SOLUTIONS - Guts Round

I2MC 2024

October 19th 2024

1. What is the area of a circle with diameter π?

   Written by: Mingyue Yang

   Answer: $\boxed{\dfrac{\pi^3}{4}}$

   Solution: If the diameter of the circle is π, then the radius is $\dfrac{\pi}{2}$. The area of the circle is then $\pi r^2 = \pi \left(\dfrac{\pi}{2}\right)^2 = \boxed{\dfrac{\pi^3}{4}}$

2. August is throwing snowballs at Daniel. He gathers snow from a rectangular area that is 4 feet wide and 5 feet long, and each snowball has a volume of 15 cubic inches. The snow accumulates at a rate of 2 inches per hour. If August can only throw whole snowballs, how many snowballs can he throw in the first minute of snowfall?

   Written by: Jeffrey Zhao

   Answer: $\boxed{6}$

   Solution: Convert the lengths 4 feet and 5 feet into 48 inches and 60 inches respectively. The total volume of snow that accumulates in an hour is equal to $48 \cdot 60 \cdot 2$ cubic inches, so the total volume that accumulates in a minute is a sixtieth of that, or $48 \cdot 2 = 96$ cubic inches. Since $6 \cdot 15 = 90$ is the largest multiple of 15 at most 96, the maximum number of snowballs that August can throw is $\boxed{6}$.

3. Jeffrey draws a card from a standard 52-card deck and rolls a fair 6-sided die. What is the probability that the number on the card is smaller than the number

on the die? (Face cards have a value of ten and aces have a value of 1.)

Written by: Jeffrey Zhao

Answer: $\boxed{\dfrac{5}{26}}$

Solution: We do casework by the number on the die.
If the die shows 1, there are no possible values for the card.
If the die shows 2, the card can only be an ace. The probability of this is $\frac{1}{6} \cdot \frac{1}{13} = \frac{1}{78}$
If the die shows 3, the card can be an ace or a 2. The probability of this is $\frac{1}{6} \cdot \frac{2}{13} = \frac{2}{78}$
If the die shows 4, the card can be an ace, 2, or 3. The probability of this is $\frac{1}{6} \cdot \frac{3}{13} = \frac{3}{78}$
If the die shows 5, the card can be an ace, 2, 3, or 4. The probability of this is $\frac{1}{6} \cdot \frac{4}{13} = \frac{4}{78}$
If the die shows 6, the card can be an ace, 2, 3, 4, or 5. The probability of this is $\frac{1}{6} \cdot \frac{5}{13} = \frac{5}{78}$
Thus the answer is $\dfrac{1}{78} + \dfrac{2}{78} + \dfrac{3}{78} + \dfrac{4}{78} + \dfrac{5}{78} = \dfrac{15}{78} = \boxed{\dfrac{5}{26}}$

4. Let the answer to this question be x. What is $x^2 + 5x + 4$?

   Written by: Mingyue Yang

   Answer: $\boxed{-2}$

   Solution:

$$\begin{aligned} x &= x^2 + 5x + 4 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \\ x &= \boxed{-2} \end{aligned}$$

5. Aryan decides that the name "I2MC" makes no sense. First, he replaces the "2" with any capital letter of the English alphabet. Then, he reorders the letters however he likes. How many four-character names could he end up with?

   Written by: Mingyue Yang

   Answer: $\boxed{588}$

Solution: Aryan can pick 26 different letters, of which 23 are not I, M, or C. In these cases there are $4! = 24$ different ways to rearrange the letters, making for $23 \cdot 24 = 552$ names. If the letter is I, M, or C, then there is a repeated letter, so there are only $\frac{4!}{2} = 12$ ways to arrange the letters, making for $3 \cdot 12 = 36$ names. Thus the answer is $552 + 36 = \boxed{588}$.

6. Each letter in the 9-digit number $INTERLAKE$ stands for a digit between 0 and 9, with different letters standing for different digits. If $I \neq 0$, what is the smallest possible value of $INTERLAKE$ that is divisible by 3?

