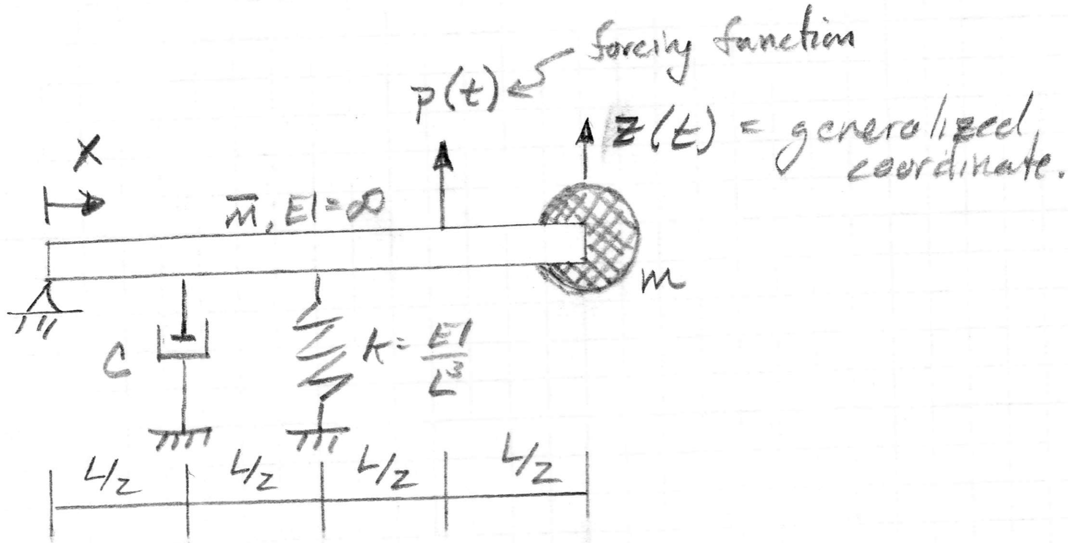
LECTURE 4: Formulation of EOM & Free Vibration

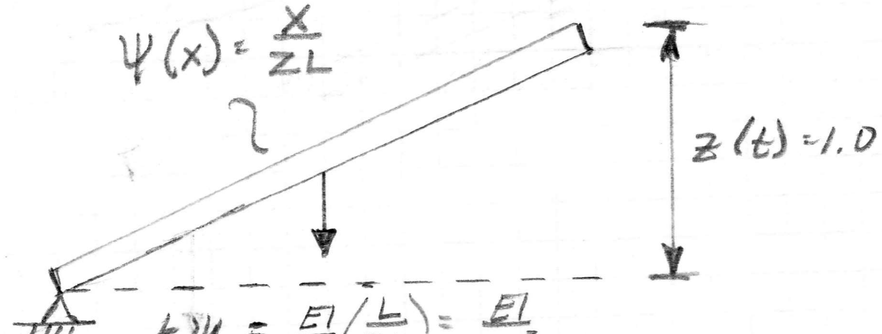
→ Formulation of the Equation of Motion (EOM)

→ Discrete Stiffness & Damping



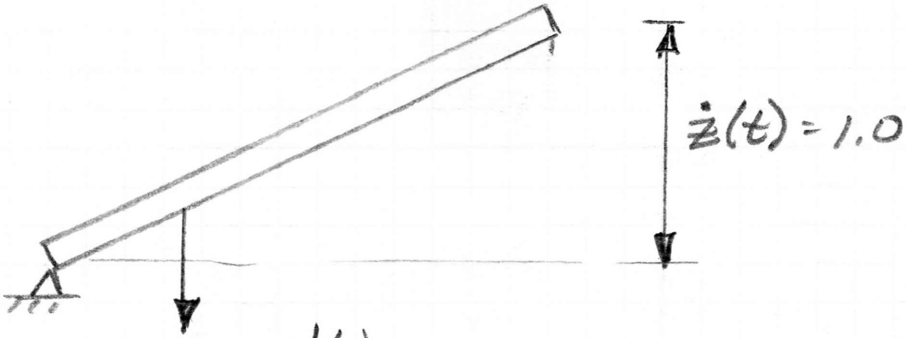
⇒ shape function $\psi(x) = \frac{x}{2L}$

→ Impose unit displacement and solve for spring forces.



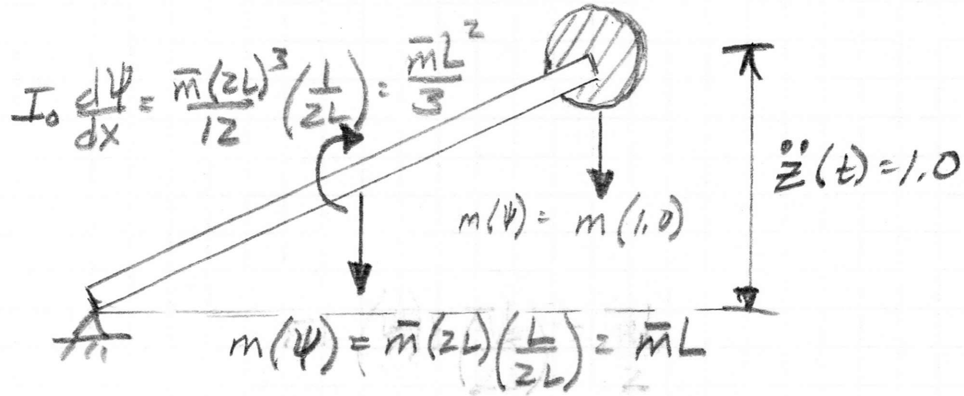
$$k\psi = \frac{EI}{L^3}\underbrace{\left(\frac{L}{2L}\right)}_{\psi(L)} = \frac{EI}{2L^3}$$

→ Impose unit velocity and solve for damping forces.

