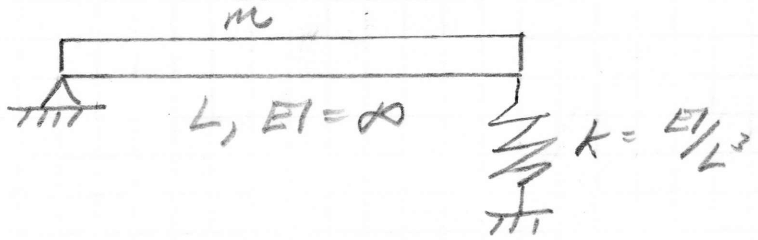
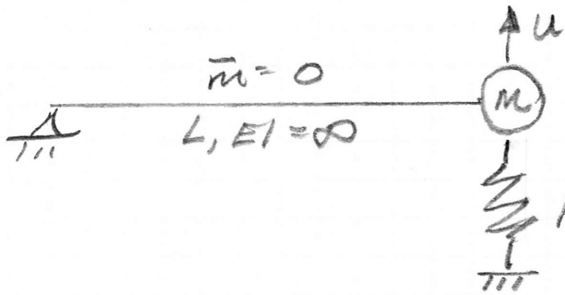
LECTURE 3: Dynamic modeling of structures w/ distributed mass.

→ Mass only associated w/ rigid members.



→ simple approach ⇒ lump mass.



$m = ? = \dfrac{\bar{m}L}{2}$ } ½ the mass of the bar.

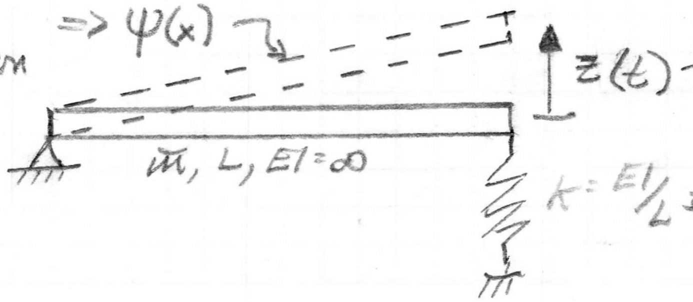
$$\omega = \sqrt{\frac{2EI}{\bar{m}L^4}} = 1.41\sqrt{\frac{EI}{\bar{m}L^4}}$$

→ How accurate is the estimate?

→ ½ $\bar{m}L$ really be lumped @ the SDOF?

→ More accurate approach ⇒ generalized coordinates.

shape function ⇒ $\psi(x)$



← choose SDOF

z = generalized coordinate.

generalized coord. → $z(t)$

shape functions → $\psi(x)$

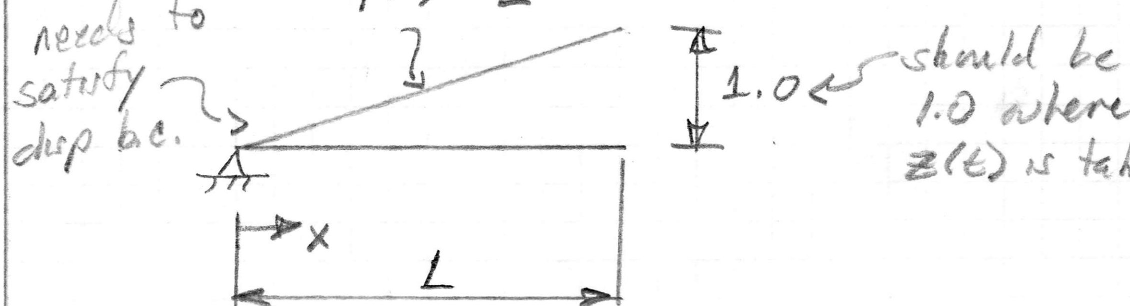
$$u(x,t) = z(t)\,\psi(x)$$

$z(t)$: only a function of time

$\psi(x)$: only a function of position

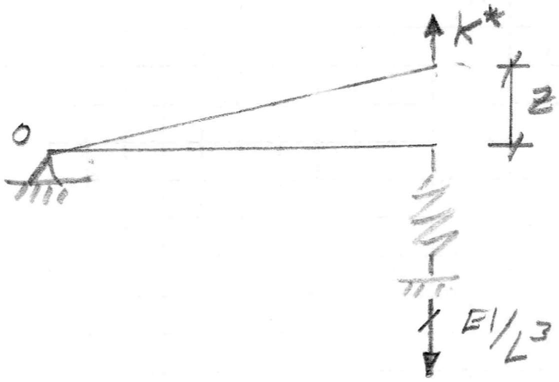
↳ e.g. the shape of deformation does not depend on time.

