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. What is the kernel of g? Well

$$\ker g = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \mid g\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}\} = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \mid \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \mid x = 0\} = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbf{R}\}$$

I.e., the kernel of g is precisely the y-axis.