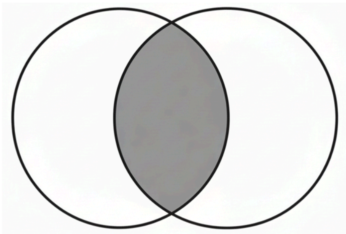
Combinatorics and Geometry Solutions

I2MC 2025

1. A Venn diagram is composed of two overlapping circles of radius 3 and covers a region of area 13π, as shown in the diagram below. Find the area of the shaded region.

   Written by: Mingyue Yang

   Answer: 5π

   Solution: The area of one circle is $3^2\pi = 9\pi$. Then, the area of shaded area is $9\pi + 9\pi - 13\pi = \boldsymbol{5\pi}$.

2. Mingyue writes the integers from 1 to 100 in order with no spaces in between. How many times will "67" appear in this sequence?

   Written by: Mingyue Yang

   Answer: 3

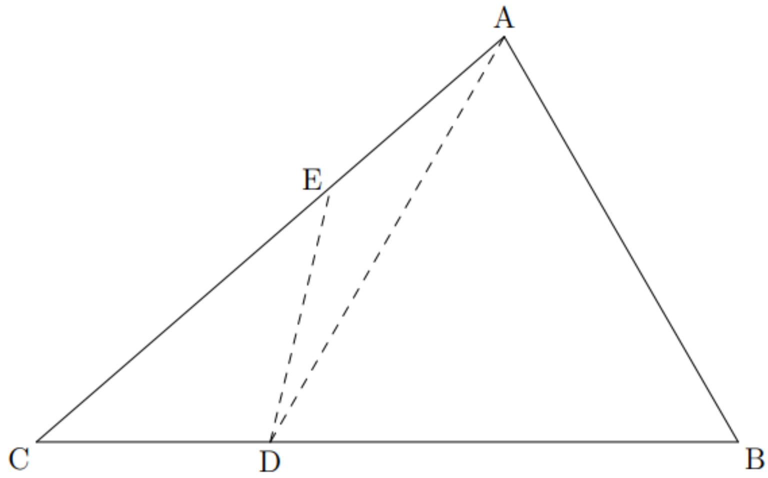
   Solution: The **3** occurrences are 6,7 or 67 or 76,77.

3. Point D is drawn on $\overline{BC}$ of $\triangle ABC$, and point E is drawn on $\overline{AC}$. Given that $AE = ED = DC$ and $AD = DB = BA$, what is $\angle BDE$?

   Written by: Mingyue Yang

   Answer: 80°

   Solution:



Observe that $\angle BDA = 60°$. Let $\angle ADE = x$. Then, $\angle CDE = 120° - x$. Since $CD = DE$, we have $\angle CED = 30° + \frac{x}{2}$. Yet we also have $\angle CED = \angle EAD + \angle ADE = 2x$. Thus, we have $2x = \frac{x}{2} + 30° \Leftrightarrow x = 20°$. Thus, we have $\angle BDE = 60° + 20° = \mathbf{80°}$.

4. Ian has four books and a bookshelf. How many ways can he arrange at least one book onto the bookshelf? If there are multiple books on the bookshelf, their order matters.

   Written by: Mingyue Yang

   Answer: 64

   Solution: Proceed by casework.

   Case 1: If there are 4 books, there are 24 ways to arrange the books.

   Case 2: If there are 3 books, there are 4 ways to choose the books on the bookshelf and 6 ways to arrange the 3 selected books, for a total of 24 ways.

   Case 3: If there are 2 books, there are 6 ways to choose the books on the bookshelf and 2 ways to arrange the 2 selected books, for a total of 12 ways.

   Case 4: If there is 1 book, there are 4 ways to choose the book and 1 way to arrange it, for a total of 4 ways.

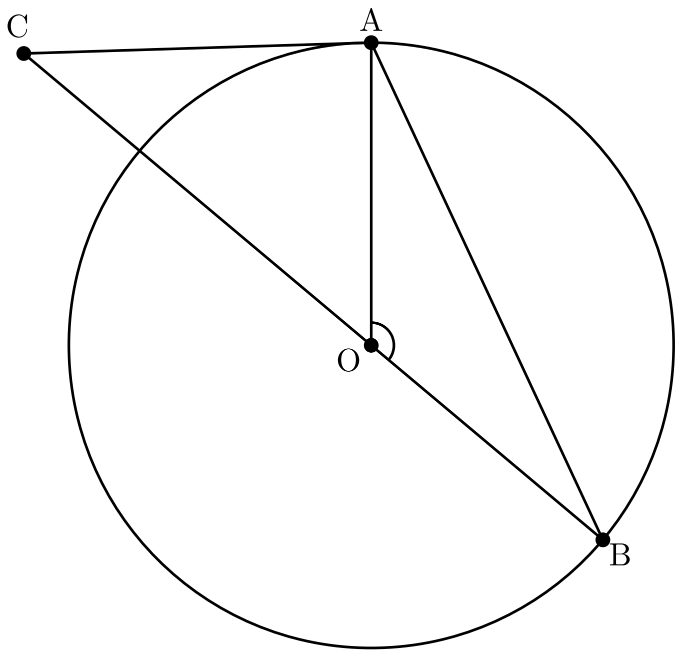
   The total number of ways is then $24 + 24 + 12 + 4 = \mathbf{64}$.