$$\psi(x) = \frac{x}{L}$$



→ Determine k^* ← stiffness @ general coordinate.

set $z(t) = 1.0$ solve for force $= k^*$

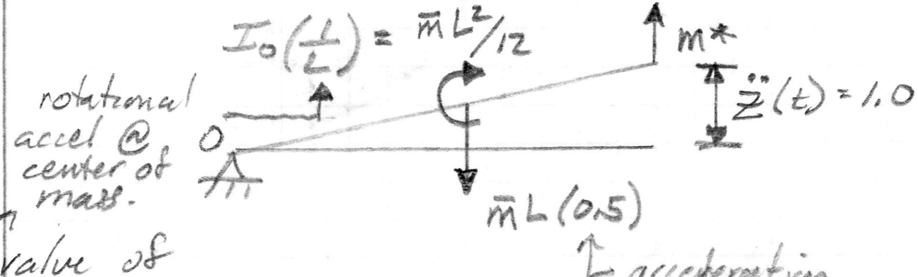


$\Sigma M_o = 0$

$\Rightarrow k^* = \dfrac{EI}{L^3}$

→ Determine m^* ← mass @ general coordinate.

set $\ddot{z}(t) = 1.0$ solve for force $= m^*$



value of $\frac{\partial\psi}{\partial x}$ @ $x = L/2$

acceleration @ CM. ← value of ψ @ $x = L/2$

$\circlearrowleft^{+} \Sigma M_0 = 0$

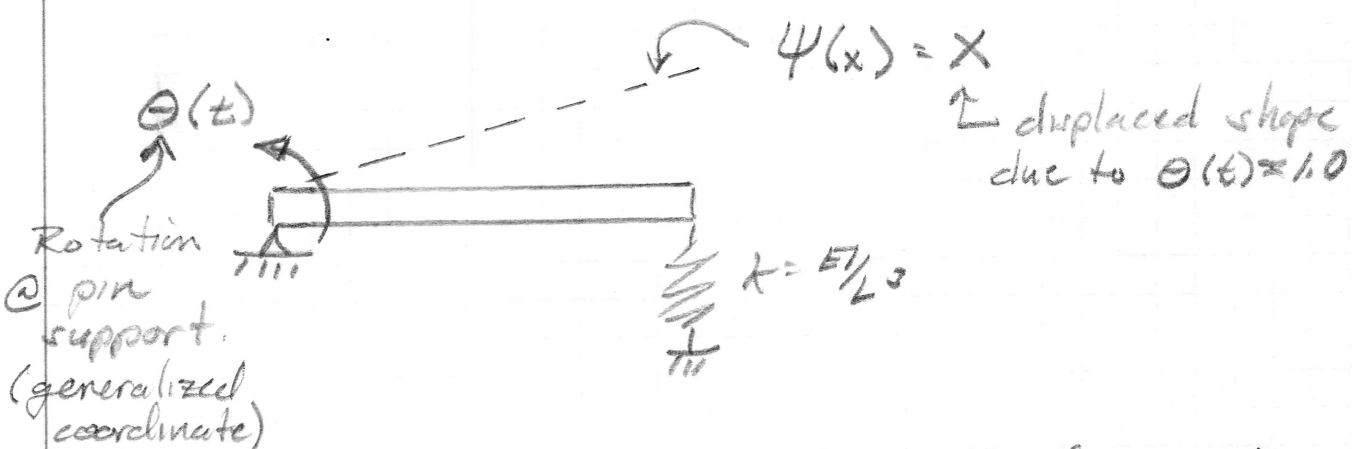
$$-\frac{\bar{m}L^2}{12} - \frac{\bar{m}L}{2}\left(\frac{L}{2}\right) + m^* L = 0$$

$\Rightarrow m^* = \frac{\bar{m}L}{3}$ as opposed to $\frac{\bar{m}L}{2}$ (lumped mass)

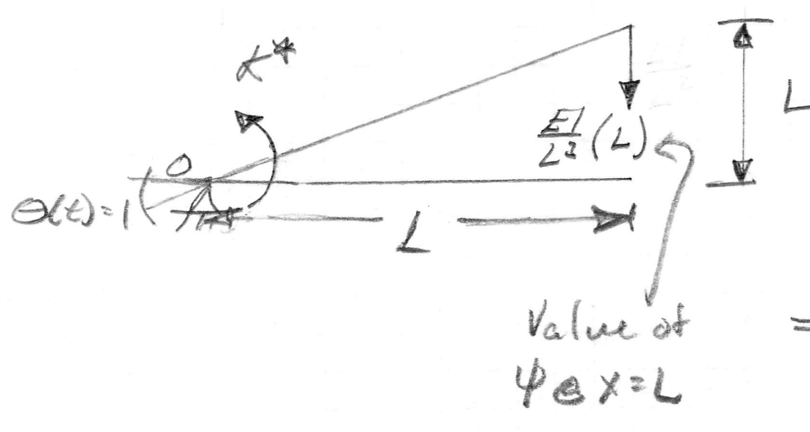
$$\omega = \sqrt{\frac{EI}{\bar{m}L^4/3}} = 1.73\sqrt{\frac{EI}{\bar{m}L^4}}$$

22% larger than lumped mass case.

