Team Round

I2MC 2024

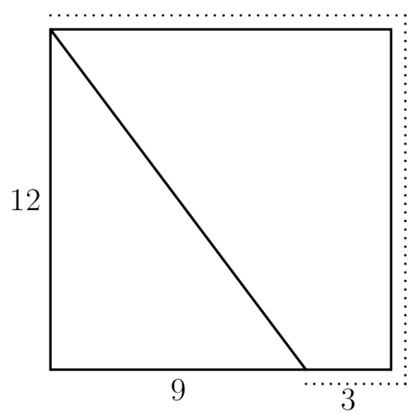
October 19th 2024

1. August is building a snow fort and is attempting to claim the territory of a 12 meter by 12 meter square by walking around the perimeter, starting at one of the corners. However, 27 meters into his walk, he sees Daniel hiding behind a nearby tree with a large arsenal of snowballs, so he runs back to the first corner, taking a straight path. In meters, what is the length of the complete route that August took?

   Written by: Mingyue Yang

   Answer: $\boxed{42}$

   Solution:



In the first 27 meters, August walks along two full sides of the square and 3 meters of the third side. As shown in the diagram, he is now 9 units away from his starting point in one direction and 12 units away in a perpendicular direction. By the Pythagorean theorem, his path back to the start has length $\sqrt{9^2+12^2} = \sqrt{225} = 15$. Thus, the total length of the path is $27 + 15 = \boxed{42}$.

2. Aryan has a $3 \times 3 \times 3$ cube. He dips it into paint so that all 6 faces of the cube are covered in paint. He then breaks the large cube into 27 cubes with side length 1. From these 27 cubes, he constructs three $2 \times 2 \times 2$ cubes and three $1 \times 1 \times 1$ cubes. What is the difference between the maximum and minimum possible area of paint on the external surfaces of the 6 cubes, in square units?

   Written by: Aishwarya Agrawal

   Answer: $\boxed{52}$

   Solution: Of the 27 small cubes, 8 have 3 faces covered in paint (those at the corners of the original cube), 1 has no faces covered in paint (the one at the center of the original cube), 6 have 1 face covered in paint (the center of each face of the original cube), and 12 have 2 faces covered in paint (everything else).

   Since each small cube in a $2 \times 2 \times 2$ cube has three faces showing on the exterior, Aryan can always ensure that the painted faces of all small cubes are showing on the exterior. Thus, all of the original painted surface area can be shown when maximizing the area of paint, making for a maximum surface area of $6 \times 3^2 = 54$ square units. Conversely, when constructing the $2 \times 2 \times 2$ cubes, it is also possible to place all of the painted faces on the interior, thus showing zero painted faces on the exterior. In this case the only painted faces showing will be on the $1 \times 1 \times 1$ cubes, which can be at best one cube with no faces painted and two cubes with one face painted each. Thus, the minimum visible area of paint is 2, so the answer is $54 - 2 = \boxed{52}$.

3. An infinite geometric sequence has a common ratio of $\dfrac{1}{\sqrt[3]{7}}$, and the sum of its first three terms is 216. What is the sum of the entire infinite geometric series?

   Written by: Mingyue Yang

   Answer: $\boxed{252}$

   Solution: Represent this geometric series as $a+ar+ar^2+ar^3+\cdots$, where $r = \dfrac{1}{\sqrt[3]{7}}$ and $a + ar + ar^2 = 216$. Then, we note that $ar^3 + ar^4 + ar^5 = (a + ar + ar^2)(r^3) = \dfrac{1}{7}(a + ar + ar^2)$. Similar logic holds for $ar^6 + ar^7 + ar^8$, and so on. Thus, this geometric series is equivalent to a geometric series with first term 216 and ratio $\frac{1}{7}$, just by combining terms into groups of 3. Therefore, the sum of the series is simply equal to $\dfrac{216}{1 - \frac{1}{7}} = \boxed{252}$.

4. A positive integer n leaves a remainder of 1 when divided by x and a remainder of x when divided by 6. What is the smallest possible value of n?

