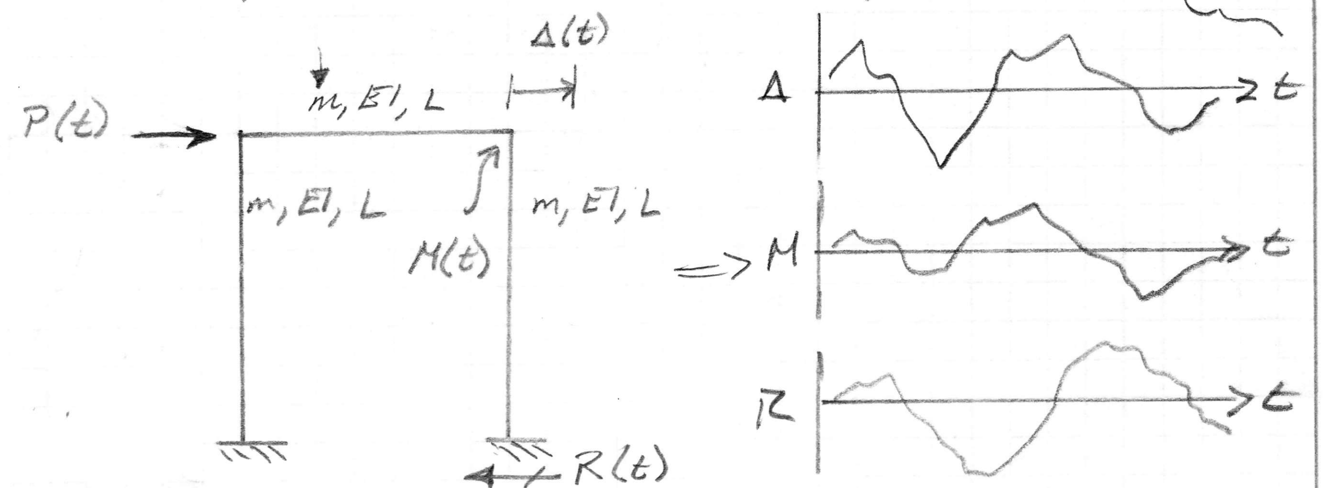
LECTURE 2: Dynamic Modeling of Structures.

→ <u>Estimation of Response</u>

- GIVEN: Structural geometry, properties. Load time history (TH)
- Compute: Response time history (TH).



If $P(t) = A\sin(\Omega t)$

↑ analytical function ⇒ continuous response TH

$P(t) = [10, -5, 6, 8, 0, -6, -10, \ldots]^T$

↑ numerical ⇒ discrete values of P @ regular intervals of t ↳ "time step".

↓ Numerical Response TH

- Two ways to estimate structural responses.

(1) Experimental

→ Testing the actual structure (Field testing)

→ Testing a prototype of model (Lab testing)

→ Very reliable, but expensive...

→ Analytical/Numerical

→ requires the construction of a mathematical model which is representative of the structure.

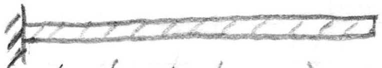
→ relatively inexpensive, but high uncertainty.

Note using both techniques together is highly recommended

→ Mathematical Modeling of Structures

Distributed Parameter (continuous)
- more realistic
- difficult to analyze

Discrete Parameters
- Idealization
- Approx.
- Ease to analyze



(actual beam).

loading → $p(x,t)$

$EI(x)$ ← flex. stiffness

$m(x)$ ← mass

⇓

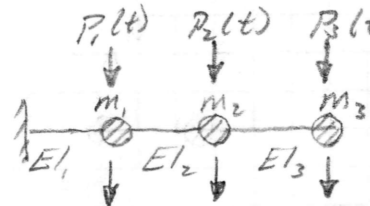
$u(x,t)$ ← *continuous function* of u in x & t. ⇒

u ← disp.

Euler-Bernoulli beam expression

$$\underbrace{m(x)\frac{\partial^2 u}{\partial t^2}}_{\text{mass * accel.}} + \underbrace{\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)}_{\text{stiffness * disp.}} = \underbrace{p(x,t)}_{\text{applied force.}}$$

⇒ Partial differential equation



$\underbrace{u_1(t) \quad u_2(t) \quad u_3(t)}$ } 3 DOF ⇒ $\underline{n=3}$

↳ degrees of freedom (DOF)

→ locate mass @ DOFs
→ apply loads @ DOFs
→ track disp. @ DOFs (compute)

$$u_i(t) : \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}$$ DOFs consider the spatial variation.