5. Let ω be a circle with center O and let A and B be points on ω. Let ℓ be the tangent to ω at A. Line $\overline{BO}$ intersects ℓ at C. If minor arc AB on ω has measure $130°$, find $\angle ACB$.

   Written by: Ian Rui

   Answer: $40°$

   Solution:



We have $\angle CAB = \frac{360°-130°}{2} = 115°$. We also have $\angle ABC = \frac{1}{2}(180° - \angle AOB) = \frac{50°}{2} = 25°$. Thus, we have $\angle ACB = 180° - 115° - 25° = \mathbf{40}°$.

6. Ryan, Aryan, Bryan, and Cryan are playing a 4-person game. On a player's turn, they roll a die. If the top face on the die is a 6, that player wins. Otherwise, they randomly give the die to another person. Given that Aryan plays first, what is the probability that either he or Cryan wins?

   Written by: Aryan Agrawal

   Answer: $\frac{13}{23}$

   Solution: Let x be the probability that Aryan wins given that it's Aryan's turn, and let y be the probability that Cryan wins given it's Aryan's turn. Observe that by symmetry, Ryan, Bryan, and Cryan each have the same probability to win; thus, $x + 3y = 1$. We can get another equation by noting that $x = \frac{1}{6} + \frac{5}{6}y$ (Aryan either wins by rolling 6 or he has to win after someone else's turn. The probability of the latter is equal to the probability Cryan wins if it is Aryan's turn, by symmetry). Thus, we get

$$x = 1 - 3y = \frac{1}{6} + \frac{5}{6}y \Leftrightarrow y = \frac{5}{23},\ x = \frac{8}{23}.$$

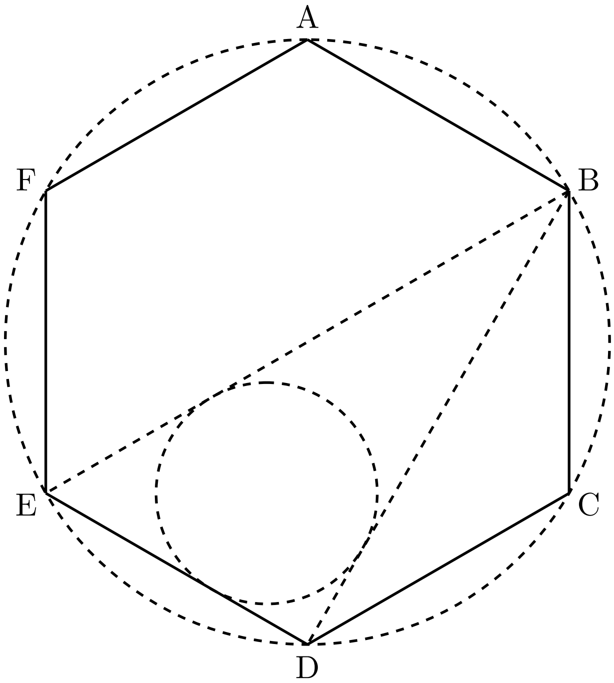
   Thus, the answer is simply $\frac{5}{23} + \frac{8}{23} = \mathbf{\frac{13}{23}}$.

7. Let $ABCDEF$ be a regular hexagon. Suppose that the area of the circumcircle of $ABCDEF$ is x, and that the area of the incircle of triangle BDE is y. What is $\frac{x}{y}$?

   Written by: Aishwarya Agrawal

   Answer: $4{+}2\sqrt{3}$

   Solution:



WLOG, let the hexagon have a side length of 2. The radius of the circumcircle is then 2 as well. Then, $\triangle BDE$ is a right triangle with side lengths 2, $2\sqrt{3}$, and 4. Thus, $\triangle BDE$ has an area of $2\sqrt{3}$ and a semiperimeter of $3+\sqrt{3}$. This implies that $\triangle BDE$ has an inradius of $\frac{2\sqrt{3}}{3+\sqrt{3}}$. The ratio of $\frac{x}{y}$ is then the square of the ratio of the radii, which is:

$$\left(\frac{2}{\frac{2\sqrt{3}}{3+\sqrt{3}}}\right)^2 = \left(\frac{6+2\sqrt{3}}{2\sqrt{3}}\right)^2 = \left(1+\sqrt{3}\right)^2 = \mathbf{4+2\sqrt{3}}.$$

8. Rivulet, a pedestrian, arrives at a traffic intersection. They would prefer to go forward or turn left, but in both directions there are red pedestrian signals. Each signal independently takes a uniformly random real number of seconds from 0 to 60 to turn white, and the moment one of the signals turns white, Rivulet will go in that direction. Rivulet also has the option to turn right, and will do so out of impatience after x seconds if neither signal has turned white. Find the value of x such that Rivulet has an equal chance of going in any of the three directions.

