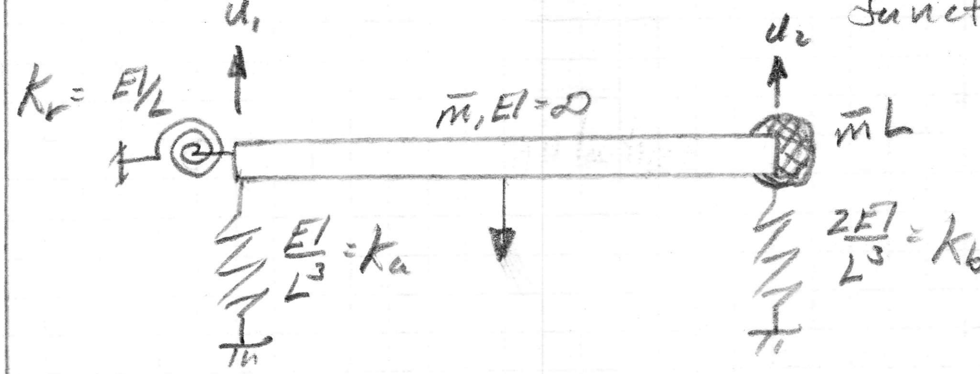
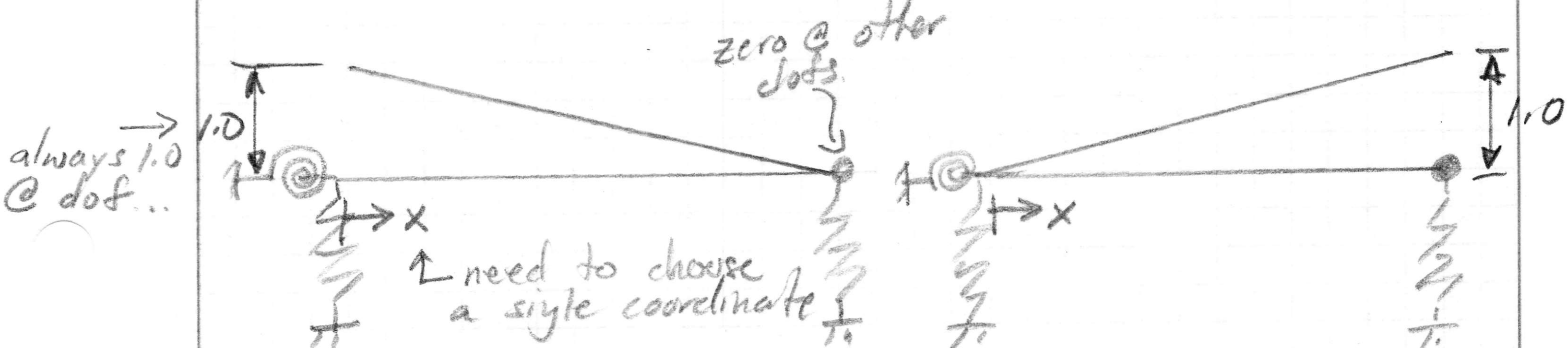
LECTURE 2 - Formulation of EOM for MDOF Sys.

(2) "Virtual Displacement" => uses generalized coordinates & shape functions.



→ Start by identifying shape functions.



$$\psi_1(x) = 1 - \frac{x}{L} \qquad \psi_2(x) = \frac{x}{L}$$

→ Similar to a sdof...