↑ only a function of time.

$$[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{P(t)\}$$

$[M]$ ← mass matrix $(n \times n)$

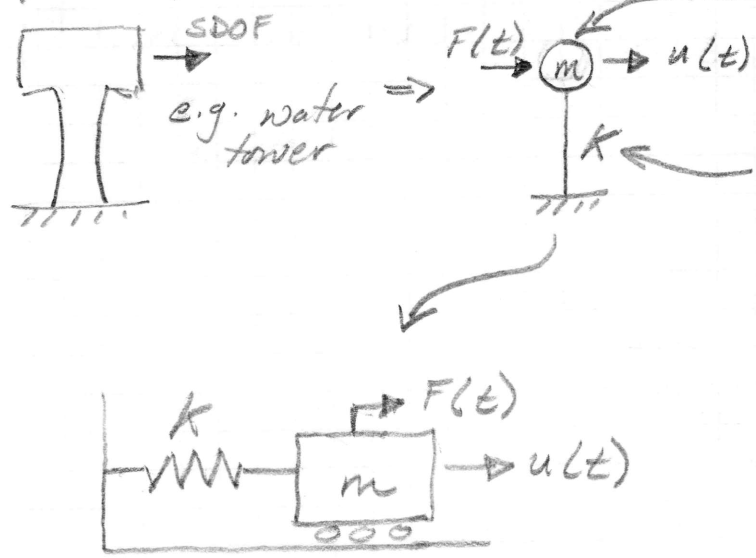
$[K]$ ← stiffness matrix $(n \times n)$

⇒ ordinary differential equation

⇒ For most civil structures, discrete models provide sufficient accuracy and thus are the most commonly used modeling approach.

Discrete Parameter Models

(1) Single degree of freedom (SDOF) models.



Mass lumped @ a single location

Stiffness lumped into a single parameter ($k = \frac{3EI}{L^3}$)

force req'd to cause a unit displacement (@ dof)

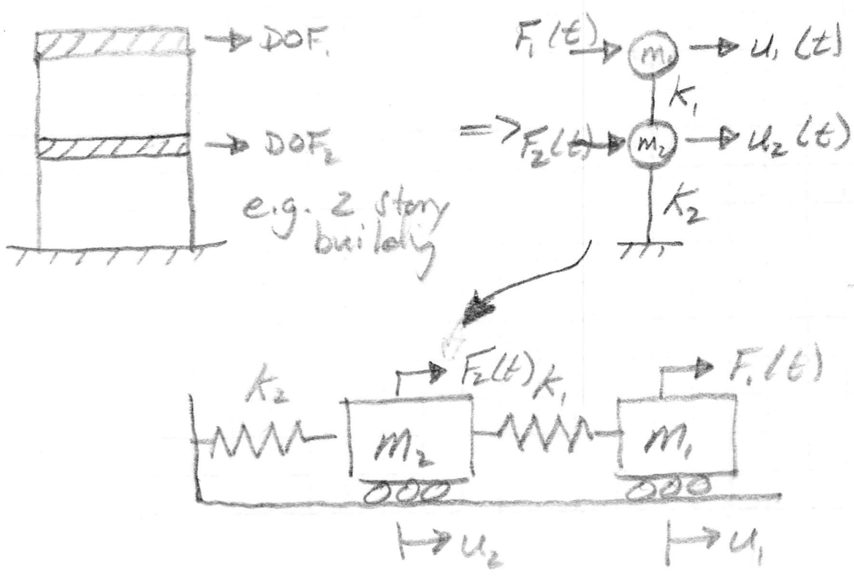
⇒ SDOF systems can be described by a single ODE

Most of CIVE 801 will focus here.

denotes 2nd der. w.r.t. time = accel.

$$m\ddot{u}(t) + k\,u(t) = F(t)$$

(2) Multiple degree of freedom (MDOF) models.



→ mass lumped @ multiple locations

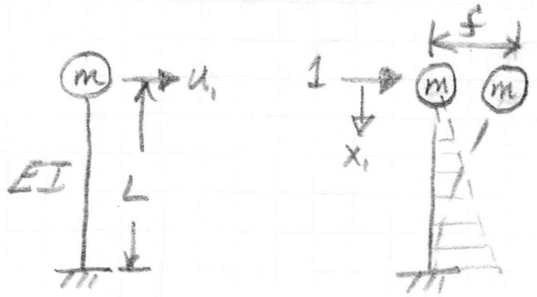
→ stiffness lumped in multiple parameters.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

MDOF systems are described by $\underbrace{\text{'n' ODEs}}_{\text{system of ODEs}}$ where n = # of DOF.

→ <u>Functional Elements of SDOF Systems</u>

(1) Spring Force (stiffness)



f = flexibility = disp. caused by a unit load.

