Guts Round

I2MC 2025

Round 1

1. Let $A_1A_2A_3...A_{12}$ be a regular 12-sided polygon. If $A_2A_4 = 8$, what is A_4A_8?

   Written by: Mingyue Yang

   Answer: $8\sqrt{3}$

   Solution: Observe that since $\triangle A_2A_4A_8$ is a right triangle, we have $A_4A_8 = \sqrt{16^2 - 8^2} = \mathbf{8\sqrt{3}}$.

2. There exists x such that $x + \frac{1}{x} = x^2 + \frac{1}{x^2}$. What is $x + \frac{1}{x}$?

   Written by: Aryan Agrawal

   Answer: 2

   Solution: Observe that $x = 1$ is a valid solution. Thus, $x + \frac{1}{x} = \mathbf{2}$.

3. Aishwarya has the number

   $$16! = 20922789888000$$

   and replaces one of its digits with X. Aryan then receives this number and the information that it was divisible by 9. How many choices does Aishwarya have for which digit to replace such that Aryan can recover the original number?

   Written by: Mingyue Yang

   Answer: 8

   Solution: All digits except for 0 and 9 can be replaced (if 0 or 9 is replaced, Aryan will not know whether X is 0 or 9). Thus, **8** digits can be replaced.

Round 2

4. Jai draws a card from a standard 52-card deck. What is the probability that the next card he draws has a higher rank? Suits do not matter in ranking cards.

   Written by: Mingyue Yang

   Answer: $\frac{8}{17}$

   Solution: Observe that there are 3 scenarios for the three cards we draw: either both cards have equal rank, the first card has a higher rank, or the second card has a higher rank. The probability that the first card has a higher rank than the second card is equal to the probability that the second card has a higher rank than the first card. The probability both cards have an equal rank is $\frac{3}{51} = \frac{1}{17}$. Hence, the probability that the second card has a higher rank than the first card is

$$\frac{1-\left(\frac{1}{17}\right)}{2}=\mathbf{\frac{8}{17}}.$$

5. Unit cube $ABCDEFGH$ is perfectly balanced such that its space diagonal $\overline{AG}$ is perpendicular to a flat table. It is then filled to 50% volume with water. Find the height of the water level.

   Written by: Jai Mukherjee

   Answer: $\frac{\sqrt{3}}{2}$

   Solution: The answer is simply half of the space diagonal by symmetry. The space diagonal is $\sqrt{3}$. Thus, the answer is $\mathbf{\frac{\sqrt{3}}{2}}$.

6. Suppose $n = 2^a 3^b$ for positive integers a and b. The *frime pactorization* of n is $a^2 b^3$. What is the smallest n which is equal to its own frime pactorization?

   Written by: Mingyue Yang

   Answer: 72

   Solution: We have $a^2 b^3 = 2^a 3^b$. One of a or b has to be divisible by 2, and one has to be divisible by 3. The smallest (a, b) are (2, 3) and (3, 2). This gives $n = 108$ or $n = 72$. As $72 < 108$, the answer is **72**.

Round 3

7. A stellated octahedron is created by adding a regular tetrahedron to each face of an octahedron. How many edges does a stellated octahedron have?

   Written by: Aishwarya Agrawal

   Answer: 36

   Solution: A stellated octahedron will have all of the edges on the original octahedron plus the extra edges added by each tetrahedron. Each tetrahedron adds 3 edges for a total of 24 edges (an octahedron has 8 faces). An octahedron itself has $8 \cdot \frac{3}{2} = 12$ edges (each of the 8 triangles contributes 3 edges and each edge gets counted twice). This gives $12 + 24 = \mathbf{36}$ edges on the stellated octahedron.

8. The positive integers 1 through 100 are written on a blackboard. Mingyue erases all 6s and replaces them with 8s. He then adds the numbers up. What does he get as the answer?

