The William Lowell Putnam Mathematical Competition

L. F. Klosinski, G. L. Alexanderson, A. P. Hillman

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. F. KLOSINSKI

Department of Mathematics, University of Santa Clara, CA 95053

G. L. ALEXANDERSON

Department of Mathematics, University of Santa Clara, CA 95053

A. P. HILLMAN

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131

The following results of the forty-first William Lowell Putnam Mathematical Competition, held on December 6, 1980, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of **Washington University**, St. Louis, Missouri. The members of its winning team were: Kevin P. Keating, Nathan E. Schroeder, and Edward A. Shpiz; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of its team were: Michael Raship, Ehud B. Reiter, and Brian F. Sheppard; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of the **University of Maryland**, College Park, Maryland. The members of its team were: Ravi B. Boppanna, Brian R. Hunt, and Eric I. Kuritzky; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of the **University of Chicago**, Chicago, Illinois. The members of its team were Daniel J. Goldstein, Nicholas F. Reingold, and Michael P. Spertus; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **University of California**, Berkeley, California. The members of its team were Randall L. Dougherty, Lin Goldstein, and Robin A. Pemantle; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **Eric D. Carlson**, Michigan State University; **Randall L. Dougherty**, University of California, Berkeley; **Daniel J. Goldstein**, University of Chicago; **Laurence E. Penn**, Harvard University; and **Michael Raship**, Harvard University. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were *Joel Friedman*, Harvard University; *Fred W. Helenius*, Massachusetts Institute of Technology; *Irwin L. Jungreis*, Cornell University; *Michael J. Larsen*, Harvard University; and *Arthur S. Parker*, University of Kansas. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *California Institute of Technology*, with team members R. Sekhar Chivukula, Peter W. Shor, John R. Stembridge; *Case Western Reserve University*, with team members Edward J. Branagan, Jr., Scott R. Fluhrer, David A. Natwick; *Massachusetts Institute of Technology*, with team members Andrew J. Bernoff, Josh D. Cohen, Lorenzo A. Sadun; *Michigan State University*, with team members Eric D. Carlson, Karl A. Dahlke, Lloyd A. Rawley; and *Princeton University*, with team members Mark P. Kleiman, Jacob Nemchyonok, and Charles H. Walter.

Honorable mention was achieved by the following thirty-three individuals, named in alphabeti-