→ Alternative Solution



→ Impose $\theta(t) = 1.0$ solve for force $= k^*$

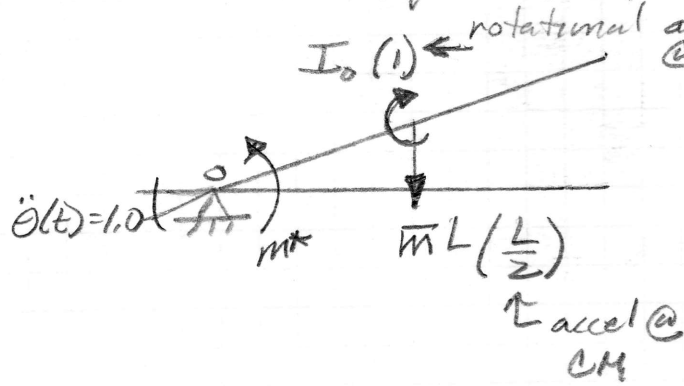


$\circlearrowleft^{+} \Sigma M_0 = 0$

$$k^* - \frac{EI}{L^2}(L) = 0$$

$$\Rightarrow k^* = \frac{EI}{L}$$

→ Impose $\ddot{\theta}(t) = 1.0$ solve for force = m^*



$\circlearrowleft^{+} \sum M_o = 0$ $\quad \frac{\bar{m}L^3}{12}$

$$m^* - I_o(1) - \frac{\bar{m}L^2}{2}\left(\frac{L}{2}\right) = 0$$

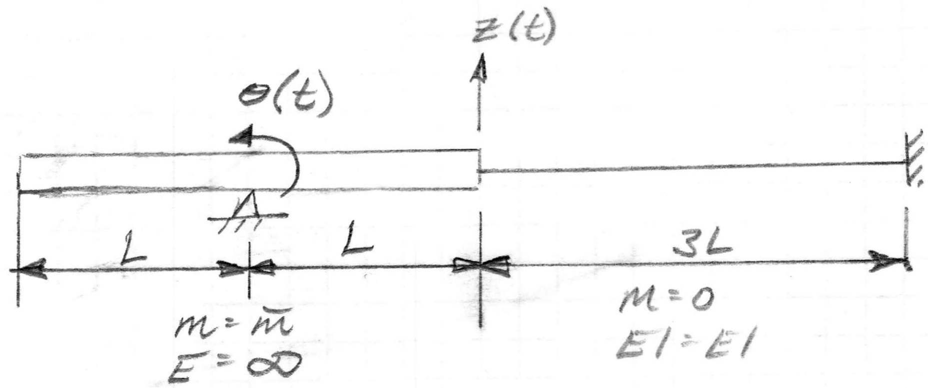
$$m^* = \frac{\bar{m}L^3}{3}$$

$$\omega = \sqrt{\frac{EI}{\bar{m}L^3/3}} = 1.73\sqrt{\frac{EI}{\bar{m}L^4}}$$

same as using $z(t)$

<u>Note</u>: In general, the selection of the SDOF can be quite influential; however, in this case it is not because the shape function was accurately known. In general, this is not the case.

