John Capetanakis

Dr. Finn

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Appendix C Notes

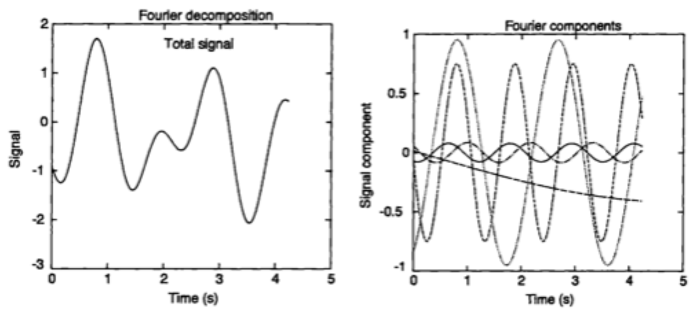
- We start off with a hypothetical signal to interpret where we can study how as a function its quantities can vary with time. In this signal y(t), it displays an oscillatory character, which is due to it consisting of the addition of five sine waves. The signal y(t) is expressed as:

$$y(t) = \sum_{j=1}^{5} y_j \sin(2\pi f_j t + \phi_j) ,$$

- Up above, we see that y_j is our amplitude, f_j is our frequency, and phi_j iis the phase of our jth sine wave
component.
- Since a lot of signals are more complicated than a simple signal over time (which partially is because there is a large sum of sine waves), we can express the y(t) function as an integral over frequency with an angular frequency of omega = 2*pi*f, and this is expressed as:

$$y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{-2\pi i f t}\, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega/2\pi)\, e^{-i\omega t}\, d\omega$$

- This is what the hypothetical signal looks like that I mentioned before in the notes (for a Fourier decomposition), as well as what a sum of individual sine waves would resemble (for the sum of the Fourier components):



- The integral for y(t) takes functions that operate in one direction, and the inverse transform takes functions in the opposite direction; these are called forward and inverse Fourier transforms. While it does not matter which one is forward or inverse, if the forward transform is followed by a backwards transform, then we return to the original function.
- When working with Fourier transforms, we need to be able to determine how to actually compute one. If we know our signal analytically (meaning we are given its functional form), then we could take the Fourier transform by doing an integral, yet this is not always the case. However, we would be able to have some knowledge of its discrete values at a time t for both computational and experimental data. When we are provided with some data, we could then make separate discrete Fourier transform equations for ym and Yn:

$$y_m = \frac{1}{N} \sum_{n=0}^{N-1} Y_n \, e^{-2\pi i mn/N}$$

$$Y_n = \sum_{m=0}^{N-1} y_m \, e^{2\pi i mn/N} ,$$

- For these transforms, if we have a number of N data points (where N values ym), then we must have N values of Yn. While we can express a majority of the physical numbers as real numbers, the Yn can be complex because of the numbers it consists of. If Yn is complex, then we have 2N pieces of information in our frequency; if not (Ym is real), then we have N pieces of information in our time domain.
- For the Fourier transform, it is important to note that 1) we need both the real and imaginary parts of Yn, we use N pieces of information, and 2) while the real part of the

transform presents a true value of ym, the imaginary part produces non-zero incorrect values. This is because of how the N pieces of information in the frequency domain cannot reconstruct 2N pieces of information in the time domain.

- The previously mentioned Fourier transforms above were a sum of exponential terms are computationally difficult to approach, where we see each term incorporating a computation of some exponential factor, each sum has a given frequency with N terms and components, which means a straightforward approach would not be ideal here, since this is messy.
- The exponential terms in the discrete Fourier transforms could all be multiplied on each other, making it possible to evaluate these transforms with an order of N log N, providing a time reduction. The specific algorithms that make this process faster are called fast fourier transforms (FFT).
- FFTs are excellent examples of improvements in calculation efficiency that makes harder, more impractical tasks easier to handle.
- The basic way to approach an FFT is by summing a small set of data points. We can see below how this can be shown below by splitting a sum into two parts (where one works with the even m number and one works with the odd m number):

$$
\begin{aligned}
Y_n &= Y_n^e + w^n Y_n^o \\
&= \sum_{m'=0}^{3} y_{2m'}\, w^{2m'n} + w^n \sum_{m'=0}^{3} y_{2m'+1}\, w^{2m'n} ,
\end{aligned}
$$

