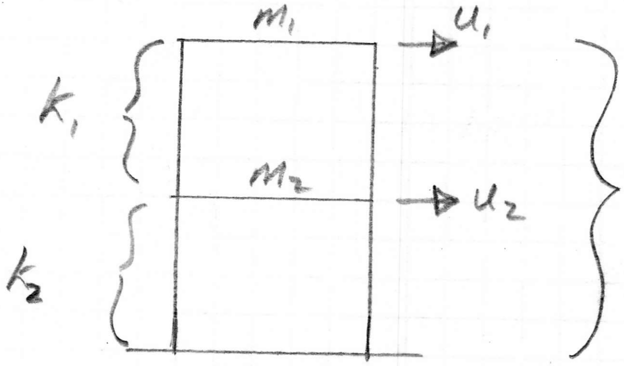
LECTURE 4: Normal Mode Method & Free Vibration of MDOF Systems



Idealized 2DOF shear building

$k_1 = k_2 = 100$ lb/in

$m_1 = m_2 = 0.5\ lb_m$

$$[K] = \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} \text{lb/in} \qquad [M] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$\Rightarrow$ Recall from lecture 1

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

$\Rightarrow$ Assume

$$u_i = \underbrace{\begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix}}_{\text{shape of vibration @ frequency } \omega_i \text{ (Mode Shape)}} \sin(\overbrace{\omega_i}^{\text{frequency of vibration}} t)$$

In general i = # DOF, i.e. a 2DOF system will have two frequencies and two mode shapes.

$\Rightarrow$ Integrating twice..

$$\ddot{u}_i = -\omega_i^2 \underbrace{\begin{Bmatrix} u_{1,i} \\ u_{2,i} \end{Bmatrix}}_{\text{same mode shape}} \sin(\omega_i t)$$

$\Rightarrow$ Subst. into EOM.

$$-\omega_i^2 [M]\{\phi_i\}\cancel{\sin(\omega_i t)} + [k]\{\phi_i\}\cancel{\sin(\omega_i t)} = 0$$

$$\{[k] - \omega_i^2 [M]\}\{\phi_i\} = 0$$

$\hookrightarrow$ Eigen problem we solved previously

$\rightarrow$ Let's look closer at this expression

Mode i $\Rightarrow$ $$[k]\{\phi_i\} = \omega_i^2 [M]\{\phi_i\} \quad (1)$$

Mode j $\Rightarrow$ $$[k]\{\phi_j\} = \omega_j^2 [M]\{\phi_j\} \quad (2)$$

$\Rightarrow$ Premultiply Eq 1 by $\{\phi_j\}^T$

$$\{\phi_j\}^T [k]\{\phi_i\} = \omega_i^2 \{\phi_j\}^T [M]\{\phi_i\} \quad (3)$$

$\Rightarrow$ Take the transpose of Eq 3

$$\{\phi_i\}^T [k]\{\phi_j\} = \omega_i^2 \{\phi_i\}^T [M]\{\phi_j\} \quad (4)$$

$\ast \Rightarrow$ Note: $\underbrace{[k] = [k]^T}_{\text{Symm}}$ & $\underbrace{[M] = [M]^T}_{\text{Symm}}$

$\Rightarrow$ Premultiply Eq 2 by $\{\phi_i\}^T$

$$\{\phi_i\}^T [k]\{\phi_j\} = \omega_j^2 \{\phi_i\}^T [M]\{\phi_j\} \quad (5)$$

$\Rightarrow$ Subtract Eq 5 from Eq. 4

$$(\omega_i^2 - \omega_j^2)\{\phi_i\}^T [M]\{\phi_j\} = 0 \quad (6)$$

for $\omega_i^2 \neq \omega_j^2$ (i.e. no repeated roots)

$$* \quad \{\phi_i\}^T[M]\{\phi_j\} = 0 \qquad i \neq j \qquad (7)$$

→ The eigen vectors (mode shapes) are said to be orthogonal w.r.t. the mass matrix

→ Subst. Eq 7 into Eq 4

$$* \quad \{\phi_i\}^T[K]\{\phi_j\} = 0 \qquad i \neq j \qquad (8)$$

⇒ Let's check to see if this is true...

