LECTURE 6: Forced vibration due to harmonic input

sin or cos

→ General Solution Procedure.

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

break $p(t)$ into components $p_1(t)$ & $p_2(t)$ w/ known solutions. superposition

→ Solve for $u_1(t)$ & $u_2(t)$

due to $p_1(t)$ due to $p_2(t)$

→ $u(t) = u_1(t) + u_2(t)$

superposition

→ Undamped Harmonic Vibration

$$m\ddot{u} + k\dot{u} = p_0 \sin(\Omega t)$$

circular frequency of the applied force.

$$p_1(t) = 0$$

$$u_1(t) = A\cos\omega t + B\sin\omega t$$

Known as transient response

will die out in the presence of damping.

response will allow the initial cond. to be satisfied.

→ Don't solve for A & B yet...

$$p_2(t) = p_0 \sin(\Omega t)$$

guess $u_2(t) = G\sin(\Omega t)$

Subst into EOM.

$$m(-\Omega^2 G \cancel{\sin \Omega t}) + kG\cancel{\sin \Omega t} = P_o \cancel{\sin \Omega t}$$

$$G(k - \Omega^2 m) = P_o$$

$$G = \frac{P_o}{k - \Omega^2 m}$$

Known as steady-state response ← $u_2 = \frac{P_o}{k - \Omega^2 m} \sin \Omega t$

↳ will continue as long as $p(t)$ is present.

→ <u>Total solution</u>: $u = u_1 + u_2$

$$u = \underbrace{A\cos wt + B\sin wt}_{\text{Transient portion vibrates @ } \omega} + \underbrace{\frac{P_o}{k - \Omega^2 m} \sin \Omega t}_{\text{Steady state portion vibrates @ } \Omega}$$

Transient portion vibrates @ ω ↳ property of structure.

Steady state portion vibrates @ Ω ↳ Property of forcing function

define $\beta = \frac{\Omega}{\omega}$

↳ frequency ratio

$$\frac{P_o}{k - \Omega^2 m} = \frac{P}{k}\left[\frac{1}{1-\beta^2}\right]$$

→ Considering initial cond. u_0 & v_0

$$u = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t + \frac{P_0}{K} \frac{1}{1-\beta^2} (\sin \Omega t - \beta \sin \omega t)$$

→ Any term w/ (ωt) is transient

→ Any term w/ (Ωt) is steady-state.

Only due to initial cond.

$$u_T = \underbrace{u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t}_{} - \underbrace{\frac{P_0}{K} \frac{\beta}{1-\beta^2} \sin \omega t}_{}$$

→ Note, the transient response is not zero even for $u_0 = v_0 = 0$

→ The last term is used to ensure the total solution satisfies initial cond... it "corrects" the steady-state portion.

→ For $u_0 = v_0 = 0$.

$$u = \frac{P_0}{K} \frac{1}{1-\beta^2} (\sin \Omega t - \beta \sin \omega t)$$

R = Dynamic influence factor $= \dfrac{u}{u_{st}} = \dfrac{u}{P_0/K}$

↳ Note: Humar defines

dynamic load factor → $D = \underbrace{\dfrac{u_{ss}}{u_{st}}}_{}$ ← steady state

↳ we'll deal with this in a minute.

$$R = \frac{1}{1-\beta^2}\left(\sin \Omega t - \beta \sin \omega t\right)$$

=> 4 Important Cases (show TH w/ Excel)

(1) $\omega \gg \Omega$

$R \Rightarrow 1.0$

$k = 1000$ lb/in
$m = 0.5$ lbm
$\Omega = 1$ rad/s

→ Extremely small transient response needed to satisfy initial cond.

and relatively high frequency

→ $\frac{1}{1-\beta^2} \Rightarrow 1.0$ as $\beta \Rightarrow 0$

→ This is the case where the load is applied very slowly, this the problem is "quasi" static

compared to what??

(2) $\Omega \gg \omega$

$R \Rightarrow 0.0$

$k = 100$ lb/in
$m = 0.5$ lbm
$\Omega = 400$ rad/s

→ Relatively large transient response (compared to SS) needed to satisfy initial cond.

