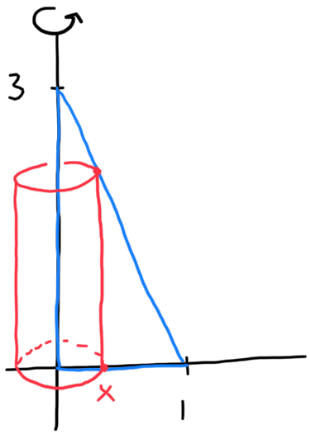
Math 221 Worksheet 27

December 10, 2020

Section 5.3: Volumes by Cylindrical Shells; Section 5.5: Average Value of a Function

1. For each of the following solids, write down and integral representing its volume. (You can use any method.)

(a) The solid obtained by rotating the region bounded by the curves $y = 0$, $x = 0$, and $y = 3 - 3x$ about the y-axis



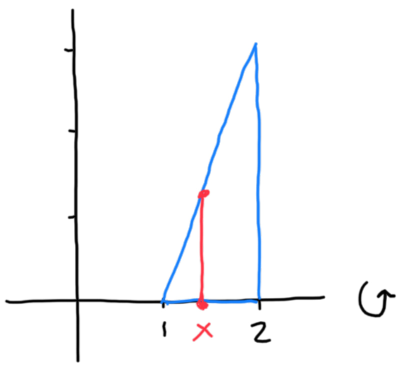
Using cylindrical shells:

The shell corresponding to x has radius x and height $3-3x$.

So

$$\text{Volume} = \boxed{\int_0^1 2\pi x(3-3x)\,dx}$$

(b) The solid obtained by rotating the region bounded by the curves $y = 0$, $x = 2$, and $y = 3x - 3$ about the x-axis



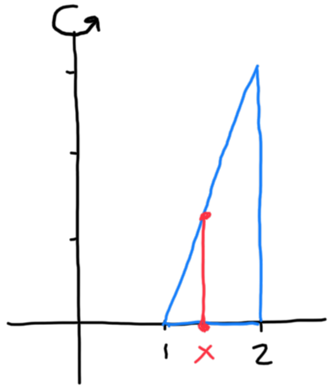
Using annuli:

The annulus corresponding to x is actually a disk of radius $3x-3$.

So

$$\text{volume} = \boxed{\int_1^2 \pi(3x-3)^2\,dx}$$

(c) The solid obtained by rotating the region from (b) about the y-axis



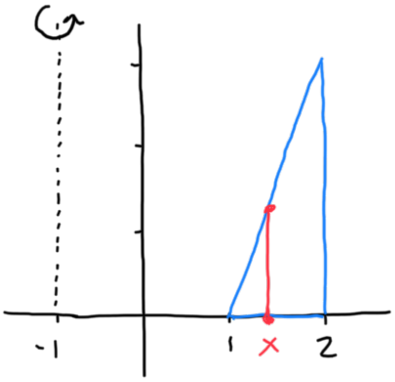
Using shells:

The shell corresponding to x has radius x and height $3x-3$.

So

$$\text{volume} = \boxed{\int_1^2 2\pi x(3x-3)\,dx}$$

(d) The solid obtained by rotating the region from (b) around the line $x = -1$



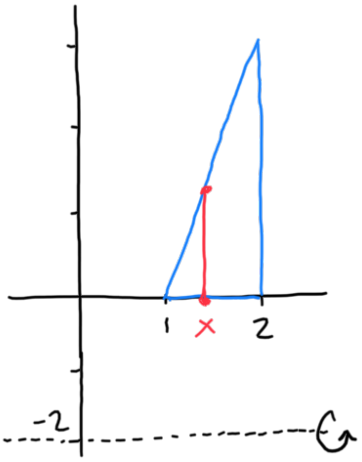
Using shells:

The shell corresponding to x has radius $x+1$ and height $3x-3$.

So

$$\text{volume} = \boxed{\int_1^2 2\pi(x+1)(3x-3)\,dx}$$

(e) The solid obtained by rotating the region from (b) around the line $y = -2$



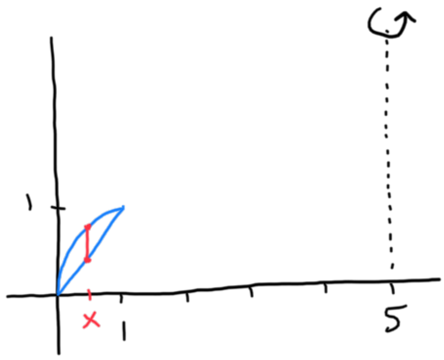
Using annuli:

The annulus corresponding to x has big radius $3x-1$ and small radius 2.