   Written by: Mingyue Yang

   Answer: $\boxed{102345693}$

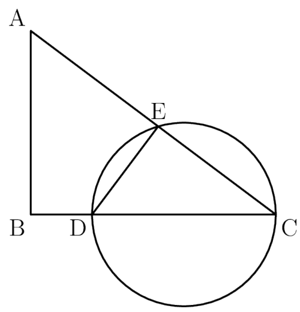
   Solution: Digits further to the left represent larger numbers and thus contribute more to the overall size of the number, so we seek to minimize those first. We fill out all digits except K with the lowest possible values from left to right, noting that I cannot be 0, to get $1023456K3$. Since the digits (excluding K) sum to 24, which is divisible by 3, we know that K must also be divisible by 3. The smallest remaining digit divisible by 3 is 9, so the answer is $\boxed{102345693}$.

7. Consider right triangle ABC with a right angle at B. $AB = 6$ and $BC = 8$. Point D lies on $\overline{BC}$ so that $DC = 6$. A circle is drawn with diameter $\overline{DC}$ and it intersects $\overline{AC}$ at point E. What is the area of $\triangle CDE$?

   Written by: Aishwarya Agrawal

   Answer: $\boxed{\frac{216}{25}}$

   Solution:



Since $\overline{DC}$ is the diameter of the circle, we know that $\angle DEC$ is a right angle. Then, since $\angle ECD = \angle BCA$, we know that $\triangle ABC$ and $\triangle DEC$ are similar. By the Pythagorean theorem we know that $AC = 10$, and since $DC = 6$ we can find that DE and EC are $\frac{18}{5}$ and $\frac{24}{5}$ respectively, so the area of $\triangle CDE$ is $\frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \boxed{\frac{216}{25}}$

8. Let $S(n)$ denote the sum of the digits of a positive integer n. What is the minimum possible value of $S(77x)$, where x is a positive integer?

   Written by: Mingyue Yang

   Answer: $\boxed{2}$

   Solution: Did you know that 1001 is equal to $7 \cdot 11 \cdot 13 = 77 \cdot 13$? If you didn't, this is not too hard to find as you only need to calculate the first 13 multiples of 77. Hopefully, you didn't quit before then.

   Once you find this value, you may realize that the only possible way to get a smaller digit sum is to have a power of 10 like 100 or 10000000000, which clearly cannot be divisible by 77. Thus the minimum digit sum is $1 + 0 + 0 + 1 = \boxed{2}$.

9. Submit any positive integer n. Your score on this question is

$$\max\left(0, \left\lfloor \frac{1}{8} n(15-n) \right\rfloor\right).$$

   Written by: Mingyue Yang

   Answer: $\boxed{7}$ and $\boxed{8}$ are optimal.

Solution: From experimentation or proof we find that the value will be greatest when n and $15 - n$ are close together. One may be tempted to submit 7.5 as the answer, which will net you exactly zero points because we literally warned you twice that it has to be a positive integer. The closest positive integers to 7.5 are $\boxed{7}$ and $\boxed{8}$ which both give the same result of 7 points.

10. Let $A_1A_2 \ldots A_{2024}$ be a regular 2024-gon. What is the degree measure of $\angle A_{20}A_2A_4$?

Written by: Mingyue Yang

Answer: $\boxed{\dfrac{360}{253}}$

Solution: Suppose we inscribe the 2024-gon in a circle with center O. Then by the inscribed angle theorem, $\angle A_{20}A_2A_4 = \frac{1}{2}\angle A_{20}OA_4$. Since the circle can be split into 2024 angles of the form $\angle A_nOA_{n+1}$, each of these angles measures $\frac{360}{2024}$ degrees. Since $\angle A_{20}OA_4$ encompasses 16 such angles, the value of $\angle A_{20}A_2A_4$ is equal to $\dfrac{1}{2} \cdot \dfrac{360}{2024} \cdot 16 = \boxed{\dfrac{360}{253}}$ degrees.

