SOLUTIONS - Individual Final Round

I2MC 2024

October 19th 2024

1. Find the sum of all integers x for which $x^2 + 9x + 27$ is a perfect square.

   Written by: Matthias Kim

   Answer: $\boxed{-18}$

   Solution 1: If $x^2 + 9x + 27$ is a perfect square, then so is $4(x^2 + 9x + 27) = (2x+9)^2 + 27$. Therefore, we get that $(2x+9)^2 + 27 = n^2$ for some integer n. Let $y = 2x + 9$. Rearranging and factoring gives that $(n+y)(n-y) = 27$.

   If $n + y = a$ and $n - y = b$, then $y = \frac{1}{2}(a - b)$. Since $ab = 27$ and both a and b are integers, there are only so many possibilities now to check:

   - If $a = 1$ and $b = 27$, then $y = -13$, giving $x = -11$.

   - If $a = 3$ and $b = 9$, then $y = -3$, giving $x = -6$.

   - If $a = 9$ and $b = 3$, then $y = 3$, giving $x = -3$.

   - If $a = 27$ and $b = 1$, then $y = 13$, giving $x = 2$.

   There are four more cases where a and b are both negative, however these will give the same values of y and x. Thus, our final answer is $-11 - 6 - 3 + 2 = \boxed{-18}$.

   Solution 2 (Courtesy of Song Han Ngo): Suppose $x^2 + 9x + 27 = n^2$. Then, $x^2 + 9x + (27 - n^2) = 0$. By the quadratic formula,

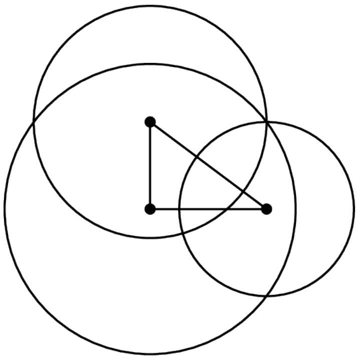
   $$x = \frac{-9 \pm \sqrt{81 - 4(27 - n^2)}}{2} = \frac{-9 \pm \sqrt{4n^2 - 27}}{2}.$$

   If x is an integer, then $\sqrt{4n^2 - 27}$ must be an integer. This means $4n^2 - 27$ must be a square number.

   The difference between two square numbers k^2 and $(k-1)^2$ is greater than 27 for all $k \geq 15$. Since $4n^2 = (2n)^2$ is a square number, it will be impossible for $(2n)^2 - 27$ to also be a square number if $2n \geq 15$. Therefore, we only have to check cases $0 \leq n \leq 7$.

Of these cases, only $n = 3$ and $n = 7$ yield square numbers. If $n = 3$, then $x = \frac{-9 \pm \sqrt{4(3)^2 - 27}}{2} = -6$ or -3. If $n = 7$, then $x = \frac{-9 \pm \sqrt{4(7)^2 - 27}}{2} = -11$ or 2. Adding these solutions up yields $\boxed{-18}$.

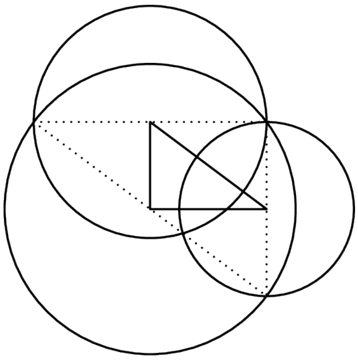
2. Aishwarya has a triangle with side lengths 3, 4, and 5. For each vertex of the triangle, she draws a circle centered around it with radius equal to the length of the side opposite it, as shown below. What is the area of the region covered by the union of the three circles?



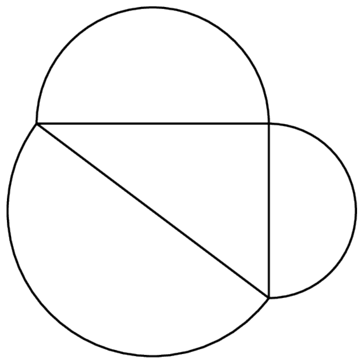
Written by: Jeffrey Zhao

Answer: $\boxed{25\pi + 24}$

Solution:



Let the center of the largest circle be $(0, 0)$. Then the other two vertices of the triangle are $(4, 0)$ and $(0, 3)$. Next, we note that the three intersection points between circles are $(4, 3)$, $(4, -3)$, and $(-4, 3)$, which form a triangle with side lengths 6, 8, and 10. The sides of this triangle are the diameters of the three circles, which means that the region covered by the circles can be split into the triangle and three semicircles, as shown below:



The semicircles have radii 3, 4, and 5, so they have a combined area of 25π. Adding the area of the triangle, we find that the total area is $\boxed{25\pi + 24}$.

3. The average number of positive factors that a positive factor of S has is 5. What is the minimum value of S?

   Written by: Mingyue Yang

   Answer: $\boxed{72}$

   Solution: This is a challenging problem and we are very impressed that three people managed to solve it. If you cannot understand this solution, don't worry. Come back in the future when you have more experience with multiplicative functions.

   The main idea is to find an explicit formula for the average number of positive factors a positive factor of a number n will have. The first step is to find a formula for the total number of positive factors over all positive factors of n.

   For a number $k = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$ written in prime factorization form, the number of factors it has is $(a_1+1)(a_2+1)\dots(a_m+1)$. Suppose $n = p_1^{x_1} p_2^{x_2} \dots p_y^{x_y}$. Then a divisor of n will have the same prime factors, with the power of each factor p_i ranging between 0 and x_i. Therefore, the total number of positive factors over all positive factors of $n = p_1^{x_1} p_2^{x_2} \dots p_y^{x_y}$ is given by

$$(1 + 2 + \dots + (x_1 + 1))(1 + 2 + \dots + (x_2 + 1) \dots (1 + 2 + \dots + (x_y + 1)).$$

   The reason is that if you expand this product, each term will count the number of positive factors of a specific divisor of n.

   Each term of this product is an arithmetic series. Therefore, we can compress this product as

$$\left(\frac{(x_1+1)(x_1+2)}{2}\right)\left(\frac{(x_2+1)(x_2+2)}{2}\right)\dots\left(\frac{(x_y+1)(x_y+2)}{2}\right).$$

That's the total. To find the average, we divide the total by the number of things being summed, which is just the number of factors of n, aka $(x_1+1)(x_2+1)\dots(x_m+1)$.

So now we have our formula for the average number of factors a factor of n has:

$$\left(\frac{x_1+2}{2}\right)\left(\frac{x_2+2}{2}\right)\dots\left(\frac{x_y+2}{2}\right)$$

Now we will casework on the number of factors.

Case 1: n has 1 factor. In this case, $\frac{x_1+2}{2}=5$. Therefore, $x_1=8$. Hence, the smallest possible value of n in this case is $2^8=256$.

Case 2: n has 2 factors. In this case, $\frac{x_1+2}{2}\cdot\frac{x_2+2}{2}=5$. Note that $x_1\geq 1$, so $x_1+2\geq 3$. The same is true for x_2+2. We are looking for $(x_1+2)(x_2+2)=20$. The only possible solution where both terms are at least 3 is $x_1+2=5$ and $x_2+2=4$. Therefore, $x_1=3$ and $x_2=2$. The minimum possible value of n in this case is $2^3 3^2=72$.

Case 3: n has more than 3 factors. In this case, $(x_1+2)(x_2+2)\dots(x_y+2)=5\cdot 2^y$. But there are not enough factors to go around - at least one x_i+2 will have to equal 2, which is not allowed. Thus there are no solutions in this case.

Therefore, our answer is $\boxed{72}$.