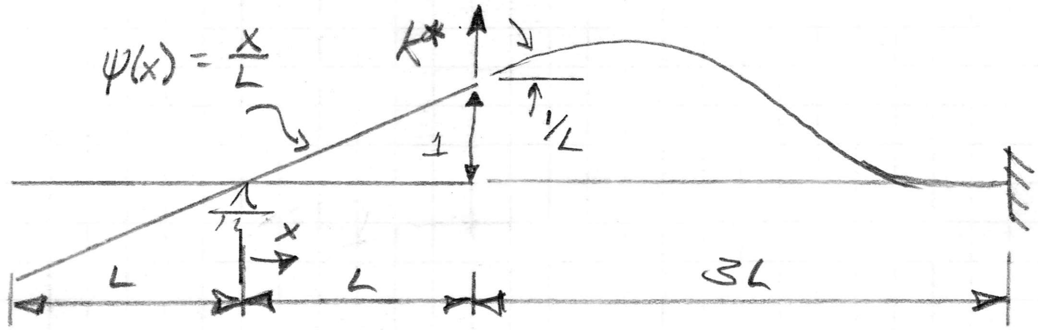
→ In Class Example

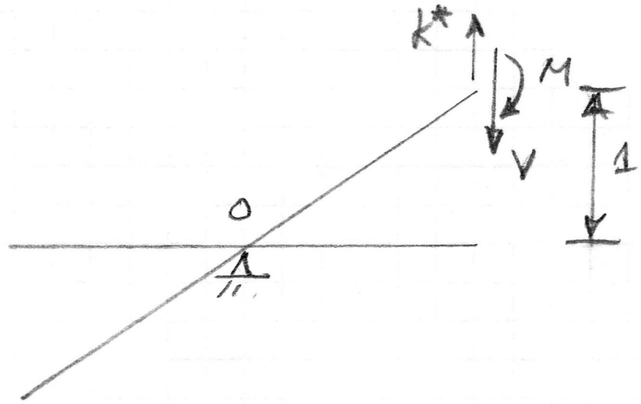


→ Estimate ω using $z(t)$ & $\theta(t)$ as the generalized coordinate.

→ Use $Z(t)$

→ solve for k^*



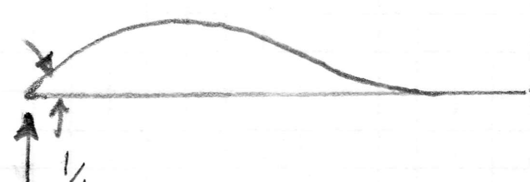


$\circlearrowleft^{+} \Sigma M_O$

$$k^* L - VL - M = 0$$

$$k^* = V + \frac{M}{L}$$

$$M_1 = \frac{4EI}{(3L)}\left(\frac{1}{L}\right)$$



$$V_1 = \frac{6EI}{(3L)^2}\left(\frac{1}{L}\right)$$

+

$$M_2 = \frac{6EI}{(3L)^2}$$

$$V_2 = \frac{12EI}{(3L)^3}$$

be careful with signs.
→ positive is opposite the signs for M & V above.

=

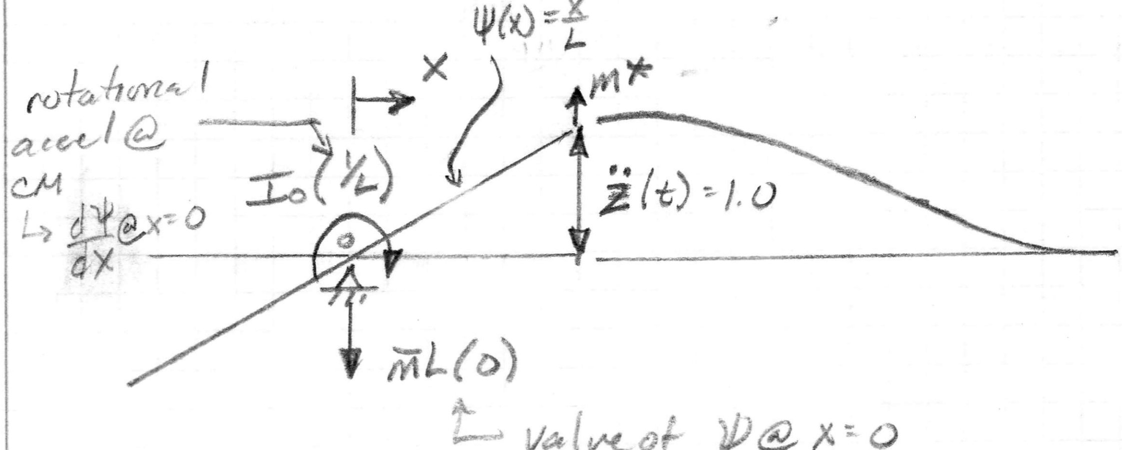
$$M = \frac{4EI}{3L^2} + \frac{6EI}{9L^2} = \frac{2EI}{L^2}$$

$$V = \frac{6EI}{9L^3} + \frac{12EI}{27L^3} = \frac{30EI}{27L^3} = \frac{10EI}{9L^3}$$

$$k^* = \frac{18EI}{9L^3} + \frac{10EI}{9L^3}$$

$$\Rightarrow k^* = \frac{28EI}{9L^3}$$

→ Solve for m^*



$\circlearrowleft + \Sigma M_0 = 0$

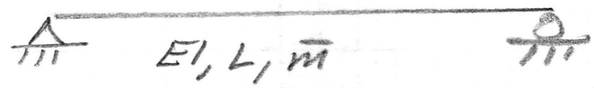
$$-I_0\left(\frac{1}{L}\right) + m^* L = 0$$