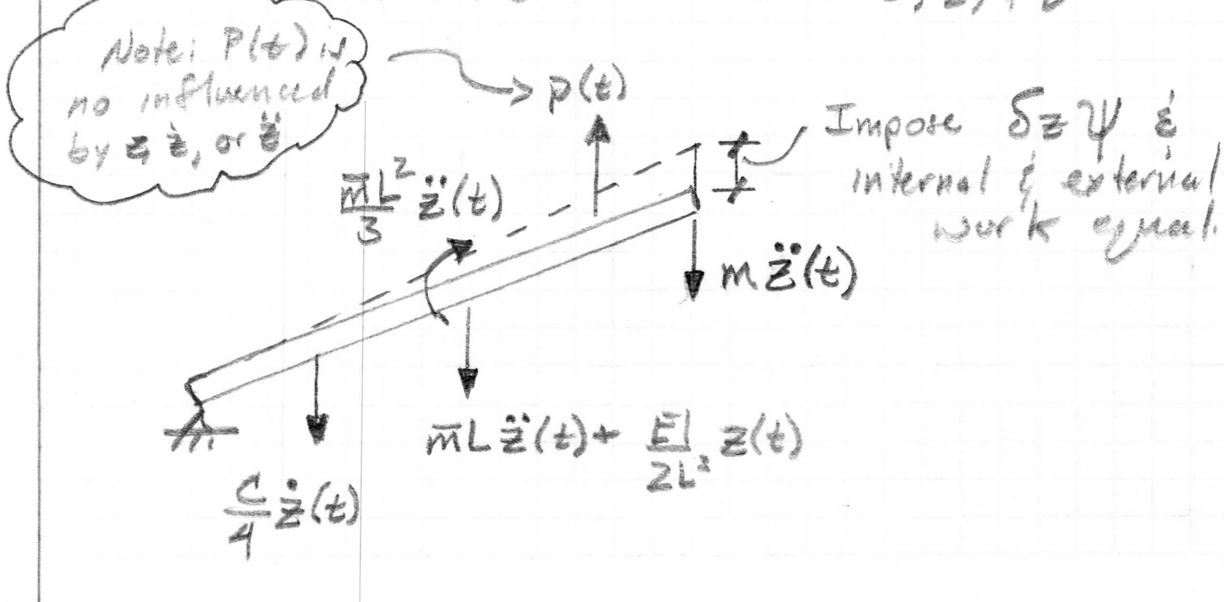


$$c\psi = c\left(\frac{L/2}{2L}\right) = \frac{c}{4}$$

→ Impose unit accel and solve for inertia forces.



→ Impose all forces on the original structure in terms of z, $\dot{z}$, & $\ddot{z}$



⇒ Balancing Internal & External virtual work

inertia forces $\left[\underbrace{\frac{\bar{m}L^2}{3}\ddot{z}(t)}_{\text{force}}\underbrace{\left[\delta z \frac{1}{2L}\right]}_{\text{virtual }\Delta} + \bar{m}L\ddot{z}(t)\left[\delta z \frac{L}{2L}\right] + m\ddot{z}(t)\left[\delta z \frac{2L}{2L}\right] + \right.$

damping forces $\left[\frac{c}{4}\dot{z}(t)\left[\delta z \frac{L/2}{2L}\right] + \right.$

spring forces $\left[\frac{EI}{2L^3}z(t)\left[\delta z \frac{L}{2L}\right] = \right.$

external forces $\left[p(t)\left[\delta z \frac{3L/2}{2L}\right] \right.$

$$\underbrace{\left[\frac{2\bar{m}L}{3} + m\right]}_{m^*}\ddot{z}(t) + \underbrace{\left[\frac{c}{16}\right]}_{c^*}\dot{z}(t) + \underbrace{\left[\frac{EI}{4L^3}\right]}_{k^*}z(t) = \underbrace{\frac{3}{4}p(t)}_{p^*}$$

distrib mass ($\frac{2\bar{m}L}{3}$) — discrete mass (m).