   Written by: Mingyue Yang

   Answer: 5270

   Solution: If the digits were unchanged, the answer would be $\frac{100 \cdot 101}{2} = 5050$. Observe that for each of the numbers from 60 to 69, the 6 in the tens digit has been replaced to 8. This increases the sum from 5050 to 5250. Then, 6, 16, ..., 96 have been transformed to 8, 18, ..., 98. This is an increase of 20. Thus, the final sum is $5250 + 20 = \mathbf{5270}$.

9. For this problem, call a volunteer over for materials and proctoring.

   Aingyue the evil dictator has just taken power, forcing you and your secret resistance team to go underground. Split your team into two groups. Each group will be handed a triangle. You are not allowed to show the other group your triangle, and you are not allowed to exchange scratch paper. You know that:

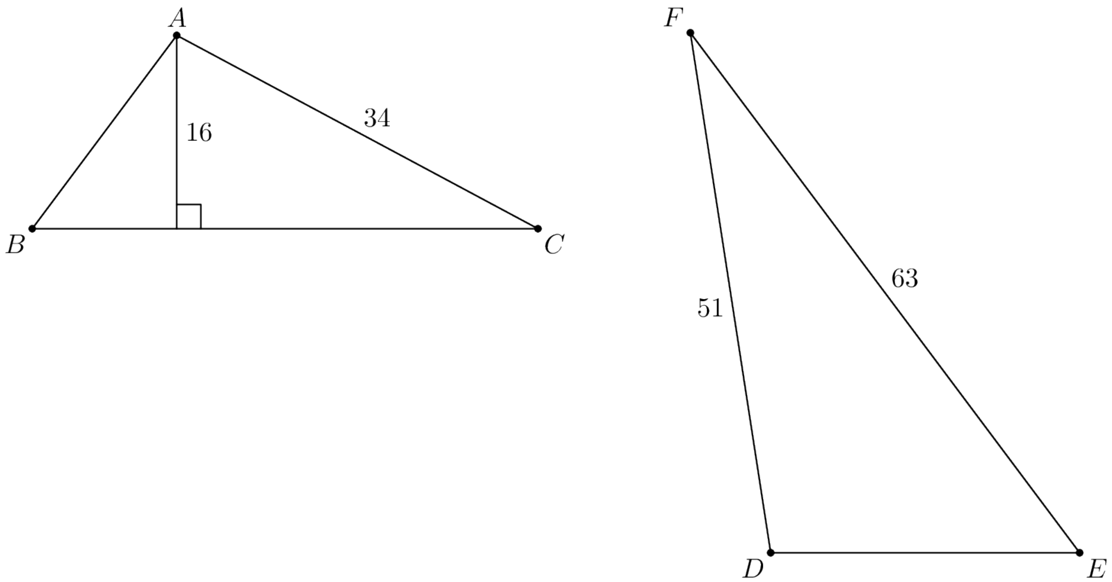
   $$\triangle ABC \sim \triangle DEF$$

   If either group can figure out the area of $\triangle ABC$, you will create a secure communication tunnel unseen by Aingyue's spies, and you will have solved this problem.

   Written by: Mingyue Yang

   Answer: 336

   Solution: The two shapes are here for reference:



   We are given that $\triangle ABC \sim \triangle DEF$. The ratio of sides is then $\frac{AC}{DF} = \frac{2}{3}$. Hence, $BC = \frac{2}{3} \cdot 63 = 42$. Thus, the area of $\triangle ABC$ is $\frac{16 \cdot 42}{2} = \mathbf{336}$.

Round 4

10. A geometric series with first term a and ratio r has a sum of 3. A geometric series with first term a and ratio $-r$ has a sum of $\frac{3}{5}$. What is the sum of a geometric series with first term a and ratio r^2?

