LECTURE 1: Formulation of EOM for MDOF Sys.

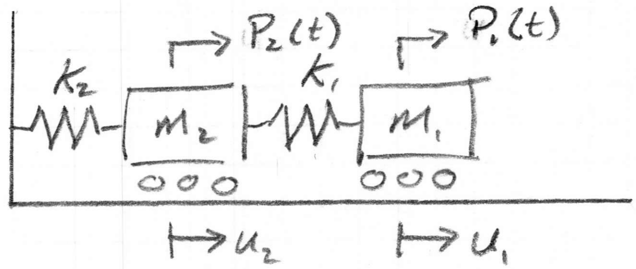
MDOF: Multiple Degree of Freedom.

→ While SDOF systems are quite useful to illustrate Str. Dyn., in many cases structures cannot be sufficiently simulated as SDOF

↳ E.g. multiple masses that can move independently of one another.

⇒ Review Distributed & Discrete Parameter Models ⇒ pp 2 of CIVE 801 Lecture 2.

→ Discrete, "Mathematical" System.



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Mass matrix — Stiffness matrix — Load or Force vector

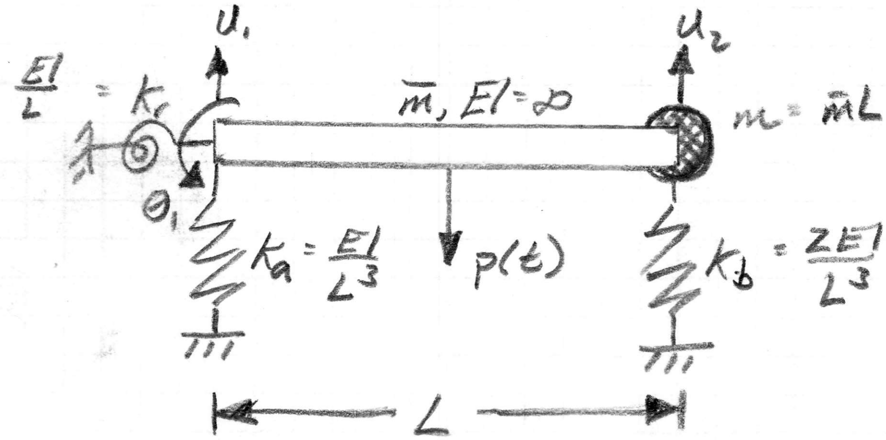
→ similarly,

$$c\dot{u} \Rightarrow \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$$

→ However due to our inability to estimate damping we will formulate the damping matrix differently... stay tuned.

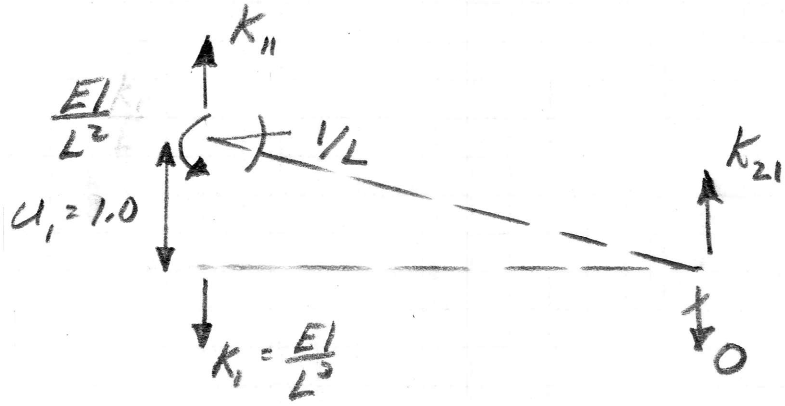
⇒ 3 Approaches to formulating EOM.

(1) "Direct" ⇒ only applicable to simple systems.



→ Start w/ DOFs u_1 & u_2

→ Solve for Stiffness Matrix.