⇒ Summary

⇒ For discrete springs.

$$k^* = \sum_{i=1}^{n}\left[\psi(x_i)\right]^2 k_i$$

location of spring 'i'

⇒ For discrete dampers

$$c^* = \sum_{i=1}^{n}\left[\psi(x_i)\right]^2 c_i$$

location of damper 'i'

=> For discrete masses.

$$m^* = \sum_{i=1}^{n} \left[\psi(x_i)\right]^2 m_i$$

location of mass "i"

=> For distributed masses associated w/ rigid members.

$$m^* = \sum_{i=1}^{n} \left[\psi(x_i)\right]^2 \bar{m} L + \sum \left[\frac{d\psi}{dx}\bigg|_{x_i}\right]^2 I_i$$

location of center of mass

location of center of mass

mass moment of inertia

=> For discrete external loads.

$$p^* = \sum_{i=1}^{n} \left[\psi(x_i)\right] p_i(t)$$

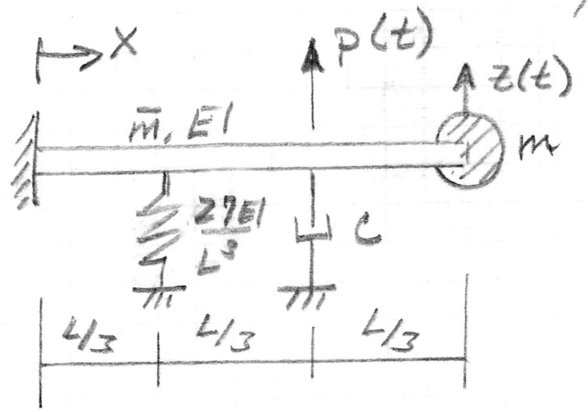
location of load.

=> For distributed loads.

$$p^* = \sum_{i=1}^{n} \int_{L_i} \psi(x) p_i(t)\, dx$$

integrate of length of applied load.

In class example. ⇒ combine w/ previous lecture...



$\Rightarrow$ use $\psi(x) = \frac{-x^3}{2L^3} + \frac{3x^2}{2L^2}$

$\psi''(x) = \frac{-3x}{L^3} + \frac{3}{L^2}$

$$K^* = \int_0^L EI\left(\frac{-3x}{L^3} + \frac{3}{L^2}\right)^2 dx + \frac{27EI}{L^3}\left(\frac{-(L/3)^3}{2L^3} + \frac{3(L/3)^2}{2L^2}\right)^2$$

$-1/54 \qquad 1/6$

$$K^* = \frac{3EI}{L^3} + \frac{27EI}{L^3}\left(\frac{4}{27}\right)^2$$

$$\Rightarrow K^* = \frac{97EI}{27L^3}$$

$-\frac{4}{27} \qquad 2/3$

$$c^* = c\left[\frac{-(2/3\,L)^3}{2L^3} + \frac{3(2/3\,L)^2}{2L^2}\right]^2 = c\left[\frac{14}{27}\right]^2$$

$$\Rightarrow c^* =$$

$$m^* = \int_0^L \bar{m}\left(\frac{-x^3}{2L^3} + \frac{3x^2}{2L^2}\right)^2 dx + m(1)^2$$

$$\Rightarrow m^* = \frac{\bar{m}L}{3} + m$$

$$p^* = p(t)\left(\frac{-(2L/3)^3}{2L^3} + \frac{3(2L/3)^2}{2L^2}\right)$$

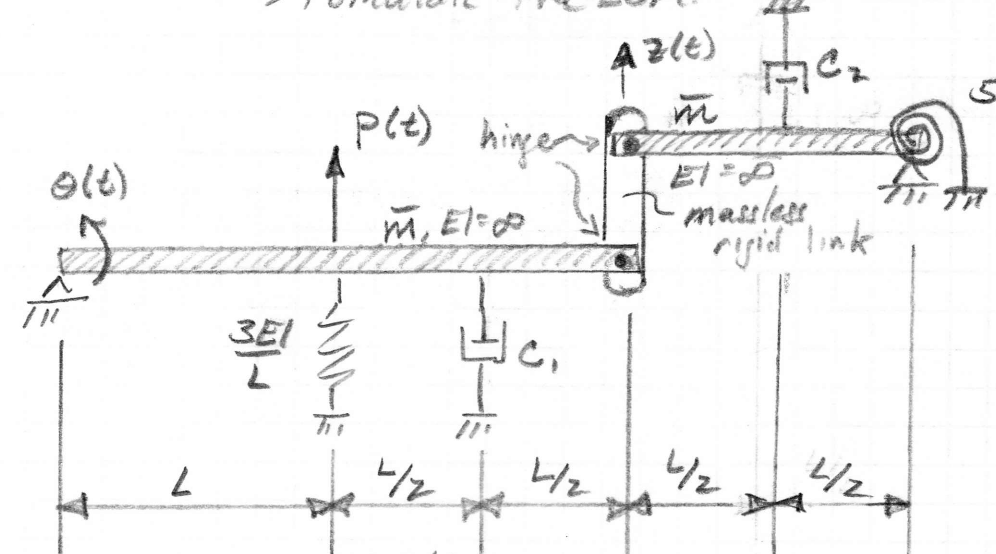
$$\Rightarrow p^* = \frac{14\,p(t)}{27}$$