    Written by: Aryan Agrawal

    Answer: $\frac{9}{5}$

Solution: Note that $\frac{a}{1-r} = 3$ and $\frac{a}{1+r} = \frac{3}{5}$. Furthermore,

$$\frac{a}{1-r} + \frac{a}{1+r} = \frac{2a}{1-r^2} = \frac{18}{5}.$$

Hence, the answer is simply $\frac{1}{2} \cdot \frac{18}{5} = \mathbf{\frac{9}{5}}$.

11. One hundred one stairs lead up to a temple. The stairs are divided into three flights, with each flight having an odd number of stairs. How many ways can the stairs be divided such that some flight has at least 67 stairs?

    Written by: Mingyue Yang

    Answer: 459

    Solution: Notice that only one of the flights can have 67 or more stairs, because there are only 101 stairs total. So, let's take away 66 of the steps, and split the remaining 35 steps into three odd numbers. Then, by choosing one of the three flights to add the 66 steps to, we guarantee that the selected flight, and only the selected flight, has at least 67 steps.

    We represent this as $(2a+1)+(2b+1)+(2c+1) = 35$ for nonnegative a, b, c, or $a+b+c = 16$. By stars and bars, the total number of ways to choose such a, b, c is $\binom{18}{2} = 153$. Finally, we choose one of the three to add 66 steps to, so the final count is $3 \cdot 153 = \mathbf{459}$.

12. The integer $3^{18} - 2^{18}$ has five prime factors. Find the largest one.

    Written by: Ian Rui

    Answer: 1009

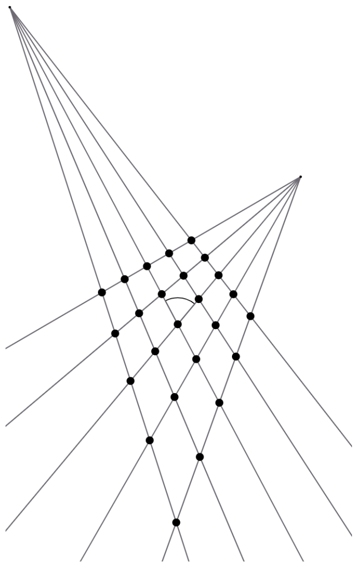
    Solution: We use difference of squares and cubes as well as sum of cubes. It is easier to apply difference of squares first, because we can easily factor the resulting expressions again.

$$\begin{aligned} 3^{18} - 2^{18} &= (3^9 - 2^9)(3^9 + 2^9) \\ &= (3^3 - 2^3)(3^6 + 3^3 \cdot 2^3 + 2^6)(3^3 + 2^3)(3^6 - 3^3 \cdot 2^3 + 2^6) \\ &= (19)(1009)(35)(577) \end{aligned}$$

    Since we are given there are 5 prime factors, they must be 5, 7, 19, 577, and **1009**.

Round 5

13. The 5 rays originating from the point on the left are separated by 1° each, and the 5 rays originating from the point on the right are separated by 2° each. The marked angle is 45°. Find the sum of the 25 acute angles at the 25 marked points.



Written by: Benjamin Fu

Answer: 1125

Solution: Notice that the marked angle is the average of all 25 angles (this can be seen by connecting the two ray origins). Thus, the answer is simply $25 \cdot 45 =$ **1125**.

14. A 6-sided die has each face labeled with a distinct positive factor of a number n with exactly 6 positive factors. When this die is rolled, the expected value is an integer k. What is the minimum possible value of k?

    Written by: Mingyue Yang

    Answer: 7

    Solution: There are two cases to consider: either $n = ab^2$ or $n = a^5$.

    Case 1: $n = ab^2$ where a and b are primes.

    In this case, the factors are 1, a, b, ab, b^2, and ab^2. The average of these factors are $k = \frac{1}{6}(1+a)(1+b+b^2)$. Note that $1+b+b^2$ is never even. Hence, for k to be an integer, a must be odd. From here, testing odd prime values of a, one can notice a minimum average of 7 at $a = 5$ and $b = 2$.

    Case 2: $n = a^5$ where a is a prime.

