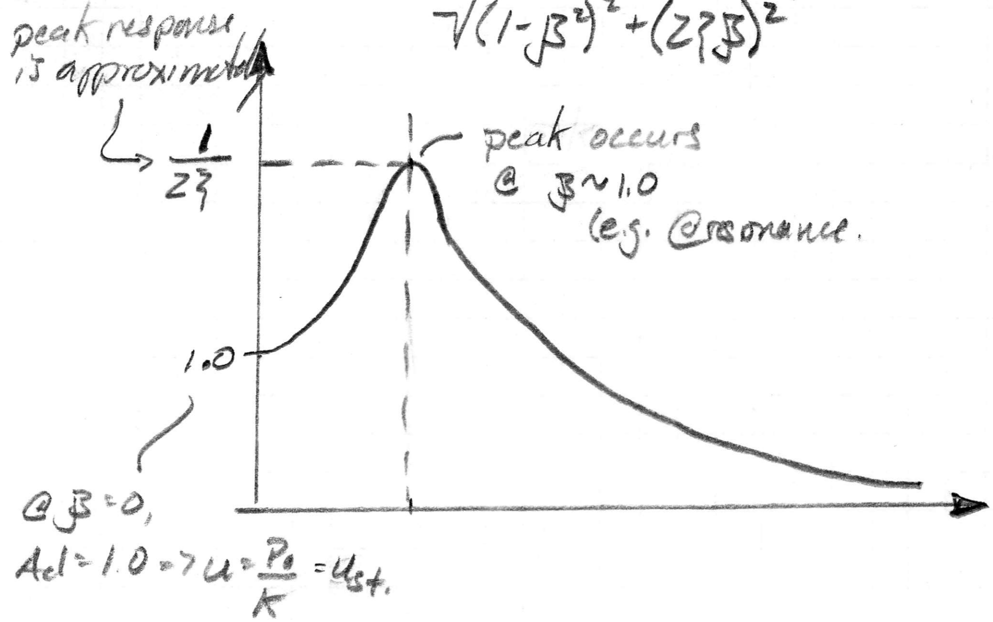
# ⇒ LECTURE 7 – Forced Vibration due to (a) Harmonic Input & (b) General loading

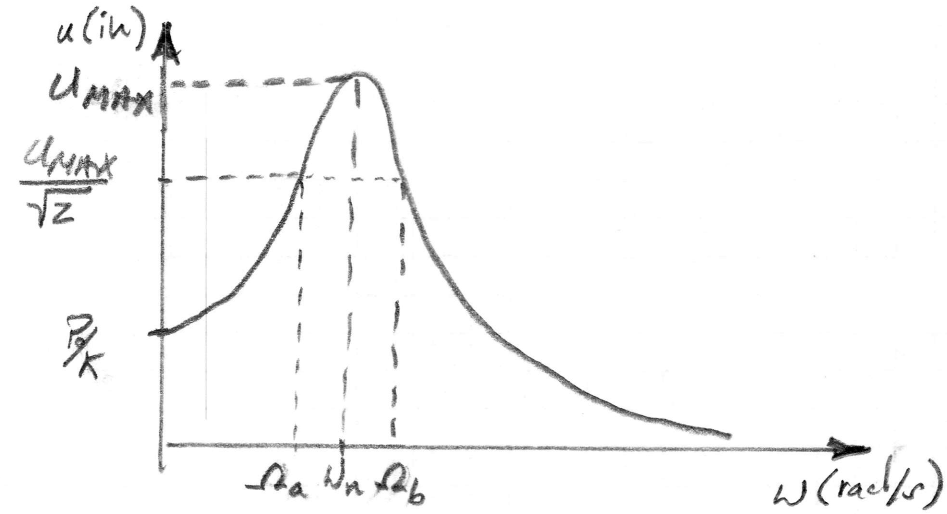
→ Recall, for $p(t) = p_0 \sin(\Omega t)$, the relative amplitude of response, $A_d$ is.

$$A_d = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$




→ Estimate System Properties From FRFs.