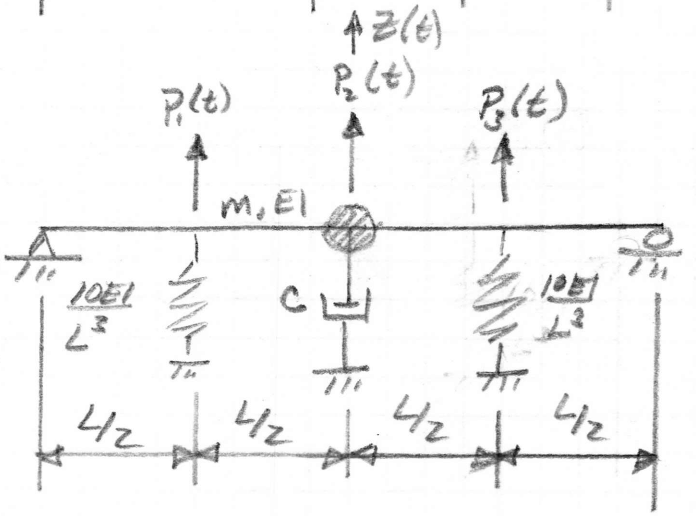
→ Total EOM:

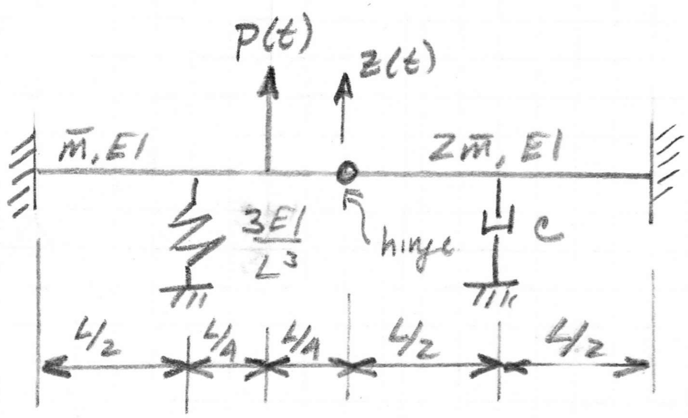
$$\left[\frac{\bar{m}L}{3}+m\right]\ddot{z}(t)+\left[\quad\right]\dot{z}(t)+\left[\frac{97EI}{27L^3}\right]z(t)=\left[\frac{14}{27}\right]p(t)$$

→ Sample HW Problems

→ Formulate the EOM.







→ <u>Free Vibration Response</u>

→ No forcing function (i.e. $p(t)=0$)

→ Vibration is caused by initial conditions.

→ <u>Undamped Systems</u> ⇒ $c=0$

$$m\ddot{u}+ku=0 \qquad \text{(EOM)}$$

⇒ assume $u = A\cos(\lambda t)+B\sin(\lambda t)$

$$\dot{u} = -A\lambda\sin(\lambda t)+B\lambda\cos(\lambda t)$$

$$\ddot{u} = -A\lambda^2\cos(\lambda t)-B\lambda^2\sin(\lambda t)$$

⇒ Sub into EOM

$$-m\lambda^2\left[A\cos(\lambda t)+B\sin(\lambda t)\right]+k\left[A\cos(\lambda t)+B\sin(\lambda t)\right]=0$$

$k-m\lambda^2=0$ ← characteristic equation

$$\lambda^2=\frac{k}{m}$$

$\lambda=\sqrt{\frac{k}{m}}=\omega$ } same frequency we obtained using energy arguments.

$$\Rightarrow u = A\cos(\omega t)+B\sin(\omega t) \quad ; \quad \omega=\sqrt{\frac{k}{m}}$$

Initial Conditions.

$u(0)=u_0$ ← initial displacement

$\dot{u}(0)=v_0$ ← initial velocity

$$u = u_0 \cos(\omega t) + \frac{V_0}{\omega} \sin(\omega t)$$

⇒ Plot using Excel

⇒ Although the response has both cosine (due to u_0) and sine (due to V_0) responses, it is only a single harmonic.

↳ this is because both components have the same frequency.

⇒ Alternative Representation.

$$u = \rho \sin(\omega t + \phi)$$

phase angle

amplitude of vibration

$$\rho = \sqrt{u_0^2 + \left(\frac{V_0}{\omega}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{u_0 \omega}{V_0}\right)$$

⇒ Using Excel show that...

(1) Both representations are identical

(2) $f = \omega / 2\pi$ (Hz)

(3) $T = 2\pi / \omega$ (sec)