    The average can now be calculated as $k = \frac{1}{6}(1 + ... + a^5)$. This implies that $1 + a + ... + a^5 \equiv 0 \pmod 6$. The minimum prime value of a for which this is true is 7, which lends itself to an average of 3268.

    Thus, by combining the results of these two cases, it follows the minimum value of k is **7**.

15. Let $f(x)$ be a cubic such that $f(1) = 2$,$f(2) = 0$, $f(3) = 2$, and $f(4) = 5$. What is the value of $f(8)$?

Written by: Aishwarya Agrawal

Answer: -33

Solution: Let $g(x) = f(x) - 2$. Note that $g(3) = g(1) = 0$. Hence, $g(x) = (x - 3)(x-1)(ax+b)$ (as $g(x)$ is a cubic). Substituting $x = 2$ and $x = 4$ yields $-2a - b = -2$ and $12a + 3b = 3$. Solving for a and b, we get $a = -\frac{1}{2}$ and $b = 3$. Thus,

$$f(8) = g(8) + 2 = (8-3)(8-1)\left(8 \cdot \left(-\frac{1}{2}\right) + 3\right) + 2 = 5 \cdot 7 \cdot (-1) + 2 = \mathbf{33}.$$

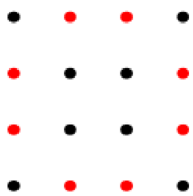
Round 6

16. Jai has a 4×4 grid of dots. What is the maximum number of dots Jai can select such that no three of them are collinear (lie on a straight line)?

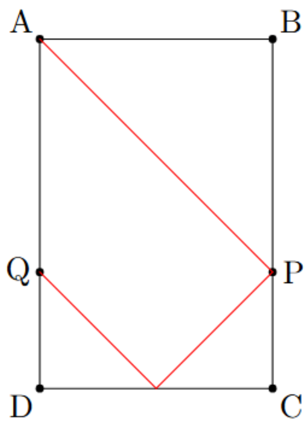
Written by: Jai Mukherjee

Answer: 8

Solution: Observe that no more than 3 dots can be selected in each row (otherwise, those 3 dots are collinear). Thus, the maximum number of dots that Jai can select is 8. An example of a valid selection of dots is provided (the red dots):



17. Let $ABCD$ be a rectangle with $AB = 20$ and $BC = 25$. A laser is shot from vertex A and first intersects a side of $ABCD$ at point P on segment BC such that $BP = 16$. The laser reflects off of the sides of the rectangle. Let Q be the point where the laser first intersects side AD. Find AQ.



Written by: Ian Rui

Answer: 18

Solution: Let the second intersection of the laser beam with the rectangle be point R. Since the laser is reflecting, we have $\angle APB = \angle RPC$, and $\angle PRC = \angle QRD$. Therefore, right triangles APB, RPC, and DQR are all similar because they all share congruent angles. Therefore, $\frac{AB}{BP} = \frac{RC}{CP}$. We have that $CP = 25 - 16 = 9$, so

$$RC = \frac{AB}{BP} \cdot CP = \frac{20}{16} \cdot 9 = \frac{45}{4}.$$

Similarly, (haha get it similar) we now have that $DR = 20 - \frac{45}{4} = \frac{35}{4}$, and $\frac{CP}{RC} = \frac{DQ}{RD}$, so

$$DQ = \frac{CP}{RC} \cdot RD = \frac{9}{\frac{45}{4}} \cdot \frac{35}{4} = 7.$$

Finally, $AQ = 25 - 7 = \mathbf{18}$.

18. Your team is pretending to be a personal circus for Aingyue the evil dictator to infiltrate his residence. Assign each of the four members of your team a different integer from 1 to 4. Aingyue is assigned the number 0.

    Since Aingyue's rule is in its *prime*, your performance will consists of your team members and Aingyue clapping in a certain order with regards to the primes. There will be 15 claps. On the nth clap, the person assigned the number equal to the remainder when the nth prime is divided by 5 must clap, and nobody else. Your performance ends after the 15th prime, 47.