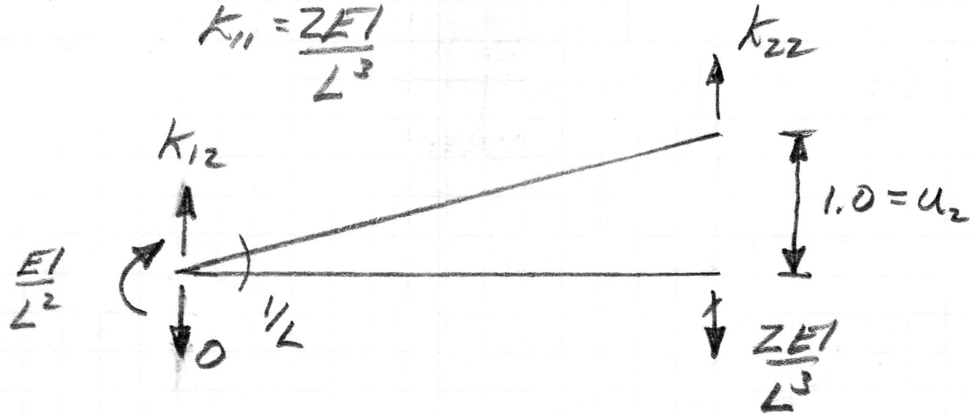
$\circlearrowright{+}\ \Sigma M_a = 0$

$$K_{21}(L) + \frac{EI}{L^2} = 0$$

$$K_{21} = -\frac{EI}{L^3}$$

$\Sigma F_y = 0$

$k_{11} = \dfrac{2EI}{L^3}$



$\circlearrowleft + \Sigma M_a = 0$

$$-\frac{EI}{L^2} - \frac{2EI}{L^3}(L) + k_{22}(L) = 0$$

$$k_{22} = \frac{3EI}{L^3}$$

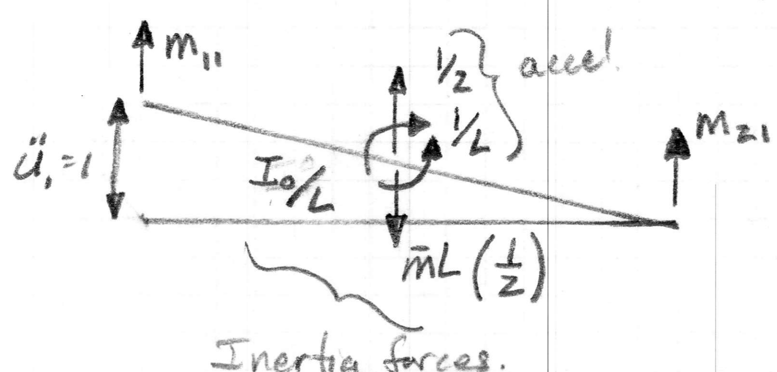
$\Sigma F_y = 0$

$$k_{12} = -\frac{EI}{L^3}$$

$$K = \begin{bmatrix} 2\frac{EI}{L^3} & -\frac{EI}{L^3} \\ -\frac{EI}{L^3} & \frac{3EI}{L^3} \end{bmatrix}$$

Symm & positive definite.

→ Solve for Mass Matrix.



$\circlearrowleft \Sigma M_a = 0$

$\frac{\bar{m}L^3}{12} \leftarrow \frac{I_0}{L} - \frac{\bar{m}L}{2}\left(\frac{L}{2}\right) + m_{21}L = 0$

$$m_{21} = -\frac{\bar{m}L}{12} + \frac{\bar{m}L}{4} = \frac{\bar{m}L}{6}$$

$\Sigma F_y = 0$

$$m_{11} - \frac{\bar{m}L}{2} + \frac{\bar{m}L}{6} = 0$$

$$m_{11} = \frac{\bar{m}L}{3}$$

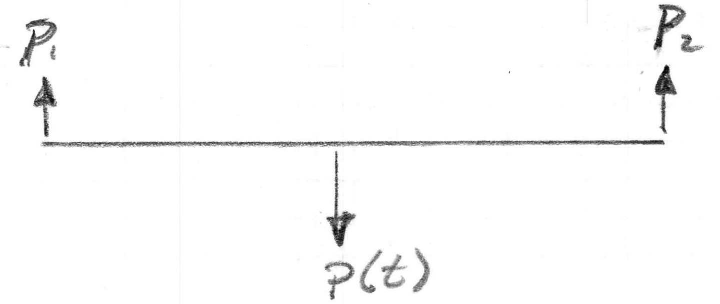
→ By inspection, $m_{22} = \frac{4\bar{m}L}{3}$ $\quad m_{12} = \frac{\bar{m}L}{6}$

$$M = \begin{bmatrix} \frac{\bar{m}L}{3} & \frac{\bar{m}L}{6} \\ \frac{\bar{m}L}{6} & \frac{4\bar{m}L}{3} \end{bmatrix}$$ symm & positive definite.

