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VIBRATIONS:
ANALYTICAL AND EXPERIMENTAL MODAL ANALYSIS

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PREFACE

An initial version of the following set of notes was originally prepared by Dave Formenti in 1977 for use in a short course at the University of Cincinnati on the subject of "Experimental Modal Analysis". The notes have since been rewritten several times to standardize the notation and add further clarification and additional topics by several authors including, R. J. Allemang, R. W. Rost, T. J. Severyn, and J. M. Leuridan, for use in other short courses and the dual level Mechanical Vibrations II (20-263-662) course at the University of Cincinnati. Any comments relative to corrections or improvements will be welcomed.

PRINTING/REVISION HISTORY

In an attempt to improve and correct these notes, several revisions have been made. Please be certain that you have the latest revision.

March 1990
Revision 1: February 1992
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Revision 5: February 1999

NOMENCLATURE

Matrix Notation

{..}	braces enclose column vector expressions
$\{..\}^T$	row vector expressions
[..]	brackets enclose matrix expressions
$[..]^H$	complex conjugate transpose, or Hermitian transpose, of a matrix
$[..]^T$	transpose of a matrix
$[..]^{-1}$	inverse of a matrix
$[..]^+$	generalized inverse (pseudoinverse)
$[\,.\,.\,]_{q \times p}$	size of a matrix: q rows, p columns
$\lceil\ .\ .\ \rfloor$	diagonal matrix

Operator Notation

A^*	complex conjugate
F	Fourier transform
$\mathbf{F}^{-1}$	inverse Fourier transform
H	Hilbert transform
$\mathbf{H}^{-1}$	inverse Hilbert transform
ln	natural logarithm
L	Laplace transform
$\mathbf{L}^{-1}$	inverse Laplace transform
Re + *j* Im	complex number: real part "Re", imaginary part "Im"
$\dot{x}$	first derivative with respect to time of dependent variable x
$\ddot{x}$	second derivative with respect to time of dependent variable x
$\bar{y}$	mean value of y
$\hat{y}$	estimated value of y

$\sum_{i=1}^{n} A_i \; B_i$	summation of $A_i \; B_i$ from i = 1 to n
$\frac{\partial}{\partial t}$	partial derivative with respect to independent variable "t"
det[. .]	determinant of a matrix
$\|\|. .\|\|_2$	Euclidean norm

Roman Alphabet

A_{pqr}	residue for response location p, reference location q, of mode r
C	damping
e	base e (2.71828...)
F	input force
F_q	spectrum of q^{th} reference†
$h(t)$	impulse response function†
$h_{pq}(t)$	impulse response function for response location p, reference location q †
$H(s)$	transfer function†
$H(\omega)$	frequency response function, when no ambiguity exist, H is used instead of $H(\omega)$†
$H_{pq}(\omega)$	frequency response function for response location p, reference location q, when no ambiguity exist, H_{pq} is used instead of $H_{pq}(\omega)$†
$[I]$	identity matrix
j	$\sqrt{-1}$
K	stiffness
K_r	modal stiffness for mode r
L	modal participation factor
M	mass
M_r	modal mass for mode r
M_{A_r}	modal A for mode r
M_{B_r}	modal B for mode r

N	number of modes
N_i	number of references (inputs)
N_o	number of responses (outputs)
p	output, or response point (subscript)
q	input, or reference point (subscript)
r	mode number (subscript)
R_I	residual inertia
R_F	residual flexibility
s	Laplace domain variable
t	independent variable of time (sec)
t_k	discrete value of time (sec)
	$t_k = k \Delta t$
T	sample period
x	displacement in physical coordinates
X	response
X_p	spectrum of p^{th} response†
z	Z domain variable

Greek Alphabet

$\delta(t)$	Dirac impulse function
Δf	discrete interval of frequency (Hertz or cycles/sec)
Δt	discrete interval of sample time (sec)
ε	small number
η	noise on the output
λ_r	r^{th} complex eigenvalue, or system pole $\lambda_r = \sigma_r + j\omega_r$
$\lceil \Lambda \rfloor$	diagonal matrix of poles in Laplace domain
ν	noise on the input
ω	variable of frequency (rad/sec)
ω_r	imaginary part of the system pole, or damped natural frequency, for mode r (rad/sec) $\omega_r = \Omega_r \sqrt{1 - \zeta_r^2}$
Ω_r	undamped natural frequency (rad/sec) $\Omega_r = \sqrt{\sigma_r^2 + \omega_r^2}$
ϕ_{pr}	scaled p^{th} response of normal modal vector for mode r
$\{\phi\}_r$	scaled normal modal vector for mode r
$[\Phi]$	scaled normal modal vector matrix
$\{\psi\}$	scaled eigenvector
ψ_{pr}	scaled p^{th} response of a complex modal vector for mode r
$\{\psi\}_r$	scaled complex modal vector for mode r
$[\Psi]$	scaled complex modal vector matrix
σ	variable of damping (rad/sec)
σ_r	real part of the system pole, or damping factor, for mode r
ζ	damping ratio
ζ_r	damping ratio for mode r

† vector implied by definition of function

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1. INTRODUCTION

Today's engineers are faced with many complex noise and vibration problems associated with the design and troubleshooting of structures. Never before have structures had so many constraints, such as legislative, cost, and durability, imposed upon them. Because of these constraints, present day structures are typically more complex in terms of design and materials. Therefore, the engineering that goes into these structures must be more exact than ever before. The structural analyst must design with accurate and realistic models and the experimentalist must be able to accurately define a structure's dynamic response to specified input forces.

Recently, tools have been developed to assist the structural engineer in these areas. The analyst now uses sophisticated finite element programs, such as Nastran, to aid in the understanding and the design of structures. The experimentalist has sophisticated digital signal analysis equipment that aids him in quantifying and understanding a structure's dynamic and input forces. However, there is little communication between the analyst and experimentalist. Due to the complexity of todays problems, the need for communication and exchange of ideas is even more critical.

The major reason for this lack of communication is that the analytical and experimental engineers do not understand each other's terminology and methods of problem solution. The discussion that will follow will attempt to fill this void.

The major point of the text is to show how frequency response function (FRF) measurements are related to the structure's mode shapes and vibrational frequencies. This overall objective will be accomplished by building a mathematical foundation from the analytical and experimental point of view.

The material is divided into two sections: single degree of freedom, and multiple degree of freedom systems. The single degree of freedom system will be used as a means for defining some standard terminology. The multiple degree of freedom material is further divided into: undamped, proportionally damped, and non-proportionally damped systems. Throughout the discussion emphasis will be placed on the relationship of frequency response measurements to modal vectors. Furthermore, the following concepts will be presented:

- Modal Frequencies, Eigenvalues, System Poles, Characteristic Roots (λ_r)
- Modal Vectors, Eigenvectors ($\{\psi_r\}$, $\{\phi_r\}$)
- Modal Coefficients (ψ_{pr}), Residues (A_{pqr})
- Real Modal Vectors, Complex Modal Vectors
- Physical Coordinates, Generalized Coordinates
- Principal Coordinates, Modal coordinates
- Modal Mass, Damping, and Stiffness (M_r, C_r, K_r)
- Modal A, Modal B (M_{A_r}, M_{B_r})

1.1 Degrees of Freedom

The development of any theoretical concept in the area of vibrations, including modal analysis, depends upon an understanding of the concept of the number of degrees of freedom (N) of a system. This concept is extremely important to the area of modal analysis since the number of modes of vibration of a mechanical system is equal to the number of degrees of freedom. From a practical point of view, the relationship between this theoretical definition of the number of degrees of freedom and the number of measurement degrees of freedom (N_o, N_i) is often confusing. For this reason, the concept of degree of freedom will be reviewed as a preliminary to the following modal analysis material.

To begin with, the basic definition that is normally associated with the concept of the number of degrees of freedom involves the following statement: The number of degrees of freedom for a mechanical system is equal to the number of independent coordinates (or minimum number of coordinates) that is required to locate and orient each mass in the mechanical system at any instant in time. As this definition is applied to a point mass, three degrees of freedom are required since the location of the point mass involves knowing the x, y, and z translations of the center of gravity of the point mass. As this definition is applied to a rigid body mass, six degrees of freedom are required since θ_x, θ_y, and θ_z rotations are required in addition to the x, y, and z translations in order to define both the orientation and location of the rigid body mass at any instant in time. This concept is represented in Figure 1-1. As this definition is extended to any general deformable body, it should be obvious that the number of degrees of freedom can now be

considered as infinite. While this is theoretically true, it is quite common, particularly with respect to finite element methods, to view the general deformable body in terms of a large number of physical points of interest with six degrees of freedom for each of the physical points. In this way, the infinite number of degrees of freedom can be reduced to a large but finite number.

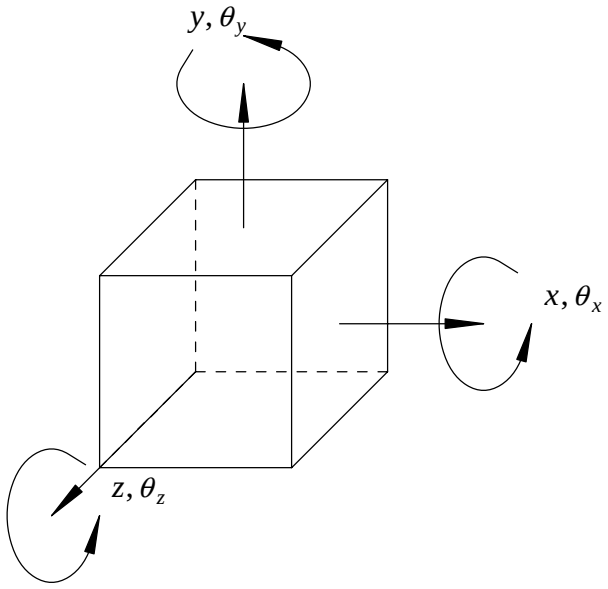

Figure 1-1. Degrees of Freedom of a Rigid Body

When measurement limitations are imposed upon this theoretical concept of the number of degrees of freedom of a mechanical system, the difference between the theoretical number of degrees of freedom (N) and the number of measurement degrees of freedom (N_o, N_i) begins to evolve. Initially, for a general deformable body, the number of degrees of freedom (N) can be considered to be infinite or equal to some large finite number if a limited set of physical points of interest is considered as discussed in the previous paragraph. The first measurement limitation that needs to be considered is that there is normally a limited frequency range that is of interest to

the analysis. For example, most dominant structural modes of vibration for an automobile would be located between 0 and 200 Hertz. As this limitation is considered, the number of degrees of freedom of this system that are of interest is now reduced from infinity to a reasonable finite number. The next measurement limitation that needs to be considered involves the physical limitation of the measurement system in terms of amplitude. A common limitation of transducers, signal conditioning and data acquisition systems results in a dynamic range of 80 to 100 db (10^4 to 10^5) in the measurement. This means that the number of degrees of freedom is reduced further due to the dynamic range limitations of the measurement instrumentation. Finally, since few rotational transducers exist at this time, the normal measurements that are made involve only translational quantities (displacement, velocity, acceleration, force) and thus do not include rotational effects, or RDOF. In summary, even for the general deformable body, the theoretical number of degrees of freedom that are of interest are limited to a very reasonable finite value ($N = 1 - 50$). Therefore, this number of degrees of freedom (N) is the number of modes of vibration that are of interest.

Finally, then, the number of measurement degrees of freedom (N_o, N_i) can be defined as the number of physical locations at which measurements are made times the number of measurements made at each physical location. For example, if x, y, and z accelerations are measured at each of 100 physical locations on a general deformable body, the number of measurement degrees of freedom would be equal to 300. It should be obvious that since the physical locations are chosen somewhat arbitrarily, and certainly without exact knowledge of the modes of vibration that are of interest, that there is no specific relationship between the number of degrees of freedom (N) and the number of measurement degrees of freedom (N_o, N_i). In general, in order to define N modes of vibration of a mechanical system, N_o, N_i must be equal to or larger than N. Note also that even though N_o, N_i is larger than N, this is not a guarantee that N modes of vibration can be found from N_o, N_i measurement degrees of freedom. The N_o, N_i measurement degrees of freedom must include physical locations that allow a unique determination of the N modes of vibration. For example, if none of the measurement degrees of freedom are located on a portion of the mechanical system that is active in one of the N modes of vibration, portions of the modal parameters for this mode of vibration can not be found.

In the development of the single and multiple degree of freedom information in the following Sections, the assumption is made that a set of N measurement degrees of freedom ($N_o, N_i = N$) exist that will allow for N modes of vibration to be determined. In reality, N_o, N_i is always

chosen much larger than N since a prior knowledge of the modes of vibration is not available. If the set of N_o, N_i measurement degrees of freedom is large enough and if the N_o, N_i measurement degrees of freedom are distributed uniformly over the general deformable body, the N modes of vibration will normally be found.

1.2 Basic Assumptions

Before proceeding, the basic assumptions must be established before the theory can be developed. The first assumption is that the structure is a linear system whose dynamics may be represented by a set of linear, second order, differential equations. The second assumption is that the structure during the test can be considered as time invariant. This assumption implies that the coefficients in the linear, second order, differential equations are constants and do not vary with time. The third assumption is that the structure is observable. While this may seem trivial, this means that the system characteristics that are affecting the dynamics can be measured and that there are sufficient sensors to adequately describe the input-output characteristics of the system. Another assumption that is often made is that the structure obeys Maxwell's reciprocity theorem. Maxwell's reciprocity theorem, in terms of frequency response function measurements, implies the following: if one measures the frequency response function between points p and q by exciting at p and measuring the response at q, the same frequency response function will be measured by exciting at q and measuring the response at p $\left(H_{pq} = H_{qp} \right)$.

2. SINGLE DEGREE OF FREEDOM SYSTEM

2.1 Theory

The general mathematical representation of a single degree of freedom system is expressed using Newton's second law in Equation 2.1:

$$M\,\ddot{x}(t) + C\,\dot{x}(t) + K\,x(t) = f(t) \tag{2.1}$$

and is represented schematically in Figure (2-1).

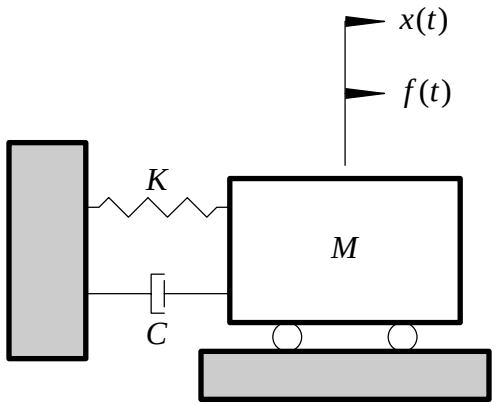

Figure 2-1. Single Degree of Freedom System

Equation 2.1 is a linear, time invariant, second order differential equation. The total solution to this problem involves two parts as follows:

$$x(t) = x_c(t) + x_p(t)$$

where:

- $x_c(t)$ = Transient portion
- $x_p(t)$ = Steady state portion

By setting $f(t) = 0$, the homogeneous (transient) form of Equation 2.1 can be solved.

$$M\ \ddot{x}\ (t) + C\ \dot{x}\ (t) + K\ x\ (t) = 0 \tag{2.2}$$

From differential equation theory, the solution can be assumed to be of the form $x_c(t) = Xe^{st}$, where s is a constant to be determined. Taking appropriate derivatives and substituting into Equation 2.2 yields:

$$(Ms^2 + C\ s + K)\ X(s)\ e^{s\ t} = 0$$

Thus, for a non-trivial solution $\left(X(s)\ e^{s\ t} \neq 0 \right)$:

$$s^2 + (\ C\ /\ M\ )\ s + (\ K\ /\ M\ ) = 0 \tag{2.3}$$

Equation 2.3 is the system's characteristic equation, whose roots λ_1 and λ_2 (λ = system pole) are:

$$\lambda_{1,2} = -\frac{C}{2M} \pm \sqrt{\left(\frac{C}{2M} \right)^2 - \left(\frac{K}{M} \right)}$$

Thus the homogeneous solution of Equation 2.1 is:

$$x_c(t) = X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t}$$

where X_1 and X_2 are constants determined from the initial conditions imposed on the system at $t = 0$.

The particular solution (steady state) is a function of the form of the forcing function. If the

forcing function is a pure sine wave of a single frequency, the response will also be a sign wave of the same frequency. If the forcing function is random in form, the response is also random.

2.2 Laplace Domain Theory

Equation 2.1 is the time domain representation of the system in Figure (2-1). An equivalent equation of motion may be determined for the Laplace or s domain. This representation has the advantage of converting a differential equation to an algebraic equation. This is accomplished by taking the Laplace transform of Equation 2.1, thus:

$$\mathbf{L}\ \{\ M\ \ddot{x} + C\ \dot{x} + K\ x\ \} = M\ (\ s^2\ X(s) - s\ x(0) - \dot{x}(0)\ )$$

$$+ C\ (\ s\ X(s) - x(0)\ ) + K\ X(s)$$

$$\mathbf{L}\ \{\ M\ \ddot{x} + C\ \dot{x} + K\ x\ \} = (\ M\ s^2 + C\ s + K\ )\ X(s) - Ms\ x(0) - M\ \dot{x}(0) - C\ x(0)$$

$$\mathbf{L}\ \{\ f(t)\ \} = F(s)$$

Thus Equation 2.1 becomes:

$$[M\ s^2 + C\ s + K]\ X(s) = F(s) + (M\ s + C)\ x(0) + M\dot{x}(0) \tag{2.4}$$

where:

- $x(0)$ is the initial displacement at time $t = 0$.
- $\dot{x}(0)$ is the initial velocity at time $t = 0$.

If the initial conditions are zero, Equation 2.4 becomes:

$$[M\ s^2 + C\ s + K]\ X(s) = F(s) \tag{2.5}$$

Let $B(s) = M\ s^2 + C\ s + K$. $B(s)$ is referred to as the system impedance. Then Equation 2.5 becomes:

$$B(s)\ X(s) = F(s) \tag{2.6}$$

Equation 2.6 is an equivalent representation of Equation 2.1 in the Laplace domain. The Laplace domain (s domain) can be thought of as complex frequency ($s = \sigma + j\ \omega$). Therefore, the quantities in Equation 2.6 can be thought of as follows:

- $F\ (s)$ - the Laplace domain (complex frequency) representation of the forcing function $f\ (t)$
- $X\ (s)$ - the Laplace domain (complex frequency) representation of the system response $x\ (t)$

Equation 2.6 states that the system response $X(s)$ is directly related to the system forcing function $F(s)$ through the quantity $B(s)$. If the system forcing function $F(s)$ and its response $X(s)$ are known, $B(s)$ can be calculated. That is:

$$B(s) = \frac{F(s)}{X(s)}$$

More frequently one would like to know what the system response is going to be due to a known input $F(s)$, or:

$$X(s) = \frac{F(s)}{B(s)} \tag{2.7}$$

By defining $H(s) = \dfrac{1}{B(s)}$, Equation 2.7 becomes:

$$X(s) = H(s)\ F(s) \tag{2.8}$$

The quantity $H(s)$ is known as the system *transfer function*. In other words, a transfer function relates the Laplace transform of the system input to the Laplace transform of the system response. From Equations 2.5 and 2.8, the transfer function can be defined as:

$$H(s) = \frac{X(s)}{F(s)} = \frac{1/M}{s^2 + (\ C\ /\ M\ )\ s + (\ K/\ M\ )} \tag{2.9}$$

assuming initial conditions are zero.

The denominator term is referred to as the system characteristic equation. The roots of the characteristic equation are:

$$\lambda_{1,2} = -(C / 2M) \pm \sqrt{(C / 2M)^2 - (K / M)} \tag{2.10}$$

Note that the above definition of the transfer function establishes a form of an analytical model that can be used to describe the transfer function. This analytical model involves a numerator and denominator polynomial with scalar coefficients. For the single degree of freedom case, the numerator polynomial is zeroeth order and the denominator polynomial is second order.

2.3 Definition Of Terms

2.3.1 Critical Damping

Critical damping C_c is defined as being the damping which reduces the radical in the characteristic equation to zero.

$$(C_c / 2M)^2 - (K / M) = 0$$

$$(C_c / 2M) = \sqrt{K / M} = \Omega_1$$

$C_c = 2M\ \Omega_1$ = critical damping coefficient

Ω_1 = undamped natural frequency (rad / sec)

2.3.2 Fraction of Critical Damping - Damping Ratio (Zeta)

The fraction of critical damping or damping ratio, ζ, is the ratio of the actual system damping to the critical system damping.

$$\zeta_1 = C / C_c$$

The roots of characteristic Equation 2.10 can now be written as:

$$\lambda_{1,2} = \left(-\zeta_1 \pm \sqrt{\zeta_1^2 - 1} \right) \Omega_1 \tag{2.11a}$$

$$\lambda_{1,2} = \left(-\zeta_1 \pm j\sqrt{1 - \zeta_1^2} \right) \Omega_1 \tag{2.11b}$$

2.4 System Classification

Systems can be classified depending on their damping ratios. That is:

- Overdamped system: $\zeta_1 > 1$
- Critically damped system: $\zeta_1 = 1$
- Underdamped system: $\zeta_1 < 1$

Figures (2-2) through (2-4) illustrate typical the time domain response of these 3 different cases. The following plots illustrate the location of the roots of the characteristic equation in the *s*-plane. Figure (2-2) (overdamped) shows two real roots that lie on the σ axis, if damping were to increase the roots would move apart. For Figure (2-3) (critically damped), there are two identical real roots. For Figure (2-4) (underdamped), there are two complex roots, complex conjugates of each other. As the damping and/or the frequency changes, these roots stay in the second and third quadrant of the graph. It should be pointed out that if any roots of the characteristic equation lie to the right of the $j\,\omega$ axis, the system would be unstable.

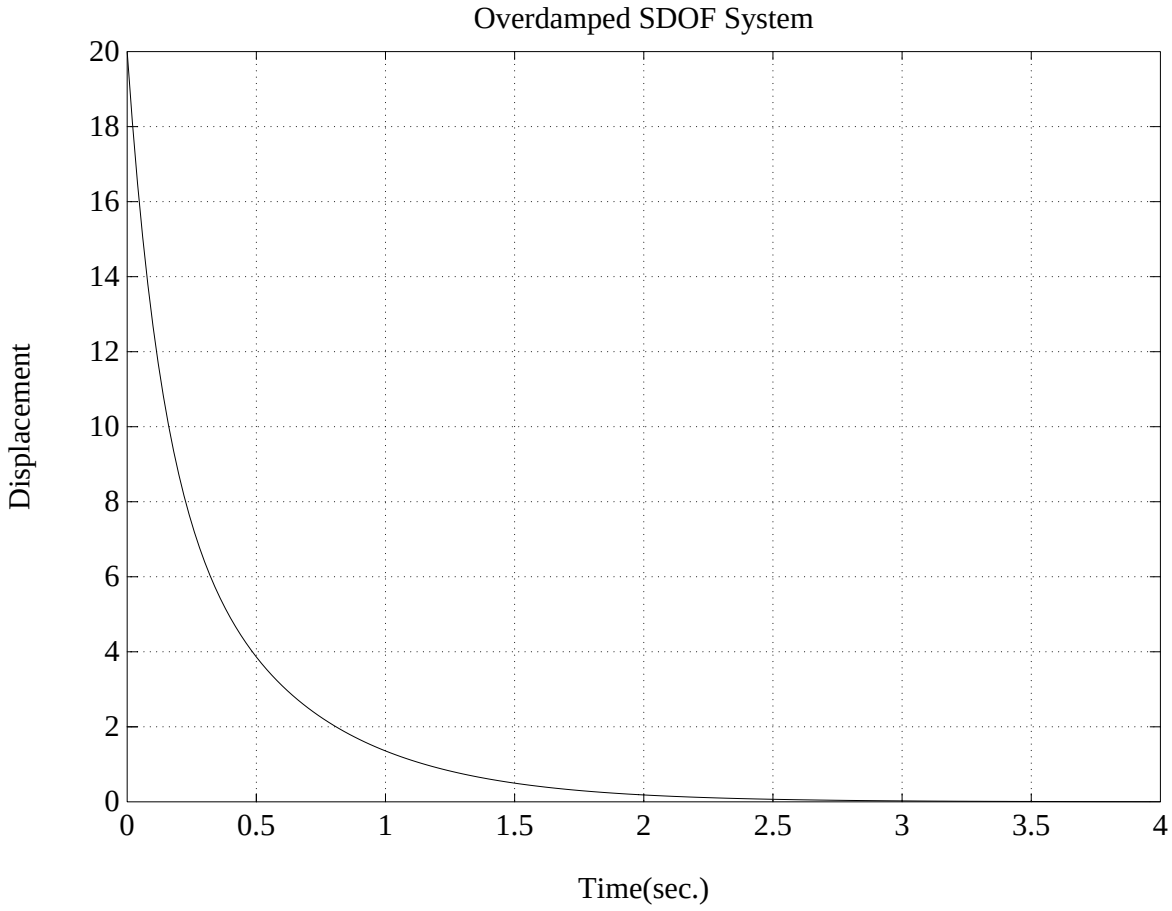

Figure 2-2. Overdamped SDOF System Response: Initial Displacement

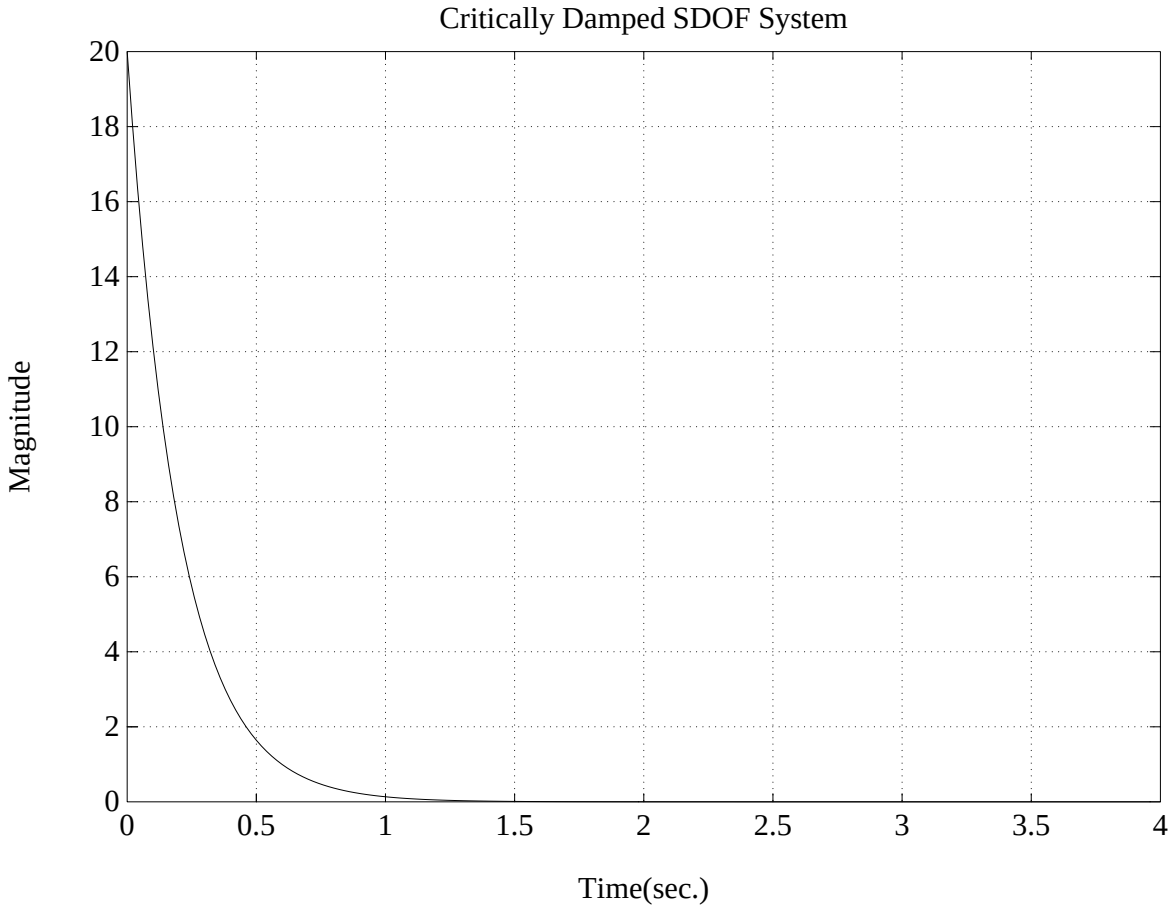

Figure 2-3. Critically Damped SDOF System Response: Initial Displacement

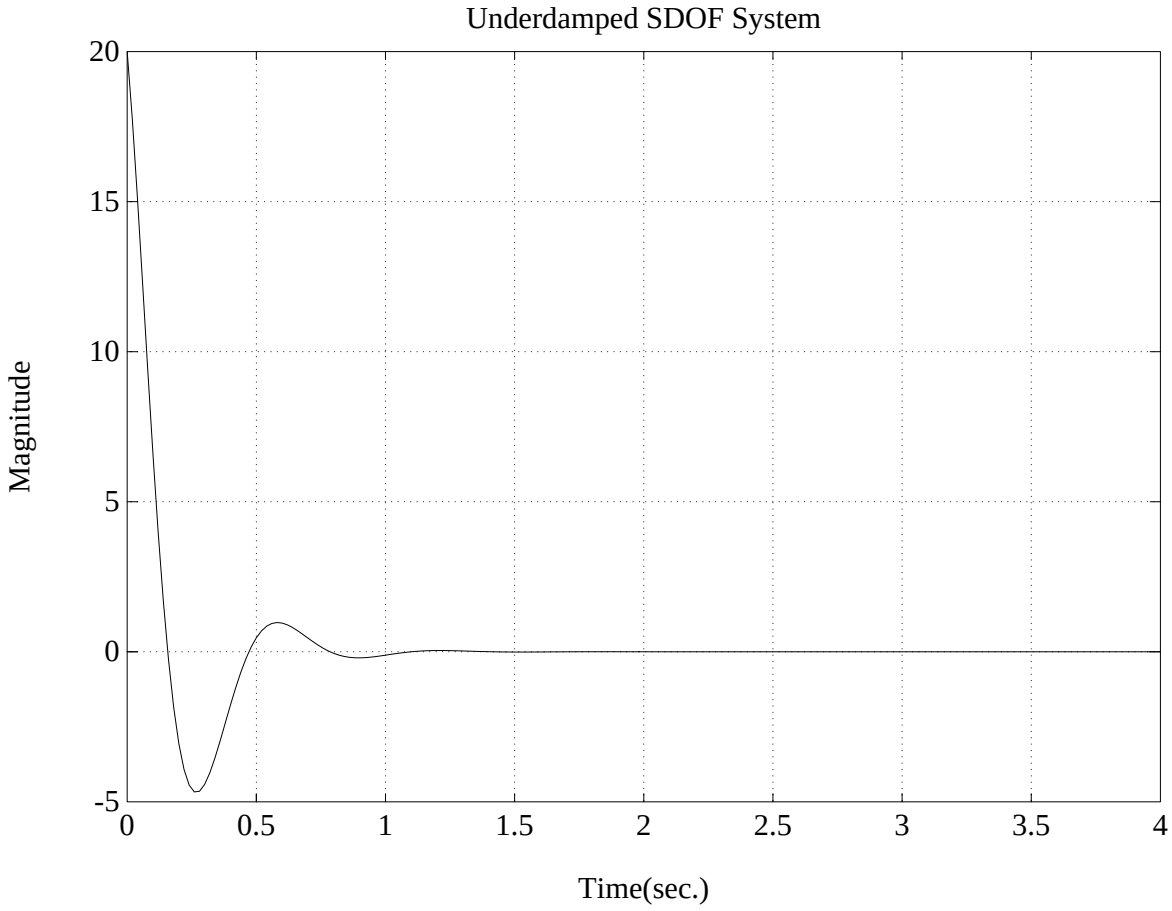

Figure 2-4. Underdamped SDOF System Response: Initial Displacement

Figures (2-5) through (2-7) show the location of the roots, in the *s* or Laplace plane, of the characteristic equation for each of the three cases.

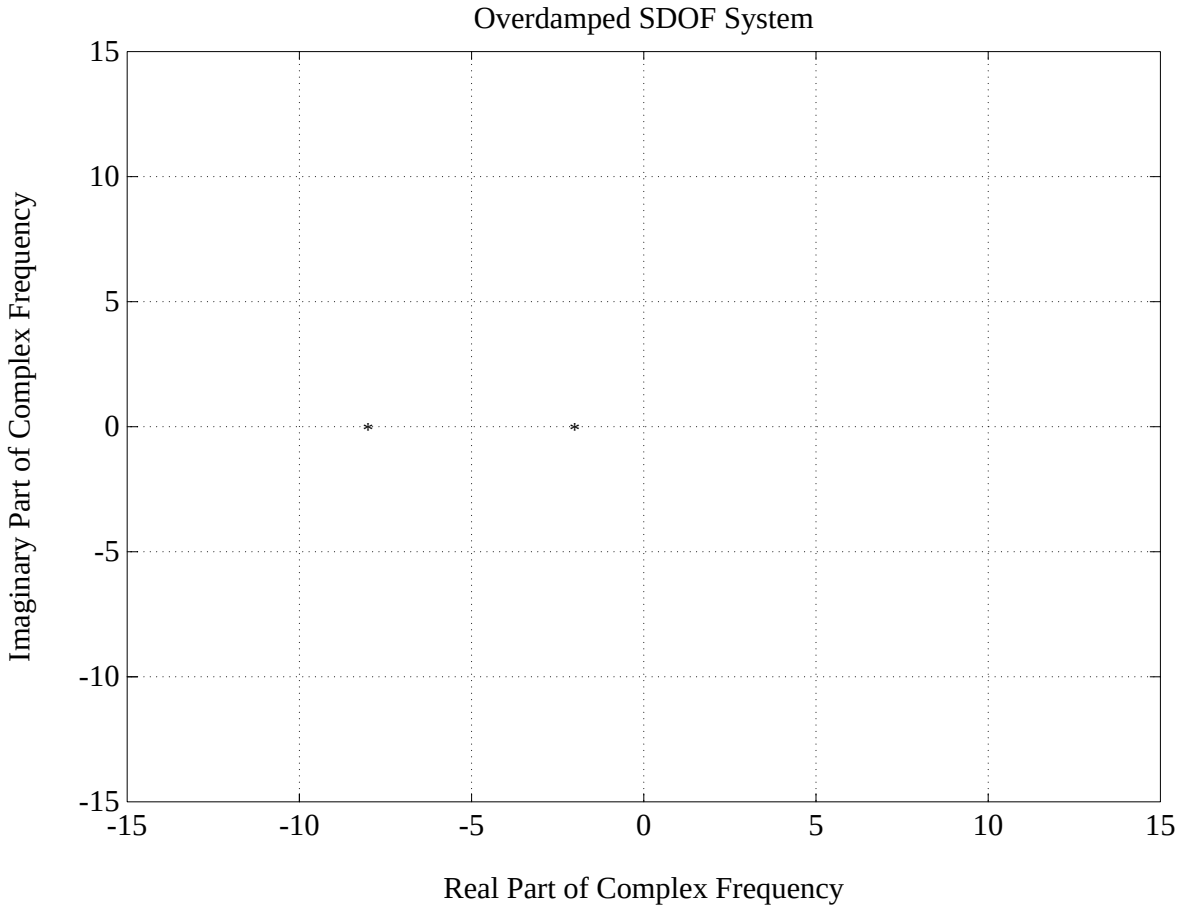

Figure 2-5. Overdamped SDOF System

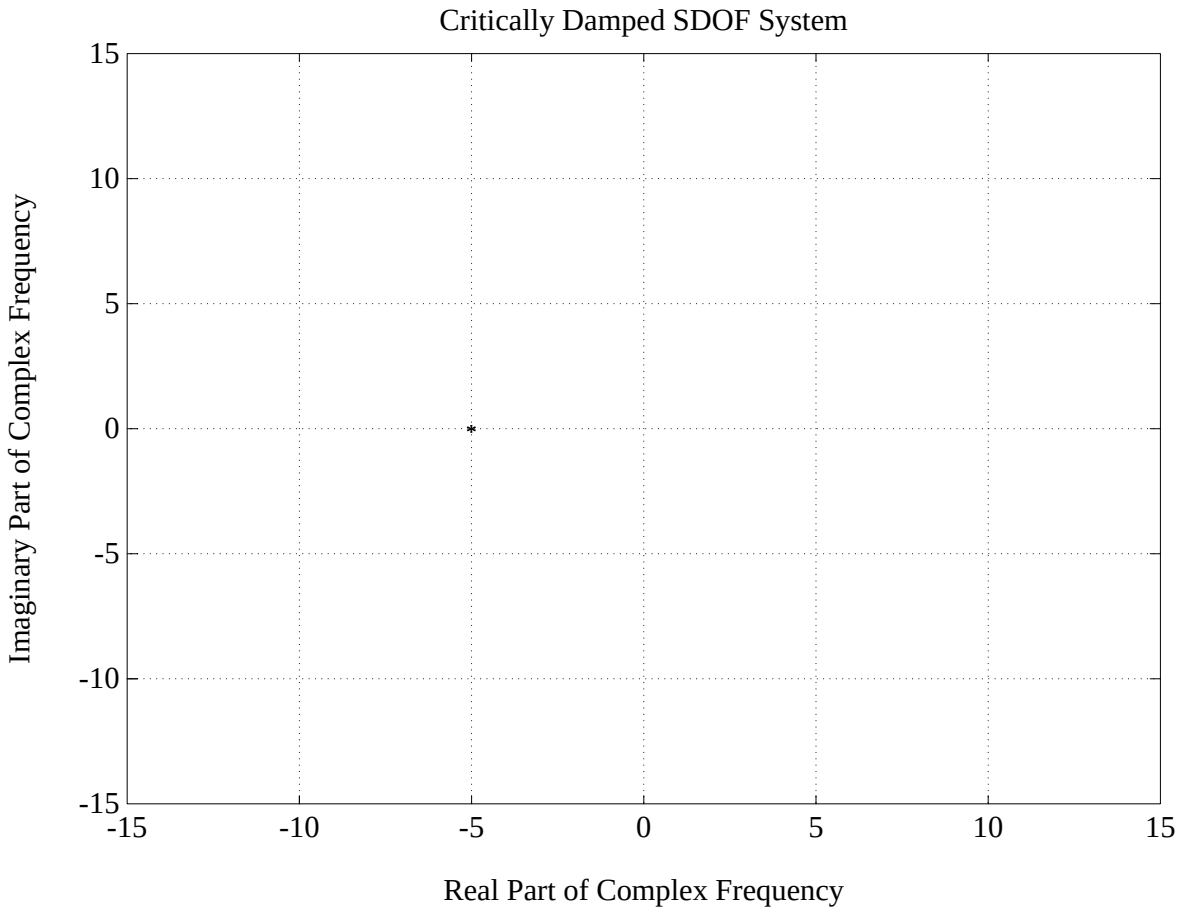

Figure 2-6. Critically Damped SDOF System

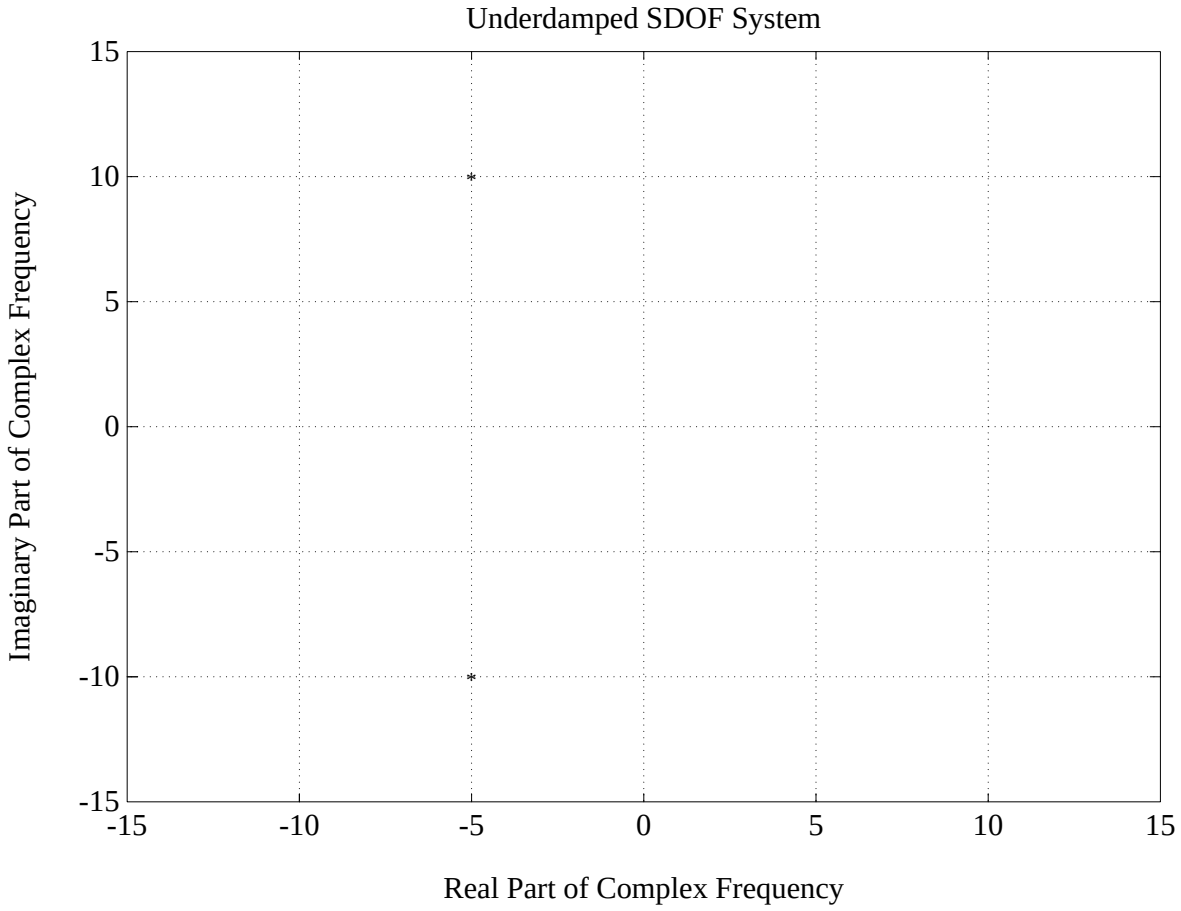

Figure 2-7. Underdamped SDOF System

For most real structures, unless active damping systems are present, the damping ratio is rarely greater than ten percent. For this reason, all further discussion will be restricted to underdamped systems $\zeta_1 < 1$. For an underdamped system, the roots of the characteristic equation can be written as:

$$\lambda_{1,2} = \sigma_1 \pm j\omega_1$$

where:

- $\sigma_1 =$ damping factor (units of rad/sec)

- ω_1 = damped natural frequency

Note that for this case λ_2 is always the complex conjugate of λ_1. Therefore, the λ_2 notation will be replaced in further equations by λ_1^*.

Using Equation 2.11 the above parameters can be related to the damping ratio (ζ_1) and the undamped natural frequency (Ω_1) as follows:

$$\zeta_1 = - \frac{\sigma_1}{\sqrt{\omega_1^2 + \sigma_1^2}}$$

$$\sigma_1 = - \zeta_1 \Omega_1$$

$$\Omega_1 = \sqrt{\omega_1^2 + \sigma_1^2}$$

The transfer function $H(s)$ can now be rewritten as a product of the roots (in factored) form as follows:

$$H(s) = \frac{1/M}{(s - \lambda_1) (s - \lambda_1^*)} \tag{2.12}$$

where:

- λ = pole of the transfer function
- $\lambda_1 = \sigma_1 + j\omega_1$
- $\lambda_1^* = \sigma_1 - j\omega_1$

The poles of the single degree of freedom system can also be viewed, looking down on the *s*-plane as shown in Figure (2-8) :

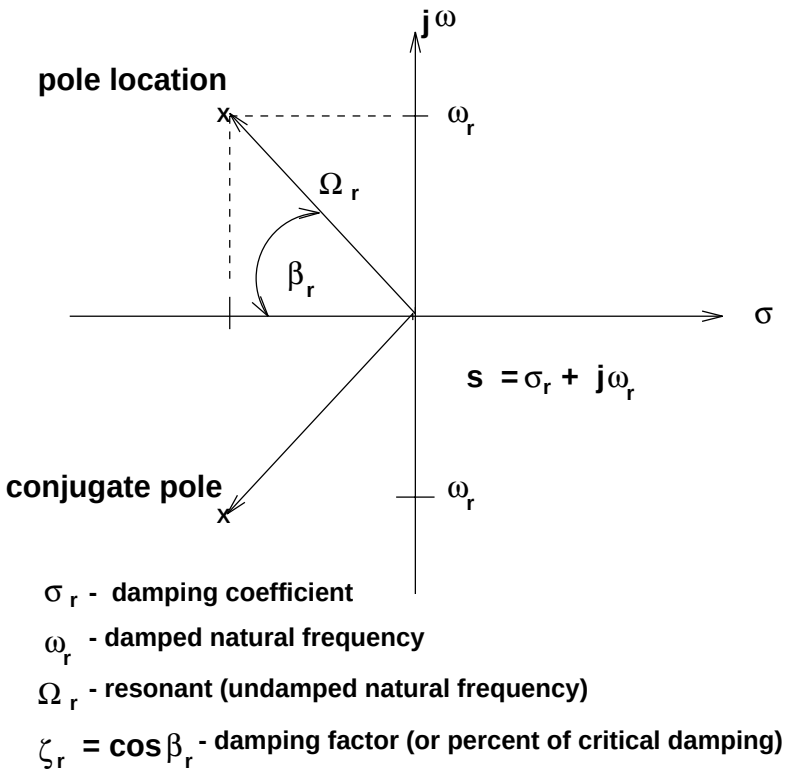

Figure 2-8. Laplace Plane Pole Location

Figures (2-9) through (2-11) illustrate a 3-dimensional plot of Equation 2.12. Figure (2-9) views the surface in real/imaginary format, Figure (2-10) represents the same data in a magnitude/phase format and Figure (2-11) uses a log magnitude/phase format. Remember that the variable *s* in Equation 2.12 is a complex variable, that is, it has a real part and an imaginary part. Therefore, it can be viewed as a function of two variables which represent a surface. Note also that the FRF measurement that is typically estimated is the slice through these surfaces where $s = j\omega$.

