LECTURE 5: Proportional Damping, Damped Free Vibration Response of MDOF Systems

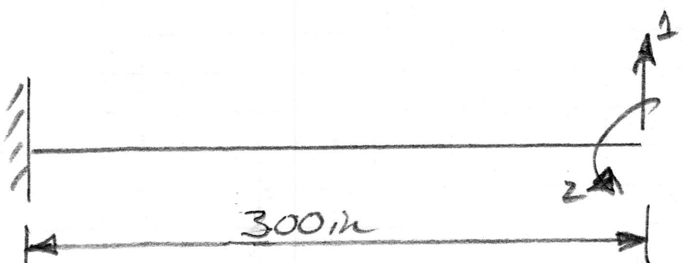
=> In class example...

For the following undamped 2DOF system:

(1) Compute the free vibration for the prescribed initial conditions.

(2) Show that if the initial conditions are proportional to one of the modes, the only that mode participates.

(3) Compare the time history to the obtained by modeling the structure as an SDOF.



$E = 29{,}000\ ksi$

$I = 1000\ in^4$

$\bar{m} = \frac{20}{386}\left(\frac{490}{12^3}\right)$

$\bar{m} = 0.0146\ lbm/in$

$$u_0 = \begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} in \qquad \dot{u}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} in/s$$

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} = \begin{bmatrix} 12{,}888 & -1{,}933{,}380 \\ -1{,}933{,}380 & 386.68\times10^6 \end{bmatrix}$$

$$[M] = \frac{\bar{m}L}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix} = \begin{bmatrix} 1.627 & -68.84 \\ -68.84 & 3{,}754.8 \end{bmatrix}$$

=> Note: SDOF $\quad k = \frac{3EI}{L^3} = 3222\ lb/in$

$m = \frac{33}{140}\bar{m}L = 1.032\ lb.m$

$\omega = \sqrt{\frac{3222}{1.032}} = 55.88\ rad/s$

→ Classical Damping

→ The damping matrix is also diagonalized by pre- and post- multiplying by the mode shape array.

⇒ 4 Methods.

(1) Mass proportional
(2) Stiffness proportional
(3) Rayleigh damping
(4) Superposition of Modal Damping Matricies

→ Mass Proportional Damping

→ Since we know ϕ_N is orthogonal w.r.t. [M], we set [C] equal to [M] times a constant

↑ damping matrix

$$[C] = \alpha_0 [M]$$

$$\phi_j^T [C] \phi_i = \alpha_0 \phi_j^T [M] \phi_i = 0$$

⇒ Orthogonality condition is preserved.

$$[C^*] = [\phi_N]^T [C][\phi_N] = \alpha_0 \underbrace{[\phi_N]^T [M][\phi_N]}_{\text{Identity matrix} = [I]}$$

$$[C^*] = \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow C_1^* = C_2^* = C_3^* \ldots = C_N^* = \alpha_0$$

$\Rightarrow$ Choose α_0 to give a specific damping ratio for a specific mode.

$$C_i^* = \alpha_0 = 2\zeta_i \omega_i$$ (definition of damping ratio)

$\Rightarrow$ All other modes have

$$C_j^* = \cancel{2}\zeta_i \omega_i = \cancel{2}\zeta_j \omega_j$$

$$\zeta_j = \frac{\omega_i}{\omega_j}\zeta_i$$

inversely proportional to ω_j

$\Rightarrow$ <u>Stiffness Proportional Damping</u>

$$[C] = \alpha_1 [K]$$

$$[C^*] = [\phi_N]^T [C][\phi_N] = \alpha_1 [\phi_N]^T [K][\phi_N]$$

$$[C^*] = \alpha_1 \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}$$

$$\Rightarrow C_1^* = \alpha_1 \omega_1^2$$
$$C_2^* = \alpha_1 \omega_2^2$$
$$C_3^* = \alpha_1 \omega_3^2$$

⇒ Again, choose α_1 to give a specific damping ratio for a specific mode.

$$C_i^* = \alpha_1 \omega_i^2 = 2\zeta_i \omega_i$$

$$\alpha_1 = \frac{2\zeta_i}{\omega_i}$$

⇒ All other modes have.

$$C_j^* = \frac{2\zeta_i}{\omega_i}\omega_j^2 = 2\zeta_j \omega_j$$

$$\zeta_j = \zeta_i \frac{\omega_j}{\omega_i}$$

↑ proportional to ω_j

⇒ Eg. 2 DOF w/ $\omega = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix}$ rad/s

⇒ Chose $\zeta_1 = 0.05$ (5% damping in mode 1)

⇒ Compute damping for mode 2.