→ For discrete springs.

rotational spring

$$K_{ij} = \sum_{l=1}^{n} \psi_i \psi_j K_l + \sum_{l=1}^{n} \left(\frac{d\psi_i}{dx}\right)\left(\frac{d\psi_j}{dx}\right) K_l^r$$

e.g. shape function ψ_i evaluated @ location of spring K_l

e.g. slope of shape function ψ_i evaluated @ location of spring K_l^r

→ 1st shape function generates force, 2nd shape function pushes the force through a virtual displacement.

	k_a	k_b		k_r
ψ_1	1.0	0	$\frac{d\psi_1}{dx}$	$-1/L$
ψ_2	0	1.1	$\frac{d\psi_2}{dx}$	$1/L$

$i = j = 1$

$$K_{11} = 1.0(1.0)\frac{EI}{L^3} + (0)(0)\frac{2EI}{L^3} + \left(-\frac{1}{L}\right)\left(-\frac{1}{L}\right)\frac{EI}{L}$$

$$K_{11} = \frac{2EI}{L^3}$$

$$K_{12} = 1.0(0)\frac{EI}{L^3} + (0)(1.0)\frac{2EI}{L^3} + \left(-\frac{1}{L}\right)\left(\frac{1}{L}\right)\frac{EI}{L}$$

$$K_{12} = -\frac{EI}{L^3}$$

$$K_{22} = (0)(0)\frac{EI}{L^3} + (1)(1)\frac{2EI}{L^3} + \left(\frac{1}{L}\right)\left(\frac{1}{L}\right)\frac{EI}{L}$$

$$K_{22} = \frac{3EI}{L^3}$$

$$K = \frac{EI}{L^3}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \checkmark$$

→ For discrete masses.

$$m_{ij} = \sum_{l=1}^{n} \underbrace{\psi_i \psi_j}_{} m_L$$

evaluated @ location of mass

→ For distributed mass associated with rigid members

$$m_{ij} = \sum_{l=1}^{n} \underbrace{\psi_i \psi_j}_{\substack{\text{evaluated} \\ \text{@ the center} \\ \text{of mass}}} \bar{m}_l L + \sum_{l=1}^{n} \underbrace{\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}}_{\substack{\text{evaluated} \\ \text{@ the center} \\ \text{of mass.}}} I_L$$

	discrete mass	cm		cm
ψ_1	0	$\frac{1}{2}$	$\frac{d\psi_1}{dx}$	$-\frac{1}{L}$
ψ_2	1.0	$\frac{1}{2}$	$\frac{d\psi_2}{dx}$	$\frac{1}{L}$

$$m_{11} = (0)(0)\,\bar{m}L + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\bar{m}L + \left(-\frac{1}{L}\right)\left(-\frac{1}{L}\right)\frac{\bar{m}L^3}{12}$$

$$m_{11} = \frac{\bar{m}L}{3}$$

$$m_{12} = (0)\left(\tfrac{1}{2}\right)\bar{m}L + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\bar{m}L + \left(-\frac{1}{L}\right)\left(\frac{1}{L}\right)\frac{\bar{m}L^3}{12}$$

$$m_{12} = \frac{\bar{m}L}{6}$$

$$m_{22} = (1)(1)\,\bar{m}L + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\bar{m}L + \left(\frac{1}{L}\right)\left(\frac{1}{L}\right)\frac{\bar{m}L^3}{12}$$

$$m_{22} = \frac{4\bar{m}L}{3}$$

$$M = \bar{m}L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{3} \end{bmatrix}$$ ✓

→ For discrete loads.

$$P_i = \sum_{l=1}^{n} \psi_i P_l$$

↑ @ location of load.

	P_1
ψ_1	1/2
ψ_2	1/2

$$P_1 = \tfrac{1}{2} \cdot P$$

$$P_2 = \tfrac{1}{2} \cdot P$$

✓

$$\bar{m}L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{2} P \\ -\frac{1}{2} P \end{Bmatrix}$$

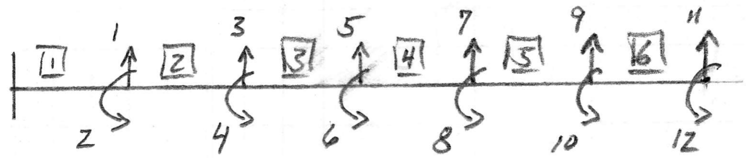
Homework #2

Using the method of virtual displacements, solve for the [k] & [M] of the structure using dofs θ_1 & u_2, and u_1 & θ_1

