SOLUTIONS - Combinatorics and Geometry Round

I2MC 2024

October 19th 2024

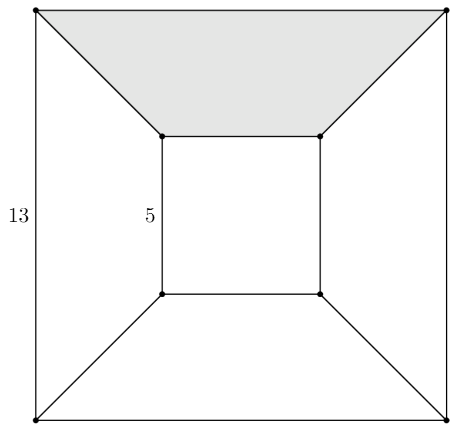
1. How many multiples of 77 are there between 7 and 777?

   Written by: Mingyue Yang

   Answer: $\boxed{10}$

   Solution: The multiples of 77 here range from 77 to 770, or $77 \cdot 1$ to $77 \cdot 10$. Thus, there are $\boxed{10}$ multiples of 77.

2. A square of side length 5 is placed perfectly at the center of a square with side length 13. What is the area of the shaded trapezoid?



   Written by: Mingyue Yang

   Answer: $\boxed{36}$

Solution: The area inside the large square but outside the small square is $13^2 - 5^2 = 144$. Since the four trapezoids are congruent, each occupies an area of $\frac{144}{4} = \boxed{36}$.

3. How many of the first 100 positive integers have the property that the sum of their digits is a perfect square?

   Written by: Matthias Kim

   Answer: $\boxed{21}$

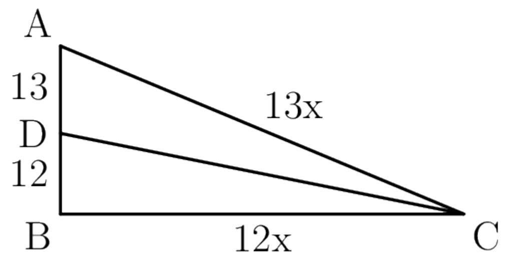
   Solution: The sum of the digits cannot exceed $9 + 9 = 18$, so we know that it can equal 1, 4, 9, or 16.
   There are 3 numbers with digit sum 1: 1, 10, and 100.
   There are 5 numbers with digit sum 4: 4, 13, 22, 31, and 40.
   There are 10 numbers with digit sum 9: 9, 18, 27, 36, 45, 54, 63, 72, 81, and 90.
   There are 3 numbers with digit sum 16: 79, 88, and 97.
   Thus, the answer is $3 + 5 + 10 + 2 = \boxed{21}$.

4. Right triangle $\triangle ABC$ has hypotenuse $\overline{AC}$. Define point D on $\overline{AB}$ such that $\overline{CD}$ bisects $\angle ACB$. Given that $AD = 13$ and $BD = 12$, find the area of $\triangle ABC$.

   Written by: Mingyue Yang

   Answer: $\boxed{750}$

   Solution:



   By the angle bisector theorem, we know that $\dfrac{BC}{AC} = \dfrac{BD}{AD} = \dfrac{12}{13}$. Let $BC = 12x$ and $AC = 13x$; then by the Pythagorean theorem, we have $AB = 5x$. Since we know $AB = 13 + 12 = 25$, we know $x = 5$. Thus, the legs of the triangle have lengths 25 and 60, so the area is equal to $\boxed{750}$.

5. You have just entered the multiverse. There are 6 universes named A, B, C, D, E, and F. You can go from any universe to another thanks to a certain Sorcerer Supreme, except you cannot go from universe B to universe F. If you are currently at universe A and want to visit all of the other universes exactly once, in how many ways can you do so?