11. Simplify the expression $14\sqrt{34\sqrt{14\sqrt{34\sqrt{\cdots}}}}$. Your answer should be in the form $a\sqrt[n]{b}$, a, b, and n are integers, n is as small as possible, and a is as large as possible.

Written by: Jeffrey Zhao

Answer: $\boxed{28\sqrt[3]{2023}}$

Solution: Let $x = 14\sqrt{34\sqrt{14\sqrt{34\sqrt{\cdots}}}}$. Then we have

$$x = 14\sqrt{34\sqrt{x}}$$

$$\frac{x^2}{14^2} = 34\sqrt{x}$$

$$\frac{x^4}{14^4} = 34^2(x)$$

$$x^3 = 14^4 34^2$$

$$x = 14\sqrt[3]{14 \cdot 34 \cdot 34} = 14\sqrt[3]{8 \cdot 7 \cdot 17^2} = \boxed{28\sqrt[3]{2023}}$$

12. What is the sum of all positive factors of 2025?

Written by: Aishwarya Agrawal

Answer: $\boxed{3751}$

Solution: The prime factorization of 2025 is $3^4 5^2$, so the sum of its positive factors is $(1+3+3^2+3^3+3^4)(1+5+5^2) = 121 \cdot 31 = \boxed{3751}$.

	1	3	3^2	3^3	3^4
1	1	3	3^2	3^3	3^4
5	5	$3 \cdot 5$	$3^2 \cdot 5$	$3^3 \cdot 5$	$3^4 \cdot 5$
5^2	5^2	$3 \cdot 5^2$	$3^2 \cdot 5^2$	$3^3 \cdot 5^2$	$3^4 \cdot 5^2$

13. You want to guess a secret 4-digit number. You know that the second and fourth digits are equal, and that the digits sum to 12. Given that it is divisible by 7 and 5, find the secret number.

Written by: Aryan Agrawal

Answer: $\boxed{9030}$

Solution: Since the number is divisible by 5, the last digit is either 0 or 5. If it is 5, then the first and third digits must sum to 2, so the number can either be 1515 or 2505, neither of which is divisible by 7. Thus, the last digit must be 0, so the number can be 3090, 4080, 5070, 6060, 7050, 8040, or 9030. Testing each of these values, we find that $\boxed{9030}$ is the only one divisible by 7, so it must be the answer.

14. Compute $\frac{1}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \frac{5}{3^5} + \cdots$, where the numerators follow the Fibonacci sequence and the sequence continues infinitely.

Written by: Owen J. Zhang

Answer: $\boxed{\frac{3}{5}}$

Solution: We know that each term of the Fibonacci sequence is calculated by summing the last two terms. Thus we write each term in the given sequence (excluding the first two) as a sum, as shown below:

$$\begin{array}{cccccccccc}
\frac{1}{3^3} & +\frac{1}{3^4} & +\frac{2}{3^5} & +\frac{3}{3^6} & +\frac{5}{3^7} & +\frac{8}{3^8} & +\frac{13}{3^9} & +\frac{21}{3^{10}} & \cdots \\
+ & + & + & + & + & + & + & + & \\
\frac{1}{3^3} & +\frac{2}{3^4} & +\frac{3}{3^5} & +\frac{5}{3^6} & +\frac{8}{3^7} & +\frac{13}{3^8} & +\frac{21}{3^9} & +\frac{34}{3^{10}} & \cdots
\end{array}$$

If we let the original sum equal x, then we can see that the top sum is equal to $\frac{x}{9}$ and the bottom sum is equal to $\frac{x-\frac{1}{3}}{3}$. If we add these two sums together along with the $\frac{1}{3}$ and $\frac{1}{3^2}$ that we excluded from the start, we will get the original sum x. Hence, $x = \frac{4x-1}{9} + \frac{4}{9} = \frac{4x+3}{9}$. Thus $4x + 3 = 9x$, so $x = \boxed{\frac{3}{5}}$.