→ $\frac{1}{1-\beta^2} \Rightarrow 0$ as $\beta \to \infty$

→ This is the case where the load is applied so rapidly that the system doesn't have time to respond.

(3) $\omega \sim \Omega$.

$k = 1000$ lb/in
$m = 0.5$ lbm
$\Omega = 50$ rad/s

$R \Rightarrow$ becomes large.

→ "Beating" is observed in the response as the SS & T components "slide" past one another.

→ Transient & SS responses are of similar magnitudes.
→ again, this is needed to correct the SS response to satisfy the initial cond.

→ $\frac{1}{1-\beta^2} \Rightarrow \infty$ as $\beta \Rightarrow 1.0$.

(4) $\omega = \Omega$ (known as resonance)

$R = \infty$, $\beta = 1.0$ }

→ This is essentially an infinitely long "beat" with an amplitude that eventually reaches infinity.

$$u = \frac{P_o}{K}\frac{1}{1-\beta^2}(\sin \underbrace{\beta\omega}_{\Omega = \beta\omega} t - \beta \sin \omega t)$$

when $\beta = 1.0$

$u = \frac{0}{0}$ } indeterminant.

⇒ Use L'Hospital's rule

$$\lim_{\beta \to 1} u = \lim_{\beta \to 1} \frac{P_o}{K}\left(\frac{\omega t \cos\beta\omega t - \sin \omega t}{-2\beta}\right)$$

$$\lim_{\beta \to 1} u = \frac{1}{2}\frac{P_o}{K}(\sin \omega t - \underbrace{\omega t \cos \omega t}_{})$$

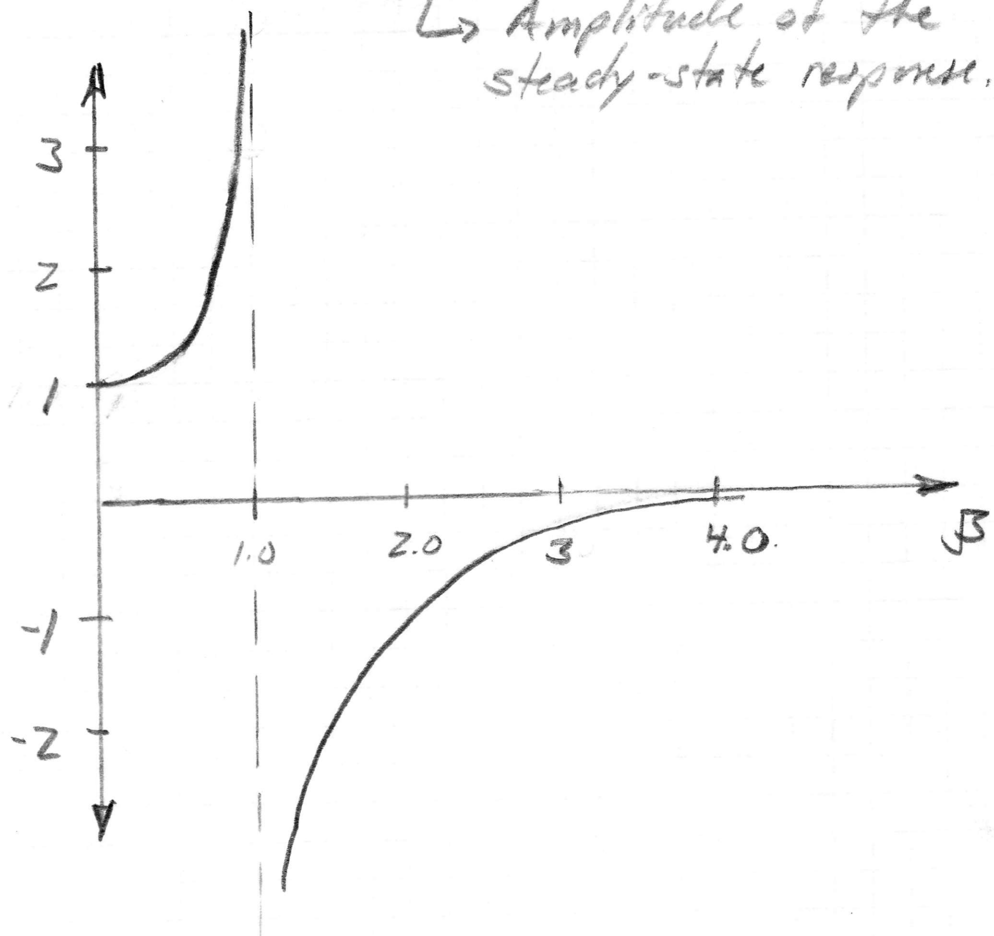
this terms drives the growth of the response.