⇒ Back to example.

$$a = (0.5)(0.5) = 0.25$$

$$b = -(100)(0.5) - (200)(0.5) = -150$$

$$c = (100)(200) - (100)(100) = 10{,}000$$

$$\omega_i^2 = \frac{150 \pm \sqrt{150^2 - 4(0.25)(10{,}000)}}{2(0.25)}$$

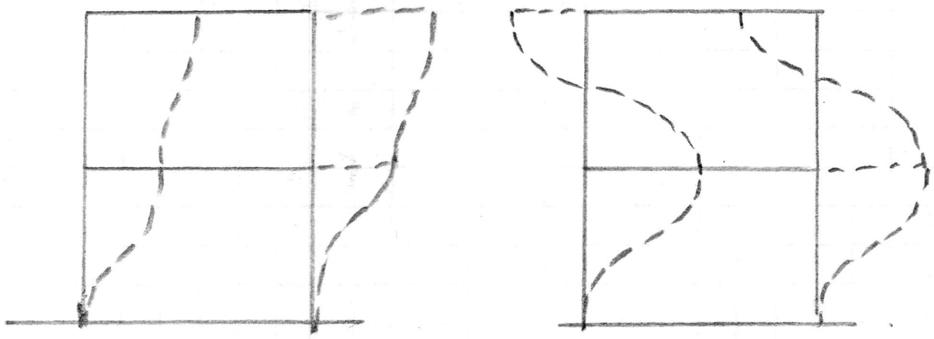
$$\omega_i^2 = 300 \pm 223.61$$

$$\omega = \begin{Bmatrix} 8.74 \\ 22.88 \end{Bmatrix} \text{ rad/s}$$

$$r_1 = \frac{100 + (76.39)(0)}{100 - (76.39)(0.5)} = 1.618$$

$$r_2 = \frac{100 + (523.5)(0)}{100 - (523.5)(0.5)} = -0.618$$

$$\phi_1 = \begin{Bmatrix} 1.0 \\ 0.618 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -0.618 \\ 1.0 \end{Bmatrix}$$ (Unit normalized)



$i=1,\ j=2$

$\phi_1^T [M]$

$$\{1.0 \quad 0.618\} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \{0.5 \quad 0.309\}$$

$\phi_1^T [M] \phi_2$

$$\{0.5 \quad 0.309\} \begin{Bmatrix} -0.618 \\ 1.0 \end{Bmatrix} = 0 \quad \checkmark$$

$\phi_1^T [k]$

$$\{1.0 \quad 0.618\} \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} = \{38.2 \quad 23.6\}$$

$\phi_1^T [k] \phi_2$

$$\{38.2 \quad 23.6\} \begin{Bmatrix} -0.618 \\ 1.0 \end{Bmatrix} = 0 \quad \checkmark$$

What does this mean??

* => Mode shapes can de-couple the stiffness & mass matricies
* => This essentially transforms a 2DOF system into two SDOF systems.

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} 1.0 & -0.618 \\ 0.618 & 1.0 \end{bmatrix}$$

$\phi^T [M]$

$$\begin{bmatrix} 1.0 & 0.618 \\ -0.618 & 1.0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.309 \\ -0.309 & 0.5 \end{bmatrix}$$

$\phi^T [M] \phi$

Note, these are rarely the same...

$$\begin{bmatrix} 0.5 & 0.309 \\ -0.309 & 0.5 \end{bmatrix} \begin{bmatrix} 1.0 & -0.618 \\ 0.618 & 1.0 \end{bmatrix} = \begin{bmatrix} 0.691 & 0 \\ 0 & 0.691 \end{bmatrix}$$

zero because of orthogonality.

$\phi^T [k]$

$$\begin{bmatrix} 1.0 & 0.618 \\ -0.618 & 1.0 \end{bmatrix} \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} = \begin{bmatrix} 38.2 & 23.6 \\ -161.8 & 261.8 \end{bmatrix}$$

$\phi^T [k] \phi$

$$\begin{bmatrix} 38.2 & 23.6 \\ -161.8 & 261.8 \end{bmatrix} \begin{bmatrix} 1.0 & -0.618 \\ 0.618 & 1.0 \end{bmatrix} = \begin{bmatrix} 52.78 & 0 \\ 0 & 361.8 \end{bmatrix}$$

⇒ This transforms the problem into "Normal" coordinates (aka "Modal" coordinates)

⇒ Somewhat (kind of, sort of) analogous to transforming stresses into principal stresses

↳ this remains in physical coordinates though...

re-writing the EOM in Normal Coordinates

uncoupled ⇒ 2 SDOF systems

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Note: z_1 & z_2 are referred to as "normal" or "modal" coordinates

Where,

M_1 = mass associated w/ mode 1

K_1 = stiffness " " " "

z_1 = generalized coordinate associated w/ mode 1 (where ϕ_1 is the associated shape function).

M_2, K_2, z_2 are associated w/ mode 2.
↳ ϕ_2 is the associated shape function.

Multiplying the EOM out...

$$M_1 \ddot{z}_1 + K_1 z_1 = 0 \qquad \omega_1 = \sqrt{\frac{K_1}{m_1}} = \sqrt{\frac{52.18}{0.691}} = 8.69 \ \tfrac{rad}{s}$$

$$M_2 \ddot{z}_2 + K_2 z_2 = 0 \qquad \omega_2 = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{361.8}{0.691}} = 22.88 \ \tfrac{rad}{s}$$

⇒ Mode Normalization

→ Thus far, we've been using unit normalized modes, where the maximum amplitude is set to 1.0.