(1) Mass proportional damping

$$C_1^* = \alpha_0 = 2\overset{0.05}{\cancel{\zeta_1}}\,\overset{2}{\cancel{\omega_1}}$$

$\alpha_0 = 0.2$ (multiply this by [M] to obtain [C] in physical coordinates)

$$\zeta_2 = \frac{\omega_1}{\omega_2}\zeta_1 = \frac{2}{12}(0.05) = 0.0083$$

↑ less than 1%...

⇒ Not realistic

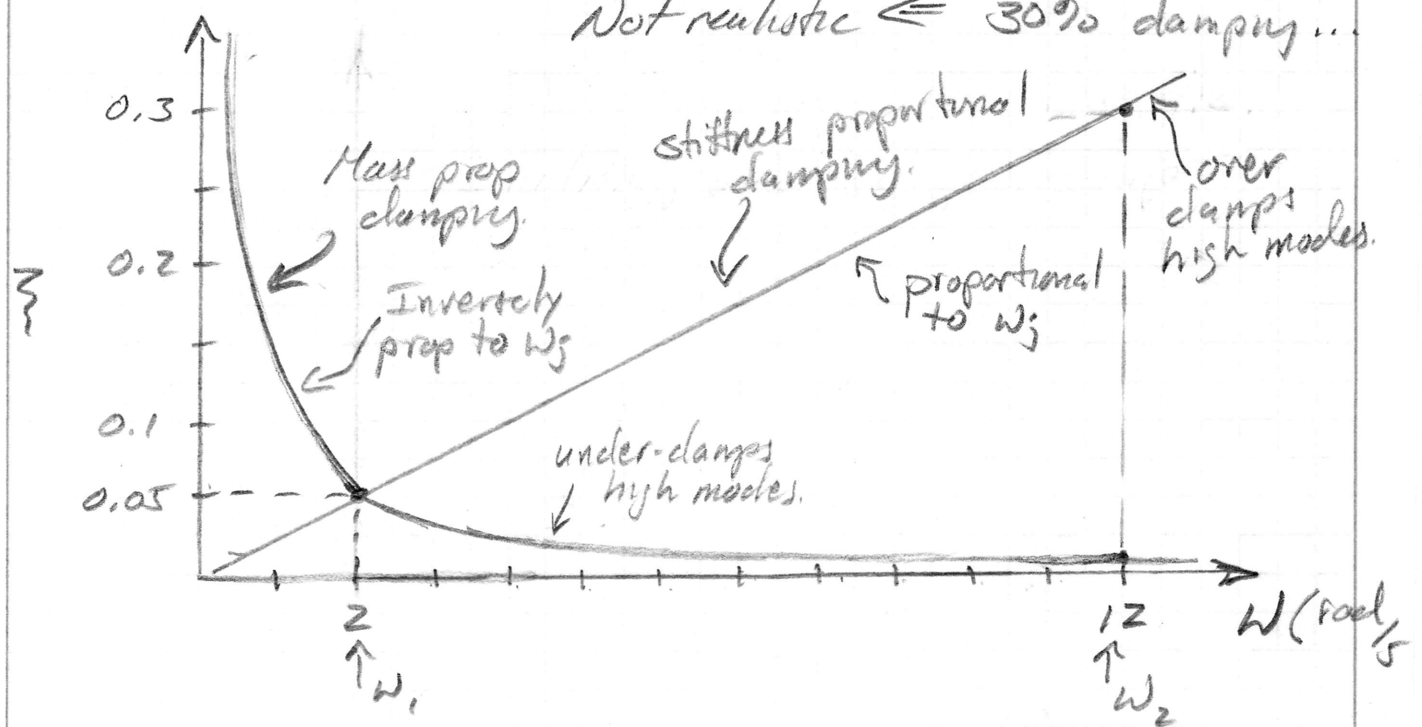
(2) Stiffness Proportional Damping.

$$\alpha_1 = \frac{2\xi_1}{\omega_1} = \frac{2(0.05)}{2} = 0.05$$

(multiply this by [k] to obtain [C] in physical coord.)

$$\xi_2 = \xi_1 \frac{\omega_2}{\omega_1} = 0.05\left(\frac{12}{2}\right) = 0.3$$

Not realistic ⇐ 30% damping...