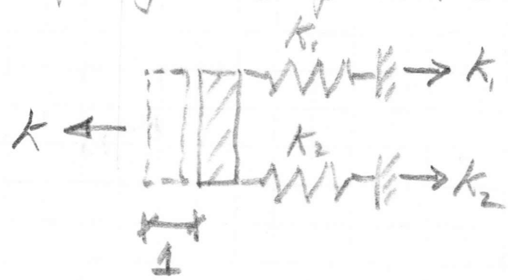
$k = \frac{1}{f}$

$m(x_1) = x_1$

$$f = \int_0^L x_1 \left(\frac{x_1}{EI}\right) dx_1 = \frac{x_1^3}{3EI}\Big|_0^L = \frac{L^3}{3EI} = \boxed{k = \frac{3EI}{L^3}} \; *$$

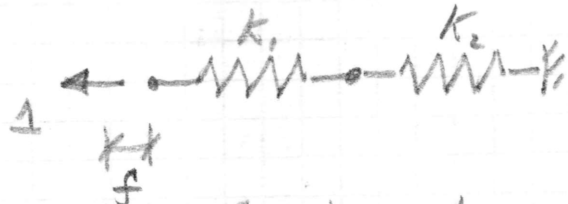
Stiffness of a cantilever beam.

→ Springs in parallel.



$\underline{\underline{k = k_1 + k_2}}$ ← stiffnesses are additive.

→ Springs in Series.

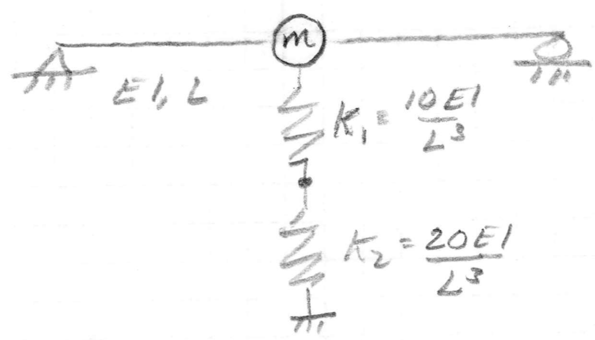


$$f = \frac{1}{k_1} + \frac{1}{k_2}$$ ← flexibilities are additive.

disp of spring 1 ↑ $\frac{1}{k_1}$; ↑ $\frac{1}{k_2}$ disp of spring 2.

$$\frac{1}{f} = k = \underline{\underline{\frac{k_1 k_2}{k_1 + k_2}}}$$

→ Example: Determine 'k'



$$f = 2\int_0^{L/2} \frac{1}{2}x\left(\frac{x}{2EI}\right)dx = \frac{x^3}{6EI}\Big|_0^{L/2} = \frac{L^3}{48EI}$$

$$k_b = \frac{48EI}{L^3}$$

→ Combine springs in series

$$k_e = \frac{k_1 k_2}{k_1 + k_2} = \frac{\frac{200(EI)^2}{L^6}}{\frac{30EI}{L^3}} = \underline{\underline{\frac{6.67EI}{L^3}}}$$

→ Combine springs w/ beam

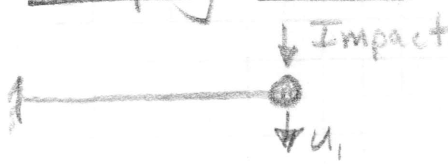
→ springs in parallel

$$k = k_b + k_e = \frac{48EI}{L^3} + \frac{6.67EI}{L^3}$$

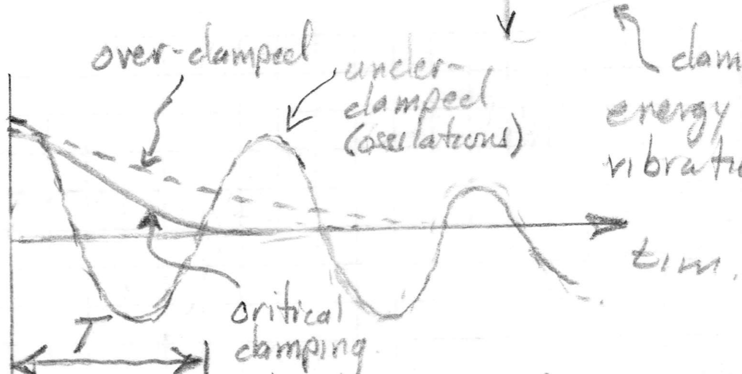
$$k = \frac{54.67EI}{L^3}$$

$\boxed{f_s = ku}$ (Spring force or "restoring" force)

(2) <u>Damping Forces</u> (Energy dissipation)



damping dissipates energy and reduces vibration w/ time.