→ Begin w/ Non-normalized FRF.



→ **Estimate Stiffness.**

→ Could use static response, but this is very difficult to get in practice.

→ Use peak response.

$$u_{MAX} = \frac{P_o}{K}\left(\frac{1}{2\zeta}\right) \Rightarrow \boxed{K = \frac{P_o}{u_{MAX}}\left(\frac{1}{2\zeta}\right)}$$

→ Note $P_o$ & $u_{MAX}$ are known

⇒ need $\zeta$ to get $K$.

→ **Estimate Damping**

→ Identify frequencies $\Omega_a$ & $\Omega_b$ that correspond to $\frac{u_{MAX}}{\sqrt{2}}$

→ Both satisfy..

$$\cancel{\frac{P_o}{K}} \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\Omega}{\omega_n}\right)\right)^2}} = \frac{1}{\sqrt{2}}\left(\cancel{\frac{P_o}{K}}\right)\frac{1}{2\zeta}$$

$\beta$

$$1 + 2\frac{\Omega^2}{\omega_n^2} + \frac{\Omega^4}{\omega_n^4} + 4\zeta^2\frac{\Omega^2}{\omega_n^2} = 8\zeta^2$$

$$\left(\frac{\Omega}{\omega_n}\right)^4 - 2(1-2\zeta^2)\left(\frac{\Omega}{\omega_n}\right)^2 + 1 - 8\zeta^2 = 0$$

use Quadratic formula to solve for $\left(\frac{\Omega}{\omega_n}\right)^2$ →

$$\frac{\Omega}{\omega_n} \approx \sqrt{1 \pm 2\zeta^2} \quad \text{(discard terms with } \zeta^2\text{)}$$

$$\frac{\Omega}{\omega_n} \approx 1 \pm \zeta$$

$$\left(\frac{\Omega_a}{\omega_n} = 1-\zeta\right) * -1$$

$$+\quad \frac{\Omega_b}{\omega_n} = 1+\zeta$$

$$* \quad \boxed{\frac{\Omega_b - \Omega_a}{\omega_n} = 2\zeta}$$

→ The only unknown is $\zeta$

→ Use this to solve for $K$.

→ Estimate Mass

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\boxed{m = \frac{K}{\omega_n^2}}$$

Example: You performed a shaker test of a structure w/ unknown $m, c, k$.

Shaker Specs – Capable of applying a constant force of 500lb over a bandwidth from 5 to 40 rad/s.

Test Method → Begin the shaker @ 5 rad/s and slowly increase the frequency by 0.05 rad/s to 40 rad/s.

→ Make sure to dwell the shaker long enough so $u_T \sim 0$

→ Measure the resulting displacement amplitude ⇒ FRF.

Solution Methods → Model Updating
→ Analytical.

→ Model Updating - see Excel

→ Analytical - From the Excel Plot.

$u_{MAX} = 2.4012 \text{ in} \ @ \ \Omega = 20.8 \text{ rad/sec} \sim \omega_n$

⇒ Half Bandwidth Method.

$$\frac{u_{MAX}}{\sqrt{2}} = \frac{2.4012}{\sqrt{2}} = 1.6979 \text{ in}$$

$$\Omega_a (u = 1.6979) \approx 19.559 \text{ rad/s}$$

$$\Omega_b (u = 1.6979) \approx 21.925 \text{ rad/sec}$$

$$\frac{21.925 - 19.559}{20.8} = 2\zeta$$

$$\zeta = 0.05688$$

$$K = \frac{(500\,lb)}{2.4012\,in}\left(\frac{1}{2(0.05688)}\right) = \underline{1830.4\ lb/in}$$

$$M = \frac{1830.4}{(20.8)^2} = \underline{4.23\ lb_m}.$$

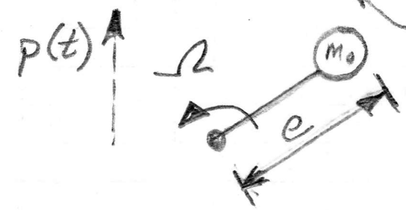
⇒ Discussion

→ How would you generate such a force to test a structure?