cal order: *Michael H. Albert*, University of Waterloo; *Richard J. Beigel*, Stanford University; *David D. Chambliss*, Princeton University; *Stephen J. Curran*, Beloit College; *Marc A. Drexler*, Johns Hopkins University; *Paul Feit*, Harvard University; *Paul N. Feldman*, Yale University; *Scott R. Fluhrer*, Case Western Reserve University; *Brian R. Hunt*, University of Maryland, College Park; *Howard J. Karloff*, University of Pennsylvania; *Kevin P. Keating*, Washington University, St. Louis; *Gary R. Lawlor*, Brigham Young University; *Franklin M. Maley*, Amherst College; *Michael P. Mattis*, Harvard University; *Victor J. Milenkovic*, Harvard University; *David J. Montana*, Harvard University; *Evan Morton*, Harvard University; *Robin A. Pemantle*, University of California, Berkeley; *Matthew J. Raw*, Washington University, St. Louis; *Zinory Reichstein*, California Institute of Technology; *Ehud B. Reiter*, Harvard University; *Subir Sachdev*, Massachusetts Institute of Technology; *Lorenzo A. Sadun*, Massachusetts Institute of Technology; *Erin J. Schram*, Michigan State University; *Nathan E. Schroeder*, Washington University, St. Louis; *Peter W. Shor*, California Institute of Technology; *Edward A. Shpiz*, Washington University, St. Louis; *Michael Spertus*, University of Chicago; *James Van Buskirk*, University of Minnesota, Duluth; *Charles H. Walter*, Princeton University; *Lawrence B. Weinstein*, Yale University; *David A. Williams*, Washington University, St. Louis; *Robert L. Zako*, Harvard University.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: University of Alberta, *Robert P. Morewood, Peter B. Weichman*; Amherst College, *George G. Watson*; Brown University, *Bruce A. Hendrickson*; California Institute of Technology, *R. Sekhar Chivukula, Lance J. Dixon, Scott R. Johnson, Christopher P. Lutz, David J. Muraki, John R. Stembridge*; University of California, Davis, *Theodore W. Jones, Carl A. Lundgren*; University of California, Santa Barbara, *John R. Rose, Dean Connable Wills*; University of California, Santa Cruz, *Stephen P. Carrier*; Case Western Reserve University, *David A. Natwick*; University of Chicago, *Nicholas F. Reingold*; George Mason University, *Stephen Billups*; Harvard University, *Anthony R. Barker, Michael V. Finn, James G. Propp, Jonathan S. Roberts, Brian F. Sheppard, Carlos T. Simpson, William A. Titus, Ron K. Unz, Robert J. Waldmann*; University of Illinois, Urbana-Champaign, *Jerome V. Walsh*; Iowa State University, *William R. Somsky*; Université Laval, *Pierre Tremblay*; University of Maryland, College Park, *Ravi B. Boppana, Eric I. Kuritzky*; Massachusetts Institute of Technology, *Josh D. Cohen, David Seibert*; Michigan State University, *Karl Dahlke*; University of Minnesota, Minneapolis, *Peter M. Thompson*; University of North Carolina, Chapel Hill, *Edward J. Rak*; University of Oklahoma, *Gary D. Köhler*; Oregon State University, *Gregory L. Larson*; University of Pittsburgh, *Randall S. Henry*; Pomona College, *William M. McGovern*; Princeton University, *David R. Grant, Mark P. Kleiman, James L. McInnes, David P. Roberts, Bruce K. Smith, Stephen A. Vavasis*; Rensselaer Polytechnic Institute, *David R. Iny, Gregory F. Taylor*; Rose-Hulman Institute of Technology, *Michael L. Call*; Stanford University, *Thomas C. Hales*; University of Virginia, *Mark G. Pleszkoch*; Washington University, St. Louis, *Bard Bloom, Chim-Chung Chan, Karl F. Narveson*; University of Waterloo, *Guy W. Hulbert, Duncan J. Murdoch*; Yale University, *Alan S. Edelman*.

There were 2043 individual contestants from 335 colleges and universities in Canada and the United States in the competition of December 6, 1980. Teams were entered by 237 institutions.

The Questions Committee of the forty-first competition consisted of E. J. Barbeau (Chairman), Joel Spencer, and K. B. Stolarsky; they proposed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Let b and c be fixed real numbers and let the ten points $(j, y_j), j = 1, 2, \ldots, 10$, lie on the parabola $y = x^2 + bx + c$. For $j = 1, 2, \ldots, 9$, let I_j be the point of intersection of the tangents to the given parabola at (j, y_j) and

$(j + 1, y_{j+1})$. Determine the polynomial function $y = g(x)$ of least degree whose graph passes through all nine points I_j.

Problem A-2

Let r and s be positive integers. Derive a formula for the number of ordered quadruples (a, b, c, d) of positive integers such that

$$3^r \cdot 7^s = \text{lcm}[a, b, c] = \text{lcm}[a, b, d] = \text{lcm}[a, c, d] = \text{lcm}[b, c, d].$$

The answer should be a function of r and s.

(Note that 1 cm $[x, y, z]$ denotes the least common multiple of x, y, z.)

Problem A-3

Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

Problem A-4

(a) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

(b) Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$

Problem A-5

Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t)\sin t \, dt = \int_0^x P(t)\cos t \, dt$$

has only finitely many real solutions x.

Problem A-6

Let C be the class of all real valued continuously differentiable functions f on the interval $0 \le x \le 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \le \int_0^1 |f'(x) - f(x)| \, dx$$

for all f in C.