$$\frac{\bar{m}(2L)^3}{12} = \frac{8\bar{m}L^3}{12}$$

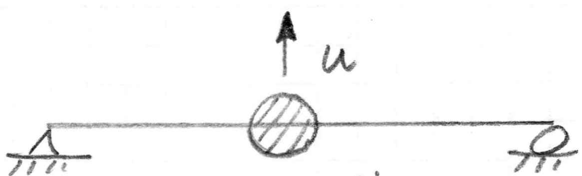
$$\Rightarrow m^* = \frac{2\bar{m}L}{3}$$

$$\omega = \sqrt{\frac{28EI}{9L^3\left(\frac{2\bar{m}L}{3}\right)}} = 2.16\sqrt{\frac{EI}{\bar{m}L^4}}$$

→ Mass associated w/ deformable members



→ Simple approach → lump mass / stiffness



$m = \frac{\bar{m}L}{2}$

$k = \frac{48EI}{L^3} \Rightarrow \omega = 9.80\sqrt{\frac{EI}{\bar{m}L^4}}$

→ How accurate is this estimate?

→ Is $\frac{1}{2}\bar{m}L$ really lumped @ mid-span?

→ Is k really associated with the stiffness due to a point load?

→ More accurate approach → generalized coordinates.

$u(x,t) = z(t)\,\psi(x)$

↑ shape function

→ For structures that have mass associated w/ deformable elements, the shape functions are rarely known precisely

→ guess shape functions

→ Rules of shape functions

(1) They must satisfy displacement boundary conditions, it is nice if the also satisfy force b.c.

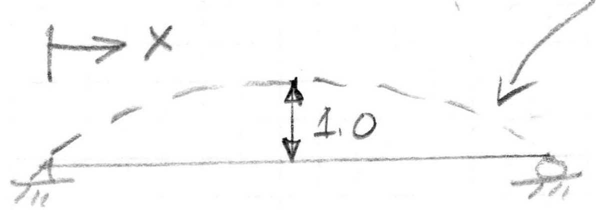
(2) Must be 1.0 @ location of the generalized coordinate.

(3) They should be as simple as possible!!!

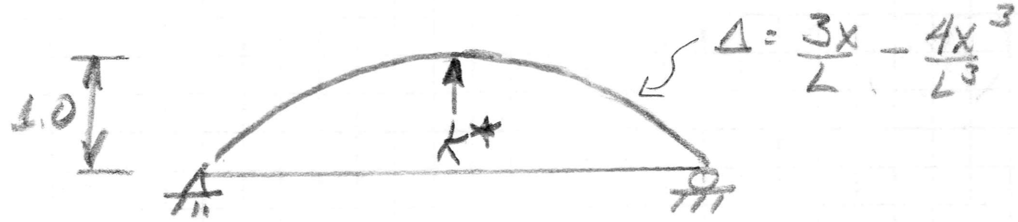
<u>Aside</u>: The shape function is a guess @ the shape of the structure as it vibrates → known as a mode shape

⇒ Back to the problem.

Assume $\psi = \frac{3x}{L} - \frac{4x^3}{L^3}$ $0 \le x \le L/2$



→ Solve for k^* ⇒ impose unit disp and solve for force = k^*



→ How can we compute k so that it is consistent with the assume deflected shape??

→ Use virtual work.

→ In the past we have used virtual forces.

Assume virtual force is so small it doesn't influence 'Δ'

e.g.

$$1\,\Delta = \int m\left(\frac{M}{EI}\right)dx$$

unit force (virtual) → 1

real external disp. of virtual force → Δ

internal virtual moment → m

internal virtual curvature → $\frac{M}{EI}$

→ For the problem at hand, the principle of virtual displacements is more convenient.

Assume virtual disp is so small it doesn't influence 'F'

$$F\,\delta z = \int M(\delta\phi)\,dx$$

externally applied force → F

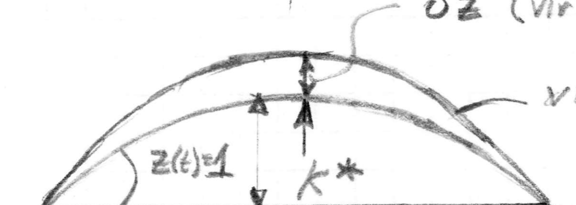
external virtual disp @ F → δz

real internal moment → M

virtual internal curvature due to δz → $\delta\phi$

For our problem...