→ How would you measure the resulting displacement?

⇒ Rotating Shakers

$$p(t) = \underbrace{m_o e \Omega^2}_{\text{force amplitude is also a function of } \Omega} \sin(\Omega t)$$




→ e.g. force amp. changes for different frequencies

⇒ Accelerometers

→ In general, it is very difficult to measure displacements due to the rigid fix reference.

→ Measuring acceleration is much easier

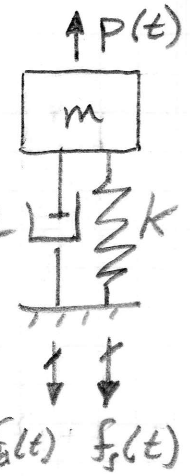
→ This requires formulating the FRF in terms of accel, not disp.

↳ good homework problem.

⇒ Error is partly due to assuming resonance occurs @ peak response, how should resonance be defined?

## → Transmissibility

(1) The ratio of the applied force transmitted to the supporting structure.



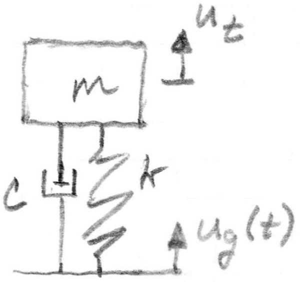

$F(t) = f_d(t) + f_s(t)$

↑ total force transmitted.

$TR = \frac{F_o}{P_o}$ ← $F_o$: amplitude of transmitted force; $P_o$: amplitude of applied force.

↑ $TR$: transmissibility.

(2) The ratio of applied support displacement to the relative displacement of the system.




$u = u_t(t) - u_g(t)$

↑ $u$: displacement of mass relative to the ground.

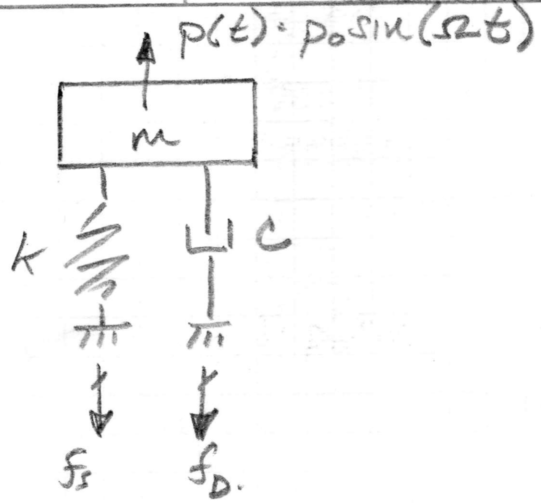
↑ $u_t(t)$: total disp of mass

↑ $u_g(t)$: ground disp

$TR = \frac{\ddot{u}_{t}}{\ddot{u}_{go}}$ ← $\ddot{u}_{t}$: amplitude of mass acceleration; $\ddot{u}_{go}$: amplitude of ground acceleration.

## ⇒ Common Scenarios

(1) Force Isolation → the goal is to control the amount of force transferred to a structure by a rotating or reciprocati[ng] piece of equipment




$$u_{ss} = \rho \sin(\Omega t - \phi)$$

$$\dot{u}_{ss} = \rho \Omega \cos(\Omega t - \phi)$$

$$F = ku + c\dot{u} = k\rho \sin(\Omega t - \phi) + c\rho\Omega\cos(\Omega t - \phi)$$

$$F = F_0 \sin(\Omega t - \phi + \eta)$$

↑ amplitude of total force transferred.

$$F_0 = p_0 \sqrt{\frac{1 + (2\xi\beta)^2}{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

$$TR = \frac{F_0}{p_0} = \sqrt{\frac{1 + (2\xi\beta)^2}{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

→ See Excel plot.

(1) Although damping reduces the response amplitude @ all $\beta$, it only reduces the transmitted force for $\beta > \sqrt{2} = 1.414$

(2) At $\beta = \sqrt{2}$, the transmitted force is equal to the applied force for all damping.