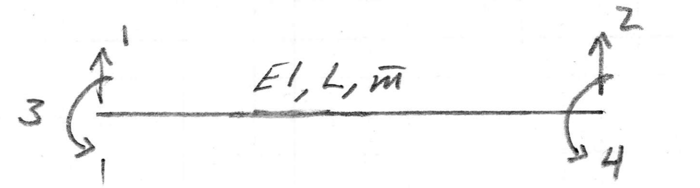
→ Compare w/ the matricies obtained using the direct approach (e.g. Method 1)

→ Derivation of Element Stiffness & Mass Matricies using Virtual Displacements.

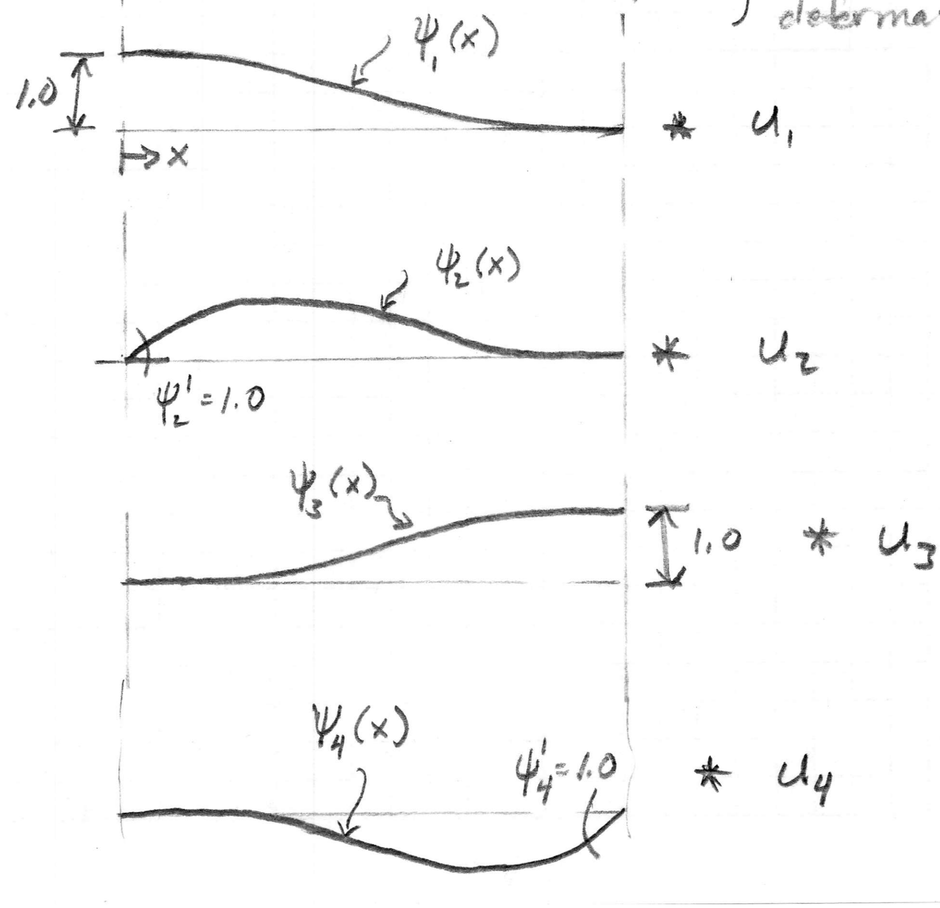
→ Intro to Finite Elements



→ Pull out element $\boxed{2}$



four dofs, ∴ we need four shape functions to describe deformation.



$\therefore \; u(x,t) = u_1(t)\,\psi_1(x) + u_2(t)\,\psi_2(x) + u_3(t)\,\psi_3(x)$
$\qquad + u_4(t)\,\psi_4(x)$

total displacement is sum of the same functions times their generalized coord.