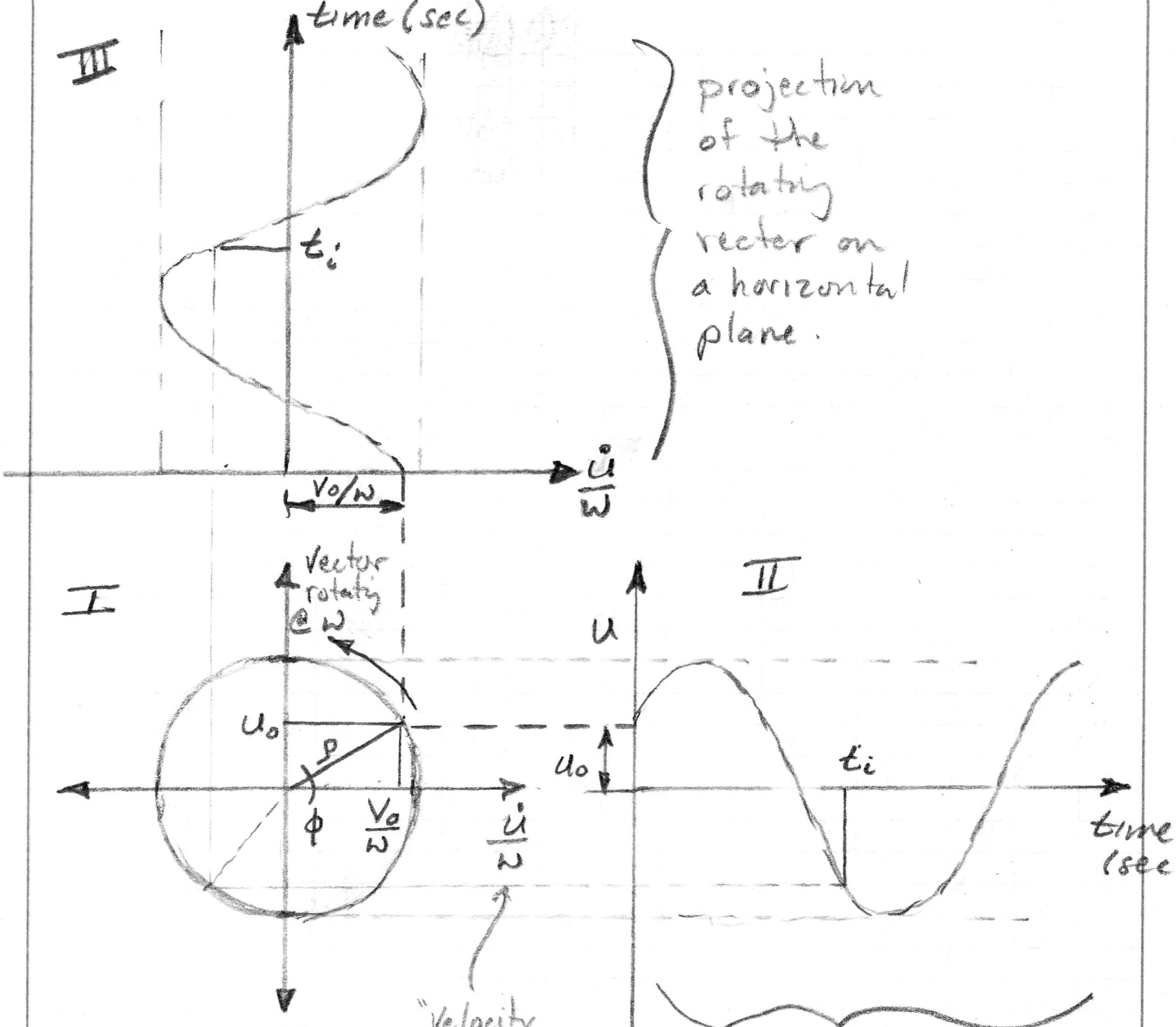
(4) Sine portion is due to initial vel.

(5) Cosine portion is due to initial disp.

(6) What happens to ϕ when $V_0 = 0$

(7) Issue w/ $\tan^{-1}$ ⇒ $\rho = \frac{|V_0|}{V} \rho$

⇒ Phase Plane Diagram



$$\rho = \sqrt{u_0^2 + \left(\frac{V_0}{\omega}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{u_0 \omega}{V_0}\right)$$

"Velocity function" ⇒ units of displacement.

projection of the rotating vecter on a vertical plane (u)

I ⇒ u vs $\dot{u}/\omega$ Vecter rotates @ ω RAD/s
↑ time

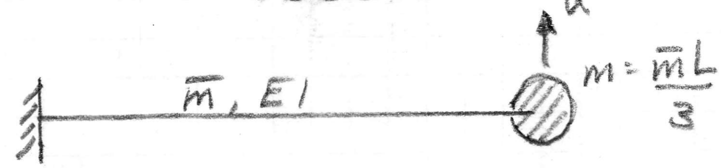
II ⇒ u vs time $\omega = \frac{2\pi}{T}$

III ⇒ $\dot{u}/\omega$ vs time $\omega = \frac{2\pi}{T}$

both oscillate @ the same frequency.

Note: Error in book $\dot{u}/\omega$ plot on pp 212

<u>Example:</u> For the following structure subjected to a 1in initial displacement @ it end, determine the displacement, velocity and moment diagram @ $t = 0.3\,sec$