So

$$\text{volume} = \boxed{\int_1^2 \pi\left[(3x-1)^2 - 4\right] dx}$$

(f) The solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 5$



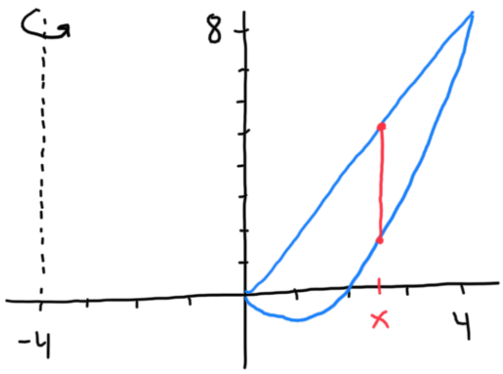
Using shells:

The shell corresponding to x has radius $5-x$ and height $\sqrt{x} - x$.

So

$$\text{volume} = \boxed{\int_0^1 2\pi(5-x)(\sqrt{x}-x)\,dx}$$

(g) The solid obtained by rotating the region bounded by the curves $y = (x-1)^2 - 1$ and $y = 2x$ about the line $x = -4$



Using shells:

The shell corresponding to x has radius $x+4$ and height $2x-(x-1)^2+1$.

So

$$\text{volume} = \boxed{\int_0^4 2\pi(x+4)\left(2x-(x-1)^2+1\right)dx}$$

2. For each of the following functions f and intervals I, compute the average value of f on I.

(a) $f(x) = \sin(2x)$, $I = [0, \pi/2]$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(2x)\,dx = \frac{2}{\pi}\left(-\frac{1}{2}\cos(2x)\right)\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{\pi}(-1-1) = \boxed{\frac{2}{\pi}}$$

(b) $f(x) = x^2 + 3$, $I = [-1, 1]$

$$\frac{1}{2}\int_{-1}^{1}(x^2+3)\,dx = \frac{1}{2}\left(\frac{x^3}{3}+3x\right)\Big|_{-1}^{1}$$

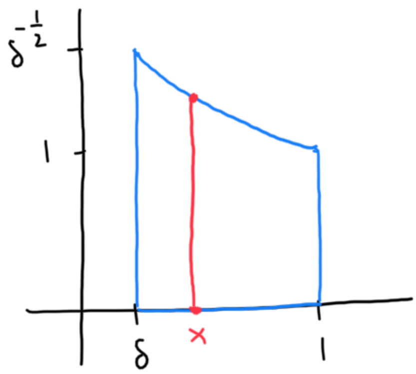
$$= \frac{1}{2}\left(\frac{1}{3}+3-\frac{-1}{3}-(-3)\right) = \boxed{\frac{10}{3}}$$

(c) $f(x) = \frac{\ln x}{x}$, $I = [1, 2]$

$$\int_1^2 \frac{\ln x}{x}\,dx = \int_0^{\ln 2} u\,du = \frac{u^2}{2}\Big|_0^{\ln 2} = \boxed{\frac{(\ln 2)^2}{2}}$$

3. Let R_δ be the region bounded by the curves $x = \delta$, $x = 1$, $y = 0$, and $y = x^{-1/2}$, where $0 < \delta < 1$. Let S_δ be the solid obtained by rotating R_δ about the x-axis. Let Area(R_δ) denote the area of R_δ and let and Vol(S_δ) denote the volume of S_δ.

(a) Determine Area(R_δ) and Vol(S_δ).



$$\text{Area}(R_\delta) = \int_\delta^1 x^{-\frac{1}{2}}\,dx$$

$$= 2x^{\frac{1}{2}}\Big|_\delta^1 = \boxed{2(1-\sqrt{\delta})}$$

Using annuli (or in this case disks),

$$\text{Vol}(S_\delta) = \int_\delta^1 \pi x^{-1}\,dx$$

$$= \pi \ln(x)\Big|_\delta^1$$

$$= \boxed{-\pi\ln\delta}$$

(b) Determine $\lim_{\delta \to 0^+} \text{Area}(R_\delta)$ and $\lim_{\delta \to 0^+} \text{Vol}(S_\delta)$.

$$\lim_{\delta \to 0^+} \text{Area}(R_\delta) = \lim_{\delta \to 0^+} 2(1-\sqrt{\delta}) = \boxed{2}$$

$$\lim_{\delta \to 0^+} \text{Vol}(S_\delta) = \lim_{\delta \to 0^+} -\pi \ln \delta = \boxed{\infty}$$

(c) If we now allow δ to be zero, what can you say about the area of R_0 and the volume of S_0?

R_0 has area 2 and S_0 has infinite volume.

It might be surprising that rotating a region with finite area can sometimes produce a solid with infinite volume!