Problem B-1

For which real numbers c is $(e^x + e^{-x})/2 \le e^{cx^2}$ for all real x?

Problem B-2

Let S be the solid in three-dimensional space consisting of all points (x, y, z) satisfying the following system of six simultaneous conditions:

$$x \ge 0, \quad y \ge 0, \quad z \ge 0,$$

$$x + y + z \le 11,$$

$$2x + 4y + 3z \le 36,$$

$$2x + 3z \le 24.$$

(a) Determine the number v of vertices of S.

(b) Determine the number e of edges of S.

(c) Sketch in the bc-plane the set of points (b, c) such that $(2, 5, 4)$ is one of the points (x, y, z) at which the linear function $bx + cy + z$ assumes its maximum value on S.

Problem B-3

For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geqslant 0$?

(Express the answer in the simplest form.)

Problem B-4

Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leqslant i \leqslant 1066$. Prove there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$.

(Here $|S|$ means the number of elements in the set S.)

Problem B-5

For each $t \geqslant 0$, let S_t be the set of all nonnegative, increasing, convex, continuous, real-valued functions $f(x)$ defined on the closed interval $[0, 1]$ for which

$$f(1) - 2f(2/3) + f(1/3) \geqslant t[f(2/3) - 2f(1/3) + f(0)].$$

Develop necessary and sufficient conditions on t for S_t to be closed under multiplication.

(This closure means that, if the functions $f(x)$ and $g(x)$ are in S_t, so is their product $f(x)g(x)$. A function $f(x)$ is convex if and only if $f(su + (1 - s)v) \leqslant sf(u) + (1 - s)f(v)$ whenever $0 \leqslant s \leqslant 1$.)

Problem B-6

An infinite array of rational numbers $G(d, n)$ is defined for integers d and n with $1 \leqslant d \leqslant n$ as follows:

$$G(1, n) = \frac{1}{n}, \qquad G(d, n) = \frac{d}{n} \sum_{i=d}^{n} G(d - 1, i - 1) \quad \text{for} \quad d > 1.$$

For $1 < d \leqslant p$ and p *prime*, prove that $G(d, p)$ is expressible as a quotient s/t of integers s and t with t *not* an integral multiple of p.

(For example, $G(3, 5) = 7/4$ with the denominator 4 not a multiple of 5.)

   On the line of the problem number for each solution, a 12-tuple $(n_{10}, n_9, \ldots, n_0, n_{-1})$ is given in which the entry n_i for $10 \geqslant i \geqslant 0$ is the number of contestants among the top 207 who achieved i points for the problem, and n_{-1} is the number not submitting a solution.

SOLUTIONS

A-1. $(52, 106, 7, 9, 0, 0, 0, 1, 8, 8, 8, 8)$

We show that $g(x) = x^2 + bx + c - (1/4)$. The equation of the tangent to the given parabola at $P_j = (j, y_j)$ is easily seen to be $y = L_j$, where $L_j = (2j + b)x - j^2 + c$. Solving $y = L_j$ and $y = L_{j+1}$ simultaneously, one finds that $x = (2j + 1)/2$ and so $j = (2x - 1)/2$ at I_j. Substituting this expression for j into L_j gives the $g(x)$ above.

A-2. $(142, 0, 0, 0, 0, 0, 0, 0, 14, 27, 13, 11)$

We show that the number is $(1 + 4r + 6r^2)(1 + 4s + 6s^2)$. Each of a, b, c, d must be of the form $3^m 7^n$ with m in $\{0, 1, \ldots, r\}$ and n in $\{0, 1, \ldots, s\}$. Also m must be r for at least two of the four numbers, and n must be s for at least two of the four numbers. There is one way to have $m = r$ for all four numbers, $4r$ ways to have one m in $\{0, 1, \ldots, r - 1\}$ and the other three equal to r, and $\binom{4}{2}r^2 = 6r^2$ ways to have two of the m's in $\{0, 1, \ldots, r - 1\}$ and the other two equal to

r. Thus there are $1 + 4r + 6r^2$ choices of allowable m's and, similarly, $1 + 4s + 6s^2$ choices of allowable n's.

