Individual Final Round

I2MC 2025

1. Determine the greatest possible volume of a tetrahedron $ABCD$ satisfying

$$AB + BC + CD = 10.$$

Written by: Benjamin Fu

Answer: $\frac{500}{81}$

Solution: Consider any choice of $\overline{AB}$ and $\overline{BC}$. After fixing these first two segments, the optimal use of the 3rd segment $\overline{CD}$ is to make it perpendicular to the plane defined by A, B, and C. Then, consider that the optimal use of the first two segments is to maximize $[ABC]$, because the volume of a tetrahedron is $\frac{1}{3}bh$, where b is the area of the base ($[ABC]$) and h is the height ($\overline{CD}$). This occurs if $\overline{AB} \perp \overline{BC}$. Thus, the volume of our tetrahedron is $\frac{1}{3} \cdot CD \cdot \frac{1}{2} \cdot AB \cdot BC$, which is maximized when $AB = BC = CD$, yielding an answer of $\mathbf{\frac{500}{81}}$.

2. A chess king is hunting on an infinite chessboard. Their target is two squares east and two squares north of them. Each minute, the king can move to any of the eight adjacent squares, or lie patiently in wait and stay still. How many ways can 4 minutes play out if the king is at their target by the end?

Written by: Mingyue Yang

Answer: 100

Solution: Consider each axis separately. Along the x-axis, the king may move right twice and stay still twice in some order, or move right thrice and move left once in some order. There are $\binom{4}{2} = 6$ ways in the first case and $\binom{4}{1} = 4$ ways in the second case. Thus, in the x-axis, there are 10 ways for the king to reach their destination. The same is true for the y-axis, so our answer is $10^2 = \mathbf{100}$.

3. Compute

$$\sqrt{1 - \sqrt{1^2 - 1}} + \sqrt{2 - \sqrt{2^2 - 1}} + \cdots + \sqrt{2025 - \sqrt{2025^2 - 1}}.$$

Written by: Benjamin Fu

Answer: $\sqrt{1013} + 22\sqrt{2}$

Solution: A common trick for nested radicals is to write them as a sum or difference of radicals. The terms in this expression are of the form $\sqrt{n - \sqrt{n^2 - 1}}$. So, let

$$\sqrt{n - \sqrt{n^2 - 1}} = \sqrt{x} - \sqrt{y}.$$

Square both sides:

$$n - \sqrt{n^2 - 1} = x + y - 2\sqrt{xy}.$$

Equate:

$$x + y = n$$

$$\sqrt{n^2 - 1} = 2\sqrt{xy}.$$

Square and rearrange the second equation:

$$\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right) = xy.$$

It is now clear that $x = \left(\frac{n+1}{2}\right)$ and $y = \left(\frac{n-1}{2}\right)$ works. Thus, $\sqrt{n - \sqrt{n^2 - 1}} = \frac{1}{\sqrt{2}}\sqrt{n+1} - \frac{1}{\sqrt{2}}\sqrt{n-1}$. When we sum from $n = 1$ to $n = 2025$, most terms will cancel, leaving only $\frac{1}{\sqrt{2}}\sqrt{2026} + \frac{1}{\sqrt{2}}\sqrt{2025} - \frac{1}{\sqrt{2}}\sqrt{1} - \frac{1}{\sqrt{2}}\sqrt{0} = \sqrt{1013} + \frac{44}{\sqrt{2}} = \boldsymbol{\sqrt{1013} + 22\sqrt{2}}$.