Team Round

I2MC 2025

1. (20) In the image below, shade some of the white cells such that the sum of the shaded cells in each row and column matches the corresponding number to the left or above it. Write as your answer the product of all unshaded integers.

	13	4	13	6	12
9	1	2	3	4	5
8	3	5	1	2	4
15	4	3	5	1	2
10	2	1	4	5	3
6	5	4	2	3	1

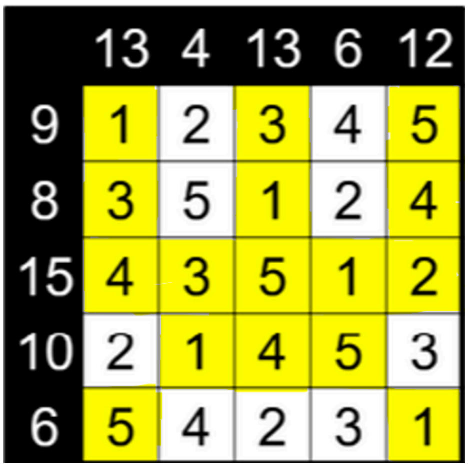
Written by: Ian Rui

Answer: 11520

Solution: All of row 3 must be shaded. In column 3, all squares except for the 2 in row 5 must be shaded. Similar logic applies in column 1. Then, we can look at column 5, and see that since the 2 is shaded, the 3 must not be shaded in order for the sum to be 12.

By doing this, we now see that rows 1, 2, and 5 all have their shaded cells sum to the required totals. Since columns 1 and 5 are complete, the 1 and 4 must be shaded in row 4 to complete the puzzle.

Here is the completed diagram:



Our final answer is $2 \cdot 4 \cdot 5 \cdot 2 \cdot 2 \cdot 3 \cdot 4 \cdot 2 \cdot 3 = \mathbf{11520}$.

2. (25) Aryan and Aishwarya have some flower petals, which they are planning to incorporate into dessert items - pastries, cakes, and ice creams, each using 3, 4, and 5 petals respectively. They realize that if they ignore the ice cream and make an equal number of pastries and cakes, they will have 1 petal left over, while if they ignore the pastry and make an equal number of cakes and ice creams, they will have 7 petals left over. What is the smallest possible number of flower petals they could have?

   Written by: Song Han Ngo

   Answer: 43

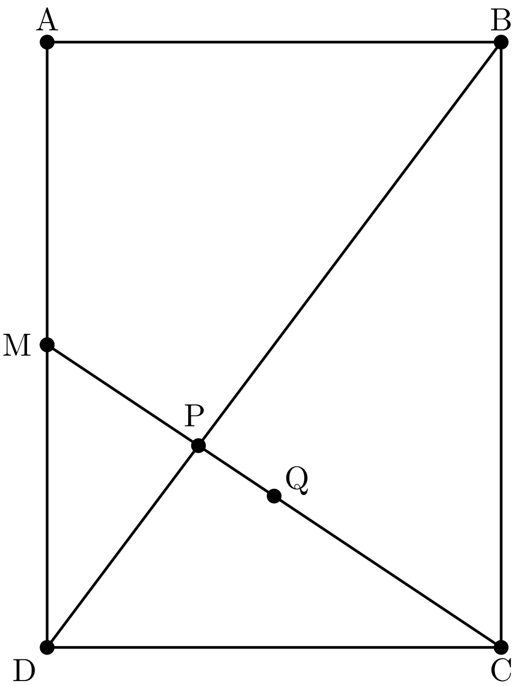
   Solution: The number of flower petals is 1 mod 7 and 7 mod 9. The smallest positive integer satisfying these conditions is **43**.

3. (25) Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 8$. Let M be the midpoint of AD and Q be the midpoint of MC. Given that P is the intersection of BD and MC, what is the ratio of the area of triangle MBQ to the area of triangle CBP?

   Written by: Aishwarya Agrawal

   Answer: $\frac{3}{4}$

   Solution:



   Observe that the desired ratio is equal to $\frac{MQ}{PC}$. Note that $MQ = \frac{1}{2}MC$. Furthermore, observe that $\frac{CP}{MP} = \frac{BC}{MD} = \frac{8}{4} = 2$. Thus, $\frac{CP}{MC} = \frac{2}{3}$, which implies that $CP = \frac{2}{3}MC$. Thus, we get an answer of $\left(\frac{1}{2}\right)/\left(\frac{2}{3}\right) = \mathbf{\frac{3}{4}}$.