$\bar{m} = 0.08\ lbm/in$
$L = 150\ in$
$E = 29000\ ksi$
$I = 100\ in^4$

=> From before

$$k^* = \frac{3EI}{L^3} = 2577.8\ lb/in$$

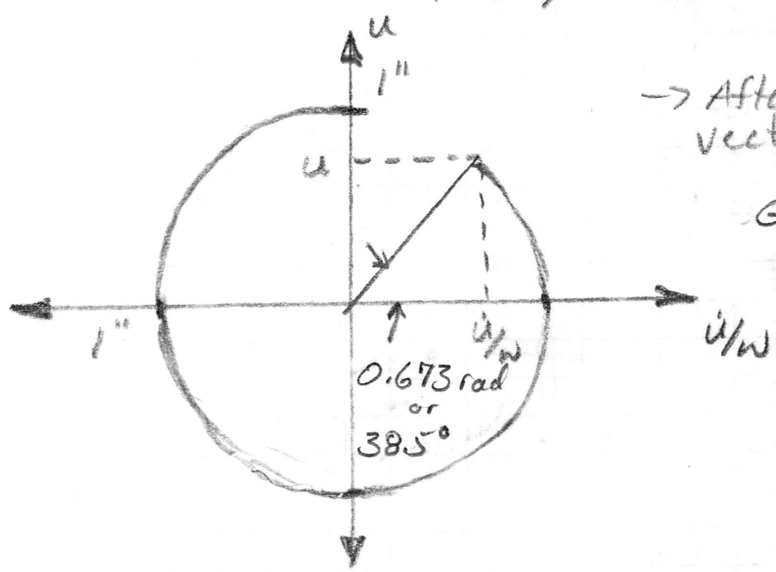
$$m^* = \frac{\bar{m}L}{3} + \frac{\bar{m}L}{3} = \frac{2\bar{m}L}{3} = 8\ lbm$$

$$\omega = \sqrt{\frac{2577.8}{8}}$$

$$\omega = 17.95\ RAD/s$$

$$\rho = \sqrt{1^2 + \left(\frac{0}{\omega}\right)^2} = 1.0$$

$$\phi = \tan^{-1}\left(\frac{1(\omega)}{0}\right) = \frac{\pi}{2} = 1.570\ rad.$$



-> After 0.3 sec the vector has rotated

$$\theta = \omega\Delta t = 17.95(0.3)$$
$$= 5.385\ rad$$
$$= 308.5^\circ$$

$$u = \sin(0.673\text{ rad})(\overset{1\text{in}}{\rho})$$

$\Rightarrow u(0.3\text{sec}) = 0.623$ in

$$\frac{\dot{u}}{w} = \cos(0.673\text{ rad})(\overset{1\text{in}}{\rho})$$

$$\dot{u} = 0.782(1\text{in})(17.95\text{ rad/s}) =$$

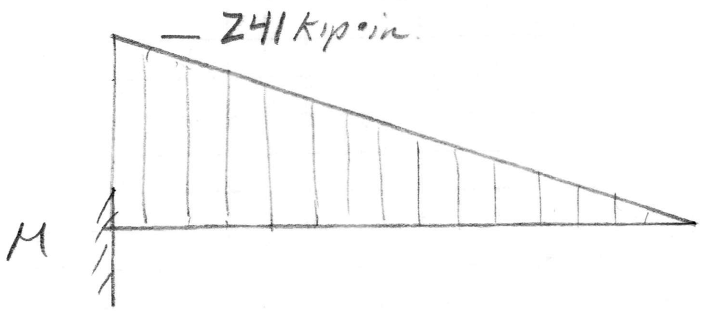
$\Rightarrow \dot{u}(0.3\text{sec}) = 14.0$ in/s

$\Rightarrow$ Moment Diagram

$$\left.\begin{aligned} \psi''(x) &= -\frac{3x}{L^3} + \frac{3}{L^2} \\ Z(0.3) &= 0.623 \end{aligned}\right\} u(0.3, x) = Z(0.3)\psi(x)$$

$$M(0.3s, x) = EI \left.\frac{d^2u}{dx^2}\right|_{\substack{t=0.3sec \\ x}} = 29{,}000\text{ ksi}(100\text{ in}^4)(0.623'')\ast \left[\frac{-3x}{(150)^3} + \frac{3}{(150)^2}\right]$$

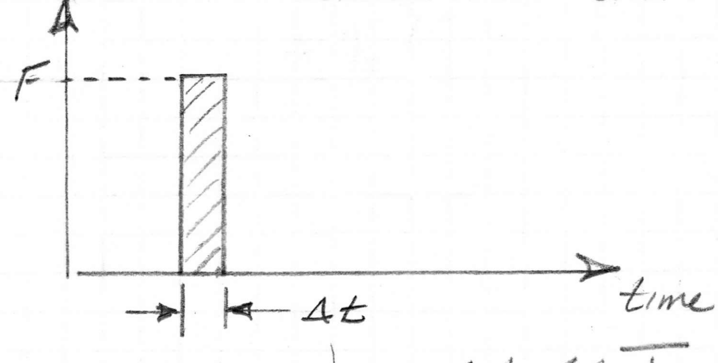
$$M(0.3s, x) = -1.606x + 240.9 \text{ [kip}\cdot\text{in]}$$



$\Rightarrow \psi$ is used to "compress" the system to a SDOF and to expand the results back out...

→ Aside from prescribed initial conditions, two realistic loadings can be addressed using free vibration response.

→ <u>Impulse</u> <u>Loading</u> ⇒ force 'F' applied over duration Δt.



↳ $\Delta t \ll T$ ← period of the structure.