15. You think that 3-dimensional geometry is too easy. Unfortunately, the math club officers overheard this and decided to give you this problem as retribution: An ant is stranded at the origin of a 4-dimensional plane with axes w, x, y, and z. If the ant is currently at (w, x, y, z), it can reach either $(w + 1, x, y, z)$, $(w, x + 2, y, z)$, $(w, x, y + 2, z)$, or $(w, x, y, z + 4)$. How many ways can the ant reach $(4, 4, 4, 4)$?

    Written by: Aryan Agrawal

    Answer: $\boxed{3780}$

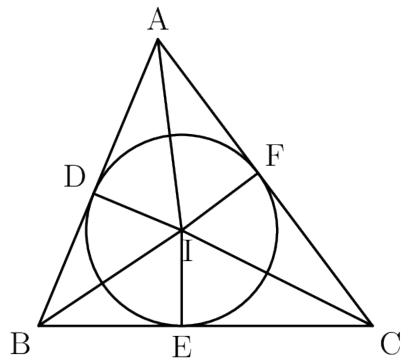
    Solution: Thankfully, this is not actually a geometry problem. The four types of moves described can be represented as W, X, Y, and Z, which correspond to the direction the move is in. To reach $(4, 4, 4, 4)$ the ant must make four W moves, two X moves, two Y moves, and one Z move. The number of routes that the ant can take is equal to the number of ways that these moves can be arranged in order, which equals $\frac{9!}{(4!)(2!)(2!)(1!)} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4} = \boxed{3780}$.

16. In triangle $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Let I be the incenter of $\triangle ABC$. Find the product of the lengths AI, BI, and CI.

    Written by: Jeffrey Zhao

    Answer: $\boxed{520}$

    Solution:



The following are true of an incircle configuration in any triangle:

-$A = rs$, where A is the area, r is the inradius, and s is the semiperimeter.

-If I is the incenter and D, E, and F are the intersections between the incircle and the sides of the triangle, then $\overline{ID}$, $\overline{IE}$, and $\overline{IF}$ are perpendicular to the sides of the triangle.

-If $BC = a$, $CA = b$, and $AB = c$, then $AD = AF = s - a$, $BD = BE = s - b$, and $CE = CF = s - c$.

Now we move on to our particular problem.

Drawing the perpendicular from A to $\overline{BC}$, we find that the triangle can be split into a 5-12-13 right triangle and a 9-12-15 right triangle, so the altitude is 12. Thus, the area is 84. By the formula $A = rs$, we may find that $r = 4$. Next, we can find that $AD = AF = 7$, $BD = BE = 6$, and $CE = CF = 8$. Therefore, we find that $AI = \sqrt{4^2 + 7^2} = \sqrt{65}$, $BI = \sqrt{4^2 + 6^2} = \sqrt{52}$, and $CI = \sqrt{4^2 + 8^2} = \sqrt{80}$. Thus the answer is $\sqrt{52 \cdot 65 \cdot 80} = \sqrt{4 \cdot 13 \cdot 5 \cdot 13 \cdot 5 \cdot 16} = 2 \cdot 13 \cdot 5 \cdot 4 = \boxed{520}$.

17. How many 10-letter strings consisting of only the letters A and B have the property that out of any three consecutive letters, exactly two of them are the same?

Written by: Mingyue Yang

Answer: $\boxed{178}$

Solution:

The main idea is to use the so-called “dynamic programming”, or recursion. Let a_i be the number of strings of length i satisfying the conditions that end with A. Let b_i be the number of strings of length i satisfying the conditions that end with B. We know that $a_1 = b_1 = 1$ and $a_2 = b_2 = 2$ (just list the strings out).

Now consider all strings of length i satisfying the conditions that end with A. Some of them will end with 1 A. If we remove that A, we get a string of length $i - 1$

that ends with B. Therefore, there are b_{i-1} such strings.

Some other of them will end with 2 As. If we remove both As, we get a string of length $i-2$ that ends with B. Therefore, there are b_{i-2} such strings.