→ Alternate approach ⇒ lump mass.

$$M = \begin{bmatrix} \frac{\bar{m}L}{2} & 0 \\ 0 & \frac{3\bar{m}L}{2} \end{bmatrix}$$ lumped mass matricies are diagonal and more efficient for computations but less accurate.

→ Solve for Load Vector.



⇒ "Statically" Equivalent Forces.

$\circlearrowleft \Sigma M_a = -P(L/2)$

$P_2 L = -P\left(\frac{L}{2}\right)$

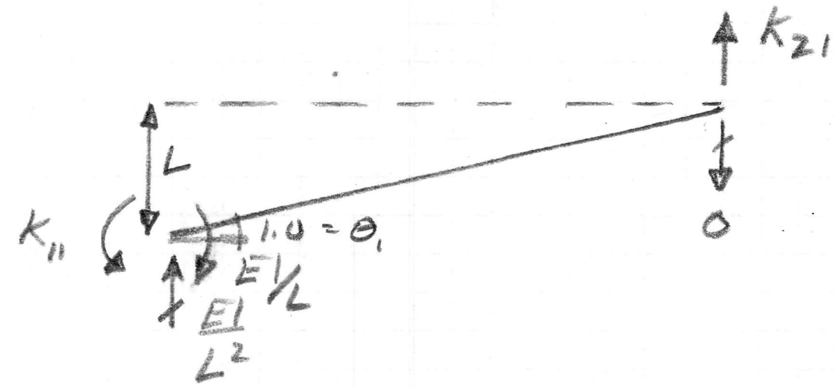
$P_2 = -\frac{P}{2}$

$\Sigma F_y = -P \Rightarrow P_1 = -\frac{P}{2}$

$$\bar{m}L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{P}{2} \\ -\frac{P}{2} \end{Bmatrix}$$

→ Solve using DOFs θ_1 & u_2

→ Stiffness Matrix.