- For discrete Fourier transforms, the two parameters we can usually control are the sampling interval and the number of points we can sample. The sampling interval determines the range of how many spectral frequencies can be represented, whereas the number of points we are able to sample is the amount of details that can be obtained from this transform. We also need to determine if we are able to see if the number of data points matches the FFT algorithm it is supposed to match, and if the sampling duration matches the whole periods of its signal.
- The easiest signal to learn how to use for a Fourier analysis is a pure sine wave, where the signal period is 4.3 s and the sampling period is .1s. When using a sampling period of 12.8 seconds (we want to use evenly space our recording over three periods) and taking in 128 signals, we produce 128 Fourier amplitudes, where half are sine wave amplitudes and the other half are cosine wave amplitudes. The only nonzero value that it produces is the original sine wave we started with.

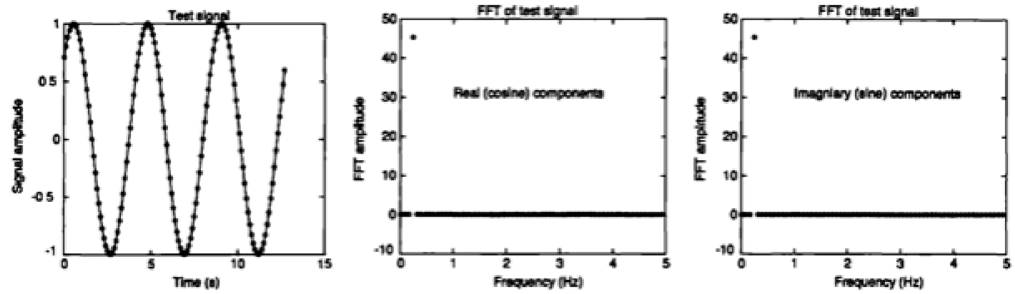


FIGURE C.3: Left Test signal that is just a sine wave that is phase shifted by $\pi/4$ The dots are the data points that were used in the FFT, center and right FFT of this test signal

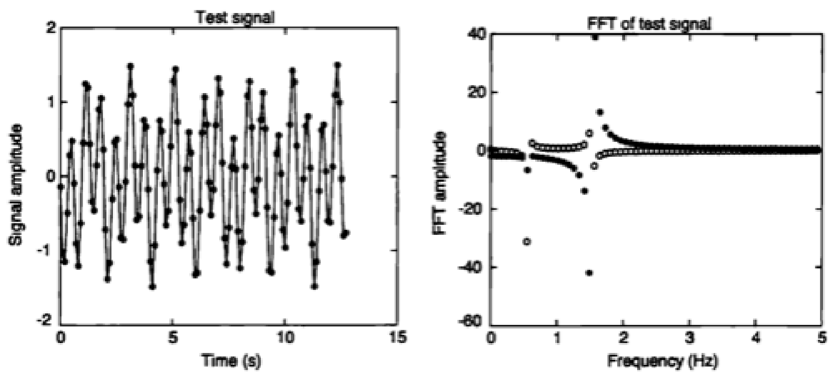


FIGURE C.4: Left Test signal which is the sum of two sine waves with different frequencies, amplitudes, and phases, right FFT of this test signal The filled circles show the real (cosine) components, while the open circles are the imaginary (sine) amplitudes