It is impossible for the string to end with 3 or more As. Therefore, $a_i = b_{i-1} + b_{i-2}$. Similarly, $b_i = a_{i-1} + a_{i-2}$. Now we calculate:

	1	2	3	4	5	6	7	8	9	10
a	1	2	3	5	8	13	21	34	55	89
b	1	2	3	5	8	13	21	34	55	89

Since the question asks for the number of 10-letter strings, the answer is simply $a_{10} + b_{10} = \boxed{178}$.

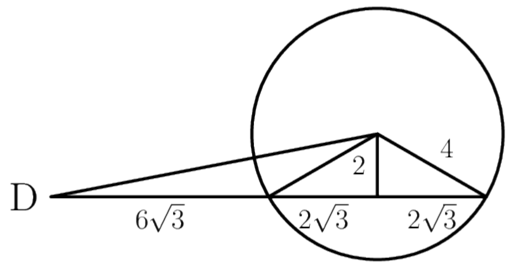
18. August is running at a constant speed around a circle at a rate of 54 seconds per lap. Daniel stands outside the circle with a snowball in his hand. Daniel notices that August is directly east of him twice every lap: once when August is $10\sqrt{3}$ meters away, and another time 18 seconds later when August is just $6\sqrt{3}$ meters away.

Submit a positive real number x. Daniel will choose the best time to throw the snowball x meters directly at August. If x is far enough, the snowball hits, and you get $\left\lfloor \frac{c}{x^3} \right\rfloor$ points, where c is some large positive real constant. If x is too close, the snowball falls harmlessly to the ground and you score 0 points.

Written by: Mingyue Yang

Answer: $\boxed{10}$ is optimal.

Solution:



If we draw an east-west line that passes through Daniel, its intersections with the circle will be $4\sqrt{3}$ meters apart. Since August takes 18 seconds to run between these points and 54 seconds to run around the entire circle, this means the two points are 120 degrees apart. Thus we can determine that the radius of the circle is 4 meters. This problem is essentially asking for the shortest possible distance from Daniel to a point on the circle, which will be along the line between Daniel and the center of the circle. The distance to the center of the cirlce is equal to $\sqrt{2^2 + (8\sqrt{3})^2} = 14$, so to find the shortest possible distance we simply subtract the radius to get $\boxed{10}$ meters.

Remark: If you literally just submitted $6\sqrt{3}$ as your answer, you would have received 10 out of 12 possible points. That's pretty gutsy.

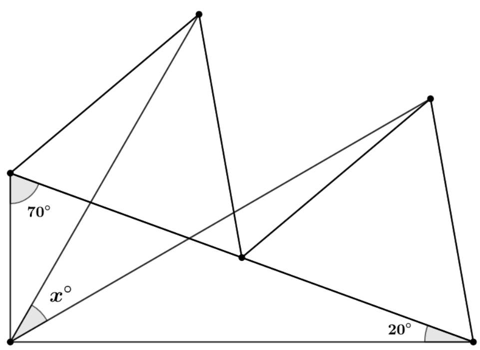
19. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2 + 3n}$.

Written by: Aryan Agrawal

Answer: $\boxed{\dfrac{11}{6}}$

Solution: We note that $n^2 + 3n = n(n+3)$, so $\dfrac{3}{n^2+3n} = \dfrac{(n+3)-n}{n(n+3)} = \dfrac{n+3}{n(n+3)} - \dfrac{n}{n(n+3)} = \dfrac{1}{n} - \dfrac{1}{n+3}$. Thus the sum is equal to $\dfrac{1}{1} - \dfrac{1}{4} + \dfrac{1}{2} - \dfrac{1}{5} + \dfrac{1}{3} - \dfrac{1}{6} + \dfrac{1}{4} - \dfrac{1}{7} + \dfrac{1}{5} - \dfrac{1}{8} + \cdots$. We can see that all of the fractions where the denominator is 4 or greater will "telescope" - there is one term with $+\frac{1}{4}$ and one term with $-\frac{1}{4}$, and so on and so forth. Thus, all of these terms simply sum to zero. Therefore, the answer is just $\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3} = \boxed{\dfrac{11}{6}}$.