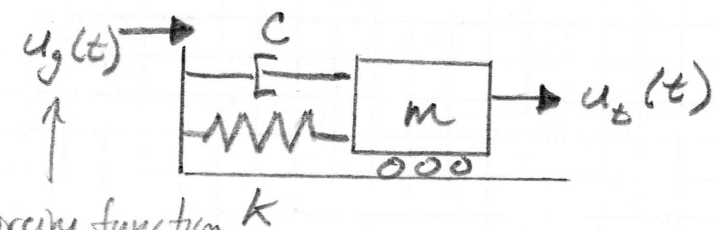
(3) For $\beta > \sqrt{2}$, no damping is actually desireable, but this is not a good idea due to transient response.

$\therefore$ To reduce the transmitted force, one should:

(1) Reduce the support stiffness or increase mass so the $\omega_n < \dfrac{\Omega}{\sqrt{2}}$

(2) Minimize damping

## (2) Base Isolation



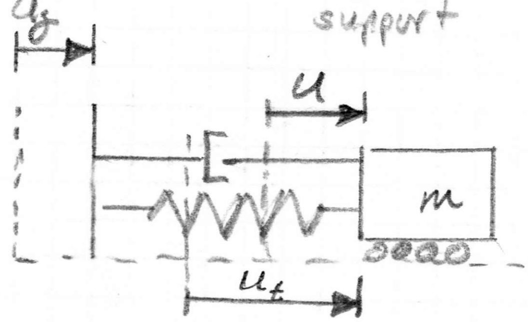

Forcing function is not a force, but a disp applied to the base → e.g. earthquake.

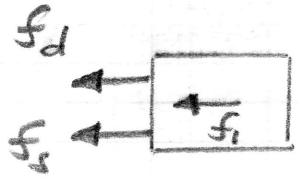
$$u_t(t) = u(t) + u_g(t)$$

$u_t(t)$ ↑ total disp of mass

$u(t)$ ↑ Relative disp of mass to support

$u_g(t)$ ↑ displacement of support.






Dynamic Equil.

$$f_I + f_D + f_s = 0$$

⇒ Note only relative motion ($u$) cause damping & stiffness forces.

⇒ Total motion ($u_t$) causes mass forces.

$$m\ddot{u}_t + c\dot{u} + ku = 0$$

or

$$m(\ddot{u} + \ddot{u}_g) + c\dot{u} + ku = 0$$

$$m\ddot{u} + c\dot{u} + ku = \underbrace{-m\ddot{u}_g}_{p(t)}$$

$p(t)$ for earthquake loading

For $u_g = G\sin(\Omega t)$

$$\ddot{u}_g = -\Omega^2 G \sin(\Omega t)$$

$$m\ddot{u} + c\dot{u} + ku = \underbrace{m\Omega^2 G}\sin(\Omega t)$$

same as force isolation with $P_0 = \underbrace{m\Omega^2 G}_{\text{constant}}$.

Example: You are hired to design a table to support a sensitive piece of equipment within an existing building. The manufacturers specs req' an accel of less than 0.005g.

You instrument the room that will house the equipment and find that the peak accel is around 0.1g at a frequency of 10 Hz.

Initial Design (COTS Table)

$k = 80 \, ^{lb}/_{in}$

$W = 100 \, lb$ (table + equipment)

$\zeta = 0.10$

$$\omega_n = \sqrt{\frac{80}{100/386}} = 17.58 \, ^{rad}/_{s}$$

$$\beta = \frac{\Omega}{\omega_n} = \frac{10(2\pi)}{17.58} = 3.575$$

$$TR = \sqrt{\frac{1 + (2(0.10)3.575)^2}{(1-(3.575)^2)^2 + (2(0.10)3.575)^2}}$$

$TR = 0.104$

$$\ddot{u}_{t_0} = 0.104(\overset{0.1g}{\ddot{u}_{g_0}}) = 0.0104g > 0.005g$$

=> Not sufficient

=> 4 options...