→ How to determine ψ_i's ???

$$EI\,\frac{d^4u}{dx^4} = 0$$ } beam bending expression for loads only @ beam ends.

↳ $\underbrace{u = Ax^3 + Bx^2 + Cx + D}_{\text{most general form} \Rightarrow \text{cubic...}}$

→ Solve for $\psi_1(x)$

$\psi_1(0) = 1.0$
$\psi_1(L) = 0$
$\psi_1'(0) = 0$
$\psi_1'(L) = 0$

$\psi_1(0) = A(0)^3 + B(0)^2 + C(0) + D = 1.0$

$D = 1.0$

$\psi_1'(0) = 3A(0)^2 + 2B(0) + C = 0$

$C = 0$

$\psi'(L) = 3AL^2 + 2BL = 0$

$$A = -\frac{2B}{3L}$$

$\psi(L) = AL^3 + BL^2 + 1 = 0$

$$-\frac{2BL^2}{3} + BL^2 = -1$$

$$B = \frac{-3}{L^2} \Rightarrow A = \frac{2}{L^3}$$

$$\psi_1(x) = \frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1$$

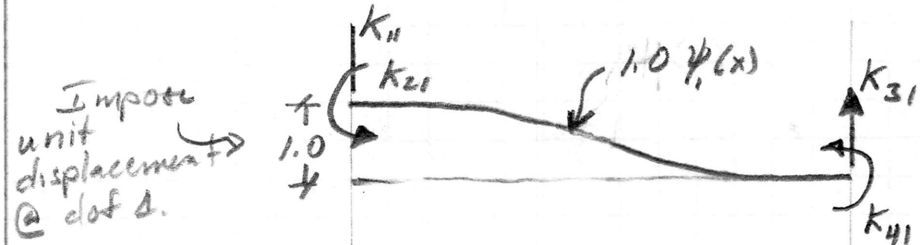
→ Similarly,

$$\psi_2(x) = \frac{x^3}{L^2} - \frac{2x^2}{L} + x$$

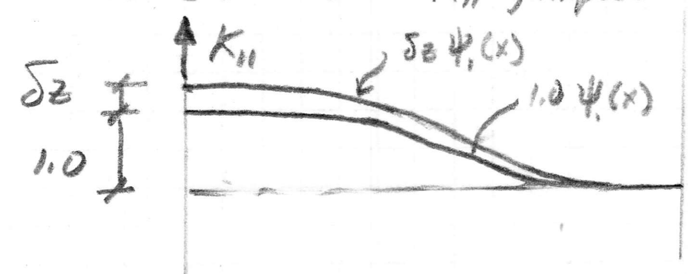
$$\psi_3(x) = -\frac{2x^3}{L^3} + \frac{3x^2}{L^2}$$

$$\psi_4(x) = \frac{x^3}{L^2} - \frac{x^2}{L}$$

→ Solve for 1st Column of Element Stiffness Matrix



To solve for K_{11}, impose a virtual $\delta z\, \psi_1(x)$



$$\cancel{\delta z}\, K_{11} = \int_0^L \underbrace{\left(EI \frac{d^2\psi_1}{dx^2}\right)}_{\text{real moment}} \underbrace{\left(\cancel{\delta z} \frac{d^2\psi_1}{dx^2}\right)}_{\text{virtual curvature.}} dx$$

$$\frac{d^2\psi_1}{dx^2} = \frac{12x}{L^3} - \frac{6}{L^2}$$

$$K_{11} = EI \int_0^L \left(\frac{12x}{L^3} - \frac{6}{L^2}\right)\left(\frac{12x}{L^3} - \frac{6}{L^2}\right) dx$$