→ minimum damping for which no oscillation occurs.

→ Most civil engineering structures are highly under-damped ~ 2-5% of critical damping

⇒ lots of oscillations

→ Exact nature of damping mechanism is not possible to determine

- Thermal
- Material
- Friction
- wind resistance
- Non-structural components.

→ Since it is difficult to assess, typically viscous damping is assumed (for convienience)

velocity dependent

$$\boxed{f_D = c\dot{u}}$$

dashpot denotes daming

experimentally determined.

(3) Mass Force (Inertia force)

Newton's 2nd Law

→ Rate of change of momentum of a mass (m) is equal to the force acting on it.

$$p(t) = \frac{\partial}{\partial t}\Big(\underbrace{m\frac{\partial u}{\partial t}}_{\text{momentum} = mv}\Big)$$

→ for constant mass (e.g. not a rocket)

$$p(t) = m\frac{\partial^2 u}{\partial t^2}$$

or

$$F = ma = m\ddot{u}$$

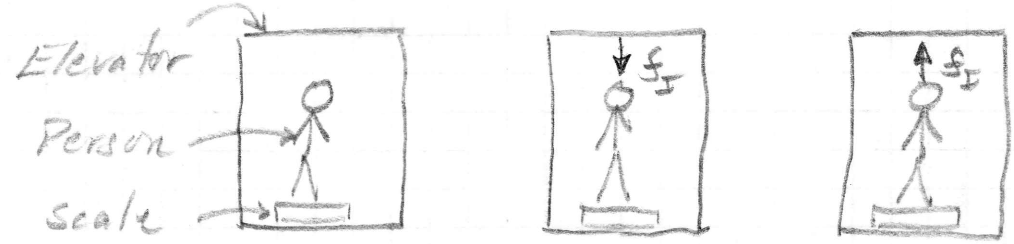
→ d'Alembert's Principle

→ Mass develops an inertial force proportional to its acceleration in an opposing direction.

$$f_I = m\ddot{u}$$

$$0 = m\ddot{u} - f_I$$

$$f_I \longleftarrow \boxed{m} \longrightarrow \ddot{u}$$



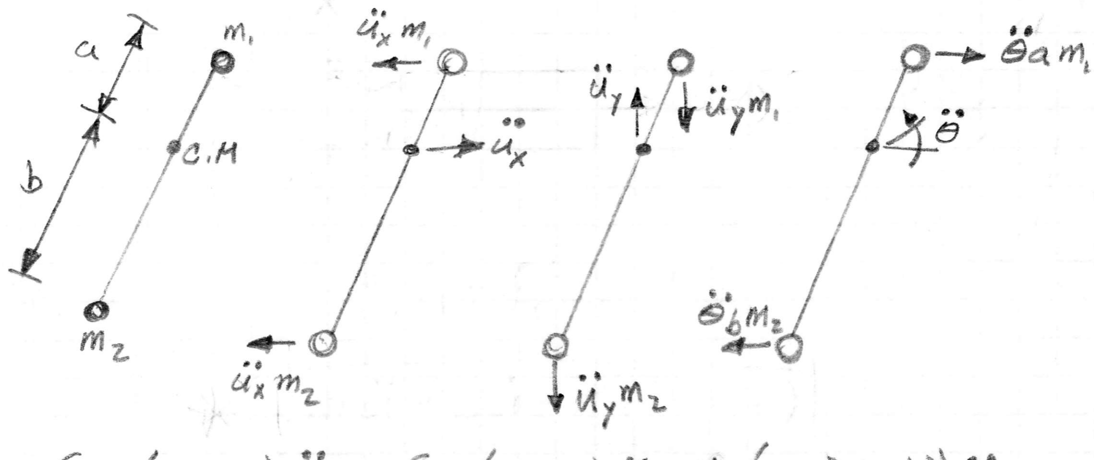
$\ddot{u} = 0$ $\quad$ $\ddot{u} = 0.5g \uparrow$ $\quad$ $\ddot{u} = -0.5g \downarrow$

$W_s = W_p$ $\quad$ $W_s = W_p + 0.5m_p$ $\quad$ $W_s = W_p - 0.5m_p$

→ Translational acceleration → translational inertia force

→ Rotational accel → rotational inertia force.