4. (30) Let $f(x) = x^2 + ax + b$ and $g(x) = x + c$ for real a, b, and c. Furthermore,

$$f(g(e)) = f(e) = g(e+1) = 0.$$

for some real e. What is $a + b - c + e$?

Written by: Aryan Agrawal

Answer: 2

Solution: Since $g(e + 1) = 0$, we have $c = -1 - e$. Then, we have $g(e) = -1$, so $f(-1) = f(e) = 0$. This implies $f(x) = (x - e)(x + 1) = x^2 + (1 - e) - e$. Thus, $a = 1 - e$ and $b = -e$. The answer is $(1 - e) + (-e) - (-1 - e) + e = \mathbf{2}$.

5. (40) Christopher and Mingyue are playing a game. The game begins with an empty bag. Mingyue goes first, and each turn, a player adds either 3 or 4 marbles into the bag. At the end of the game, there are 26 marbles in the bag. There are x possible ways the game could have gone and Christopher moves last in y of them. What is $\frac{y}{x}$?

   Written by: Aishwarya Agrawal

   Answer: $\frac{4}{7}$

   Solution: The game will end in either 8 or 7 moves. If the game ends in 7 moves, 4 marbles were added 5 times while 3 marbles were added twice. Thus, there are $\binom{7}{3} = 21$ possible games, and Mingyue moves last in all of them. If the game ends in 8 moves, 3 marbles were added 6 times while 4 marbles were added twice. In this case, $\binom{8}{3} = 28$ is the number of possible games, and Christopher moves last in all of them. Our answer is $\frac{28}{28+21} = \mathbf{\frac{4}{7}}$.

6. (45) Christopher has a triangle with area and perimeter both equal to 100. He draws a segment L parallel to one of its sides, splitting the triangle into two regions of equal area and equal perimeter. What is the length of segment L?

   Written by: Mingyue Yang

   Answer: $50\sqrt{2} - 50$

   Solution: Observe that the larger triangle is similar to the smaller triangle formed by L with a ratio of $\sqrt{2}$, as its area is $\sqrt{2}^2 = 2$ times that of the smaller triangle. Since the large triangle has a perimeter of 100, the small triangle must have a perimeter of $50\sqrt{2}$.

   The small triangle and trapezoid have the same perimeter. Since they both share L, we may ignore it. Note this means that the other two sides of the small triangle have length summing to 50.

   Our answer is thus $\mathbf{50\sqrt{2} - 50}$.

7. (50) Let x be an integer. Given that $x^{2025} - 1$ is divisible by $(x - 1)^2$, what is the largest possible value of x?

   Written by: Aishwarya Agrawal

   Answer: 2026

Solution: This implies that $\frac{x^{2025}-1}{x-1} = 1 + x + ... + x^{2024}$ is divisible by $(x-1)$. By the factor theorem, we have

$$1 + x + ... + x^{2024} \equiv 1 + 1 + 1^2 + ... + 1^{2024} \equiv 2025 \equiv 0 \pmod{x-1}.$$

It follows that the largest value of x is **2026**.

8. (55) Aryan loves square numbers a bit too much. In an effort to increase Aryan's love for other numbers a bit more, Ian comes up with "quare" numbers, integers that can be written as a difference of two squares and are not squares themselves. How many "quare" numbers are there between 1 and 2025 inclusive?

   Written by: Aishwarya Agrawal

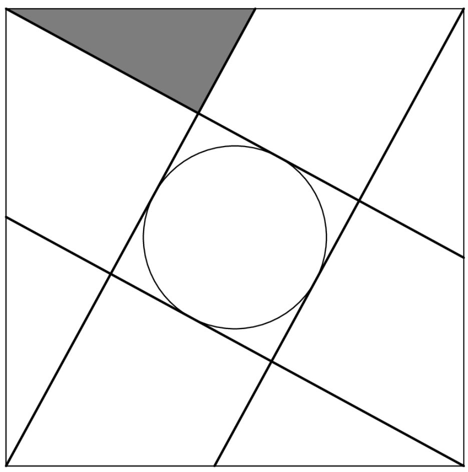
   Answer: 1474

   Solution: Suppose that a number n can be written as a difference of squares $x^2 - y^2$. Then, $n = x^2 - y^2 = (x+y)(x-y)$. Thus, consider any two factors that multiply to n. As long as they are of the same parity, there will exist corresponding x and y.