Written by: Mingyue Yang

Answer: $\boxed{11}$

Solution: Since n leaves a remainder of x when divided by 6, we know that x must be between 0 and 5.

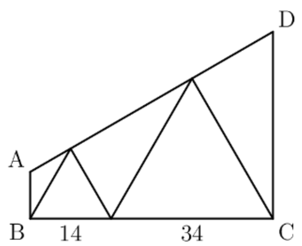
Obviously, n cannot leave a remainder of 1 when divided by 0 or 1. Therefore, x cannot be 0 or 1.

Suppose that x is 2 or 4. Then, n leaving a remainder of x when divided by 6 would imply that n is even. However, an even number cannot leave a remainder of 1 when divided by another even number.

Similarly, if x were equal to 3, then n would have to be a multiple of 3, and thus could not leave a remainder of 1 when divided by 3.

Thus, $x = 5$. The smallest number to leave a remainder of 1 when divided by 5 and a remainder of 5 when divided by 6 is $\boxed{11}$.

5. Trapezoid $ABCD$ has right angles $\angle ABC$ and $\angle BCD$. Two equilateral triangles with side lengths 14 and 34 are inscribed in $ABCD$ as shown.

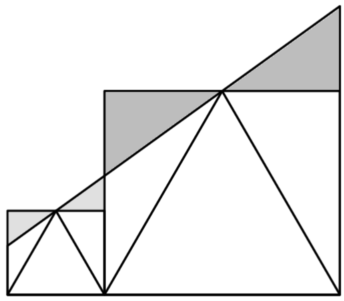


If the area of $ABCD$ is $k\sqrt{3}$, find $\lfloor \sqrt{k} \rfloor$.

Written by: Jeffrey Zhao

Answer: $\boxed{26}$

Solution:



We draw two rectangles as shown in the diagram above. Note that each pair of shaded triangles in the diagram above is congruent. Thus, the two rectangles together have the same area as the trapezoid. To find the area of the rectangles, note that they have the same bases and heights as the equilateral triangles inscribed within them. One rectangle has base 14 and height $7\sqrt{3}$ while the other has base 34 and height $17\sqrt{3}$, so the sum of their areas equals $98\sqrt{3} + 578\sqrt{3} = 676\sqrt{3}$. Thus, the answer equals $\lfloor\sqrt{676}\rfloor = \boxed{26}$.

6. A sentient plastic knight is placed at the origin on the Cartesian plane. The knight may jump two units right and one unit up, or one unit right and two units up. How many ways can the knight reach $(23, 16)$?

   Written by: Mingyue Yang

   Answer: $\boxed{286}$

   Solution: Suppose the knight takes a moves of the form 2 right 1 up and b moves of the form 1 right 2 up. Then, the total distance the knight moves to the right is $2a + b$, and the total distance it moves up is $a + 2b$. We know that these values must equal 23 and 16 respectively, so solving the system of equations gives that $a = 10$ and $b = 3$. Thus, the number of paths the knight can take is equal to the number of ways to pick three of the 13 total moves to be 1 right and 2 up. Therefore, the answer is $\binom{13}{3} = \dfrac{13 \cdot 12 \cdot 11}{6} = \boxed{286}$.

7. Benjamin writes down all 142 fractions with denominator 143 that are strictly between 0 and 1. He then simplifies all the fractions. What is the sum of all of the denominators of the 142 simplified fractions?

   Written by: Mingyue Yang

Answer: $\boxed{17426}$

Solution: The number 143 can be written as $11 \cdot 13$. Out of the numbers 1 to 142, there are 12 numbers divisible by 11, 10 numbers divisible by 13, and 120 numbers divisible by neither. None are divisible by both since 143 is the smallest such number. All of the numerators divisible by 11 will have denominator 13 when simplified, and similarly all the numerators divisible by 13 will have denominator 11 when simplified. All other values in the numerator will result in a denominator of 143. Thus, the sum of the denominators is equal to $(12\cdot 13)+(10\cdot 11)+(120\cdot 143) = (12\cdot 13+1)(10\cdot 11+1)-1 = 157\cdot 111-1 = \boxed{17426}$.

