SOLUTIONS - Algebra and Number Theory Round

I2MC 2024

October 19th 2024

1. Four years ago, Alice's age was three times Bob's age. One year ago, Alice's age was two times Bob's age. How old is Bob currently?

   Written by: Aishwarya Agrawal

   Answer: $\boxed{7}$

   Solution: Let Bob's current age be b, and let Alice's current age be a. From the first statement, we get $a - 4 = 3(b - 4)$. From the second statement, we get $a - 1 = 2(b - 1)$. Simplifying, we get $a = 3b - 8$ and $a = 2b - 1$, so $3b - 8 = 2b - 1$. This gives us $b = \boxed{7}$.

2. What is the greatest integer less than 100 that can be written as the product of two composite integers?

   Written by: Mingyue Yang

   Answer: $\boxed{96}$

   Solution: Every composite number can be written as the product of 2 or more prime numbers which are not necessarily distinct. This means that the product of two composite numbers can always be written as the product of 4 or more (not necessarily distinct) prime numbers. The numbers 99 and 98 each have 3 total prime factors ($3^2 \cdot 11$ and $2 \cdot 7^2$), and the number 97 is prime, so none of them can be the product of two composite numbers. The number 96 is equal to $2^5 \cdot 3$ and thus has 6 total prime factors, so it can be written as the product of two composite numbers (some examples are $4 \cdot 24$, $6 \cdot 16$, or $8 \cdot 12$). Thus $\boxed{96}$ is the answer.

3. Daniel and August were making snowballs together. When they split apart, Daniel finds that he takes 50% more time to make the same number of snowballs, and August finds that he takes $x\%$ more time to make the same number of snowballs. What is x?

   Written by: Mingyue Yang

   Answer: $\boxed{200}$

   Solution: Suppose Daniel and August can produce 3 snowballs in one minute while working together. If Daniel takes 50% more time when working alone, that means he can produce 3 snowballs in one and a half minutes, or 2 snowballs each minute. This means that August produces 1 snowball each minute when working alone, so he would take 3 minutes to produce 3 snowballs. Thus August takes $\boxed{200\%}$ more time compared to when he and Daniel work together.

4. Find the value of $\dfrac{2025!}{2023! + 2024!}$.

   Written by: Mingyue Yang

   Answer: $\boxed{2024}$

   Solution: By the definition of factorials we have that $2024! = 2024 \cdot 2023!$ and $2025! = 2025 \cdot 2024 \cdot 2023!$. Thus the expression can be written as $\dfrac{2025 \cdot 2024 \cdot 2023!}{2023! + (2024 \cdot 2023!)} = \dfrac{(2023!)(2025 \cdot 2024)}{(2023!)(1 + 2024)} = \dfrac{2025 \cdot 2024}{2025} = \boxed{2024}$.

5. The 2024 Interlake yearbook was labeled “Volume 57”. The two numbers 2024 and 57 do not share any factors greater than 1. Assuming that the “Volume” number increases by 1 each year, when is the next year in which the year and volume number share a common divisor greater than 30?

   Written by: Mingyue Yang

   Answer: $\boxed{2248}$

   Solution: If two numbers a and b share a common factor, then $a - b$ must also share that factor. Each year, the year number increases by one (wow!) and so does the yearbook’s volume number, which means that the difference between the year and volume number is always equal to $2024 - 57 = 1967$. Thus any common factor shared by the year and volume number must also be a factor of 1967. We note that the prime factorization of 1967 is $7 \cdot 281$, so the only possible factors greater than 30 are 281 and 1967, both of which will first occur when the volume number equals 281 or 1967 respectively. We want the next year (and thus the smallest possible year number) where this occurs, which would occur when the volume number is 281. This means the year number is $281 + 1967 = \boxed{2248}$.