→ Focus on SS response.

$$R_{ss} = \left(\frac{1}{1-\beta^2}\right)\sin\Omega t.$$

Note: no phase angle ⇒ phase follows the sign of Ad.

↳ Amplitude of the steady-state response. (Ad)



→ For small β ⇒ Ad ~ 1.0

→ No amplification.

→ For β ~ 1.0 ⇒ Ad is large

→ For large β ⇒ Ad is near zero

same as seen in cases 1 through 4

→ For β < 1.0, Ad is positive

For β > 1.0, Ad is negative.

what does this mean??

A: For $\beta < 1.0$, the response is "in phase" with the load.

e.g. When the force is pushing to the left, the structure is moving to the left.

For $\beta > 1.0$, the response is "out-of-phase" with the load.

e.g. When the force is pushing to the left, the structure is moving to the right.

$\Rightarrow$ Using Excel.

$k = 100$ lb/in.

$m = 0.5$ lb-m

$\Omega = 14$ rad/sec & 15 rad/sec

$\Rightarrow$ look @ direction of SS response.

$\rightarrow$ Damped Harmonic Vibration

$$m\ddot{u} + c\dot{u} + ku = p_o \sin \Omega t$$

$\rightarrow$ Complimentary solution (Transient Response)

$$m\ddot{u} + c\dot{u} + ku = 0$$

from before.

$$u_t = e^{-\omega_n \xi t}(A\cos(\omega t) + B\sin(\omega t))$$

Dependent on I.C. & forcing function.

⇒ Particular Solution (Steady-State Response)

$$m\ddot{u} + c\dot{u} + ku = p_0 \sin(\Omega t)$$

assume.

$$u_{ss} = G_1 \cos \Omega t + G_2 \sin \Omega t.$$

⇒ Subst. u_{ss} into EOM & solve for G_1 & G_2.

Steady state. $$u_{ss} = \frac{p_0}{k} \frac{1}{(1-\beta^2)^2 + (2\zeta\beta)^2}\left[(1-\beta^2)\sin \Omega t - 2\zeta\beta \cos \Omega t\right]$$

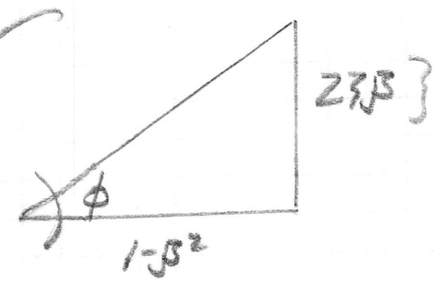
re-writing

$$u_{ss} = \rho \sin(\Omega t - \phi)$$

$$\rho = \frac{p_0}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$ } constant amplitude ⇒ steady-state.

$$\phi = \tan^{-1}\left(\frac{2\zeta\beta}{1-\beta^2}\right)$$

∴ ϕ is always between 0° & 180°



$2\zeta\beta$ } always positive.