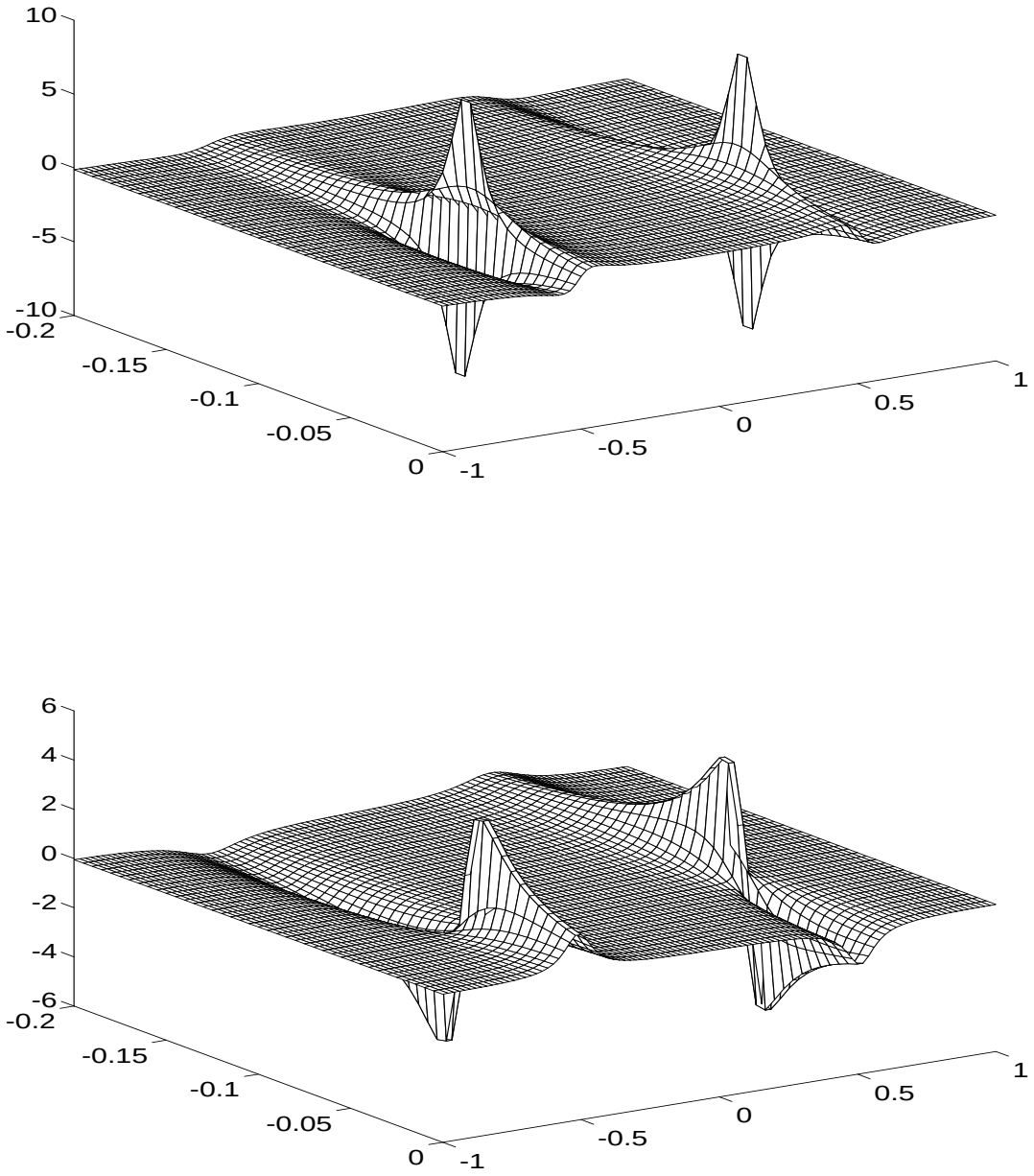

Figure 2-9. Transfer Function (Real-Imaginary), Surface Representation

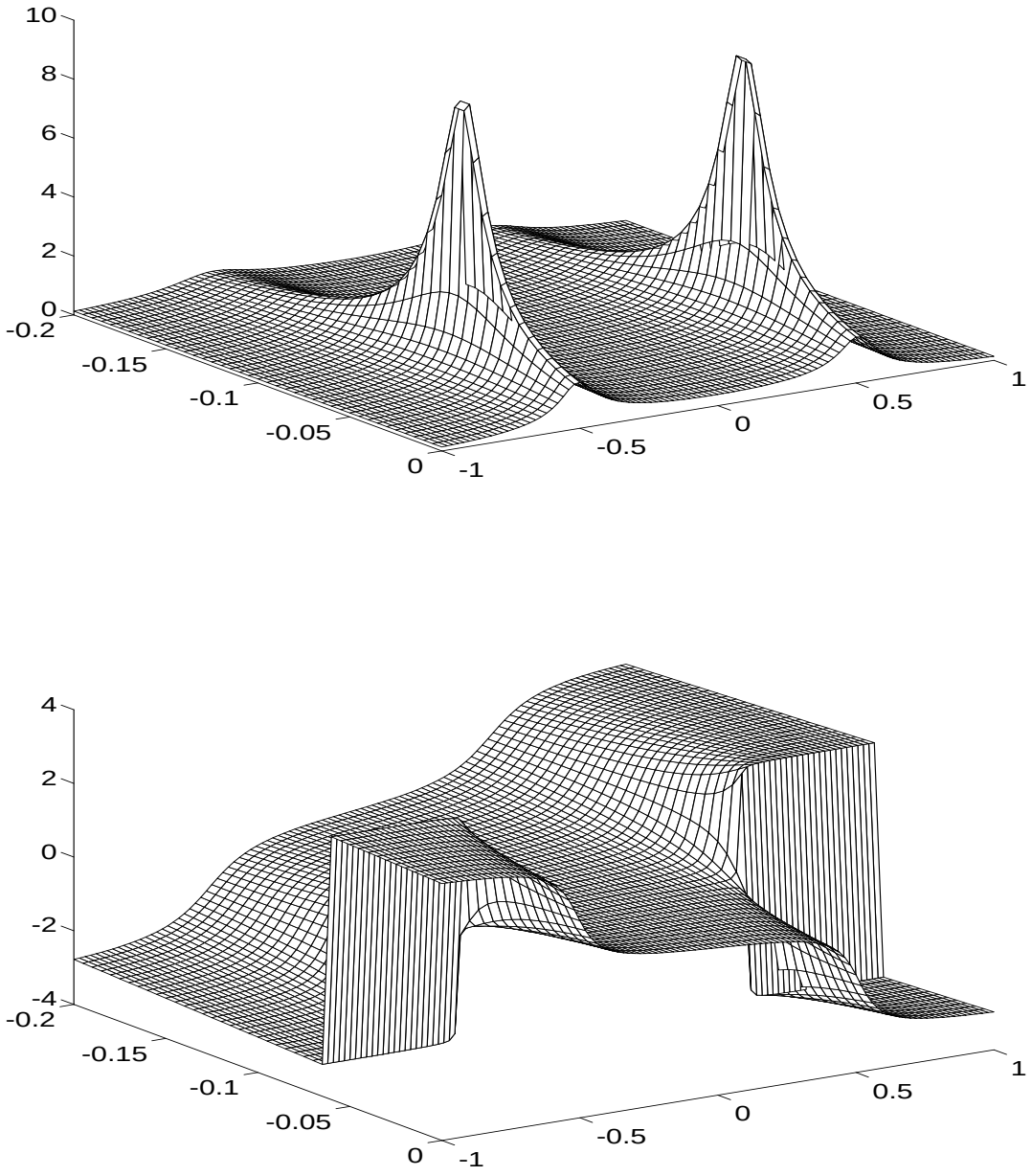

Figure 2-10. Transfer Function (Magnitude-Phase), Surface Representation

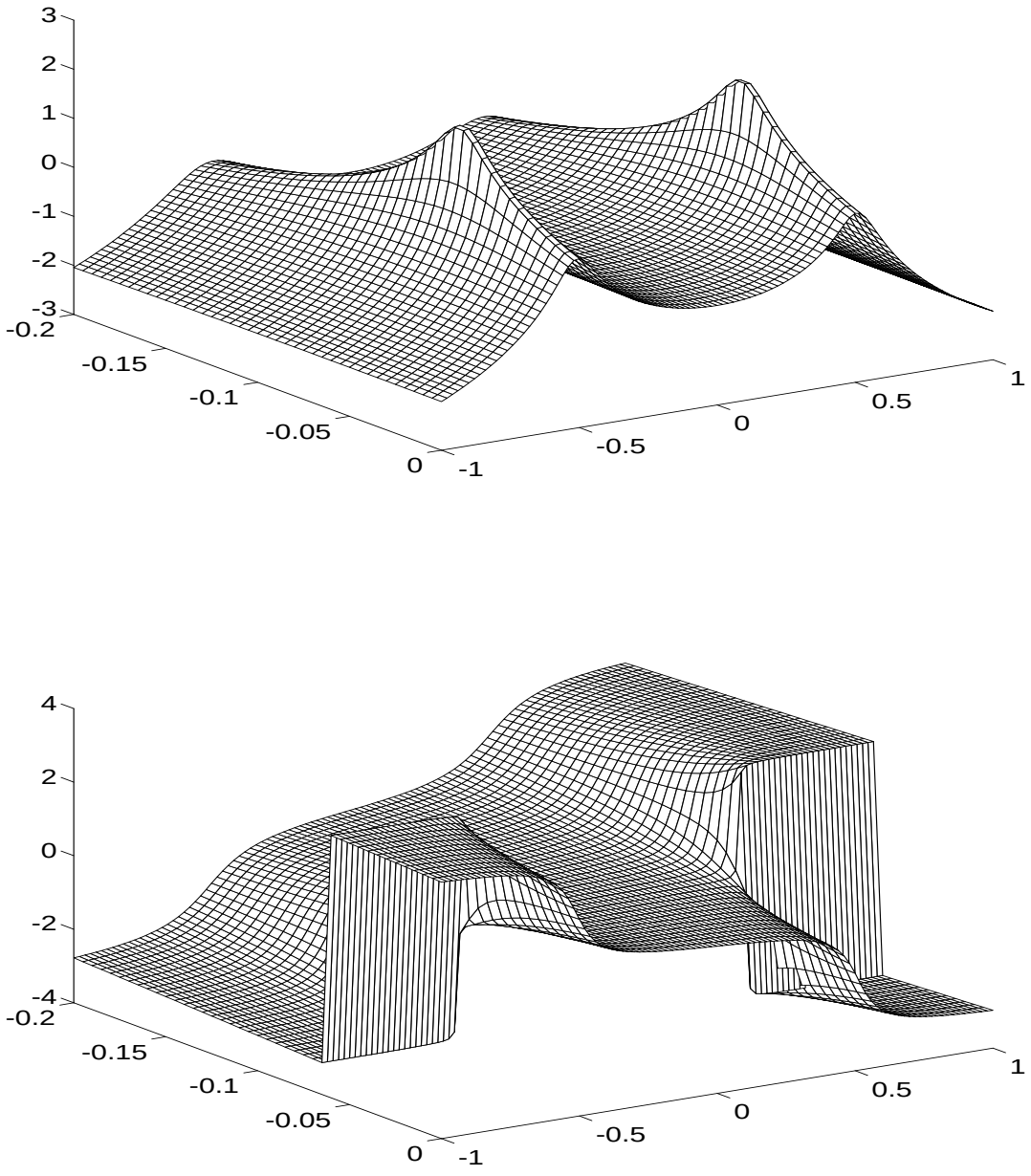

Figure 2-11. Transfer Function (Log Magnitude-Phase), Surface Representation

2.5 Analytical Model - Scalar Polynomial

One popular method of representing the transfer function involves a scalar polynomial representation in the numerator and denominator. For the single degree of freedom case, this is very simple concept that is directly based upon the physical characteristics (M,C,K) of the system. Generalizing Equation (2.9) yields:

$$H(s) = \frac{\beta_0}{\alpha_2 (s)^2 + \alpha_1 (s)^1 + \alpha_0 (s)^0} \tag{2.13}$$

This can be rewritten:

$$H(s) = \frac{\beta_0}{\sum_{k=0}^{2} \alpha_k (s)^k} \tag{2.14}$$

This model serves as the basis for many modal parameter estimation methods and is a common formulation utilzed in control theory applications..

2.6 Analytical Model - Partial Fraction

The concept of residues can now be discussed in terms of the partial fraction expansion of the transfer function equation. This is just one common approach to determining the residues. Another popular method involves a polynomial representation in the numerator and denominator.

Equation 2.12 can be expressed in terms of partial fractions:

$$H(s) = \frac{1/M}{(s - \lambda_1)(s - \lambda_1^*)} = \frac{c_1}{(s - \lambda_1)} + \frac{c_2}{(s - \lambda_1^*)} \tag{2.15}$$

The residues of the transfer function are defined as being the constants c_1 and c_2 . The

terminology and development of residues comes from the evaluation of analytic functions in complex analysis. As will be shown later, the residues of the transfer function are directly related to the amplitude of the impulse response function. The constants c_1 and c_2 (residues) can be found by multiplying both sides of Equation 2.15 by $(s - \lambda_1)$ and evaluating the result at $s = \lambda_1$. Thus:

$$\frac{1/M}{(s - \lambda_1^*)} \Big|_{s = \lambda_1} = c_1 + \left[\frac{c_2(s - \lambda_1)}{(s - \lambda_1^*)} \right] \Big|_{s = \lambda_1}$$

$$\frac{1/M}{(\lambda_1 - \lambda_1^*)} = c_1$$

Thus:

$$c_1 = \frac{1/M}{(\sigma_1 + j\ \omega_1) - (\sigma_1 - j\omega_1)} = \frac{1/M}{j\ 2\ \omega_1} = A_1$$

Similarly:

$$c_2 = \frac{1/M}{-j2\omega_1} = A_1^*$$

In general, for a multiple degree of freedom system, the residue A_1 can be a complex quantity. But, as shown for a single degree of freedom system A_1 is purely imaginary.

Therefore:

$$H(s) = \frac{A_1}{(\ s - \lambda_1\ )} + \frac{A_1^*}{(\ s - \lambda_1^*\ )} \tag{2.16}$$

2.7 Frequency Response Function Representation

The frequency response function is the transfer function (surface) evaluated along the $j\,\boldsymbol{\omega}$ (frequency) axis. Thus, from the previously dervied equations:

Polynomial Model

$$H(s)|_{s=j\omega} = H(\omega) = \frac{\beta_0}{\alpha_2 (j\omega)^2 + \alpha_1 (j\omega)^1 + \alpha_0 (j\omega)^0} \tag{2.17a}$$

Partial Fraction Model

$$H(s)|_{s=j\omega} = H(\omega) = \frac{A_1}{(j\omega - \lambda_1)} + \frac{A_1^*}{(j\omega - \lambda_1^*)}$$

$$H(\omega) = \frac{A_1}{(j\omega - \sigma_1 - j\omega_1)} + \frac{A_1^*}{(j\omega - \sigma_1 + j\omega_1)}$$

$$H(\omega) = \frac{A_1}{j(\omega - \omega_1) - \sigma_1} + \frac{A_1^*}{j(\omega + \omega_1) - \sigma_1} \tag{2.17b}$$

From an experimental point of view, when one talks about measuring a transfer function, the frequency response function is actually being measured.

The value of the frequency response function at the damped natural frequency of the system is:

$$H(\omega_1) = - \frac{A_1}{\sigma_1} + \frac{A_1^*}{j\ 2\ \omega_1 - \sigma_1} \tag{2.18}$$

which can be approximated as:

$$H(\omega_1) = - \frac{A_1}{\sigma_1}$$

The second term on the right of Equation 2.18 approaches zero as ω_1 gets large. In other words, the contribution of the negative frequency portion of the frequency response function is negligible.

Therefore, many single degree of freedom models are represented as:

$$H(\omega) \approx \frac{A_1}{(\ j\omega - \lambda_1)} \tag{2.19}$$

Another way of interpreting Equation 2.17 is that the value of the transfer function, for a single degree of freedom system, at a particular frequency (ω) is a function of the residue, damping, and damped natural frequency.

2.8 Impulse Response Function Representation

The impulse response function of the single degree of freedom system can be determined from Equation 2.17 assuming that the initial conditions are zero and that $F(s) = 1$ for a system impulse. Thus:

$$X(s) = \frac{A_1}{(s - \lambda_1)} + \frac{A_1^*}{(s - \lambda_1^*)}$$

$$x(t) = \mathbf{L}^{-1} \{ X(s) \}$$

$$x(t) = A_1 e^{\lambda_1 t} + A_1^* e^{\lambda_1^* t} = h(t) = \text{impulse response}$$

$$x(t) = e^{\sigma_1 t} \left[A_1 e^{j \omega_1 t} + A_1^* e^{-j \omega_1 t} \right]$$

Thus, using Euler's formula for $e^{j\omega_1 t}$ and $e^{-j\omega_1 t}$, the residue A_1 controls the initial amplitude of the impulse response, the real part of the pole is the decay rate and the imaginary part of the pole is the frequency of oscillation. Figures (2-12) and (2-13) illustrate the frequency response and impulse response functions respectively, for a single degree of freedom system.

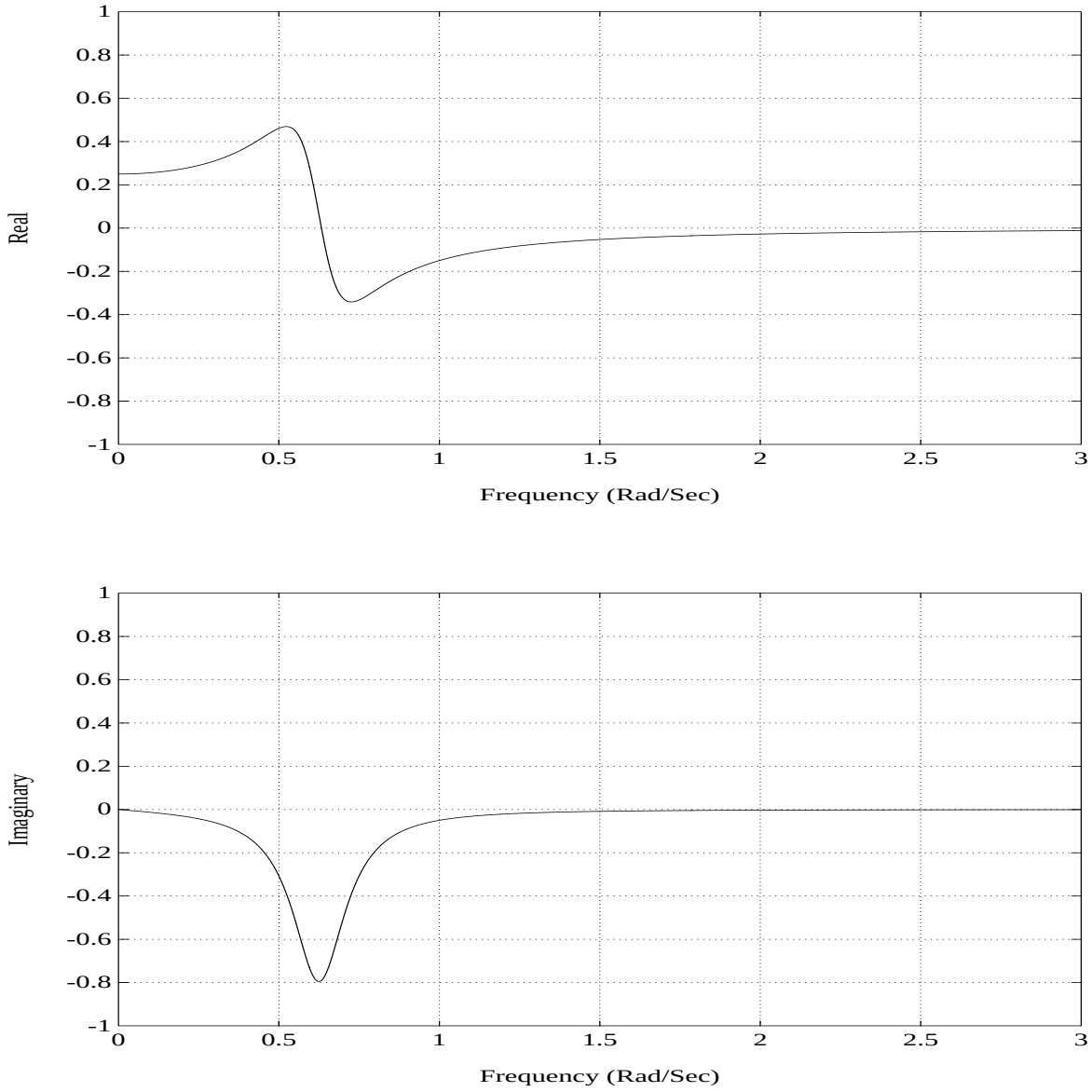

Figure 2-12. Frequency Response Function: SDOF System

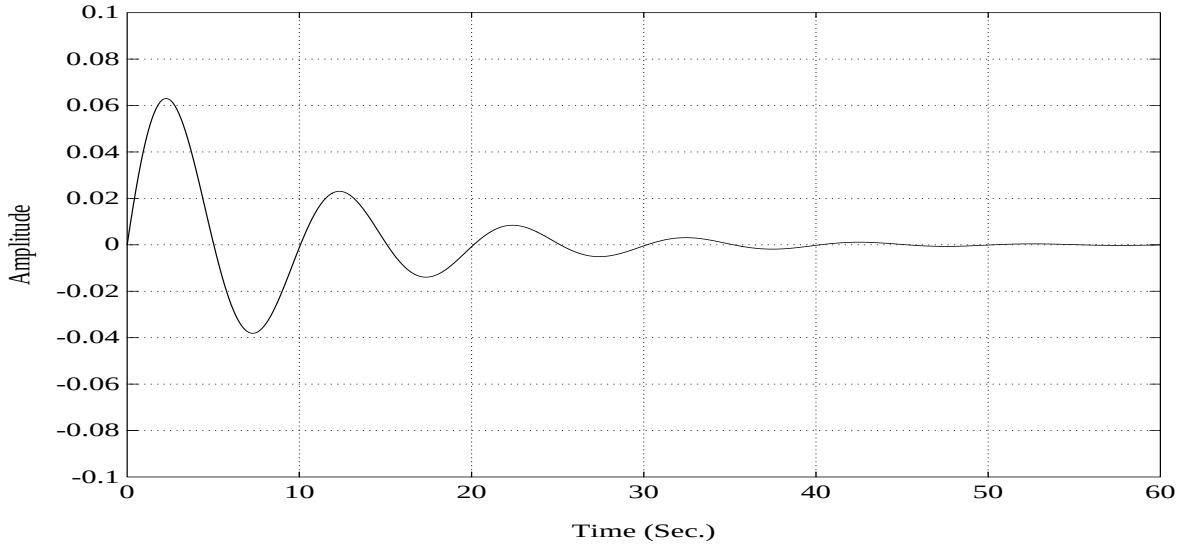

Figure 2-13. Impulse Response Function: SDOF System

2.9 Change of Physical Parameters

While it is not always possible to alter the physical parameter (mass, stiffness and/or damping) of a system and is very difficult to practically alter one physical parameter (mass, for example) without altering another physical parameter (stiffness, for example), it is still important to understand how a change in physical parameter will affect the system characteristics. Figures (2-14), (2-15) and (2-16) show how the frequency response function will be affected due to a change in one physical parameter at a time.

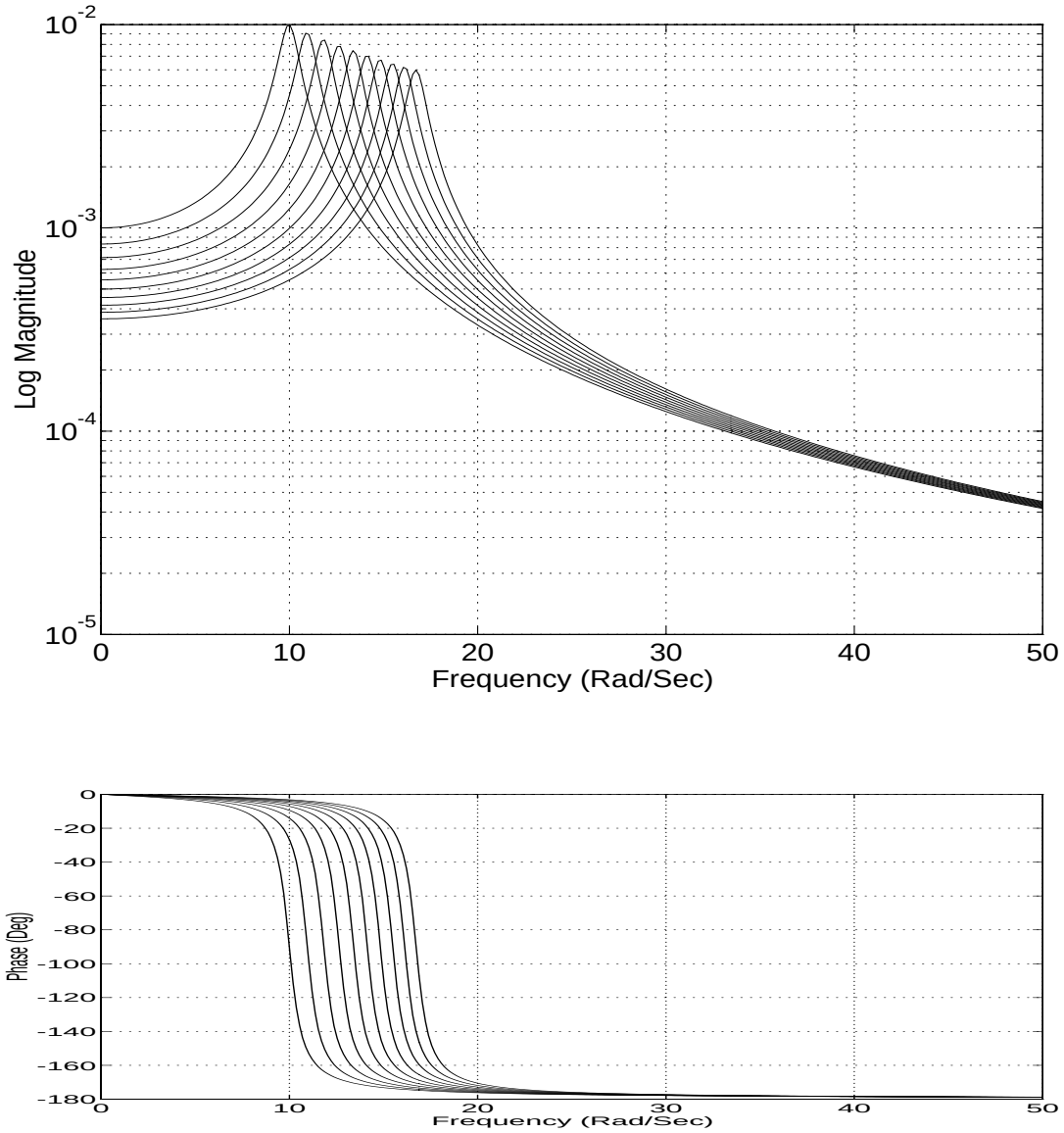

Figure 2-14. Change of Stiffness: SDOF System

Note that a change in stiffness affects both the resonant frequency as well as the system characteristic at low frequency. This dominance of stiffness at low frequency is the reason that this region of the frequency response function is known as the stiffness, or more accurately, compliance line.

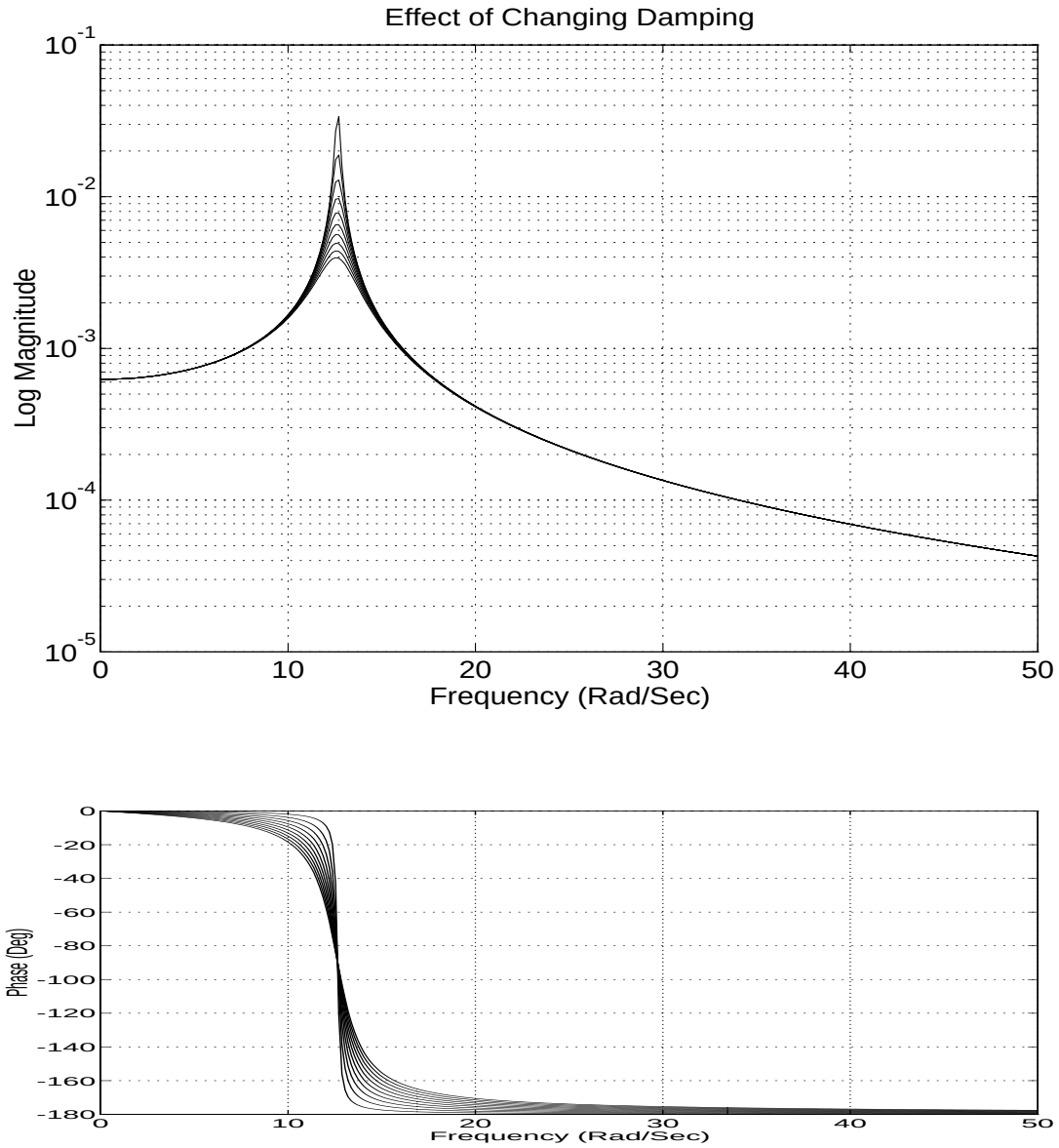

Figure 2-15. Change of Damping: SDOF System

Note that a change in damping has no apparant effect on the resonant frequency. The only noticable change involves a change in frequency response function in the region of the resonant frequency.

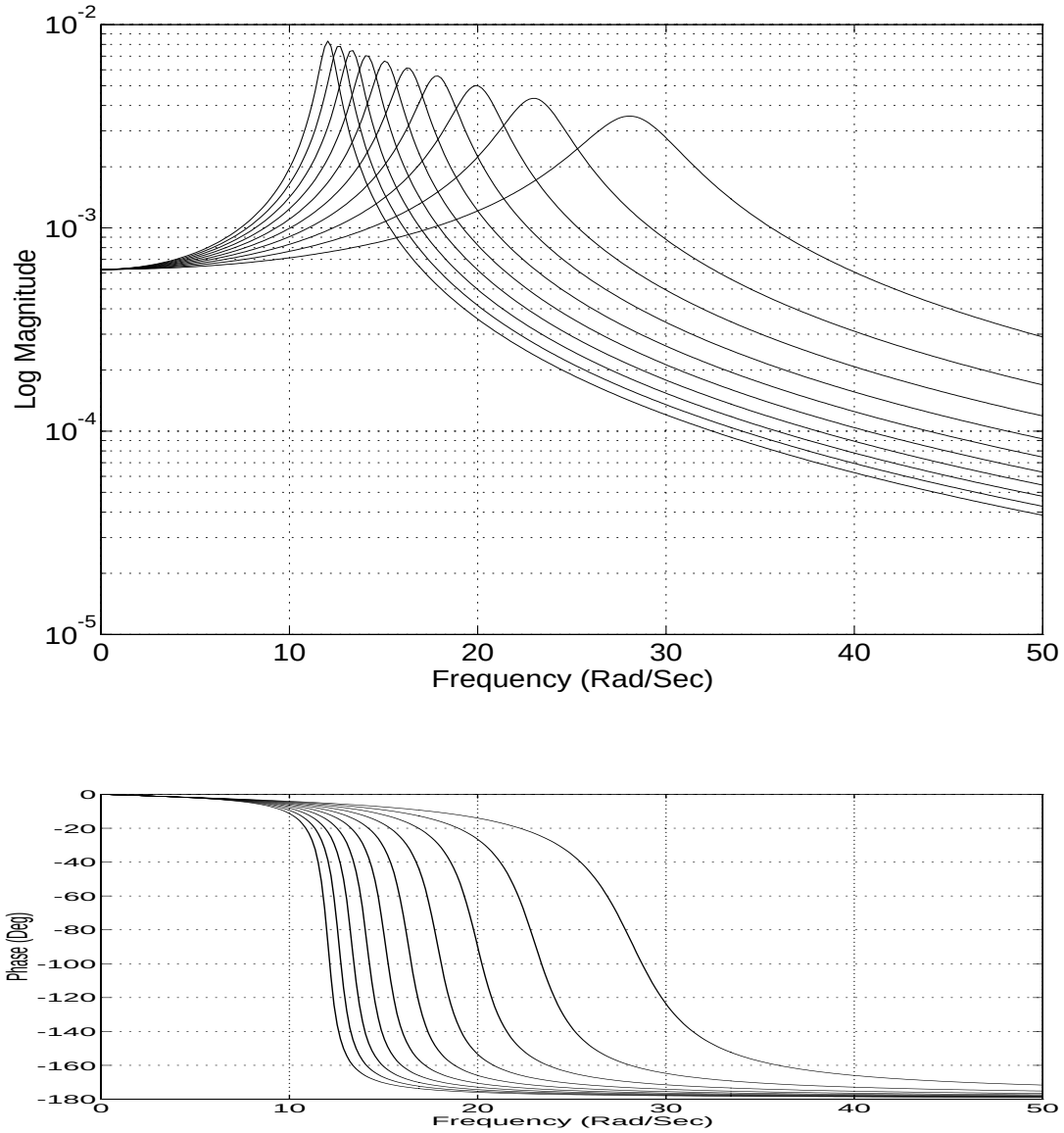

Figure 2-16. Change of Mass: SDOF System

Note that a change in mass affects both the resonant frequency as well as the system characteristic at high frequency. This dominance of mass at high frequency is the reason that this region of the frequency response function is known as the mass line. Also note that as the mass changes, the apparant damping (sharpness of the resonant frequency) changes accordingly. A change in mass affects both the resonant frequency, the system characteristic at high frequency as

well as the fraction of critical damping ($\zeta_1 = \frac{C}{C_c} = \frac{C}{2\ M\ \Omega_1}$).

2.10 Estimating Partial Fraction Parameters

Assuming that a lightly damped single degree of freedom (SDOF) system is being evaluated, the parameters needed for a partial fraction model can be quickly estimated directly from the measured frequency response function. While this approach is based upon a SDOF system, as long as the modal frequencies are not too close together, the method can be used for multiple degree of freedom (MDOF) systems as well.

Starting with the partial fraction model formulation of a SDOF frequency response function

$$H(\omega) = \frac{A_1}{(\ j\omega - \lambda_1\ )} + \frac{A_1^*}{(\ j\omega - \lambda_1^*\ )} \tag{2.20}$$

only the constants λ_1 and A_1 must be estimated. Since $\lambda_1 = \sigma_1 + j\omega_1$, the estimation process begins by estimating ω_1. The damped natural frequency ω_1 is estimated in one of three ways:

- The frequency where the magnitude of the FRF reaches a maximum.
- The frequency where the real part of the FRF crosses zero.
- The frequency where the imaginary part of the FRF reaches a relative minima (or maxima).

Of these three methods, the last approach gives the most reliable results under all conditions.

Once the damped natural frequency ω_1 has been estimated, the real part of the modal frequency, the damping factor σ_1, can be estimated. The damping factor σ_1 can be estimated by using the *half-power bandwidth* method. This method uses the data from the FRF in the region of the resonance frequency to estimate the fraction of critical damping from the following formula:

$$\zeta_1 = \frac{{\omega_b}^2 - {\omega_a}^2}{(2\ \omega_1)^2} \tag{2.21}$$

In the above equation, ω_1 is the damped natural frequency as previously estimated. ω_a is the frequency, below ω_1, where the magnitude is 0.707 of the peak magnitude of the FRF. This corresponds to a half power point. ω_b is the frequency, above ω_1, where the magnitude is 0.707 of the peak magnitude of the FRF. This also corresponds to a half power point.

For lightly damped systems, the above equation can be approximated by the following:

$$\zeta_1 \approx \frac{\omega_b - \omega_a}{2\,\omega_1} \tag{2.22}$$

Once ζ_1 is estimated, the damping factor σ_1 can be estimated from the following equation.

$$\sigma_1 = -\zeta_1\,\Omega_1 \tag{2.23}$$

Again, assuming that the system is lightly damped, $\Omega_1 \approx \omega_1$, the damping factor can be estimated from the following equation:

$$\sigma_1 \approx -\zeta_1\,\omega_1 \tag{2.24}$$

Once the modal frequency λ_1 has been estimated, the residue A_1 can be estimated by evaluating the partial fraction model at a specific frequency. If the specific frequency is chosen to be ω_1, the following result is obtained.

$$H(\omega_1) = \frac{A_1}{(\,j\omega_1 - (\sigma_1 + j\omega_1\,)} + \frac{A_1^*}{(\,j\omega_1 - (\sigma_1 - j\omega_1\,)} \tag{2.25}$$

$$H(\omega_1) = \frac{A_1}{(-\sigma_1\,)} + \frac{A_1^*}{(\,2\,j\,\omega_1 - \sigma_1\,)} \tag{2.26}$$

As long as ω_1 is not too small, the above equation can be approximated by:

$$H(\omega_1) \approx \frac{A_1}{(-\sigma_1\,)} \tag{2.27}$$

Therefore, the residue A_1 can be estimated from the following relationship:

$$A_1 \approx (-\sigma_1)\, H(\omega_1) \tag{2.28}$$

In the above relationship, $H(\omega_1)$ is very close to being a purely imaginary value for the displacement over force FRF. This means that the residue A_1 will be very close to a purely imaginary value as well.

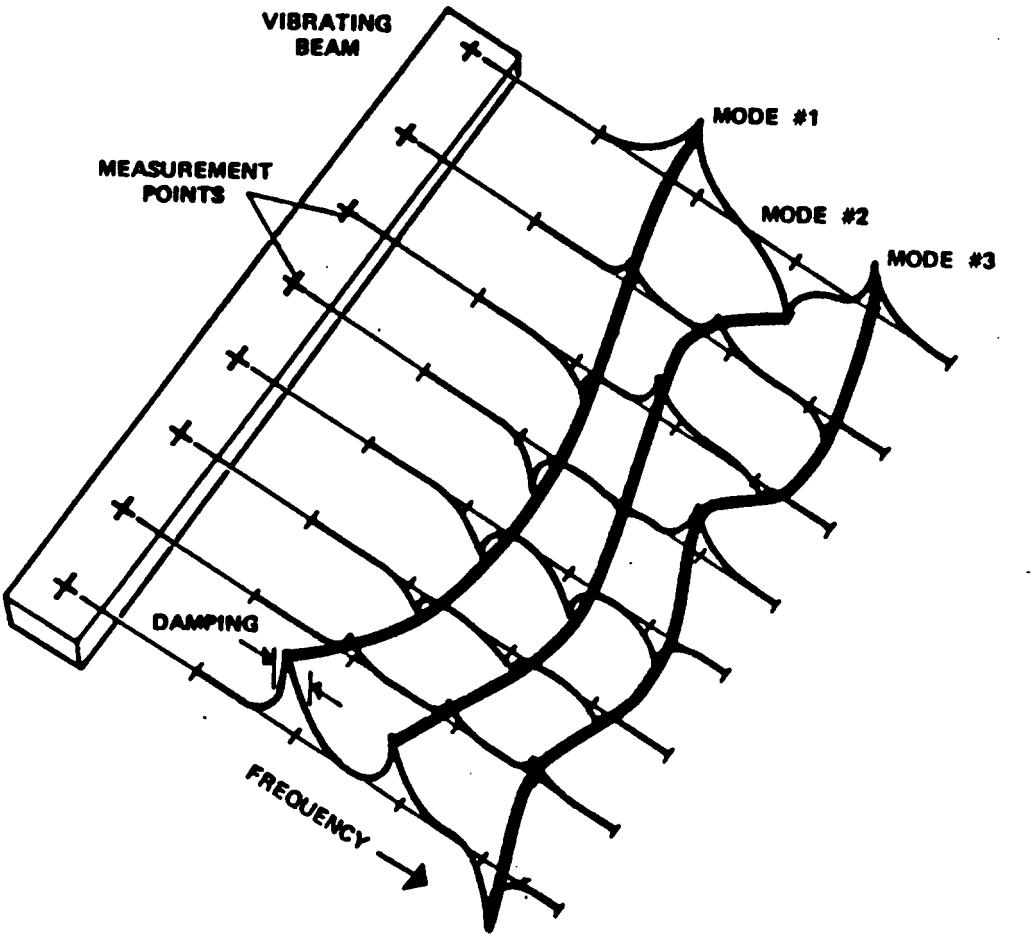

Figure 2-17. Modal Vectors from the Imaginary Part of the FRF

2.10.1 Example

For the following SDOF case, (M=10, K=1600, C=2), the data can be estimated from the FRF as shown in the following plots. The exact answers are λ_1 = -0.1000 + j 12.6487 and A_1 = -j 0.0040. Note that the digitized data, in the neighborhood of the damped natural frequency, is tabulated in Table 2-1.