20. A right triangle has angles 20° and 70°. Two congruent equilateral triangles sharing a vertex are constructed along its hypotenuse as shown below. If the marked angle measures x°, find x.



Written by: Benjamin Fu

Answer: $\boxed{30}$

Solution: Label the points A, B, C, D, E, and M such that $\triangle ABC$ is right with the right angle at B, and the two equilateral triangles are $\triangle AMD$ and $\triangle EMC$. Then, M is the midpoint of $\overline{AC}$, and is equidistant from A, B, C, D, and E, so it is the center of a circle that passes through the other five points. Thus, the angle we are looking for ($\angle DEC$) is inscribed in this circle, meaning it is equal to half of angle $\angle DMC$, which means the answer is $\frac{1}{2} \cdot 60 = \boxed{30}$ degrees.

21. The *median-mode-mean* of a set of positive integers is the mean of its median and its unique mode. Given the data set 8, 1, 4, b, 10, c, 8 has a *median-mode-mean* of 6 but does not contain the number 6, what is the mean of this data set?

    Written by: Mingyue Yang

    Answer: $\boxed{\frac{36}{7}}$

    Solution: Since there are two 8's in the data set, the mode can either be 1, 4, 10 (if b and c are both equal to one of those values), or 8. We also note that the median can only be between 4 and 8, so for the *median-mode-mean* to be 6, the mode must either be 4 or 8. If the mode is 4, then $b = c = 4$, which would make the median also 4. This is bad, so the mode must be 8.

    Suppose now that the mode is 8. Note that if at least one of b or c is 8, then the median will be 8, but we need the median to be 4, so this is also bad. Therefore, neither b nor c is 8. All the numbers in the set (except 8) must be distinct in order for 8 to be the mode, so for the median to be 4 we must have that b and c are less than 4. Thus b and c must equal 2 and 3, so the mean of the data set is $\frac{1+2+3+4+8+8+10}{7} = \boxed{\frac{36}{7}}$

22. The number $\overline{aba}$ (where a is the hundreds digit, b is the tens digit, etc) in base 10 is equal to $\overline{7aa}$ in base b. Convert $\overline{7aa}$ in base 10 to base b.

    Written by: Jeffrey Zhao

    Answer: $\boxed{1350}$

Solution:

$$101a + 10b = 7b^2 + ab + a$$
$$100a - ab = 7b^2 - 10b$$
$$a = \frac{b(7b-10)}{100-b}$$

Since b is a digit in base 10 we know that $2 \le b \le 9$ (we can't have base 0 or base 1), and since a and 7 are digits in base b, we know that $a < b$ and $7 < b$. Thus, b must either be 8 or 9. If $b = 9$, then $a = \frac{9 \cdot 53}{91}$, which is not an integer. If $b = 8$, then $a = \frac{8 \cdot 46}{92} = 4$. Thus, $a = 4$ and $b = 8$ is the only solution. Finally, 744 in base 10 is equal to $512 + 192 + 40 = 8^3 + 3 \cdot 8^2 + 5 \cdot 8$. Thus, 744 in base 10 is equal to $\boxed{1350}$ in base 8.

23. Find all real x satisfying

$$3\lfloor x \rfloor^2 + 2\{x\}^2 = \lceil x \rceil^2$$

$\{x\}$ denotes the fractional part of x.

Written by: Matthias Kim

Answer: $\boxed{0, \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}}$

Solution: We will use the fact that $x = \lfloor x \rfloor + \{x\}$. Let $a = \lfloor x \rfloor$ and $b = \{x\}$.

Suppose first that x is not an integer. Then $\lceil x \rceil = \lfloor x \rfloor + 1$, so the problem becomes $3a^2 + 2b^2 = (a+1)^2 = a^2 + 2a + 1$, or $2b^2 = -2a^2 + 2a + 1$. Since a must be an integer, the right side of this equation is an odd integer. Since b is between 0 and 1, the left side of the equation is between 0 and 2. There is only one odd integer between 0 and 2, thus the right side of the equation has to equal 1. This yields $2a^2 = 2a$. Thus, a is either 0 or 1, and $b = \frac{\sqrt{2}}{2}$.

