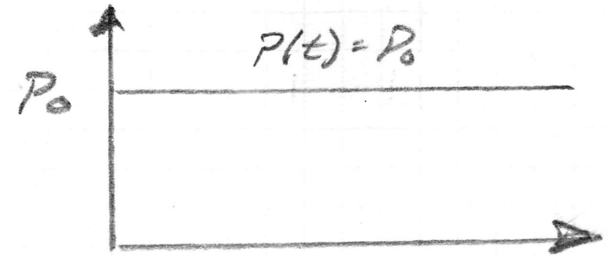
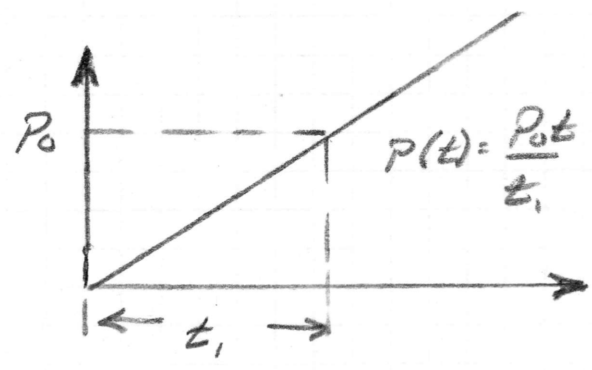
LECTURE 8 - Combinations of Ramp & Step Loadings (cont'd)
Numerical Techniques

→ Solutions.



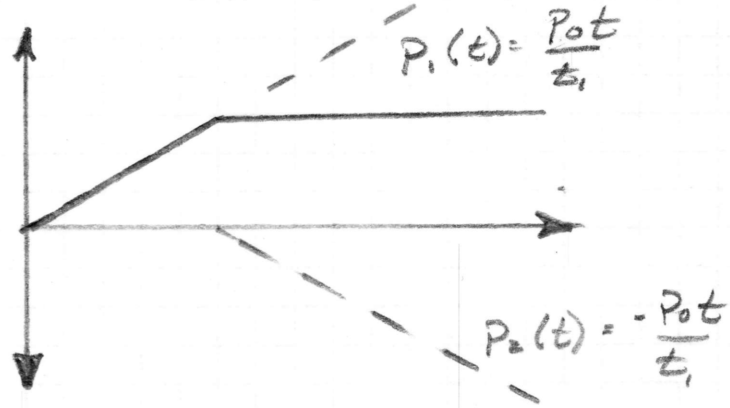
$$u = \frac{P_0}{K}(1-\cos\omega t)$$

$$u = \frac{P_0}{K}\left(1-e^{-\xi\omega t}(\cos\omega t + \xi\sin\omega t)\right)$$



$$u = \frac{P_0}{K}\left(\frac{t}{t_1} - \frac{\sin\omega t}{\omega t_1}\right)$$

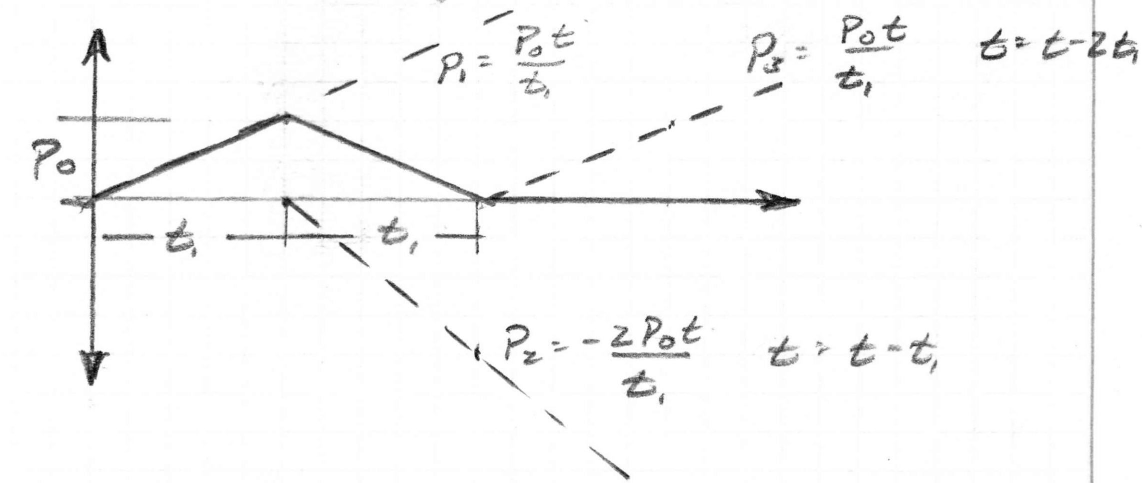
Example → Step Loading w/ Finite Rise Time (t_1)