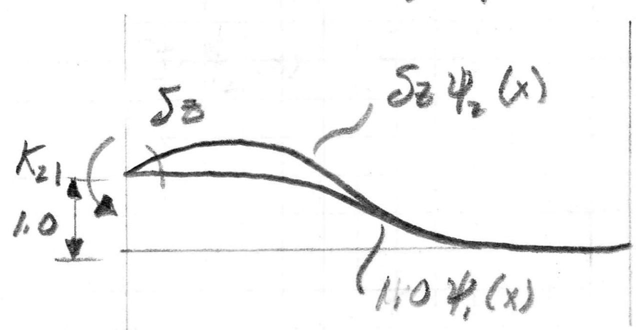
$$K_{11} = EI \int_0^L \frac{144x^2}{L^6} - \frac{144x}{L^5} + \frac{36}{L^4}\, dx$$

$$K_{11} = EI \left[\frac{144L^3}{3L^6} - \frac{144L^2}{2L^5} + \frac{36L}{L^4} \right]$$

$$K_{11} = EI \left[\frac{48}{L^3} - \frac{72}{L^3} + \frac{36}{L^3} \right]$$

$$K_{11} = \frac{12EI}{L^3}$$

To solve for K_{21}, impose a virtual $\delta z\, \psi_2(x)$



$$\cancel{\delta z}\, K_{21} = \int_0^L \underbrace{\left(EI \frac{d^2\psi_1}{dx^2}\right)}_{\text{real moment}} \underbrace{\left(\cancel{\delta z} \frac{d^2\psi_2}{dx^2}\right)}_{\text{virtual curvature}} dx$$

$$\frac{d^2\psi_2}{dx^2} = \frac{6x}{L^2} - \frac{4}{L}$$

$$K_{21} = EI \int_0^L \left(\frac{12x}{L^3} - \frac{6}{L^2}\right)\left(\frac{6x}{L^2} - \frac{4}{L}\right) dx$$

$$K_{21} = EI \int_0^L \frac{72x^2}{L^5} - \frac{36x}{L^4} - \frac{48x}{L^4} + \frac{24}{L^3}\, dx$$

$$K_{21} = EI \left[\frac{24}{L^2} - \frac{18}{L^2} - \frac{24}{L^2} + \frac{24}{L^2} \right]$$

$$K_{21} = \frac{6EI}{L^2}$$

→ Generalizing the expression

$$* \quad K_{ij} = EI \int_0^L \left(\frac{d^2\psi_i}{dx^2}\right)\left(\frac{d^2\psi_j}{dx^2}\right) dx$$

using this produces.

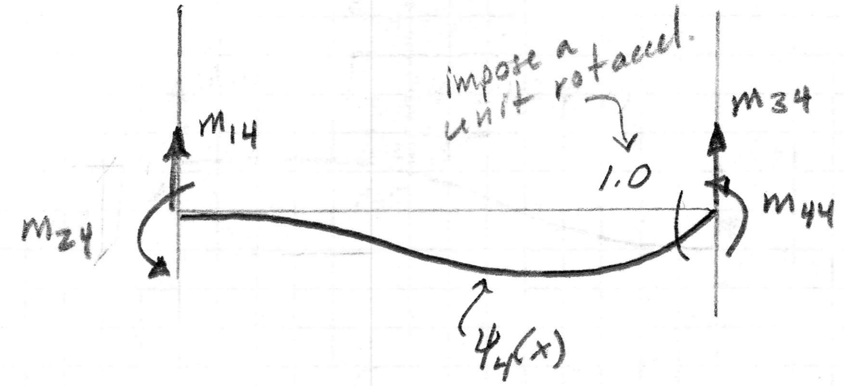
$$* \quad \left[K^e\right] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

→ Note that this is the same element stiffness matrix derived using S-D equations