    Once your performance begins, you may not communicate with each other, nor may you consult any scratch paper until your performance ends. When you are ready to perform, ask your grader. You have unlimited attempts to satisfy Aingyue with your performance. Good luck!

    Written by: Mingyue Yang

    Answer: 230213243412132

    Solution: The number listed above represents the clapping order. The first 15 primes are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

    An interesting coincidental observation is that no two people clap the same number of times.

Round 7

19. There exists exactly one four digit perfect square of the form $\overline{AABB}$. Find $A \cdot B$.

    Written by: Aryan Agrawal

    Answer: 28

Solution: Note that $\overline{AABB}$ is divisible by 11. Since $\overline{AABB}$ is a perfect square, it's actually divisible by 121. Then, $\overline{A0B}$ is divisible by 11. This implies that $A + B = 11$. Iterate through values of A and B to check if $\frac{1}{11} \cdot (100A + B)$ is divisible by 11. This gives us $A = 7$ and $B = 4$, and an answer of $7 \cdot 4 = \mathbf{28}$.

20. The mean, median, and unique mode of 100 positive integers are increasing consecutive integers, in that order. Determine the least possible value of the mean.

    Written by: Benjamin Fu

    Answer: 4

    Solution: The answer is **4**, and we can construct such a set, for example with 32 1s, 2 2s, 16 4s, and 50 6s. To prove the answer cannot be lower, note that if a mean of 1 or 2 was possible, we could increase each number by 2 or 1 respectively to get a construction for a mean of 3. Therefore, if we disprove the existence of such a set with mean 3, then we would be done.

    Note that a median of 4 means at least 50 numbers are at least 4. Note that since 5 is the unique mode, the number of 5s must be at least the number of 1s.

    Let the number of 1s be x and the number of 5s be z. Then, the sum of the set is at least $x + 5z + (50 - z) \cdot 4 + (50 - x) \cdot 2 = 300 + z - x$. However, $z > x$, so the sum is greater than 300, a contradiction.

21. A *Latin square* puzzle consists of a grid of size $n \times n$. The solver must place the digits 1 through n inclusive into the grid, such that there is one digit in each cell and no two copies of the same digit appear in the same row or column.

    How many ways can a Latin square puzzle of size 4×4 be solved?

    Written by: Jai Mukherjee

    Answer: 576

    Solution: Proceed by deciding the first row and first column. There are $4! = 24$ ways to permute the numbers in the first row, and then $3! = 6$ ways to permute the remaining numbers in the first column, since one has already been chosen.

    Now, without loss of generality, assume that the first row and first column are both in the order "1234." We will refer to positions by the notation $r\#c\#$, where the $\#$'s refer to the row number and column number, from 1 to 4. We will begin by placing the remaining two $2'$s. Either they can go on the main diagonal (r3c3 and r4c4), or they don't (i.e. r4c3 and r3c4). If the $2'$s are on the main diagonal, then we can place the two remaining $3'$s instantly into r4c2 and r2c4. Then, the two remaining $4'$s go into r2c3 and r3c2, and finally we can place the $1'$s.

    If, on the other hand, the $2'$s are in r4c3 and r3c4, then we have two cases: either the $3'$s go along the main diagonal in r2c2 and r4c4, or they go into r2c4 and r4c2. If they are on the main diagonal, $4'$s and $1'$s are forced ($4'$s go into r2c3 and r3c2). If they aren't, 4′s and 1′s can go in two different ways (either 4′s are in r2c2 and r3c3 or r2c3 and r3c2, then 1′s follow suit).

So finally, there are 4 ways to place the other 9 numbers into the Latin square. Therefore, our answer is $24 \times 6 \times 4 = \mathbf{576}$, since all of these placements are independent.