A-3. $(20, 2, 1, 0, 0, 0, 0, 0, 2, 2, 39, 141)$

Let I be the given definite integral and $\sqrt{2} = r$. We show that $I = \pi/4$. Using $x = (\pi/2) - u$, one has

$$ I = \int_{\pi/2}^{0} \frac{-du}{1 + \cot^r u} = \int_{0}^{\pi/2} \frac{\tan^r u \, du}{\tan^r u + 1}. $$

Hence

$$ 2I = \int_{0}^{\pi/2} \frac{1 + \tan^r x}{1 + \tan^r x} dx = \int_{0}^{\pi/2} dx = \pi/2 \quad \text{and} \quad I = \pi/4. $$

A-4. $(4, 6, 2, 0, 0, 0, 0, 1, 23, 10, 29, 132)$

(a). Let S be the set of the 10^{18} real numbers $r + s\sqrt{2} + t\sqrt{3}$ with each of r, s, t in $\{0, 1, \ldots, 10^6 - 1\}$ and let $d = (1 + \sqrt{2} + \sqrt{3})10^6$. Then each x in S is in the interval $0 \le x < d$. This interval is partitioned into $10^{18} - 1$ "small" intervals $(k - 1)e \le x < ke$ with $e = d/(10^{18} - 1)$ and k taking on the values $1, 2, \ldots, 10^{18} - 1$. By the pigeonhole principle, two of the 10^{18} numbers of S must be in the same small interval and their difference $a + b\sqrt{2} + c\sqrt{3}$ gives the desired a, b, c since $c < 10^{-11}$.

(b) Let $F_1 = a + b\sqrt{2} + c\sqrt{3}$ and F_2, F_3, F_4 be the other numbers of the form $a \pm b\sqrt{2} \pm c\sqrt{3}$. Using the irrationality of $\sqrt{2}$ and $\sqrt{3}$ and the fact that a, b, c are not all zero, one easily shows that no F_i is zero. (The demonstration of this was Problem A-1 of the 15th Competition, held on March 5, 1955. For the proof, see page 402 of *The William Lowell Putnam Mathematical Competition Problems and Solutions: 1938–1964*, published by the MAA, or see this MONTHLY, 62 (1955) 561.) One also sees readily that the product $P = F_1 F_2 F_3 F_4$ is an integer. Hence $|P| \ge 1$. Then $|F_1| \ge 1/|F_2 F_3 F_4| > 10^{-21}$ since $|F_i| < 10^7$ and thus $1/|F_i| > 10^{-7}$ for each i.

A-5. $(8, 6, 6, 4, 3, 3, 0, 2, 2, 11, 55, 107)$

Let $Q = P - P'' + P^{IV} - \cdots$. Using repeated integrations by parts, the equations of the given system become

$$ \int_{0}^{x} P(t) \sin t \, dt = -Q(x) \cos x + Q'(x) \sin x + Q(0) = 0, $$

$$ \int_{0}^{x} P(t) \cos t \, dt = Q(x) \sin x + Q'(x) \cos x - Q'(0) = 0. $$

These imply that

$$ Q(x) \doteq Q'(0) \sin x + Q(0) \cos x. \tag{E} $$

Since P' and, hence, Q are polynomials of positive degree and the right side of (E) is bounded, equation (E) has all of its solutions in some interval $|x| \le M$. In such an interval, $P(x) \sin x$ has only finitely many zeros and $\int_{0}^{x} P(t) \sin t \, dt = 0$ has at most one more zero by Rolle's Theorem.

Q.E.D.