   Written by: Ian Rui

   Answer: $60 - 20\sqrt{3}$

   Solution: Observe that the probability that none of the other signals occur before x seconds is $\frac{60-x}{60}$. Since Rivulet has an equal chance of going in each direction, the probability Rivulet goes to the right is $\frac{1}{3}$. Then, we have

$$\left(\frac{60-x}{60}\right)^2 = \frac{1}{3} \Leftrightarrow (60-x)^2 = 1200 \Leftrightarrow 60 - x = 20\sqrt{3} \Leftrightarrow x = \mathbf{60 - 20\sqrt{3}}.$$

9. Aishwarya wants to place penguins in a 3×3 grid. As she wants no penguin to be lonely, she decides that every penguin must either be in the same row or same

column as another penguin. Given that at most one penguin can be placed in each cell, how many ways can Aishwarya place penguins into the grid? She may place zero penguins.

Written by: Aishwarya Agrawal

Answer: 398

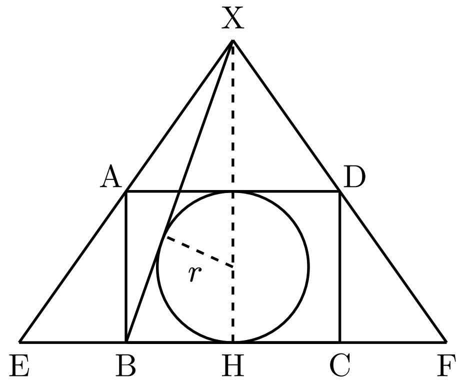
Solution: The main idea is to use the Principle of Inclusion-Exclusion. There are 512 ways to place the penguins without any restrictions. We subtract $9 \cdot 2^4 = 144$ scenarios where there is at least one lonely penguin (9 ways to place the lonely penguin and 2^4 ways to place however many penguins in the 4 remaining cells). We then add back $\frac{9 \cdot 4 \cdot 2}{2} = 36$ where there are at least 2 lonely penguins (9 ways to choose the first, 4 ways to choose the second, 2 for whether or not to keep a penguin in the 9th cell and dividing by 2 because the first and second penguins are interchangeable). Finally, we subtract $3! = 6$ for the scenario of 3 lonely penguins (one penguin in each row and column). This gets us $512 - 144 + 36 - 6 = \mathbf{398}$ ways to place the penguins.

10. A sphere is inscribed within a cube inscribed within a cone, such that one face of the cube lies on the base of the cone. A segment connecting the vertex of the cone to a vertex of the cube on the base of the cone is tangent to the sphere. Compute the ratio of the cone's height to the radius of its base.

    Written by: Benjamin Fu

    Answer: $\sqrt{2}$

    Solution:



We take the cross section which is perpendicular to the base of the cone and contains the diagonal of the bottom face of the cube.

In the diagram, the rectangle is the cube, the triangle is the cone, the circle is the sphere, the segment is $\overline{XB}$, and H is the foot of the vertex of the cone onto the base of the cone).

Now, let the height of the cone be $h = XH$ and the radius of the sphere be r. Since the sphere is inscribed in the cube, we know that $AB = 2r$. Furthermore, as BC is a face diagonal of the cube, we know that $BC = \sqrt{2}AB = 2\sqrt{2}r$. Since H

is the midpoint of BC, we know that $BH = \frac{1}{2}BC = \sqrt{2}r$. To compute XH, we note that the area of XBH is $\frac{1}{2}BH \cdot XH = \frac{\sqrt{2}}{2}rh$. However, the area of XBH can also be expressed as $\frac{1}{2}r \cdot XB + \frac{1}{2}rBH$. Using the Pythagorean theorem, we know that $XB = \sqrt{BH^2 + XH^2} = \sqrt{2r^2 + h^2}$. Thus, we have

$$\begin{aligned}
\frac{\sqrt{2}}{2}rh &= \frac{1}{2}r\sqrt{2r^2 + h^2} + \frac{\sqrt{2}}{2}r^2 \\
\frac{\sqrt{2}}{2}(rh - r^2) &= \frac{1}{2}r\sqrt{2r^2 + h^2} \\
\sqrt{2}(h - r) &= \sqrt{2r^2 + h^2} \\
2(h - r)^2 &= 2r^2 + h^2.
\end{aligned}$$

By expanding and simplifying, we get $h(h - 4r) = 0$ which implies $h = 4r$ (notice that $h = 0$ is an extraneous solution).

To finish this problem, we need to find EH in terms of r. Notice that $\triangle EBA \sim \triangle EHX$. Thus,

$$\frac{EB}{AB} = \frac{EH}{XH} \Leftrightarrow \frac{EB}{2r} = \frac{EB + \sqrt{2}r}{4r} \Leftrightarrow EB = \sqrt{2}r.$$

Since we have all the necessary lengths, the answer is simply

$$\frac{XH}{EH} = \frac{AB}{EB} = \frac{2r}{\sqrt{2}r} = \boldsymbol{\sqrt{2}}.$$