# LECTURE 5: Proportional Damping, Damped Free Vibration Response of MDOF Systems

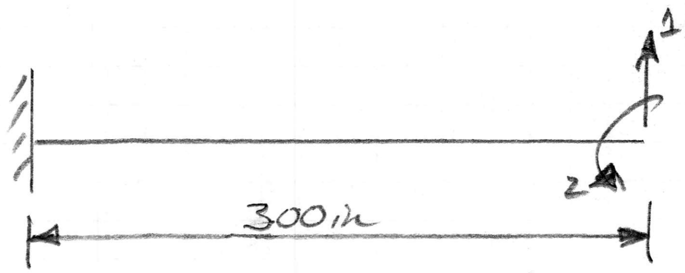
=> In class example...

For the following undamped 2DOF system:

(1) Compute the free vibration for the prescribed initial conditions.

(2) show that if the initial conditions are proportional to one of the modes, the only that mode participates.

(3) Compare the time history to the obtained by modeling the structure as an SDOF.




$E = 29{,}000\ ksi$

$I = 1000\ in^4$

$\bar{m} = \frac{20}{386}\frac{(490)}{12^3}$

$\bar{m} = 0.0146\ ^{lbm}/_{in}$

$$u_0 = \begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} in \quad \dot{u}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} {}^{in}/_{s}$$

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} = \begin{bmatrix} 12{,}888 & -1{,}933{,}380 \\ -1{,}933{,}380 & 386.68\times10^6 \end{bmatrix}$$

$$[M] = \frac{\bar{m}L}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix} = \begin{bmatrix} 1.627 & -68.84 \\ -68.84 & 3{,}754.8 \end{bmatrix}$$

=> Note: SDOF $\quad k = \frac{3EI}{L^3} = 3222\ ^{lb}/_{in}$

$$m = \frac{33}{140}\bar{m}L = 1.032\ lbm$$

$$\omega = \sqrt{\frac{3222}{1.032}} = 55.88\ ^{rad}/_{s}$$

→ Classical Damping

→ The damping matrix is also diagonalized by pre- and post- multiplying by the mode shape array.

⇒ 4 Methods.

(1) Mass proportional

(2) Stiffness proportional

(3) Rayleigh damping

(4) Superposition of Modal Damping Matricies

→ Mass Proportional Damping

→ Since we know $\phi_N$ is orthogonal w.r.t. [M], we set [C] equal to [M] times a constant

$$\underset{\substack{\uparrow \\ \text{damping matrix}}}{[C]}$$

$$[C] = \alpha_0 [M]$$

$$\phi_j^T [C] \phi_i = \alpha_0 \phi_j^T [M] \phi_i = 0$$

⇒ Orthogonality condition is preserved.

$$[C^*] = [\phi_N]^T [C] [\phi_N] = \alpha_0 \underbrace{[\phi_N]^T [M] [\phi_N]}_{\text{Identity matrix} = [I]}$$

$$[C^*] = \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $C_1^* = C_2^* = C_3^* \ldots = C_N^* = \alpha_0$

$\Rightarrow$ Choose $\alpha_0$ to give a specific damping ratio for a specific mode.

$C_i^* = \alpha_0 = 2\xi_i \omega_i$ (definition of damping ratio)

$\Rightarrow$ All other modes have

$$C_j^* = \cancel{2}\xi_i \omega_i = \cancel{2}\xi_j \omega_j$$

$$\xi_j = \frac{\omega_i}{\omega_j}\xi_i$$

inversely proportional to $\omega_j$

$\Rightarrow$ <u>Stiffness</u> <u>Proportional</u> <u>Damping</u>.

$$[C] = \alpha_1 [K]$$

$$[C^*] = [\phi_N]^T [C][\phi_N] = \alpha_1 [\phi_N]^T [K][\phi_N]$$

$$[C^*] = \alpha_1 \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}$$

$\Rightarrow$ $C_1^* = \alpha_1 \omega_1^2$

$C_2^* = \alpha_1 \omega_2^2$

$C_3^* = \alpha_1 \omega_3^2$

⇒ Again, choose $\alpha_1$ to give a specific damping ratio for a specific mode.

$$C_i^* = \alpha_1 \omega_i^2 = 2 \zeta_i \omega_i$$

$$\alpha_1 = \frac{2 \zeta_i}{\omega_i}$$

⇒ All other modes have.

$$C_j^* = \frac{2\zeta_i}{\omega_i} \omega_j^2 = 2 \zeta_j \omega_j$$

$$\zeta_j = \zeta_i \frac{\omega_j}{\omega_i}$$

↑ proportional to $\omega_j$

⇒ Eg. 2 DOF w/ $\omega = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix}$ rad/s

⇒ Chose $\zeta_1 = 0.05$ (5% damping in mode 1)

⇒ Compute damping for mode 2.

(1) Mass proportional damping

$$C_1^* = \alpha_0 = 2 \overset{0.05}{\cancel{\zeta_1}} \overset{2}{\cancel{\omega_1}}$$

$\alpha_0 = 0.2$ (multiply this by [M] to obtain [C] in physical coordinates)

$$\zeta_2 = \frac{\omega_1}{\omega_2} \zeta_1 = \frac{2}{12}(0.05) = 0.0083$$

↑ less than 1%...