Round 8

22. Points O and A are in the plane with $OA = 1$. Six equally spaced lines $\ell_1, ..., \ell_6$ pass through O, with A on ℓ_1. An immortal bug begins at A. On each step, the bug walks in a straight line from its current position to one of the nearest points in the plane that is on a different line. The bug is condemned to perform infinitely many steps to repent for its sins. Find the total length of the bug's path.

    Written by: Ian Rui

    Answer: $2 + \sqrt{3}$

    Solution: Six equally spaced lines means they are each 30° apart from each other. Thus, the bug's first step is $\frac{1}{2}$ units and every subsequent step is $\frac{\sqrt{3}}{2}$ as long. This is a geometric series, and thus our answer is

$$\frac{\frac{1}{2}}{1 - \frac{\sqrt{3}}{2}} = \mathbf{2 + \sqrt{3}}.$$

23. Find the number of ordered positive integer triples $a, b, c \leq 100$ such that $a^2 + b^2 + c^2$ is a multiple of 20.

    Written by: Jai Mukherjee

    Answer: 25000

    Solution: Squares are either 0 or 1 mod 4. Thus, if $a^2 + b^2 + c^2$ is divisible by 20 and thus 4, all three squares must be 0 mod 4. This equates to a, b, and c being all even.

    Squares are either 0, 1, or 4 mod 5. Therefore, if $a^2 + b^2 + c^2$ is divisible by 20 and thus 5, either a^2, b^2, and c^2 are all 0 mod 5 or they are 0, 1, and 4 mod 5 in some permutation.

    There are 10 positive even integers a at most 100 such that $a^2 \equiv 0 \bmod 5$. There are 20 such that $a^2 \equiv 1 \bmod 5$, and similarly 20 such that $a^2 \equiv 4 \bmod 5$. Our final answer is $10^3 + 6 \cdot 10 \cdot 20 \cdot 20 = \mathbf{25000}$.

24. An unordered set of positive integers $\{a, b, c\}$ is considered to be a *friend group* if every pair of numbers has a common factor greater than 1. Given that a, b, and c are all unique positive integers at most 15, how many friend groups are there?

Written by: Aishwarya Agrawal

Answer: 48

Solution: Split the problem into two cases:

Case 1: all three numbers share a common factor.

Subcase A: they share a common factor of 2:

There are 7 multiples of 2 at most 15, thus there are $\binom{7}{3} = 35$ friend groups in this subcase.

Subcase B: they share a common factor of 3:

In this case, $\binom{5}{3} = 10$.

Subcase C: they share a common factor of 5:

In this case, $\binom{3}{3} = 1$.

It can be seen that these cases cover all bases but do not overlap.

Case 2: $\gcd(a, b, c) = 1$.

This case only happens with $(6, 10, 15)$ or $(12, 10, 15)$.

Our answer is $35 + 10 + 1 + 2 = \mathbf{48}$

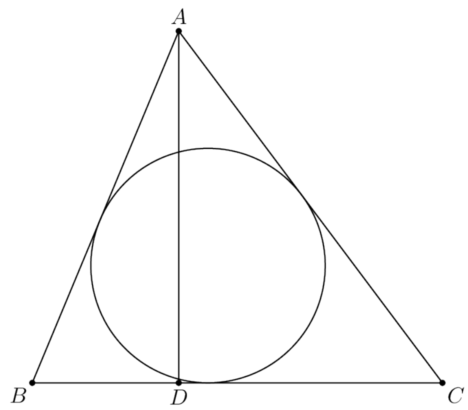
Round 9

25. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. The altitude $\overline{AD}$ is drawn, intersecting the incircle at X and Y. What is $AX - DY$?

Written by: Mingyue Yang

Answer: $\frac{16}{3}$

Solution:



Note it does not matter which intersection point is X or Y. The semiperimeter of the triangle is 21. Thus, the length of the tangent from A to the incircle is $21 - 14 = 7$.