(1) Reduce damping (since $\beta > \sqrt{2}$)

(2) Increase 'm' (i.e. $\beta\uparrow$)

(3) Decrease 'k' (i.e. $\beta\uparrow$)

(4) Combinations of 1, 2 & 3

=> Choose (2) (easiest & most reliable)

-> guess @ additional weight.

$$W = 100\,lb + \overset{150\,lb}{\vec{m}_{add}} = 250\,lb$$

$$\omega_n' = \sqrt{\frac{80}{250/386}} = 11.11 \text{ rad/s}$$

$$\beta = \frac{2\pi(10)}{11.11} = 5.655$$

=> Don't forget to modify damping ratio

$$c = \zeta(2m\omega_n)$$

$$c = 0.1\left(2\left(\frac{100}{386}\right)17.58\right) = \underbrace{0.911 \frac{lb \cdot s}{in}}_{\text{original damping}}$$

$$c' = 0.911 = \zeta\left(2\left(\frac{250}{386}\right)11.11\right)$$

$$\zeta = 0.063$$

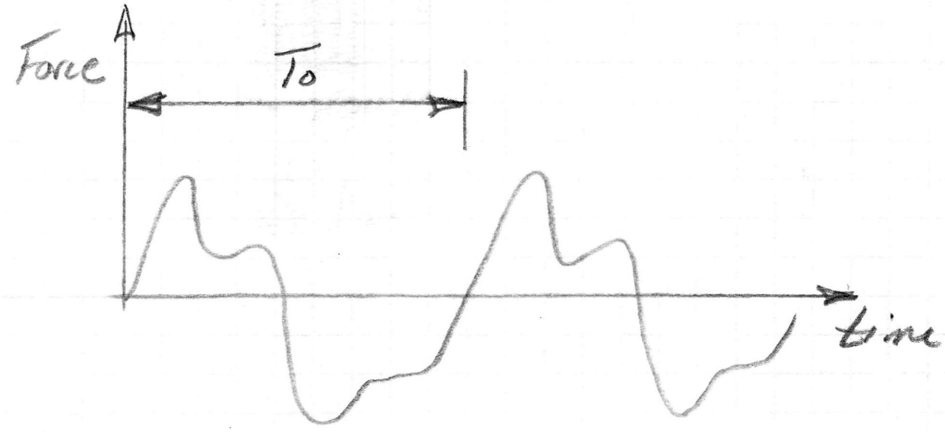
$$TR = \sqrt{\frac{1 + (2(0.063)5.655)^2}{(1-5.655^2)^2 + (2(0.063)5.655)^2}}$$

$$TR = 0.04$$

$$\ddot{u}_t = 0.04(0.1g) = 0.004g < 0.005g$$

**ok**

# → Periodic Loads




⇒ Fourier Series Representation.

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\omega_0 t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_0 t)$$

periodic function.

where, $\omega_0 = \frac{2\pi}{T_0}$

period of the periodic excitation.

⇒ Breaks any periodic function into an infinite number of harmonic components.

↳ sin & cos.

⇒ Typically used to determine steady-state response

↳ of primary interest for harmonic & periodic loads.