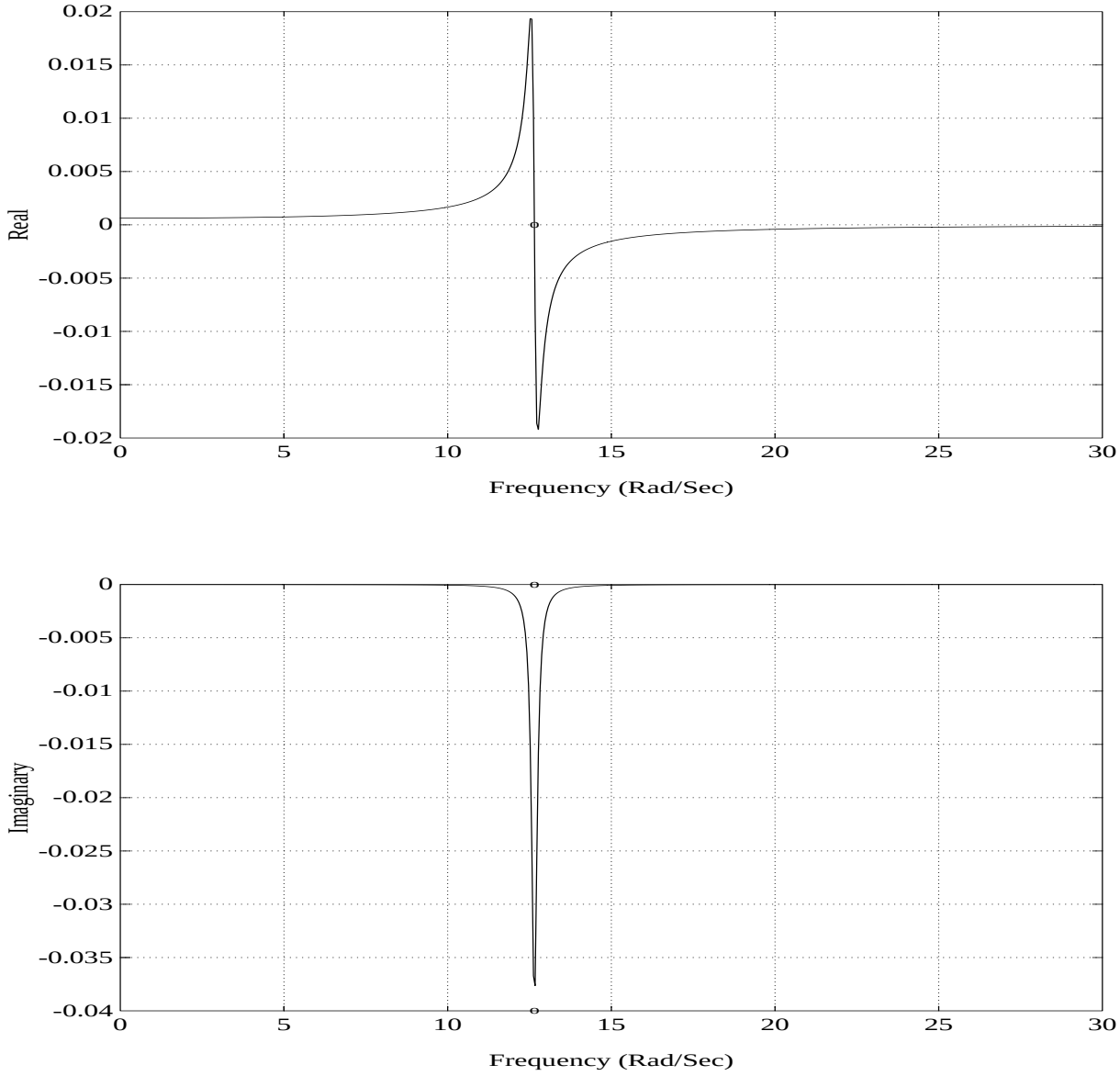

Figure 2-18. Frequency Response Function: SDOF System

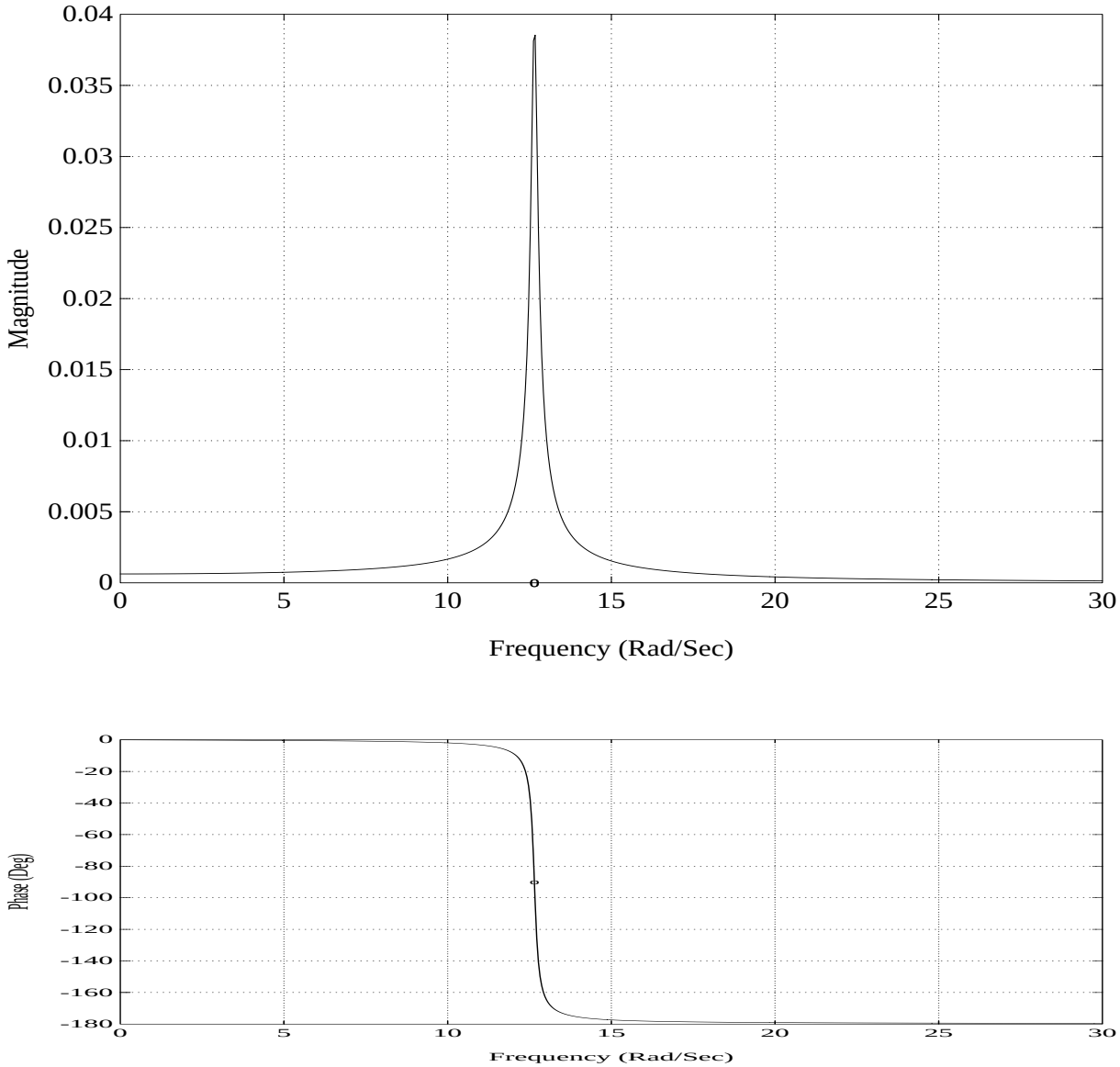

Figure 2-19. Frequency Response Function: SDOF System

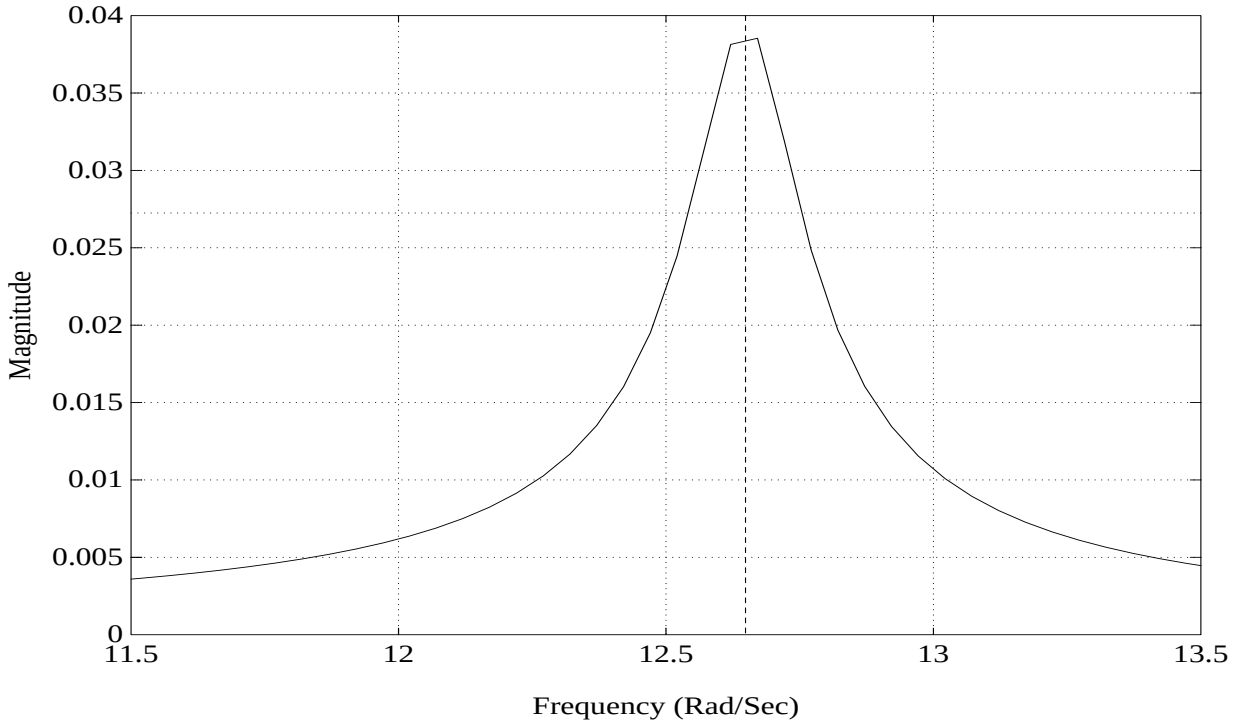

Figure 2-20. Frequency Response Function: SDOF System

Frequency (rad/sec)	Real	Imag	Magnitude	Phase (deg)
12.2705	0.0099	-0.0026	0.0103	-14.578
12.3205	0.0112	-0.0034	0.0117	-16.717
12.3706	0.0127	-0.0045	0.0135	-19.549
12.4207	0.0147	-0.0064	0.0160	-23.452
12.4708	0.0170	-0.0095	0.0195	-29.109
12.5209	0.0193	-0.0150	0.0245	-37.804
12.5710	0.0193	-0.0246	0.0313	-51.903
12.6210	0.0103	-0.0367	0.0381	-74.301
12.6711	-0.0083	-0.0376	0.0385	-102.401
12.7212	-0.0186	-0.0259	0.0319	-125.711
12.7713	-0.0192	-0.0158	0.0249	-140.565
12.8214	-0.0170	-0.0099	0.0197	-149.696
12.8715	-0.0146	-0.0066	0.0160	-155.597
12.9215	-0.0126	-0.0047	0.0135	-159.646
12.9716	-0.0110	-0.0035	0.0115	-162.569

TABLE 2-1. Discrete SDOF Data from Plot

This example also illustrates a common problem with simplified modal parameter estimation. In this example, referring to Figure 2-20, it is apparent that the damped natural frequency occurs between two of the measured frequencies in the frequency respone function. In Figure 2-20, the apparent truncation, or clip, of the frequency response function near the peak frequency is a result of this lack of resolution. For a lightly damped situation, the true magnitude at the resonance may be 2 to 20 times higher. This means that finding the half-power frequencies in order to estimate damping will be impossible since the true magnitude of the resonance is unknown. In this situation, the damping estimate will be in error (too high) which will cause the residue to be in error (too high).

Note that there are a number of other more robust SDOF, and MDOF, modal parameter estimation algorithms that do not require knowledge of the half-power frequencies. These techniques depend only upon the data being accurate at the measured frequencies in order for an accurate estimate of the damping estimate. These methods do not have the accuracy problem of the simplified SDOF case utilized in this example.

2.11 Why Study/Emphasize SDOF Systems?

Frequently, there is some concern that the amount of time studying SDOF systems is not warranted. Many multiple degree-of-freedom (MDOF) systems can be simplified as single degree-of-freedom systems. More often, the MDOF system can be broken down, on a frequency range basis, into frequency regions that are dominated by only one degree-of-freedom. This is the situation with SDOF modal parameter esimation algorithms.

Even though a continuous beam has an infinite number of modes, the evaluation of these modes (estimation of frequency, damping, modal vector and modal scaling) can often be accomplished with essentially single degree-of-freedom (SDOF) concepts. The primary assumption is that each mode of vibration is well separated in frequency from the other modes. This is often the case for lightly damped structures. Different modes of vibration of the beam can be visulaized in Figure 2-17 by noting the solid black line connecting the peaks of the imaginary parts of each frequency response. Normally, these modes are plotted in a wireframe model showing the extrema of the modal vector so that the motion can be easily understood. Figures 2-21 and 2-22 show the first two bending modes of a uniform beam that is pinned at each end.

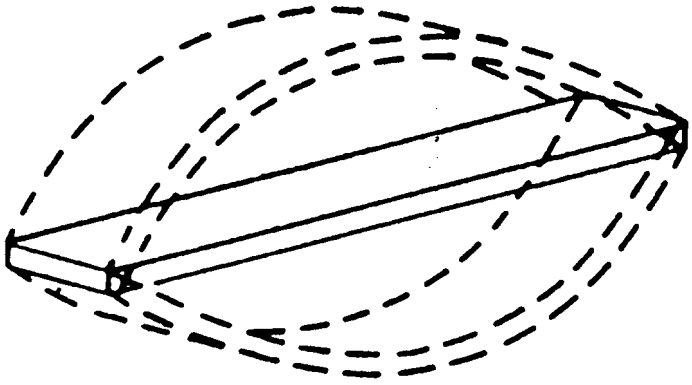

Figure 2-21. First Bending Mode at First Damped Natural Frequency (ω_1)

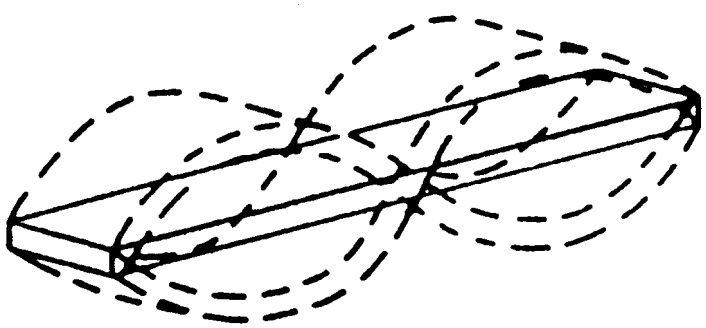

Figure 2-22. Second Bending Mode at Second Damped Natural Frequency (ω_2)

In order to understand why this information can be determined from the imaginary part of the frequency response functions, SDOF theory must be reviewed and extended slightly, primarily from a notational point of view.

The general mathematical representation of a single degree of freedom system is expressed using Newton's second law in Equation 2-29:

$$M\ \ddot{x}(t)\ +\ C\ \dot{x}(t)\ +\ K\ x(t)\ =\ f(t) \tag{2-29}$$

For the general case with a forcing function that can be represented as a summation of sin and cosine terms, the forcing function can be represented as:

$$f(t)\ =\ \sum_{\omega=0}^{\infty} F(\omega)\ e^{j\omega t} \tag{2-30}$$

Assuming that the system is underdamped and that enough time has passed that any transient response of the system due to initial condition or startup of the excitation has decayed to zero, the response of the system can be represented as:

$$x(t)\ =\ \sum_{\omega=0}^{\infty} X(\omega)\ e^{j\omega t} \tag{2-31}$$

Note that, while $x(t)$ and $f(t)$ are real valued functions, $X(\omega)$ and $F(\omega)$ are complex valued. Working with any arbitrary frequency term in Equations 2-30 and 2-31, Equation 2-30, Equation 2-31 and the derivatives of Equation 2-31 can substituted into Equation 1 yielding the following frequency response function (FRF) relationship for a SDOF system:

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{-M\ \omega^2 + C\ j\omega + K} \tag{2-32}$$

Note the characteristic of the above frequency response function when it is evaluated (measure) at the undamped natural frequency. At the undamped natural frequency, the mass and stiffness terms cancel each other and the FRF is purely imaginary valued.

The first extension that is necessary provides a description for the case where $x(t)$ and $f(t)$ are not located at the same point. On a single degree-of-freedom system, this would provide redundant information (no new information) but it becomes important as the extension to multple degrees-of-freedom occurs. For example, assume that the particular point (and direction) on the mass where the force is applied is referred to as DOF p and the particular point (and direction) on the mass where the response is measured is referred to as DOF q. Equation 2-32 now can be written as follows to note this information.

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} = \frac{1}{-M\ \omega^2 + C\ j\omega + K} \tag{2-33}$$

The system is still a SDOF system so $H_{qp} = H_{pp} = H_{qq} = H_{qs} = \cdots$ but the input and output location can now be described. This clearly demonstrates that the number of modes (one in this case) is unrelated to the number of input and output sensors that are used to measure the system.

The second extension that is necessary provides a way to indicate that the modal characteristics (modal coefficients) of both the input and output are represented in the frequency response function model. The modal frequency is already represented by noting that the denominator is related to the characteristic equation. A form of modal scaling is already represented by noting the the mass term in the denominator scales the equation. Modal coefficient information, which is relative not absolute information, can be added by changing the numerator to reflect this.

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} = \frac{\psi_q\ \psi_p}{-M\ \omega^2 + C\ j\omega + K} \tag{2-34}$$

Note that, since the system is still a SDOF system, the relative motion at each DOF would be normalized to 1 such that $\psi_p = \psi_q = \psi_s = 1$ which shows that Equation 2-33 and 2-34 still represent the same information. Note as before, if the FRF is evaluated (measured) at the undamped natural frequency), the FRF is once again imaginary valued and is a function of the modal coefficients and damping. Assuming the damping is unknown but constant means that the product of the modal coefficients is proportional to the imaginary part of the FRF.

Finally, the third extension that is necessary provides for the change from SDOF to MDOF. Note that for a linear system, linear superposition can be used in the frequency domain to add the information associated with each mode together to represent the frequency response function of a MDOF system. To desribe this, every term in Equation 2-34 will need a subcript (r) to indicate which mode the information is associated with. The final form of the frequency response function is:

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} = \sum_{r=1}^{\infty} \frac{\psi_{qr}\ \psi_{pr}}{-M_r\ \omega^2 + C_r\ j\omega + K_r} \tag{2-35}$$

Equation 2-35 is one common representation of the FRF of a MDOF system. Note that the M_r, C_r and K_r terms in the denominator are the modal or generalized mass, damping and stifness parameters, not the physical mass, damping and stiffness parameters. The modal or generalized parameters can be found analytically from the physical mass, damping and stiffness parameters or experimentally using more complicated parameter estimation algorithms.

Note that, as long as the modes are well separated in frequency, the information in the neighborhood of the undamped natural frequency for a given mode can be found from:

$$H_{qp}(\omega) = \frac{X_q(\omega)}{F_p(\omega)} \approx \frac{\psi_{qr}\ \psi_{pr}}{M_r\ \omega^2 + C_r\ j\omega + K_r} \tag{2-36}$$

Note that, if the output DOF (point and location) is held fixed while the input DOF is moved, the only information that changes in Equation 2-36 as different FRFs are measured is the information relative to the modal coefficient for the particular mode of interest. If Equation 2-36 is evaluated (measured) near the undamped natural frequency, this means that the imaginary part of the FRF will be proportional to the modal coefficient. The proportionality constant α_r) is:

$$\alpha_r \approx \frac{\psi_{pr}}{-M_r\ \omega^2 + C_r\ j\omega + K_r} \tag{2-37}$$

Since mode shapes are relative patterns, not absolute motions, the value of the constant is not important unless damping or modal scaling is required.

Therefore, modal vectors can be estimated from the imaginary part of the frequency response functions at the damped natural frequencies (or from the magnitude and phase information of the frequency response functions at the damped natural frequencies). This will be reasonably accurate as long as the undamped natural frequencies are well separated and the damping is small (undamped and damped natural frequencies nearly equal).

This result is consistent with the expansion theorem concept (the response of the system at any instant in time or at any frequency is a linear combination of the modal vectors):

Expansion Theorem - Time Domain:

$$\{\ x(t_i)\ \} = \sum_{r=1}^{N} \beta_r\ \{\ \psi_r\ \} \tag{2-38}$$

Expansion Theorem - Frequency Domain:

$$\{\ X(\omega_i)\ \} = \sum_{r=1}^{N} \beta_r\ \{\ \psi_r\ \} \tag{2-39}$$

Using the frequency domain form of the expansion theorem, if the response is evaluated at the undamped natural frequency of mode r, the expansion coefficient β_r will dominate and be approximately equal to alpha defined in Equation 2-37.

3. MULTIPLE DEGREE OF FREEDOM SYSTEMS

3.1 Theory

Generally, most structures are more complicated than the single mass, spring, and damper system discussed in the previous section. The general case for a multiple degree of freedom system will be used to show how the frequency response functions of a structure are related to the modal vectors of that structure. Throughout the following section the following two degree of freedom system will be used to illustrate the concepts discussed.

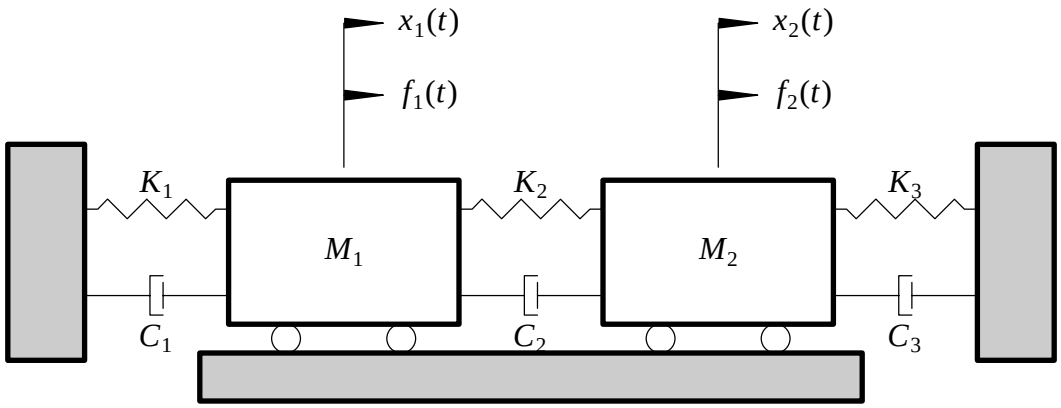

Figure 3-1. Two Degree of Freedom System

The equations of motion for the above system are:

$$M_1\ddot{x}_1(t) + (C_1 + C_2)\,\dot{x}_1(t) - C_2\,\dot{x}_2(t) + (K_1 + K_2)\,x_1(t) - K_2\,x_2(t) = f_1(t)$$

$$M_2\ddot{x}_2(t) + (C_2 + C_3)\,\dot{x}_2(t) - C_2\,\dot{x}_1(t) + (K_2 + K_3)\,x_2(t) - K_2\,x_1(t) = f_2(t)$$

In matrix notation:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} + \begin{bmatrix} (C_1 + C_2) & -C_2 \\ -C_2 & (C_2 + C_3) \end{bmatrix}\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix}$$

$$+ \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \tag{3.1}$$

The above equations are still second order, linear, time invariant, differential equations, but are now coupled by the coordinate choice. Therefore, this system of equations must be solved simultaneously. The process of solving the set of equations in Equation 3.1 will now be reviewed in an analytical sense. The modal vectors and frequencies will result as the solution to the homogeneous portion of the differential equations summarized in Equation 3.1.

The solution of the above system of second order differential equations is first obtained for the undamped system. Assuming that $C_1 = C_2 = C_3 = 0$.

$$[\ M\ ]\{\ \ddot{x}(t)\ \} + [\ K\ ]\{\ x(t)\ \} = \{\ f(t)\ \} \tag{3.2}$$

where:

- $[\ M\ ] = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$ = Mass Matrix

- $[\ K\ ] = \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix}$ = Stiffness Matrix

- $\{\ f\ \} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$ = Forcing Vector

- $\{ x \} = \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\}$ = Response Vector

Since the forcing and response vectors are always functions of time, the functional notation (t) will be dropped in further equations.

The system of equations represented by Equation 3.2 has the general solution of:

$$\{ x \} = \{ X \} e^{s t}$$

Thus:

$$\{ \dot{x} \} = s \{ X \} e^{s t} = s x$$

$$\{ \ddot{x} \} = s^2 \{ X \} e^{s t} = s^2 x \tag{3.3}$$

where:

- $s = \sigma + j \omega$ = complex valued frequency

Substituting Equation 3.3 into Equation 3.2 yields:

$$s^2 [M] \{ X \} + [K] \{ X \} = \{ f \}$$

If there are no forcing function so that $\{ f \} = \{ 0 \}$, then:

$$s^2 [M] \{ X \} + [K] \{ X \} = \{ 0 \}$$

$$\left(s^2 [M] + [K] \right) \{ X \} = \{ 0 \} \tag{3.4}$$

Equation 3.4 is nothing more than a set of simultaneous algebraic equations in X_i. The unknowns are the X's and the s's. From the theory of differential equations, in order for Equation 3.4 to have other than the trivial solution, $\{ X \} = \{ 0 \}$, the determinant of the coefficients must equal zero. The determinant of the coefficients will be a polynominal in s^2. The roots of this polynominal are called eigenvalues.

In order to manipulate Equation 3.4 into a standard eigenvalue-eigenvector form, Equation 3.4 can be reformulated in a couple of different ways. First, divide Equation 3.4 by s^2 and premultiply by $[K]^{-1}$.

$$\left[[\ K\ ]^{-1} [\ M\ ] + \frac{1}{s^2} [\ I\ ] \right] \{\ X\ \} = \{\ 0\ \} \tag{3.5}$$

A different way of formulating the eigenvalue problem would be to premultiply Equation 3.4 by $[M]^{-1}$. Note that by doing this, the resulting dynamic matrix, $[\ K\ ]^{-1}[\ M\ ]$ in Equation 3.5 or $[\ M\ ]^{-1}[\ K\ ]$ in Equation 3.6, is no longer symmetric.

$$\left[[\ M\ ]^{-1} [\ K\ ] + s^2 [\ I\ ] \right] \{\ X\ \} = \{\ 0\ \} \tag{3.6}$$

In Equation 3.5 the eigenvalues are $\frac{1}{s^2}$ and in Equation 3.6 the eigenvalues are s^2. Equations 3.5 and 3.6 are really just the inverse of each other. In Equation 3.5 or Equation 3.6, the matrix on the left hand side of the equation is often referred to as the dynamic matrix. Note that the multiplication of Equation 3.4 by a matrix to obtain Equation 3.5 or 3.6 amounts to a coordinate transformation.

The frequency of a mode of vibration is defined in terms of the eigenvalue. The solution vector $\{X\}$ of Equation 3.5 or 3.6 corresponding to a particular eigenvalue is called an eigenvector, characteristic vector, mode shape, or modal vector. The X's represent a deformation pattern of the structure for a particular frequency of vibration. Since Equations 3.5 or 3.6 are homogeneous there is not a unique solution for the X's; only a relative pattern or ratio among the X's can be obtained. In other words, the X's can only be solved for in terms of one of the X's, which in turn can be given any arbitrary value. Mathematically, the rank of the equation systems represented by Equation 3.5 or 3.6 is always one less than the number of equations.

Therefore, the deflected deformation of a structure, which describes a natural mode of vibration, is defined by known ratios of the amplitude of motion at the various points on the structure. Thus, the actual amplitude of vibration of a structure is a combination of the modal vector and the level, location, and characteristic of excitation forces and not a direct property of a natural mode of vibration. The amplitude of vibration is really dependent on the placement and

amplitude of the systems forcing functions along with any initial conditions of the system together with the properties of the structure described by the eigenvalues and eigenvectors.

3.2 Solution of the Eigenvalue Problem

The solution of either Equation 3.5 or 3.6 is obtained by recognizing that these equations are a set of homogeneous equations. Therefore, for a non-trivial solution, the determinant of the coefficients must equal zero.

$$\left| [\ K\ ] + s^2\ [\ M\ ] \right| = 0. \tag{3.7a}$$

$$\left| [\ M\ ]^{-1}\ [\ K\ ] + s^2\ [\ I\ ] \right| = 0. \tag{3.7b}$$

$$\left| [\ K\ ]^{-1}\ [\ M\ ] + \frac{1}{s^2}\ [\ I\ ] \right| = 0. \tag{3.7c}$$

The determinant in Equation 3.7 is referred to as the characteristic determinant. The expansion of the characteristic determinant results in the *characteristic equation* or the *frequency equation*.

Equation 3.7 may be rewritten as:

$$\alpha^n + a_1\ \alpha^{n-1} + a_2\ \alpha^{n-2} + \ldots\ + a_n = 0. \tag{3.8}$$

Equation 3.8 is the characteristic equation of a N-degree of freedom system, where $\alpha = s^2$ for Equation 3.7a or 3.7b or $\alpha = \frac{1}{s^2}$ for Equation 3.7c. The roots of Equation 3.8 are the eigenvalues of the system. Note that the values of s corresponding to the roots of Equation 3.8 are the complex-valued modal frequencies ($\lambda_r = \sigma_r + j\ \omega_r$).

3.2.1 Two Degree of Freedom Example: Undamped, Unforced

Given a two degree of freedom system (Equation 3.1), find its eigenvalues (undamped natural frequencies) and the respective eigenvectors (modal vectors) for the undamped system.

Referring to Figure (3-1), let:

- $M_1 = 5 \qquad M_2 = 10$
- $K_1 = 2 \qquad K_2 = 2 \qquad K_3 = 4$

Substituting into Equation 3.1:

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The eigenvalue problem then becomes (Equation 3.5):

$$\left[\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} + \frac{1}{s^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[\begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} + \frac{1}{s^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or:

$$\begin{bmatrix} \frac{3}{2} + \frac{1}{s^2} & 1 \\ \frac{1}{2} & 2 + \frac{1}{s^2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{ \ 0 \ \} \tag{3.9}$$

The determinant of the coefficient matrix of Equation 3.9 must equal zero for a non-trivial solution.

$$\left(\frac{3}{2} + \frac{1}{s^2} \right) \left(2 + \frac{1}{s^2} \right) - \frac{1}{2} = 0.$$

Using $\alpha = \frac{1}{s^2}$ as a change of variable, the characteristic equation becomes:

$$\alpha^2 + \frac{7}{2}\,\alpha + \frac{5}{2} = 0. \tag{3.10}$$

The roots of Equation 3.10 are:

$$\alpha_{1,2} = \frac{\frac{-7}{2} \pm \sqrt{49/4 - 10}}{2} = \frac{-7}{4} \pm \frac{\sqrt{9/4}}{2}$$

$$\alpha_1 = \frac{-5}{2}$$

$$\alpha_2 = -1$$

Noting the change of variable $\alpha = \frac{1}{s^2}$:

$$\alpha_1 = \frac{1}{\lambda_1^2} \qquad\qquad \alpha_2 = \frac{1}{\lambda_2^2}$$

Since $\lambda_r = \sigma_r \pm j\ \omega_r$, the complex-valued modal frequencies are:

$$\lambda_1 = \sigma_1 \pm j\ \omega_1 = \pm j\ \omega_1 = \pm j\ \sqrt{2/5}$$

$$\lambda_2 = \sigma_2 \pm j\ \omega_2 = \pm j\ \omega_2 = \pm j\ 1$$

Now the frequencies ω_1 and ω_2 can be used in Equation 3.9 to determine the modal vectors.

The modal vector for $\lambda_1 = \pm j\ \omega_1$ is determined using the following equations:

$$\begin{bmatrix} -1 & 1 \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{ \quad 0 \quad \}$$

or

$$-X_1 + X_2 = 0$$

$$X_2 = X_1$$

Thus, the modal vector corresponding to the natural frequency ω_1 is:

$$\{ \ \psi \ \}_1 = \begin{Bmatrix} X_1 \\ X_1 \end{Bmatrix}_1$$

where:

- X_1 is arbitrary (depends on scaling method)

Similarly for $\lambda_2 = \pm \ j \ \omega_2$, the modal vector is:

$$\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{ \quad 0 \quad \}$$

or

$$\frac{1}{2} X_1 + X_2 = 0$$

$$X_2 = \frac{-1}{2} X_1$$

or:

$$\{ \ \psi \ \}_2 = \begin{Bmatrix} X_1 \\ \frac{-X_1}{2} \end{Bmatrix}_2$$

If the deformation of $X_1 = 1$, which is an arbitrary choice depending on the scaling method, then:

For $\omega_1 = \sqrt{2/5}$:

$$\{ \psi \}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_1$$

For $\omega_2 = 1$:

$$\{ \psi \}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}_2$$

3.3 Weighted Orthogonality of Modal Vectors

The solution of the eigenvalue problem as formulated in Equation 3.5 yields N natural frequencies, λ_r, and N modal vectors $\{\psi\}_r$ where N is the number of degrees of freedom of the system.

Note that any particular undamped natural frequency and the associated modal vector $\{\psi\}_r$ satisfy Equation 3.4. Thus, substituting into Equation 3.4 $s = \lambda_r$ and $\{X\} = \{\psi\}_r$ yields:

$$\lambda_r^2 [M] \{ \psi \}_r = - [K] \{ \psi \}_r \tag{3.11}$$

Now pre-multiply Equation 3.11 by a different modal vector, $\{\psi\}_s^T$, thus:

$$\lambda_r^2 \{ \psi \}_s^T [M] \{ \psi \}_r = - \{ \psi \}_s^T [K] \{ \psi \}_r \tag{3.12}$$

where the superscript T denotes a matrix transpose.

Using a rule of matrix algebra for the transpose of a product of matrices:

$$[[C] [D]]^T = [D]^T [C]^T$$

Taking the transpose of both sides of Equation 3.12 yields:

$$\lambda_r^2 \{ \psi \}_r^T [M] \{ \psi \}_s = - \{ \psi \}_r^T [K] \{ \psi \}_s \tag{3.13}$$

where:

- $[M]^T = [M]$ since $[M]$ is a symmetric matrix.
- $[K]^T = [K]$ since $[K]$ is a symmetric matrix.

Next, substitute $s = \lambda_s$ and $\{X\} = \{\psi\}_s$ into Equation 3.4 and pre-multiply both sides by $\{\psi\}_s^T$. This yields:

$$\lambda_s^2 \{ \psi \}_r^T [M] \{ \psi \}_s = - \{ \psi \}_r^T [K] \{ \psi \}_s \tag{3.14}$$

Subtracting Equation 3.14 from Equation 3.13 gives:

$$(\lambda_r^2 - \lambda_s^2) \{ \psi \}_r^T [M] \{ \psi \}_s = 0. \tag{3.15}$$

If $r \neq s$ (implying two different frequencies), it follows that:

$$\{ \psi \}_r^T [M] \{ \psi \}_s = 0. \tag{3.16}$$

From Equation 3.14, it follows that:

$$\{ \psi \}_r^T [K] \{ \psi \}_s = 0. \tag{3.17}$$

Equations 3.16 and 3.17 are statements of the weighted orthogonality properties of the modal vectors with respect to the system mass and stiffness matrices. The concept of orthogonality can be looked at from a vector analysis standpoint. In vector analysis, two vectors are orthogonal if their dot product equals zero. This means that the projection of one vector on the other is zero. Therefore, the two vectors are perpendicular to each other. An obvious example is the 3-dimensional cartesian coordinate system. The *i*, *j*, and *k* unit vectors for the cartesian coordinate system are orthogonal to each other. Modal vectors of an n-degree of freedom system can be viewed as being just a vector in n-dimensional space, which unfortunately cannot be

visualized. In order for modal vectors to be orthogonal, though, a simple dot product will not suffice. The concept of a weighted dot product, where the weighting matrix is the theoretical mass or stiffness matrix, must be used. If, for instance, the mass matrix in Equation 3.16 was the identity matrix, the weighted dot product would reduce to the simple dot product and result in a direct analog of the orthogonality condition for the unit vectors in the cartesian coordinate system. Because the mass and stiffness matrices in Equation 3.16 and 3.17 are not generally the identity matrix, the orthogonality relationships in Equation 3.16 and Equation 3.17 are generally referred to as weighted orthogonality.

If two modal vectors happen to have the same frequency $\lambda_r = \lambda_s$ (Equation 3.15) their corresponding modal vectors are not necessarily orthogonal to one another. This condition is known as a *repeated root* or *repeated pole* and will be discussed further in a later section. For this condition, the modal vectors associated with the repeated roots will be orthogonal to the other modal vectors and independent of one another.

In Equation 3.15, if the same modal vector is used to pre- and post-multiply the mass matrix, then Equation 3.16 is equal to some scalar constant other than zero, commonly noted as M_r. Thus:

$$\{ \psi \}_r^T [M] \{ \psi \}_r = M_r = \text{Modal Mass} \tag{3.18}$$

Similarly, Equation 3.14 yields:

$$\{ \psi \}_r^T [K] \{ \psi \}_r = \omega_r^2 M_r = K_r = \text{Modal Stiffness} \tag{3.19}$$

Since, as previously shown, the amplitude of any particular modal vector (eigenvector) is completely arbitrary, the modal vector can be normalized in an arbitrary way. This means that M_r is not unique.

For instance, one common criteria used to normalize the modal vector is to scale the modal vector such that M_r in Equation 3.18 is equal to unity.

The resulting scaled modal vectors normalized in this manner are generally referred to as orthonormal modal vectors (eigenvectors).

3.4 Modal Vector Scaling Example

Using the previous two degree of freedom example, normalize the modal vectors $\{\psi\}_1$ and $\{\psi\}_2$ such that:

$$\{\ \psi\ \}_1^T\ [\ M\ ]\ \{\ \psi\ \}_1 = M_1 = 1. \tag{3.20}$$

and:

$$\{\ \psi\ \}_2^T\ [\ M\ ]\ \{\ \psi\ \}_2 = M_2 = 1. \tag{3.21}$$

From the previous example:

$$\{\ \psi\ \}_1 = \left\{ \begin{array}{c} X_1 \\ X_1 \end{array} \right\}_1 \qquad\qquad \{\ \psi\ \}_2 = \left\{ \begin{array}{c} X_1 \\ \frac{-X_1}{2} \end{array} \right\}_2$$

Substituting $\{\psi\}_1$ into Equation 3.20 yields:

$$\left\{ \begin{array}{c} X_1 \\ X_1 \end{array} \right\}^T \left[\begin{array}{cc} 5 & 0 \\ 0 & 10 \end{array} \right] \left\{ \begin{array}{c} X_1 \\ X_1 \end{array} \right\} = 1$$

$$\left\{ \begin{array}{c} 5\ X_1 \\ 10\ X_1 \end{array} \right\}^T \left\{ \begin{array}{c} X_1 \\ X_1 \end{array} \right\} = 1$$

$$5\ X_1^2 + 10\ X_1^2 = 1$$

$$X_1^2 = \frac{1}{15}$$

$$X_1 = \pm \sqrt{1/15}$$

Using the positive root, the modal vector $\{\psi\}_1$, normalized for unity modal mass, results:

$$\{\ \psi\ \}_1 = \left\{ \begin{array}{c} X_1 \\ X_1 \end{array} \right\}_1 = \left\{ \begin{array}{c} \sqrt{1/15} \\ \sqrt{1/15} \end{array} \right\}_1$$

Similarly for $\{\psi\}_2$:

$$\begin{Bmatrix} X_1 \\ \dfrac{-X_1}{2} \end{Bmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} X_1 \\ \dfrac{-X_1}{2} \end{Bmatrix} = 1$$

$$\begin{Bmatrix} 5\,X_1 \\ -5\,X_1 \end{Bmatrix}^T \begin{Bmatrix} X_1 \\ \dfrac{-X_1}{2} \end{Bmatrix} = 1$$

$$5\,X_1^2 + \frac{5}{2}\,X_1^2 = 1$$

$$X_1^2 = \frac{2}{15}$$

$$X_1 = \pm\sqrt{2/15}$$

Thus, $\{\psi\}_2$ normalized to unity modal mass is:

$$\{\ \psi\ \}_2 = \begin{Bmatrix} \sqrt{2/15} \\ \dfrac{-\sqrt{2/15}}{2} \end{Bmatrix}_2$$

The normalized modal vectors will give $M_r = 1$ for all the modes of vibration. The significance of this normalization will be obvious later.

3.5 Principal Coordinates - Modal Coordinates

With reference to the equations of motion for an undamped system (Equation 3.2), the major obstacle encountered when trying to solve for the system response $\{x\}$, due to a particular set of forcing functions and initial conditions, is the coupling between the equations. In terms of the system's mass and stiffness matrices, coupling is represented in terms of non-zero off diagonal elements. Generally two types of coupling can exist for an undamped system; (1) Static coupling

(non-diagonal stiffness matrix); or (2) dynamic coupling (non-diagonal mass matrix). Equation 3.2 represents a system which is only statically coupled. If the system of equations in Equation 3.2 could be uncoupled, that is diagonal mass and stiffness matrices, then each equation in Equation 3.2 could be solved independent of the other equations. Another way of looking at this would be that each uncoupled equation would look just like the equation for a single degree of freedom, whose solution can very easily be obtained. Therefore, if a set of coupled system equations could be reduced to an uncoupled system, the solution would become straightforward. Indeed, from an analytical sense, this is the whole point of what has become known as modal analysis.

The procedure used to uncouple a set of coupled system equations is basically a coordinate transformation. In other words, the goal is to find a coordinate transformation that transforms the original coordinates $\{x\}$ into another equivalent set of coordinates $\{q\}$ that renders the system statically and inertially decoupled. This new set of coordinates $\{q\}$ is typically referred to as principal coordinates, normal coordinates or modal coordinates.

A similar benefit of a coordinate transformation occurs in many other engineering problems. One example of this situation is in the calculation of moments and products of inertia when the inertia properties of a complex structure need to be defined. The first step in the calculation of the inertia properties is to choose a set of axis to base the inertia properties on. Then, the following properties would be measured or calculated: I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{xz}. In general, both moments of inertia and products of inertia are required. However, if a different set of axis with respect to the structure were defined such that these axis happened to coincide with the structures principle axis, the result would be moments of inertia I_x, I_y, and I_z but the products of inertia would all be zero ($I_{xy} = I_{yz} = I_{xz} = 0$). Therefore, by changing the coordinate system, the products of inertia have been eliminated.

Another example of the benefit of a coordinate transformation is noticed when computing principle strains at a point on a structure. Typically, a strain gage rosette is used to determine the normal and shearing strains at a point of interest. From this information, a new coordinate system can be determined (strain element orientation) such that only principal normal strains exist; the shear strains are equal to zero for the new coordinate system. To determine the orientation of this new coordinate system that renders the shearing strain to zero, MOHR's circle techniques are commonly used. Once again, a simple coordinate transformation is used to

eliminate the shearing strains.

The problem of finding a coordinate transformation that uncouples our original equations of motion is very straightforward. It turns out that, due to the unique orthogonality properties of the modal vectors, the required coordinate transformation is already available. Referring to Equations 3.16-3.19, if either the mass or stiffness matrix is pre- and post- multiplied by different modal vectors (Equations 3.16 and 3.17), the result is zero. However, if the same modal vector is used to pre- and post- multiply the mass or stiffness matrix (Equations 3.18 and 3.19), the result is a constant.

Therefore, the new coordinate system can be defined by the following transformation:

$$\{ x \} = \left[\psi \right] \{ q \} \tag{3.22}$$

where:

- $[\psi]$ is the transformation matrix (matrix whose columns are the modal vectors of the original system).

This matrix is generally referred to as the *modal matrix* or *matrix of modal vectors*. Recall the general form of the undamped system of equations with forcing functions:

$$[M] \{ \ddot{x} \} + [K] \{ x \} = \{ f \} \tag{3.23}$$

Substituting Equation 3.22 into Equation 3.23 gives:

$$[M] \left[\psi \right] \{ \ddot{q} \} + [K] \left[\psi \right] \{ q \} = \{ f \} \tag{3.24}$$

Pre-multiplying by $\left[\psi\right]^T$ yields:

$$\left[\psi \right]^T [M] \left[\psi \right] \{ \ddot{q} \} + \left[\psi \right]^T [K] \left[\psi \right] \{ q \} = \left[\psi \right]^T \{ f \}$$

Equation 3.25 is the equivalent of Equation 3.23 but in a different coordinate system. Analyzing

Equation 3.25, noting the orthogonality properties of the modal vectors (Equation 3.16-3.19):

$$\left[\psi \right]^T \left[M \right] \left[\psi \right] = \lceil M \rfloor$$

and:

$$\left[\psi \right]^T \left[K \right] \left[\psi \right] = \lceil K \rfloor$$

where:

- $\lceil M \rfloor$ is a diagonal matrix.
- $\lceil K \rfloor$ is a diagonal matrix.

Therefore, Equation 3.25 becomes:

$$\lceil M \rfloor \{ \ddot{q} \} + \lceil K \rfloor \{ q \} = \left[\psi \right]^T \{ f \} \tag{3.26}$$

From inspection, since both the new mass and stiffness matrices are diagonal, the coordinate transformation $\{x\} = [\psi] \{q\}$ has completely uncoupled the set of equations. Now each equation in Equation 3.26 is an equation for a single degree of freedom oscillator which is easily solved.

The r-th Equation of Equation 3.26 is:

$$M_r \ddot{q}_r + K_r q_r = \{ \psi \}_r^T \{ f \} = f_r \tag{3.27}$$

This is the equation of motion for the single degree of freedom system shown below.

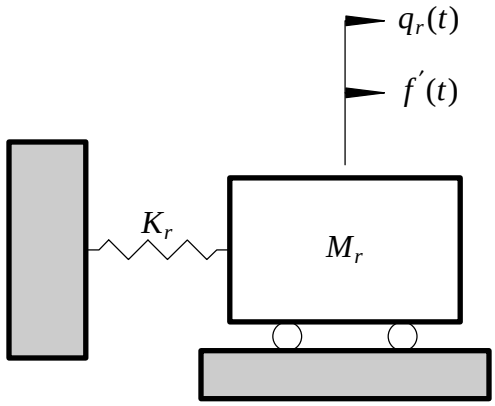

Figure 3-2. Single Degree of Freedom System

The quantity M_r is called the modal mass or generalized mass for the r-th mode of vibration. The quantity K_r is called the modal stiffness or generalized stiffness for the r-th modal vector of vibration. While these quantities are viewed as mass and stiffness related, it is important to remember that the magnitude of these quantities depends upon the scaling of the modal vectors. Therefore, although both the modal vectors and the modal mass/stiffness quantities are computed in a relative manner, only the combination of a modal vector together with the associated modal mass represents a unique absolute characteristic concerning the system being described.

It has been shown previously that the modal vectors may be normalized such that $M_r = 1$. If this has been done, then Equation 3.27 can be rewritten for the r-th modal vector as:

$$\ddot{q}_r + \Omega_r^2 \, q_r = f' \tag{3.28}$$

where:

- $M_r = 1.0$
- $K_r = \Omega_r{}^2$

Once the solution (time responses) of Equation 3.26 for all q's has been computed, the solution in terms of the original coordinates can then be obtained through the use of the coordinate transformation equation in Equation 3.22.

3.6 Two Degree of Freedom Example: Undamped, Forced

Referring to the previous example, some forcing functions can now be included in the system of equations.