→ Since we can only know the shape of the mode, we can scale them in any way we wish.

→ Mass Normalized Modes.

$$\phi_{N_1} = \frac{\phi_1}{\sqrt{M_1}}$$

→ For the example problem.

$$\phi_{N_1} = \frac{1}{\sqrt{0.691}} \begin{Bmatrix} 1.0 \\ 0.618 \end{Bmatrix} = \begin{Bmatrix} 1.203 \\ 0.743 \end{Bmatrix}$$

$$\phi_{N_2} = \frac{1}{\sqrt{0.691}} \begin{Bmatrix} -0.618 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} -0.743 \\ 1.203 \end{Bmatrix}$$

Again, in most cases $M_1 \neq M_2$

→ Using mass normalized modes gives.

$$\phi_N^T [M] \phi_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

$$\phi_N^T [k] \phi_N = \underbrace{\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}}_{\text{this is known as the spectral matrix}} = \Lambda$$

→ <u>Normal Mode Method for Free Vibration</u>
(Mode superposition)

→ General approach.

(1) Transform the problem from physical coordinates to modal or normal coordinates.

(2) Solve the resulting uncoupled eqn as a series of SDOF systems.

(3) Transform the SDOF responses back to physical coordinates and sum them (using super-position).

→ Recall from the derivation of element stiffness & mass matricies.

$$u(x,t) = \psi_1(x)\, u_1(t) + \psi_2(x)\, u_2(t) \ldots$$

→ Similarly,

second mode shape.

$$\{u(t)\} = \phi_1 z_1(t) + \phi_2 z_2(t) + \ldots$$

1st mode shape — response of mode 1 — response of mode 2

→ More explicity...

$$\{u\} = [\phi_N]\{z\}$$

response @ all dofs. — mode shape matrix (Mass normalized) — responses of each mode.

$$[\phi_N]^T[M][\phi_N] + [\phi_N^T][K][\phi_N] = 0$$

↳ Produces 'n' uncoupled equations or 'n' SDOF systems.

Initial disp in physical coord.

$$\{u_0\} = [\phi_N]\{z_0\}$$

Inital disp in modal coord.

or

$$\{z_0\} = [\phi_N]^T[M]\{u_0\}$$

Initial vel in physical coord.

$$\{\dot{u}_0\} = [\Phi_N]\{\dot{z}_0\}$$

Initial vel in modal coord.

or

$$\{\dot{z}_0\} = [\Phi_N]^T[M]\{\dot{u}_0\}$$

→ Back to example...

→ Solve for the free vibration response of the 2 DOF System subjected to the following initial cond.

$$u_0 = \begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} \text{in} \qquad \dot{u}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

→ Bring I.C. into Modal Coordinates.

$$z_0 = [\Phi_N]^T[M]\{u_0\} = \begin{bmatrix} 1.203 & 0.743 \\ -0.743 & 1.203 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix}$$

$$z_0 = \begin{bmatrix} 0.6015 & 0.3715 \\ -0.3715 & 0.6105 \end{bmatrix}\begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix}$$

$$z_0 = \begin{Bmatrix} 0.6015 \\ -0.3715 \end{Bmatrix} \text{in}$$

initial disp of mode 1

initial disp of mode 2

Check (Aside)

Mode 1	+	Mode 2	=	Total

$$0.6015\begin{Bmatrix} 1.203 \\ 0.743 \end{Bmatrix} = \begin{Bmatrix} 0.724 \\ 0.447 \end{Bmatrix}\text{in} + -0.3715\begin{Bmatrix} -0.743 \\ 1.203 \end{Bmatrix} = \begin{Bmatrix} 0.276 \\ -0.447 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} ✓$$

⇒ Since $\dot{u}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ → No velocity in physical coord.

$\dot{z}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ → No velocity in Modal coord.

⇒ Recall from CIVE 801, Lecture, the solution for an undamped SDOF system subjected to u_0 & $\dot{u}_0$ is.

$$u(t) = u_0 \cos(\omega t) + \frac{V_0}{\omega} \sin(\omega t)$$

⇒ Mode 1

$$z_1 = \overset{z_{10}}{0.6015} \cos(\overset{\omega_1}{8.69}\, t)$$

⇒ Mode 2

$$z_2 = \overset{z_{20}}{-0.3715} \cos(\overset{\omega_2}{22.88}\, t)$$

⇒ Using superposition.

$$u = \phi_{N_1} z_1 + \phi_{N_2} z_2$$

$$u = \begin{Bmatrix} 0.724 \\ 0.447 \end{Bmatrix} \cos(8.69t) + \begin{Bmatrix} 0.276 \\ -0.447 \end{Bmatrix} \cos(22.88t)$$