Solution:

$$u(t) = \begin{cases} u(t) = \dfrac{P_0}{K}\left(\dfrac{t}{t_1} - \dfrac{\sin\omega t}{\omega t_1}\right) & t < t_1 \\[2ex] u(t) = \dfrac{P_0}{K}\left(\dfrac{t}{t_1} - \dfrac{\sin\omega t}{\omega t_1}\right) - \dfrac{P_0}{K}\left(\dfrac{t-t_1}{t_1} - \dfrac{\sin\omega(t-t_1)}{\omega(t_1)}\right) & t \geq t_1 \end{cases}$$

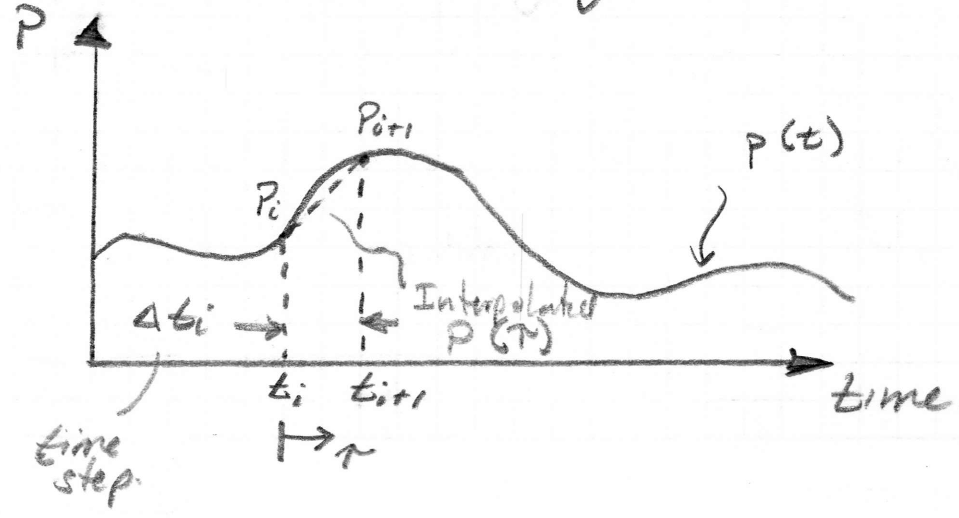
Example → Triangular Pulse



Solution:

$$u(t) = \begin{cases} u(t) = \frac{P_0}{k}\left(\frac{t}{t_1} - \frac{\sin \omega t}{\omega t_1}\right) & t < t_1 \\ u(t) = \frac{P_0}{k}\left(\frac{t}{t_1} - \frac{\sin \omega t}{\omega t_1}\right) - \frac{2P_0}{k}\left(\frac{t-t_1}{t_1} - \frac{\sin \omega (t-t_1)}{\omega t_1}\right) & t_1 \le t < 2t_1 \\ u(t) = \frac{P_0}{k}\left(\frac{t}{t_1} - \frac{\sin \omega t}{\omega t_1}\right) - \frac{2P_0}{k}\left(\frac{t-t_1}{t_1} - \frac{\sin \omega (t-t_1)}{\omega t_1}\right) + \frac{P_0}{k}\left(\frac{t-2t_1}{t_1} - \frac{\sin \omega (t-2t_1)}{\omega t_1}\right) & t \ge 2t_1 \end{cases}$$

→ General Loading.



→ Essentially breaks a continuous forcing function into a vector of forces at each time step.

→ Numerically defines the forcing function.

→ Between t_i & t_{i+1} there are four excitations that define the response.