δz (virtual disp @ external force)



virtual disp $= \underbrace{\delta z\,(\Psi(x))}_{\text{use same shape function}}$

Real deformed shape $= 1.0\,(\Psi(x))$

definition of stiffness

Recall:

$$\phi = \frac{d^2\Delta}{dx^2} = \frac{d^2\psi}{dx^2}$$

$$\phi = \frac{M}{EI} \quad \text{or} \quad M = EI\phi = EI\frac{d^2\psi}{dx^2}$$

$$\underset{\substack{\uparrow \\ \text{generalized} \\ \text{stiffness}}}{K^*}\ \underset{\substack{\uparrow \\ \text{virtual} \\ \text{disp.}}}{\cancel{\delta z}} = \int_0^L \underbrace{EI\left(\frac{d^2\psi}{dx^2}\right)}_{\text{Real moment}} \underbrace{\left[\cancel{\delta z}\left(\frac{d^2\psi}{dx^2}\right)\right]}_{\text{Virtual curvature.}} dx$$

$$\ast \Longrightarrow K^* = \int_0^L EI\left(\frac{d^2\psi}{dx^2}\right)^2 dx$$

Note: for bending ψ must be differentiable @ least twice.

$$\begin{cases} \dfrac{d\psi}{dx} = \dfrac{3}{L} - \dfrac{12x^2}{L^3} & 0 \le x \le L/2 \\ \dfrac{d^2\psi}{dx} = -\dfrac{24x}{L^3} & 0 \le x \le L/2 \end{cases}$$

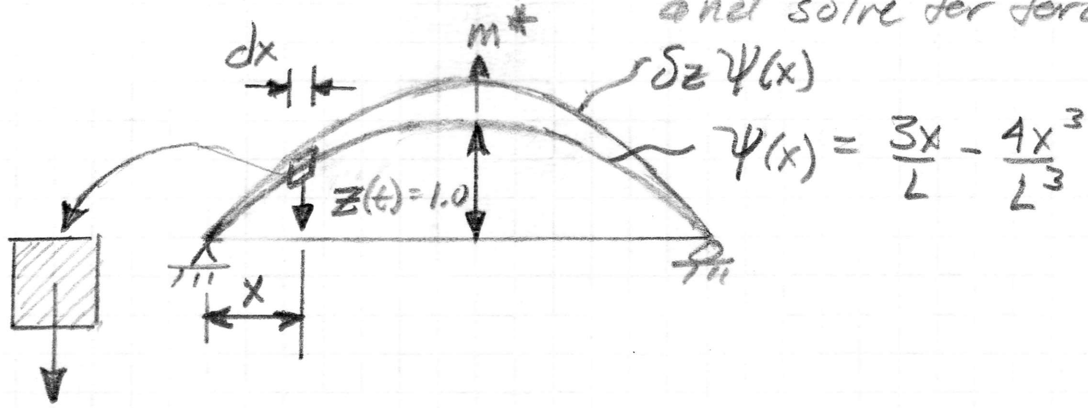
$$K^* = 2EI\int_0^{L/2} \frac{576x^2}{L^6}\,dx$$

$$K^* = 2EI\left(\frac{192\left(\frac{L^3}{8}\right)}{L^6}\right)$$

Same as before. $\longrightarrow K^* = \dfrac{48EI}{L^3}$

(simple case), why? $\Rightarrow$ I chose the deflected shape of a beam w/ a point load as my ψ

→ Solve for m^* ⇒ impose unit accel and solve for force $= m^*$



$\bar{m}\,\psi(x)\,dx$ } inertia force developed by element dx.

↑ acceleration @ x.

→ Use the principle of virtual displacements again.

$$\underset{\text{real force}}{m^*}\;\underset{\text{virtual disp.}}{\cancel{\delta z}} = \int_0^L \underbrace{\bar{m}\,\psi(x)}_{\text{real inertia force on element }dx}\overbrace{\left[\cancel{\delta z}\,\psi(x)\right]}^{\text{virtual disp.}}dx$$

$$\Longrightarrow\; m^* = \int_0^L \bar{m}\left(\psi(x)\right)^2 dx$$

$$m^* = 2\bar{m}\int_0^{L/2}\left(\frac{3x}{L} - \frac{4x^3}{L^3}\right)^2 dx$$

$$= 2\bar{m}\int_0^{L/2} \frac{9x^2}{L^2} - \frac{24x^4}{L^4} + \frac{16x^6}{L^6}\,dx$$

$$m^* = 2\bar{m}\left[\frac{9x^3}{3L^2} - \frac{24x^5}{5L^4} + \frac{16x^7}{7L^6}\Big|_0^{L/2}\right]$$