⇒ Niether one is realistic except for closely spaced modes

↳ Experimental testing indicates structures exhibit similar levels of damping across their modes of interest.

→ Rayleigh Damping

$$[C] = \underbrace{\alpha_0 [M] + \alpha_1 [k]}_{\text{both stiffness and mass proportionate..}}$$

⇒ As before

$$[C^*] = \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}$$

$$C_1^* = \alpha_0 + \alpha_1 \omega_1^2 = 2\xi_1 \omega_1$$
$$C_2^* = \alpha_0 + \alpha_1 \omega_2^2 = 2\xi_2 \omega_2$$
$$C_3^* = \alpha_0 + \alpha_1 \omega_3^2 = 2\xi_3 \omega_3$$

⇒ Choose α_0 & α_1 to give a specific damping ratio for __two__ modes

⇒ E.g. 3 DOF w/ $\omega = \begin{Bmatrix} 2 \\ 12 \\ 14 \end{Bmatrix}$ rad/sec

⇒ Choose $\underbrace{\xi_1 = \xi_3}_{\text{typically choose the highest and lowest modes of interest}} = 0.05$

you'll see why shortly.

2 equations, 2 unknowns. $\begin{cases} \alpha_0 + \alpha_1 (2)^2 = 2(0.05)(2) \\ \alpha_0 + \alpha_1 (14)^2 = 2(0.05)(14) \end{cases}$

$$(\alpha_0 + 4\alpha_1 = 0.2) * -1$$

$$\begin{array}{r} + \ \alpha_0 + 196\alpha_1 = 1.4 \\ \hline 192\alpha_1 = 1.2 \end{array}$$

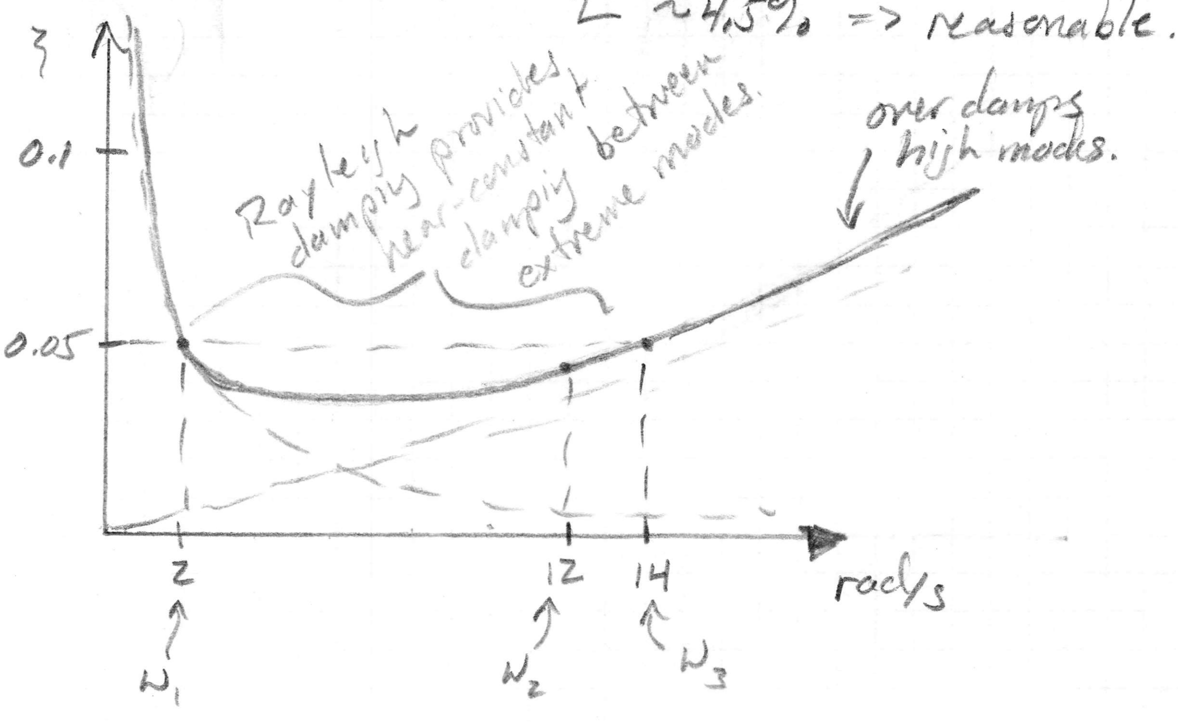
(Multiply these by [K] & [M] to obtain [C] in physical coord.)
$$\begin{cases} \alpha_1 = 0.00612 \\ \alpha_0 = 0.1755 \end{cases}$$