(1) Initial displacement u_i

(2) Initial velocity $\dot{u}_i$

(3) A step function $p(\tau) = p_i$

(4) A ramp function $p(\tau) = \frac{P_{i+1} - P_i}{\Delta t_i}\tau$
$= \frac{\Delta P_i}{\Delta t_i}\tau$

→ Solution approach

(1) Set up an expression for u_{i+1} in terms of u_i & $\dot{u}_i$

(2) Set up an expression for $\dot{u}_{i+1}$ in terms of $\dot{u}_i$ & u_i

(3) Identify initial conditions $(t=0)$ to start the solution.

(4) "Time step" by solving to new disp & vel @ t_{i+1}, then use these to solve for disp & vel @ t_{i+2}, etc., etc.

$\Rightarrow$ **Using superposition.** $(\zeta = 0)$

$$u(\tau) = \underbrace{u_i \cos \omega \tau}_{\text{due } u_0} + \underbrace{\frac{\dot{u}_i}{\omega} \sin \omega \tau}_{\text{due to } \dot{u}_0} + \underbrace{\frac{P_i}{k}(1 - \cos \omega \tau)}_{\text{due to step.}}$$

$$+ \underbrace{\frac{\Delta P_i}{k}\left(\frac{\tau}{\Delta t_i} - \frac{\sin \omega \tau}{\omega \Delta t_i}\right)}_{\text{due to ramp.}}$$

$\Rightarrow$ for $\tau = \Delta t_i \Rightarrow u(\tau) = u_{i+1}$

$$* \quad u_{i+1} = u_i \overbrace{\cos(\omega \Delta t_i)}^{\text{constant for all } u_i} + \frac{\dot{u}_i}{\omega} \underbrace{\sin(\omega \Delta t_i)}_{\text{constant for all } \dot{u}_i}$$

$$+ \frac{P_i}{k}\left(1 - \cos(\omega \Delta t_i)\right) + \frac{\Delta P_i}{k} \frac{1}{\omega \Delta t_i}\left(\omega \Delta t_i - \sin(\omega \Delta t_i)\right)$$

$\rightarrow$ **Similarly**

$$* \quad \dot{u}_{i+1} = -u_i \omega \sin(\omega \Delta t_i) + \dot{u}_i \cos(\omega \Delta t_i) + \frac{P_i \omega}{k} \sin(\omega \Delta t_i)$$

$$+ \frac{\Delta P}{k \Delta t_i}\left(1 - \cos(\omega \Delta t_i)\right)$$

$\Rightarrow$ These expressions can be rewritten in terms of constants and u_i, $\dot{u}_i$, P_i & P_{i+1}

$$u_{i+1} = A u_i + B \dot{u}_i + C p_i + D p_{i+1}$$

$$\dot{u}_{i+1} = A' u_i + B' \dot{u}_i + C' p_i + D' p_{i+1}$$

$\Rightarrow$ See pp. 402 to A-D & A'-D' expression w/ damping.

→ Pros of P-W Method

→ Exact except for forcing function approx.

↑ No way around this for EQ → digital record.

→ No problems w/ time step length.

→ Extremely efficient → only need to compute A-D' once.

→ Cons of P-W Method

→ Only applicable to linear structures.

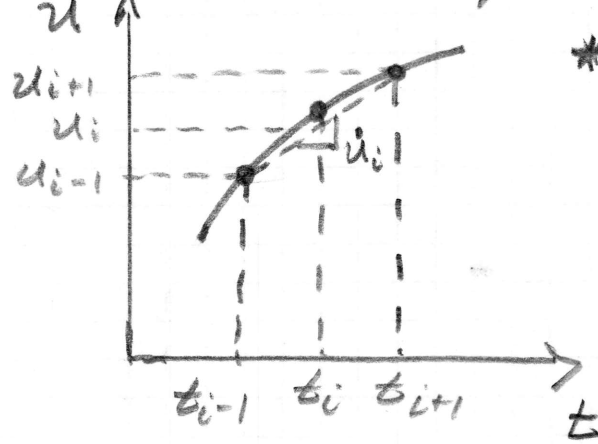
↳ based on superposition

↳ sum of responses.

→ More General Numerical Integration Approaches.

→ Central Difference Method

→ Basic Assumptions.