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \{ \ddot{x} \} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \{ x \} = \{ f \} \tag{3.29}$$

The natural frequencies and normalized modal vectors of the above system are:

For $\omega_1 = \sqrt{2/5}$:

$$\{ \psi \}_1 = \begin{Bmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{Bmatrix}_1$$

For $\omega_2 = 1$:

$$\{ \psi \}_2 = \begin{Bmatrix} \sqrt{2/15} \\ -\dfrac{\sqrt{2/15}}{2} \end{Bmatrix}_2$$

Forming the modal matrix:

$$\left[\ \psi\ \right]=\left[\ \{\ \psi\ \}_1\ \ \{\ \psi\ \}_2\ \right]=\begin{bmatrix}\sqrt{1/15} & \sqrt{2/15}\\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}$$

Now make the following coordinate transformation:

$$\{\ x\ \}=\left[\ \psi\ \right]\{\ q\ \} \tag{3.30}$$

$$\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{bmatrix}\sqrt{1/15} & \sqrt{2/15}\\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}$$

Substituting Equation 3.30 into Equation 3.29 and pre-multiplying by $\left[\ \psi\ \right]^T$ yields:

$$\left[\ \psi\ \right]^T[\ M\ ]\left[\ \psi\ \right]\{\ q\ \}+\left[\ \psi\ \right]^T[\ K\ ]\left[\ \psi\ \right]\{\ q\ \}=\left[\ \psi\ \right]^T\{\ f(t)\ \}$$

$$\left[\ \psi\ \right]^T[\ M\ ]\left[\ \psi\ \right]=\begin{bmatrix}\sqrt{1/15} & \sqrt{1/15}\\ \sqrt{2/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 10\end{bmatrix}\begin{bmatrix}\sqrt{1/15} & \sqrt{2/15}\\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}$$

$$\left[\ \psi\ \right]^T[\ M\ ]\left[\ \psi\ \right]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\left[\ \psi\ \right]^T\left[\ K\ \right]\left[\ \psi\ \right]=\begin{bmatrix}\sqrt{1/15} & \sqrt{1/15}\\ \sqrt{2/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}\begin{bmatrix}4 & -2\\ -2 & 6\end{bmatrix}\begin{bmatrix}\sqrt{1/15} & \sqrt{2/15}\\ \sqrt{1/15} & -\dfrac{\sqrt{2/15}}{2}\end{bmatrix}$$

$$\left[\ \psi\ \right]^T\left[\ K\ \right]\left[\ \psi\ \right]=\begin{bmatrix}2/5 & 0\\ 0 & 1\end{bmatrix}$$

Therefore, the new equations of motion are:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix}+\begin{bmatrix}2/5 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}=\begin{Bmatrix}\sqrt{1/15}\,f_1+\sqrt{1/15}\,f_2\\ \sqrt{2/15}\,f_1-\dfrac{\sqrt{2/15}}{2}\,f_2\end{Bmatrix}=\begin{Bmatrix}f_1'\\ f_2'\end{Bmatrix}$$

The matrix equation of Equation 3.31 can now be written in terms of algebraic differential equations:

$$\ddot{q}_1+\frac{2}{5}\,q_1=f_1' \tag{3.32}$$

$$\ddot{q}_2+q_2=f_2' \tag{3.33}$$

Hence, the system equations have been uncoupled by using the modal matrix as a coordinate transformation.

The original system looked like:

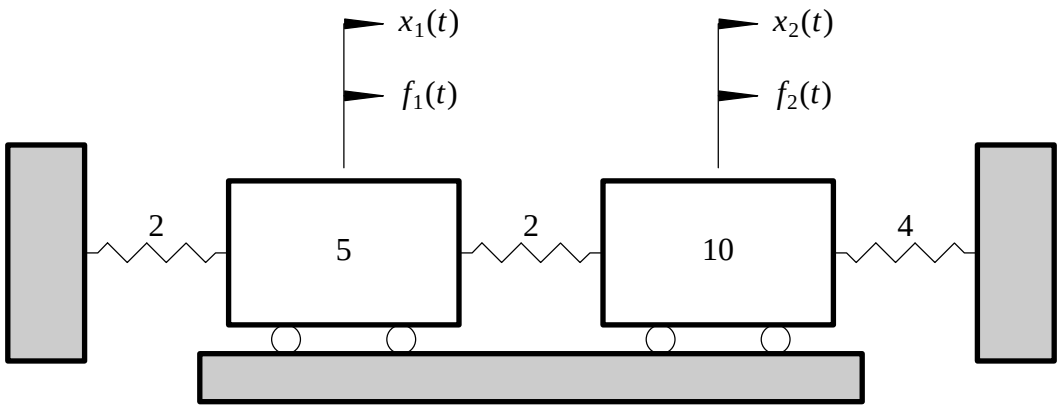

Figure 3-3. Original System

The transformed system can be pictured as:

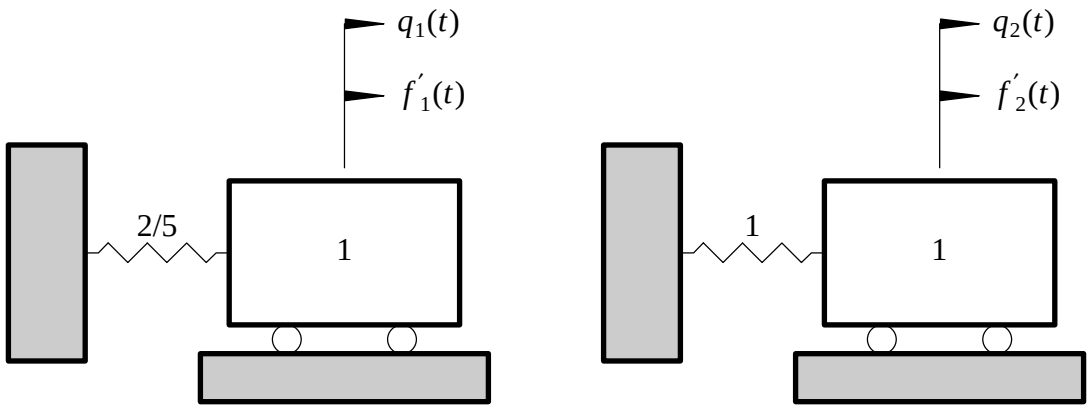

Figure 3-4. Transformed System

Once $q_1(t)$ and $q_2(t)$ are known, Equation 3.30 can be used to compute $x_1(t)$ and $x_2(t)$. Thus:

$$x_1(t) = \sqrt{1/15}\, q_1(t) + \sqrt{2/15}\, q_2(t)$$

$$x_2(t) = \sqrt{1/15}\, q_1(t) - \frac{\sqrt{2/15}}{2}\, q_2(t)$$

Many points should be emphasized from the previous discussion. Modal vectors, along with their frequencies, are a dynamic property of a structure. The amplitudes of a modal vector are completely arbitrary; that is, only the ratios between the components of a particular modal vector are unique. Because of the orthogonality properties of the modal vectors, with respect to the system's mass and stiffness matrices, modal mass and modal stiffness can be defined. These quantities depend upon the scaling of the modal vectors, so that the absolute magnitudes of these quantities are also arbitrary. Finally, a simple coordinate transformation (modal matrix) can be

used to represent a complicated interconnection of springs and masses as a collection of single degree of freedom oscillators.

3.7 Proportional Damping

In order to evaluate multiple degree of freedom systems that are present in the real world, the effect of damping on the complex frequencies and modal vectors must be considered. Many physical mechanisms are needed to describe all of the possible forms of damping that may be present in a particular structure or system. Some of the classical types are:

- Structural Damping
- Viscous Damping
- Coulomb Damping
- Hysteretic Damping

It is generally difficult to ascertain which type of damping is present in any particular structure. Indeed most structures exhibit damping characteristics that result from a combination of all the above, plus others that have not been described here.

It will suffice to say that whenever a structure is modeled with a particular form of damping, for example, viscous, that the damping model is an equivalent model to whatever type of damping that may actually be present.

Rather than consider the many, different physical mechanisms, the probable location of each mechanism, and the particular mathematical representation of the mechanism of damping that is needed to describe the dissipative energy of the system, a model will be used that is only concerned with the resultant mathematical form. This model will represent a hypothetical form of damping, that is proportional to the system mass or stiffness matrix. Therefore:

$$[\ C\ ] = \alpha\ [\ M\ ]$$

or :

$$[\ C\ ] = \beta\ [\ K\ ]$$

The most common formulation for proportional damping is:

$$[\ C\ ] = \alpha\,[\ M\ ] + \beta\,[\ K\ ]$$

where:

- $[\ C\ ]$ = damping matrix
- α , β = constants

Note that the case of no damping is the trivial proportional damped case with both coefficients equal to zero. While the above definition is sufficient for most cases, the theoretical relationship between mass, stiffness and damping matrices can be somewhat more complicated and still qualify as proportional damping. Theoretically, any damping matrix that satisfies the following relationship will yield proportional damping with all the qualifications (normal modes) involved in subsequent discussion.

$$\left[[M]^{-1}[C]\right]^s\left[[M]^{-1}[K]\right]^r = \left[[M]^{-1}[K]\right]^r\left[[M]^{-1}[C]\right]^s$$

where:

- r and s = integers.

For the purposes of most practical problems, the simpler relationship will be sufficient.

3.8 Modal Vectors from the System Matrix

The modal vectors can be determined in a somewhat more direct manner through a manipulation of the system matrix. Understanding this approach to the evaluation of modal vectors is very useful in relating measured frequency response function data to the system modal vectors.

Starting with Equation 3.4:

$$\left[[\ M\ ]s^2 + [\ C\ ]s + [\ K\ ]\right]\ \{\ X\ \} = \{\ 0\ \} \tag{3.34}$$

Define:

$$[\ B(s)\ ] = \left[[\ M\ ]s^2 + [\ C\ ]s + [\ K\ ] \right]$$

where:

- $[\ B(S)\ ]$ = System Impedance Matrix

From matrix algebra:

$$[\ B(s)\ ][\ B(s)\ ]^{-1} = [\ I\ ] \tag{3.35}$$

$$[\ B(s)\ ]^{-1} = \frac{[\ B(s)\ ]^A}{|\ [\ B(s)\ ]\ |} \tag{3.36}$$

where:

- $[\ B(s)\ ]^A$ is the adjoint of matrix $[\ B(s)\ ]$.

Substituting Equation 3.36 into Equation 3.35 yields:

$$[\ B(s)\ ][\ B(s)\ ]^A = |[\ B(s)\ ]|\ [\ I\ ] \tag{3.37}$$

If λ_r is a root of the characteristic equation from Equation 3.34, then $|\ [\ B(\lambda_r)\ ]\ | = 0$.

Evaluating Equation 3.37 at $s = \lambda_r$ gives:

$$\left[\ B(\lambda_r)\ \right]\left[\ B(\lambda_r)\ \right]^A = [\ 0\ ] \tag{3.38}$$

Equation 3.38 can be rewritten using any column of $[\ B(\lambda_r)\ ]^A$, the $i-th$ column for example $\{\ B(\lambda_r)\ \}_i^A$. Therefore:

$$\left[\ B(\lambda_r)\ \right]\{\ B(\lambda_r)\ \}_i^A = \{\ 0\ \} \tag{3.39}$$

Equation 3.39 represents a set of homogeneous equations in $\{ B(\lambda_r) \}_i^A$ which determines each element of $\{ B(\lambda_r) \}_i^A$ to within an arbitrary constant. Note that the constant will be different depending upon the column that is used.

Evaluating Equation 3.34 at one of the eigenvalues of the system (λ_r):

$$\left[\; B (\lambda_r) \; \right] \{ \; X \; \}_r = \{ \; 0 \; \} \tag{3.40}$$

Equation 3.40 (formerly Equation 3.34), just like Equation 3.39, represents a set of homogeneous equations in $\{ X \}$ Equation 3.34 is evaluated at a specific eigenvalue, the resulting solution is the eigenvector coresponding to the specific eigenvalue. This eigenvector is determined to within an arbitrary constant. Therefore, from Equation 3.39 and Equation 3.40, $\{ B(\lambda_r) \}_i^A$ and $\{X\}_r$ are proportional and both represent the eigenvector corresponding to the eigenvalue λ_r. Recall that $\{X\}_r$ (Equation 3.40) has been previously shown to be the r-th modal vector of the system. Therefore:

$$\{ \; X \; \}_r = \beta_{ir} \{ \; B (\lambda_r) \; \}_i^A$$

where:

- β_{ir} is a proportionality constant.

Note: One of the major points is that the columns of the adjoint matrix $[B(\lambda_r)]^A$ are all proportional to the r-th modal vector.

Since the mass, damping and stiffness matrices are assumed to be symmetric when absolute coordinates are used (and proportional damping is present), the system impedance matrix $[B(s)]$ is symmetric. Therefore, the adjoint matrix of $[B(\lambda_r)]$ is also symmetric. Thus, the rows of the adjoint matrix are also proportional to the modal vector. The adjoint matrix can now be expressed for the r-th mode in terms of the r-th modal vector.

$$\left[\; B (\lambda_r) \; \right]^A = \gamma_r \{ \; \psi \; \}_r \{ \; \psi \; \}_r^T \tag{3.41}$$

$$\left[B(\lambda_r) \right]^A = \gamma_r \left[\begin{array}{ccccccc} \psi_1\psi_1 & \psi_1\psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_1\psi_N \\ \psi_2\psi_1 & \psi_2\psi_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_N\psi_1 & \psi_N\psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_N\psi_N \end{array} \right]_r$$

where:

- $\gamma_r =$ constant associated with the scaling of $\{\psi\}_r$ relative to the absolute scaling (units) of the adjoint matrix.

Note that the adjoint matrix is not the same as the modal matrix since each column of the adjoint matrix is proportional to the same modal vector. Therefore, the adjoint matrix needs to be evaluated for each of the N eigenvalues to determine the N eigenvectors. Also note that, due to the symmetry of the adjoint matrix, if one of the modal coefficients is zero, then a complete row and column of the corresponding adjoint matrix will be zero. This phenomenon is normal and corresponds to physically trying to excite (force) the system at the node (modal coefficient equal to zero) of one of the modal vectors of the system. Theoretically, the corresponding mode of vibration will not be observed in this situation. Analytically, this problem can be overcome by evaluating a different row or column of the adjoint matrix. Experimentally, the configuration of the input and/or output sensors may have to be altered to detect this case.

Equation 3.41 is extremely important and will be used in the next section to show that the residues of a frequency response function for a particular pole (λ_r) are directly related to the elements of a modal vector. Also, the symmetry of the adjoint matrix is the justification for not needing to evaluate the complete frequency response function matrix.

4. FREQUENCY RESPONSE FUNCTION DEVELOPMENT

4.1 Theory

All of the techniques discussed previously are useful if an analytical model of the system already exists. From an experimental point, this is rarely the case. Typically, solving problems on real systems or pieces of systems must be accomplished without the aid of a theoretical model or in order to verify a theoretical model.

In this chapter, frequency response function measurements will begin to be used as the basis for defining modal frequencies and damping values, modal vectors, modal mass, modal stiffness, and modal damping of real life structures. To accomplish this task, an analytical model will be developed to represent the transfer function between any possible measurement locations on the structure. Frequency response functions will be directly related to the transfer functions that have been theoretically developed.

The transfer function representation of an undamped multiple degree of freedom system can be formulated by starting with the differential equations of motion in terms of mass, stiffness, and damping matrices.

$$[\ M\ ]\{\ \ddot{x}\ \} + [\ K\ ]\{\ x\ \} = \{\ f\ \} \tag{4.1}$$

Taking the Laplace transform of Equation 4.1, assuming all initial conditions are zero, yields:

$$\left[\ s^2\ [\ M\ ] + [\ K\ ]\ \right]\{\ X\ (s)\ \} = \{\ F\ (s)\ \} \tag{4.2}$$

Let:

$$[\ B\ (s)\ ] = \left[\ s^2\ [\ M\ ] + [\ K\ ]\ \right]$$

Then Equation 4.2 becomes:

$$[\ B\ (s)\ ]\{\ X\ (s)\ \} = \{\ F\ (s)\ \} \tag{4.3}$$

where [$B(s)$] is referred to as the *system impedance matrix* or just the *system matrix.*

Pre-multiplying Equation 4.3 by [$B(s)$]$^{-1}$ yields:

$$[\ B\ (s)\ ]^{-1}\ \{\ F\ (s)\ \} = \{\ X\ (s)\ \}$$

Defining:

$$[\ H\ (s)\ ] = [\ B\ (s)\ ]^{-1}$$

Then:

$$[\ H\ (s)\ ]\ \{\ F\ (s)\ \} = \{\ X\ (s)\ \} \tag{4.4}$$

Equation 4.4 relates the system response $\{ X(s) \}$ to the system forcing functions $\{ F(s) \}$ through the matrix [$H(s)$]. The matrix [$H(s)$] is generally referred to as the *transfer function matrix.*

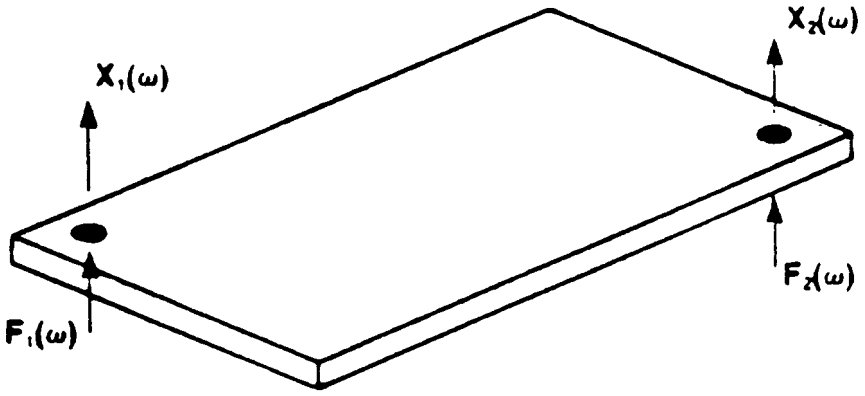

Figure 4-1. Two Input-Output Model

Equations 4.2-4.4 can be expanded for a two degree of freedom system in order to view in detail the components of each position in the system matrix.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{4.5}$$

Taking the Laplace transform of Equation 4.5 yields:

$$\left[\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^2 + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right] \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix}$$

Thus,

$$[\ B(s)\ ]\ \{\ X(s)\ \} = \{\ F(s)\ \}$$

where:

$$\bullet\ [\ B(s)\ ] = \begin{bmatrix} M_{11}\, s^2 + K_{11} & M_{12}\, s^2 + K_{12} \\ M_{21}\, s^2 + K_{21} & M_{22}\, s^2 + K_{22} \end{bmatrix}$$

Defining the transfer function as the inverse of the impedance matrix:

$$[\ B(s)\ ]^{-1} = [\ H(s)\ ]$$

The inverse of the impedance matrix can be found for this analytical case as the adjoint of the impedance matrix divided by the determinant of the impedance matrix as follows:

$$[\ B(s)\ ]^{-1} = \frac{\begin{bmatrix} M_{22}\, s^2 + K_{22} & -(\, M_{12}\, s^2 + K_{12}\,) \\ -(\, M_{21}\, s^2 + K_{21}\,) & M_{11}\, s^2 + K_{11} \end{bmatrix}}{(M_{11}\, s^2 + K_{11})\,(M_{22}\, s^2 + K_{22}) - (M_{21}\, s^2 + K_{21})\,(M_{12}\, s^2 + K_{12})} \qquad (4.6)$$

Note that the denominator of Equation 4.6 is $|\ B(s)\ |$ which is the characteristic or frequency equation for the system. This highlights the fact that the complex-valued modal frequencies (λ_r) are global properties of the system since this characteristic equation appears in every term of $[\ H(s)\ ]$. This characteristic equation can be expressed as a product of its roots, thus:

$$|\ B(s)\ | = E\,(\, s - \lambda_1\,)\,(\, s - \lambda_2\,)\,(\, s - \lambda_3\,)\,(\, s - \lambda_4\,)$$

where:

- E = Constant coeficient of the highest order term in the polynomial (Product/Sum of the mass terms)
- λ_1, λ_2, λ_3 and λ_4 are the 4 roots of the characteristic equation.

Equation 4.6 for a two degree of freedom system can then be written as:

$$\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \frac{\begin{bmatrix} M_{22}\, s^2 + K_{22} & -(\, M_{12}\, s^2 + K_{12}\,) \\ -(\, M_{21}\, s^2 + K_{21}\,) & M_{11}\, s^2 + K_{11} \end{bmatrix}}{E\,(\, s - \lambda_1\,)\,(\, s - \lambda_2\,)\,(\, s - \lambda_3\,)\,(\, s - \lambda_4\,)} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} \tag{4.7}$$

[$H(s)$], the transfer function matrix, can be defined as:

$$[\ H\,(s)\ ] = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

where, for instance:

$$H_{11}(s) = \frac{M_{22}\, s^2 + K_{22}}{E\,(\, s - \lambda_1\,)\,(\, s - \lambda_2\,)\,(\, s - \lambda_3\,)\,(\, s - \lambda_4\,)} \tag{4.8}$$

Equation 4.4 can now be expressed as:

$$\begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} = \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} \tag{4.9}$$

Multiplying out Equation 4.9 yields:

$$\begin{aligned} H_{11}(s)\, F_1(s) + H_{12}(s)\, F_2(s) &= X_1(s) \\ H_{21}(s)\, F_1(s) + H_{22}(s)\, F_2(s) &= X_2(s) \end{aligned} \tag{4.10}$$

If, in Equation 4.10, $F_2(s) = 0$, then:

$$H_{1\,1}(s)\ F_1(s) = X_1(s)$$

$$H_{2\,1}(s)\ F_1(s) = X_2(s)$$

This results in the familiar relationships for the transfer function (output over input):

$$H_{11}(s) = \frac{X_1(s)}{F_1(s)}$$

$$H_{21}(s) = \frac{X_2(s)}{F_1(s)}$$

In general

$$H_{pq} = \frac{X_p}{F_q}$$

where :

- p is the output degree of freedom (physical location and orientation).
- q is the input degree of freedom (physical location and orientation).

Therefore, measuring a column of the [H] matrix is accomplished by using a single, fixed input (excitor system) with a roving response and measuring a row is accomplished by using a roving input (hammer) and a single fixed response. It should be reiterated that the subscript notation of p or q refers to both a physical location and also direction or orientation.

Thus, $H_{11}(s)$ is the transfer function measured by exciting the system with $F_1(s)$ and measuring the response $X_1(s)$. Similarly $H_{21}(s)$ is the transfer function measured by exciting the system with $F_1(s)$ and measuring the response at $X_2(s)$. Likewise, $H_{12}(s)$ and $H_{22}(s)$ can be measured by exciting the system with $F_2(s)$, letting $F_1(s) = 0$, and measuring the responses $X_1(s)$ and $X_2(s)$.

As in the single degree of freedom case, the denominator polynominal in Equation 4.8 is called the characteristic equation. Notice that all of the transfer functions that are represented in Equation 4.6 have the same denominator polynomial. The roots of this denominator polynomial (characteristic equation) are the modal frequencies of the system.

Since the coefficients of the characteristic equation are real, the roots will appear as complex conjugate pairs. Rewriting Equation 4.8 with this in mind gives the following result:

$$H_{11}(s) = \frac{M_{22}\, s^2 + K_{22}}{E\,(\,s - \lambda_1\,)\,(\,s - \lambda_1^*\,)\,(\,s - \lambda_2\,)\,(\,s - \lambda_2^*\,)} \tag{4.11}$$

where λ_1, λ_1^*, λ_2, and λ_2^* are the *roots of the characteristic equation.* The roots of the characteristic equation are also referred to as the *poles of the transfer function* $H_{11}(s)$.

4.2 Analytical Model - Scalar/Matrix Polynomial (MDOF)

The general formulation of Equation 4.11 or of any of the transfer functions defined by an anlytical mass, damping and stiffness matrix model can be written as a numerator polynomial of the independent variable *s* divided by a denominator polynomial of the independent variable *s*. Both polynomials involve coefficients that are different numerical combinations of the discrete values of mass, damping, stiffness of the system. The roots of the denominator polynomial are the modal frequencies of the system and are considered global properties of the system. The roots of the numerator polynomial are the zeroes of the system and are local properties of the system that depend upon the specific input-output relationship of the transfer function. The general polynomial model for a single transfer function can be written as follows using scalar coefficients:

$$\frac{X_p(s)}{F_q(s)} = H_{pq}(s) = \frac{\beta_n\,(s)^n + \beta_{n-1}\,(s)^{n-1} + \cdots + \beta_1\,(s)^1 + \beta_0\,(s)^0}{\alpha_m\,(s)^m + \alpha_{m-1}\,(s)^{m-1} + \cdots + \alpha_1\,(s)^1 + \alpha_0\,(s)^0} \tag{4.12}$$

The previous model can be rewritten in a more concise form as follows:

$$\frac{X_p(s)}{F_q(s)} = \frac{\sum_{k=0}^{n} \beta_k\,(s)^k}{\sum_{k=0}^{m} \alpha_k\,(s)^k}$$

Further rearrranging yields the following equation that is linear in the unknown α and β terms:

$$\sum_{k=0}^{m} \alpha_k\,(s)^k X_p(s) = \sum_{k=0}^{n} \beta_k\,(s)^k F_q(s) \tag{4.13}$$

This model can be generalized to represent the general multiple input, multiple output case as

follows using a matrix polynomial formulation:

$$\sum_{k=0}^{m}\left[[\alpha_k](s)^k\right]\{X(s)\} = \sum_{k=0}^{n}\left[[\beta_k](s)^k\right]\{F(s)\} \tag{4.14}$$

The previous models can be used to represent frequency response function (FRF) data by limiting the s variable ($s = j\,\omega$) and applying a few matrix operations. If both sides of the above equation are post multiplied by the hermitian (complex conjugate transpose) of the force vector ($\{F(s)\}^H$), the following equation results.

$$\sum_{k=0}^{m}\left[[\alpha_k](s)^k\right]\{X(s)\}\{F(s)\}^H = \sum_{k=0}^{n}\left[[\beta_k](s)^k\right]\{F(s)\}\{F(s)\}^H$$

In the above equation note that the vector products are the definition of the cross power ($[G_{xf}(s)] = \{X(s)\}\{F(s)\}^H$) and auto power spectrum ($[G_{ff}(s)] = \{F(s)\}\{F(s)\}^H$) when $s = j\omega$ (after averaging).

$$\sum_{k=0}^{m}\left[[\alpha_k](s)^k\right]\left[G_{xf}(s)\right] = \sum_{k=0}^{n}\left[[\beta_k](s)^k\right]\left[G_{ff}(s)\right]$$

The above equation can be post multipled by the inverse of the auto power spectrum matrix.

$$\sum_{k=0}^{m}\left[[\alpha_k](s)^k\right]\left[G_{xf}(s)\right]\left[G_{ff}(s)\right]^{-1} = \sum_{k=0}^{n}\left[[\beta_k](s)^k\right][I]$$

Finally, the above equation can be put in final form by noting that the product of the cross spectrum matrix and the inverse of the auto spectrum matrix ($[G_{xf}(s)]\,[G_{ff}(s)]^{-1}$) is the definition of the FRF matrix ($[H(s)]$) for the multiple input, multiple output case when $s = j\omega$.

$$\sum_{k=0}^{m}\left[[\alpha_k](j\omega)^k\right][H(\omega)] = \sum_{k=0}^{n}\left[[\beta_k](j\omega)^k\right][I] \tag{4.15}$$

For a single input, single output case the above equation yields:

$$\sum_{k=0}^{m}\alpha_k\,(j\omega)^k H_{pq}(\omega) = \sum_{k=0}^{n}\beta_k\,(j\omega)^k \tag{4.16}$$

4.3 Analytical Model - Partial Fraction (Residue)

Equation 4.11 can be represented very generally by expansion in terms of its partial fractions. That is:

$$H_{11}(s) = \frac{X_1(s)}{F_1(s)} = \frac{c_1}{(\,s - \lambda_1\,)} + \frac{c_2}{(\,s - \lambda_1^*\,)} + \frac{c_3}{(\,s - \lambda_2\,)} + \frac{c_4}{(\,s - \lambda_2^*\,)} \tag{4.17}$$

The constants $c_1 \rightarrow c_4$ can be found in a similar fashion as in the single degree of freedom case. Equating Equation 4.11 and Equation 4.17 yields:

$$\frac{M_{22}\, s^2 + K_{22}}{E\,(s - \lambda_1)\,(s - \lambda_1^*)\,(s - \lambda_2)\,(s - \lambda_2^*)} = \frac{c_1}{(s - \lambda_1)} + \frac{c_2}{(s - \lambda_1^*)} + \frac{c_3}{(s - \lambda_2)} + \frac{c_4}{(s - \lambda_2^*)} \tag{4.18}$$

Note that c_1 can be evaluated by multiplying Equation 4.18 by $s - \lambda_1$ and evaluating the expression at $s = \lambda_1$. Thus:

$$c_1 = \frac{M_{22}\, \lambda_1^2 + K_{22}}{E\,(\,\lambda_1 - \lambda_1^*\,)\,(\,\lambda_1 - \lambda_2\,)\,(\,\lambda_1 - \lambda_2^*\,)} = A_{111} \tag{4.19}$$

In a similar fashion:

$$c_2 = \frac{M_{22}\, \lambda_1^{*2} + K_{22}}{E\,(\,\lambda_1^* - \lambda_1\,)\,(\,\lambda_1^* - \lambda_2\,)\,(\,\lambda_1^* - \lambda_2^*\,)} = c_1^* = A_{111}^*$$

$$c_3 = \frac{M_{22}\, \lambda_2^2 + K_{22}}{E\,(\,\lambda_2 - \lambda_1\,)\,(\,\lambda_2 - \lambda_1^*\,)\,(\,\lambda_2 - \lambda_2^*\,)} = A_{112}$$

$$c_4 = \frac{M_{22}\, \lambda_2^{*2} + K_{22}}{E\,(\,\lambda_2^* - \lambda_1\,)\,(\,\lambda_2^* - \lambda_1^*\,)\,(\,\lambda_2^* - \lambda_2\,)} = c_3^* = A_{112}^*$$

Equation 4.17 becomes:

$$H_{11}(s) = \frac{A_{111}}{(\,s - \lambda_1\,)} + \frac{A_{111}^*}{(\,s - \lambda_1^*\,)} + \frac{A_{112}}{(\,s - \lambda_2\,)} + \frac{A_{112}^*}{(\,s - \lambda_2^*\,)} \tag{4.20}$$

Thus, the transfer function of a two degree of freedom system has been represented by the sum of two single degree of freedom systems (Section 2.5). This result can now be extrapolated to apply to any number of degrees of freedom.

Rewriting Equation 4.20 in terms of this summation:

$$H_{11}(s) = \sum_{r=1}^{2} \left[\frac{A_{11r}}{s - \lambda_r} + \frac{A_{11r}^*}{s - \lambda_r^*} \right] \tag{4.21}$$

As in a single degree of freedom case, the A_{11r}'s are again referred to as the residues associated with the poles λ_r.

The rest of the transfer functions of the system can be expressed, using the same logic as the development of Equation 4.21.

$$H_{21}(s) = \sum_{r=1}^{2} \left[\frac{A_{21r}}{s - \lambda_r} + \frac{A_{21r}^*}{s - \lambda_r^*} \right] \tag{4.22}$$

$$H_{12}(s) = \sum_{r=1}^{2} \left[\frac{A_{12r}}{s - \lambda_r} + \frac{A_{12r}^*}{s - \lambda_r^*} \right] \tag{4.23}$$

$$H_{22}(s) = \sum_{r=1}^{2} \left[\frac{A_{22r}}{s - \lambda_r} + \frac{A_{22r}^*}{s - \lambda_r^*} \right] \tag{4.24}$$

Equation 4.9 can now be rewritten in terms of the above partial fraction expansion.

$$\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{bmatrix} \sum_{r=1}^{2} \left[\frac{A_{11r}}{s - \lambda_r} + \frac{A_{11r}^*}{s - \lambda_r^*} \right] & \sum_{r=1}^{2} \left[\frac{A_{12r}}{s - \lambda_r} + \frac{A_{12r}^*}{s - \lambda_r^*} \right] \\ \sum_{r=1}^{2} \left[\frac{A_{21r}}{s - \lambda_r} + \frac{A_{21r}^*}{s - \lambda_r^*} \right] & \sum_{r=1}^{2} \left[\frac{A_{22r}}{s - \lambda_r} + \frac{A_{22r}^*}{s - \lambda_r^*} \right] \end{bmatrix} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} \tag{4.25}$$

The transfer function matrix [$H(s)$] can also be rewritten in terms of partial fractions combining Equations 4.21 through 4.24.

$$[\ H\ (s)\ ] = \frac{\begin{bmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{bmatrix}}{(\ s - \lambda_1\ )} + \frac{\begin{bmatrix} A^*_{111} & A^*_{121} \\ A^*_{211} & A^*_{221} \end{bmatrix}}{(\ s - \lambda^*_1\ )} + \frac{\begin{bmatrix} A_{112} & A_{122} \\ A_{212} & A_{222} \end{bmatrix}}{(\ s - \lambda_2\ )} + \frac{\begin{bmatrix} A^*_{112} & A^*_{122} \\ A^*_{212} & A^*_{222} \end{bmatrix}}{(\ s - \lambda^*_2\ )} \quad . \tag{4.26}$$

The numerator matrices in each term of the above equation are called the residue matrices. Note that there is a separate matrix associated with each of the modal frequencies (poles) of the system. Note also that the residue matrices associated with complex conjugate modal frequencies are also complex conjugates.

Previously, the constant A_{111} has been evaluated (Equation 4.19). The constants A_{211}, A_{121}, and A_{221} can now be evaluated in a similar fashion. Once this is done, the form of the first residue matrix can be evaluated for pole λ_1.

$$A_{211} = \frac{-(\ M_{21}\ \lambda_1^2 + K_{21}\ )}{E\ (\ \lambda_1 - \lambda^*_1\ )\ i(\ \lambda_1 - \lambda_2\ )\ (\lambda_1 - \lambda^*_2\ )}$$

$$A_{121} = \frac{-(\ M_{12}\ \lambda_1^2 + K_{12}\ )}{E\ (\ \lambda_1 - \lambda^*_1\ )\ (\ \lambda_1 - \lambda_2\ )\ (\lambda_1 - \lambda^*_2\ )}$$

$$A_{221} = \frac{-(\ M_{11}\ \lambda_1^2 + K_{11}\ )}{E\ (\ \lambda_1 - \lambda^*_1\ )\ (\ \lambda_1 - \lambda_2\ )\ (\lambda_1 - \lambda^*_2\ )}$$

The first term on the right of Equation 4.27 can be rewritten as follows:

$$\frac{\begin{bmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{bmatrix}}{(\ s - \lambda_1\ )} = \frac{\begin{bmatrix} M_{22}\ \lambda_1^2 + K_{22} & -(\ M_{12}\ \lambda_1^2 + K_{12}\ ) \\ -(\ M_{21}\ \lambda_1^2 + K_{21}\ ) & M_{11}\ \lambda_1^2 + K_{11} \end{bmatrix}}{E\ (\ \lambda_1 - \lambda^*_1\ )\ (\ \lambda_1 - \lambda_2\ )\ (\ \lambda_1 - \lambda^*_2\ )\ (\ s - \lambda_1\ )} \tag{4.27}$$

Note that the numerator matrix on the right side of Equation 4.27 is the adjoint of the system matrix discussed in Section 3.8. Therefore, from Equation 3.41:

$$\begin{bmatrix} M_{22}\ \lambda_1^2 + K_{22} & -(\ M_{12}\ \lambda_1^2 + K_{12}\ ) \\ -(\ M_{21}\ \lambda_1^2 + K_{21}\ ) & M_{11}\ \lambda_1^2 + K_{11} \end{bmatrix} = \gamma_1 \begin{bmatrix} \psi_1\ \psi_1 & \psi_1\ \psi_2 \\ \psi_2\ \psi_1 & \psi_2\ \psi_2 \end{bmatrix}_1 \tag{4.28}$$

Plugging Equation 4.28 into Equation 4.27 gives:

$$\frac{\begin{bmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{bmatrix}}{(s-\lambda_1)} = \frac{\gamma_1 \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_1}{E(\lambda_1-\lambda_1^*)(\lambda_1-\lambda_2)(\lambda_1-\lambda_2^*)(s-\lambda_1)} \tag{4.29}$$

Finally, the relationship between the transfer function data (via the residue matrices) and the modal vectors of the system can be established. Equation 4.29 shows that the residue matrix has the same structure as the adjoint matrix: each column contains a redundant estimate of the same modal vector, different only by a constant. This relationship is normally stated as follows:

For pole λ_1 :

$$\begin{bmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{bmatrix} = Q_1 \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_1$$

In general, for pole λ_r :

$$\begin{bmatrix} A_{11r} & A_{12r} \\ A_{21r} & A_{22r} \end{bmatrix} = Q_r \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_r \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_r^T = Q_r \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_r \tag{4.30}$$

where:

- Q_1 is a constant that is a function of the modal vector scaling and the absolute units of the residue matrix.
- $\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 =$ mode shape for pole λ_1

Equation 4.30 indicates that, with respect to the residue matrix, every row and column of the residue matrix contains the same modal vector multiplied by a component of the modal vector and the scaling constant. This is an important result for the experimental case. Therefore, assuming that all of the residues within a row or column are not perfectly zero, the modal vector can be estimated from only one row or column of the residue matrix. This will completely define

the modal vector for that particular pole. In terms of the experimental requirements, in order to estimate a particular element in the residue matrix, a frequency response function must be measured. Under the assumption that every element in a particular row or column is not zero, only N measurements will be required rather than $N \times N$. Once this information is measured, all other terms in the residue matrix could be synthesized.

This previous discussion represents the MINIMUM requirements in terms of measurements. In order to be certain that modal vectors are not missed or to utilize the redundant information in different rows or columns, more measurements than one row or column are typically taken.

It should be pointed out that the constant of proportionality Q_r in Equation 4.30 is not unique since the constant will depend upon the choice of how the modal vector is scaled. This is consistent with the concept that modal vectors represent relative motion between the degrees of freedom. Note however that the residues are scaled quantities that depend upon the units of the frequency response function(s).

4.4 Modal Vector Example

The transfer function matrix for the two degree of freedom system example used previously can now be used to determine the modal vectors.

Substituting mass and stiffness values into Equation 4.6 gives:

$$[\ H\ (s)\ ] = \frac{\begin{bmatrix} (\ 10\ s^2 + 6\ ) & 2 \\ 2 & (\ 5\ s^2 + 4\ ) \end{bmatrix}}{(\ 5\ s^2 + 4\ )\ (\ 10\ s^2 + 6\ ) - (-2)\ (-2)}$$

$$[\ H\ (s)\ ] = \frac{\begin{bmatrix} (\ 10\ s^2 + 6\ ) & 2 \\ 2 & (\ 5\ s^2 + 4\ ) \end{bmatrix}}{50\ (\ s^4 + 7/5\ s^2 + 2/5\ )}$$

The roots of the characteristic equation have previously been calculated as:

$$\lambda_1 = j\ \sqrt{2/5}\ \ (\ rad/sec\ ) \qquad\qquad \lambda_1^* = -\ j\ \sqrt{2/5}\ \ (\ rad/sec\ )$$

$$\lambda_2 = j \ \ (\ rad/sec\ ) \qquad\qquad \lambda_2^* = -\ j \ \ (\ rad/sec\ )$$

Therefore:

$$[\ H\ (s)\ \ ] = \frac{\begin{bmatrix} (\ 10\ s^2 + 6\ ) & 2 \\ 2 & (\ 5\ s^2 + 4\ ) \end{bmatrix}}{50\ (\ s - j\ \sqrt{2/5}\ )\ (\ s + j\ \sqrt{2/5}\ )\ (\ s - j\ )\ (\ s + j\ )}$$

The transfer function $H_{11}(s)$ can now be represented in terms of its partial fraction expansion:

$$H_{11}(s) = \frac{10\ s^2 + 6}{50\ (\ s - j\ \sqrt{2/5}\ )\ (\ s + j\ \sqrt{2/5}\ )\ (\ s - j\ )\ (\ s + j\ )}$$

$$H_{11}(s) = \frac{A_{111}}{(\ s - j\ \sqrt{2/5}\ )} + \frac{A_{111}^*}{(\ s + j\ \sqrt{2/5}\ )} + \frac{A_{112}}{(\ s - j\ )} + \frac{A_{112}^*}{(\ s - j\ )}$$

$$A_{111} = -\ \frac{j\ \sqrt{2/5}}{12} \qquad\qquad A_{111}^* = \frac{j\ \sqrt{2/5}}{12}$$

$$A_{112} = \frac{-\ j}{15} \qquad\qquad A_{112}^* = \frac{j}{15}$$

In a similar fashion, the rest of the residues for the remaining transfer functions can also be determined.

The system transfer function matrix [$H(s)$] can now be expressed in terms of partial fractions.

$$[\ H\ (s)\ ] = \frac{\begin{bmatrix} -\frac{j\sqrt{2/5}}{12} & -\frac{j\sqrt{2/5}}{12} \\ -\frac{j\sqrt{2/5}}{12} & -\frac{j\sqrt{2/5}}{12} \end{bmatrix}}{(\ s - j\sqrt{2/5}\ )} + \frac{\begin{bmatrix} \frac{j\sqrt{2/5}}{12} & \frac{j\sqrt{2/5}}{12} \\ \frac{j\sqrt{2/5}}{12} & \frac{j\sqrt{2/5}}{12} \end{bmatrix}}{(\ s + j\sqrt{2/5}\ )}$$

$$+ \frac{\begin{bmatrix} \frac{-j}{15} & \frac{j}{30} \\ \frac{j}{30} & \frac{-j}{60} \end{bmatrix}}{(\ s - j\ )} + \frac{\begin{bmatrix} \frac{j}{15} & \frac{-j}{30} \\ \frac{-j}{30} & \frac{j}{60} \end{bmatrix}}{(\ s + j\ )}$$

Recall that the modal vector associated with the pole frequency λ_1 is proportional to the residue matrix for pole λ_1. Notice that the first two residue matrices are just the complex conjugate of each other. Therefore, the modal vector for pole λ_1^* is just the complex conjugate of the modal vector for pole λ_1. The same is true for the last two residue matrices. This will always be the case for conjugate pairs of poles.

The modal vector for the pole $\lambda_1 = \sqrt{2/5}$ can be extracted from the first residue matrix.

Using Equations 4.29 and 4.30, the modal vectors can be related to the residue matrix for the first pole.

$$Q_1\ \{\ \psi\ \}_1\ \{\ \psi\ \}_1^T = Q_1 \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix}_1$$

$$Q_1\ \{\ \psi\ \}_1\ \{\ \psi\ \}_1^T = \begin{bmatrix} -\frac{j\sqrt{2/5}}{12} & -\frac{j\sqrt{2/5}}{12} \\ -\frac{j\sqrt{2/5}}{12} & -\frac{j\sqrt{2/5}}{12} \end{bmatrix}_1$$

The constant $Q_1 = \pm\ \frac{j\sqrt{2/5}}{12}$ can now be factored out of the residue matrix. The choice of this

constant at this point is purely arbitrary by virtue of the fact that the absolute amplitudes of the modal vectors are purely arbitrary.

$$\left[\begin{array}{cc} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{array} \right]_1 = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]_1$$

Hence, the modal vector for the first mode is:

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_1 = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}_1$$

Similarly, the modal vector for the second mode:

$$Q_2 \{ \ \psi \ \}_2 \{ \ \psi \ \}_2^T = Q_2 \left[\begin{array}{cc} \psi_1 \ \psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{array} \right]_2 = \left[\begin{array}{cc} \frac{-j}{15} & \frac{j}{30} \\ \frac{j}{30} & \frac{-j}{60} \end{array} \right]_2$$

The constant $Q_2 = \pm \frac{-j}{60}$ can now be factored from this residue matrix in the same arbitrary manner as before.

$$\left[\begin{array}{cc} \psi_1 \ \psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{array} \right]_2 = \left[\begin{array}{cc} 4 & -2 \\ -2 & 1 \end{array} \right]_2$$

Equating elements of the two matrices yields:

$$\psi_1\psi_1 = \psi_1^2 = 4 \qquad \psi_1 = 2$$

$$\psi_2\psi_1 = -2 \qquad \psi_2 = -1$$

Thus:

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_2 = \left\{ \begin{array}{c} 2 \\ -1 \end{array} \right\}_2$$

Both of these modal vectors could have been obtained directly by just using either a row or a column of the residue matrix as the modal vector directly.

4.5 Analytical Model - General Partial Fraction (Residue)

Recalling Equation 4.21 for a two degree of freedom system:

$$H_{11}(s) = \sum_{r=1}^{2} \frac{A_{11r}}{(s - \lambda_r)} + \frac{A^*_{11r}}{(s - \lambda^*_r)}$$

Equation 4.21 can now be generalized for an N-degree of freedom system as the following:

$$H_{11}(s) = \sum_{r=1}^{N} \frac{A_{11r}}{(s - \lambda_r)} + \frac{A^*_{11r}}{(s - \lambda^*_r)} \tag{4.31}$$

$$\left[\; H_{pq}(s) \; \right] = \sum_{r=1}^{N} \frac{\left[\; A_{pqr} \; \right]}{(s - \lambda_r)} + \frac{\left[\; A^*_{pqr} \; \right]}{(s - \lambda^*_r)} \tag{4.32}$$

Furthermore, the entire system transfer function matrix [$H(s)$] can be generalized as:

$$[\; H(s) \;] = \sum_{r=1}^{N} \frac{[\; A \;]_r}{s - \lambda_r} + \frac{[\; A^* \;]_r}{s - \lambda^*_r} \tag{4.33}$$

In terms of the modal vectors of the system directly:

$$[\; H(s) \;] = \sum_{r=1}^{N} \frac{Q_r \{ \psi \}_r \{ \psi \}_r^T}{(s - \lambda_r)} + \frac{Q^*_r \{ \psi \}^*_r \{ \psi \}^{*T}_r}{(s - \lambda^*_r)} \tag{4.34}$$

Equation 4.34 is the general form of the system transfer function matrix. As will be shown later, this form does not change when a system with damping is considered. Remember, though, that the complete transfer function can not be measured. Not once again that the frequency response function measurement, which is just Equation 4.34 evaluated at $s = j\,\omega$, is actually what is measured.

4.6 Residue Relationship to Modal Vectors

The relationship established in Section 4.3 between the residue matrix and the modal vector can be developed in a more formal or rigorous manner. It is important to understand that the residue defined in the previous section is the key to the relationship between modal vectors and modal scaling (modal mass) for the experimental case. At the present time, the following discussion is limited to the undamped and/or proportionally damped cases. The relationship for the case of general damping is discussed in a later section.