A-6. $(1, 0, 1, 0, 0, 10, 0, 0, 3, 3, 73, 116)$

We show that $u = 1/e$. Since $f' - f = (fe^{-x})'e^x$ and $e^x \ge 1$ for $x \ge 0$,

$$ \int_{0}^{1} |f' - f| dx = \int_{0}^{1} |(fe^{-x})'e^x| dx \ge \int_{0}^{1} (fe^{-x})' dx = [fe^{-x}]_{0}^{1} = 1/e. $$

To see that $1/e$ is the largest lower bound, we use functions $f_a(x)$ defined by

$$f_a(x) = (e^{a-1}/a)x \quad \text{for} \quad 0 \leqslant x \leqslant a, \qquad f_a(x) = e^{x-1} \quad \text{for} \quad a \leqslant x \leqslant 1.$$

Let $m = e^{a-1}/a$. Then

$$\int_0^1 |f_a'(x) - f_a(x)| \, dx = \int_0^a |m - mx| \, dx = m\left(a - \frac{a^2}{2}\right) = e^{a-1}\left(1 - \frac{a}{2}\right).$$

As $a \to 0$, this expression approaches $1/e$. The function $f_a(x)$ does not have a continuous derivative, but one can smooth out the corner, keeping the change in the integral as small as one wishes, and thus show that no number greater than $1/e$ can be an upper bound.

B-1. $(27, 19, 20, 0, 0, 0, 0, 41, 15, 20, 34, 31)$

The inequality holds if and only if $c \geqslant 1/2$. For $c \geqslant 1/2$,

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \leqslant \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2} \leqslant e^{cx^2}$$

for all x since $(2n)! \geqslant 2^n n!$ for $n = 0, 1, \ldots$.
    Conversely, if the inequality holds for all x, then
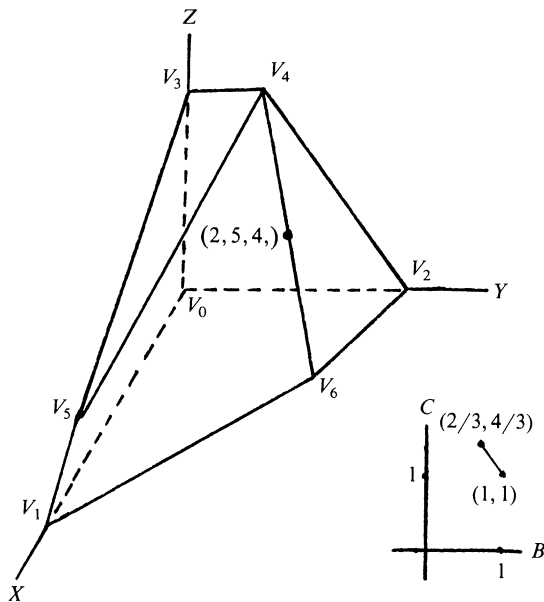
$$0 \leqslant \lim_{x \to 0} \frac{e^{cx^2} - \frac{1}{2}(e^x + e^{-x})}{x^2} = \lim_{x \to 0} \frac{(1 + cx^2 + \cdots) - (1 + \frac{1}{2}x^2 + \cdots)}{x^2} = c - \frac{1}{2}$$

and so $c \geqslant 1/2$.

B-2. $(19, 4, 11, 4, 2, 0, 3, 8, 10, 41, 53, 52)$

(a) $v = 7$. The seven vertices are $V_0 = (0,0,0)$, $V_1 = (11,0,0)$, $V_2 = (0,9,0)$, $V_3 = (0,0,8)$, $V_4 = (0,3,8)$, $V_5 = (9,0,2)$, and $V_6 = (4,7,0)$.

(b) $e = 11$. The eleven edges are V_0V_1, V_0V_2, V_0V_3, V_1V_5, V_1V_6, V_2V_4, V_2V_6, V_3V_4, V_3V_5, V_4V_5, and V_4V_6.

