Algebra and Number Theory Solutions

I2MC 2025

1. The Algebra and Number Theory Round is a 10-question test taken by 72 competitors. Once the test is completed, 15 graders simultaneously begin grading the questions, each at the same constant rate. If instead there were three fewer graders, how many more questions would each grader have to grade?

   Written by: Jai Mukherjee

   Answer: 12

   Solution: There are $72 \cdot 10 = 720$ total questions. If 15 graders were to grade them, each grader would grade $\frac{720}{15} = 48$ questions. If 12 graders were to grade them, each grader would now grade $\frac{720}{12} = 60$ questions. As a result, the graders each graded **12** extra questions.

2. Aishwarya and Aryan are standing in a line. Aishwarya counts 43 people in front of her, and Aryan counts 14 people in between him and Aishwarya. Given that one of them is at the end of the line, find the sum of all possibilities for the number of people in line.

   Written by: Aishwarya Agrawal

   Answer: 103

   Solution: There are two cases: Either Aishwarya is behind Aryan or Aryan is behind Aishwarya. If Aishwarya is behind Aryan, Aishwarya is at the end of the line. Hence, there are 44 people in this case. In the other case, Aryan is at the end of the line. Aishwarya is the still the 44th person in line, so Aryan is the 59th person in line (note that there are 14 people from the 45th to the 58th position). Hence, the sum of all possibilities is $44 + 59 = \mathbf{103}$.

3. The least common multiple of three positive integers is 45. What is the least possible value of their sum?

   Written by: Mingyue Yang

   Answer: 15

   Solution: Observe that there must be a number divisible by 5 and a number divisible by 9. Clearly, these should be different numbers and as small as possible, giving $\text{lcm}(9, 5, 1) = 45$. This leads to an answer of $9 + 5 + 1 = \mathbf{15}$.

4. Call a date of the form '$m/d/y$' *additive* if the sum of d and m is a factor of y. For example, June 19th, 2025 is additive since $6 + 19 = 25$ is a factor of 2025. Find the next year in which February 29th is an additive day.

Written by: Jai Mukherjee

Answer: 2108

Solution: Observe that $d + m = 29 + 2 = 31$. Hence, y must be divisible by 31. However, February 29th only occurs in leap years. Thus, y has to be divisible by 4. This implies y is divisible by 124. The next year after 2025 that is divisible by 124 is **2108**.

5. In a non-constant arithmetic sequence of positive integers, the 1st, 5th, and 2025th terms form a geometric sequence. Find the common ratio of this geometric sequence.

   Written by: Ayden Lee

   Answer: 505

   Solution: Let a be the first term of the sequence and d be the common difference. The 5th term is then $a + 4d$ and the 2025th term is $a + 2024d$. For these terms to form a geometric sequence, we must have

$$\frac{a + 2024d}{a + 4d} = \frac{a + 4d}{a} \Leftrightarrow a^2 + 2024ad = a^2 + 8ad + 16d^2 \Leftrightarrow 2016ad = 16d^2.$$

   Since d is nonzero, dividing by d gives $126a = d$. The common ratio of the geometric sequence is

$$\frac{a + 4d}{a} = \frac{a + 4(126a)}{a} = \frac{505a}{a} = \mathbf{505}.$$

6. Let the representation of a positive integer N in base 6 be A, and let its representation in base 7 be B. What is the minimum possible value of N such that if we treat A and B instead as integers written in base 10, their sum is divisible by $6 + 7$?

   Written by: Jai Mukherjee

   Answer: 22

   Solution: At the end of the day, this problem is unfortunately just a finite case check. However, here are some strategies that can speed up the search:

   - It might sound obvious, but in this case an ordered search from $N = 1$ up is best.
   - Rather than finding $A + B$ for every choice of N, we only need to focus on checking whether multiples of 13, such as 13, 26, 39, 42, and 65 are achievable. This allows us to skip many choices of N which are far from these numbers.
   - When N is increased by 1, most of the time $A + B$ increases by 2. Only when N reaches a multiple of 6 or 7 is this not the case.