The development of the relationship between the residue matrix and the modal vectors proceeds along a similar path as the development of the relationship between the residue matrix and the adjoint of the system matrix (Section 3.8). Beginning with the definition of the impedance matrix:

$$[B(s)] = \left[[\ M\ ]s^2 + [\ C\ ]s + [\ K\ ] \right] \tag{4.35}$$

where:

- $[\ B(s)\ ]$ = System Impedance Matrix

From matrix algebra:

$$[B(s)]\,[B(s)]^{-1} = [I] \tag{4.36}$$

Noting that the inverse of the impedance matrix is the transfer function matrix gives:

$$[B(s)]\,[H(s)] = [I] \tag{4.37}$$

Replacing the transfer function matrix with the equivalent partial fraction representation from Equation 4.33 yields:

$$[I] = \sum_{r=1}^{N} \frac{[\ B(s)\ ]\,[\ A\ ]_r}{s - \lambda_r} + \frac{[B(s)]\,[\ A^*\ ]_r}{s - \lambda_r^*} \tag{4.38}$$

Premultiplying each term of the above equation by $(s - \lambda_r)$ and evaluating the equation at $s = \lambda_r$

for any specific mode r (all of the terms drop out except the term associated with λ_r):

$$[\ 0\ ] = \Big[B(\lambda_r) \Big] [\ A\ ]_r \tag{4.39}$$

Note that the above equation proves that each column of the residue matrix $[A]_r$ must be proportional to the modal vector associated with λ_r just as it did in the adjoint matrix case (Section 3.8). Likewise the structure of the residue matrix must be the same as the structure of the adjoint matrix.

$$[A]_r = Q_r \{\ \psi\ \}_r \{\ \psi\ \}_r^T \tag{4.40}$$

$$[A]_r = Q_r \begin{bmatrix} \psi_1\ \psi_1 & \psi_1\ \psi_2 & . & . & . & . & \psi_1\ \psi_N \\ \psi_2\ \psi_1 & \psi_2\ \psi_2 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \psi_N\ \psi_1 & \psi_N\ \psi_2 & . & . & . & . & \psi_N\ \psi_N \end{bmatrix}_r$$

where:

- $Q_r =$ constant associated with the scaling of $\{\psi\}_r$ relative to the absolute scaling (units) of the residue matrix.
- $Q_r =$ is proportional, but not generally equal to, the proportionality constant γ_r defined in Section 3.8.

4.7 Residue Relationship to Modal Mass

For the proportionally damped case, which includes the undamped case as a trivial form, the relationship between the residue and the modal mass can also be established consistent with the modal mass found analytically from the mass matrix. Starting with Equation 4.38 and 4.40:

$$[I] = \sum_{r=1}^{N} \frac{[\ B(s)\ ]\ Q_r \{\psi\}_r\ \{\psi\}_r^T}{s - \lambda_r} + \frac{[B(s)]\ Q_r^* \left\{\psi^*\right\}_r \left\{\psi^*\right\}_r^T}{s - \lambda_r^*} \tag{4.41}$$

Note that, for proportionally damped systems, the modal vectors are always real (normal) modes. Therefore, the conjugate of a modal vector is the same as the modal vector ($\{\psi\}_r = \left\{\psi^*\right\}_r$). Making this substitution and premultiplying both sides of Equation 4.41 by $\{\psi\}_t^T$:

$$\{\psi\}_t^T = \sum_{r=1}^{N} \frac{\{\psi\}_t^T\ [\ B(s)\ ]\ Q_r \{\psi\}_r\ \{\psi\}_r^T}{s - \lambda_r} + \frac{\{\psi\}_t^T\ [B(s)]\ Q_r^* \{\psi\}_r\ \{\psi\}_r^T}{s - \lambda_r^*} \tag{4.42}$$

Substitute Equation 4.35 into Equation 4.42:

$$\{\psi\}_t^T = \sum_{r=1}^{N} \frac{Q_r\ \{\psi\}_t^T \left[\ [\ M\ ]\ s^2 + [\ C\ ]\ s + [\ K\ ]\ \right] \{\psi\}_r\ \{\psi\}_r^T}{s - \lambda_r} +$$

$$\frac{Q_r^*\ \{\psi\}_t^T \left[\ [\ M\ ]\ s^2 + [\ C\ ]\ s + [\ K\ ]\ \right] \{\psi\}_r\ \{\psi\}_r^T}{s - \lambda_r^*} \tag{4.43}$$

Applying the orthogonality relationships between the modal vectors and the mass, damping and stiffness matrices eliminates all terms except for those associated with mode *t*:

$$\{\psi\}_t^T = \frac{Q_t\ (\ M_t\ s^2 + C_t\ s + K_t\ )\{\psi\}_t^T}{s - \lambda_t} + \frac{Q_t^*\ (\ M_t\ s^2 + C_t\ s + K_t\ )\{\psi\}_t^T}{s - \lambda_t^*} \tag{4.44}$$

Eliminating $\{\psi\}_t^T$ from each term of Equation 4.44 leaves the following scalar equation:

$$1 = \frac{Q_t\ (\ M_t\ s^2 + C_t\ s + K_t\ )}{s - \lambda_t} + \frac{Q_t^*\ (\ M_t\ s^2 + C_t\ s + K_t\ )}{s - \lambda_t^*} \tag{4.45}$$

Clearing the fractions from the previous equation:

$$(s - \lambda_t)\ (s - \lambda_t^*) = Q_t\ (\ M_t\ s^2 + C_t\ s + K_t\ )\ (s - \lambda_t^*) + Q_t^*\ (\ M_t\ s^2 + C_t\ s + K_t\ )(s - \lambda_t)$$

. (4.46)

Note that, by definition:

$$(M_t\ s^2 + C_t\ s + K_t\ ) = M_t\ (s - \lambda_t)\ (s - \lambda_t^*) \tag{4.47}$$

Substituting Equation 4.47 into Equation 4.46 and eliminating the common terms on both sides of the equation $(s - \lambda_t)\ (s - \lambda_t^*)$:

$$1 = Q_t\ M_t\ (s - \lambda_t^*) + Q_t^*\ M_t\ (s - \lambda_t) \tag{4.48}$$

Evaluating Equation 4.49 at $s = \lambda_t$ gives the final relationship:

$$1 = Q_t\ M_t\ (\lambda_t - \lambda_t^*) \tag{4.49}$$

$$1 = Q_t\ M_t\ (2\ j\ \omega_t) \tag{4.50}$$

$$M_t = \frac{1}{2\ j\ \omega_t\ Q_t} \tag{4.51}$$

Equation 4.51 represents the relationship between modal mass and the scaling involved between the residues and the modal vectors (Recall Equation 4.40). Therefore, once the residue information is found for mode *t* and some convenient form of modal vector scaling is chosen for mode *t*, the scaling constant Q_t can be determined. Equation 4.51 can then be used to determine modal mass for mode *t* consistent with the modal vector scaling. This means that as long as the modal vector is chosen consistently, modal mass can be compared between different solution approaches (analytical versus experimental, for example).

5. GENERAL DAMPED SYSTEMS

5.1 Non-Proportionally Damped Systems

In reality, physical structures or systems are generally comprised of many substructures tied together in various fashions. These substructures can be made-up of a variety of materials, i.e. metals, plastics, and wood. Furthermore, these substructures may be connected to one another by rivets, bolts, screws, dampers, springs, weldments, friction, etc. Also, the spatial geometry of the structure may be very complicated, such as an exhaust system of an automobile. All of these factors influence the inherent dynamical properties of the structure. For these structures, mass, damping, and stiffness distribution (matrices) of the system are rather complicated. In general, for real life structures, the damping matrix for such a system will not always be proportional to the mass and/or stiffness matrix. Therefore, the damping of this system can be classified as non-proportional.

In an analytical sense, modal analysis of this general type of damped system cannot be described using the formulation of the eigenvalue problem as discussed previously for an undamped system. Remember, that in an analytical sense, the purpose of modal analysis is to find a coordinate transformation that uncoupled the original equations of motion. This coordinate transformation turned out to be a matrix comprised of the modal vectors of the system. These modal vectors were determined from the solution of the eigenvalue problem for that system. The coordinate transformation diagonalized the system mass, damping, and stiffness matrices, for an undamped or proportionally damped system. When a system contains non-proportional damping, the previously used formulation of the eigenvalue problem will not yield modal vectors (eigenvectors) that uncouple the equations of motion of the system. A technique used to circumvent this problem was first documented by Duncan and Collar and involves the reformulation of the original equations of motion, for an N-degree of freedom system, into an equivalent set of $2N$ first order differential equations known as Hamilton's Canonical Equations. The solution of these equations can be carried out in a similar manner that has been discussed previously.

For nonproportional damping, the coordinate transformation discussed previously, that diagonalizes the system mass and stiffness matrices, will not diagonalize the system damping matrix. Therefore, when a system with nonproportional damping exists, the equations of motion are coupled when formulated in N dimension physical space. Fortunately, the equations of motion can be uncoupled when formulated in $2N$ dimension *state space.* This is accomplished

by augmenting the original N dimension physical space equation by a N dimension identity as follows.

Assume a viscous, nonproportionally damped system can be represented by Equation 5.1.

$$[\ M\ ]\{\ \ddot{x}\ \} + [\ C\ ]\{\ \dot{x}\ \} + [\ K\ ]\{\ x\ \} = \{\ f\ \} \tag{5.1}$$

This system of equations can be augmented by the identity shown in Equation 5.2:

$$[\ M\ ]\{\ \dot{x}\ \} - [\ M\ ]\{\ \dot{x}\ \} = \{\ 0\ \} \tag{5.2}$$

Equations 5.1 and 5.2 can be combined as follows to yield a new system of $2N$ equations. Note that all the matrices in Equation 5.3 are symmetric and Equation 5.3 is now in a classical eigenvalue solution form. The notation used in Equation 5.3 is consistent with the notation used in many mathmatics and/or controls textbooks.

$$[\ A\ ]\{\ \dot{y}\ \} + [\ B\ ]\{\ y\ \} = \left\{\ f'\ \right\} \tag{5.3}$$

where:

- $[A\ ] = \begin{bmatrix} [\ 0\ ] & [\ M\ ] \\ [\ M\ ] & [\ C\ ] \end{bmatrix} \qquad [B\ ] = \begin{bmatrix} -[\ M\ ] & [\ 0\ ] \\ [\ 0\ ] & [\ K\ ] \end{bmatrix}$

- $\{\ \dot{y}\ \} = \begin{Bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{Bmatrix} \qquad \{\ y\ \} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} \qquad \begin{Bmatrix} f' \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{f\} \end{Bmatrix}$

Forming the homogeneous equation from Equation 5.3 yields:

$$[\ A\ ]\{\ \dot{y}\ \} + [\ B\ ]\{\ y\ \} = \{\ 0\ \} \tag{5.4}$$

The solution of Equation 5.4 yields the complex-valued natural frequencies (eigenvalues) and complex-valued modal vectors (eigenvectors) for the augmented $2N$ equation system. Note that in this mathematical form, the eigenvalues will be found directly (not the square of the

eigenvalue) and the $2N$ eigenvectors will be $2N$ in length. The exact form of the eigenvectors can be seen from the eigenvector matrix (state space modal matrix) for the $2N$ equation system. Note that the notation $\{ \phi \}$ is used for an eigenvector in the $2N$ equation system and that the notation $\{ \psi \}$ is used for an eigenvector of the original N equation system.

The state space modal matrix $[\phi]$ for this nonproportionally damped system can now be assembled.

$$\left[\phi \right] = \left[\{\phi\}_1 \quad \{\phi\}_2 \quad \{\phi\}_3 \quad \{\cdots\} \quad \{\phi\}_r \quad \{\cdots\} \quad \{\phi\}_{2N} \right] \tag{5.5a}$$

Based upon the change of coordinate applied in Equation 5.3, each column of the eigenvector matrix (each eigenvector) is made up of the derivative of the desired modal vector above the desired modal vector. This structure is shown in Equation 5.5b.

$$\left[\phi \right] = \begin{bmatrix} \lambda_1\{\psi\}_1 & \lambda_2\{\psi\}_2 & \lambda_3\{\psi\}_3 & \cdots & \lambda_r\{\psi\}_r & \cdots & \lambda_{2N}\{\psi\}_{2N} \\ \{\psi\}_1 & \{\psi\}_2 & \{\psi\}_3 & \cdots & \{\psi\}_r & \cdots & \{\psi\}_{2N} \end{bmatrix} \tag{5.5b}$$

5.1.1 Weighted Orthogonality of the Eigenvectors

Similar to the case for undamped systems, a set of weighted orthogonality relationships are valid for the system matrices $[A]$ and $[B]$.

$$\{\phi\}_r^T [A] \{\phi\}_s = 0 \tag{5.6}$$

$$\{\phi\}_r^T [B] \{\phi\}_s = 0 \tag{5.7}$$

The terms *modal A* and *modal B* can now be defined as follows. Note that these quantities have the same properties as *modal mass* and *modal stiffness* for the undamped and proportionally damped cases.

$$\{\phi\}_r^T [A] \{\phi\}_r = M_{A_r} \tag{5.8}$$

$$\{\phi\}_r^T [B] \{\phi\}_r = M_{B_r} \tag{5.9}$$

The terms *modal A* and *modal B* are modal scaling factors for the nonproportional case just as modal mass and modal stiffness can be used for the undamped and proportionally damped cases. Whenever complex modal vectors are present, *modal A* and *modal B* should be used to provide the modal scaling. Note that *modal A* and *modal B* could be used to provide modal scaling even for the undamped and proportionally damped cases.

Note that the eigenvector matrix (state space modal matrix) provides a coordinate transformation from the physical state space coordinate system to the uncoupled principl state space coordinate system.

$$[\ A\ ]\{\ \dot{y}\ \} + [\ B\ ]\{\ y\ \} = \left\{\ f'\ \right\} \tag{5.10}$$

$$[\ A\ ]\left[\ \phi\ \right]\{\ \dot{q}\ \} + [\ B\ ]\left[\ \phi\ \right]\{\ q\ \} = \left\{\ f'\ \right\} \tag{5.11}$$

$$\left[\ \phi\ \right]^T [\ A\ ]\left[\ \phi\ \right]\{\ \dot{q}\ \} + \left[\ \phi\ \right]^T [\ B\ ]\left[\ \phi\ \right]\{\ q\ \} = \left[\ \phi\ \right]^T \left\{\ f'\ \right\} \tag{5.12}$$

For the $r-th$ eigenvalue/eigenvector:

$$M_{A_r}\ \dot{q}_r + M_{B_r} = f_r' \tag{5.13}$$

This uncoupled equation has a characteristic equation of the form:

$$M_{A_r}\ s + M_{B_r} = 0 \tag{5.14}$$

This means that Modal A and Modal B for each mode are related.

$$M_{A_r}\ \lambda_r + M_{B_r} = 0 \tag{5.15}$$

$$M_{A_r}\ \lambda_r = -\ M_{B_r} \tag{5.16}$$

This concept will not be pursued further here. The interested reader is referred to Chapter 6 of *Mechanical Vibrations* by Tse, Morse and Hinkle or Chapter 9 of *Analytical Methods in*

Vibrations by Leonard Meirovitch.

In an experimental sense, the approach to the problem is the same for non-proportionally damped system as for an undamped or a proportionally damped system. No matter what type of damping a structure has, proportional or non-proportional, the frequency response functions of the system can be measured. For this reason, the modal vectors of the system can be found by using the residues determined from the frequency response function measurements. While the approach is the same as before, the results in terms of modal vectors will be somewhat more complicated for a system with non-proportional damping. As an example of the differences that result for the non-proportional case, the same two degree of freedom system, used in previous examples, will be used again except that the damping matrix will be made non-proportional to the mass and/or stiffness matrices.

5.2 Proportionally Damped Systems

For the class of physical damping mechanisms that can be mathematically represented by the proportional damping concept, the coordinate transformation discussed previously for the undamped case, that diagonalizes the system mass and stiffness matrices, will also diagonalize the system damping matrix. Therefore, when a system with proportional damping exists, that system of coupled equations of motion can be transformed as before to a system of equations that represent an uncoupled system of single degree of freedom systems that are easily solved. The procedure to accomplish this follows.

Assume a viscously damped system can be represented by Equation 5.17.

$$[\ M\ ]\{\ \ddot{x}\ \} + [\ C\ ]\{\ \dot{x}\ \} + [\ K\ ]\{\ x\ \} = \{\ f\ \} \tag{5.17}$$

The eigenvalue problem associated with the undamped system can be solved as a first step to understanding the problem.

$$[\ M\ ]\{\ \ddot{x}\ \} + [\ K\ ]\{\ x\ \} = \{\ 0\ \}$$

This yields the system's natural frequencies (eigenvalues) and modal vectors (eigenvectors).

The modal matrix $[\psi]$ for this undamped system can now be assembled.

$$\left[\psi \right] = \left[\{ \psi \}_1 \{ \psi \}_2 \ldots \{ \psi \}_N \right]$$

The coordinate transformation can now be applied to Equation 5.10:

$$[x] = \left[\psi \right] \{ q \}$$

$$[M] \left[\psi \right] \{ \ddot{q} \} + [C] \left[\psi \right] \{ \dot{q} \} + [K] \left[\psi \right] \{ q \} = \{ f \}$$

Now pre-multiply Equation 5.18 by $[\psi]^T$.

$$\left[\psi \right]^T [M] \left[\psi \right] \{ \ddot{q} \} + \left[\psi \right]^T [C] \left[\psi \right] \{ \dot{q} \} + \left[\psi \right]^T [K] \left[\psi \right] \{ q \} = \left[\psi \right]^T \{ f \} \qquad (5.19)$$

Due to the orthogonality properties of the modal vectors :

$$\left[\psi \right]^T [M] \left[\psi \right] = \lceil M \rfloor$$

$$\left[\psi \right]^T [K] \left[\psi \right] = \lceil K \rfloor$$

Since the assumed form of the damping matrix is proportional to the mass and/or stiffness matrix, the damping matrix will also be diagonalized.

$$[C] = \alpha [M] + \beta [K]$$

The application of the orthogonality condition yields:

$$\left[\psi \right]^T [C] \left[\psi \right] = \left[\psi \right]^T \{ \alpha [M] + \beta [K] \} \left[\psi \right]$$

$$\left[\psi \right]^T [C] \left[\psi \right] = \alpha \left[\psi \right]^T [M] \left[\psi \right] + \beta \left[\psi \right]^T [K] \left[\psi \right]$$

$$\left[\psi \right]^T [C] \left[\psi \right] = \alpha \lceil M \rfloor + \beta \lceil K \rfloor$$

Therefore:

$$\left[\psi \right]^T [C] \left[\psi \right] = \lceil C \rfloor$$

where:

- $\lceil C \rfloor$ is a diagonal matrix.

Therefore, Equation 5.19 becomes:

$$\lceil M \rfloor \{ \ddot{q} \} + \lceil C \rfloor \{ \dot{q} \} + \lceil K \rfloor \{ q \} = \left[\psi \right]^T \{ f(t) \} \qquad (5.20)$$

Equation 5.20 represents an uncoupled set of damped single degree of freedom systems. The r-th equation of Equation 5.20 is:

$$M_r \ddot{q}_r + C_r \dot{q}_r + K_r q_r = f_r'(t) \qquad (5.21)$$

Equation 5.21 is the equation of motion for a system represented below.

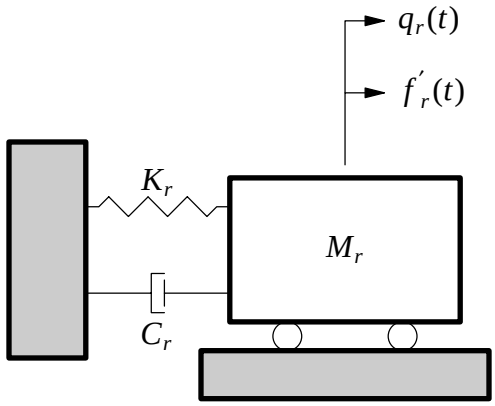

Figure 5-1. Proportional Damped SDOF Equivalent Model

The solution of this damped single degree of freedom system has been discussed previously.

5.3 Example with Proportional Damping

In order to understand the concept of proportional damping, the same two degree of freedom example, worked previously for the undamped case, can be reworked for the proportionally damped case.

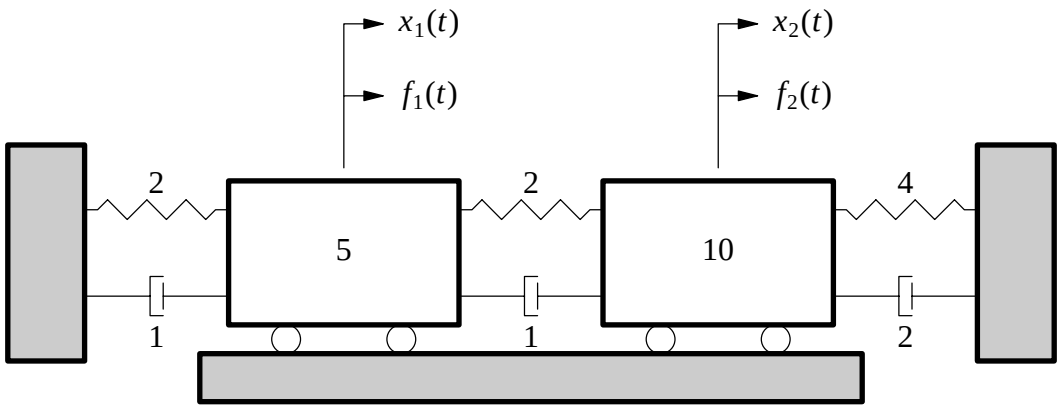

Figure 5-2. Two Degree of Freedom Model with Proportional Damping

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Note that the damping matrix [C] is proportional to the stiffness matrix [K]:

$$[\ C\ ] = \left(\frac{1}{2}\right)[\ K\ ]$$

While this form of the damping matrix is quite simple, the solution that will result will yield a general characteristic that is common to all problems that can be described by the concept of proportional damping.

Using the previously calculated modal vectors and natural frequencies for the undamped system, a coordinate transformation can be performed:

$$\{ x \} = [\psi] \{ q \}$$

Noting the results of the previous example in Section 3.6, the uncoupling of the equations of motion for the mass and stiffness matrices has already been shown. Therefore, for the damping matrix:

$$[\psi]^T [C] [\psi] = \begin{bmatrix} \sqrt{1/15} & \sqrt{1/15} \\ \sqrt{2/15} & -\frac{1}{2}\sqrt{2/15} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{1/15} & \sqrt{2/15} \\ \sqrt{1/15} & -\frac{1}{2}\sqrt{2/15} \end{bmatrix}$$

$$[\psi]^T [C] [\psi] = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/2 \end{bmatrix}$$

The transformed system in terms of modal coordinates would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 1/5 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1'(t) \\ f_2'(t) \end{Bmatrix}$$

or pictorially:

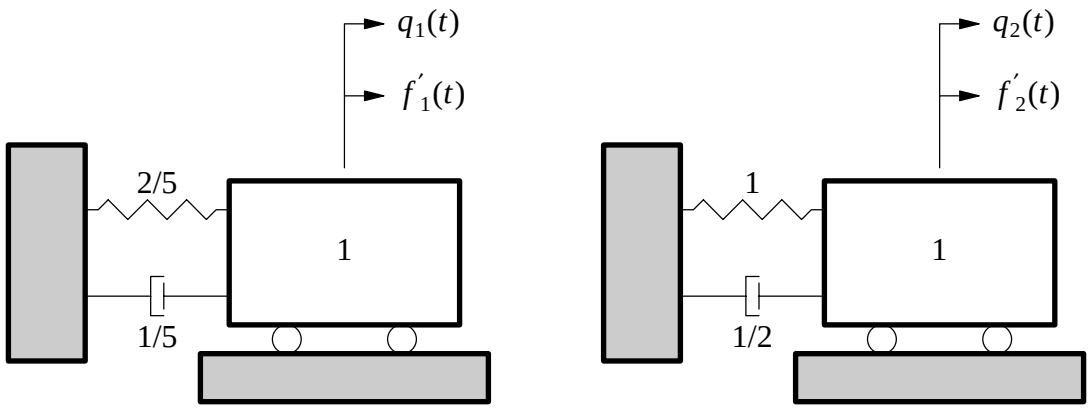

Figure 5-3. Proportionally Damped MDOF Equivalent Model

5.3.1 Frequency Response Function Implications

This same problem can be evaluated from a transfer function point of view by determining the modal frequencies and modal vectors. The transfer function matrix of the system is:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} (M_{22}\,s^2 + C_{22}\,s + K_{22}) & -(M_{12}\,s^2 + C_{12}\,s + K_{12}) \\ -(M_{21}\,s^2 + C_{21}\,s + K_{21}) & (M_{11}\,s^2 + C_{11}\,s + K_{11}) \end{bmatrix}}{|\ B\ (s)\ |}$$

$$|\ B\ (s)\ | = \left(M_{11}\,s^2 + C_{11}\,s + K_{11}\right)\left(M_{22}\,s^2 + C_{22}\,s + K_{22}\right) - \left(M_{21}\,s^2 + C_{21}\,s + K_{21}\right)\left(M_{12}\,s^2 + C_{12}\,s + K_{12}\right)$$

Upon substituting the values for the mass, damping, and stiffness matrices from the example problem:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} \left(10\,s^2 + 3\,s + 6\right) & (s+2) \\ (s+2) & \left(5\,s^2 + 2\,s + 4\right) \end{bmatrix}}{50\,s^4 + 35\,s^3 + 75\,s^2 + 20\,s + 20}$$

The roots of the characteristic equation are:

$$\lambda_1 = -\frac{1}{10} + j\,\frac{\sqrt{39}}{10}\ (\,rad/sec\,) \qquad \lambda_1^* = -\frac{1}{10} - j\,\frac{\sqrt{39}}{10}\ (\,rad/sec\,)$$

$$\lambda_2 = -\frac{1}{4} + j\,\frac{\sqrt{15}}{4}\ (\,rad/sec\,) \qquad \lambda_2^* = -\frac{1}{4} - j\,\frac{\sqrt{15}}{4}\ (\,rad/sec\,)$$

Note that the poles of a damped system now contain a real part. The pole of a transfer function has been previously defined (Chapter 2) as:

$$\lambda_r = \sigma_r + j\,\omega_r$$

where:

- σ_r = damping factor
- ω_r = damped natural frequency.

Remember that in the previous undamped case, the poles could be written in the following form:

$$\lambda_r = j\,\omega_r$$

where:

- ω_r = damped natural frequency = undamped natural frequency

The system modal vectors can be determined by just using the minimum transfer function data, assuming that all elements of the row or column are not perfectly zero. If this is true, then the

modal vectors can be found by using one row or column out of the system transfer function matrix. For example, choose $H_{11}(s)$ and $H_{21}(s)$, which amounts to the first column.

First, $H_{11}(s)$ and $H_{21}(s)$ can be expanded in terms of partial fractions.

$$H_{11}(s) = \frac{10\,s^2 + 3\,s + 6}{50\,(s - \lambda_1)(s - \lambda_1^*)(s - \lambda_2)(s - \lambda_2^*)}$$

After some work:

$$H_{11}(s) = \frac{-j\,\frac{\sqrt{39}}{117}}{(s - \lambda_1)} + \frac{j\,\frac{\sqrt{39}}{117}}{(s - \lambda_1^*)} + \frac{-j\,\frac{4\sqrt{15}}{225}}{(s - \lambda_2)} + \frac{j\,\frac{4\sqrt{15}}{225}}{(s - \lambda_2^*)}$$

$$H_{21}(s) = \frac{s + 2}{50\,(s - \lambda_1)(s - \lambda_1^*)(s - \lambda_2)(s - \lambda_2^*)}$$

$$H_{21}(s) = \frac{-j\,\frac{\sqrt{39}}{117}}{(s - \lambda_1)} + \frac{j\,\frac{\sqrt{39}}{117}}{(s - \lambda_1^*)} + \frac{j\,\frac{2\sqrt{15}}{225}}{(s - \lambda_2)} + \frac{-j\,\frac{2\sqrt{15}}{225}}{(s - \lambda_2^*)}$$

Recall that the residues have been shown to be proportional to the modal vectors. Since only a column of the transfer function has been used from the transfer function matrix, only a column of the residue matrix for each modal frequency has been calculated.

Thus, by looking at the residues for the first pole λ_1, the modal vector that results is:

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_1 \rightarrow \left\{ \begin{array}{c} \frac{-j\sqrt{39}}{117} \\ \frac{-j\sqrt{39}}{117} \end{array} \right\}_1 = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}_1 \tag{5.23}$$

For the modal frequency λ_2, the modal vector that results is:

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_2 \rightarrow \left\{ \begin{array}{c} \frac{-j\,4\sqrt{15}}{225} \\ \frac{2\,j\sqrt{15}}{225} \end{array} \right\}_2 = \left\{ \begin{array}{c} 1 \\ -1/2 \end{array} \right\}_2 \tag{5.24}$$

Thus, the modal vectors calculated from the transfer function matrix of this proportionally damped system are the same as for the undamped system. This will always be the case as long as the system under consideration exhibits proportional type of damping.

Notice that for the undamped and proportionally damped systems studied, not only were their modal vectors the same, but their modal vectors, when normalized, are real valued. This may not be the case for real systems. Since the modal vectors are real valued, they are typically referred to as *real modes or normal modes.*

5.3.2 Impulse Response Function Considerations

The impulse response function of a proportionally damped multiple degree of freedom system can now be discussed. Remember that the impulse response function is the time domain equivalent to the transfer function. Once the impulse response function for a multiple degree of freedom system has been formulated, it can be compared to the previous single degree of freedom system. Recall Equation 4.24.

$$H_{pq}(s) = \sum_{r=1}^{N} \frac{A_{pqr}}{(s - \lambda_r)} + \frac{A^*_{pqr}}{(s - \lambda^*_r)} \tag{5.25}$$

Recall the definition of $H_{pq}(s)$:

$$H_{pq}(s) = \frac{X_p(s)}{F_q(s)}$$

Therefore:

$$X_p(s) = F_q(s) \sum_{r=1}^{N} \frac{A_{pqr}}{(s - \lambda_r)} + \frac{A^*_{pqr}}{(s - \lambda^*_r)} \tag{5.26}$$

If the force used to excite the DOF (q) is an impulse, then the Laplace transform will be unity. While this concept is used to define the impulse response function, once the transfer function is formulated and the system is considered to be linear, this type of assumption can be made with complete generality.

$$F_q\,(s) \;=\; 1$$

Thus the system impulse response at DOF (p) is the inverse Laplace transform of Equation 5.19.

$$h_{pq}\,(t) \;=\; \mathbf{L}^{-1}\left\{\; X_p\,(s) \;\right\} \;=\; \mathbf{L}^{-1}\left\{\; \sum_{r\,=\,1}^{N} \frac{A_{pqr}}{(\,s \;-\; \lambda_r\,)} \;+\; \frac{A^{*}_{pqr}}{(\,s \;-\lambda^{*}_{r}\,)} \;\right\}$$

If $p = q$, this would be the driving point impulse response function. For simplicity, Equation 5.27 can be expanded for the two degree of freedom case (N = 2).

$$X_p\,(s) \;=\; \frac{A_{pp1}}{(\,s \;-\; \lambda_1\,)} \;+\; \frac{A^{*}_{pp1}}{(\,s \;-\; \lambda^{*}_{1}\,)} \;+\; \frac{A_{pp2}}{(\,s \;-\; \lambda_2\,)} \;+\; \frac{A^{*}_{pp2}}{(\,s \;-\; \lambda^{*}_{2}\,)} \qquad (5.28)$$

$$h_{pp}\,(t) \;=\; \mathbf{L}^{-1}\left\{\; X_p\,(s) \;\right\} \;=\; A_{pp1}\,e^{\lambda_1 t} \;+\; A^{*}_{pp1}\,e^{\lambda^{*}_{1} t} \;+\; A_{pp2}\,e^{\lambda_2 t} \;+\; A_{pp2}\,e^{\lambda^{*}_{2} t} \qquad (5.29)$$

Recall that:

$$\lambda_r \;=\; \sigma_r \;+\; j\,\omega_r$$

$$\lambda^{*}_{r} \;=\; \sigma_r \;-\; j\,\omega_r$$

Also, since the system is proportionally damped, the residues are all purely imaginary. Therefore, the following definition can be made arbitrarily. This definition will make certain trigonometric identities obvious in a later equation.

$$A_{pp1} \;=\; \frac{R_{pp1}}{2\;j}$$

$$A_{pp2} \;=\; \frac{R_{pp2}}{2\;j}$$

Using Euler's formula, Equation 5.22 then becomes:

$$h_{pp}(t) = \frac{R_{pp1}}{2j}\left(e^{(\sigma_1 + j\omega_1)t} - e^{(\sigma_1 - j\omega_1)t} \right) + \frac{R_{pp2}}{2j}\left(e^{(\sigma_2 + j\omega_2)t} - e^{(\sigma_2 - j\omega_2)t} \right)$$

$$h_{pp}(t) = R_{pp1}\, e^{\sigma_1 t}\, \frac{(e^{j\omega_1 t} - e^{j-\omega_1 t})}{2j} + R_{pp2}\, e^{\sigma_2 t}\, \frac{(e^{j\omega_2 t} - e^{-j\omega_2 t})}{2j}$$

$$h_{pp}(t) = R_{pp1}\, e^{\sigma_1 t}\, \sin(\omega_1 t) + R_{pp2}\, e^{\sigma_2 t}\, \sin(\omega_2 t)$$

Finally:

$$h_{pp}(t) = R_{pp1}\, e^{\sigma_1 t}\, \sin(\omega_1 t) + R_{pp2}\, e^{\sigma_2 t}\, \sin(\omega_2 t) \qquad (5.30)$$

Comparing this to the single degree of freedom case done previously, note that the impulse response function is nothing more than the summation of two single degree of freedom responses. Note also that the amplitude of the impulse response function is directly related to the residues for the two modal vectors. This means that the impulse response function amplitude is directly related to the modal vectors of the two modes, since the residues have been shown to be directly proportional to the modal vectors.

In summary, a transfer function, and therefore a frequency response function, can be expressed as a sum of single degree of freedom systems (Equation 4.24). A typical frequency response function is illustrated in Figure (5-4), in terms of its real and imaginary parts. This frequency response function represents the response of the system at degree of freedom 1 due to a force applied to the system at degree of freedom 2 $H_{12}(\omega)$. Figure (5-5) is the equivalent impulse response function of Figure (5-4). Likewise, this impulse response function represents the response of the system at degree of freedom 1 due to a unit impulse applied to the system at degree of freedom 2 $h_{12}(t)$. Notice that the impulse response function starts out at zero for t = 0 as it should for a system with real modes of vibration. Since the transfer function is the sum of single degree of freedom systems, the frequency response functions for each of the single degree of freedom systems can be plotted independently as is illustrated in Figures (5-6) and (5-8).

Note that by adding together Figures (5-6) and (5-8), the same plot as in Figure (5-4) would result. In Figures (5-6) and (5-8) the imaginary parts peak where the real parts cross zero, where as, in Figure (5-4) this is not exactly true. Similarly, the impulse response functions for each single degree of freedom system can be plotted separately as in Figures (5-7) and (5-9). The sum of these two figures would be the same as Figure (5-5). Therefore, the impulse response of a two degree of freedom system is just the sum of two damped sinusoids.