   If n is odd, simply consider $n = 1 \cdot n$. If n is divisible by 4, consider $n = 2 \cdot \frac{n}{2}$. However, if n is even but not divisible by 4, there do not exist two factors of the same parity that multiply to n.

   This means that all numbers 0, 1, or 3 mod 4 are either square or quare. There are 1519 such numbers at most 2025. Since no squares are 2 mod 4, we may subtract them directly. Our answer is $1519 - 45 = 1474$.

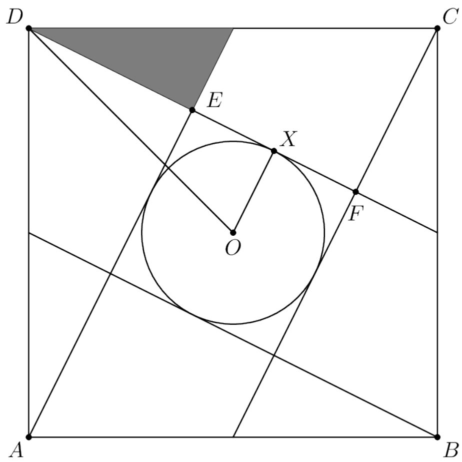
9. (55) Four tangents are drawn from the vertices of a square to a circle concentric with the square, as shown. The square has area 100 and the circle has area π. What is the area of the shaded triangle?

   Written by: Mingyue Yang

   Answer: $\frac{27}{2}$

   Solution:



Consider the right triangle $\triangle DXO$. We know $DO = 5\sqrt{2}$ and $OX = 1$. Thus, $DX = 7$ by the Pythagorean Theorem. This means that $DE = 6$ and $DF = 8$.

Consider that $[DCF] = \frac{1}{4}(100 - 2^2) = 24$, since the small square has area 2^2 and $\triangle DCF$ is a quarter of the remaining area.

Finally, the shaded region is similar to $\triangle DCF$ with ratio $\frac{6}{8}$. Thus, our answer is $24 \cdot \left(\frac{6}{8}\right)^2 = \frac{27}{2}$.

10. (60) Each square on an 8×8 chessboard has some cake. Specifically, if the square is in column m and row n, place $\frac{m}{n}$ cakes there. A "hungry pawn" starts on row 1 and can either move forward one space or diagonally forward one space. Whenever it is on a square, it eats the cake there. It keeps moving until it can't anymore. A path is called "satisfying" if the hungry pawn devours a whole number amount of cake. How many satisfying paths are there?

    Written by: Mingyue Yang

    Answer: 15

    Solution: Denote a square (m, n) if it is in the mth column and nth row. Note that the hungry pawn passes through exactly one square on each row. This means that we must pick one fraction with each denominator from 1 to 8. Since 5 is the only denominator involving the prime 5, we must pick $\frac{5}{5}$ to end up with an integer sum. The same is true for the prime 7, and thus we must pass through both $(5, 5)$ and $(7, 7)$.

    Due to the hungry pawn's movement, it must also pass through $(6, 6)$. Thus, consider row 3. It is the only undecided row left involving the prime 3, and thus we must pick either $\frac{3}{3}$ or $\frac{6}{3}$.

    Case 1: Go through $(3, 3)$:

    In this case, we must go through $(4, 4)$, which also forces $(8, 8)$ to be the last square (as $\frac{6}{8}$ and $\frac{7}{8}$ would both lead to non-integer sums). From here, we can either choose $(2, 2)$ or $(4, 2)$, each of which lead to three options for where to start on the first row. Thus, there are 6 valid paths in this case.

Case 2: Go through $(6, 3)$:

Subcase A: The pawn goes $(6, 3) \to (5, 4) \to (5, 5)$:

This means we must end the path with $(6, 8)$ to counteract the $(5, 4)$. We must then select $(6, 2)$ and any starting square on the first row, leading to 3 valid paths.

Subcase B: The pawn goes $(6, 3) \to (6, 4) \to (5, 5)$:

This means we must end the path at $(8, 8)$. On the second row, we must select $(5, 2)$ or $(7, 2)$ to counteract the $(6, 4)$, and for each case we have three valid options for the starting square in the first row. This leads to 6 valid paths.

The final answer is thus $6 + 3 + 6 = \mathbf{15}$.