We get then that $AX \cdot AY = 7^2$.

The length of the tangent from B to the incircle is $21 - 15 = 6$. Note that $BD = 5$ by a well-known fact of the $13 - 14 - 15$ triangle. Thus, the tangent from D to the incircle has length 1. Therefore, $DX \cdot DY = 1$.

Observe $AY = 12 - DY$, and similarly $DX = 12 - AX$. Thus, $AX \cdot (12 - DY) = 49$ and $(12 - AX) \cdot DY = 1$. Combining these equations yields $AX - DY = \mathbf{\frac{16}{3}}$.

26. Some number of horses are galloping around a circular racetrack with circumference 1600 meters. They all begin at the same point on the track and run counterclockwise with constant integer speeds (in meters per second). Suppose the first time that a horse passes another horse, one of those two horses has run 6000 meters in 400 seconds. Let $t > 0$ be the time in seconds between the first time and the second time a horse passes another horse. Find the sum of all possible values of t.

Written by: Ian Rui

Answer: $\frac{1600}{3}$

Solution: Notice that we can treat all horses travelling at the same speed as one horse, because whenever one of those horses passes another horse, all of them do. From now on, we can assume all horses have different speeds.

We know that the first time a horse passes another horse, it must be the fastest horse passing the slowest horse by travelling exactly one lap extra. $\frac{6000}{400} = 15$, so either the fastest horse or the slowest horse runs at 15 meters per second.

We also know that the fastest horse travels exactly one lap, or 1600 meters, farther than the slowest horse, over the span of the 400 seconds. Therefore, the difference between their speeds must be $\frac{1600}{400} = 4$ meters per second.

Notice that the time it takes for two horses to meet depends only on the difference in their speeds and not the speeds themselves. If two horses have speeds differing by d, then the time it takes for the faster one to pass the slower one is $\frac{1600}{d}$.

Also, notice that $d \leq 4$ because the speed difference between the fastest and slowest horses is 4. Our t depends on the second largest d between two horses (where the largest is 4), since it is the second time that two horses pass each other.

Therefore, $d = 3$ (e.g., if the horses have speeds $n, n+1, n+4$) or $d = 2$ (if the horses have speeds n, $n+2$, $n+4$). Or, if there is only one value of d, then $d = 4$. Notice that $d = 1$ cannot be our second largest value of d, because it would force two horses to have a difference of 3.

If $d = 2$ then it takes $\frac{1600}{2} = 800$ seconds after the start to meet up, or $t = 800 - 400 = 400$ seconds after the first pass.

If $d = 3$ then it takes $\frac{1600}{3}$ seconds from the start, or $t = \frac{400}{3}$ seconds.

If $d = 4$ then we only have the two horses, so it takes another 400 seconds to meet up.

Thus, the two possible values of t are 400 and $\frac{400}{3}$, for a total of $\mathbf{\frac{1600}{3}}$.

27. Your team has gone through trials and tribulations, and you are now fighting the evil dictator Aingyue himself in a battle of wits. Here is how the game goes:

    Players take turns writing either a '1' or a '−1' onto a shared piece of paper. Writing a '1' takes one stroke, while writing a '−1' takes two. The game ends once 11 strokes are drawn onto the paper (you may not draw more). If the product of all written numbers is positive, you will best the evil dictator, and save the world. If the product of all written numbers is negative, then Aingyue will unfortunately win, and continue to oppress his subjects.

    Aingyue plays with perfect strategy and has given you the mercy of choosing to go first or second. How can you win? Try it out against him in real life! Ask your grader to call Aingyue over, for up to 2 attempts.

    Written by: Mingyue Yang

    Answer: N/A

    Solution: Go first and play -1. There are 9 strokes remaining. Whatever Aingyue does, play the opposite. This way, the rest of the 9 strokes will be divided into three sets of three, with each set resulting in a product of -1. No matter what, the final product will be $(-1)^4 = 1$.