Now suppose that x is an integer, in which case the floor and ceiling are both equal to x. In this case, the equation becomes $3x^2 = x^2$, which has one integer solution $x = 0$. Thus, the possible values of x are $\boxed{0, \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}}$.

24. Let S be a set of 5 randomly selected (distinct) elements from the set $\{1, 2, 3, \ldots, 8\}$. What is the probability that the mean of S is greater than the median of S?

Written by: Matthias Kim

Answer: $\boxed{\dfrac{23}{56}}$

Solution: The mean of S can either be less than, equal to, or greater than the median. Define S to have a "sister set" where we replace every element x with $9-x$. Note that this divides the collection of possible S into sister pairs. Also, if the mean of S is less than the median of S, then the mean of its sister set will be greater than the median of the sister set.

Therefore, there will be the same number of sets S where the mean is less than the median as the number of sets where the mean is greater than the median. Thus, if we subtract the number of sets where the mean is equal to the median from the total and divide by two, we will get our desired answer.

Observe that there are $\binom{8}{5} = 56$ possible sets S. Now we find the number of sets where the mean is equal to the median:

a) The median and mean equals 1: Not possible. Hence, 0 cases.

b) The median and mean equals 2: Not possible. Hence, 0 cases.

c) The median and mean equals 3: There is only one set $\{1,2,3,4,5\}$. Hence, there is 1 case.

d) The median and mean equals 4: Carefully listing sets based on the smallest elements gives $\{1,2,4,6,7\}$, $\{1,2,4,5,8\}$, $\{1,3,4,5,8\}$, and $\{2,3,4,5,6\}$. Hence, we have 4 cases.

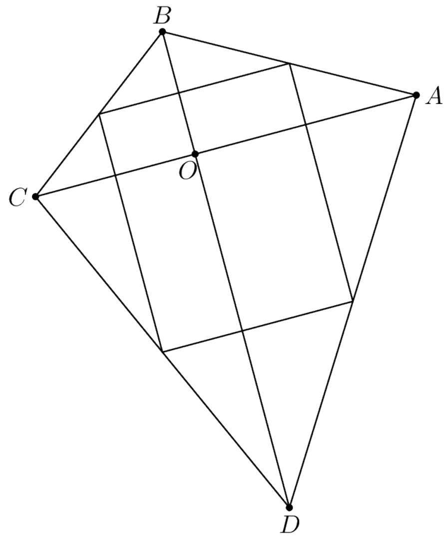
The cases where the mean and median are equal to 5, 6, 7, or 8 are symmetrical to the ones we already have, so we can just double our count. Therefore, there are 10 sets where the mean and median are equal. The other 46 will have unequal means and medians, and half of those will have a greater mean than their median. This gives us a probability of $\boxed{\dfrac{23}{56}}$.

25. Let $ABCD$ be a convex quadrilateral with $AB = 4$, $BC = 5$, and $CD = 6$. Given that the quadrilateral formed by the midpoints of the sides of $ABCD$ is a square, determine the length of side $\overline{DA}$.

Written by: Benjamin Fu

Answer: $\boxed{3\sqrt{3}}$

Solution: The fact that the formed quadrilateral is a square is somewhat irrelevant; we'll only use the fact that it's a rectangle.



The diagonals of the quadrilateral are parallel to the lines formed by the midpoints. Since these lines formed by the midpoints form a rectangle, they are perpendicular to each other. Therefore, the diagonals of the quadrilateral are perpendicular to each other.

Now note by the Pythagorean Theorem applied multiple times that $AB^2+CD^2 = OA^2+OB^2+OC^2+OD^2 = BC^2+DA^2$. Therefore, $16+36 = 25+DA^2$, resulting in $DA = \boxed{3\sqrt{3}}$.