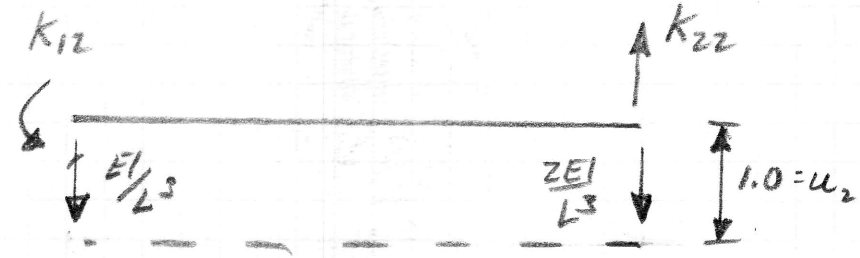


$\circlearrowleft + \Sigma M_b = 0$

$-\frac{EI}{L^2}(L) - \frac{EI}{L} + K_{11} = 0 \quad K_{11} = \frac{2EI}{L}$

$\Sigma F_y = 0$

$K_{21} = -\frac{EI}{L^2}$



$$\Sigma F_y = 0$$

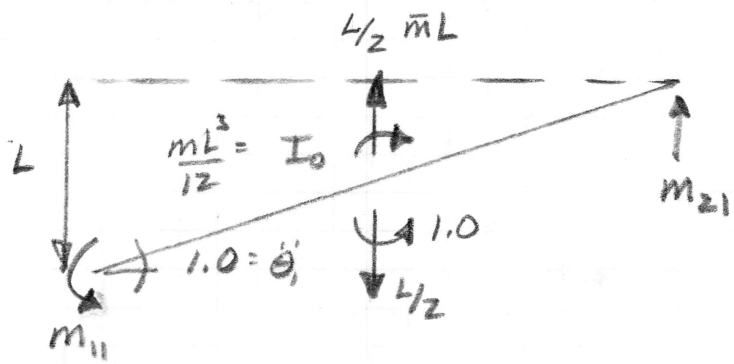
$$K_{22} = \frac{3EI}{L^3}$$

$$\Sigma M_a = 0$$

$$K_{12} + \frac{3EI}{L^2}(L) - \frac{2EI}{L^3}(L) = 0$$

$$K_{12} = -\frac{EI}{L^3}$$

→ Solve for Mass Matrix



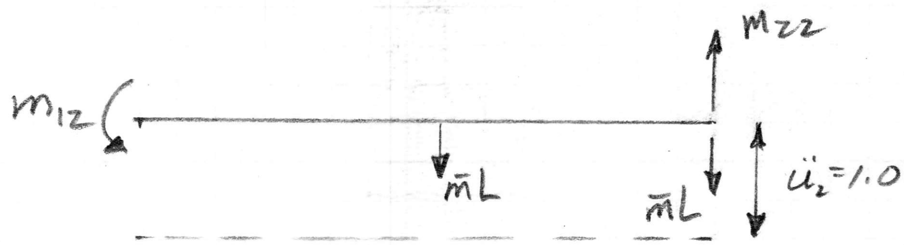
$$\Sigma M_b = 0$$

$$m_{11} - \frac{\bar{m}L^3}{12} - \frac{\bar{m}L^2}{2}\left(\frac{L}{2}\right) = 0$$

$$m_{11} = \frac{\bar{m}L^3}{3}$$

$$\Sigma F_y = 0$$

$$m_{21} = -\frac{\bar{m}L^2}{2}$$



$$\sum F_y = 0$$

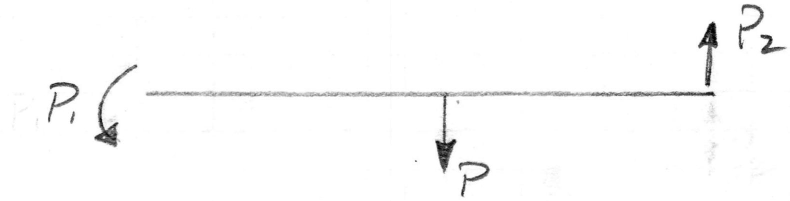
$$m_{22} = 2\bar{m}L$$

$$\circlearrowleft^{+} \sum M_b = 0$$

$$\bar{m}L\left(\frac{L}{2}\right) + m_{12} = 0$$

$$m_{12} = -\frac{\bar{m}L^2}{2} \quad \checkmark$$

→ Solve for Load Vector.



$$\sum F_y = -P$$

$$P_2 = -P$$

$$\circlearrowleft^{+} \sum M_c = -PL/2$$

$$-PL + P_1 = -\frac{PL}{2}$$

$$P_1 = \frac{PL}{2}$$

$$\bar{m}L \begin{bmatrix} \frac{L^2}{3} & -\frac{L}{2} \\ -\frac{L}{2} & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 2L^2 & -L \\ -L & 3 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} PL/2 \\ -P \end{Bmatrix}$$

Are the two representations (i.e. EOMs) equivalent?

→ Recall for SDOF

$$m\ddot{u} + ku = 0$$

$$u = A\sin(\omega t) \quad \text{(assume harmonic motion)}$$

$$\ddot{u} = -A\omega^2\sin(\omega t)$$

$$-mA\omega^2\cancel{\sin(\omega t)} + kA\cancel{\sin(\omega t)} = 0$$

$$k - \omega^2 m = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

→ Use a similar approach for MDOF (start w/ 2DOF)

$$[M]\{\ddot{u}\} + [k]\{u\} = 0$$

$$u_i = \begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix} \sin(\omega_i t)$$

can have multiple frequencies.

can have multiple shape functions

↳ "mode" shapes.

$$\ddot{u}_i = -\omega_i^2 \begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix} \sin(\omega_i t)$$

shape of disp is the same as the shape of acceleration.