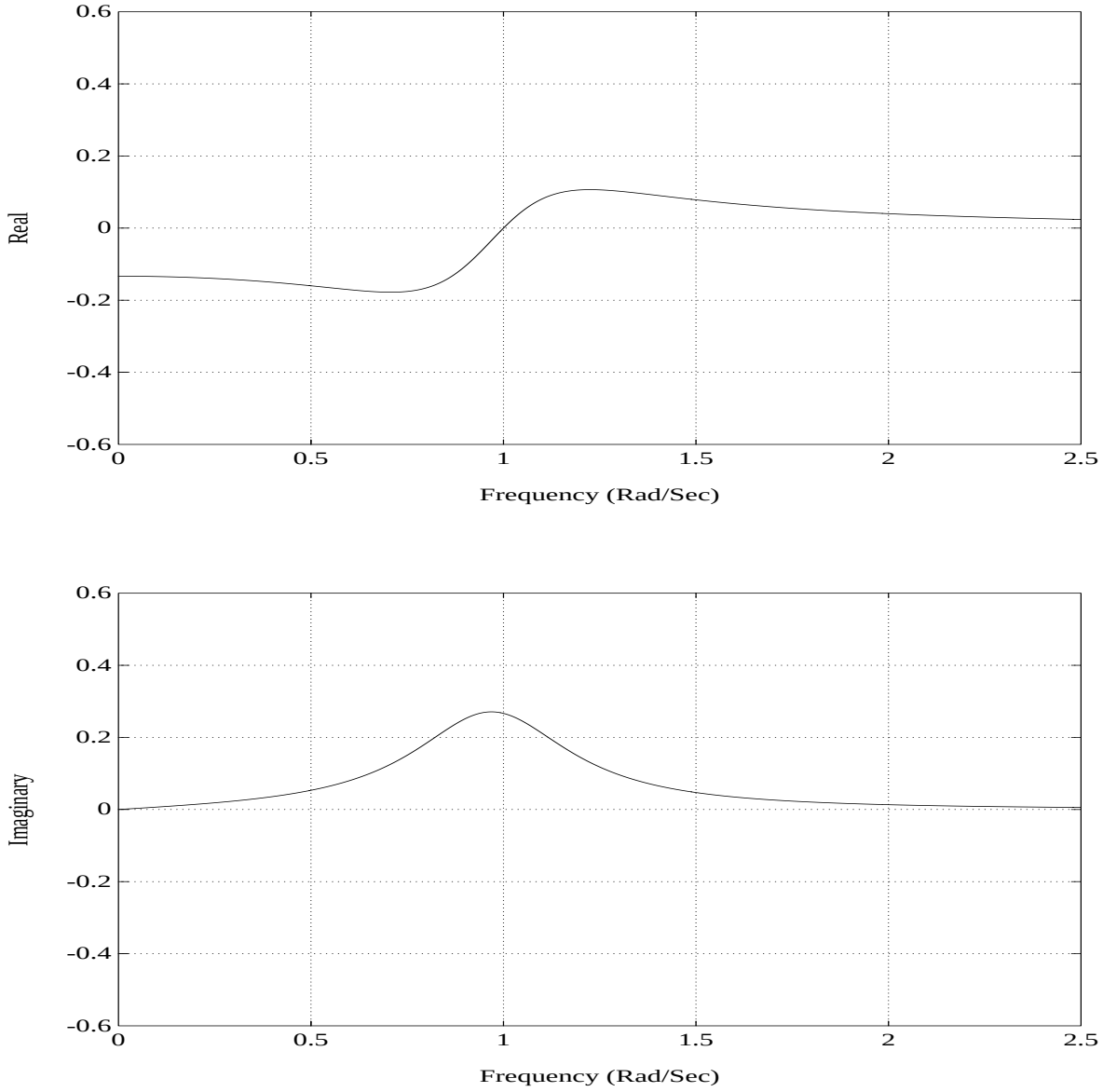

Figure 5-4. FRF, 2 DOF System, Proportionally Damped

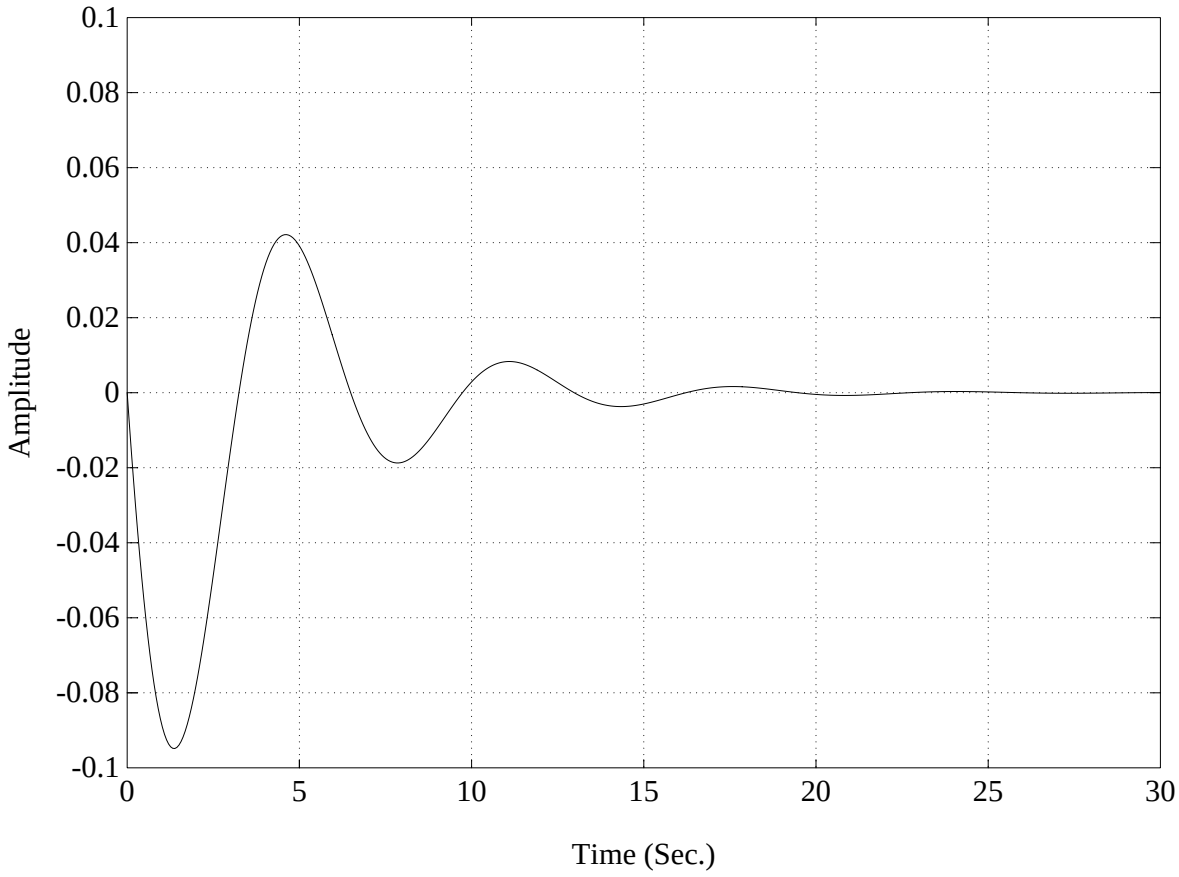

Figure 5-5. IRF, 2 DOF System, Proportionally Damped

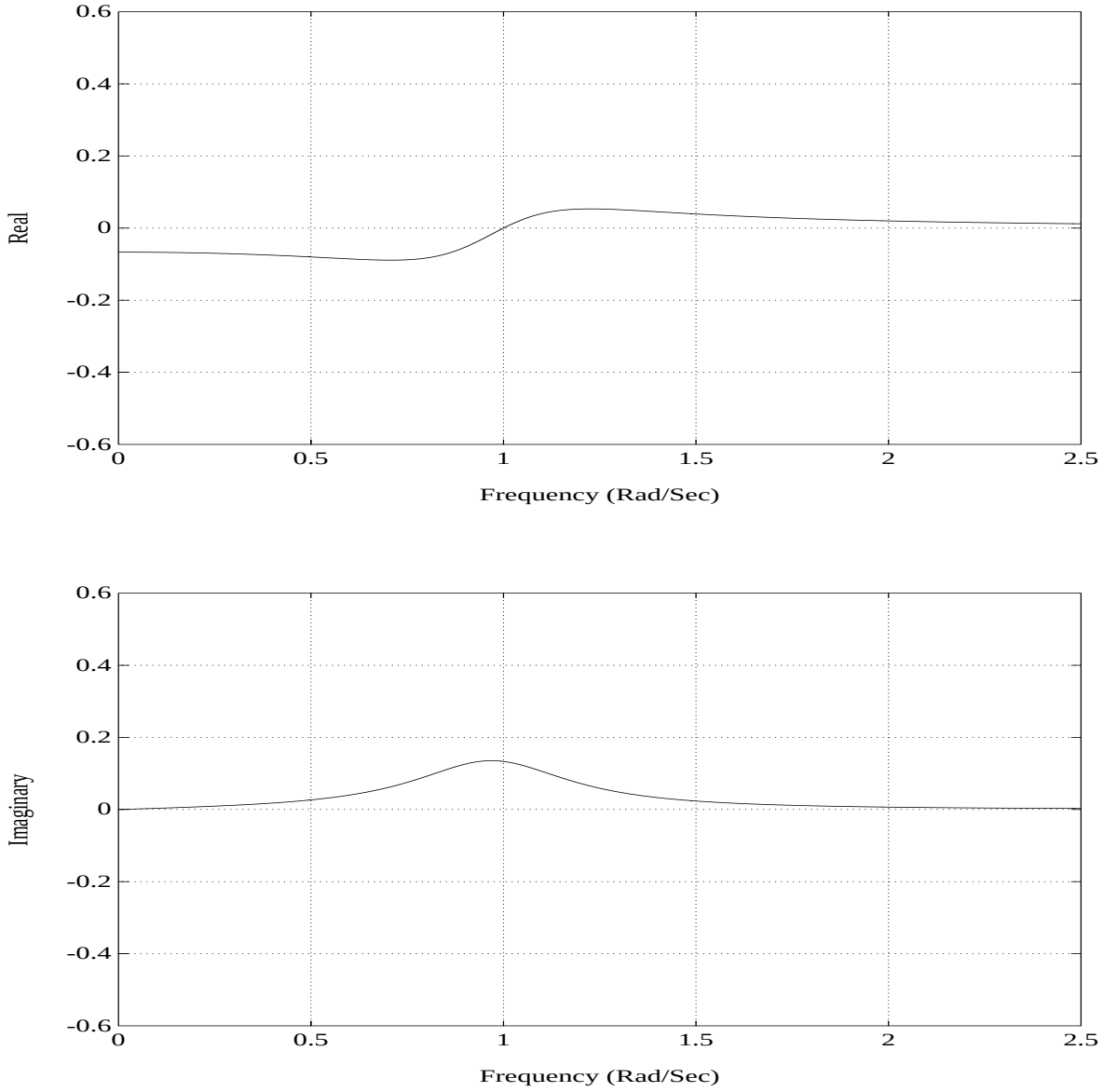

Figure 5-6. FRF, 2 DOF System, Proportionally Damped, First Mode

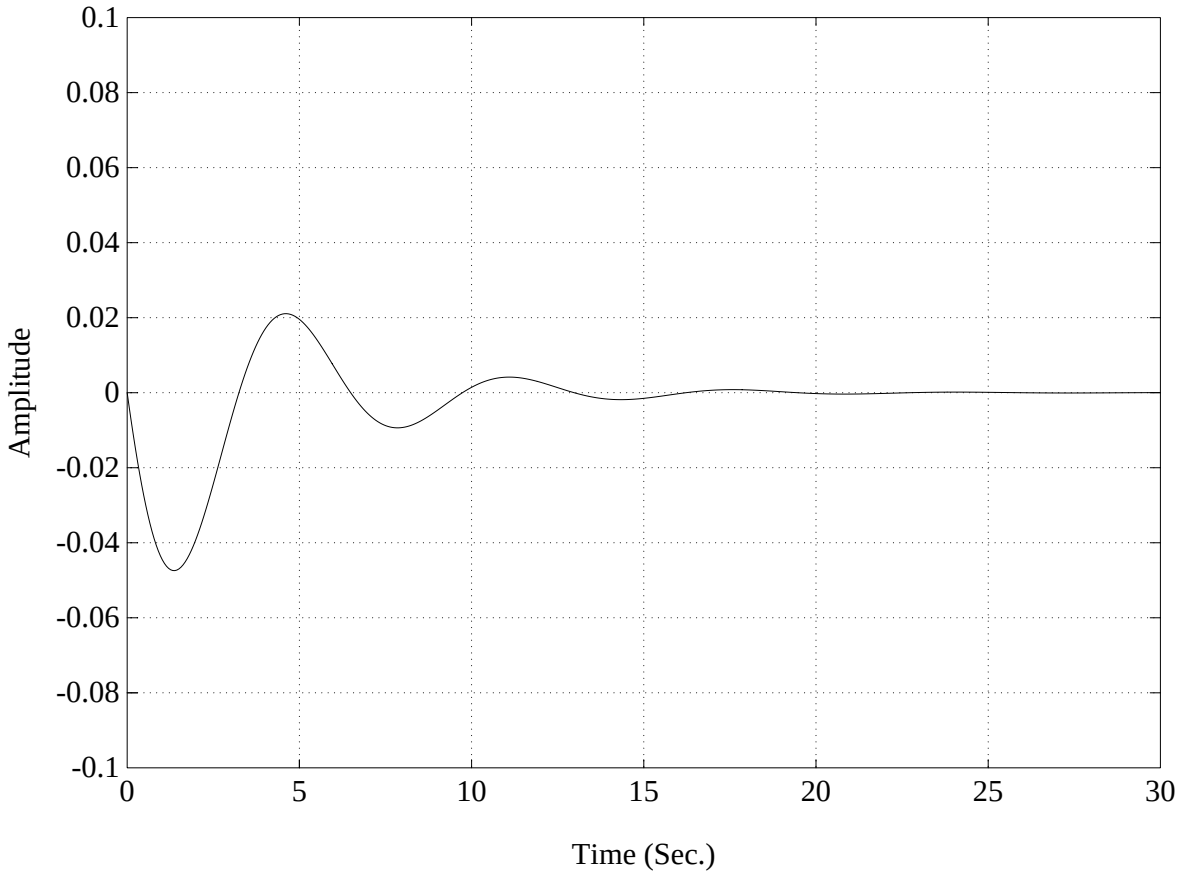

Figure 5-7. IRF, 2 DOF System, Proportionally Damped, First Mode

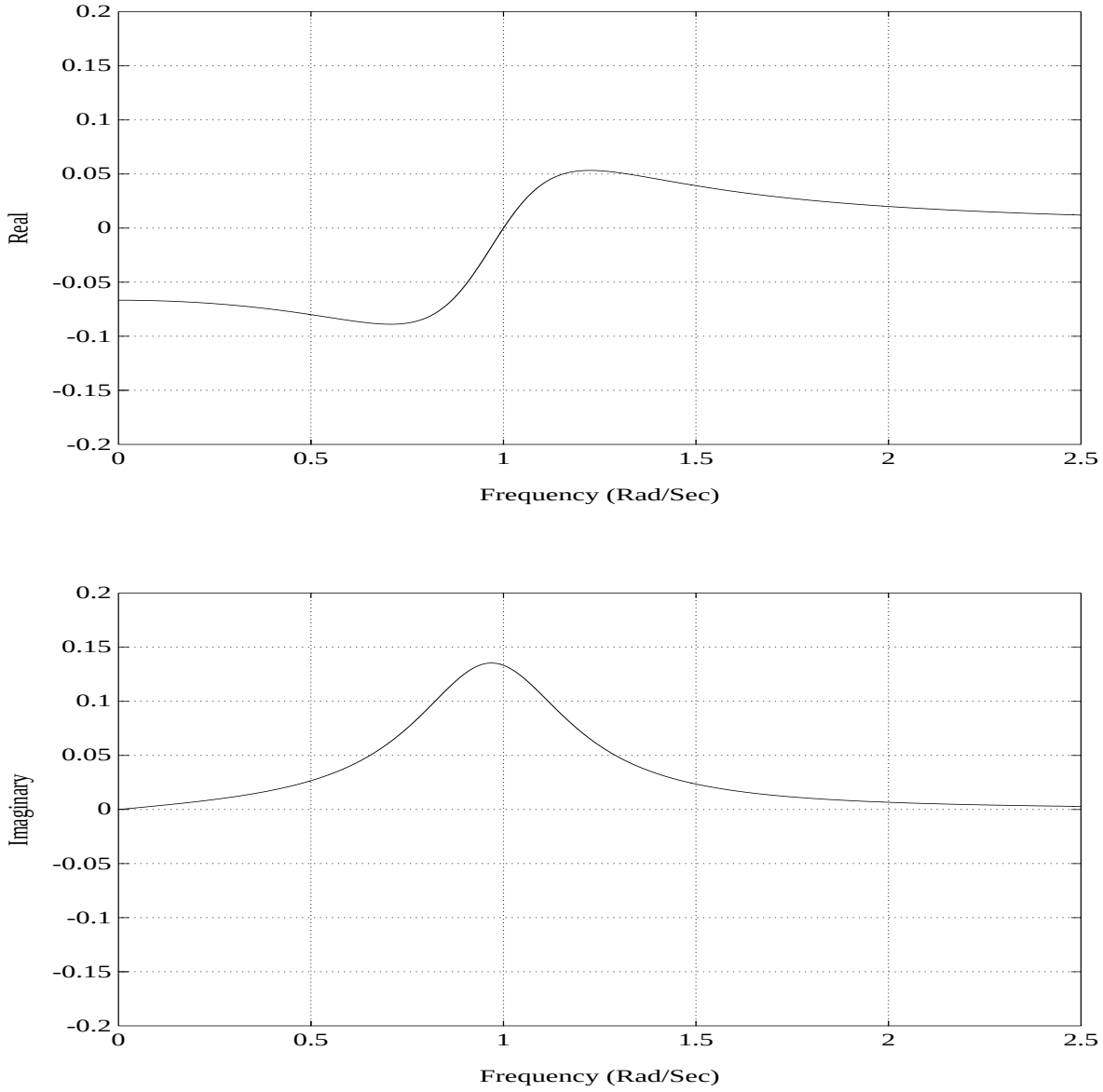

Figure 5-8. FRF, 2 DOF System, Proportionally Damped, Second Mode

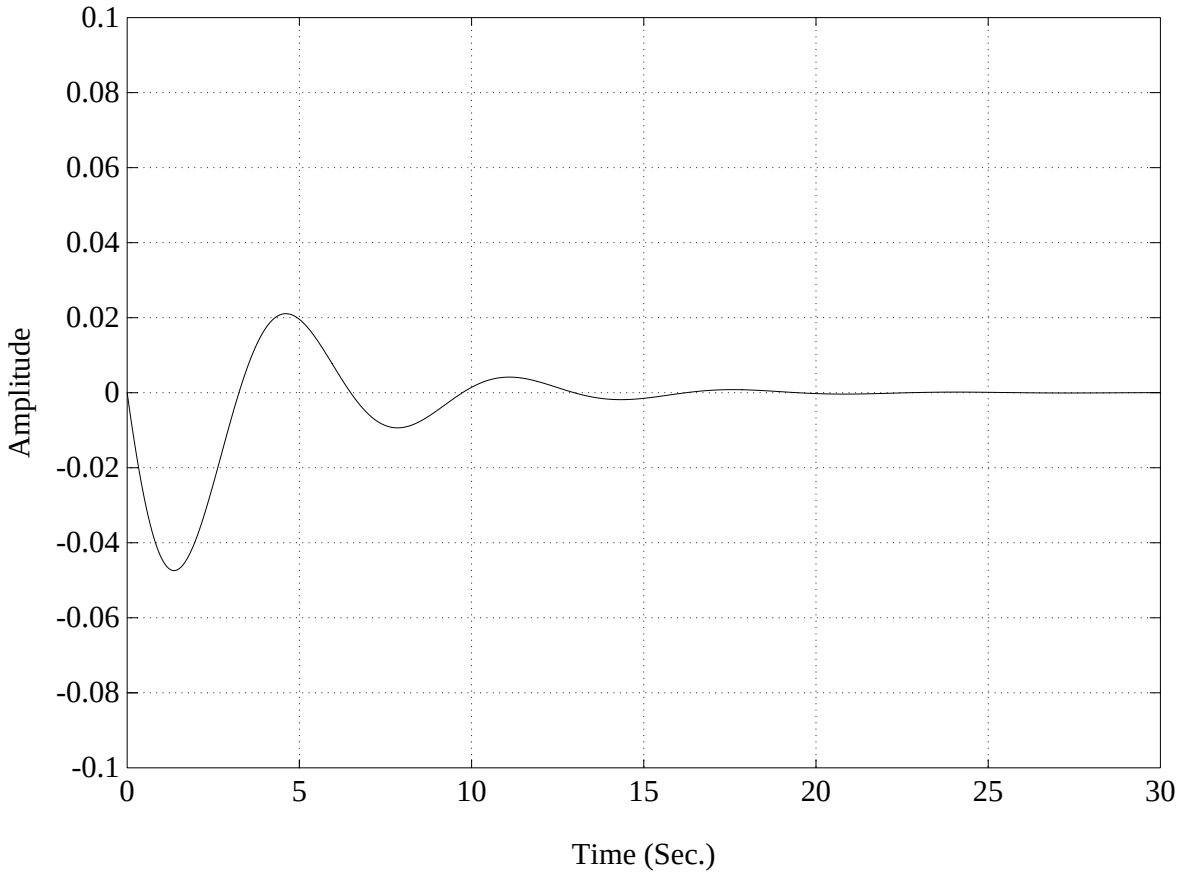

Figure 5-9. IRF, 2 DOF System, Proportionally Damped, Second Mode

5.4 Non-Proportional Damping Example

The same two degree of freedom system, in terms of mass and stiffness, as before will now be used to explain the effect of non-proportional damping.

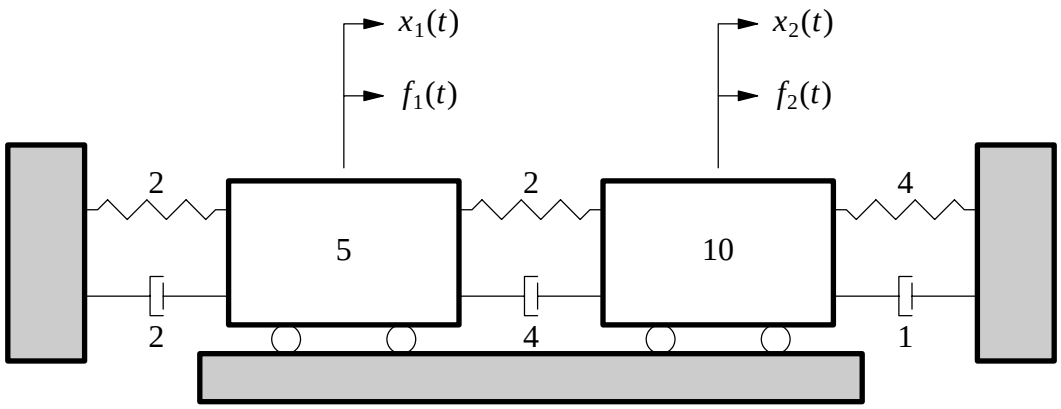

Figure 5-10. Two Degree of Freedom System

The damping coefficients have been chosen such that they are not proportional to the mass and/or stiffness. Using Equation 5.22, the transfer function matrix of the resulting system is:

$$[\ H(s)\ ] = \frac{\begin{bmatrix} 10s^2 + 5s + 6 & 4s + 2 \\ 4s + 2 & 5s^2 + 6s + 4 \end{bmatrix}}{50[s^4 + (17/10)\ s^3 + (42/25)\ s^2 + (4/5)\ s + 2/5]} \tag{5.31}$$

The poles of the transfer function are the roots of the characteristic equation:

$$s^4 + \frac{17}{10}\ s^3 + \frac{42}{25}\ s^2 + \frac{4}{5}\ s + \frac{2}{5} = 0 \tag{5.32}$$

The roots are:

$$\lambda_1 = -0.095363 + j\ 0.629494 \ \ (\ rad/sec\ )$$
$$\lambda_1^* = -0.095363 - j\ 0.629494 \ \ (\ rad/sec\ )$$

$$\lambda_2 = -0.754635 + j\ 0.645996 \ \ (\ rad/sec\ )$$
$$\lambda_2^* = -0.754635 - j\ 0.645996 \ \ (\ rad/sec\ )$$

These poles are the damping and frequency parameters for the system's two modes of vibration. Recall: $\lambda_r = \sigma_r + j\ \omega_r$.

As before, the modal vectors of the system can be determined by evaluating one row or column of the transfer function matrix in terms of partial fractions. In a previous example, the first column of the transfer function matrix was utilized in order to determine the modal vectors. This time, to illustrate the pervasiveness of the modal vectors with respect to all elements of [$H\ (s)$], the last row of the transfer function matrix will be utilized. That is:

$$H_{21}\ (\mathrm{s}) \ \text{ and } \ H_{22}\ (\mathrm{s})$$

H_{21} (s) can be measured by exciting the system at mass 1 and measuring the response at mass 2. Similarly, H_{22} (s) can be measured by exciting the system at mass 2 and measuring the response at mass 2. H_{22}(s) and H_{21} (s) can now be expanded in terms of partial fractions.

$$H_{22}\ (s) = \frac{5s^2 + 6s + 4}{50\ (s - \lambda_1)\ (s - \lambda_1^*)\ (s - \lambda_2)\ (s - \lambda_2^*)}$$

After some work:

$$H_{22}\ (s) = \frac{0.003707 - j\ 0.058760}{(s - \lambda_1)} + \frac{0.003707 + j\ 0.058760}{(s - \lambda_1^*)}$$

$$+ \frac{-0.003707 - j\ 0.016358}{(s - \lambda_2)} + \frac{-0.003702 + j\ 0.016358}{(s - \lambda_2^*)}$$

Also:

$$H_{21}(s) = \frac{4s^2 + 2}{50\,(s-\lambda_1)\,(s-\lambda_1^*)\,(s-\lambda_2)\,(s-\lambda_2^*)}$$

After some more work:

$$H_{21}(s) = \frac{-0.003473 - j\,0.050100}{(s-\lambda_1)} + \frac{-0.003473 + j\,0.050100}{(s-\lambda_1^*)}$$

$$+ \frac{0.003473 + j\,0.045270}{(s-\lambda_2)} + \frac{0.003473 - j\,0.045280}{(s-\lambda_2^*)}$$

Using the residues as the modal vectors directly (without any normalization or scaling), the following vectors result:

Mode 1: $\lambda_1 = -0.095363 + j\,0.629494$

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_1 = \left\{ \begin{array}{l} -0.003473 - j\,0.050100 \\ +0.003707 - j\,0.058760 \end{array} \right\}$$

Mode 2: $\lambda_2 = -0.754635 + j\,0.645996$

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_2 = \left\{ \begin{array}{l} +0.003473 + j\,0.045270 \\ -0.003707 - j\,0.016358 \end{array} \right\}$$

The modal vectors are obviously no longer purely real. In both modal vectors, regardless of how the modal vectors are normalized, the description of the modal vector will now require a vector of complex numbers. This will in general be true for any system with non-proportional damping.

The above modal vectors can be converted to amplitude and phase.

Thus, for modal vector one:

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_1 = \left\{ \begin{array}{ll} 0.050220, & -93.96^\circ \\ 0.058877, & -86.39^\circ \end{array} \right\}_1$$

For modal vector two:

$$\left\{ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\}_2 = \left\{ \begin{array}{ll} 0.045403, & +85.61^\circ \\ 0.016765, & -102.77^\circ \end{array} \right\}_2$$

For mode one, as in the previous examples, the two masses are moving approximately in phase. Likewise, for mode two, as in the previous examples, the two masses are moving approximately out of phase. Thus, the non-proportionally damped system, which resulted in complex mode shapes, has phase terms which are not generally 0° or 180°.

Another way of comparing the differences between real modal vectors and complex modal vectors is as follows: A real modal vector represents a mode of vibration where all the points on the structure pass through their equilibrium positions at the same time. A complex modal vector represents a mode of vibration where the points tested do not pass through their equilibrium positions at the same time. In other words, a real mode appears as a standing wave and a complex mode appears as a traveling wave with respect to the structure. The nodal point or line for real modes will be stationary with respect to the structure. Conversely, the nodal point or line for a complex mode will be non-stationary with respect to the structure.

Many structures tested will have nearly real mode shapes, that is, the phase term will be very close to being 0° or 180°. Structures in general can have both real and complex mode shapes. In the previous example, the first modal vector is within ± 4° of being totally real, whereas the second modal vector is within ± 13° of being real. Thus, the second modal vector is more complex than the first modal vector.

5.4.1 Impulse Response Function Example

The impulse response function is computed in a similar fashion as for the proportionally damped system. From Equation 5.27, the basic definition of the impulse response function is:

$$H_{pq}(t) = \mathbf{L}^{-1} \left\{ \sum_{r=1}^{N} \frac{A_{pqr}}{(s-\lambda_r)} + \frac{A_{pqr}^*}{(s-\lambda_r^*)} \right\} \tag{5.33}$$

The only difference now is that the residues A_{pqr} are no longer purely imaginary but are, in general, complex. For a two-degree of freedom system, the impulse response function becomes:

$$h_{pq}(t) = A_{pq1}\, e^{\lambda_1 t} + A^*_{pq1}\, e^{\lambda^*_1 t} + A_{pq2}\, e^{\lambda_2 t} + A^*_{pq2}\, e^{\lambda^*_2 t} \tag{5.34}$$

The only difference between the impulse response function of a system with complex modal vectors (Equation 5.34) and a system with real modal vectors (Equation 5.29 and 5.30) is that the individual contributions from each mode of vibration to the total impulse response function do not begin with a value of zero response at time zero. Note that the sum of these individual contributions must total zero at time zero for the system to be causal. Again note that the residues, which are proportional to the product of the modal coefficients of the input and response degrees of freedom, appear as the amplitude of the damped sinusoids.

A typical frequency response function is once again illustrated in Figure (5-11), in terms of its real and imaginary parts. This frequency response function represents the response of the system at degree of freedom 1 due to a force applied to the system at degree of freedom 2 $H_{12}(\omega)$. Figure (5-12) is the equivalent impulse response function of Figure (5-11). Likewise, this impulse response function represents the response of the system at degree of freedom 1 due to a unit impulse applied to the system at degree of freedom 2 $h_{12}(t)$. Figure (5-11) depicts a frequency response function of a two degree of freedom system that has complex modal vectors. The only difference between this plot and that of Figure (5-4) is that the residues in this case have a phase angle of other than 0° or 180°. The impulse response function of this system is shown in Figure (5-11). Notice that the impulse response function does start at zero for t=0. This is always the case for any causal system with or without complex modal vectors. Notice, though, that the impulse response function contribution from each individual mode does not start at zero for this case of complex modal vectors. Nevertheless, for a causal system, the sum of the residues, and therefore the combination of magnitudes and phase angles in Equation 5.27, will always yield an impulse response function that is zero at time zero. (The sum of the residues will be equal to zero for any system where the order of the numerator polynomial is two or more less than the order of the denominator polynomial.) As before, the frequency response function can be represented as (Figure (5-11)) is the sum of two single degree of freedom systems (Figures (5-13) and (5-15)). Another indication of a complex modal vector is the asymmetric characteristic of the imaginary part of the frequency response function for such a simple system. In Figure (5-6) and (5-8), note that the imaginary parts are symmetrical, whereas in Figures

(5-13) and (5-15) the imaginary parts are not symmetrical. The degree to which they are not symmetrical is an indication of how complex that particular component of the modal vector is. Figures (5-14) and (5-16) are the impulse response functions of Figures (5-13) and (5-15).

The modal vectors and modal frequencies of the undamped, proportionally damped, and non-proportionally damped system can now be compared for the simple two degree of freedom system used in all the examples. Remember that the stiffnesses and masses are the same throughout the examples. Therefore, the modal vectors and modal frequencies can only be different as a result of the damping elements. For comparison purposes, the unscaled residues from the standard form of the transfer function will be used as the modal vectors for each case.

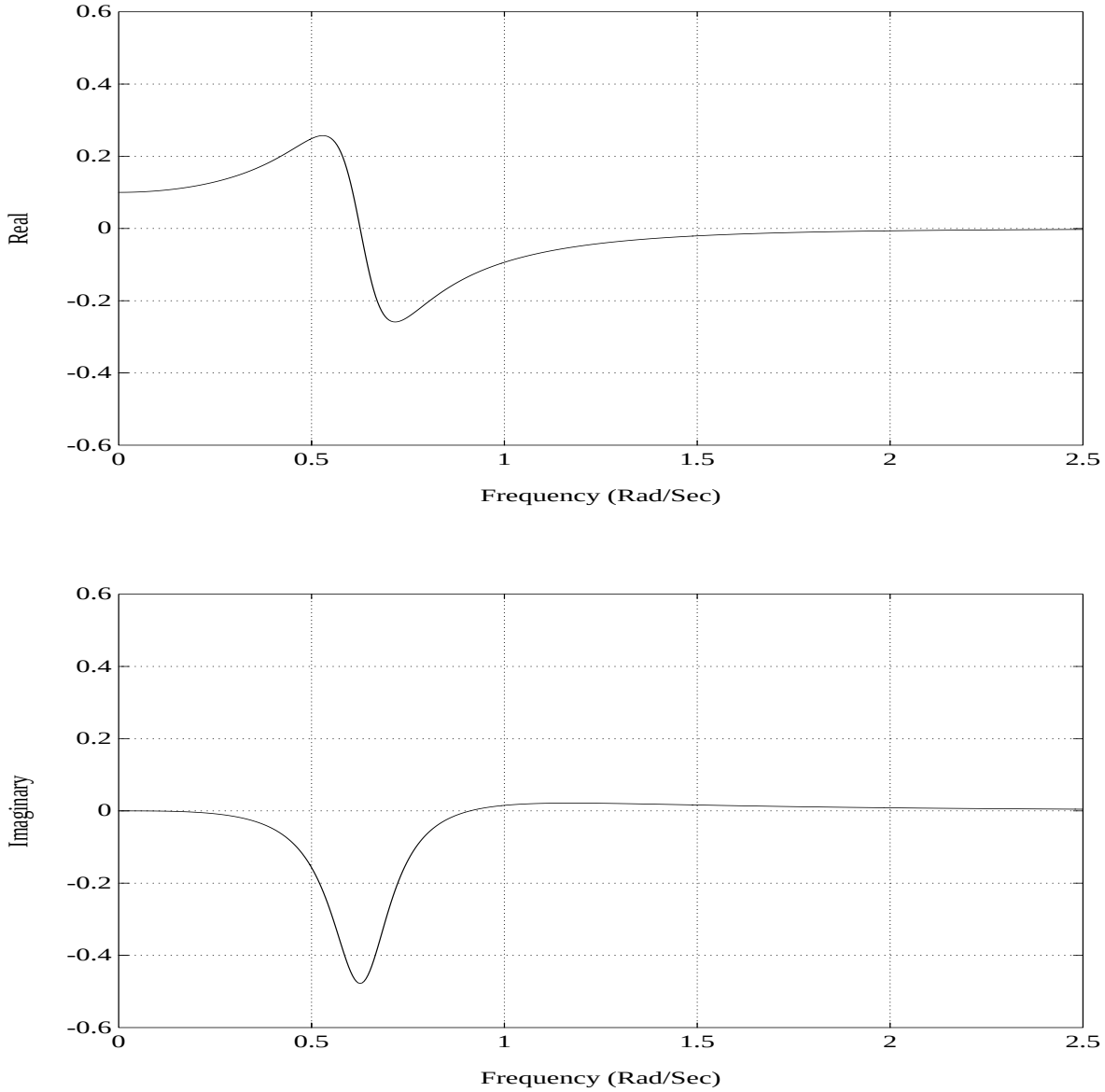

Figure 5-11. FRF, 2 DOF System, Non-Proportionally Damped

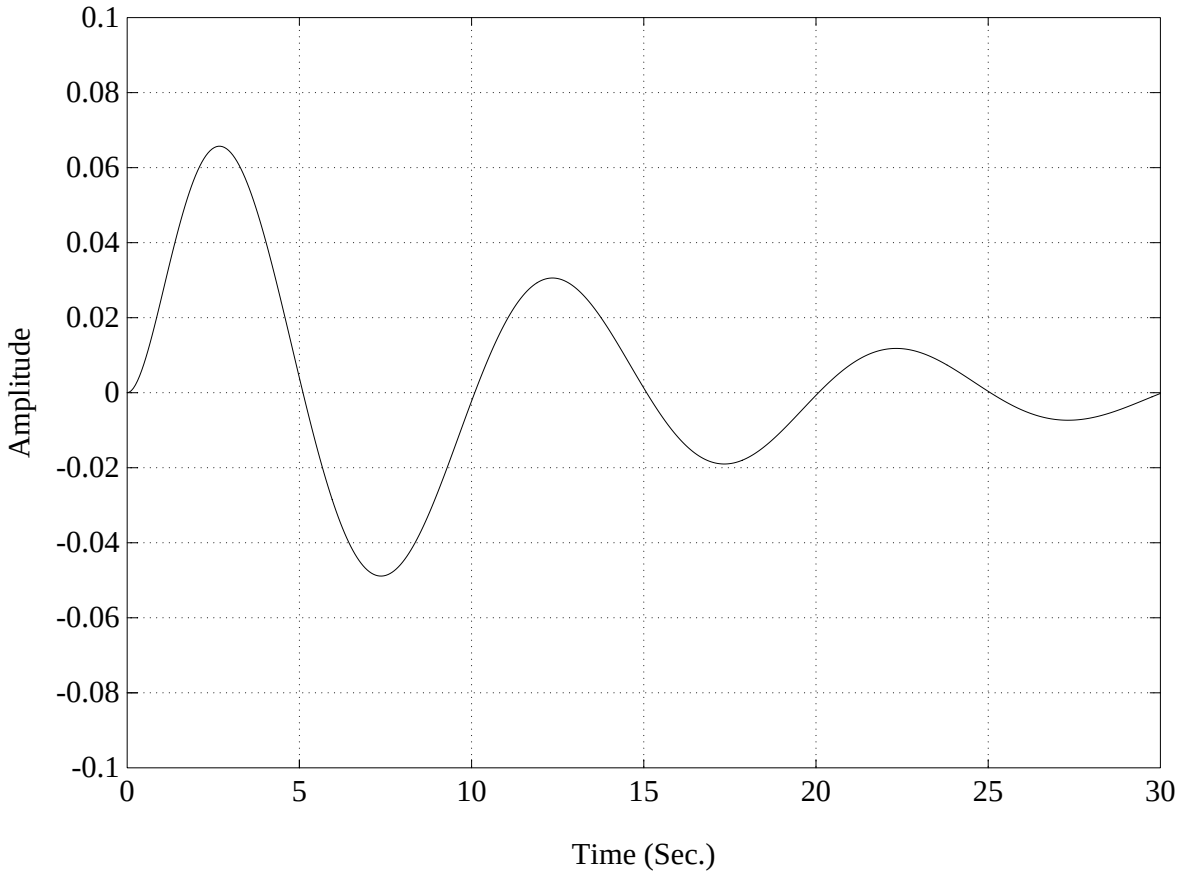

Figure 5-12. IRF, 2 DOF System, Non-Proportionally Damped

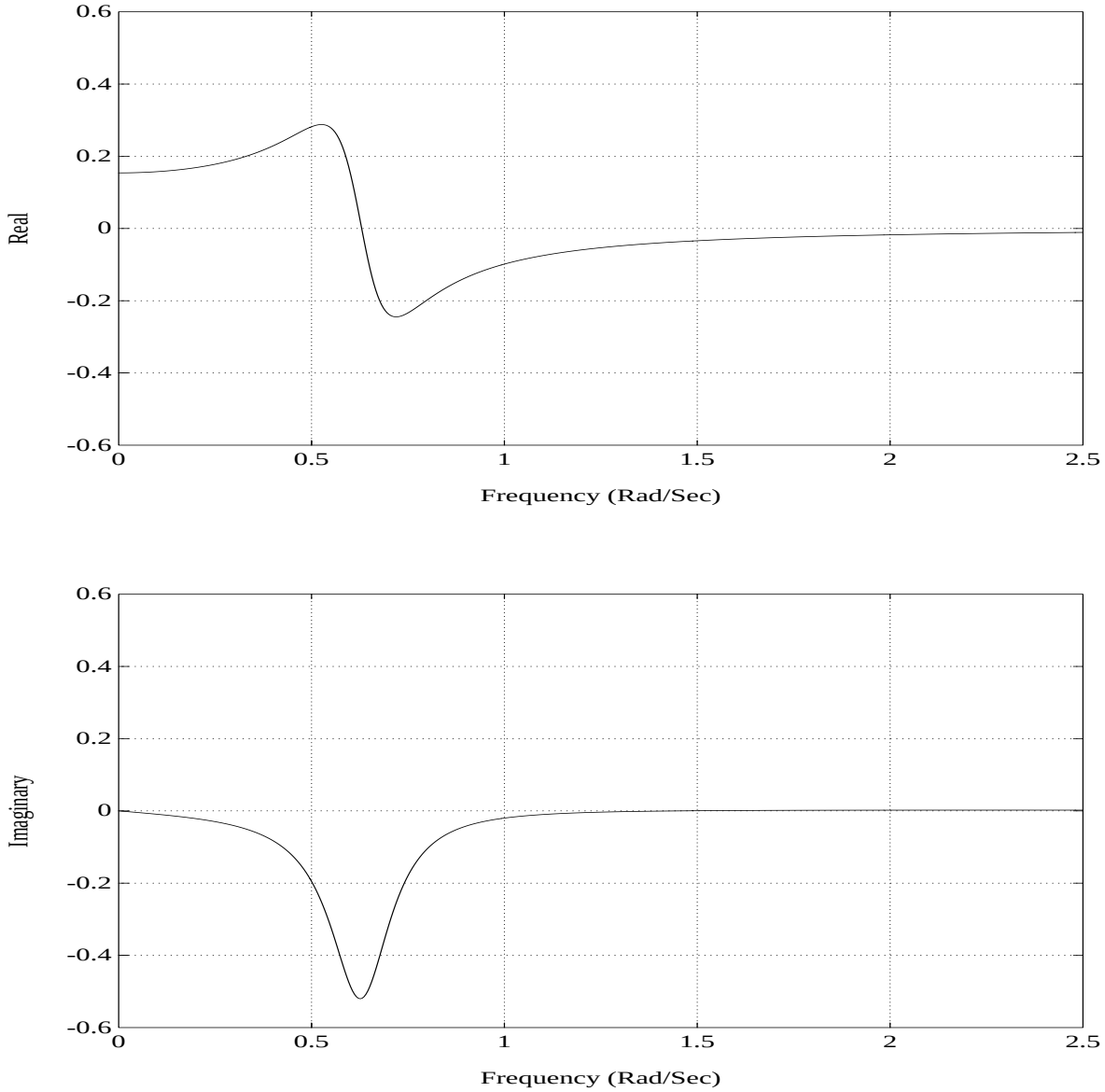

Figure 5-13. FRF, 2 DOF System, Non-Proportionally Damped, First Mode

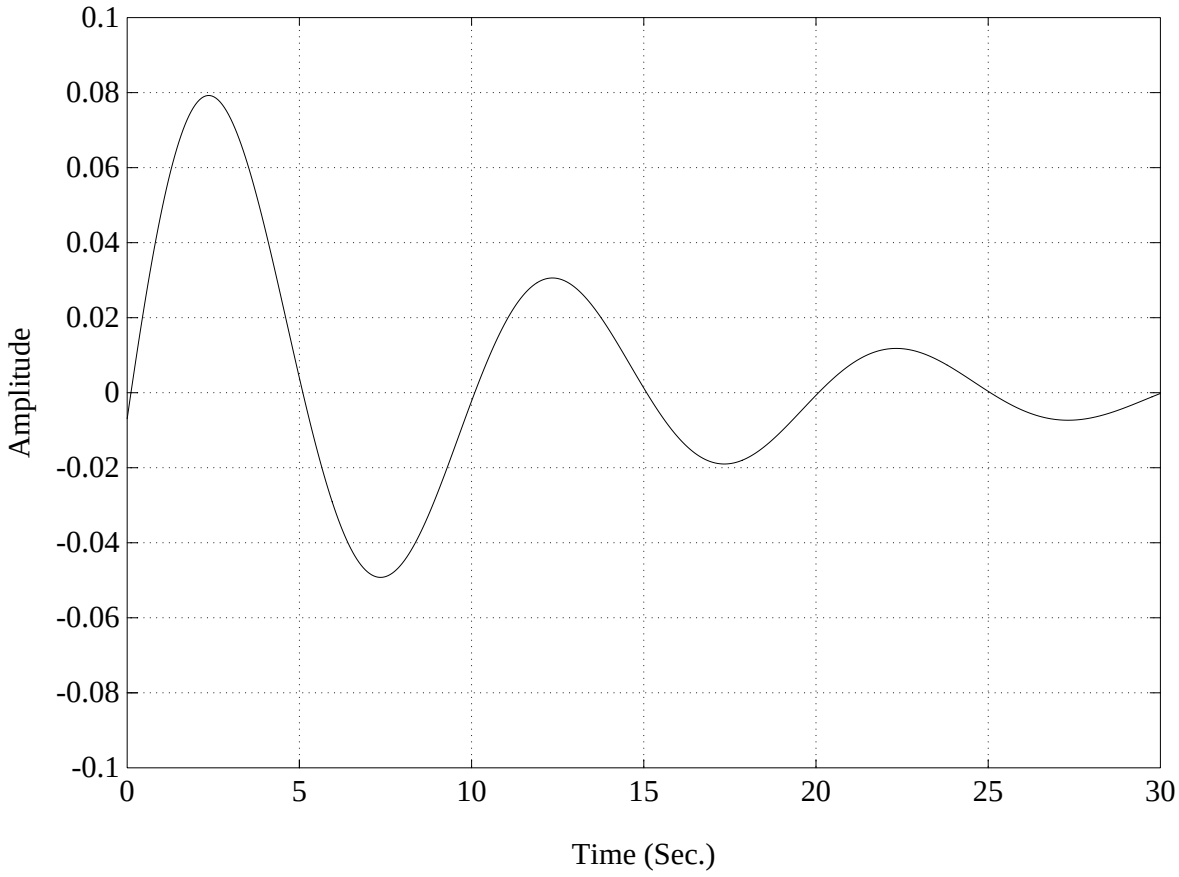

Figure 5-14. IRF, 2 DOF System, Non-Proportionally Damped, First Mode

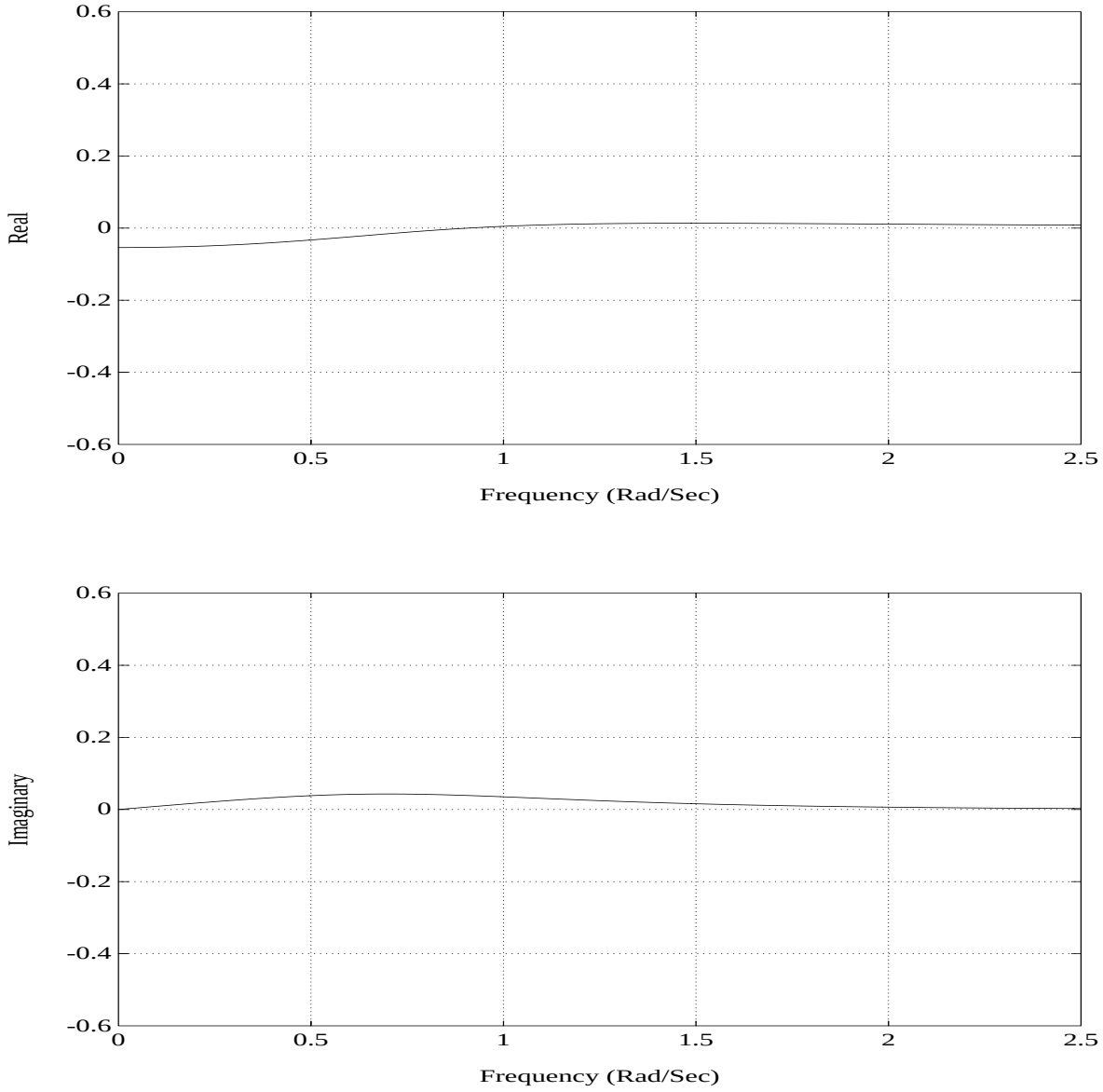

Figure 5-15. FRF, 2 DOF System, Non-Proportionally Damped, Second Mode

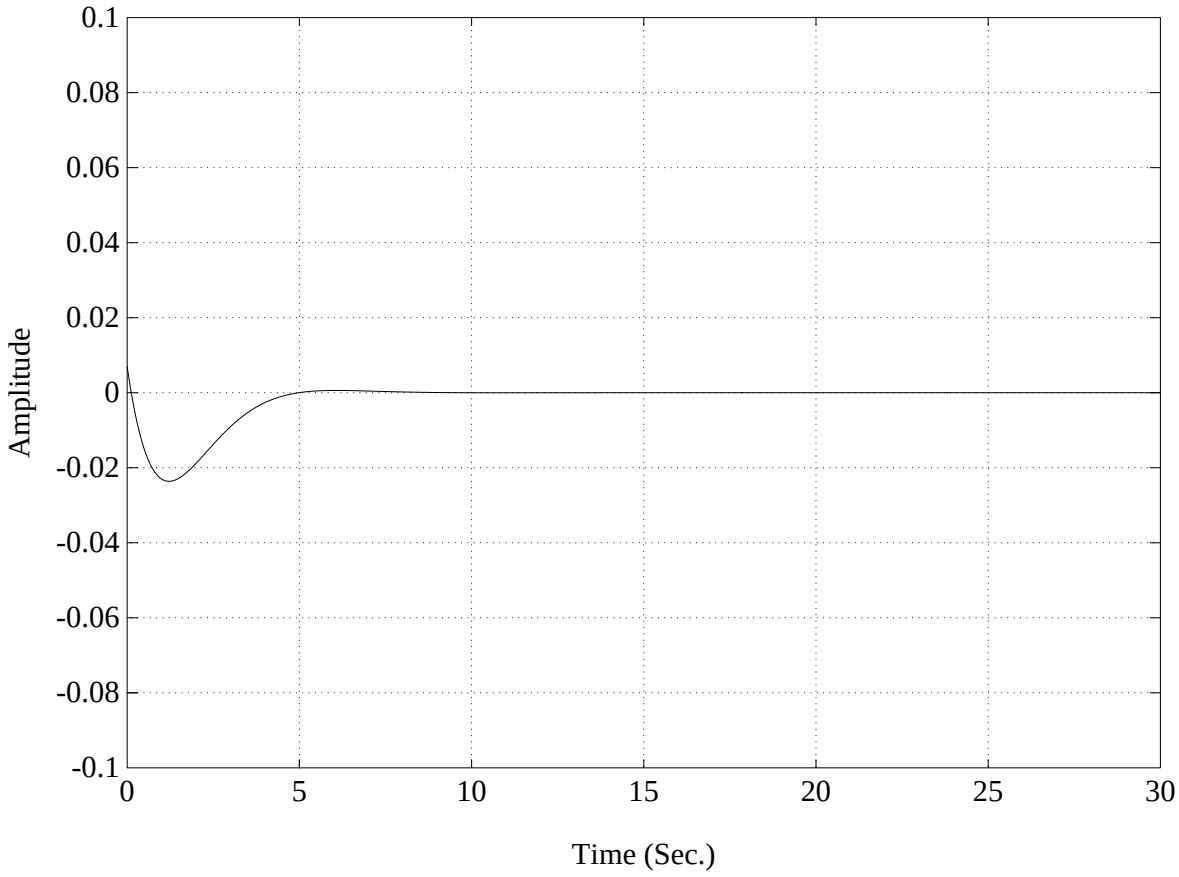

Figure 5-16. IRF, 2 DOF System, Non-Proportionally Damped, Second Mode

5.5 Summary of Modal Parameters (Undamped and Damped Systems)

Recalling the modal vectors from the previous examples:

Mode 1 Mode 2

Undamped Case:

$$\lambda_1 = j\sqrt{2/5} \qquad \lambda_2 = j$$

$$\psi_1 = \begin{Bmatrix} -j\,\dfrac{\sqrt{2/5}}{12} \\ -j\,\dfrac{\sqrt{2/5}}{12} \end{Bmatrix} \qquad \psi_2 = \begin{Bmatrix} -j\,\dfrac{1}{15} \\ j\,\dfrac{1}{30} \end{Bmatrix}$$

Proportionally Damped Case:

$$\lambda_1 = \frac{1}{10} + j\,(\frac{\sqrt{39}}{10}) \qquad \lambda_2 = -\frac{1}{4} + j\,(\frac{\sqrt{15}}{4})$$

$$\psi_1 = \begin{Bmatrix} -j\,\dfrac{\sqrt{39}}{117} \\ -j\,\dfrac{\sqrt{39}}{117} \end{Bmatrix} \qquad \psi_2 = \begin{Bmatrix} -j\,\dfrac{4\sqrt{15}}{225} \\ j\,\dfrac{2\sqrt{15}}{225} \end{Bmatrix}$$

Non-proportionally Damped Case:

$$\lambda_1 = -0.095363 + j\,0.629494 \qquad \lambda_2 = -0.754635 + j\,0.645996$$

$$\psi_1 = \begin{Bmatrix} -0.008569 - j\,0.041970 \\ -0.003473 - j\,0.050100 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} 0.008569 - j\,0.122700 \\ 0.003473 + j\,0.045270 \end{Bmatrix}$$

Note that for the non-proportional damped case, the residues given in the table are for the same frequency response functions used in the undamped and proportionally damped cases (the first column of the frequency response function matrix). This required the computation of the residues for $H_{11}(s)$.

Recall that a pole (λ_r) may be written as:

$$\lambda_r = \sigma_r + j\ \omega_r = (-\ \zeta_r + j\ \sqrt{1-\zeta_r^2})\ \Omega_n \tag{5.35}$$

where:

- ζ_r = damping ratio
- Ω_r = undamped natural frequency
- σ_r = damping factor
- ω_r = damped natural frequency

From Equation 5.35, the following relationships can also be noted:

$$\zeta_r = \frac{-\sigma_r}{\sqrt{\omega_r^2 + \sigma_r^2}}$$

$$\Omega_r = \frac{-\sigma_r}{\zeta_r}$$

The first modal vector for the 3 different cases can now be compared. Note that the modal vectors have been converted to amplitude and phase.