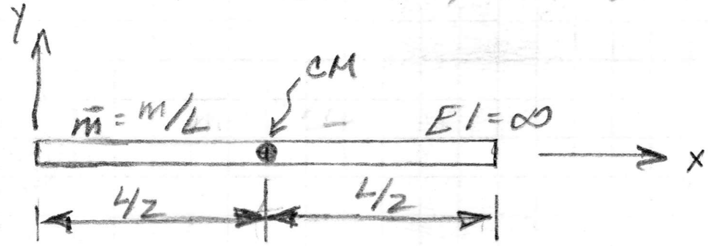
⇒ Massless rigid bar w/ 2 pt masses.



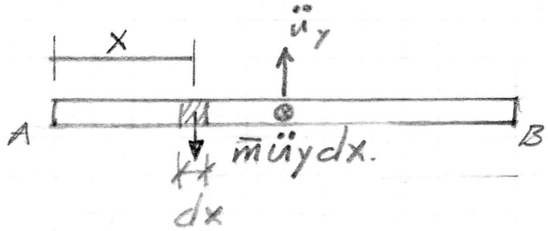
$f_{I_x} = (m_1 + m_2)\ddot{u}_x$ $\quad$ $f_{I_y} = (m_1 + m_2)\ddot{u}_y$ $\quad$ $M_{I_z} = (m_1 a^2 + m b^2)\ddot{\theta}$

$\Sigma F_x = 0$ $\quad$ $\Sigma F_y = 0$ $\quad$ $\Sigma M_{cM} = 0$

→ What happens if the mass is not concentrated, but distributed?



→ Y accel.



$\sum F_y = 0$

$$f_{I_y} = \int_0^L \bar{m}\ddot{u}_y\,dx$$

$$f_{I_y} = \bar{m}L\ddot{u}_y$$

$\sum M_A = 0$

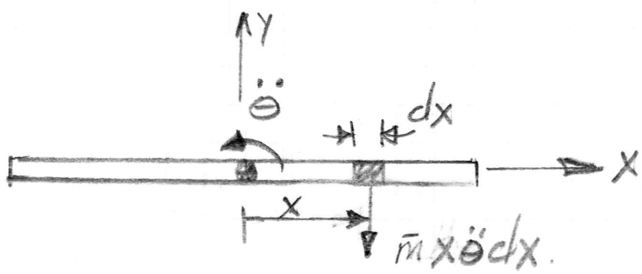
$$f_I\bar{x} = \int_0^L \bar{m}\ddot{u}_y x\,dx$$

$$f_I\bar{x} = \frac{\bar{m}\ddot{u}_y L^2}{2}$$

resultant force acts @ the C.M. → $\bar{x} = \frac{L}{2}$

If you sum forces @ the C.M. ⇒ translational accel doesn't cause rotational inertia.

→ Rotational accel.



$\sum F_y = 0$

$$f_I = \int_{-L/2}^{L/2} \bar{m}x\ddot{\theta}\,dx = 0$$

↑ Rotational accel applied @ C.M. cause no translational accell

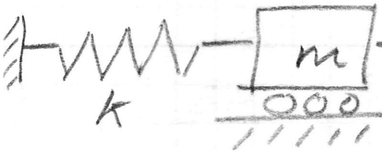
$\sum M_{CM} = 0$

$$M_I = \int_{-L/2}^{L/2} \bar{m}x^2\ddot{\theta}\,dx = \left.\frac{\bar{m}x^3\ddot{\theta}}{3}\right|_{-L/2}^{L/2}$$

$$M_I = \frac{\bar{m}L^3\ddot{\theta}}{12}$$

→ Estimation of Natural Frequency for SDOF System

Mathematical model of a SDOF (no damping)



SDOF are always an idealization

→ As shown using conservation of energy

$$\omega = \sqrt{\frac{k}{m}}$$

circular frequency (RAD/s)

→ 2 important questions.

(1) What degree of freedom do you select?

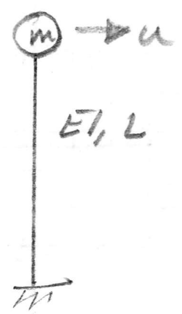
→ we'll come back to this.

(2) What is k & m?

→ Problems w/ a single point mass.

→ Basic Problems.