   Eventually, we may find that $N = \mathbf{22}$ yields $A = 34$ and $B = 31$, which satisfies the problem.

7. How many positive integers less than or equal to 210 are divisible by exactly one of 2, 3, 5, or 7?

   Written by: Aryan Agrawal

   Answer: 92

   Solution: By the Chinese Remainder Theorem, each integer from $1 \le x \le 210$ can be written uniquely as a tuple $(x \bmod 2, x \bmod 3, x \bmod 5, x \bmod 7)$. Thus, we are tasked with counting the number of tuples (a, b, c, d) where exactly one of a, b, c, or d equals 0. Therefore, the answer is

$$210 \cdot \left(\left(\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) + \left(1-\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\right.$$
$$\left. + \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(\frac{1}{5}\right)\left(1-\frac{1}{7}\right) + \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(\frac{1}{7}\right)\right) = \mathbf{92}$$

8. Suppose that b and c are positive integers such that $x^2 + bx + c$ and $x^2 + bx + (c + 5)$ both have integer roots. What is the minimum possible value of $b + c$?

   Written by: Mingyue Yang

   Answer: 15

   Solution: Observe that the discriminants of both $x^2 + bx + c$ and $x^2 + bx + (c + 5)$ are integers. This implies that $\sqrt{b^2 - 4c}$ and $\sqrt{b^2 - 4c - 20}$ are both integers. Let $m = \sqrt{b^2 - 4c}$ and $n = \sqrt{b^2 - 4c - 20}$. Then, $m^2 - n^2 = 20$, meaning that $(m + n)(m - n) = 20$. Since $m + n$ and $m - n$ must be the same parity, $m + n = 10$ and $m - n = 2$ is the only valid option, yielding $m = 6$ and $n = 4$.

   Thus, we get $b^2 - 4c = 36 \Leftrightarrow b^2 = 4c + 36$. Since c is nonzero and we want to minimize b, we should have $b^2 = 64 \Rightarrow b = 8$. This gives $c = 7$. The minimum possible sum of $b + c$ is then **15**.

9. For each positive factor d of 60, Jeffrey calculates the sum of the positive factors of d. What is the sum of all of these sums?

   Written by: Mingyue Yang

   Answer: 385

   Solution: Observe that each of the divisors is of the form $2^a 3^b 5^c$ where $0 \le a \le 2$, $0 \le b \le 1$, and $0 \le c \le 1$. The sum of the divisors of these divisors is

$$(1 + 2 + ... + 2^a) \cdot \left(1 + ... + 3^b\right) \cdot (1 + ... + 5^c).$$

   The sum of the sum of the divisors of these divisors is then

$$(1 + (1 + 2) + (1 + 2 + 4)) \cdot (1 + (1 + 3))) \cdot (1 + (1 + 5)) = \mathbf{385}.$$

   One can verify that if this product were expanded, each term would be the sum of the divisors of one divisor of 60.

10. Let a positive integer be *5-ish* if the sum of its digits is divisible by 5. Find the sum of all *5-ish* integers less than 1000.

Written by: Mingyue Yang

Answer: 99900

Solution: Let $\overline{abc}$ represent a 3 digit number. It is *5-ish* if $a + b + c$ is divisible by 5. There are 10 choices for a and 10 choices for b. After choosing a and b, there are 2 choices for c to ensure $a + b + c$ is divisible by 5. Thus, there are 200 *5-ish* numbers.

Now, fix a. As b goes from 0 to 9, c also rotates modulo 5. Thus, the number of times $c = 0$ is 20, the number of times $c = 1$ is 20, etc. Observe that the same logic can be applied to a: fix b and rotate c. This same logic can be applied to b: fix c and rotate a. Thus, each digit appears in each place equally often.

The average of all digits from 0 to 9 is 4.5. Therefore, the average value of $\overline{abc} = 100a + 10b + c$ is $(111)(4.5)$. Hence, the answer is $200 \cdot 4.5 \cdot 111 = \mathbf{99900}$.