=> Compute ζ_2

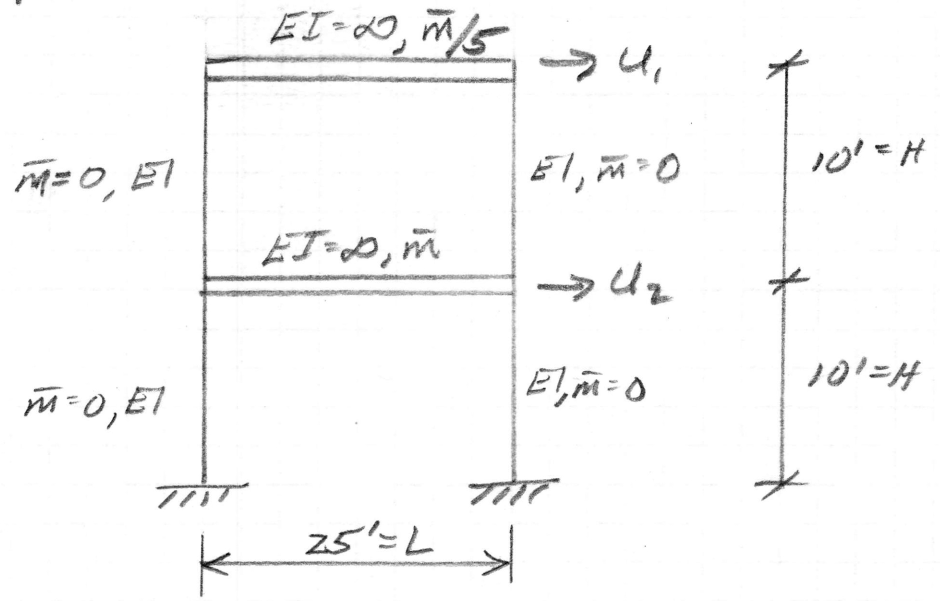
$$0.1755 + 0.00612(12)^2 = 2\zeta_2(12)$$

$$\zeta_2 = 0.044$$

~4.5% => reasonable.



Example:



$E = 29{,}000\ ksi$
$I = 1{,}000\ in^4$
$\bar{m} = (20ft)(1ft)(145^{lb}/ft^3)(1/12)\left(\frac{1}{386.4}\right) = 0.625\ ^{lbm}/in$

=> Damping: Mode 1 => $\xi_1 = 0.04$ (4%)
Mode 2 => $\xi_2 = 0.06$ (6%)

=> Initial Cond:

$$u_0 = \begin{Bmatrix} 2\ in \\ 0\ in \end{Bmatrix} \quad \dot{u}_0 = \begin{Bmatrix} 2\ in/sec \\ -1\ in/sec \end{Bmatrix}$$

=> Compute the total response...

(1) Formulate K & M

$$[K] = \begin{bmatrix} \frac{24EI}{H^3} & \frac{-24EI}{H^3} \\ -\frac{24EI}{H^3} & \frac{48EI}{H^3} \end{bmatrix} = \begin{bmatrix} 402.8 & -402.8 \\ -402.8 & 805.6 \end{bmatrix} kip/in$$

$$[M] = \begin{bmatrix} \frac{\bar{m}L}{5} & 0 \\ 0 & \bar{m}L \end{bmatrix} = \begin{bmatrix} 0.094 & 0 \\ 0 & 0.188 \end{bmatrix} kipm$$

(2) Solve the Eigen problem

$$a = \det[M] = 0.0176$$

$$b = -(402.8(0.188) + 805.6(0.094)) = -151.5$$

$$c = \det[k] = 162{,}248$$

$$\omega_i^2 = \frac{151.5 \pm \sqrt{151.5^2 - 4(0.0176)(162{,}248)}}{2(0.0176)}$$

$$\omega_i^2 = \frac{151.5 \pm 107.4}{0.0352} = (1252.8,\ 7355.1)$$

* $$\omega = \begin{Bmatrix} 35.4 \\ 85.8 \end{Bmatrix} \text{rad/s} = \begin{Bmatrix} 5.63 \\ 13.65 \end{Bmatrix} \text{Hz}$$

$$r_1 = \frac{+402.8}{402.8 - (1253)(0.094)} = 1.41$$

$$r_2 = \frac{+402.8}{402.8 - (7355)(0.094)} = -1.40$$

* unit normalized modes.