→ Subst m...

$$-\omega_i^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix} \cancel{\sin \omega_i t} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix} \cancel{\sin \omega_i t} = 0$$

$$\left[\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} - \omega_i^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \right] \begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix} = 0$$

$$\left\{ [K] - \omega_i^2 [M] \right\} \{ \phi_i \} = 0$$

natural frequency i mode shape i

⇒ Eigen problem

ω_i^2 = eigen values (square of freq.)

ϕ_i = eigen vectors (mode shapes)

⇒ Trivial Solutions.

$\phi_i = 0$ ⇒ no vibration

⇒ Non-trivial Solution

$$\det \left| [K] - \omega_i^2 [M] \right| = 0$$

$$\det \begin{vmatrix} k_{11} - \omega_i^2 m_{11} & k_{12} - \omega_i^2 m_{12} \\ k_{12} - \omega_i^2 m_{12} & k_{22} - \omega_i^2 m_{22} \end{vmatrix} = 0$$

$$k_{11}k_{22} + \omega_i^4 m_{11}m_{22} - \omega_i^2(k_{11}m_{22} + k_{22}m_{11})$$

$$-k_{12}^2 + \omega_i^4 m_{12}^2 - 2\omega_i^2 k_{12}m_{12} = 0$$

$$\omega_i^4(m_{11}m_{22} + m_{12}^2) + \omega_i^2(k_{11}m_{22} + k_{22}m_{11} - 2k_{12}m_{12})$$

$$+ k_{11}k_{22} - k_{12}^2 = 0$$

=> Quadratic in terms of ω_i^2

$$\omega_i^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = \det[M]$

$b = -(k_{11}m_{22} + k_{22}m_{11} - 2k_{12}m_{12})$

$c = \det[K]$

=> Once you solve for ω_i^2, you can sub back in & solve for ϕ

$$(k_{11} - \omega_i^2 m_{11})(\phi_{1,i}) + (k_{12} - \omega_i^2 m_{12})(\phi_{2,i}) = 0$$

$$r_i = \frac{\phi_{1,i}}{\phi_{2,i}} = \frac{-k_{12} + \omega_i^2 m_{12}}{k_{11} - \omega_i^2 m_{11}}$$

} can only solve for mode shapes in a relative sense...

modal amplitude ratio.

$$\underbrace{\phi_i = \overset{r_i < 1.0}{\begin{Bmatrix} r_i \\ 1.0 \end{Bmatrix}} \text{ or } \overset{r_i > 1}{\begin{Bmatrix} 1.0 \\ 1/r_i \end{Bmatrix}}}_{}$$

This provides "unit normalized" mode shapes..

→ Solve for ω_i & ϕ_i for dof u_1 & u_2

$$a = \bar{m}^2 L^2 \left(\frac{4}{9} - \frac{1}{36} \right) = \frac{15}{36} \bar{m}^2 L^2$$

$$b = -\frac{EI\bar{m}}{L^2} \left(\frac{8}{3} + 1 + \frac{1}{3} \right) = -4 \frac{EI\bar{m}}{L^2}$$

$$c = \left(\frac{EI}{L^3} \right)^2 (6 - 1) = 5 \frac{E^2 I^2}{L^6}$$

$$\omega_i^2 = \frac{4 \frac{EI\bar{m}}{L^2} \pm \sqrt{16 \frac{E^2I^2\bar{m}^2}{L^4} - 4\left(\frac{15}{36}\right)(5)\frac{E^2I^2\bar{m}^2}{L^4}}}{2\left(\frac{15}{36}\right)\bar{m}^2 L^2}$$

$$\omega_i^2 = (4.8 \pm 3.3) \frac{EI}{\bar{m} L^4}$$

* $\omega_1 = 1.22 \sqrt{\frac{EI}{\bar{m} L^4}}$

* $\omega_2 = 2.85 \sqrt{\frac{EI}{\bar{m} L^4}}$

$$r_1 = \frac{\cancel{\frac{EI}{L^3}} + 1.488 \frac{\cancel{EI}}{\cancel{\bar{m}L^4}} \frac{\cancel{\bar{m}L}}{6}}{2\cancel{\frac{EI}{L^3}} - 1.488 \frac{\cancel{EI}}{\cancel{\bar{m}L^4}} \frac{\cancel{\bar{m}L}}{3}} = \frac{1.248}{1.504} = 0.83$$