   Written by: Aryan Agrawal

   Answer: $\boxed{96}$

   Solution: We first count the number of ways to visit all of the universes without restriction, then subtract the routes that include a move from B to F. There are 5 universes to visit, so the total number of paths is $5! = 120$. There are 4 different places where the B to F move can be (for example, you could move from A to B, then immediately to F, or you could visit F last and have B right before it), and for each there are $3! = 6$ ways to arrange the other three universes, which means 24 paths include a move from universe B to universe F. Thus the answer is $120 - 24 = \boxed{96}$.

6. Mingyue randomly places a square of side length 2 inside a square of side length 4 with the same orientation (it's not rotated) such that the entire smaller square is contained within the bigger square. What is the probability that when Mingyue rotates the smaller square 45° around its center that it is still entirely contained within the bigger square?

   Written by: Mingyue Yang

   Answer: $\boxed{6 - 4\sqrt{2}}$

   Solution: Consider where the center O of the square can be. If the unrotated square of side length 2 fits inside the larger square, then O must be at least 1 unit away from each of the edges of the larger square. This gives an area of $(4-2)^2$ where it can be.

   Now consider the rotated square. For it to still fit within the larger square, O must be at least $\sqrt{2}$ units away from each of the edges of the larger square, as that is how far away the corners of the square stick away from the center. Thus, there is a smaller area of $(4-2\sqrt{2})^2$ where O can be such that after the rotation, the smaller square is still inside the larger square.

This gives a final probability of

$$\frac{(4-2\sqrt{2})^2}{(4-2)^2} = \boxed{6-4\sqrt{2}}.$$

7. Forty percent of Interlake High School students are in the IB program, and one out of every 6 non-IB students is sad. Given that 54% of all students are not sad, find the probability that a randomly selected sad student is in the IB program.

   Written by: Jeffrey Zhao

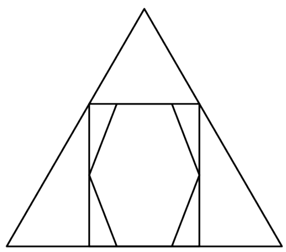
   Answer: $\boxed{\frac{18}{23}}$

   Solution:

	Sad	Not Sad	TOTAL
IB	36	4	40
Not IB	10	50	60
TOTAL	46	54	100

   For ease of calculation, assume there are 100 students at Interlake. Then 40 students are in IB and 60 students are not. Out of the non-IB students, 10 are sad, so 50 are not sad. Since 54 total students are not sad, 4 IB students are not sad, which means 36 IB students are sad and 46 total students are sad. Thus, the probability that a random sad student is in IB equals $\frac{36}{46} = \boxed{\frac{18}{23}}$.

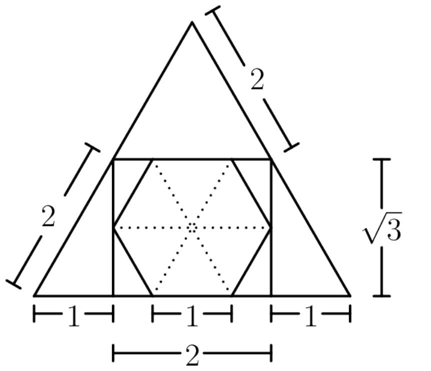
8. A regular hexagon is inscribed in a rectangle, which is inscribed in an equilateral triangle, as shown. What fraction of the triangle does the hexagon occupy?



   Written by: Mingyue Yang

Answer: $\boxed{\dfrac{3}{8}}$

Solution:



The idea is to work inside-out, taking advantage of every 30-60-90 triangle along the way. Suppose the hexagon has side length 1. Then, its area equals $\frac{3\sqrt{3}}{2}$. Then, the rectangle has width 2 and height $\sqrt{3}$. Then, the triangle has side length 4 and area $4\sqrt{3}$, so the hexagon occupies $\boxed{\dfrac{3}{8}}$ of the triangle.

9. Daniel is building a snow fort, which is divided into a 4×4 grid. Daniel wants to place 8 very large snowballs in the fort for defense purposes, and in order to optimally deter August, each row and column of the grid must contain 2 snowballs. How many ways can Daniel defend his fort?