$$m^* = 2\bar{m}\left[\frac{3L}{8} - \frac{3L}{20} + \frac{L}{56}\right]$$

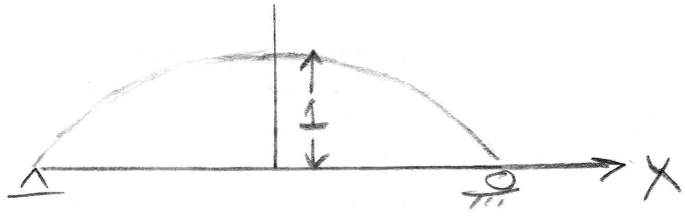
$$m^* = 2\bar{m}\left[\frac{22L}{56} - \frac{3L}{20}\right]$$

$$m^* = \frac{17}{35}\left[\bar{m}L\right] = 0.486\,\bar{m}L$$

$$\omega = \sqrt{\frac{48EI}{\frac{17}{35}\bar{m}L^4}} = 9.94\sqrt{\frac{EI}{\bar{m}L^4}}$$

only 1.4% diff.

→ What happens with a second order shape function?



$$\psi(x) = -\frac{4x^2}{L^2} + 1$$

$$\frac{d^2\psi}{dx^2} = -\frac{8}{L^2}$$

$$k^* = \int_0^{L/2} 2EI\left(\frac{64}{L^4}\right)dx$$

$$= 128EI\left(\frac{x}{L^4}\right)\Big|_0^{L/2} = \frac{64EI}{L^3} = k^* \Leftarrow$$

(~33% error)

$$m^* = 2\bar{m}\int_0^{L/2} \frac{16x^4}{L^4} - \frac{8x^2}{L^2} + 1\,dx$$

$$m^* = 2\bar{m}\left[\frac{16x^5}{5L^4} - \frac{8x^3}{3L^2} + x\right]_0^{L/2}$$

$$m^* = 2\bar{m}\left[\frac{L}{10} - \frac{L}{3} + \frac{L}{2}\right]$$

$$m^* = \frac{16}{30}\bar{m}L = 0.533\bar{m}L \quad (\sim 9\% \text{ error})$$

$$\omega = \sqrt{\frac{64EI}{\frac{16}{30}\bar{m}L^4}} = 10.95\sqrt{\frac{EI}{\bar{m}L^4}}$$

~10% diff.

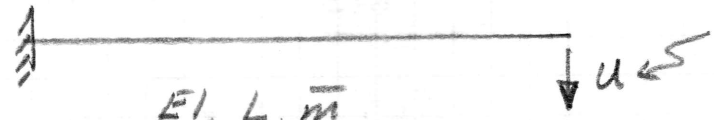
→ Why is mass estimate better than stiffness estimate? → disp vs curvature..

→ Why is 'ω' estimate not further off? ⇒ errors compensate & square root...

⇒ What happens when you use:

$$\psi(x) = \sin\left(\frac{x\pi}{L}\right)$$

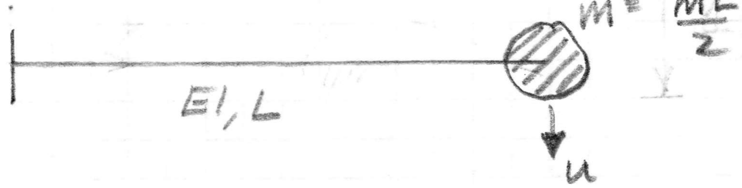
Example: Cantilever beam



choose because it undergoes the maximum disp.

→ Determine natural frequency

→ Simple approach (lump mass/stiffness)



$$k = \frac{3EI}{L^3}$$

$$\omega = \sqrt{\frac{6EI}{\bar{m}L^4}} = 2.449\sqrt{\frac{EI}{\bar{m}L^4}}$$

⇒ VA example. $\begin{pmatrix} I = 301\,in^4, & E = 29{,}000\,ksi \\ L = 240\,in, & \bar{m} = 0.00564\,\frac{lbm}{in} \end{pmatrix}$

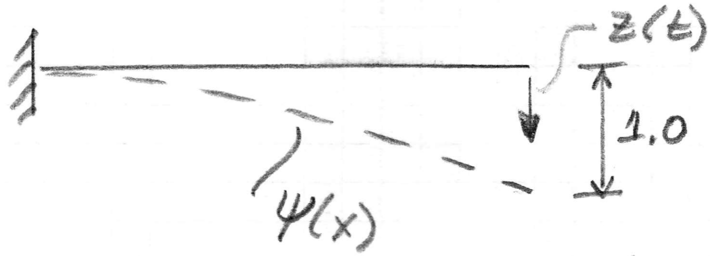
$$\hookrightarrow \frac{7.68\,in^2\left(\frac{1}{12}\right)^2\left(490\,{}^{lb}/_{ft^3}\right)\left(\frac{1}{12}\right)}{386\,{}^{in}/_{sec^2}}$$

$$\omega = \sqrt{\frac{6(29{,}000{,}000)(301)}{(0.00564)(240)^4}} = 52.90\,{}^{rad}/_{sec}$$

$$f = \frac{\omega}{2\pi} = 8.420\,Hz \quad \text{(same as VA)}$$

→ Isn't is accurate enough???