26. One day during lunch, Jeffrey grabbed J fries from the school lunch line, planning to eat them all. However, Christopher swiftly snatched and gobbled groups of C fries from Jeffrey's tray until there were not enough fries to make another full group. This left poor Jeffrey with just 3 fries. If the least common multiple of C and J is 150, what is the value of $C + J$?

Written by: Christopher Yu

Answer: $\boxed{81}$

Solution: Since $150 = 2 \cdot 3 \cdot 5^2$, at least one of C or J must be a multiple of 25. Suppose C is a multiple of 25. Then, we know that $J = 25k + 3$ for some positive k. However, no number of the form $25k + 3$ is a factor of 150. Therefore, C is not a multiple of 25.

Thus, J is a multiple of 25. Now note that at least one of C or J has to be a multiple of 3. However, if C is a multiple of 3, then $J = kC + 3$ is also a multiple of 3. This means that no matter what, J is a multiple of 3.

Thus, J is a multiple of 75. Therefore, it can only be 75 or 150. If $J = 150$, then $150 = kC + 3$ gives that C is a factor of 147. However, C is also a factor of 150. This gives that C is a factor of 3, which is impossible because C must be greater than 3 in order for 3 fries to be left on Jeffrey's tray. Therefore, J is not 150.

This leaves $J = 75$. In this case, $75 = kC + 3$ gives that C is a factor of 72. Since it is also a factor of 150, C is a factor of $\gcd(72, 150) = 6$. The only factor of 6 greater than 3 is 6, so $C = 6$.

Finally, this gives us an answer of $75 + 6 = \boxed{81}$.

27. Find is the greatest positive integer n, such that it is possible to do the following on an 8×8 chessboard:

- Place 6 "blockers" on the chessboard on any squares you like, then
- Place n rooks on the chessboard such that none of the rooks attack each other

Two rooks attack each other if they are in the same row or same column, and there is not a blocker in between them.

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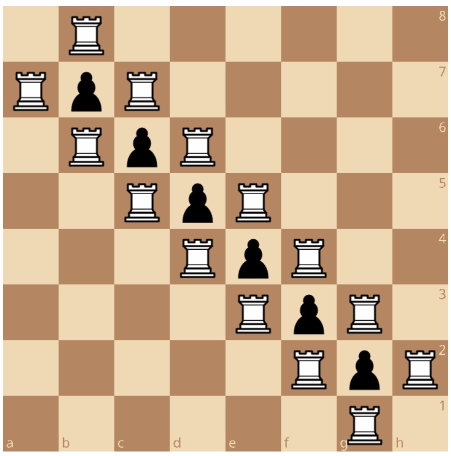
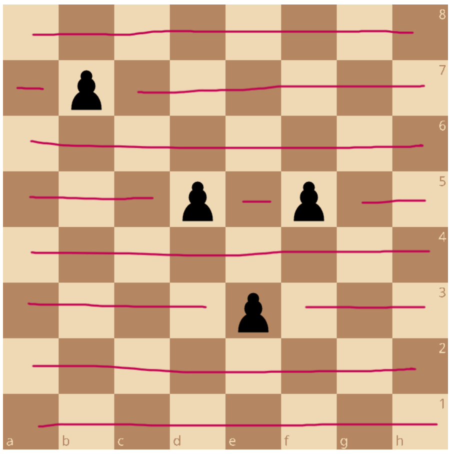
Answer: $\boxed{14}$ is optimal.

Solution: Consider the following construction for 14:



To prove that 14 is optimal, define a "horizontal segment" to be a horizontal line of squares with either the edge of the board or a blocker on either end. For example, the setup below has 12 segments:



Note that a board with no blockers on it has 8 segments. Every time we add a blocker, we either do not change the number of segments or split one segment into two segments. Therefore, adding 6 blockers adds at most 6 segments, resulting in at most 14 segments overall.

Since each segment can only host 1 rook without the rooks attacking each other, 14 is the theoretical maximum.