I2MC 2019 - Combinatorics and Geometry

Interlake Math Club

December 15, 2019

Problems

1. Ben wants to build a wall to surround his house. His house is a 3000 mile by 1500 mile rectangle. If each mile of the wall costs 20 million dollars, how many billions of dollars will the wall cost?

2. A deck of 32 cards consists of four cards of each integer from 1 to 8, inclusive. A matching pair (two cards with the same number) is removed from the deck. Given that these cards are not returned to the deck, what is the probability that two randomly selected cards also form a pair, as a common fraction?

3. Let ω be a circle with diameter $AD = 4$, and let B and C be two other points on ω such that the quadrilateral $ABCD$ is an isosceles trapezoid and $AD \parallel BC$. If the height of the trapezoid is $\sqrt{3}$, find BC.

4. Coach Chad has 6 identical red hats and 10 identical green hats. As team gear for I2MC, he would like give his hats to his 4 students such that each student receives at least one hat of each color. How many ways can he do so?

5. Gregory decides that for lunch, he will eat the numbers between 1 and 100, inclusive. However, ancient tradition forbids him from eating numbers that are either multiples of 2 or 5. If Gregory eats as many numbers as possible without breaking tradition, how many numbers will he eat for lunch?

6. How many ordered tuples of integers $(x_1, x_2, x_3, x_4, x_5)$ are there such that $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq 5$?

7. Two parallel chords of a circle ω have lengths 10 and 14. The distance between the chords is 6. Find the length of the chord on ω that is parallel and equidistant from the two given chords (in other words, the distances from the chord to each of the original chords are equal).

8. Geronimo has a fair 10 sided dice and and Stilton a fair 15 sided die. If they both roll at the same time, what is the probability that Stilton rolls higher than Geronimo?

9. Let isosceles trapezoid $ABCD$ have side lengths $AB = BC = CD = 8$ and $AD = 10$. Let B' be the reflection of B across AC and let the circle that passes through B, B', and D meet the line AC at the point X. Find the length AX.

10. Let O_1 and O_2 be the centers of externally tangent circles ω_1 and ω_2, respectively, and let that point of tangency be T. Let points A and B be on ω_1 and ω_2, respectively, such that T is on $\overline{AB}$ and $AO_2 = BO_1$. Given that the radii of ω_1 and ω_2 are 3 and 5, respectively, what is AO_2?