$$[\phi] = \begin{bmatrix} 1.0 & 1.0 \\ 0.709 & -0.709 \end{bmatrix}$$

$$\quad\ \uparrow \qquad\quad \uparrow$$

$$\{\phi_1\} \qquad \{\phi_2\}$$

(3) Bring the problem into modal space (coordinates)

⇒ Use mass normalized mode shapes.

$$[\phi]^T[M][\phi] = [M^*]$$

$$[\phi]^T[M] = \begin{bmatrix} 1.0 & 0.709 \\ 1.0 & -0.709 \end{bmatrix}\begin{bmatrix} 0.094 & 0 \\ 0 & 0.188 \end{bmatrix} = \begin{bmatrix} 0.094 & 0.133 \\ 0.094 & -0.133 \end{bmatrix}$$

$$[\phi]^T[M][\phi] = \begin{bmatrix} 0.094 & 0.133 \\ 0.094 & -0.133 \end{bmatrix}\begin{bmatrix} 1.0 & 1.0 \\ 0.709 & -0.709 \end{bmatrix} = \begin{bmatrix} 0.188 & 0 \\ 0 & 0.188 \end{bmatrix}$$

$$\phi_{N_1} = \frac{\phi_1}{\sqrt{M_1^*}} = \begin{Bmatrix} 2.31 \\ 1.64 \end{Bmatrix}$$

$$\phi_{N_2} = \frac{\phi_2}{\sqrt{M_2^*}} = \begin{Bmatrix} 2.31 \\ -1.64 \end{Bmatrix}$$

$$[\phi_N] = \left[\begin{array}{c|c} 2.31 & 2.31 \\ 1.64 & -1.64 \end{array}\right]$$

By definition, using $[\phi_N]$ to bring the system into modal coord. yields.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Free vibration.

or, plugging in.

$$\ddot{z}_1 + 1253 z_1 = 0$$ ← $SDOF_1$, associated with mode 1

$$\ddot{z}_2 + 7355 z_2 = 0$$ ← $SDOF_2$, associated with mode 2.

(4) Compute damping matrix.
→ use Rayleigh damping

$$[C^*] = \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} \overset{\omega_1^2}{1253} & 0 \\ 0 & \underset{\omega_2^2}{7355} \end{bmatrix}$$

Modal damping coeff.

$$\alpha_0 + 1253\alpha_1 = 2\zeta_1\omega_1 = 2(0.04)(35.4) = 2.83$$

$$-\quad \alpha_0 + 7355\alpha_1 = 2\zeta_2\omega_2 = 2(\underset{\zeta_2}{0.06})(85.8) = 10.30$$

Mode 2 damping coeff.

$$-6102\alpha_1 = -7.47$$

$$\alpha_1 = 0.00122$$

$$\alpha_2 = 1.301$$

0.491
0.983

In physical coord.

$$[C] = \alpha_0[M] + \alpha_1[K]$$

$$[C] = \begin{bmatrix} 0.613 & -0.491 \\ -0.491 & 1.228 \end{bmatrix}$$

↑ Needed if you directly integrate EOM → for nonlinear problems.

If you bring this into modal space w/ $[\phi_N]$

$$\Rightarrow \begin{bmatrix} 2.83 & 0 \\ 0 & 10.30 \end{bmatrix}$$

(5) Bring I.C. into modal coord.

$\Rightarrow$ Be sure to use $[\Phi_N]$!!!

$$[\Phi_N]^T[M] = \begin{bmatrix} 2.31 & 1.64 \\ 2.31 & -1.64 \end{bmatrix} \begin{bmatrix} 0.094 & 0 \\ 0 & 0.188 \end{bmatrix}$$

$$= \begin{bmatrix} 0.217 & 0.308 \\ 0.217 & -0.308 \end{bmatrix}$$

$$\{Z_0\} = [\Phi_N]^T[M]\{U_0\} = \begin{bmatrix} 0.217 & 0.308 \\ 0.217 & -0.308 \end{bmatrix} \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

mode 1 initial disp.

$$* \quad \{Z_0\} = \begin{Bmatrix} 0.434 \\ 0.434 \end{Bmatrix} \text{ in}$$

mode 2 initial disp.

$$\{\dot{Z}_0\} = [\Phi_N]^T[M]\{\dot{U}_0\} = \begin{bmatrix} 0.217 & 0.308 \\ 0.217 & -0.308 \end{bmatrix} \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

Mode 1 initial velocity.

$$* \quad \{\dot{Z}_0\} = \begin{Bmatrix} 0.126 \\ 0.742 \end{Bmatrix} \text{ in/sec}$$

mode 2 initial Velocity.