$\underbrace{1-\beta^2}$ negative for $\beta > 1.0$

$\Rightarrow u = u_T + u_{ss}$

$\Rightarrow$ Solving for 'A' & 'B' w/ u_0 & V_0.

due to u_0 & V_0

$$u_T = e^{-\zeta\omega t}\left(u_0 \cos\omega t + \frac{V_0 + u_0\omega\zeta}{\omega}\sin\omega t\right)$$

Assumes $\omega_n = \omega_d (= \omega)$

$$+ \frac{P_0}{k}\,\frac{e^{-\omega\zeta t}}{\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]}\Big[2\zeta\beta\cos\omega t + \left(2\beta\zeta^2 - \beta(1-\beta^2)\right)\sin\omega t\Big]$$

due to forcing function

$\Rightarrow$ this corrects steady-state response to have $u_0 = V_0 = 0$.

$\Rightarrow$ See Excel (Plot $u_{ss} + u_T$.)

$\Rightarrow$ The transient response dies out and the ss response becomes the total response.

$\Rightarrow$ "Beating" still observed for ω close to Ω, but not as pronounced.

$\Rightarrow$ Focus on u_{ss}.

$$u_{ss} = \rho\sin(\Omega t - \phi)$$

$\Rightarrow$ Examine amplitude ρ

$$A_d = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$ ⇒ see Excel plots for different ξ's.

⇒ Peak amplitude decreases rapidly with an increase in ξ.

⇒ For $\beta = 0$, $A_d = 1.0$ (static response)

⇒ For $\beta \gg$, $A_d \approx 0$

⇒ Let's take a closer look.

$$\rho = \frac{P_o}{K}\left(\frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}\right)$$

Case 1: Small β.

→ denominator of ρ ⇒ 1.0

$$\therefore \rho \approx \underbrace{\frac{P_o}{K}}$$

displacement amplitude is dependent on stiffness.

→ This response is termed "stiffness controlled"

Case 2: Large β.

$$\rho_{\beta \Rightarrow \infty} = \frac{P_o}{K} \frac{1}{\sqrt{\underbrace{(1-\beta^2)^2}_{\sim \beta^4} + (2\xi\beta)^2}} \approx \frac{P_o}{K\beta^2}$$

neglect since its only 2nd order

$$\rho = \frac{P_o}{k\left(\frac{\Omega^2}{\omega^2}\right)} = \frac{P_o}{k\,\frac{\Omega^2}{k/m}} = \underbrace{\frac{P_o}{m\Omega^2}}$$

displacement amplitude is dependent on mass

→ Response is termed "mass controlled"

Case 3: $\beta = 1.0$ (resonance).

$$\rho = \frac{P_o}{k}\,\frac{1}{\sqrt{(1-1^2)^2+(2\xi(1))^2}} = \underbrace{\frac{P_o}{k}\left(\frac{1}{2\xi}\right)}$$

* For a damped structure; disp do not go to ∞ @ resonance

⇐ For a given static displacement, response is dependent on damping

→ Response is termed "damping controlled"

→ Aside: Does the peak response occur @ resonance?

↳ e.g. $\beta = 1.0$

Examine trend & importance using Excel

$$\frac{dA_d}{d\beta} = 0 \rightarrow \beta = \sqrt{1-2\xi^2}$$

→ subst. into A_d

$$A_{d_{MAX}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

⇒ Discuss the implications of these observations in the case of earthquake loading

→ Low-rise sheer wall buildings.

→ High-rise buildings.

→ 5 story frame building.

⇒ Discuss the influence of k & m on ω (and thus β) as well.

⇒ <u>Examine Phase Angle, ϕ</u>

$$\phi = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

or

$$\phi = \cos^{-1}\left(\frac{(1-\beta^2)}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}\right)$$

⇒ See Excel plot.

(1) For $\beta << 1.0$

$\phi \approx 0°$ ⇒ response is in phase w/ force

(2) For $\beta >> 1.0$

$\phi \approx 180°$ ⇒ response is out of phase w/ force

(3) For $\beta = 1.0$ (resonance)

$\phi = 90°$ => response "lags" the force by $\Delta t = \dfrac{\phi}{\Omega}$

=> Show using TH plot in Excel.