⇒ using superposition

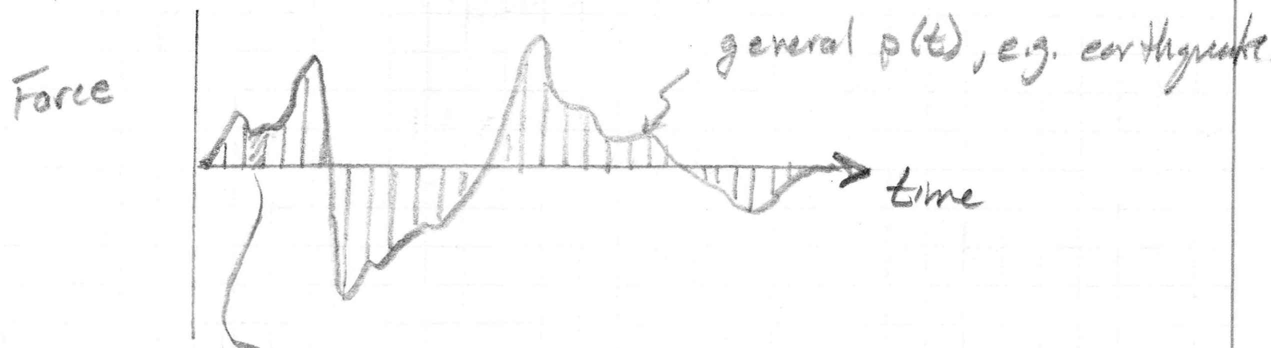
steady state →

$$u(t) = u_0(t) + \underbrace{\sum_{j=1}^{\infty} u_j^c(t)}_{\text{response to cosine forcing functions } (a_j)} + \underbrace{\sum_{j=1}^{\infty} u_j^s(t)}_{\text{response to sine forcing functions } (b_j)}$$

response to $a_0$ ⇒ average load $= \frac{a_0}{K}$

# → Response to General Loading

→ Solution Approach.




(1) break the general loading into a series of impulses $= F\Delta t$

$\uparrow$ $\Delta t$ is really small.

(2) solve for the response to each impulse

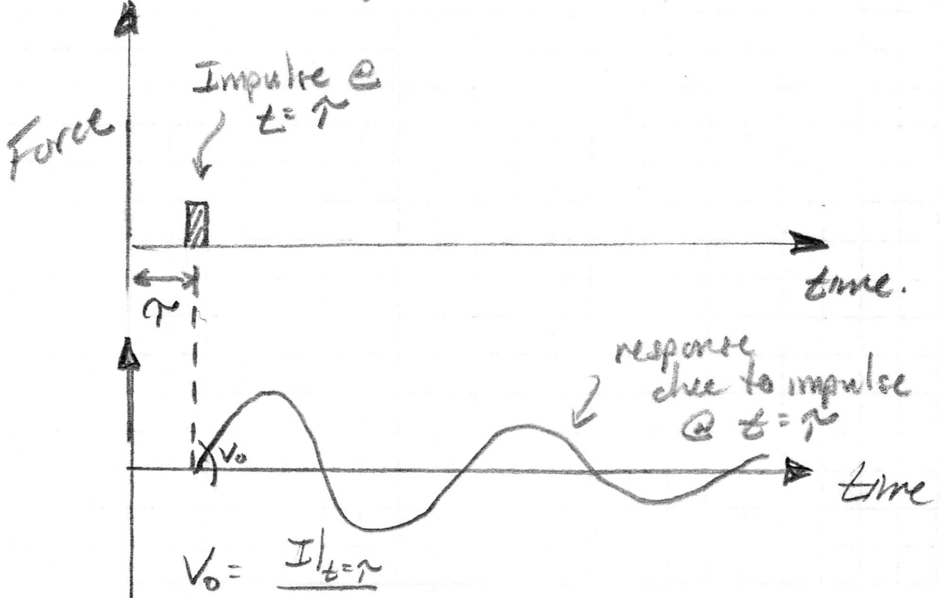
$$V_0 = \frac{F\Delta t}{m} = \frac{I}{m}$$

$$u_0 = 0$$

(3) add the responses together

→ principle of superposition

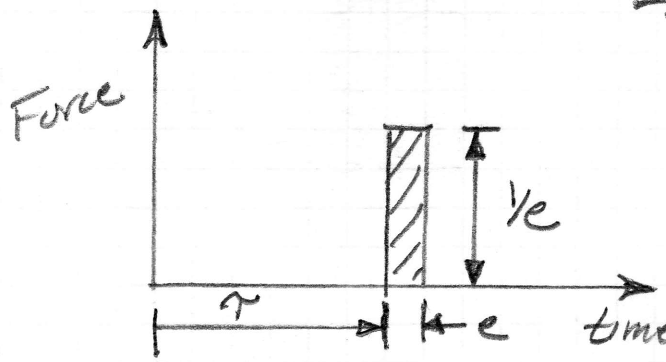
→ requires linearity, stationarity.




$$V_0 = \frac{I|_{t=\tau}}{m}$$

## → Response to a Unit Impulse

↳ aka Impulse Response Function (IRF)




$$I = F\Delta t = \frac{1}{e}(e) = 1.0$$

⇒ unit impulse.