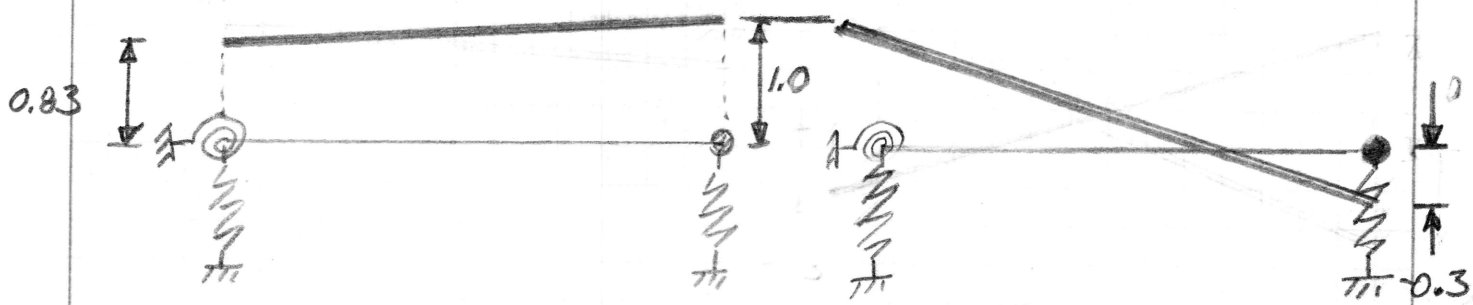
* $\phi_1 = \begin{Bmatrix} 0.83 \\ 1.0 \end{Bmatrix}$

$$r_2 = \frac{1 + 8.123\left(\frac{1}{6}\right)}{2 - 8.123\left(\frac{1}{3}\right)} = \frac{2.35}{-0.707} = -3.32$$

* $\phi_2 = \begin{Bmatrix} 1.0 \\ -0.30 \end{Bmatrix}$

→ Plot Mode Shapes.

<u>Mode 1</u> $\omega_1 = 1.22\sqrt{\frac{EI}{\bar{m}L^4}}$ <u>Mode 2</u> $\omega_2 = 2.85\sqrt{\frac{EI}{\bar{m}L^4}}$



→ Solve for ω_i & ϕ_i for θ_1 & u_2

$$a = \bar{m}^2L^2\left(\frac{2L^2}{3} - \frac{L^2}{4}\right) = \frac{5\bar{m}^2L^4}{12}$$

$$b = -\frac{EI\bar{m}}{L^2}\left(4L^2 + L^2 - L^2\right) = -4EI\bar{m}$$

$$c = \frac{E^2I^2}{L^6}\left(6L^2 - L^2\right) = 5\frac{E^2I^2}{L^4}$$

$$\omega_i^2 = \frac{4EI\bar{m} \pm \sqrt{16E^2I^2\bar{m}^2 - 4\left(\frac{5}{12}\right)(5)E^2I^2\bar{m}^2}}{(2)\frac{5\bar{m}^2L^4}{12}}$$

$$\omega_i^2 = (4.8 \pm 3.3)\frac{EI}{\bar{m}L^4}$$

$\ast$ $\omega_1 = 1.22\sqrt{\frac{EI}{\bar{m}L^4}}$

$\ast$ $\omega_2 = 2.85\sqrt{\frac{EI}{\bar{m}L^4}}$

$$r_1 = \frac{\frac{EI}{L^2} + 1.488\,\frac{EI}{\bar{m}L^4}\left(-\frac{\bar{m}L^2}{2}\right)}{\frac{2EI}{L} - 1.488\,\frac{EI}{\bar{m}L^4}\,\frac{\bar{m}L^3}{3}} = \frac{0.256/L}{1.504} = \frac{0.17}{L}$$