(1)

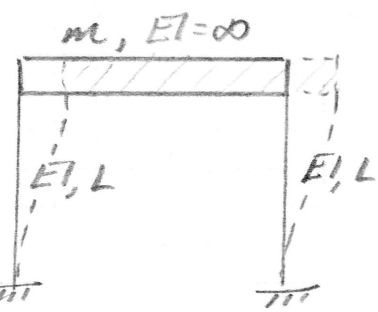


$K = \frac{3EI}{L^3}$

$m = m$

$\omega = \sqrt{\frac{3EI}{mL}}$

(2)



$K = 2\left(\frac{12EI}{L^3}\right)$

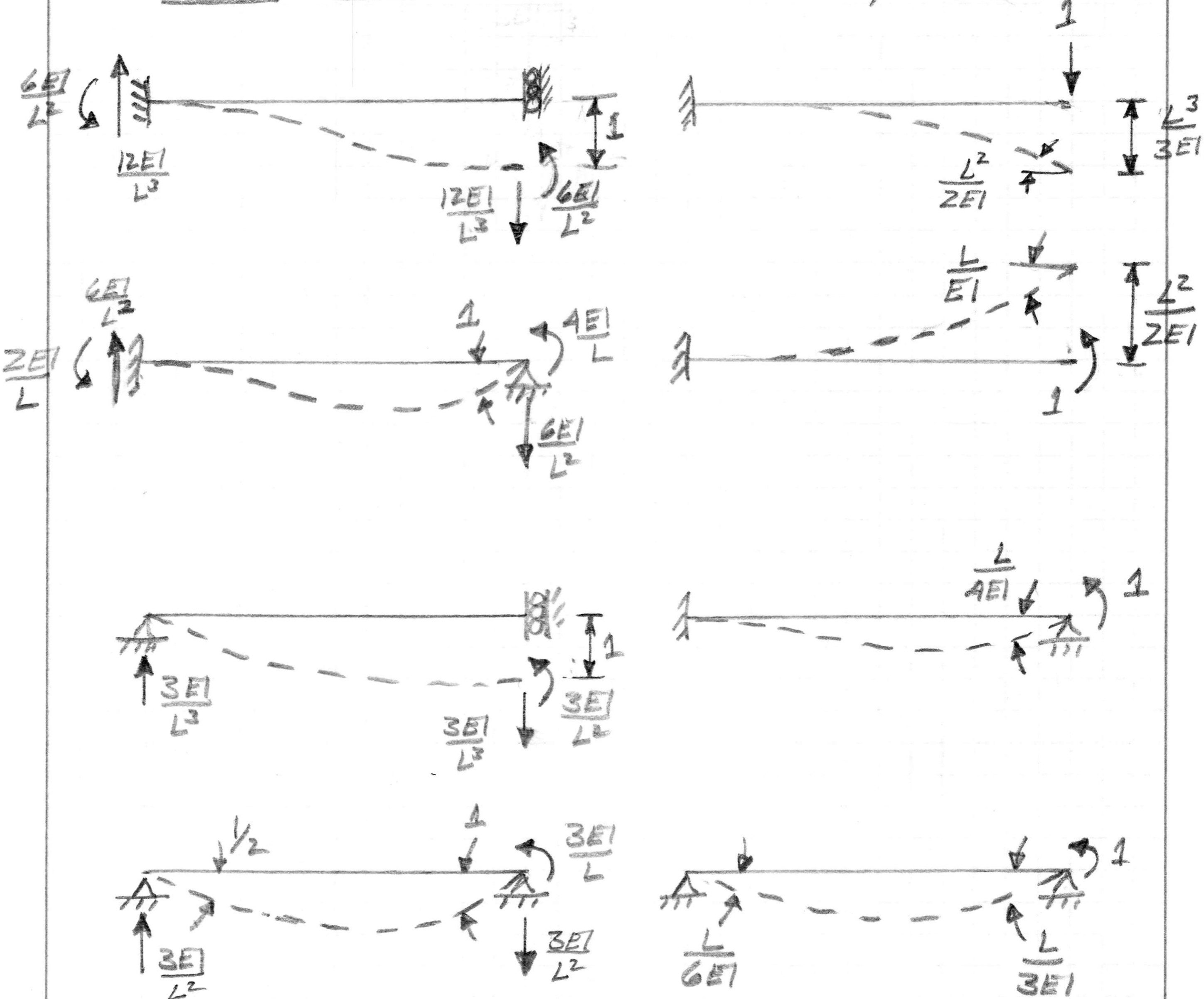
$m = m$

$\omega = \sqrt{\frac{24EI}{mL^3}}$

→ Stiffness & Flexibility Coefficients.

- Common Cases.