Using Newton's 2nd Law.

$$F = \frac{d}{dt}(m\dot{u})$$

⇒ for constant mass.

$$F = m\frac{d\dot{u}}{dt} \Rightarrow \frac{m(\Delta\dot{u})}{\Delta t}$$

$$\Delta\dot{u} = \frac{\overset{\rightarrow 1.0}{\cancel{F\Delta t}}}{m} = \frac{1}{m}$$

→ For a system @ rest.

$$V_0 = \frac{1}{m}$$

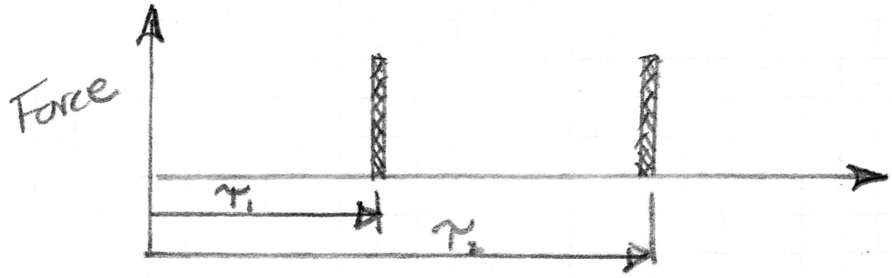
$$u_0 = 0$$

$$\left[\begin{array}{ll} u(t) = 0 & t < \tau \\ u(t) = h(t-\tau) = \dfrac{1}{m\omega_n}\sin(\omega_n(t-\tau)) & t \geq \tau \\ \quad \text{or.} & \\ u(t) = h(t-\tau) = \dfrac{e^{-\xi\omega_n(t-\tau)}}{m\omega_D}\sin(\omega_D(t-\tau)) & t \geq \tau \end{array}\right.$$

$h(t-\tau)$ = Impulse response function.

= response to a unit impulse that occurs @ $t=\tau$.

=> What about 2 unit Impulses?




$$\left[\begin{array}{ll} u(t) = 0 & t << \tau_1 \\ u(t) = h(t-\tau_1) & t \geq \tau_1 \\ u(t) = \underbrace{h(t-\tau_1) + h(t-\tau_2)}_{\text{superposition}} & t \geq \tau_2 \end{array}\right.$$

=> Response to an Arbitrary Force.

$$\underbrace{du(t)}_{\substack{\text{Incremental portion of response} \\ \text{due to impulse @ } t=\tau}} = \underbrace{\underbrace{I(\tau)}_{\substack{\text{Impulse @} \\ t=\tau}} \underbrace{h(t-\tau)}_{\substack{\text{Response to a} \\ \text{unit impulse @ } t=\tau}}}_{\substack{\text{Response to impulse} \\ I(\tau) \text{ @ } t=\tau}} \qquad \underbrace{t>\tau}_{\substack{\text{Valid only} \\ \text{after impulse}}}$$

=> Note

$$I(\tau) = \underbrace{P(\tau)}_{\text{force}} \underbrace{d\tau}_{\text{duration}}$$

Integrating from $\tau=0$ to $\tau=t$.

This is known as Duhamel's integral. → $$u(t)=\int_0^t p(\tau)\,h(t-\tau)\,d\tau$$

arbitrary forcing function.

$$* \quad u(t)=\frac{1}{m\omega_D}\int_0^t p(\tau)\,e^{-\xi\omega_n(t-\tau)}\sin(\omega_D(t-\tau))\,d\tau$$

or

$$* \quad u(t)=\frac{1}{m\omega_n}\int_0^t p(\tau)\sin(\omega_n(t-\tau))\,d\tau$$

→ Pros
- Closed form solution to any loading condition.