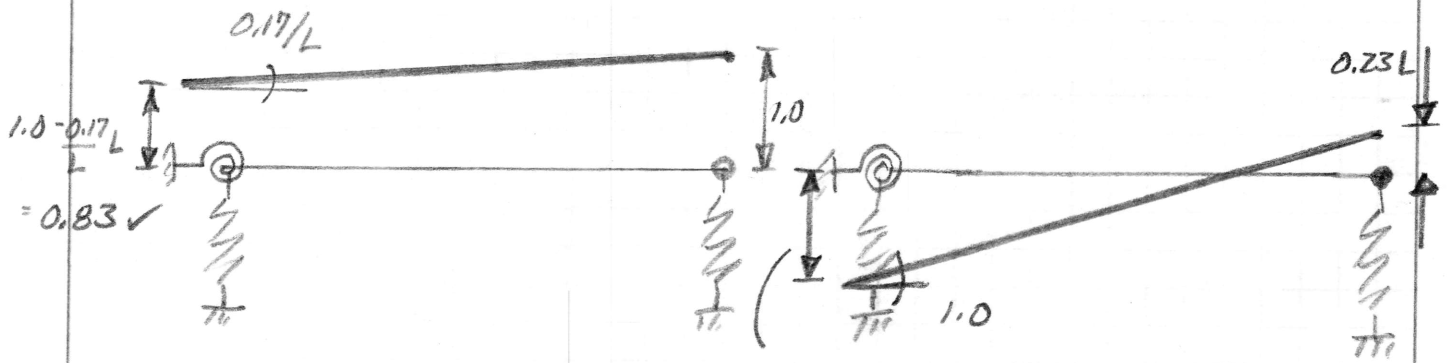
$$* \;\phi_1 = \begin{Bmatrix} \frac{0.17}{L} \\ 1.0 \end{Bmatrix}$$ ← Note, when you mix dofs (rot & trans), the mode shapes are in terms of L.

$$r_2 = \frac{1 + 8.122\left(-\frac{1}{2}\right)}{L\left(2 - 8.122\left(\frac{1}{3}\right)\right)} = \frac{-3.06/L}{-0.707} = \frac{4.33}{L}$$

$$* \;\phi_2 = \begin{Bmatrix} 1.0 \\ 0.23L \end{Bmatrix}$$

→ Plot Mode Shapes

Mode 1 $\omega_1 = 1.22\sqrt{\frac{EI}{\bar{m}L^4}}$ Mode 2 $\omega_2 = 2.85\sqrt{\frac{EI}{\bar{m}L^4}}$



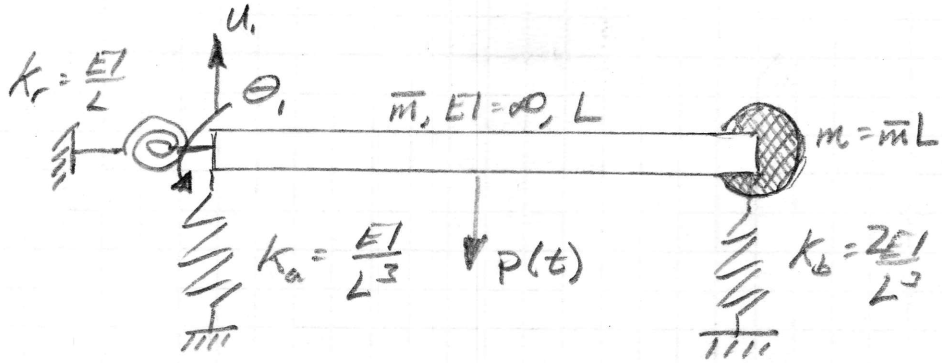
$0.23L - 1.0L$
$= -0.77L$

$$\phi_2 = \begin{Bmatrix} -0.77L \\ 0.23L \end{Bmatrix}$$

or divide by $-0.77L$

$$\checkmark\; \phi_2 = \begin{Bmatrix} 1.0 \\ -0.3 \end{Bmatrix}$$

Homework #1



(1) Solve for [k] & [M] using u_1 & θ_1

(2) Solve for ω_i & ϕ_i and show these are equivalent to the ones obtain in class

(3) Write the equation of motion.