(6) Solve each SDOF seperately
↳ in modal space.

→ $SDOF_1$ (Mode 1)

$$(1)\ \ddot{z}_1 + 2\xi_1\omega_1\dot{z}_1 + \omega_1^2 z_1 = 0$$

↑ mass ↑ damping ↑ stiffness

$$\left.\begin{array}{l} z_{1_0} = 0.434 \text{ in} \\ \dot{z}_{1_0} = 0.126 \text{ in/sec} \end{array}\right\} \text{initial conditions}$$

From CIVE 801

$$z_1 = e^{-\omega_1\xi_1 t}\left(z_{1_0}\cos(\omega_1 t) + \left(\frac{\dot{z}_{1_0} + \omega_1\xi_1 z_{1_0}}{\omega_1}\right)\sin(\omega_1 t)\right)$$

Note $\omega_1 = 35.4$

$\xi_1 = 0.04$

$$* \quad z_1 = e^{-1.416t}\left(0.434\cos(35.4t) + 0.021\sin(35.4t)\right)$$

→ $SDOF_2$ (Mode 2)

$$(1)\ \ddot{z}_2 + 2\xi_2\omega_2\dot{z}_2 + \omega_2^2 z_2 = 0$$

$z_{2_0} = 0.434$ in $\qquad \xi_2 = 0.06$

$\dot{z}_{2_0} = 0.742$ in $\qquad \omega_2 = 85.8$ rad/s

$$* \quad z_2 = e^{-5.148t}\left(0.434\cos(85.8t) + 0.035\sin(85.8t)\right)$$

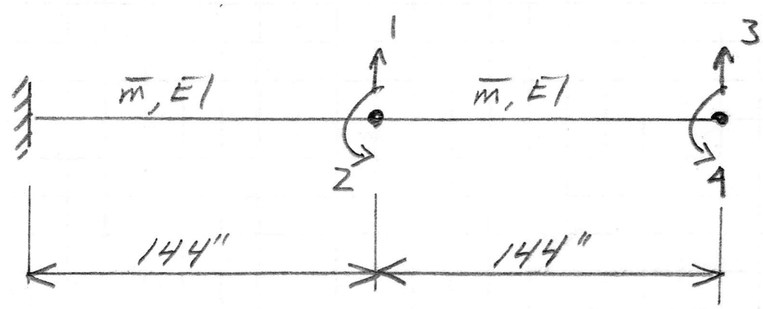
(7) Bring the solution for each mode back into physical coord. and sum them.
↳ modal superposition

$$\{u\} = \{\phi_{N_1}\} z_1 + \{\phi_{N_2}\} z_2$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 2.31 \\ 1.64 \end{Bmatrix} e^{-1.416t}\left(0.434\cos(35.4t) + 0.021\sin(35.4t)\right)$$

$$+ \begin{Bmatrix} 2.31 \\ -1.64 \end{Bmatrix} e^{-5.148t}\left(0.434\cos(85.8t) + 0.025\sin(85.8t)\right)$$

<u>Example 2:</u>



$$\bar{m} = (20\,in^2)(490\,^{lb}/_{ft^3})\left(\frac{1}{12}\right)^3\left(\frac{1}{386.4}\right) = 0.0147\,^{lbm}/_{in}$$

$E = 29{,}000{,}000$ psi

$I = 1000\ in^4$

⇒ Damping Mode 1 $\zeta_1 = 0.05$ } Rayleigh damping
Mode 3 $\zeta_3 = 0.05$

⇒ Initial Cond.

$$u_0 = \begin{Bmatrix} 1.0 \\ 0 \\ 1.0 \\ 0 \end{Bmatrix} in \quad \dot{u}_0 = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} {}^{in}/_{sec}$$

→ Solve for the response of the structure using

(1) All four modes

(2) The first three modes

(3) The first two modes

(4) The first mode

→ What errors do you observe?