- Following this procedure, we use the same signal but shift it by adding a phase factor of pi/4. This FFT is more complicated, as it produces one nonzero sine component and one nonzero cosine component.
- The first two FFT results have been simple to interpret because of how the period of the signal has matched the sampling time, allowing it to make three **complete** periods. If this was not the case, we would have a more complicated appearance. For example, if we had a signal with two sine waves that had different frequencies, we would have large components of the sine waves used to make the original signal, but then we would also find that our amplitudes could be small or nonzero over a range of frequencies.
- In the sampling method, the FFT gives a near perfect analysis of the Fourier components so long as the frequencies are below the Nyquist frequency ($1/2\Delta t$). If we are met with a condition where this case is not satisfied, then we have a frequency that is much lower than our signal frequency.

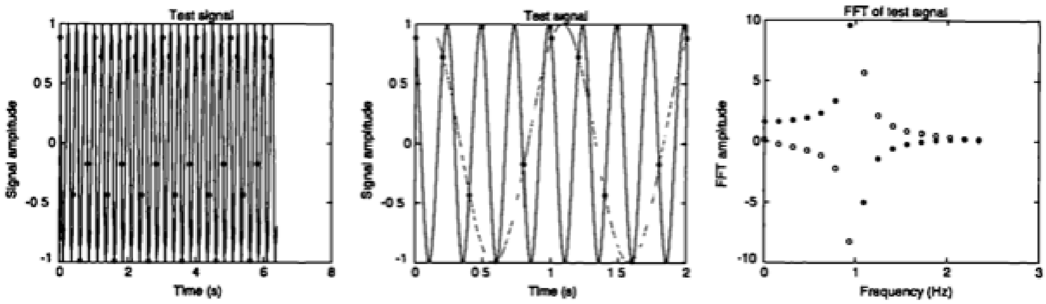


FIGURE C.6: Left test signal that is a single sine wave The solid curve is what the signal would look like if it were sampled at a very large number of points, while the solid symbols show the 32 data points that were analyzed in the FFT Center an expanded view of the signal The dotted curve shows a second sine wave, which has a much lower frequency (1 Hz) and which is seen to also pass through all of the data points Right FFT. The filled circles are the real (cosine) components, while the open circles are the imaginary (sine) components

- The folding back of frequencies that are above the Nyquist frequency is referred to as aliasing. We use aliasing when trying to determine the discrete elements to capture a continuous signal.
- It is common to have the Nyquist frequency be higher than the Fourier components that are present. If the frequency components are below other components in the signal, then we get exposed to more problems. For this reason, it is important to use a small sampling interval to enable us to have a Nyquist frequency lie above certain present frequencies, but we need to have a reasonable number of points in a small region when compared to a whole spectrum.
- In the case where we want to observe the relative amplitudes and frequencies of the Fourier transforms (but do not care about their phases), we can display the Fourier transform with its autocorrection (it measures how well a signal y is correlated across times with a periodicity of τ. This autocorrection, along with its power spectrum can be shown as:

$$Corr[y](\tau) = \int_{-\infty}^{\infty} y(t)^{*} y(t+\tau) d\tau \ ,$$

and the power spectrum is given by

$$PS[y](f) = \int_{-\infty}^{\infty} y(t)^{*} y(t+\tau) e^{2\pi i f \tau} d\tau = |Y(f)|^2 \ .$$

- As a result of measuring the signal y over the periodicity, we can observe that the power spectrum will also have its peaks conjugate at similar values of frequencies equivalent to $1/\tau$.

- For a discrete case, we can observe that the average corresponding power PJ at a frequency fj is proportional to the sum of the squares of the amplitudes of the cosine (real) and the sine (imaginary components at frequency fj. This can be listed as:
    - **Pj** = Yj(real)^2 + Yj(imaginary)^2.
- Note that the FFT results for Yj(real) and Yj(imaginary) are used to compute powers at each frequency.

Questions

- Which transform method is the easiest to use?
- Is there ever a case where we can observe Nyquist frequency of 0?
- Does a FFT ever produce more than one nonzero sine/cosine component?