For modal vector one (λ_1):

Undamped	Prop. Damped	Non-prop. Damped
$\left\{ \begin{array}{l} 0.052705, -90° \\ 0.052705, -90° \end{array} \right\}_1$	$\left\{ \begin{array}{l} 0.053376, -90° \\ 0.053376, -90° \end{array} \right\}_1$	$\left\{ \begin{array}{l} 0.042836, -101.53° \\ 0.050220, -93.96° \end{array} \right\}_1$
$\Omega_1 = 0.632456$	$\Omega_1 = 0.632456$	$\Omega_1 = 0.63667$
$\zeta_1 = 0\%$	$\zeta_1 = 15.8114\%$	$\zeta_1 = 14.9782\%$

For modal vector two (λ_2):

Undamped	Prop. Damped	Non-prop. Damped
$\left\{ \begin{array}{l} 0.066667, -90° \\ 0.033333, +90° \end{array} \right\}_2$	$\left\{ \begin{array}{l} 0.068853, -90° \\ 0.034427, +90° \end{array} \right\}_2$	$\left\{ \begin{array}{l} 0.122999, -86.00° \\ 0.045403, +85.61° \end{array} \right\}_2$
$\Omega_2 = 1.0$	$\Omega_2 = 1.0$	$\Omega_2 = 0.99337$
$\zeta_2 = 0\%$	$\zeta_2 = 25\%$	$\zeta_2 = 75.9671\%$

In conclusion, note that the addition of damping (proportional and non-proportional) has not affected the undamped natural frequencies for either modal vector. This will always be the case. Furthermore, the overall behavior of the modal vectors remained the same. That is, in the first mode the 2 masses are moving in the same direction and in the second mode they are moving in opposite directions. The major difference arises in the phase of the mode shapes. Note that for the non-proportionally damped system, the phase of the components is other than 0° or 180°. These types of modes are again referred to as complex modes. How much the phase differs from 0° or 180° is directly related to the systems mass, damping, and stiffness distributions. Note that as the phase approaches 0° or 180° for a particular mode, the modal vector can be approximated as a real or normal mode by setting the phase equal to 0° or 180°.

6. FREQUENCY RESPONSE FUNCTION SYNTHESIS

In this section, the approach for constructing the entire frequency response function matrix [$H(\omega)$] will be explained. The frequency response function matrix [$H(\omega)$], which is made up of $N_0 \times N_i$ frequency response functions, can be synthesized by using the data from N_o frequency response functions where N_o is the number of measured response degrees of freedom (physical response points times number of direction(s) at each physical point) on the structure. For the following discussion, N_o is assumed to be larger than N_i and N_o is assumed to include the N_i measurement degrees of freedom. For the case of fixed response degrees of freedom with a large number of applied inputs (impact testing, for example), N_i will be much larger than N_o but since the [$H(\omega)$] is reciprocal, the same assumption can be made concerning N_i. These are reasonable assumptions for all testing situations. There are two restrictions that apply to being able to synthesize the complete matrix accurately from only N_o or N_i frequency response functions. First of all, the N_o or N_i frequency response functions must consist of either a complete row or complete column of frequency response function measurements from the frequency response function matrix. Secondly, the N_o or N_i frequency response functions must contain non-zero residue information for every modal vector present in the structure within the frequency range of interest. This means that if the modal vector is entirely zero (due to excitation at the node of a modal vector) proper frequency response function synthesis will not be possible.

Recall the general form of the frequency response function matrix [$H(\omega)$] (Equation 4.25) in terms of partial fractions.

$$[\ H(\ \omega\ )\ ] = \begin{bmatrix} H_{11}(\ \omega\ ) & H_{12}(\ \omega\ ) & . & . & H_{1m}(\ \omega\ ) \\ H_{21}(\ \omega\ ) & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ H_{m1}(\ \omega\ ) & . & . & . & H_{mm}(\ \omega\ ) \end{bmatrix} \tag{6.1}$$

Assuming that only the k-th column of [$H(\omega)$] has been measured by way of frequency response functions, the k-th column can be represented as follows:

$$\begin{Bmatrix} H_{1k}(\omega) \\ H_{2k}(\omega) \\ . \\ . \\ H_{ik}(\omega) \\ . \\ . \\ H_{mk}(\omega) \end{Bmatrix} = \sum_{r=1}^{N} \frac{\{A_r\}_k}{(j\,\omega - \lambda_r)} + \frac{\{A_r^*\}_k}{(j\,\omega - \lambda_r^*)} \tag{6.2}$$

where:

- m = the number of measurements (N_o or N_i).

Equation 6.2 is the mathematical description of the N_o frequency response function measurements that were obtained during the test of the structure. For a particular mode of vibration, the N modal frequencies and the N residues for each of the N_o frequency response functions can be determined through the use of a modal parameter estimation algorithm.

Once these modal parameters are known, frequency response functions from column k or any other column in the frequency response function can be synthesized using a partial fraction model with the correct residues for the particular input and output degrees of freedom desired. The synthesis is formulated on a frequency by frequency basis for each mode - the response for each mode is summed together to get the total frequency response function as is shown in the following figures:

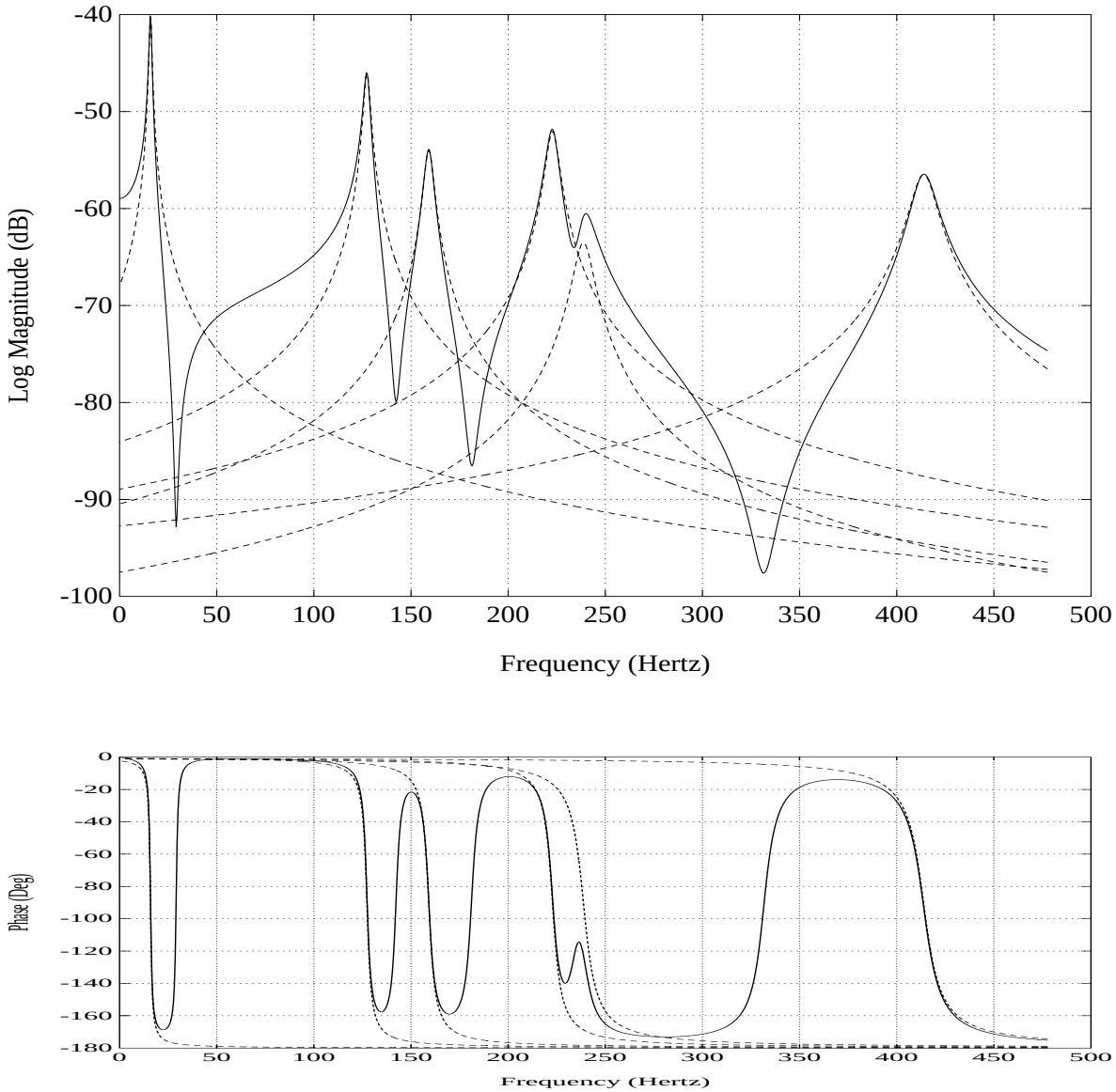

Figure 6-1. Frequency Response Function Synthesis

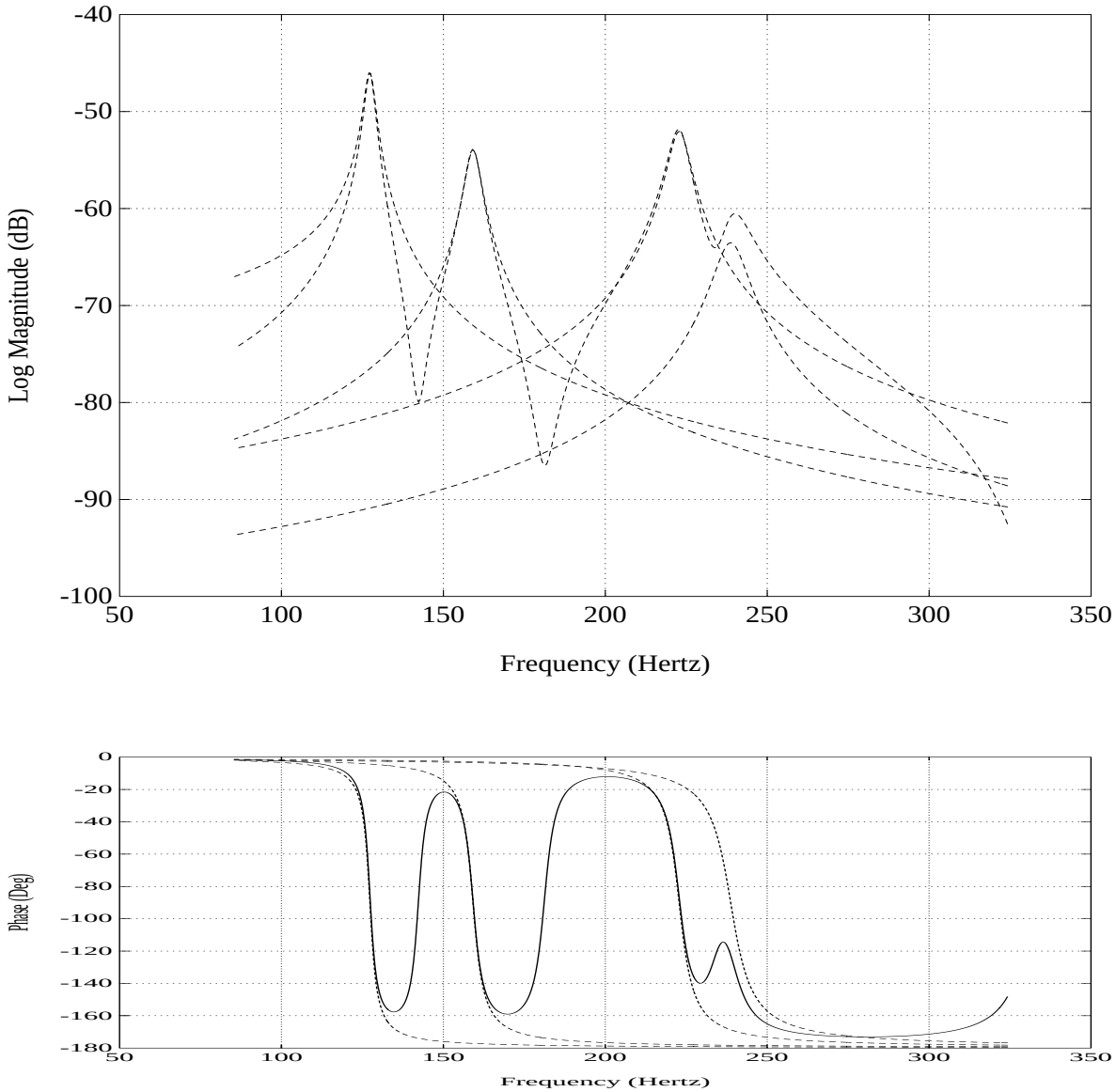

Figure 6-2. Frequency Response Function Synthesis

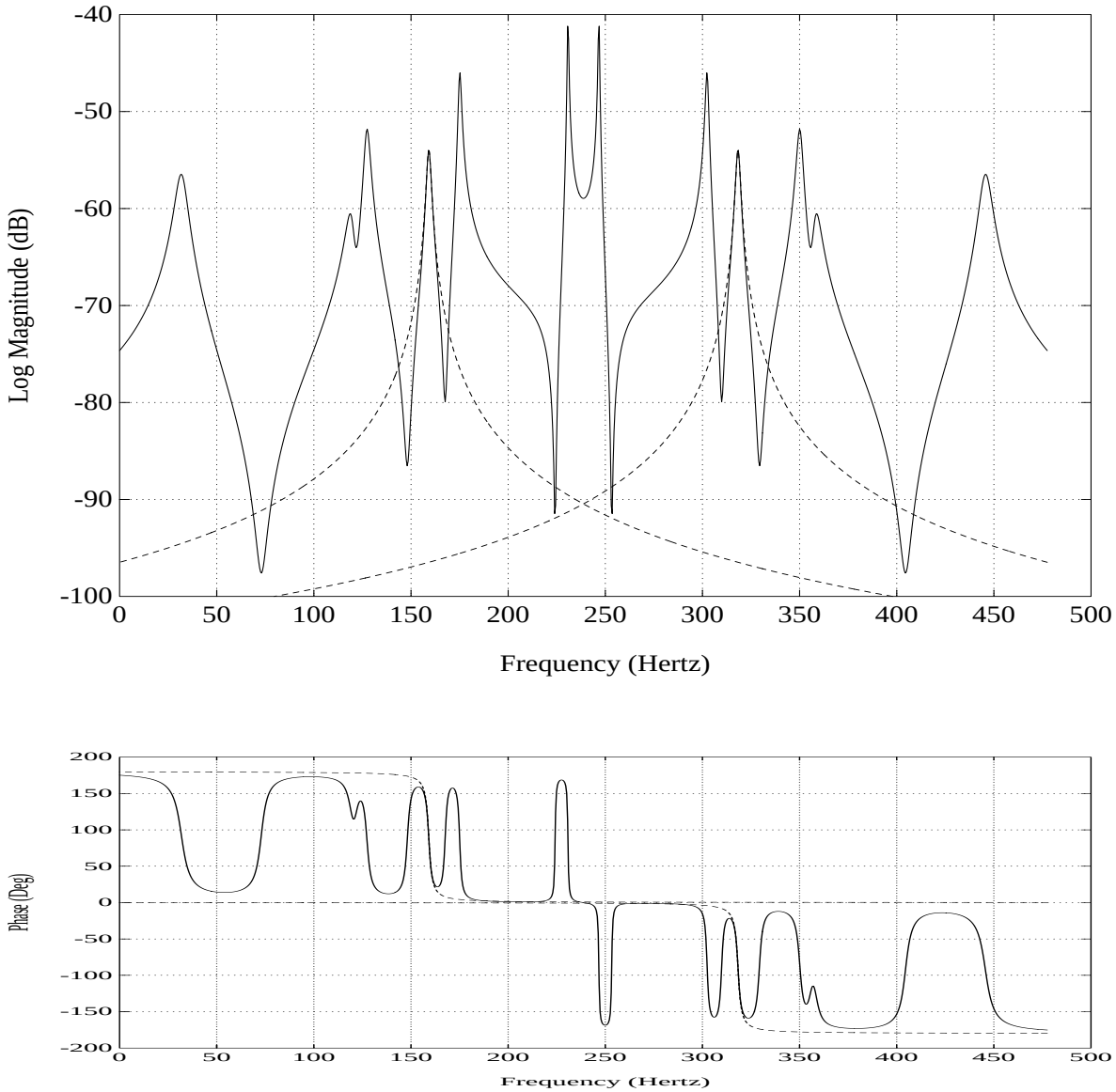

Figure 6-3. Frequency Response Function Synthesis

In order to construct all of the elements that make up [$H(\omega)$], all of the elements of $[A]_r$ for each mode will be required. Previously, the residue matrix $[A]_r$ for a particular mode has been shown to be directly related to the modal vector according to the following equation:

$$\left[A_r \right] = Q_r \{ \psi \}_r \{ \psi \}_r^T = Q_r \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_1\psi_m \\ \psi_2\psi_1 & \psi_2\psi_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_m\psi_1 & \psi_m\psi_2 & \cdot & \cdot & \cdot & \cdot & \psi_m\psi_m \end{bmatrix}_r \tag{6.3}$$

When the k-th column of [$H(\omega)$] has been measured, the k-th column of each residue matrix $[A]_r$ can be defined. Equation 6.3 can now be rewritten for only the k-th column of of the r-th mode of vibration.

$$\begin{Bmatrix} A_{1k} \\ A_{2k} \\ \cdot \\ \cdot \\ A_{ik} \\ \cdot \\ \cdot \\ A_{mk} \end{Bmatrix}_r = Q_r \begin{Bmatrix} \psi_1\psi_k \\ \psi_2\psi_k \\ \cdot \\ \cdot \\ \psi_i\psi_k \\ \cdot \\ \cdot \\ \psi_m\psi_k \end{Bmatrix}_r = Q_r \, \psi_{kr} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_i \\ \cdot \\ \cdot \\ \psi_m \end{Bmatrix}_r \tag{6.4}$$

where:

- m = Number of measurements (N_o or N_i).
- r = Mode number

In order to compute the residue for any position in the residue matrix for each mode of vibration, Equation 6.4 can be used to compute the values of Q_r and $\{\psi\}_r$. In order to do this, either Q_r or ψ_{kr} must be chosen according to a scaling criteria. At this point the individual elements of $\{\psi\}_r$ can now be found. Once these individual elements are known, any residue for a given mode can be computed using the following equation:

$$A_{pqr} = Q_r \, \psi_{pr} \, \psi_{qr} \tag{6.5}$$

If modal mass will be calculated, the individual values of Q_r and $\{\psi\}_r$ will be required. Therefore, Equation 6.5 can be used easily to synthesize any arbitrary residue. In reality, there is no need to know the individual values of Q_r and $\{\psi\}_r$ in order to synthesize another residue. If the k-th column of the residue matrix $[A]_r$ is again used to synthesize any arbitrary residue from another location in the residue matrix, the following equation is all that is required:

$$A_{pqr} = \frac{A_{pkr} \; A_{qkr}}{A_{kkr}} = \frac{Q_r \; \psi_{pr} \; \psi_{kr} \;\; Q_r \; \psi_{qr} \; \psi_{kr}}{Q_r \; \psi_{kr} \; \psi_{kr}} \tag{6.6}$$

Note that with Equation 6.6, any arbitrary residue can be synthesized if the proper elements of the frequency response function matrix and, therefore, the proper elements of each residue matrix $[A]_r$ have been measured. For example, the driving point information for the k-th column of any residue matrix need not be measured directly if the following elements have been measured:

$$A_{kkr} = \frac{A_{pkr} \; A_{qkr}}{A_{pqr}} \tag{6.7}$$

When the errors involved with measuring the driving point frequency response function are taken into account, Equation 6.7 may be a preferable way to estimate the driving point information even though this method requires information from another column of the residue matrix. Note that this will require that another excitation location be taken during the test of the structure.

In summary, one row or column of the frequency response function matrix, including driving point information, must be measured in order to synthesize any arbitrary measurement in the frequency response function matrix. A modal parameter estimation algorithm is then used on these measurements to determine the modal parameters. With the driving point residue information for a particular mode of vibration, any other required residue information can be synthesized by Equation 6.5 or Equation 6.6.

6.1 Displacement, Velocity and/or Acceleration Data

In practice, frequency response functions are often measured using an accelerometer to measure the response of the system and a load cell to measure the input to the system. Velocity or

displacement transducers could also be used to measure the response but have disadvantages with respect to size, configuration or datum. Regardless, the mathematical model that has been developed is a representation of displacement normalized to the force that caused the displacement. Note that if velocity normalized to force is actually measured, this data can be synthetically integrated to displacement over force by dividing the velocity over force measurement by $j\omega$. Likewise, if acceleration normalized to force is actually measured, this data can be synthetically integrated to displacement over force by dividing the acceleration over force measurement by $(j\omega)^2$. This approach has numerical problems near zero frequency but is generally satisfactory elsewhere.

6.2 Skewed Sensor Orientation

While it is common and somewhat desirable to align all sensors (input and output) in directions that are colinear with a set of orthogonal axes (x, y, z), this is not a requirement. Directional cosine information can be used to reorient the sensor information according to any desirable set of axes. Note that the spatial subscripts used to this point (pq for example) do not imply any specific constraint in this sense.

However, if modal scaling or the ability to synthsize arbitrary functions that may or may not have been measured is desired, the measurment of the driving point information is very important. If a skewed excitation is used with a set of essentially orthogonal response sensors, the simplest solution to this problem is to add a skewed response sensor, at the excitation location, so that a true driving point measurement can be estimated. With this in mind, the Equations 6.1 through 6.7 still apply when synthesis of residues and frequency response functions from data taken from a skewed input is the data represented by column k.

6.3 Simulation of Structural Response

Once all of the frequency response functions (transfer functions) for a particular system have been measured or synthesized, the response of the structure to any arbitrary set of inputs can be simulated or predicted. To illustrate the process, the two degree of freedom system that is represented by the following equation can be used as an example.

$$\left\{ \begin{array}{c} X_1(\omega) \\ X_2(\omega) \end{array} \right\} = \left[\begin{array}{cc} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{array} \right] \left\{ \begin{array}{c} F_1(\omega) \\ F_2(\omega) \end{array} \right\} \tag{6.8}$$

Equation 6.8 is the frequency response function form of Equation 4.9 ($s = j\,\omega$). The matrix equations in Equation 6.8 can be expanded as follows:

$$X_1(\omega) = H_{11}(\omega)\, F_1(\omega) + H_{12}(\omega)\, F_2(\omega)$$

$$X_2(\omega) = H_{21}(\omega)\, F_1(\omega) + H_{22}(\omega)\, F_2(\omega)$$

Note that $X_1(\omega)$ and $X_2(\omega)$ are the response of the system (in the frequency domain) due to the forces $F_1(\omega)$ and $F_2(\omega)$. Thus, $X_1(\omega)$ and $X_2(\omega)$ can be defined by the process of multiplication and addition of the known frequency response function measurements and forcing functions. The time domain response of the system, $x_1(t)$ and $x_2(t)$, can then be computed by taking the inverse Fourier transform of $X_1(\omega)$ and $X_2(\omega)$.

This synthesis procedure can be extended to systems with more than two degrees of freedom. In general, for an N-degree of freedom system:

$$\left\{ \begin{array}{c} X_1(\omega) \\ X_2(\omega) \\ . \\ . \\ . \\ X_N(\omega) \end{array} \right\} = \left[\begin{array}{cccc} H_{11}(\omega) & H_{12}(\omega) & \ldots . & H_{1N}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & \ldots . & H_{2N}(\omega) \\ . & . & \ldots . & . \\ . & . & \ldots . & . \\ . & . & \ldots . & . \\ H_{N1}(\omega) & H_{N2}(\omega) & \ldots . & H_{NN}(\omega) \end{array} \right] \left\{ \begin{array}{c} F_1(\omega) \\ F_2(\omega) \\ . \\ . \\ . \\ F_N(\omega) \end{array} \right\} \tag{6.9}$$

While Equation 6.9 is written in terms of the number of degrees of freedom N, this is the limiting theoretical consideration. In general, Equation 6.9 can be written in terms of any arbitrary number of inputs and responses such that [$H(s)$] is no longer a square matrix.

For instance, the time domain response $x_1(t)$ is just the inverse Fourier transform of $X_1(\omega)$ where:

$$X_1(\omega) = H_{11}\, F_1(\omega) + H_{11}\, F_2(\omega) + \ldots . \qquad + H_{1N}\, F_N(\omega)$$

Theoretically, the transfer function matrix [$H(s)$] consists of $N \times N$ transfer functions for an N-degree of freedom system. From a practical point of view, the measured portion of the transfer function matrix (frequency response function matrix) will consist of $N_o \times N_i$ frequency response functions regardless of the number of modes N.

7. MODAL SCALING

7.1 Proportionally Damped Systems (Modal Mass)

The modal matrix (matrix of modal vectors) has been previously used as a coordinate transformation in order to diagonalize the mass, damping (if proportional), and stiffness matrices. The diagonalization of these matrices leads to the analytical definition of modal mass, modal damping, and modal stiffness. From an experimental view point, the mass, damping, and stiffness matrices are generally not known. Therefore, this theoretical approach to the definition of modal mass, modal damping, and modal stiffness is not useful. Even so, the modal mass, modal damping, and modal stiffness can be computed directly from the measured frequency response functions without the benefit of prior knowledge of the mass, damping, and stiffness matrices. Note that the modal mass, modal damping, and modal stiffness are, in general, not physical properties but are generalized, or normalized, properties related to the physical properties.

Recall that, analytically, the value of the modal mass is completely dependent on the scaling chosen for the modal vectors. Any development from the frequency response function (experimental) must be consistent with this concept. The development in Section 4.7 (Equation 4.51) satisfies this constraint. Equation 4.51 is repeated here as Equation 7.1.

$$M_r = \frac{1}{j\, 2\, Q_r\, \omega_r} \tag{7.1}$$

The scaling coefficient Q_r in the above definition can only be found after the choice of modal vector scaling is made.

Recal that for the r-th mode of an N degree of freedom system:

$$[\ A\ ]_r = Q_r \{\ \psi\ \}_r \{\ \psi\ \}_r^T \tag{7.2}$$

$$[\ A\ ]_r = Q_r \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 & . & . & . & . & . & \psi_1\psi_m \\ \psi_2\psi_1 & \psi_2\psi_2 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \psi_m\psi_1 & . & . & . & . & . & . & \psi_m\psi_m \end{bmatrix}_r \tag{7.3}$$

where:

- Q_r is the scaling constant that is a function of the scaling of the modal vectors.

Using only the q-th column of the residue matrix:

$$\begin{bmatrix} A_{1q} \\ A_{2q} \\ \cdot \\ \cdot \\ A_{iq} \\ \cdot \\ \cdot \\ A_{mq} \end{bmatrix}_r = Q_r \begin{bmatrix} \psi_1\psi_q \\ \psi_2\psi_q \\ \cdot \\ \cdot \\ \psi_i\psi_q \\ \cdot \\ \cdot \\ \psi_m\psi_q \end{bmatrix}_r = Q_r \; \psi_{qr} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_i \\ \cdot \\ \cdot \\ \psi_m \end{bmatrix}_r \tag{7.4}$$

The relationship between the residue, the scaling constant and the scaling of the modal vector is most clear when the FRF measurement $H_{qq}(\omega)$ is utilized. Note that $H_{qq}(\omega)$ is obtained by exciting at point q and measuring the response at point q. This is typically referred to as the *driving point* frequency response function. Therefore, the A_{qqr} residue for all modes $r = 1 \rightarrow N$ can be determined from the $H_{qq}(\omega)$ measurement.

Now, note the q-th element of Equation 7.4:

$$A_{qqr} = Q_r \; \psi_{qr} \; \psi_{qr} = Q_r \; \psi_{qr}^2 \tag{7.5}$$

Therefore, the r-th modal mass of a multi-degree of freedom system is defined as:

Modal Mass

$$M_r = \frac{1}{j \; 2 \; Q_r \; \omega_r} \tag{7.6}$$

In general:

$$M_r = \frac{\psi_{pr} \; \psi_{qr}}{j \; 2 \; A_{pqr} \; \omega_r} \tag{7.7}$$

where:

- M_r = Modal mass
- Q_r = Modal scaling constant
- ω_r = Damped natural frequency

While this concept of modal mass being a relative quantity (relative to scaling) is at odds with the engineering view of mass being an absolute quantity, this is consistent with the analytical definition of modal mass.

While many choices of modal vector scaling exist, choosing to scale the modal vector such that the largest modal coefficient will be equal to 1.0 gives a result where the modal mass will always be bounded between zero and the physical mass of the system. For the case where the modal vector of the system describes the rigid body translation (bounce) of the system, this particular choice will give the mass of the system as the modal mass. *Therefore, if the largest scaled modal coefficient is equal to unity, Equation 7.7 will compute a quantity of modal mass that has physical significance. The physical significance is that the quantity of modal mass computed under these conditions will be a number between zero and the total mass of the system.* Therefore, under this scaling condition, the modal mass can be viewed as the amount of mass that is participating in each mode of vibration.

Note that the *modal mass* defined in Equation 7.6 or 7.7 is developed in terms of displacement over force units. If measurements, and therefore residues are developed in terms of any other units (velocity over force or acceleration over force), either the measurements or Equation 7.6 and 7.7 will have to be altered accordingly.

Once the modal mass is known, the modal damping and stiffness can be obtained through the following SDOF relationships:

Modal Damping

$$C_r = 2\,\sigma_r\,M_r \tag{7.8}$$

Modal Stiffness

$$K_r = (\sigma_r^2 + \omega_r^2) M_r = \Omega_r^2 M_r \tag{7.9}$$

7.1.1 Modal Vector Scaling

There are many ways to scale the modal vectors, however the following approaches are typically used:

- Unity Modal Mass
- Unity Modal Coefficient
- Unity Modal Vector Length

The scaling value, Q_r, can be calculated for each of the three cases.

7.1.1.1 Unity Modal Mass

$$Q_r = \frac{1}{j\ 2\ M_r\ \omega_r} = \frac{1}{j\ 2\ \omega_r}$$

The scaled modal coefficient at the driving point can now be computed as follows:

$$Q_r\ \psi_{qr}\ \psi_{qr} = A_{qqr}$$

Dividing both sides by Q_r:

$$\psi_{qr}\ \psi_{qr} = \frac{A_{qqr}}{Q_r}$$

The scaled modal coefficient at the driving point can now be found. For the proportionally damped case, this is obviously trivial since the square root of both sides of the above equation involves the square root of a real-valued number. For the general case, the modal coefficient can be complex, and the above equation must be solved for the complex modal coefficient. In this case the square root cannot be used.

Thus, the scaled modal vector is:

$$\{ \psi \}_r = \frac{1}{Q_r \psi_{qr}} \{ A \}_r$$

7.1.1.2 Unity Modal Coefficient

Assume that the i-th component ψ_i of a modal vector must be set equal to unity ($\psi_i = 1.0$). Then:

$$\{ \psi \}_r = \frac{1}{A_{iq}} \{ A \}_r$$

Thus:

$$Q_r = \frac{A_{iqr}}{\psi_{ir} \psi_{qr}}$$

7.1.1.3 Unity Modal Vector Length

$$\{ \psi \}_r = \frac{1}{\|\{A\}_r\|_2} \{ A \}_r$$

where:

- $\|\{A\}_r\|_2$ = vector norm of $\{A\}_r$
- $\|\{A\}_r\|_2 = \sqrt{\sum_{i=1}^{m} A_{iqr} A_{iqr}^*}$

Thus:

$$Q_r = \frac{A_{iqr}}{\psi_{ir} \psi_{qr}}$$

7.1.2 Modal Vector Scaling Example

Using the modal vectors from the example in Chapter 5, the modal vector scaling required for unity modal mass can be determined.

The modal vectors for modes 1 and 2 are:

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}_1 = \begin{bmatrix} -j\,\dfrac{\sqrt{39}}{117} \\ -j\,\dfrac{\sqrt{39}}{117} \end{bmatrix} \qquad \lambda_1 = -\frac{1}{10} + j\,\frac{\sqrt{39}}{10}$$

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}_2 = \begin{bmatrix} -j\,\dfrac{4\sqrt{15}}{225} \\ +j\,\dfrac{2\sqrt{15}}{225} \end{bmatrix} \qquad \lambda_2 = -\frac{1}{4} + j\,\frac{\sqrt{15}}{4}$$

For unity modal mass, the scaling factor can be evaluated from Equation 7.6.

$$Q_r = \frac{1}{j\,2\,\omega_r}$$

Thus, the scaled modal vector for mode 1 for a modal mass of unity is:

$$\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}_1 = Q_1\,\psi_1 \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 \qquad \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 = \begin{Bmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{Bmatrix}_1$$

The scaled modal vector for mode 2 for a modal mass of unity is:

$$\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}_2 = Q_2\,\psi_1 \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 \qquad \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 = \begin{Bmatrix} \sqrt{2/15} \\ -\dfrac{1}{2}\sqrt{2/15} \end{Bmatrix}_2$$

Notice that these modal vectors that have been scaled with respect to unity modal mass, are identical to the modal vectors previously computed (Section 3.2). The important concept,

though, is that these scaled modal vectors were determined directly from the frequency response functions. In other words, the mass, damping, and stiffness of the system did not have to be known in order to determine the same information.

The modal damping and modal stiffnesses can now be calculated using Equations 7.8.and 7.9.

$$C_1 = 2\,\sigma_1\,M_1 \qquad = 2\,(\frac{1}{10})\,1 \qquad = \frac{1}{5}$$

$$C_2 = 2\,\sigma_2\,M_2 \qquad = 2\,(\frac{1}{4})\,1 \qquad = \frac{1}{5}$$

$$K_1 = (\sigma_1^2 + \omega_1^2)\,M_1 \quad = (\frac{1}{100} + \frac{39}{100})\,1 \qquad = \frac{2}{5}$$

$$K_2 = (\sigma_2^2 + \omega_2^2)\,M_2 \quad = (\frac{1}{16} + \frac{15}{16})\,1 \qquad = 1$$

These are exactly the same parameters as those calculated previously.

7.2 Non-Proportionally Damped Systems (Modal A)

Consistent with the definition of *modal A* as the modal scaling factor used for the theoretical case of nonproportionally damped systems, the *modal A* scaling factor is also the basis for the relationship between the scaled modal vectors and the residues determined from the measured frequency response functions. In general, for most experimental work, *modal A* is used as the default scaling approach. If modal mass needs to be estimated, under the contraint of real (normal) modes, modal mass can be estimated from *modal A.* The following development explains the relationship between *modal A* and modal mass. Starting with the anlytical definition of *modal mass* in 2N space:

$$M_{A_r} = \{\phi\}_r^T \begin{bmatrix} [\,0\,] & [\,M\,] \\ [\,M\,] & [\,C\,] \end{bmatrix} \{\phi\}_r \tag{7.10}$$

$$M_{A_r} = \begin{Bmatrix} \lambda_r\{\psi\}_r \\ \{\psi\}_r \end{Bmatrix}^T \begin{bmatrix} [\,0\,] & [\,M\,] \\ [\,M\,] & [\,C\,] \end{bmatrix} \begin{Bmatrix} \lambda_r\{\psi\}_r \\ \{\psi\}_r \end{Bmatrix} \tag{7.11}$$

Multiplying Equation 7.11 out in terms of N space yields:

$$M_{A_r} = \lambda_r \{\psi\}_r^T [M] \{\psi\}_r + \lambda_r \{\psi\}_r^T [M] \{\psi\}_r + \{\psi\}_r^T [C] \{\psi\}_r \tag{7.12}$$

If the system is proportionally damped, the weighted orthogonality relationships between the mass matrix and the modal vectors can now be applied.

$$M_{A_r} = \lambda_r M_r + \lambda_r M_r + C_r = 2 \lambda_r M_r + C_r \tag{7.13}$$

Applying the SDOF relationship between the modal damping and modal mass ($C_r = -\sigma_r M_r$):

$$M_{A_r} = 2 \lambda_r M_r - 2\sigma_r M_r \tag{7.14}$$

$$M_{A_r} = 2 (\sigma_r + j \omega_r) M_r - 2\sigma_r M_r \tag{7.15}$$

$$M_{A_r} = j2\omega_r M_r \tag{7.16}$$

Equation 7.16 indicates that, for proportionally damped systems, if the modal vectors are scaled to give real valued normal modes and the modal mass is, therefore, real valued, the associated *modal A* for a proportionally damped system is imaginary valued.

While Equation 7.16 is valid only for proportionally damped systems, another form of Equation 7.16 gives a more general result that includes any type of damping. Equation 4.51 is repeated here for convenience.

$$M_r = \frac{1}{j2\omega_r Q_r} \tag{7.17}$$

Plugging Equation 7.17 into 7.16 yields:

$$M_{A_r} = \frac{1}{Q_r} \tag{7.18}$$

Since the basic development of the relationship between the residue and the modal vector coefficients and associated modal scaling ($A_{pqr} = Q_r \psi_{pr} \psi_{qr}$) did not depend upon the assumption of proportional damping, Equation 7.18 is valid for any damping condition and is the most

general form of modal scaling. Note that in general, *modal A* will be complex valued. Equation 7.18 can also be written in an equivalent form that clearly expresses the dependence of *modal A* on the modal vector scaling.

$$M_{A_r} = \frac{1}{Q_r} = \frac{\psi_{pr}\ \psi_{qr}}{A_{pqr}} \tag{7.19}$$

Note that the above development of *modal A* is in terms of displacement over force units. If measurements, and therefore residues are developed in terms of any other units (velocity over force or acceleration over force), Equation 7.19 or the FRF data will have to be altered accordingly.

Consistent with the *modal B* scaling value defined previously from an analytical viewpoint, *modal B* can be determined once *modal A* is known by way of the modal frequency.

$$M_{B_r} = -\lambda_r\ M_{A_r} \tag{7.20}$$

7.3 Modal Mass, Modal A Units Discussion

In general, the modal vectors are considered to be dimensionless since they represent relative patterns of motion. Therefore, the modal mass or modal A scaling terms are considered to carry the units of the respective measurement. For example, the development of the frequency response is based upon displacement over force units. The residue must therfore, have units of length over force-seconds. Since the modal A scaling coefficient is inversely related to the residue, modal A will have units of force-seconds over length. This unit combination is the same as mass over seconds. Likewise since modal mass is related to modal A, for porportionally damped systems, through a direct relationship involvinng the damped natural frequency, the units on modal mass are mass units as expected.

The following table summarizes the units of modal A and modal mass for typical consistent unit applications.

Consistent Units Relationships				
Mass	Force	Length	Modal A (M_{Ar})	Modal Mass (M_r)
M	F	L	M/S	M
KG	NT	Meter	KG/S	KG
KG	KGf	g-S-S	KG/S	KG
LBm	LBf	g-S-S	LBm/S	LBm
Slug	LBf	Feet	Slug/S	Slug

8. ADVANCED MODAL ANALYSIS CONCEPTS

8.1 Introduction

As the theoretical basis of expermental modal analysis is extended to real world problems, several clarifications of the theory developed to the present time must be made. The development of more general and/or concise models to represent the entire frequency response function matrix $[H(\omega)]$ or impulse response function matrix $[h(t)]$ is the primary concern. These models must be consistent with the single reference concepts developed previously but must be compatible with multiple reference concepts as well. Therefore, the general concept of measurement degree of freedom must be extended to account for the multiple input, multiple output nature of the problem. Two other concepts must also be discussed in order to fully develop the theoretical basis for experimental modal analysis. These concepts include systems that have repeated modal frequencies (repeated roots) and systems that can not be considered reciprocal.

8.2 Measurement Degrees of Freedom

For the general situation of a multiple input, multiple output model of a system, the experimental definition of the mechanical system is generated from the frequency, or impulse, response function matrix. The size of this matrix is a function of the locations where forces are applied to the mechanical system (inputs) and a function of the locations where responses of the mechanical system (outputs) are measured. This general concept is often referred to as *measurement degrees of freedom* to distinguish the size of the matrix from the number of modal frequencies N of the mechanical system. Obviously, since there is no reason to assume that the number of inputs will be the same as the number of outputs, this general concept of measurement degrees of freedom needs to be extended to properly reflect that the dimension of the frequency, or impulse, response function matrix is rectangular. With this in mind, the number of inputs can be defined by N_i and the number of outputs can be defined by N_o. Therefore, the dimension of the frequency, or impulse, response function matrix is $N_o \times N_i$.

8.3 Mathematical Models

The mathematical model that represents the relationship between the modal parameters and the measured frequency, or impulse, response functions can be represented as follows:

Frequency Response Function Model

Single Measurement:

$$\sum_{k=0}^{m} \alpha_k (j\omega)^k H_{pq}(\omega) = \sum_{k=0}^{n} \beta_k (j\omega)^k \tag{8.1}$$

$$H_{pq}(\omega) = \sum_{r=1}^{N} \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A^*_{pqr}}{j\omega - \lambda^*_r} \tag{8.2}$$

Multiple Measurement:

$$\sum_{k=0}^{m} \left[[\alpha_k] (j\omega)^k \right] [H(\omega)] = \sum_{k=0}^{n} \left[[\beta_k] (j\omega)^k \right] [I] \tag{8.3}$$

$$[H(\omega)]_{N_o \times N_i} = \sum_{r=1}^{N} \frac{\left[A_r \right]}{j\omega - \lambda_r} + \frac{\left[A^*_r \right]}{j\omega - \lambda^*_r} \tag{8.4}$$

Impulse Response Function Model

Single Measurement:

$$h_{pq}(t) = \sum_{r=1}^{N} A_{pqr}\ e^{\lambda_r t} + A^*_{pqr}\ e^{\lambda^*_r t} \tag{8.5}$$

Multiple Measurement:

$$[h(t)\ ]_{N_o \times N_i} = \sum_{r=1}^{N} \left[A_r \right] e^{\lambda_r t} + \left[A^*_r \right] e^{\lambda^*_r t} \tag{8.6}$$

where:

- $s =$ Laplace variable
- $s = \sigma + j\omega =$ Angular damping variable (rad/sec)
- $\omega =$ Angular frequency variable (rad/sec)
- $p =$ Measured degree-of-freedom (response)
- $q =$ Measured degree-of-freedom (input)
- $r =$ Modal vector number
- $m =$ Number of poles or modal frequencies (2N)
- $n =$ Number of zeroes (2N-2 or less)
- $N =$ Number of positive modal frequencies
- $A_{pqr} =$ Residue $= Q_r \psi_{pr} \psi_{qr}$
- $Q_r =$ Complex modal scaling coefficient for mode r
- $\psi_{pr} =$ Modal coefficient for measured degree-of-freedom p and mode r
- $[A_r] =$ Residue matrix for mode r ($N_o \times N_i$)
- $\lambda_r =$ System pole $= \sigma_r + j\omega_r$

While these models are perfectly appropriate for the multiple input, multiple output case, by a

slight alteration of these models a more appropriate form of the models can be developed which will facilitate the development of parameter estimation algorithms.

First of all, the summation form of the equations can be simplified from two terms to one term as follows:

$$[H(\omega)\,] = \sum_{r=1}^{2N} \frac{\left[A_r \right]}{j\omega - \lambda_r} \tag{8.7}$$

$$[h(t)\,] = \sum_{r=1}^{2N} \left[A_r \right] e^{\lambda_r t} \tag{8.8}$$

If these forms of the mathematical models are used, no assumption is made in the model concerning the complex conjugate nature of the solution for modal frequencies (λ_r) or modal vectors ($\{ \psi_r \}$). When the modal parameters are estimated, the evaluation of modal parameters can include a comparison of these terms to determine whether the complex conjugate nature of the solution is found.

Finally, the summation in the mathematical models can be eliminated completely if a different form of the residue is used. In order to do this, the concept of *modal participation factor* (L_{qr}) is introduced. Physically, the modal participation factor is a relative indication of how well a particular mode of vibration is excited from a specific measurement degree of freedom. If all of the modal participation factors for a specific modal vector are represented in a row, this vector is referred to as the modal participation vector and has dimension of $1 \times N_i$. The modal participation vector is not unique (has properties of an eigenvector) but in combination with the modal coefficient defines the residue in the following way.

$$A_{pqr} = Q_r \psi_{pr} \psi_{qr} \tag{8.9}$$

$$A_{pqr} = L_{qr} \psi_{pr} \tag{8.10}$$

$$\{A\,\}_{qr} = L_{qr} \{\psi\,\}_r \tag{8.11}$$

$$L_{qr} = Q_r \psi_{qr} \tag{8.12}$$

Equations 8.10 and 8.11, therefore, are the general statements relating the residue, the modal participation factor, and the modal coefficient. For the general case, the modal coefficient can be thought of as an element from the right eigenvector of the system; the modal participation factor can be thought of as an element from the left eigenvector of the system. With this in mind, when a system obeys Maxwell's Reciprocity, the right and left eigenvectors represent the same modal vector, Equations 8.9 and 8.12 are valid. For the general (nonsymmetric) case, though, only Equations 8.10 and 8.11 will be valid.

From the above equations it is important to note that only the residue can be absolutely and uniquely defined. Both the modal coefficient ψ_{pr} and the modal participation factor L_{qr} are a function of one another and can take on any value; only the combination of the two terms is unique.

Since the term *modal mass* can be defined in terms of the modal scaling Q_r, the modal mass can also be defined in terms of the modal participation factor L_{qr}.

$$M_r = \frac{\psi_{qr}}{2jL_{qr}\omega_r} \tag{8.13}$$

If this simplification is made, the frequency response function model can now be written in the following form:

$$[H(\omega)]_{N_o \times N_i} = \left[\psi\right]_{N_o \times 2N} [\Lambda]_{2N \times 2N} [L]_{2N \times N_i} \tag{8.14}$$

where:

$$[\Lambda]_{2N \times 2N} = \begin{bmatrix} \frac{1}{j\omega - \lambda_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{j\omega - \lambda_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{j\omega - \lambda_r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{j\omega - \lambda_{2N}} \end{bmatrix}$$

Likewise, a similar simplification can be developed for the impulse response function matrix:

$$[h(t)]_{N_o \times N_i} = \left[\psi \right]_{N_o \times 2N} \left[e^{\lambda_r t} \right]_{2N \times 2N} [L]_{2N \times N_i} \tag{8.15}$$

where:

$$\left[e^{\lambda_r t} \right]_{2N \times 2N} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_r t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\lambda_{2N} t} \end{bmatrix}$$

Since the modal participation vector $\{ L \}_r$ is unique only to within a complex scaling constant, several other definitions of the modal participation factors are possible. One common form can be defined by scaling the modal participation such that a specific element of the vector, L_{pr}, is unity. If this normalization is used, the following definition of modal participation factor will apply:

$$\overline{L_{pqr}} = \frac{L_{qr}}{L_{pr}} = \frac{Q_r \psi_{qr}}{Q_r \psi_{pr}} = \frac{\psi_{qr}}{\psi_{pr}} \tag{8.16}$$

This form of the definition of modal participation factor is used in the development of the theory for the Polyreference Time Domain modal parameter estimation algorithm.