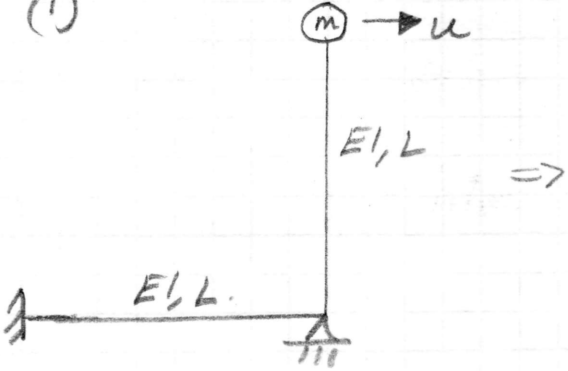
<u>Stiffness</u> <u>Coefficients</u>   <u>Flexibility</u> <u>Coefficient</u>

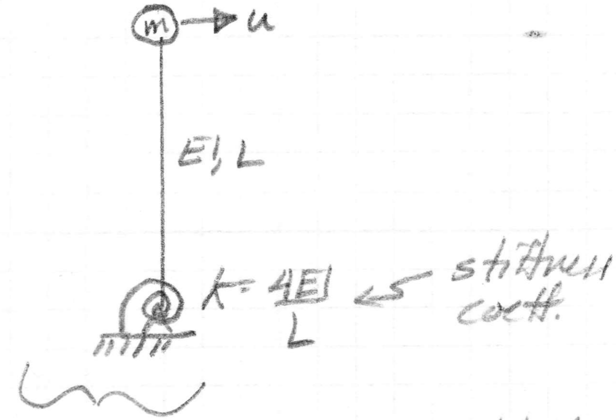


⇒ All of these coefficients are based on the assumption of no axial or shear deformation.

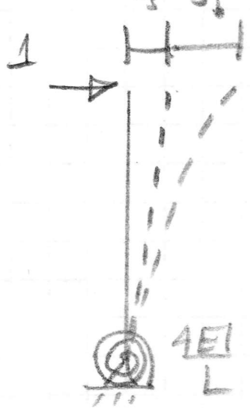
→ Problems using springs in parallel & series.

(1)





Springs in series => add flex.



$$f_s = \underbrace{(L)}_{\text{moment in spring}}\overbrace{\left(\frac{L}{4EI}\right)}^{\text{rotation of spring}}(L) = \frac{L^3}{4EI}$$

$$f_b = \frac{L^3}{3EI}$$

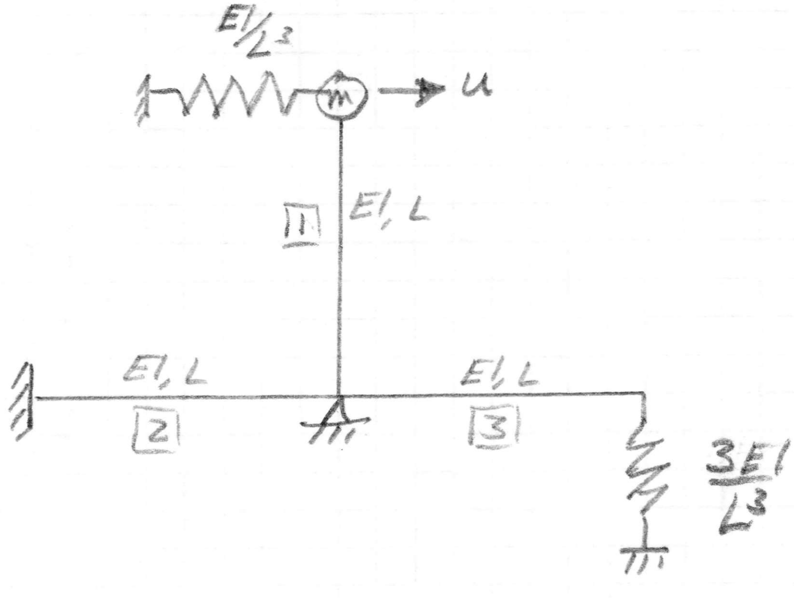
$$f = \frac{3L^3}{12EI} + \frac{4L^3}{12EI}$$

$$f = \frac{7L^3}{12EI}$$

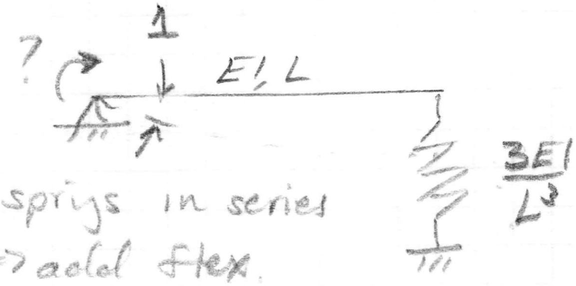
$$K = \frac{12EI}{7L^3}$$

$$\omega = \sqrt{\frac{12EI}{7mL^3}}$$

(2)



→ Start by reducing member 3 & spring to a rotational spring.