Remark: the final manipulation is to make the calculation much faster. It is very easy to multiply a number by 111.

8. There are 430 prime numbers less than 3000. How many numbers less than 3000 have a prime number of divisors?

   Written by: Jeffrey Zhao

   Answer: $\boxed{453}$

   Solution: The only numbers that have 2 divisors are prime numbers. To have 3 divisors, a number must be the square of a prime number, of which there are 16 less than 3000: 2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2, 37^2, 41^2, 43^2, 47^2, and $53^2 = 2809$. To have 5 divisors, a number must be the 4th power of a prime number, of which there are 4 less than 3000: $2^4 = 16$, $3^4 = 81$, $5^4 = 625$, and $7^4 = 2401$. To have 7 divisors, a number must be the 6th power of a prime number, of which there are 2 less than 3000: $2^6 = 64$ and $3^6 = 729$. To have 11 divisors, a number must be the 10th power of a prime number. The only such number under 3000 is $2^{10} = 1024$. To have 13 divisors a number must be the 12th power of a prime number, of which none are less than 3000. Thus, the number of divisors cannot be a prime number greater than 11 if the number must be less than 3000, so the answer is $430 + 16 + 4 + 2 + 1 = \boxed{453}$.

9. Let $ABCD$ be a parallelogram, and let M, N, and P denote the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{AD}$, respectively. Given that the region bounded by lines $\overline{AC}$, $\overline{BP}$, and $\overline{MN}$ is an equilateral triangle of side length 1, determine the area of $ABCD$.

   Written by: Benjamin Fu

   Answer: $\boxed{12\sqrt{3}}$

Solution: Rotate the parallelogram such that $\overline{BC}$ and $\overline{AD}$ are horizontal. Let the height of the parallelogram (the distance from $\overline{BC}$ to $\overline{AD}$) be h. Let the intersection of $\overline{AC}$ and $\overline{BP}$ be X. Let the intersection of $\overline{AC}$ and $\overline{MN}$ be Y and the intersection of $\overline{BP}$ and $\overline{MN}$ be Z.

Since $\triangle APX \sim \triangle CBX$ with a ratio $1:2$, the distance from X to $\overline{BC}$ is $\frac{2h}{3}$. Now note that the distance from $\overline{MN}$ to $\overline{BC}$ is $\frac{h}{2}$. Therefore, the height of the equilateral triangle is $\frac{2h}{3} - \frac{h}{2} = \frac{h}{6}$. Since we already know the height of the equilateral triangle is $\frac{\sqrt{3}}{2}$, $h = 3\sqrt{3}$.

Now note that $\triangle XYZ \sim \triangle XCB$ with ratio $\frac{h}{6} : \frac{2h}{3} = 1:4$. Since $YZ = 1$, $BC = 4$. Therefore, the area of the parallelogram is $4 \cdot 3\sqrt{3} = \boxed{12\sqrt{3}}$.

10. Let b and c be positive real numbers that satisfy $(b-1)^2 + (c-1)^2 = 7$ and $bc = 5$. Find the maximum possible value of b.

Written by: Matthias Kim

Answer: $\boxed{\dfrac{5+\sqrt{5}}{2}}$

Solution:

$$
\begin{gathered}
(b-1)^2 + (c-1)^2 = 7 \\
b^2 + c^2 - 2b - 2c = 5 \\
b^2 + 2bc + c^2 - 2b - 2c = 5 + 2bc \\
(b+c)^2 - 2(b+c) = 15 \\
(b+c)^2 - 2(b+c) + 1 = 16 \\
(b+c-1)^2 = 16 \\
b + c - 1 = 4 \text{ (since } b \text{ and } c \text{ are positive)}
\end{gathered}
$$

Since $b+c = 5$ and $bc = 5$, by Vieta's formula we have that b and c are simply the roots of $x^2 - 5x + 5 = 0$. Using the quadratic formula, the two roots are $\dfrac{5 \pm \sqrt{5}}{2}$, so the maximum possible value of b is $\boxed{\dfrac{5+\sqrt{5}}{2}}$.