Written by: Mingyue Yang

Answer: $\boxed{90}$

Solution: There are $\binom{4}{2} = 6$ ways that Daniel can place the snowballs in the 1st row. Since we can swap the orders of the columns with no repercussions, we can assume without loss of generality that the snowballs are in the 1st and 2nd columns in row 1 and multiply our answer by 6 later.

Now proceed by casework:

Case 1: Row 2 has snowballs in columns 3 and 4.

In this case, there are **6** ways again to fill in row 3, and row 4 is forced to take the ones that row 3 didn't take to make sure there are two snowballs in every column.

Case 2: Row 2 has snowballs in columns 1 and 3.

In this case, notice that a snowball must be placed in column 4 of both rows 3 and 4. There are **2** ways to place the remaining 2 snowballs, by either placing the other snowball in row 3 in column 2 or column 3.

Variants of Case 2:

Note that if row 2 had snowballs in columns 1 and 4, 2 and 3, or 2 and 4, the exact same logic of Case 2 applies, so counting all of these, there are **8** total solutions in case 2 and its variants.

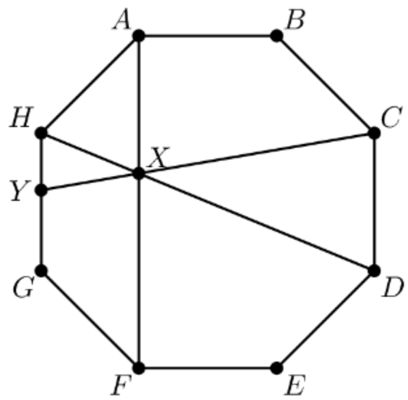
Case 3: Row 2 has snowballs in columns 1 and 2.

In this case, we are forced to place the remaining 4 snowballs in the bottom 2×2 corner.

This yields **1** case.

Thus, our answer is $6(6 + 8 + 1) = 90$.

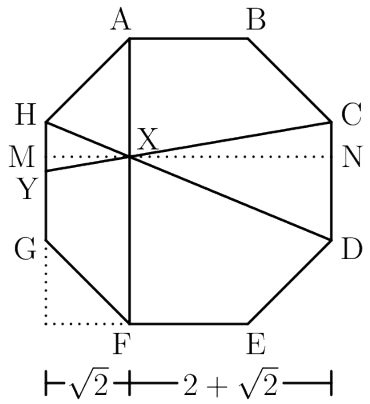
10. Consider regular octagon $ABCDEFGH$ with side length 2. Let X be defined as the intersection of $\overline{AF}$ and $\overline{DH}$. Let Y be defined as the intersection of $\overline{CX}$ and $\overline{HG}$. What is XY?



Written by: Aishwarya Agrawal

Answer: $\boxed{2\sqrt{6} - 2\sqrt{3}}$

Solution:



Draw a line through X parallel to $\overline{AB}$. Let M be its intersection with $\overline{HG}$ and N be its intersection with $\overline{CD}$. Then $MX = \sqrt{2}$ and $NX = 2 + \sqrt{2}$. We also note that the following pairs of triangles are similar: $\triangle HXY$ and $\triangle DXC$; $\triangle HXM$ and $\triangle DXN$; and $\triangle MXY$ and $\triangle NXC$. We know from this that $\dfrac{YH}{CD} = \dfrac{MX}{XN} = \dfrac{\sqrt{2}}{2+\sqrt{2}} = \sqrt{2}-1$, and since $CD = 2$, we know $YH = 2\sqrt{2}-2$. Since $HC = 2\sqrt{2}+2$ and $\angle CHY$ is right, $CY = 2\sqrt{6}$. Thus, $XY = CY\dfrac{\sqrt{2}}{2+2\sqrt{2}} = (2\sqrt{6})\dfrac{2-\sqrt{2}}{2} = \boxed{2\sqrt{6}-2\sqrt{3}}$.