→ Cons
- Difficult to evaluate except for very simple forcing functions.
- Only valid for linear systems since it is based on superposition.

→ We'll examine two cases.

Step load ⇒ $p(t)=p_0$ (w/ damping)

Ramp load ⇒ $p(t)=\dfrac{p_0 t}{t_1}$ (w/o damping)

→ Step Load.

→ $p(t) = p_o$

→ Duhamel's Integral Becomes

$$u = \frac{1}{m\omega}\int_0^t P_o e^{-\xi\omega(t-\tau)} \sin(\omega(t-\tau))\,d\tau$$

$$u = \frac{P_o}{k}\left[1 - e^{-\omega\xi t}\left(\cos\omega t + \xi \sin\omega t\right)\right]$$

⇒ This valid for small amounts of damping $\omega_d = \omega_n = \omega$

→ for $\xi = 0$

$$u = \frac{P_o}{k}\left(1 - \cos\omega t\right)$$

→ See Excel.

→ Structure oscillates about it's new "equil" position of $P_o/k$

→ For zero damping the total displacement is $\frac{2 P_o}{k}$

↑ theoretical max, why?

~ similar to a initial displacement of $P_o/k$

## → Ramp Load

→ $p(t) = \frac{P_o t}{t_1}$

→ Duhamel's Integral Becomes ($\xi = 0$)

$$u(t) = \frac{1}{m\omega} \int_0^t \frac{P_o \tau}{t_1} \sin(\omega(t-\tau))\, d\tau$$

$$u(t) = \frac{P_o}{K}\left(\frac{t}{t_1} - \frac{\sin \omega t}{\omega t_1}\right)$$

⇒ for $\xi = 0.0$

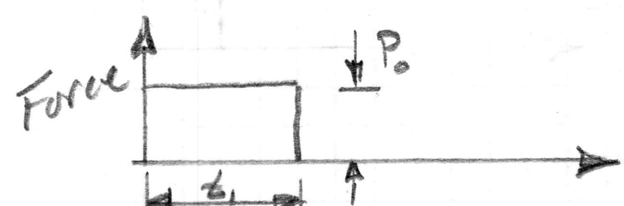
→ See Excel

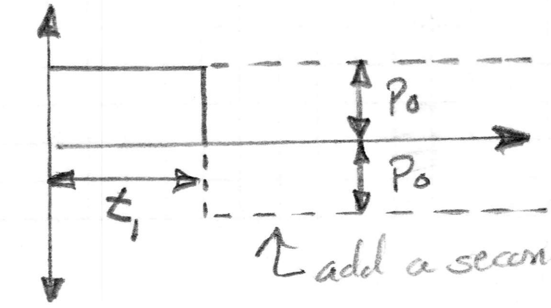
→ Structure oscillates about it's new "equil" position of $\frac{P_o t}{K t_1}$

## → Combination of Step & Ramp Loading

→ Similar to concept behind Duhamel's Integral, add these solutions together to generate more complex, generalized loading

**Example** → Pulse Loading




↑ add a second step function at $t = t_1$ w/ $P_o = -P_o$

Solution:

$$u(t) = \begin{cases} u(t) = \frac{P_0}{k}\left[1 - e^{-\omega\xi t}(\cos\omega t + \xi \sin\omega t)\right] & t < t_1 \\ u(t) = \frac{P_0}{k}\left[1 - e^{-\omega\xi t}(\cos\omega t + \xi \sin\omega t)\right] \\ \quad - \frac{P_0}{k}\left[1 - e^{\omega\xi(t-t_1)}\left(\cos(\omega(t-t_1)) + \xi \sin(\omega(t-t_1))\right)\right] & t \geq t_1 \end{cases}$$

→ See Excel

(1) Notice the apparent influence of $\omega$

↳ this is actually the influence of $\frac{t_1}{T}$ ← period of the structure.

(2) What happens when $\frac{t_1}{T}$ = integer.

(3) What is the "worst" $\frac{t_1}{T}$ ratio?

↳ One that stops the pulse when the structure is at it's peak displacement.

(4) Any ideas for alternate solutions??