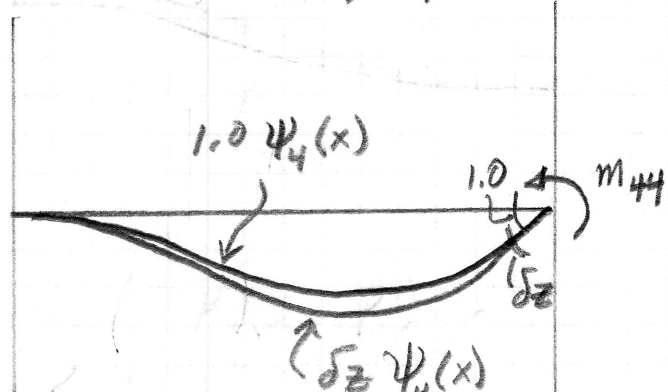
Homework #3

Compute the remaining stiffness coefficients of $[K^e]$ using the shape functions derived.

→ Solve for the 4^{th} column of the element mass matrix.



→ Solve for m_{44}, impose a virtual $\delta z\, \psi_4(x)$



$$m_{44}\, \delta z = \int_0^L \underbrace{(\bar{m}\, \psi_4)}_{\text{real inertia force}} \underbrace{(\delta z\, \psi_4)}_{\text{virtual displacement}} dx$$

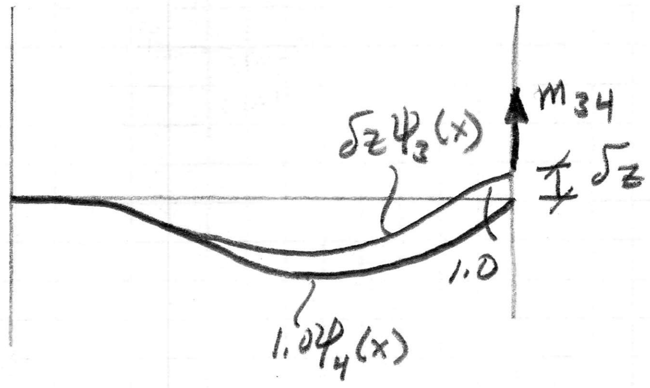
$$m_{44} = \bar{m} \int_0^L \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)\left(\frac{x^3}{L^2} - \frac{x^2}{L}\right) dx$$

$$m_{44} = \bar{m} \int_0^L \frac{x^6}{L^4} - \frac{2x^5}{L^3} + \frac{x^4}{L^2}\, dx$$

$$m_{44} = \bar{m}\left[\frac{L^3}{7} - \frac{L^3}{3} + \frac{L^3}{5}\right]$$

$$m_{44} = \frac{L^3}{105}$$

→ Solve for m_{34}, impose a virtual $\delta z\, \psi_3(x)$



$$m_{34}\, \cancel{\delta z} = \int_0^L \underbrace{(\bar{m}\, \psi_4)}_{\text{real inertia force}} \underbrace{(\cancel{\delta z}\, \psi_3)}_{\text{virtual displacement}} dx$$

$$m_{34} = \bar{m} \int_0^L \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)\left(-\frac{2x^3}{L^3} + \frac{3x^2}{L^2}\right) dx$$

$$m_{34} = \bar{m} \int_0^L -\frac{2x^6}{L^5} + \frac{2x^5}{L^4} + \frac{3x^5}{L^4} - \frac{3x^4}{L^3}\, dx$$

$$m_{34} = \bar{m}\left[-\frac{2L^2}{7} + \frac{L^2}{3} + \frac{L^2}{2} - \frac{3L^2}{5}\right]$$

$$m_{34} = \frac{-11\bar{m}L^2}{210}$$

→ Generalizing the expressions

$$* \quad m_{ij} = \bar{m} \int_0^L \psi_i\, \psi_j\, dx$$

using this produces.

$$[m^e] = \frac{\bar{m}L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Homework #4

Compute the remaining mass coefficients of $[m^e]$ using the shape functions derived.

