I2MC 2019 - Algebra and Number Theory

Interlake Math Club

December 15, 2019

Problems

1. I have 10 liters of a mix of juice and water which is 50% juice by volume. How much 75% juice solution, in liters, do I have to add to my mix in order to obtain a final mix which is $66\frac{2}{3}\%$ juice by volume?

2. Let $\mathcal{S}(x)$ be the sum of the positive divisors of a positive integer x. Find $\mathcal{S}(\mathcal{S}(12))$.

3. Bob walks to school at 5 miles per hour in order to arrive on time. A third of his way to school, he realizes that he forgot his glasses. He runs back home at 15 miles per hour, grabs his glasses, and sprints to school at 30 miles per hour. If Bob arrives at school 10 minutes early, how far away is school from Bob's house, in miles? Express your answer as a common fraction.

4. Dave has a life-long ambition to soak all of the water in Lake Washington in wooden chairs. If one chair can absorb 4800 cubic centimeters of water, and Lake Washington is 3 cubic kilometers, how many wooden chairs would it take to soak up all the water? Express your answer as $a \times 10^b$, where b is an integer and $1 \le a < 10$.

5. In the mighty jungle, elephants weigh 27 pounds and cats weigh 4 pounds. An explorer notes that the total weight of all the animals in the jungle is one pound more than the maximum total weight that is impossible to obtain. What is the average weight of an animal?

6. You are given that $10^{0.3010} \approx 2$. Find the number of digits in 25^{52}.

7. Find the remainder when $4^{1009} + 1$ is divided by 2018.

8. Let $0 \le k \le 2019$ be an integer. Define a sequence (a_i) such tha $a_1 = 2019$, $a_2 = k$, and for all positive integers $n \ge 3$, we have

$$a_n = \left\lfloor \sqrt{a_{n-1} a_{n-2}} \right\rfloor.$$

Let m be the least positive integer such that $a_m = a_{m+1}$. How many values of k are there such that $a_m \ne k$?

9. What is the largest prime factor of $2^{14} + 1$?

10. Given that the polynomial $x^5 - 60x^3 - 10x^2 + 699x - 630$ is divisible by the polynomial $x^3 - 11x^2 + 31x - 21$, find the value of $\frac{x^5 - 60x^3 - 10x^2 + 699x - 630}{x^3 - 11x^2 + 31x - 21}$ when $x = 4$.