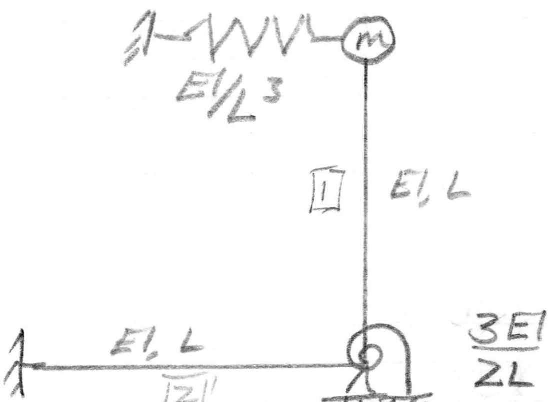


→ springs in series
⇒ add flex.

$$f_s = \frac{1}{L}\left(\frac{L^3}{3EI}\right)\left(\frac{1}{L}\right) \stackrel{!}{=} \frac{L}{3EI}$$

$$f_b = \frac{L}{3EI}$$

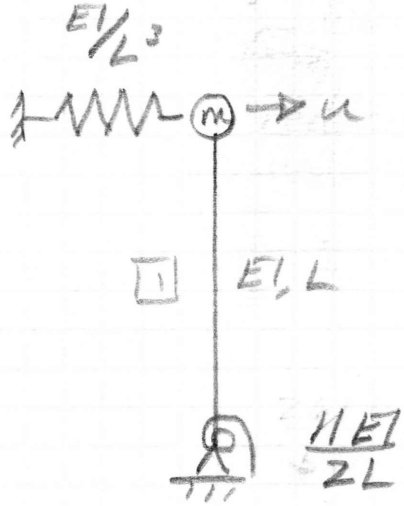
$$f = \frac{2L}{3EI}$$



→ Combine member 2 & rot sp.

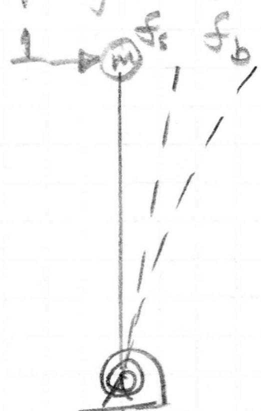
→ springs in parallel
⇒ add stiffness.

$$\frac{4EI}{L} + \frac{3EI}{2L} = \frac{11EI}{2L}$$



→ Combine member 1 & rot. spring

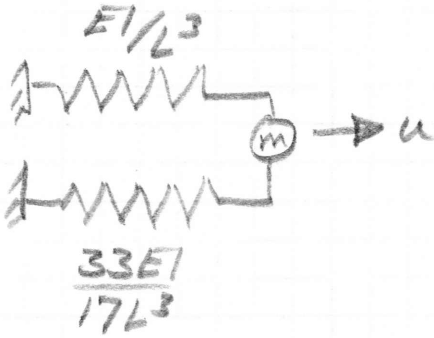
→ spring in series ⇒ add flex.



$$f_s = L\left(\frac{2L}{11EI}\right)L = \frac{2L^3}{11EI}$$

$$f_b = \frac{L^3}{3EI}$$

$$f = \frac{17L^3}{33EI} \Rightarrow K = \frac{33EI}{17L^3}$$



→ Combine remaining 2 springs

→ Springs in parallel

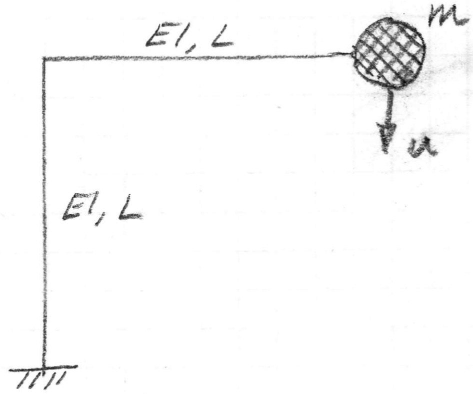
⇒ add stiffness.

$$K = \frac{17EI}{17L^3} + \frac{33EI}{17L^3} = \frac{50EI}{17L^3}$$

$$\omega = \sqrt{\frac{50EI}{17mL^3}}$$

→ Sample HW Problems.

(1)



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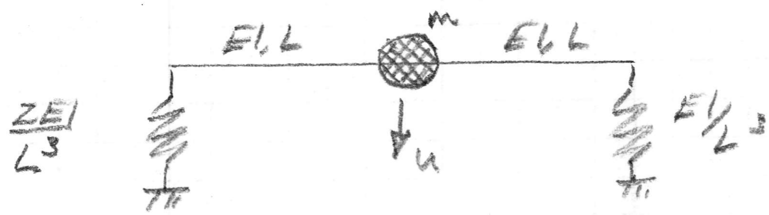

(3)