→ More accurate approach.



$u(x,t) = \psi(x) z(t)$

→ What do we know about $\psi(x)$?

$\psi(0) = 0$ (1)

$\psi'(0) = 0$ (2)

$\psi(L) = 1.0$ (3)

$\psi''(L) = 0$ → zero moment @ free end. (4)

→ Assume a polynomial

$$\psi(x) = Ax^3 + Bx^2 + Cx + D$$

→ using (1) & (2) $\Rightarrow C = D = 0$

$$\psi''(L) = 6AL + 2B = 0.0$$

$$A = -\frac{B}{3L}$$

$$\psi(L) = AL^3 + BL^2 = 1.0$$

$$-\frac{BL^2}{3} + BL^2 = 1.0$$

$$B = \frac{3}{2L^2}, \quad A = -\frac{1}{2L^3}$$

$$\Rightarrow \psi(x) = -\frac{x^3}{2L^3} + \frac{3x^2}{2L^2}$$

$$\psi''(x) = -\frac{3x}{L^3} + \frac{3}{L^2}$$

→ Compute k^*

$$k^* = \int_0^L EI\left(-\frac{3x}{L^3} + \frac{3}{L^2}\right)^2 dx$$

$$k^* = EI \int_0^L \frac{9x^2}{L^6} - \frac{18x}{L^5} + \frac{9}{L^4}\, dx$$

$$k^* = EI\left[\frac{3x^3}{L^6} - \frac{9x^2}{L^5} + \frac{9x}{L^4}\right]_0^L$$

$$k^* = \frac{3EI}{L^3}$$ → same as cantilever

↳ $\psi(x)$ = deflected shape of cantilever

→ Compute m^*

$$m^* = \bar{m}\int_0^L \underbrace{\left(\frac{x^6}{4L^6} - \frac{6x^5}{4L^5} + \frac{9x^4}{4L^4}\right)}_{(\psi(x))^2} dx$$

$$m^* = \bar{m}\left[\frac{L}{28} - \frac{L}{4} + \frac{9L}{20}\right]$$

$$m^* = \frac{33}{140}\bar{m}L$$ } roughly 23% of the mass.

→ Compute ω

$$\omega = \sqrt{\frac{3EI}{\frac{33}{140}\bar{m}L^4}} = 3.568\sqrt{\frac{EI}{\bar{m}L^4}}$$

→ Back to numerical example.

$$\omega = 3.568\sqrt{\frac{29000000(301)}{0.00564(240)^4}} = 77.05 \text{ Rad/sec}$$

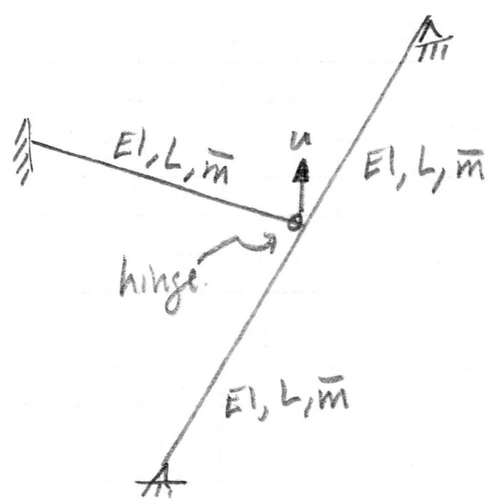
$$f = \frac{\omega}{2\pi} = 12.263 \text{ HZ}$$

1 element = 8.42 Hz
2 elements = 10.845 Hz
4 elements = 11.795 Hz
8 elements = 11.995 Hz
80 elements = 12.080 Hz
800 elements = 12.090 Hz

→ Why doesn't it converge to our hand solution?

→ Which one do you believe?

In class example,



→ Estimate the natural frequency of the plane grid.

$$k^* = \frac{3EI}{L^3} + \frac{48EI}{(2L)^3} = \frac{9EI}{L^3}$$

$$m^* = \frac{33}{140}\bar{m}L + \frac{17}{35}\bar{m}2L = \frac{169}{140}\bar{m}L$$

$$\omega = \sqrt{\frac{9EI}{\frac{169}{140}\bar{m}L^4}} = 2.730\sqrt{\frac{EI}{\bar{m}L^4}}$$

$$\omega = 2.730\sqrt{\frac{29000{,}000(301)}{0.00564(240)^4}} = 58.962$$

$$f = 9.384 \text{ Hz}$$

→ Compare w/ SAP & VA for

→ Do you see any problems.