6. Find the sum of all integers n such that the degree measure of an interior angle of a regular n-gon is odd.

   Written by: Jeffrey Zhao

   Answer: $\boxed{624}$

   Solution: We know that each exterior angle of a polygon and its corresponding interior angle must sum to 180 degrees. Thus the interior angle being odd is equivalent to the exterior angle being odd. We also know that the sum of the exterior angles in a polygon is always $360°$, and since we are dealing with regular n-gons, the exterior angle is $\frac{360}{n}$. The prime factorization of 360 is $2^3 \cdot 3^2 \cdot 5$, so in order for $\frac{360}{n}$ to be odd, n must be a multiple of $2^3 = 8$ and a factor of 360. Thus, the possible values of n are 8, $8 \cdot 3 = 24$, $8 \cdot 5 = 40$, $8 \cdot 3^2 = 72$, $8 \cdot 3 \cdot 5 = 120$, and $8 \cdot 3^2 \cdot 5 = 360$. Summing the possible values, we get $360 + 120 + 72 + 40 + 24 + 8 = \boxed{624}$.

7. The fraction $\dfrac{503}{12}$ is not an integer in base 10, but it is an integer for the Flaming Martians, who interpret both 503 and 12 in base k. What is k?

   Written by: Mingyue Yang

   Answer: $\boxed{21}$

   Solution: The fraction $\frac{503}{12}$ in base k is equal to $\frac{5k^2+3}{k+2}$ in base 10. Through polynomial division, we can write $5k^2+3$ as $(k+2)(5k-10)+23$. Thus, $\frac{23}{k+2}$ must be an integer in base 10, and since 23 is prime, the only positive value of k is $\boxed{21}$.

8. The number of letters in Edward's favorite word is divisible by 41. Also, adding 16 to this number results in a perfect square. What is the smallest possible number of letters in Edward's favorite word?

Written by: Jeffrey Zhao

Answer: $\boxed{1353}$

Solution: Since the number is 16 less than a square, we can write it as $k^2 - 16 = (k+4)(k-4)$. Since 41 is a prime number, either $k+4$ or $k-4$ must have a factor of 41. To minimize k we want one of these to equal 41 exactly, so the smallest value occurs when $k+4 = 41$ and $k = 37$. Thus, the number of letters is $37^2 - 16 = \boxed{1353}$.

9. Call a number of the form $\frac{1}{n}$ where n is a positive integer *delectable* if it terminates when written in decimal form. What is the sum of all *delectable* numbers?

   Written by: Mingyue Yang

   Answer: $\boxed{\frac{5}{2}}$

   Solution: A number of the form $\frac{1}{n}$ is delectable if an only if n has only 2 and 5 as prime factors. Otherwise, we end up with a repeating decimal.

   To find the sum of all delectable numbers, consider the product:

$$\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right)\left(\frac{1}{1}+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\dots\right).$$

   If we expand this product using the distributive property, we see that each fraction of the form $\frac{1}{n}$ where n only has 2 and 5 as prime factors appears exactly once. Thus, the sum we are looking for is equal to this product. But each of the terms is just a geometric series, so our answer is simply

$$\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{5}} = \boxed{\frac{5}{2}}.$$

10. When the polynomial $(x-1)(x-2)(x-3)\dots(x-19)(x-20)$ is expanded, what is the coefficient of x^{18}?

    Written by: Aishwarya Agrawal

    Answer: $\boxed{20615}$

Solution: Let $P(x)$ be the said polynomial. Let c be the coefficient of x^{18}. When we expand the polynomial, we select either x or the number from each of the 20 terms being multiplied. Since c is the coefficient of x^{18}, we select x from 18 terms and select the number from 2 terms. Thus, the value of c can be expressed as

$$\begin{aligned} c = {} & 1 \times 2 + 1 \times 3 + \cdots + 1 \times 20 \\ & + 2 \times 3 + 2 \times 4 + \cdots + 2 \times 20 \\ & + \dots \\ & + 18 \times 19 + 18 \times 20 \\ & + 19 \times 20 \end{aligned}$$

Note that this product is equal to

$$\frac{1}{2}\left((1 + 2 + \cdots + 20)(1 + 2 + \cdots + 20) - 1^2 - 2^2 - \cdots - 20^2\right)$$

(to see this, expand!)

We know that $1+2+\cdots+20 = \frac{20 \cdot 21}{2} = 210$, and we know that $1^2+2^2+\cdots+20^2 = \frac{20 \cdot 21 \cdot 41}{6} = 2870$ (by the sum of squares formula)

Therefore, the answer is $\frac{1}{2}(210^2 - 2870) = \boxed{20615}$.