→ This formulation is known as the "consistent" mass matrix

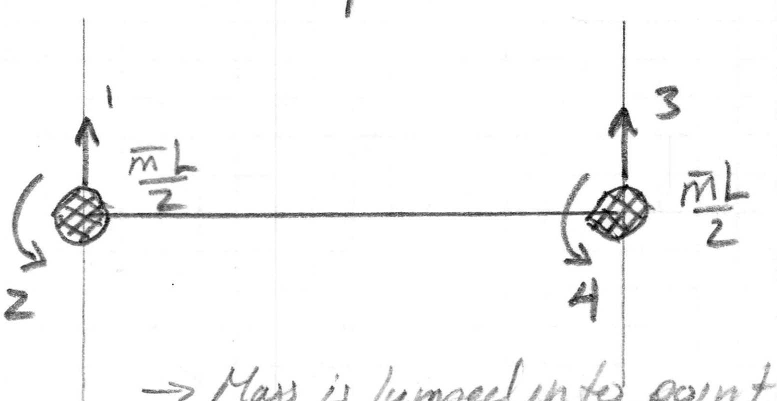
↳ It is "consistent" with the stiffness matrix in that they are derived using the same shape functions

→ Pros: Accuracy

→ Cons: Computationally inefficient.

→ Alternative Formulation

→ Lumped Mass Matrix.



→ Mass is lumped into point masses @ the beam ends.

using this produces

$$[m^e] = \frac{\bar{m}L}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Pros: Computationally efficient

→ Diagonal => no coupling mass terms.

→ No rotational mass

↳ $m_{22} = m_{44} = 0$

∴ Dof 2 & 4 are not "dynamic" dofs => they do not have both mass and stiffness

→ More on this later....

→ Cons: Potentially large errors.

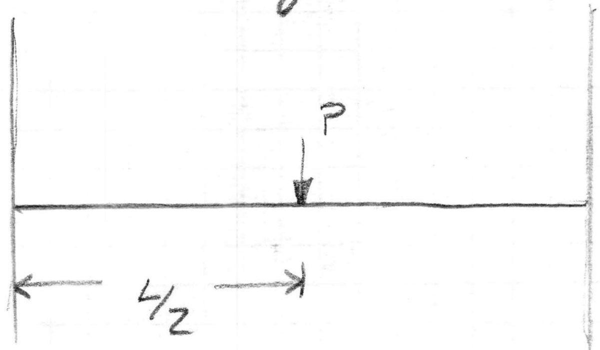
→ If coupling mass and/or rotational mass is influence.

→ Fix => discretize the structure into more elements (Recall 801 lecture 3)

<u>Homework #5</u>

Compute this mass matrix using the "direct" method.

→ Solve for Equivalent Nodal Forces.



Solve for P_1, impose virtual $\delta z\, \psi_1(x)$

$$P_1 \cancel{\delta z} = \cancel{\delta z}\, \psi_1(L/2) - P$$

$$P_1 = \left(\frac{2}{\cancel{L^3}}\left(\frac{\cancel{L^3}}{8}\right) - \frac{3}{\cancel{L^2}}\left(\frac{\cancel{L^2}}{4}\right) + 1 \right)(-P)$$

$$P_1 = -\frac{P}{2}$$

Solve for P_2, impose virtual $\delta z\, \psi_2(x)$

$$P_2 \cancel{\delta z} = \cancel{\delta z}\, \psi_2(L/2)(-P)$$

$$P_2 = \left(\frac{1}{\cancel{L^2}}\left(\frac{\cancel{L^3}}{8}\right) - \frac{2}{\cancel{L}}\left(\frac{\cancel{L^2}}{4}\right) + \frac{L}{2} \right)(-P)$$

$$P_2 = -\frac{PL}{8}$$

Generalizing the expression

$$P_i = \underbrace{\psi_i \Big|_{x=x_P} (P)}_{\text{point load}} + \int_0^L \underset{\uparrow \text{ distributed load}}{w}\, \psi_i(x)\, dx$$

Applying this expression

$$P = \begin{Bmatrix} -P/2 \\ -PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix}$$

Homework #6

(1) Relate these forces to the "fixed-end" forces used within the S-D method of analyses.