$k = 100$ lb/in
$m = 0.5$ lbm
$\Omega = \omega$ (resonance) $= 14.1421356\,2$ rad/s

=> time lag $= \Delta t = \dfrac{\pi/2}{\Omega} = 0.11107$ sec

from TH...

$$\Delta t = 0.758\left(\frac{0.115 - 0.110}{2.776 + 0.758}\right) + 0.11$$

$\Delta t = 0.11107$ sec.

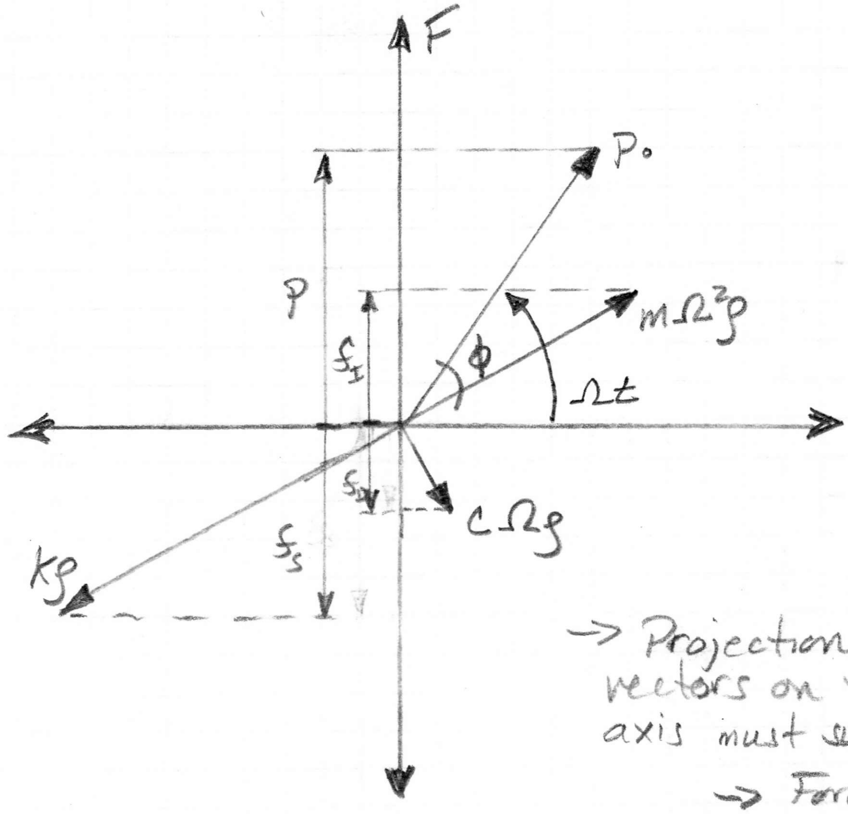
=> <u>Force Balance of SS Response.</u>

<u>EOM</u>

$$P_0 \sin(\Omega t) - \underbrace{f_I}_{m\ddot{u}} - \underbrace{f_D}_{c\dot{u}} - \underbrace{f_S}_{ku} = 0$$

$$u_{ss} = \rho \sin(\Omega t - \phi)$$
$$\dot{u}_{ss} = \Omega \rho \cos(\Omega t - \phi)$$
$$\ddot{u}_{ss} = -\Omega^2 \rho \sin(\Omega t - \phi)$$

$$f_I = -m\Omega^2 \rho \sin(\Omega t - \phi)$$
$$f_D = c\Omega\rho \cos(\Omega t - \phi)$$
$$f_S = k\rho \sin(\Omega t - \phi)$$



→ Projections of the vectors on the vertical axis must sum to zero
→ Force balance.

→ Interpretation.

→ In phase → $\phi = 0°$

→ Stiffness force opposes applied load

⇒ "Stiffness controlled"

→ Out-of-Phase → $\phi = 180°$

→ Mass force opposes applied load.

⇒ "Mass controlled"

→ At resonance → $\phi = 90°$

→ Mass & stiffness ⊥ to applied load
→ Damping opposed to applied load
⇒ "damping controlled"