(c) The desired (b, c) are those with $b + c = 2$ and $2/3 \leqslant b \leqslant 1$. Let $L(x, y, z) = bx + cy + z$. Since L is linear and $(2, 5, 4)$ is on edge V_4V_6, the maximum of L on S must be assumed at V_4 and at V_6 and the conditions on b and c are obtained from $L(0, 3, 8) = L(4, 7, 0) \geqslant L(x, y, z)$, with (x, y, z) ranging over the other five vertices.

B-3. $(62, 27, 29, 10, 2, 6, 5, 6, 16, 27, 2, 15)$

We show that $u_n > 0$ for all $n \geqslant 0$ if and only if $a \geqslant 3$. Let $\Delta u_n = u_{n+1} - u_n$. Then the recursion (i.e., difference equation) takes the form $(1 - \Delta)u_n = n^2$. Since n^2 is a polynomial, a particular solution is

$$u_n = (1 - \Delta)^{-1}n^2 = (1 + \Delta + \Delta^2 + \cdots)n^2 = n^2 + (2n + 1) + 2 = n^2 + 2n + 3.$$

(This is easily verified by substitution.) The complete solution is $u_n = n^2 + 2n + 3 + k \cdot 2^n$, since $v_n = k \cdot 2^n$ is the solution of the associated homogeneous difference equation $v_{n+1} - 2v_n = 0$. The desired solution with $u_0 = a$ is $u_n = n^2 + 2n + 3 + (a - 3)2^n$. Since $\lim_{n \to \infty}[2^n/(n^2 + 2n + 3)] = +\infty$, u_n will be negative for large enough n if $a - 3 < 0$. Conversely, if $a - 3 \geqslant 0$, it is clear that each $u_n > 0$.

Alternatively, one sees that $u_0 = a$ and $u_1 = 2a$ and one can prove by mathematical induction that

$$u_n = 2^n a - \sum_{k=1}^{n-1} 2^{n-1-k}k^2 \quad \text{for} \quad n \geqslant 2.$$

Hence $u_n > 0$ for $n \geqslant 0$ if and only if $a > \sum_{k=1}^{n-1} 2^{-1-k}k^2$ and this holds if and only if $a \geqslant L$, where $L = \sum_{k=1}^{\infty} 2^{-1-k}k^2$. Let D mean d/dx. Then for $|x| < 1$,

$$(1 - x)^{-1} = \sum_{k=0}^{\infty} x^k$$

$$D(1 - x)^{-1} = (1 - x)^{-2} = \sum_{k=1}^{\infty} kx^{k-1}$$

$$D(1 - x)^{-2} = 2(1 - x)^{-3} = \sum_{k=2}^{\infty} k(k - 1)x^{k-2}.$$

Let $g(x) = 2x^3(1 - x)^{-3} + x^2(1 - x)^{-2}$. Then $L = g(1/2) = 3$ and the answer is all $a \geqslant 3$.

B-4. $(1, 38, 14, 11, 1, 0, 0, 2, 4, 1, 40, 95)$

The result we are asked to prove is clearly not true if $|X| < 10$. Hence we assume that $|X| \geqslant 10$ or that the A_j are distinct, which implies that $|X| \geqslant 10$.

Let $X = \{x_1, \ldots, x_m\}$, with $m = |X|$, and let n_i be the number of j such that x_i is in A_j. Let N be the number of ordered pairs (i, j) such that x_i is in A_j. Then

$$N = n_1 + n_2 + \cdots + n_m = |A_1| + |A_2| + \cdots + |A_{1066}| > 1066(m/2) = 533m.$$

Hence one of the n_i, say n_1, exceeds 533.

Let $B_1, \ldots, B_s$ be those sets A_j not containing x_1 and $Y = \{x_2, x_3, \ldots, x_m\}$. Then $s = 1066 - n_1 \leqslant 532$ and each $|B_j| > |Y|/2$. We can assume that x_2 is in at least as many B_j as any other x_i, and let $C_1, \ldots, C_t$ be the B_j not containing x_2. As before, one can show that $t \leqslant 265$.