⇒ Not realistic

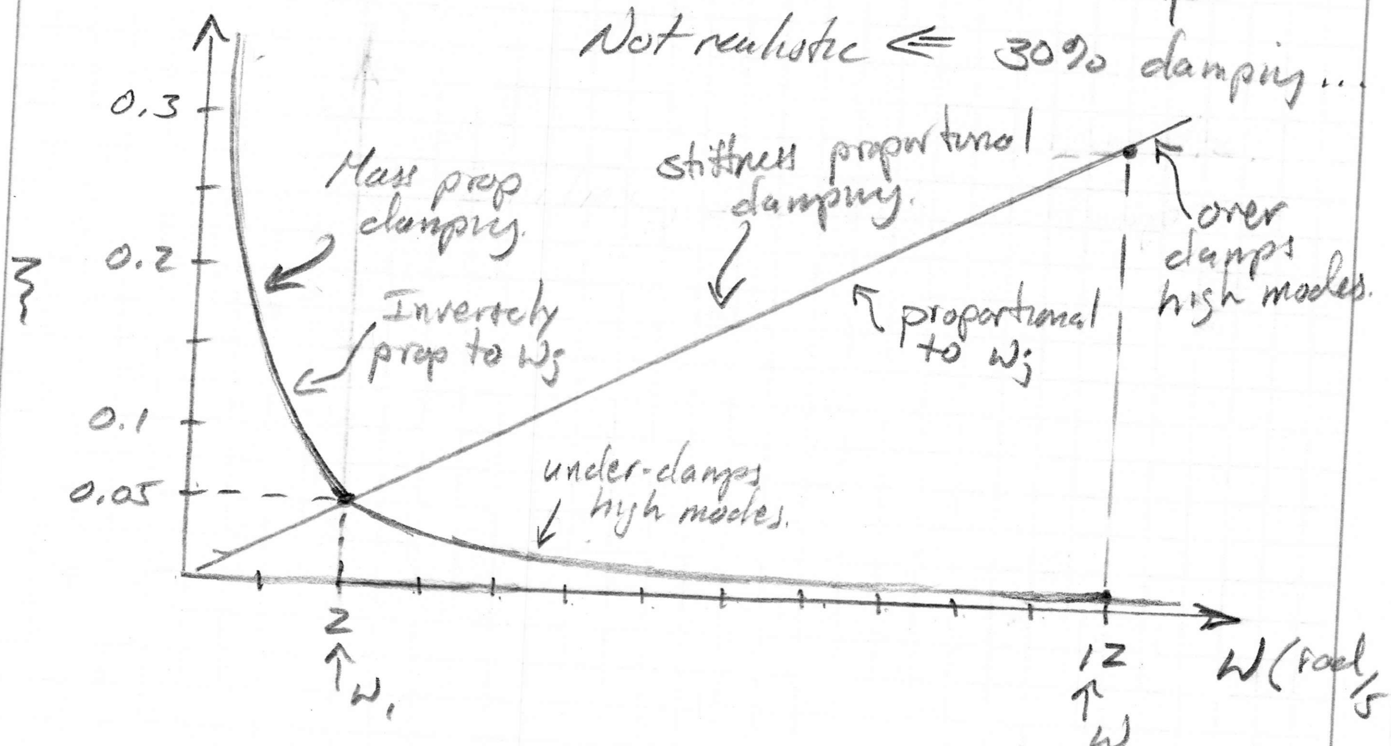
(2) Stiffness Proportional Damping.

$$\alpha_1 = \frac{2\xi_1}{\omega_1} = \frac{2(0.05)}{2} = 0.05$$

(multiply this by [k] to obtain [c] in physical coord

$$\xi_2 = \xi_1 \frac{\omega_2}{\omega_1} = 0.05\left(\frac{12}{2}\right) = 0.3$$

Not realistic ⇐ 30% damping...




⇒ Niether one is realistic except for closely spaced modes

↳ Experimental testing indicates structures exhibit similar levels of damping across their modes of interest.

→ Rayleigh Damping

$$[C] = \underbrace{\alpha_0 [M] + \alpha_1 [k]}_{\text{both stiffness and mass proportionate..}}$$

⇒ As before

$$[C^*] = \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}$$

$$C_1^* = \alpha_0 + \alpha_1 \omega_1^2 = 2\xi_1 \omega_1$$
$$C_2^* = \alpha_0 + \alpha_1 \omega_2^2 = 2\xi_2 \omega_2$$
$$C_3^* = \alpha_0 + \alpha_1 \omega_3^2 = 2\xi_3 \omega_3$$

⇒ Choose $\alpha_0$ & $\alpha_1$ to give a specific damping ratio for **two** modes

⇒ E.g. 3 DOF w/ $\omega = \begin{Bmatrix} 2 \\ 12 \\ 14 \end{Bmatrix}$ rad/sec

⇒ Choose $\underbrace{\xi_1 = \xi_3}_{\text{typically choose the highest and lowest modes of interest}} = 0.05$

you'll see why shortly

2 equations, 2 unknowns. $\begin{cases} \alpha_0 + \alpha_1 (2)^2 = 2(0.05)(2) \\ \alpha_0 + \alpha_1 (14)^2 = 2(0.05)(14) \end{cases}$

$$(\alpha_0 + 4\alpha_1 = 0.2) * -1$$

$$\frac{+ \quad \alpha_0 + 196\alpha_1 = 1.4}{192\alpha_1 = 1.2}$$

(Multiply these by [K] & [M] to obtain [C] in physical coord.)

$$\left\{ \begin{array}{l} \alpha_1 = 0.00612 \\ \alpha_0 = 0.1755 \end{array} \right.$$

=> Compute $\zeta_2$

$$0.1755 + 0.00612(12)^2 = 2\zeta_2(12)$$

$$\zeta_2 = 0.044$$

↑ ~4.5% => reasonable.