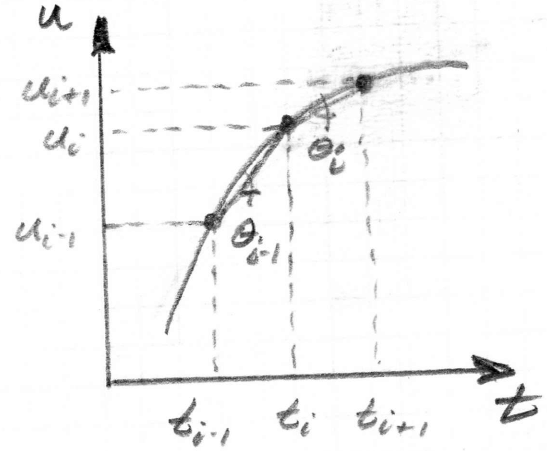
→| Δt |←

└ time step.

$$* \quad \dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \quad (1)$$

↑ slope between u_{i+1} & u_{i-1}

$$\approx \left.\frac{du}{dt}\right|_i$$



$$\ddot{u}_i = \frac{\theta_i - \theta_{i-1}}{\Delta t}$$

→ "curvature" of u is the change in slope divided by length.

→ $\approx \left.\dfrac{d^2u}{dt^2}\right|_i$

$$\ddot{u}_i = \frac{\dfrac{u_{i+1}-u_i}{\Delta t} - \dfrac{u_i - u_{i-1}}{\Delta t}}{\Delta t}$$

$$* \quad \ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} \qquad (2)$$

⇒ We now have expressions for $\dot{u}$ & $\ddot{u}$ in terms of u

→ Sub into EOM.

$$m\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}\right) + c\left(\frac{u_{i+1} - u_{i-1}}{2\Delta t}\right) + ku_i = p_i$$

→ Solve for u_{i+1} (only unknown).

$$* \quad \left\{ \begin{aligned} u_{i+1} &= \frac{\hat{p}}{\hat{k}} \qquad (3) \\ \hat{p} &= p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right] u_{i-1} - \left[k - \frac{2m}{\Delta t}\right] u_i \\ \hat{k} &= \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t} \end{aligned} \right.$$

$\Rightarrow$ Problem: How to start the time stepping?

To determine u_1 we need $\underset{\substack{\uparrow \\ \text{known}}}{u_0}$ & $\underset{\substack{\uparrow \\ \text{unknown}}}{u_{-1}}$

both unknown

$$\dot{u}_0 = \frac{u_1 - u_{-1}}{2\Delta t} \Rightarrow u_1 = \underbrace{2\Delta t\,\dot{u}_0 + u_{-1}}$$

$$\ddot{u}_0 = \frac{u_1 - 2u_0 - u_{-1}}{(\Delta t)^2}$$

$$* \quad u_{-1} = \underset{\substack{\uparrow \\ \text{given}}}{u_0} - \Delta t(\underset{\substack{\uparrow \\ \text{given}}}{\dot{u}_0}) + \frac{(\Delta t)^2}{2}\underset{\substack{\uparrow \\ \text{unknown}}}{\ddot{u}_0} \qquad (4)$$

Writing EOM @ $i=0$

$$m\ddot{u}_0 + c\dot{u}_0 + ku_0 = p_0$$

$$* \quad \ddot{u}_0 = \frac{p_0 - c\dot{u}_0 - ku_0}{m} \qquad (5)$$

$\rightarrow$ Solution Procedure.

- $\rightarrow$ Use Eqn (4) & (5) + u_0 & $\dot{u}_0$ to solve for the starting point $\Rightarrow u_{-1}$ & $\ddot{u}_0$

- $\rightarrow$ Use Eqn (3) to step from $i=1$ to n $\leftarrow$ total record length.

- $\rightarrow$ Use Eqn (1) & (2) to solve for $\dot{u}_i$ & $\ddot{u}_i$ if desired.

→ Pros of CDM

→ Not dependent on linearity!

↳ superposition was not used.

⇒ $ku \rightarrow \underbrace{f(u, \dot{u})}_{\text{Nonliner function of stiffness}} u$

→ Cons of CDM

→ Approx representation of $p(t)$

→ Approx representation of $\underbrace{u, \dot{u} \,\&\, \ddot{u}}$

→ Conditionally stable

Accuracy depends on size of Δt

↳ the smaller the better

$$\underbrace{\frac{\Delta t}{T_n} < \frac{1}{\pi}}$$

In reality it should be much smaller than this.