→ Newton's 2nd Law

$$F = \frac{d}{dt}\left(m \frac{du}{dt}\right) \Rightarrow \overset{\text{constant mass}}{m} \underbrace{\frac{\Delta V}{\Delta t}}_{\text{average accel over } \Delta t}$$

$$F \Delta t = m \Delta V$$

$$\Delta V = \frac{F(\Delta t)}{m}$$

$F(\Delta t)$ ← impulse

ΔV ↑ change in velocity

→ If this occurs when the structure is at rest (e.g. $t=0$), then

$$V_0 = \frac{F(\Delta t)}{m}$$

V_0 ↑ initial condition

→ Impact with a Mass

m_i – impact mass
V_{i_0} – impact velocity

⇒ Use conservation of momentum
(note: kinetic energy is only conserved during an elastic impact) ← very rare.

velocity of m @ impact.
velocity of m_i following impact.

$$m V_{s_0} + m_i V_{i_0} = m_i V_{i_f} + m V_{s_f}$$

impact mass

velocity of m_i @ impact.

velocity of m following impact.

In this case $m = m + m_i$

⇒ If m_i & m stick together, then

$$V_{i_f} = V_{s_f} = V_f$$

$$m V_{s_0} + m_i V_{i_0} = (m_i + m) V_f$$

$$-m_i V_{s_0} + m_i V_{i_0} = (m_i + m) V_f - m V_{s_0} - m_i V_{s_0}$$

$$m_i (V_{i_0} - V_{s_0}) = (m_i + m)\underbrace{(V_f - V_{s_0})}_{\Delta V}$$

$$\Delta V = \frac{m_i (V_{i_0} - V_{s_0})}{(m_i + m)}$$

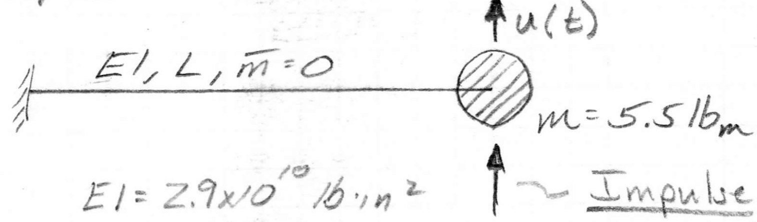
If this occurs when the structure is @ rest (e.g. $t=0$), then

$$V_0 = \frac{m_i V_{i_0}}{(m_i + m)}$$

initial condition

Example:

Initial Cond: $u_0 = -1 \text{ in}$
$V_0 = -10 \text{ in/s}$



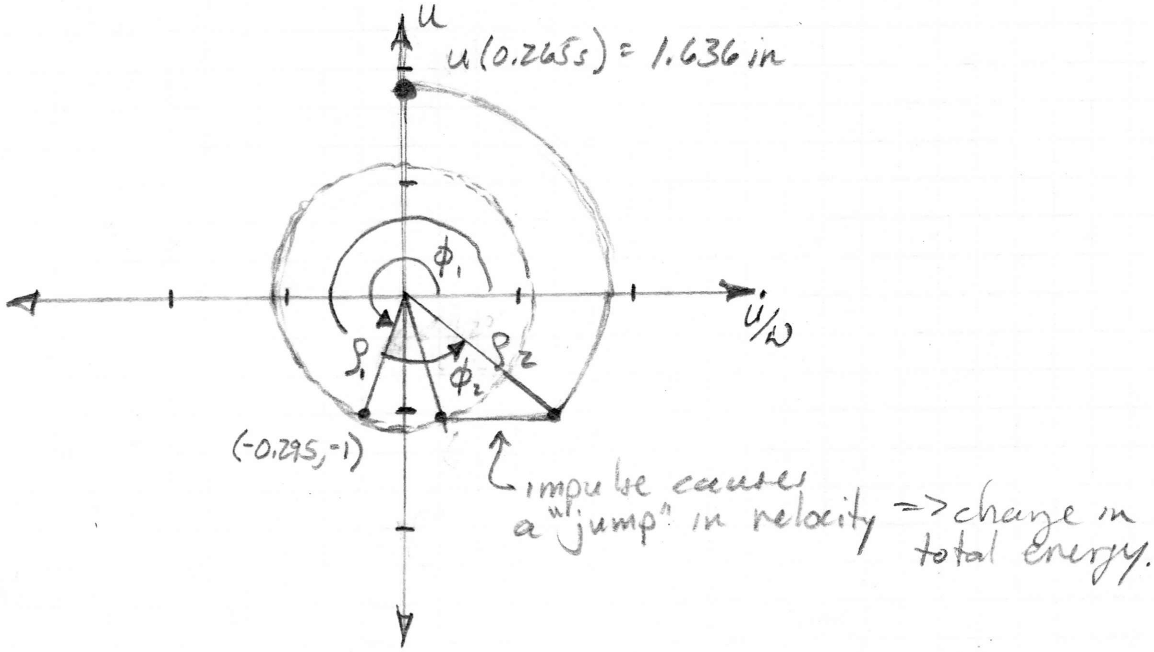
$EI = 2.9 \times 10^{10} \text{ lb}\cdot\text{in}^2$
$L = 240 \text{ in}$

Impulse @ $t = 0.2s$
$F = 20{,}000 \text{ lb}$
$\Delta t = 0.01 s$

Determine: u, v, & $M(x)$ @ $t = 0.265$ sec

$$k = \frac{3EI}{L^3} = 6293.4 \text{ lb/in}$$

$$\omega = \sqrt{\frac{k}{m}} = 33.83 \text{ rad/s}$$



$$\rho_1 = \sqrt{(0.295)^2 + (1)^2} = 1.043$$

$$\phi = \tan\left(\frac{-1}{-0.295}\right) + \pi = 4.425 \text{ rad} \quad (253.5°)$$