We continue in this way. The 4th sequence of sets $D_1, \ldots, D_u$ will number no more than 132. The numbers of sets in the 5th through 10th sequences will number no more than 65, 32, 15, 7, 3, and 1, respectively. Thus we obtain the desired elements $x_1, \ldots, x_{10}$ unless X has fewer than 10 elements.

B-5. $(0, 1, 0, 0, 0, 0, 0, 0, 1, 8, 31, 166)$

The answer is $1 \geqslant t$ (or $0 \leqslant t \leqslant 1$). The product fg of two nonnegative increasing continuous

real-valued functions has the same properties. Using the fact that $0 \leqslant a \leqslant c$ and $0 \leqslant b \leqslant d$ imply $ad + bc \leqslant cb + cd$, one shows that fg is convex when f and g are convex. The function $f(x) = x$ is in S_t for all t. If S_t is closed under multiplication, x^2 is in S_t and so $2/9 = 1 - 2(4/9) + (1/9) \geqslant t[4/9 - 2(1/9)] = 2t/9$ or $1 \geqslant t$. Conversely, when $1 \geqslant t$, a lengthy, straightforward computation verifies that S_t is closed.

B-6. $(1, 0, 0, 0, 0, 0, 0, 0, 1, 17, 20, 168)$

Let $F_d(x) = \Sigma_{n=d}^{\infty} G(d, n)x^n$. Then $F_1(x) = \Sigma_{n=1}^{\infty} x^n/n$ and $F_1'(x) = \Sigma_{n=0}^{\infty} x^n$. One sees that $F_d'(x) = dF_{d-1}(x)F_1'(x)$ by finding the coefficients of x^{n-1} on both sides and using $nG(d, n) = d\Sigma_{i=d}^{n} G(d - 1, i - 1)$. Then an induction gives us $F_d(x) = [F_1(x)]^d$. Now, for $1 < d \leqslant p$, the coefficient $G(d, p)$ of x^p in $F_d(x)$ is the coefficient of x^p in $[\Sigma_{n=1}^{p-d+1} x^n/n]^d$, and hence $G(d, p) = s/t$ with s and t integers and t a product of primes less than p.

MATHEMATICAL NOTES

ON McCARTY'S QUEEN SQUARES

CYRIL W. L. GARNER
Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario K1S 5B6

AGNES M. HERZBERG
Department of Mathematics, Imperial College, London, England

1. Introduction. McCarty [3] introduced queen squares and posed questions regarding their existence. These squares arise as an extension of the well-known problem of placing n sets of n nonattacking queens on an $n \times n$ chess-board; see, for example, Rouse Ball and Coxeter [1].

According to McCarty, a *queen square* of order n is a square arrangement of the elements from $\{1, 2, \ldots, n\}$ in an $n \times n$ array such that each element occurs at most once in each row, column, and diagonal, and elements i and j may not be placed $|i - j|$ entries apart in any row, column, or diagonal. Note that there may be empty grid squares.

McCarty represented the maximum number of nonempty entries that a queen square of order n contains by $M(n)$ and denoted by $R(n) = n^{-2}M(n)$ the ratio of $M(n)$ to the number of grid squares of an $n \times n$ board. He gave a table of values of lower bounds of $M(n)$ and $R(n)$ for from 3 through 18.

In a recent paper [2], the authors defined Latin queen squares. These are the same as queen squares except that no empty grid squares are allowed in the $n \times n$ array. They showed that Latin queen squares exist for every prime number p greater than or equal to 11.

The purpose of this brief note is to exhibit the relationship between queen squares and Latin queen squares and to show that, for n a prime greater than or equal to 11, $M(n) = n^2$ and $R(n) = 1$. This answers the first of the following questions posed by McCarty:

(1) Can a latin square be constructed that is also a queen square? If yes, for which n is it possible? If no, is there an upper bound on $R(n)$?

(2) Does there exist an algorithm for the maximal placement of the queens into the n-cube?