$\Rightarrow$ Rotation of vector from $t=0$ to $t=0.2s$.

$$\Delta\theta = 33.83(0.2s) = 6.766 \text{ rad.} \quad (387.6°)$$

$$\theta = 4.425 + 6.766 \text{ rad} = 4.901 \text{ rad} + 2\pi$$

(281.2°)

1 full rotation

$$\left.\frac{\dot{u}}{\omega}\right|_{t=0.2} = \rho \sin(0.195\,\text{rad}) = 0.2025\text{ in}$$

$$\left.\dot{u}\right|_{t=0.2} = (33.83\ \text{rad/s})\,0.2025\text{ in} = 6.850\ \text{in/s}$$

$$\left.u\right|_{t=0.2} = -\rho \cos(0.195\,\text{rad}) = -1.023\text{ in}$$

⇒ Change in velocity due to impact.

$$\Delta V = \frac{(20{,}000\text{ lb})(0.01\text{ s})}{5.5\text{ lb}} = 36.36\ \text{in/sec}$$

⇒ "New" initial conditions

$$\left.u\right|_{t=0.2} = -1.023\text{ in}$$

$$\left.\dot{u}\right|_{t=0.2} = 6.850 + 36.36 = 43.21\ \text{in/sec.}$$

$$\phi_2 = \tan^{-1}\left(\frac{-1.023\,(33.83)}{43.21}\right) = -0.676\text{ rad}$$

$$= 5.604\text{ rad } (321^\circ)$$

$$\rho_2 = \sqrt{(-1.023)^2 + \left(\frac{43.21}{33.83}\right)^2} = 1.636$$

$$\left.\frac{\dot{u}}{\omega}\right|_{t=0.2} = 1.277\text{ in}$$

⇒ Rotation from $t = 0.2\text{s}$ to $t = 0.265$ sec

$$\Delta\theta = 0.065(33.83) = 2.199\text{ rad. } (126.0^\circ)$$

$$\theta = 5.604 + 2.199 = 1.523 + 2\pi$$

1 full rotation

87.23°

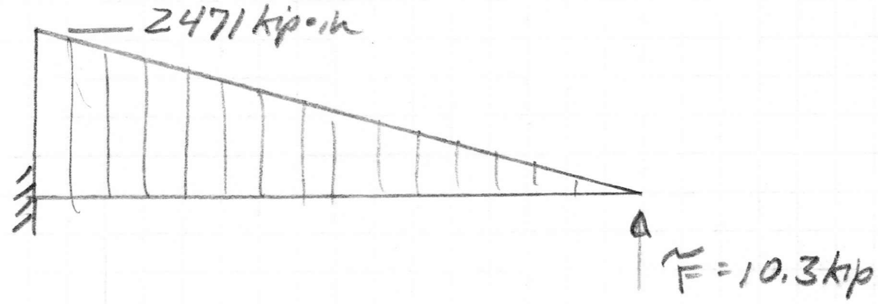
$\sim \xi_2$

$$u(0.265s) = 1.636 \sin(1.523) = 1.634 \text{ in}$$

$$\dot{u}(0.265s) = 1.636 \cos(1.523)(33.83) = 2.644 \text{ in/s}$$

$\Rightarrow$ Effective force

$$\tilde{F} = K\Delta = 6293.4(1.636) = 10,296 \text{ lb.}$$



$\Rightarrow$ <u>Alternative Solution</u>

$$u = u_o \cos(\omega t) + \frac{\dot{u}_o}{\omega} \sin(\omega t)$$

$$u(0.2s) = -1 \cos(33.83(0.2)) + \frac{-10}{33.83} \sin(33.83(0.2))$$

$$\Rightarrow u(0.2s) = -1.023 \text{ in}$$

$$\dot{u} = -u_o \omega \sin(\omega t) + \dot{u}_o \cos(\omega t)$$

$$\dot{u}(0.2s) = -1(33.83)\sin(33.83(0.2)) + -10 \cos(33.83(0.2))$$

$$\Rightarrow \dot{u}(0.2s) = 6.84 \text{ in/s}$$

"New" Initial conditions

$$u(0.2s) = -1.023 \text{ in}$$

$$\dot{u}(0.2s) = 6.84 + 36.36 = 43.21 \text{ in/sec}$$

↑ due to impact

need to subtract out t @ impact
⇒ restart clock.

$$u(0.265s) = -1.023\cos(33.83(0.265-0.2)) + \frac{43.21}{33.83}\sin(33.83(0.265-0.2))$$

$$\Rightarrow u(0.265s) = 1.634\,in \checkmark$$

$$\dot{u}(0.265s) = -1.025(33.83)\sin(33.83(0.265-0.2)) + 43.21\cos(33.83(0.265-0.2))$$

$$\Rightarrow \dot{u}(0.265s) = 2.632\,in/s. \checkmark$$

⇒ See additional examples work out on exel.