8.4 Repeated Modal Frequencies

Repeated modal frequencies occur whenever two or more modes of the system occur at exactly the same modal frequency λ_r. This condition is often also referred to as repeated roots or repeated poles. Analytically, the presence of repeated roots is determined directly from the characteristic equation just as in any other case. The modal vectors, associated with the repeated roots, now come from a system of equations (the homogeneous equations evaluated at a repeated root) which will be rank deficient by more than one. This means that rather than choosing one of the physical coordinates, as is the case for the non-repeated root situation, a number of physical

coordinates, equal to the number of repeated roots, must be assumed. This process is repeated once for each of the repeated roots. Each of the vectors found in this manner will in general be independent to one another and orthogonal to all other modal vectors. Each vector within this set, however, will not necessarily be orthogonal to each other. While this is not a problem mathematically, it may be more consistent to find a set of vectors that are also orthogonal to one another. This can be accomplished by additionally imposing the cross-orthogonality constraints between the vectors. Using any mathematical procedure to determine a set of orthogonal vectors from a set of independent vectors will also work as long as the weighting matrix (mass and/or stiffness) is utilized in the procedure. Note that there is an infinite number of modal vector sets that will satisfy repeated root situation, whether the orthogonality constraint is enforced or not.

Experimentally, it is important to detect the presence of repeated roots in order to build a complete modal model of the mechanical system that will accurately represent the dynamic response of the mechanical system to any set of forcing conditions. The most common cause of this situation is symmetry in the mechanical system. For example, in the case of a flagpole, there are two modes of vibration occurring at the same frequency for each lateral bending mode of the flagpole. Any time one or more axes of symmetry exist in the mechanical system, this condition will exist. The important consideration, though, is that, in order to detect the repeated root condition, more than one row or column of the frequency, or impulse, response function matrix must be measured. Therefore, to detect a repeated root of order two, two *independent* rows or columns of the frequency, or impulse, response function matrix must be used. Note that it is possible to choose two rows or columns that are not independent and thus miss the repeated root.

The modal vectors that are associated with repeated modal frequencies are independent of one another and each is orthogonal (weighted) to the other modal vectors of the set. Even so, the modal vectors associated with the repeated modal frequencies do not individually have fixed patterns even in the relative sense that modal vectors associated with nonrepeated modal frequencies do. Only when the set of modal vectors associated with repeated modal frequencies are considered as a set, are the modal vectors unique in any way. In the case of a repeated modal frequency of multiplicity two, two independent modal vectors will be required to describe the modal space but any two independent modal vectors will do. An analogy in three dimensional graphics involves using two vectors to define a plane. There is an infinite set of vectors that can be used to describe the same plane. Any two vectors lying in that plane can be used to uniquely define the plane and yet the two vectors are not unique but only independent from one another.

In this case, independent simply means that the two vectors are not scalar multiples of one another.

Therefore, there are an infinite number of combinations of modal vectors which will serve as the modal vectors for the repeated modal frequencies. In this case, the exact characteristic of each modal vector is unimportant since the set of modal vectors will always be considered together. The modal vectors associated with the repeated modal frequencies often appear to be exactly the same at first glance. This is due to the symmetry and on close inspection using physical coordinates and directions, the distinction can be easily detected. Again consider the flagpole example. Since there is no way to define a unique x and y direction with respect to the circular cross section of the flagpole, there will not be a single set of modal vectors that will describe the modal deformation at a repeated modal frequency.

At this point, repeated modal frequencies may seem to be only a theoretical concept that does not have much impact on real structures that are not symmetric. Actually, due to the discrete nature of the frequency response function, a very real problem often exists where several modal frequencies occur between the frequency resolution that is used. Since *a priori* knowledge of the modal density is not normally possible, this condition happens quite frequently. This is referred to as *psuedo-repeated modal frequencies.* While the modal vectors do not theoretically have the same attributes as in the repeated root case, for all practical purposes, the result is the same.

For the case of a mechanical system with repeated roots, or psuedo-repeated roots, Leuridan [1] has shown that the same mathematical model can be used to represent the relationship between measured frequency, or impulse, response function data and modal parameters. For a mechanical system with repeated modal frequencies of order N_r the following basic relationship for frequency response functions will apply.

$$H_{pq}(\omega) = \sum_{r=1}^{2N} \frac{A_{pqr}}{j\omega - \lambda_r} \tag{8.17}$$

$$H_{pq}(\omega) = \frac{A_{pq1}}{j\omega - \lambda_1} + \frac{A_{pq2}}{j\omega - \lambda_2} + \ldots\ldots\ldots + \sum_{s=1}^{N_r} \frac{A_{pqs}}{j\omega - \lambda_s} + \ldots\ldots\ldots + \frac{A_{pqr}}{j\omega - \lambda_r} + \ldots\ldots\ldots \tag{8.18}$$

.

With the above model, several poles could be repeated; the multiplicity of any pole can be at most N.

In order to understand the implications of the repeated modal frequency on the residue information, the above equation can be rewritten in terms of column q of the frequency response function matrix.

$$\{H\}_q = \frac{Q_1\psi_{q1}\{\psi\}_1}{j\omega-\lambda_1} + \frac{Q_2\psi_{q2}\{\psi\}_2}{j\omega-\lambda_2} + \ldots\ldots\ldots + \sum_{s=1}^{N_r}\frac{Q_s\psi_{qs}\{\psi\}_s}{j\omega-\lambda_s} + \ldots\ldots\ldots + \frac{Q_r\psi_{qr}\{\psi\}_r}{j\omega-\lambda_r} + \ldots\ldots\ldots \quad (8.19)$$

.

The implication of this representation is that if only one column of the frequency response function matrix is measured, the residue that will be estimated for the repeated modal frequencies will be the linear combination represented by the summation in the above equation [1]. Note that this linear combination is not unique; if a different column of the frequency response function is used, the residue column for the repeated modal frequencies will not be the same. This observation yields the simplest procedure for the detection of repeated or psuedo-repeated modal frequencies. If the modal vectors that are estimated from different reference positions (different inputs) are not the same, the modal vectors are probably the result of repeated modal frequencies. Note that the modal vectors associated with the repeated modal frequencies are independent of one another and not simply a scalar multiple of one another as would be the case for a nonrepeated modal frequency.

Therefore, if the repeated modal frequency condition is not detected, the residue vector, associated with the repeated modal frequency, that will be estimated from the frequency response function data taken from only one reference will represent a linear combination of the modal vectors associated with the repeated modal frequencies. If frequency response function data from a second reference is observed without knowledge of the first reference (and the repeated modal frequency condition is still not detected) the residue vector will again represent a linear

combination of the modal vectors associated with the repeated modal frequencies. Unfortunately, this residue vector will not represent the same modal vector when compared to the previous estimate. This means that knowledge of any single reference set of data will be insufficient to describe the repeated root situation.

Nevertheless, if data from several references is available, a characteristic of the modal vectors associated with the repeated root is possible. In general, for a repeated root of order N_r, at least N_r references will be required to make this determination. Unlike in the non-repeated condition, though, the individual modal vectors determined by this process will not be unique. Only the combination of vectors will represent a unique characteristic. For example, if a system contains a repeated modal frequency λ_s of order 2, the two residue columns that will be estimated for columns p and q can always be represented as follows:

For column p:

$$\{A\}_p = Q_{11}\psi_{p1}\{\psi\}_1 + Q_{22}\psi_{p2}\{\psi\}_2 \tag{8.20}$$

For column q:

$$\{A\}_q = Q_{11}\psi_{q1}\{\psi\}_1 + Q_{22}\psi_{q2}\{\psi\}_2 \tag{8.21}$$

Therefore, in matrix notation:

$$\left[\{A\}_p\{A\}_q\right] = \left[\{\psi\}_1\{\psi\}_2\right]\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}\begin{bmatrix} \psi_{p1} & \psi_{q1} \\ \psi_{p2} & \psi_{q2} \end{bmatrix} \tag{8.22}$$

The above equation states that, when the modal vectors, $\{\psi\}_1$ and $\{\psi\}_2$, are excited in *independent* combinations in column p and column q of the frequency response function matrix, then the residue vectors, $\{A\}_p$ and $\{A\}_q$, are independent and in turn define two independent modal vectors that always can be normalized such that the modal scaling matrix, $[Q]$, is diagonal.

For the general case of a repeated modal frequency of order N_r, there will be a set of N_r

independent modal vectors that will satisfy Eq. 8.22. Note that any set of N_r modal vectors that are independent linear combinations of the modal vectors defined by Eq. 8.22 can be used. In this case, the modal scaling matrix, [Q], will not be diagonal. This case is represented by the following equation and will be demonstrated in a later example.

$$\left[\{A\}_p \{A\}_q \right] = \left[\{\psi\}_1 \{\psi\}_2 \right] [Q] \begin{bmatrix} \psi_{p1} & \psi_{q1} \\ \psi_{p2} & \psi_{q2} \end{bmatrix} \tag{8.21}$$

8.4.1 Repeated Modal Frequency Example: Residue Synthesis

In order to fully understand the nature of the problem that can arise due to repeated modal frequencies, a simple example may be used. Consider a repeated modal frequency, λ_s, of order two in a three degree of freedom system. The following represents two independent modal vectors associated with the two repeated modal frequencies.

$$\{\psi\}_1 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$\{\psi\}_2 = \begin{Bmatrix} 2 \\ 1 \\ 3 \end{Bmatrix}$$

For this example, assume that the modal vectors given above have been scaled such that the modal scaling matrix, [Q], is diagonal with elements on the diagonal equal to j.

$$[Q] = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$$

For the following discussion, only the portion of the frequency response function matrix, [H], that depends upon the repeated modal frequencies will be synthesized.

To begin with, all of the columns of the frequency response function matrix will be synthesized

from the theoretical modal data (Case 1, 2, 3). This data represents the true answer or the answer that would be generated from the measured frequency response functions if the repeated root condition was not detected.

Case Number One: Column 1 of $[H]$

Case One represents the proper synthesis of the repeated root portion of Column 1 of the frequency response function matrix.

$$\{H\}_1 = \ldots\ldots\ldots + \frac{j1\begin{Bmatrix}1\\2\\3\end{Bmatrix}_1}{j\omega - \lambda_s} + \frac{j2\begin{Bmatrix}2\\1\\3\end{Bmatrix}_2}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_1 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}5\\4\\9\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_1 = \ldots\ldots\ldots + \frac{j\sqrt{5}\begin{Bmatrix}\sqrt{5}\\ \dfrac{4}{\sqrt{5}}\\ \dfrac{9}{\sqrt{5}}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Notice that if only this column of the frequency response function matrix is measured, there is no reason to suspect that a repeated modal frequency exists. Therefore, the number of modal frequencies and modal vectors would be reduced accordingly.

Case Number Two: Column 2 of $[H\ ]$

Case Two represents the proper synthesis of the repeated root portion of Column 2 of the frequency response function matrix.

$$\{H\ \}_2 = \ldots\ldots\ldots + \frac{j2\left\{\begin{matrix}1\\2\\3\end{matrix}\right\}_1}{j\omega - \lambda_s} + \frac{j1\left\{\begin{matrix}2\\1\\3\end{matrix}\right\}_2}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_2 = \ldots\ldots\ldots + \frac{j\left\{\begin{matrix}4\\5\\9\end{matrix}\right\}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\ \}_2 = \ldots\ldots\ldots + \frac{j\sqrt{5}\left\{\begin{matrix}\frac{4}{\sqrt{5}}\\\sqrt{5}\\\frac{9}{\sqrt{5}}\end{matrix}\right\}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Once again notice that if only this column of the frequency response function matrix is measured, there is no reason to suspect that there is a repeated modal frequency. If the modal vector for the modal frequency λ_s is compared to that found from column 1 of the frequency response function matrix, it is obvious that a different modal vector has been estimated. This comparison is the simplest method of detecting a repeated modal frequency.

Case Number Three: Column 3 of $[H]$

Case Three represents the proper synthesis of the repeated root portion of Column 3 of the frequency response function matrix.

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j3\begin{Bmatrix}1\\2\\3\end{Bmatrix}_1}{j\omega - \lambda_s} + \frac{j3\begin{Bmatrix}2\\1\\3\end{Bmatrix}_2}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}9\\9\\18\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j\sqrt{18}\begin{Bmatrix}\frac{9}{\sqrt{18}}\\\frac{9}{\sqrt{18}}\\\sqrt{18}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Notice that the modal vector determined from this column of the frequency response function is once again different from either of the first two columns. This is a characteristic of the modal vector resulting from a repeated modal frequency; if the repeated modal frequency is not detected, the modal vector appears to be different as a function of the reference (input) location.

If only the first column of the frequency response function matrix $[H]$ has been measured and the repeated modal frequency has not been identified, the synthesis of column 2 and column 3 can be attempted from the measured first column as follows:

Case Number Four: Column 2 of $[H]$

Case Four represents the improper synthesis Column 2 of the frequency response function if only frequency response function data from Column 1 is used (repeated root not detected).

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j\,\frac{4}{\sqrt{5}}\begin{Bmatrix}\sqrt{5}\\ \frac{4}{\sqrt{5}}\\ \frac{9}{\sqrt{5}}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_2 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}4\\ \frac{16}{5}\\ \frac{36}{5}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Note that this synthesis does not agree with Case 2 which is the theoretical result. Therefore, if only column 1 of the frequency response function matrix is measured, the repeated modal frequency cannot be detected and the proper dynamic characteristics of the system, when excited at location 2, cannot be predicted.

Case Number Five: Column 3 of $[H\ ]$

Case Five represents the improper synthesis Column 3 of the frequency response function if only frequency response function data from Column 1 is used (repeated root not detected).

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j\dfrac{9}{\sqrt{5}}\begin{Bmatrix}\sqrt{5}\\ \dfrac{4}{\sqrt{5}}\\ \dfrac{9}{\sqrt{5}}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j\begin{Bmatrix}9\\ \dfrac{36}{5}\\ \dfrac{81}{5}\end{Bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Once again the result for this case should compare to Case 3 (the theoretical case) but does not. Therefore, if only column 1 of the frequency response function matrix is measured, the repeated modal frequency cannot be detected and the proper dynamic characteristics of the system, when excited at location 3, cannot be predicted.

8.4.2 Repeated Modal Frequency Example: Modal Vector Solution

Using the residue data generated from two independent columns of the frequency response function matrix, the set of independent modal vectors for the repeated modal frequency can be determined if the order of the repeated modal frequency is known. This can be estimated via singular value decomposition techniques. For the previous example, assuming that the multiplicity of the repeated modal frequency is 2, the results can be determined by using the first

two columns of the frequency response function matrix (Case 1 and Case 2).

$$\left[\{H\}_1 \{H\}_2 \right] = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Since the residue columns found from the two columns are independent, these residue columns can be used as the [ψ] matrix in Eq. 8.21. In this case, though, there is no reason to assume that the modal scaling matrix [Q] is diagonal. With this in mind, Eq. 8.21 can be used in Eq. 8.2 as follows:

$$\left[\{H\}_1 \{H\}_2 \right] = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

While it may not be immediately obvious, the modal scaling matrix in the above equation must be chosen so that the product of the three matrices and the scalar j yields the proper residues for the first two columns. The inverse of this matrix is the only matrix that satisfies this condition.

$$\left[\{H\}_1 \{H\}_2 \right] = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

Now using these two residue columns and the modal scaling matrix [Q] the synthesis of the residue for column 3 is possible.

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

$$\{H\}_3 = \ldots\ldots\ldots + \frac{j \begin{bmatrix} 9 \\ 9 \\ 18 \end{bmatrix}}{j\omega - \lambda_s} + \ldots\ldots\ldots$$

This is the correct answer as previously defined by Case 3. Note that it is possible to find a set of modal vectors such that the corresponding modal scaling matrix [Q] is diagonal. This is done by substituting the eigenvalue decomposition of [Q] into the above equation. While this is possible and makes the theory consistent, it is unnecessary as the previous example proved.

Question?

If the modal vectors corresponding to a repeated modal frequency with multiplicity of two are:

$$\{\psi\}_1 = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

$$\{\psi\}_2 = \begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix}$$

Is it possible to correctly identify the modal vectors from the first and third columns of the frequency response function matrix if the multiplicity of the repeated modal frequency is known? (Hint: Are the first and third columns of the frequency response function matrix independent?)

8.5 Left and Right Eigenvectors

The relationship between the modal participation vector and the modal vector is particularly interesting when the theoretical matrix equation of motion is examined. For the case of a system that obeys Maxwell's reciprocity, the mass, stiffness, and damping matrices will be symmetric matrices. With this in mind, the basic eigenvalue-eigenvector problem that is generated is of the following form:

$$\left[\lambda_r^2[M] + \lambda_r[C] + [K] \right] \{\psi\}_r = \{0\} \tag{8.22}$$

In the above equation, note that the dimension of $\{\psi\}_r$ is the same as the dimension of the

square mass, stiffness, or damping matrix. Since this theoretical problem is of dimension $N \times N$, there is no difference between N, N_o, *or* N_i. Therefore, the modal participation vector is also of the same dimension.

The eigenvector $\{ \psi \}_r$ is also referred to as the right eigenvector of the system. If the eigenvalue-eigenvector problem is reformulated, the eigenvector is also the left eigenvector of the system.

$$\{\psi\}_r^T \left[\lambda_r^2[M] + \lambda_r[C] + [K] \right] = \{0\} \tag{8.23}$$

Note that in both Eqs. 8.22 and 8.23, the modal vector $\{ \psi \}_r$ could be replaced by the modal participation vector $\{ L \}_r$ without loss of generality.

If the system does not satisfy reciprocity (the mass, stiffness, and/or damping matrices are not symmetric), then the right and left eigenvectors will be different. For the right eigenvector, Eq. 8.24 will be appropriate.

$$\left[\lambda_r^2[M] + \lambda_r[C] + [K] \right] \{\psi\}_r = \{0\} \tag{8.24}$$

For the left eigenvector, Eq. 8.25 will now be the appropriate form.

$$\{L\}_r^T \left[\lambda_r^2[M] + \lambda_r[C] + [K] \right] = \{0\} \tag{8.25}$$

Note that the modal participation vector is the same as the left eigenvector. At this point it is important to note that the residue for a nonreciprocal system can still be defined by the product of the modal vector and the modal participation factor as in Eqs. 8.8 and 8.9. For the nonreciprocal case, the relationship between the modal vector and modal participation vector, as defined by Eq. 8.10, no longer is true.

This concept of the relationship between left and right eigenvectors and modal vectors and modal participation vectors is true for systems that have repeated modal frequencies as well.

8.6 Summary/Conclusions

Whether a system contains repeated modal frequencies or nonreciprocal characteristics, the general equations relating modal parameters to the measure frequency, or impulse, response function matrices will properly predict the dynamics of the system as long as sufficient elements of the matrix are measured. The extension of Eqs. 8.1 and 8.3 to Eqs. 8.12 and 8.13 provide the generality and flexibility to describe these characteristics.

8.7 References

[1] Leuridan, J., *Some Direct Parameter Model Identification Methods Applicable for Multiple Input Modal Analysis*, Doctoral Dissertation, Department of Mechanical Engineering, University of Cincinnati, 1984, 384 pp.

Appendix A: MEASUREMENT DISPLAY FORMATS

The following measurement display formats are all from the same theoretical system and represent the information associated with either the impulse response function ($h_{12}(t)$) or the frequency response function ($H_{12}(\omega)$). The theoretical system is a two degree of freedom system with damping and the measurement represents the system characteristics measured by forcing the system at the second degree of freedom and measuring the response at the first degree of freedom.

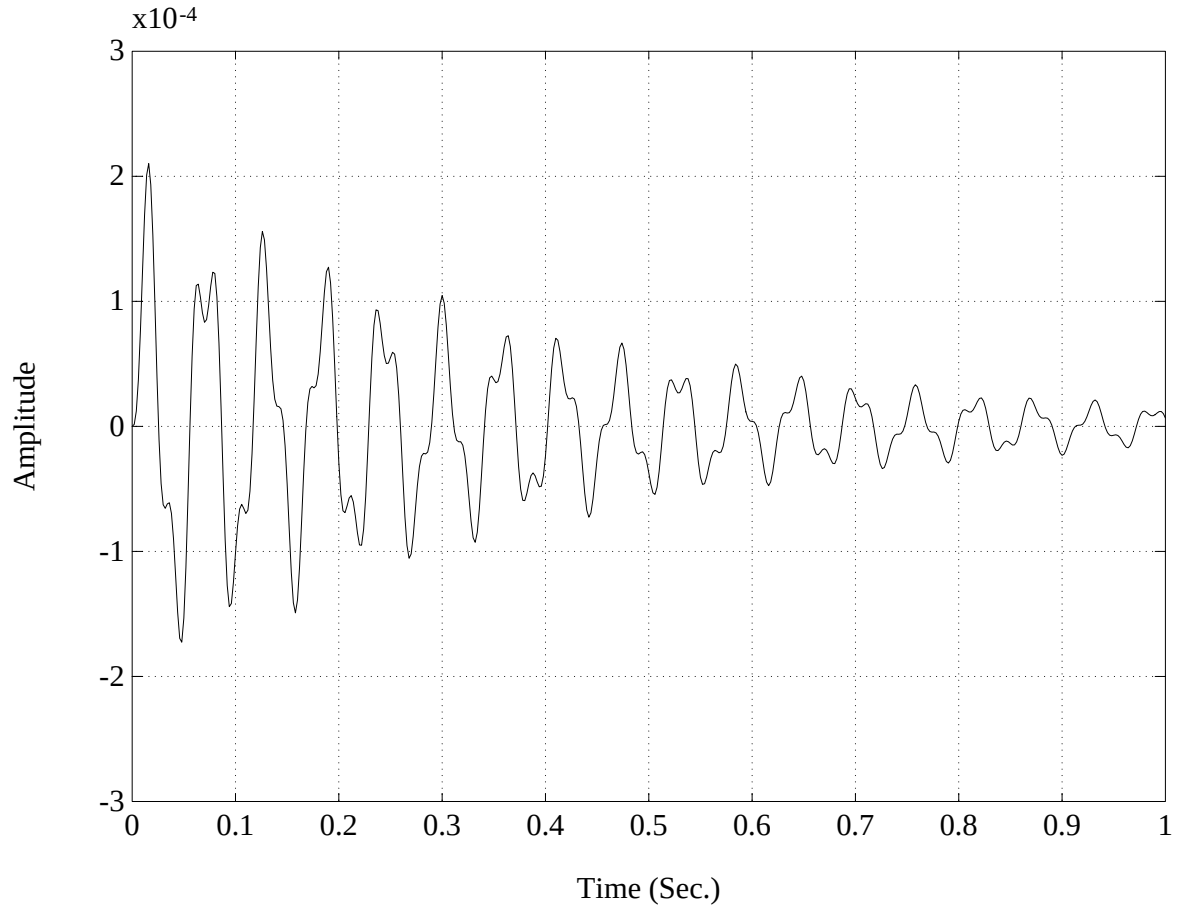

Figure A-1. Impulse Response Function Measurement Format

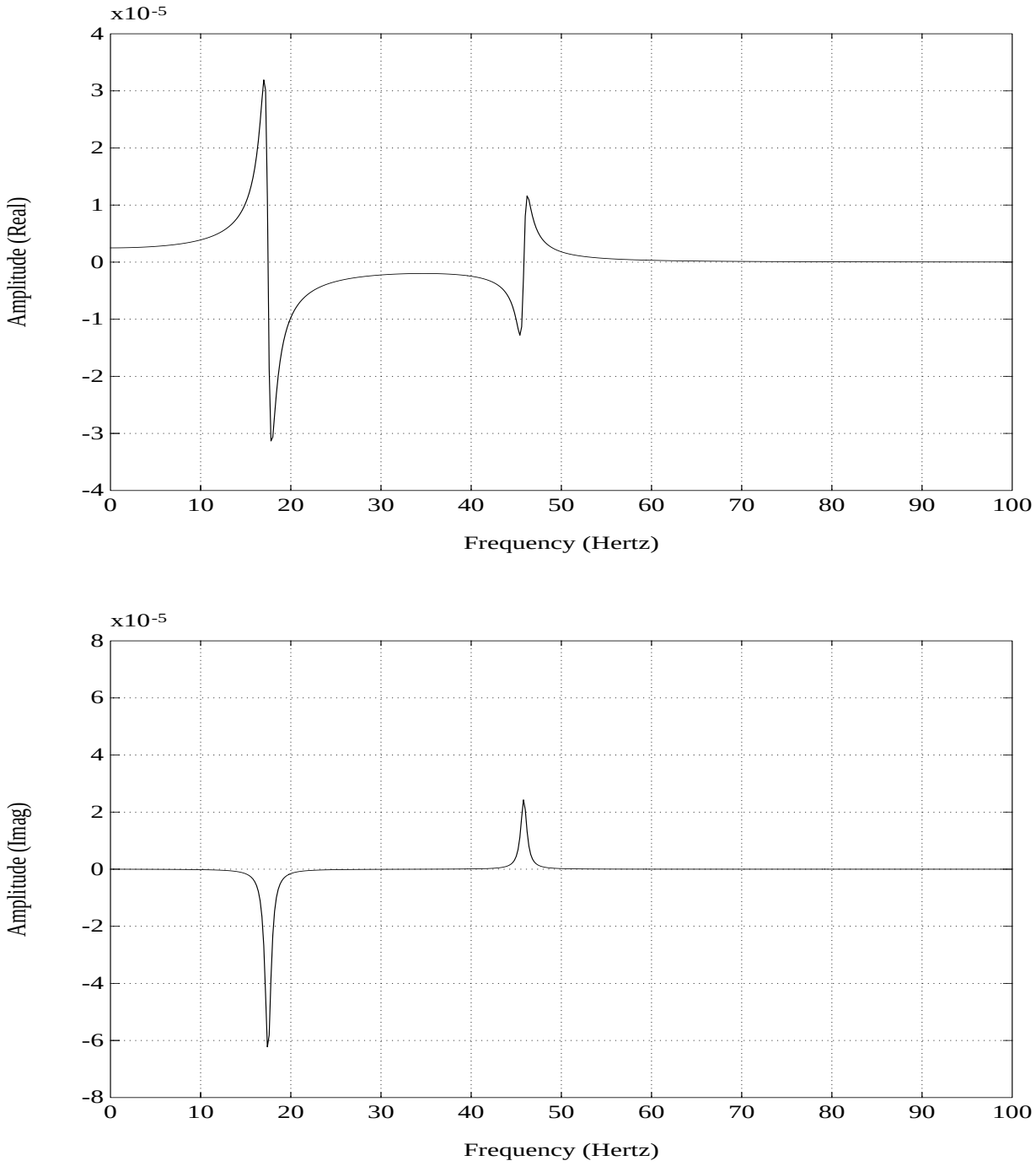

Figure A-2. Frequency Response Function Measurement Format

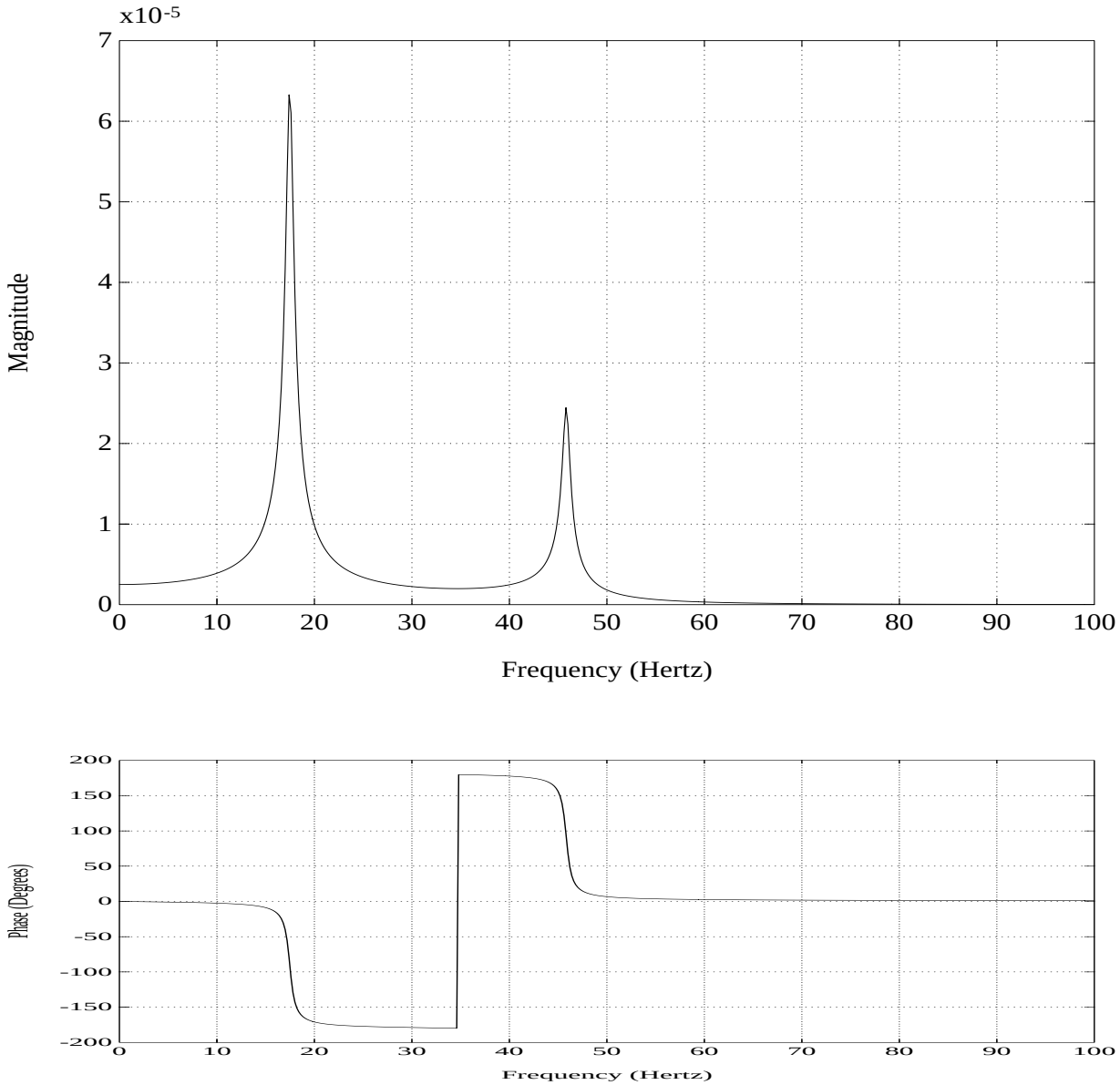

Figure A-3. Frequency Response Function Measurement Format

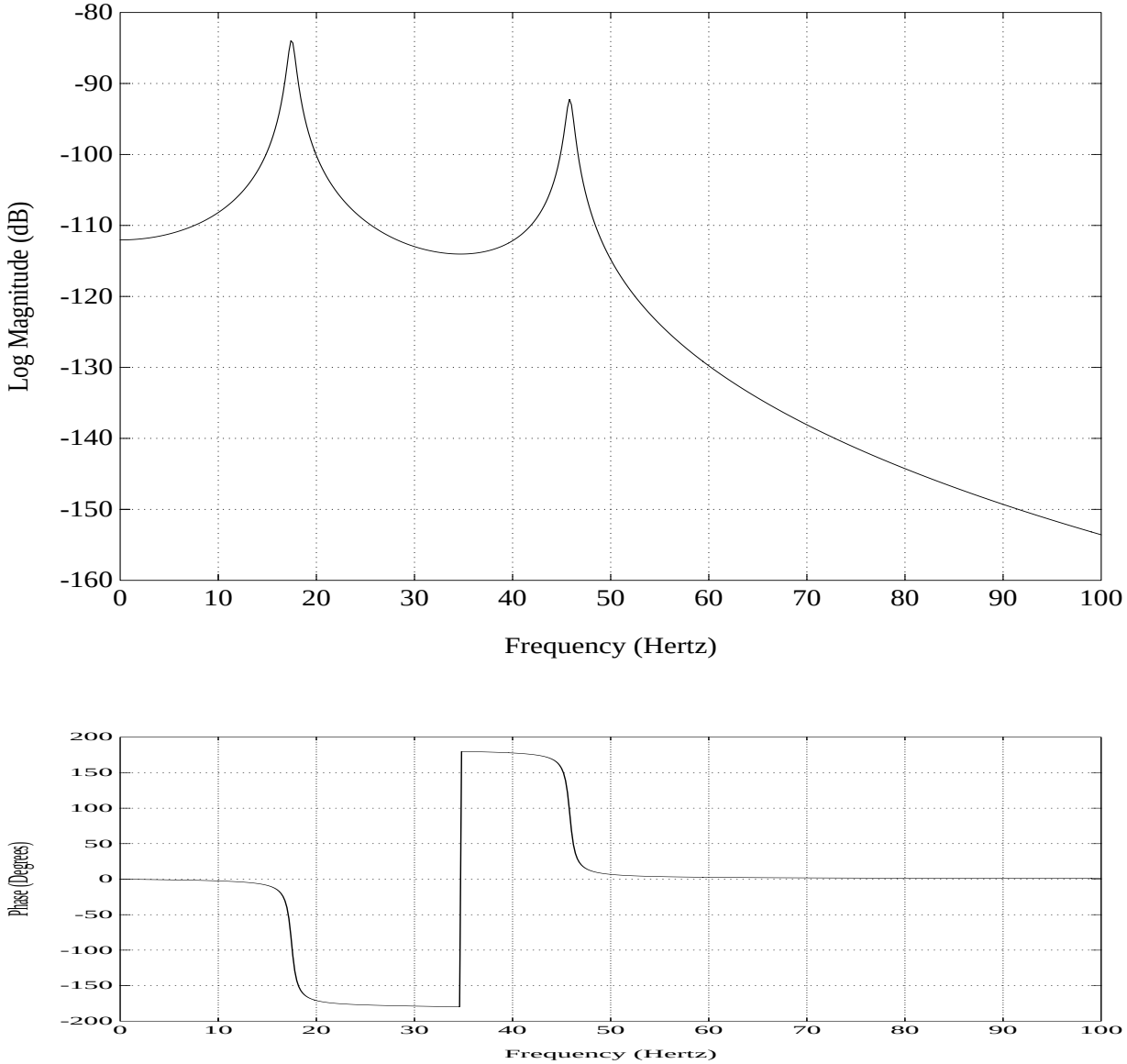

Figure A-4. Frequency Response Function Measurement Format

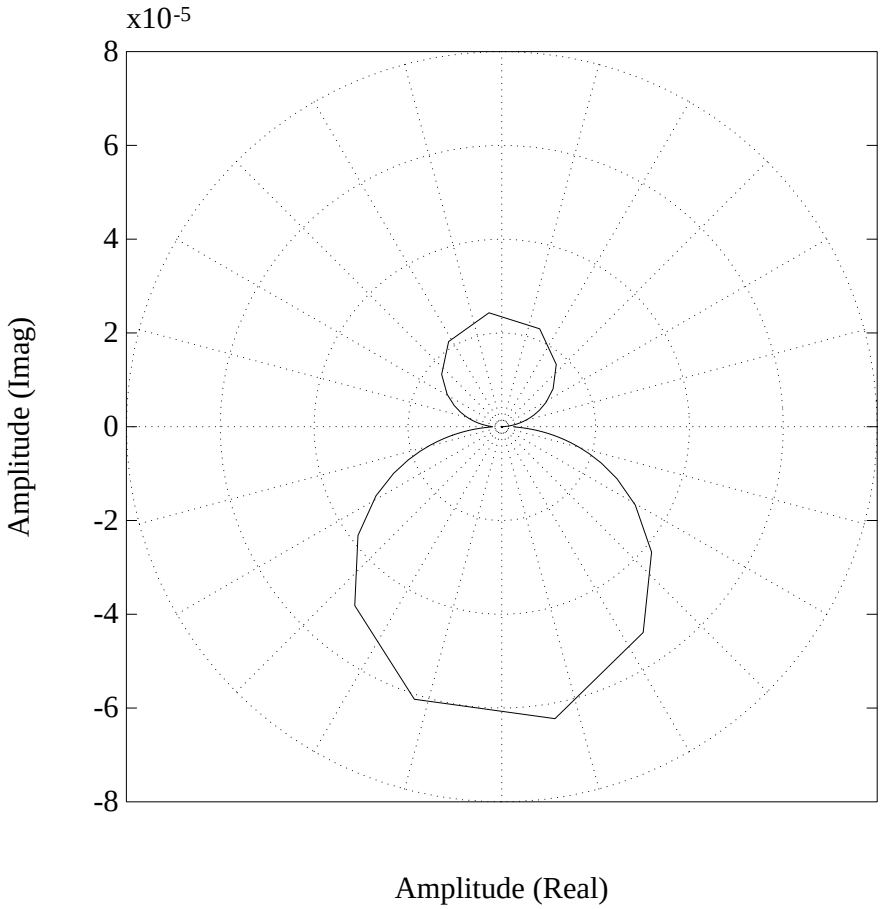

Figure A-5. Frequency Response Function Measurement Format

Appendix B: MATLAB Script Files

The following script files have been used to generate the solutions and plots for the examples shown in the text.

```
%     VIBS2 brings up a menu of the available exercises designed
%       for the Vibrations II course at the University of
%       Cincinnati.
%
%***********************************************************************
% Author: Randall J. Allemang
% Date:     18-Apr-94
% Structural Dynamics Research Lab
% University of Cincinnati
% Cincinnati, Ohio  45221-0072
% TEL:  513-556-2725
% FAX:  513-556-3390
% E-MAIL: randy.allemang@uc.edu
%***********************************************************************
%

echo off
clear
while 1
      vibs2= ['v2_001'
                'v2_002'
                'v2_009'
                'v2_011'
                'v2_015'
                'v2_016'
                'v2_017'
                'v2_020'
                'v2_052'
                'v2_055'
                'v2_056'
                'v2_057'
                'v2_090'
                'v2_095'
                'v2_099'];
      clc
      help v2list
      n = input('Select a VIBS2 program number: ');
      if ((n <= 0) | (n > 17))
                clear
            break
      end
      vibs2 = vibs2(n,:);
      eval(vibs2)
      clear
end
clc
```

```
%
%        Vibrations 2 Scripts
%
%        1)  v2_001   - SDOF System
%        2)  v2_002   - SDOF System, Under-Critical-Over Damped
%        3)  v2_009   - SDOF System, Transfer Function (3-D)
%        4)  v2_011   - SDOF System
%        5)  v2_015   - Changing Stiffness, SDOF System
%        6)  v2_016   - Changing Damping, SDOF System
%        7)  v2_017   - Changing Mass, SDOF System
%        8)  v2_020   - SDOF System
%        9)  v2_052   - SDOF System
%        10) v2_055   - 2DOF System-Proportional Damping
%        11) v2_056   - 2DOF System-Non-Proportional Damping
%        12) v2_057   - 2DOF System-Different Plot Formats
%        13) v2_090   - MDOF(3) System
%        14) v2_095   - MDOF(2-3) Addition of MKC Subsystem
%        15) v2_099   - MDOF(3) System, FRF by Inversion
%
%        0)  EXIT
%
%
```

```
% v2_001.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units.  The output includes
% poles, residues (modal coefficients) and both
% time and frequency domain plots of the impulse
% frequency response functions.

%*********************************************************************
% Author: Randall J. Allemang
% Date:     18-Apr-94
% Structural Dynamics Research Lab
% University of Cincinnati
% Cincinnati, Ohio  45221-0072
% TEL:  513-556-2725
% FAX:  513-556-3390
% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clg, clear
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
mass=input('Mass value: (10)');if isempty(mass), mass=10;end;
stiff=input('Stiffness value: (16000)');if isempty(stiff),stiff=16000;end;
damp=input('Damping value: (10)'); if isempty(damp),damp=10;end;
f=linspace(0,150,500);
H=1.0./((-mass.*f.*f+stiff)+j.*damp.*f);
plot(f,real(H))
pause
plot(f,imag(H))
pause
clear f,H;
pause
a=[mass,damp,stiff]
b=[0,0,1]
[r,p,k]=residue(b,a)
residue = r(1)
lambda=p(1)
whos
pause(2)
t=linspace(0,5,500);
xt=residue./2*exp(lambda.*t) + residue'./2*exp(lambda'.*t);
plot(t,xt)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -deps v2_001a.eps, end;
pause
clear t,xt;
f=linspace(0,150,500);
xf=residue./(j.*f-lambda) + residue'./(j.*f-lambda');
plot(f,abs(xf))
xlabel('Frequency (Hz)'),ylabel('Magnitude'),grid
if plt==1,print -deps v2_001b.eps, end;
pause
semilogy(f,abs(xf))
xlabel('Frequency (Hz)'),ylabel('Log Magnitude'),grid
```

```
if plt==1,print -deps v2_001c.eps, end;
pause
plot(f,360./(2.*pi).*angle(xf))
xlabel('Frequency (Hz)'),ylabel('Phase (Deg)'),grid
if plt==1,print -deps v2_001d.eps, end;
pause
plot(f,360./(2.*pi).*angle(xf))
xlabel('Frequency (Hz)'),ylabel('Phase (Deg)'),grid
if plt==1,print -deps v2_001e.eps, end;
pause
plot(f,real(xf))
xlabel('Frequency (Hz)'),ylabel('Real'),grid
if plt==1,print -deps v2_001f.eps, end;
pause
plot(f,imag(xf))
xlabel('Frequency (Hz)'),ylabel('Imaginary'),grid
if plt==1,print -deps v2_001g.eps, end;
```

```
% v2_002.m
%
% This is a script file to illustrate the response
% of an underdamped, critically damped and overdamped
% SDOF system to an initial displacement.
%
% SDOF System
% Underdamped, critically damped overdamped response
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

%*********************************************************************
% Author: Randall J. Allemang
% Date:     18-Apr-94
% Structural Dynamics Research Lab
% University of Cincinnati
% Cincinnati, Ohio  45221-0072
% TEL:  513-556-2725
% FAX:  513-556-3390
% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end
mag=10;
V=[-15,15,-15,15];
%
%     Overdamped Solution
%
lambda1=-2;
lambda2=-8;
t=linspace(0,4.0,201);
x=mag.*exp(lambda1.*t)+mag.*exp(lambda2.*t);
plot(t,x)
xlabel('Time(sec.)'),ylabel('Displacement'),grid
title('Overdamped SDOF System')
if plt==1,print -deps v2_002a.eps, end;
pause
clg
axis(V);
plot(real(lambda1),imag(lambda1),'*',real(lambda2),imag(lambda2),'*')
axis;
xlabel('Real Part of Complex Frequency')
ylabel('Imaginary Part of Complex Frequency'),grid
title('Overdamped SDOF System')
if plt==1,print -deps v2_002b.eps, end;
pause
clear x
lambda1=-5;
lambda2=-5;
x=mag.*exp(lambda1.*t)+mag.*exp(lambda2.*t);
plot(t,x)
xlabel('Time(sec.)'),ylabel('Magnitude'),grid
title('Critically Damped SDOF System')
if plt==1,print -deps v2_002c.eps, end;
pause
```

```
clg
axis(V);
plot(real(lambda1),imag(lambda1),'*',real(lambda2),imag(lambda2),'*')
axis;
xlabel('Real Part of Complex Frequency')
ylabel('Imaginary Part of Complex Frequency'),grid
title('Critically Damped SDOF System')
if plt==1,print -deps v2_002d.eps, end;
pause
clg
clear x
lambda1=-5+j*10;
lambda2=-5-j*10;
x=mag.*exp(lambda1.*t)+mag.*exp(lambda2.*t);
plot(t,x)
xlabel('Time(sec.)'),ylabel('Magnitude'),grid
title('Underdamped SDOF System')
if plt==1,print -deps v2_002e.eps, end;
pause
clg
axis(V);
plot(real(lambda1),imag(lambda1),'*',real(lambda2),imag(lambda2),'*')
axis;
xlabel('Real Part of Complex Frequency')
ylabel('Imaginary Part of Complex Frequency'),grid
title('Underdamped SDOF System')
if plt==1,print -deps v2_002f.eps, end;
pause
```

```
% v2_009.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units.  The output is a three
% dimensional plot in the s domain (complex independent
% variable.
%
% SDOF System, Laplace Domain (3D) plot
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

%*********************************************************************
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% FAX:  513-556-3390
% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end
mass=10;
a=input('Real Part of Pole: (-0.0625)'); if isempty(a),a=-0.0625;end
b=input('Imaginary Part of Pole: (0.51)'); if isempty(b),b=0.51;end
rot=input('Rotation View Angle: (60)'); if isempty(rot),rot=60;end
lambda(1)=a+j*b;
lambda(2)=a-j*b;
%
%        Set up mesh for only quadrants two and three
%
[sigma,omega]=meshdom(-0.2:0.005:0,-1:0.02:1);
s=sigma+j*omega;
H=(1.0./mass).*(1.0./((s-lambda(1)).*(s-lambda(2))));
view=[rot,30];
mesh(real(H),view)
%     title('Transfer Function (SDOF):  Real Part')
if plt==1,print -f1 -deps v2_009a,end;
pause
mesh(imag(H),view)
%     title('Transfer Function (SDOF):  Imaginary Part')
if plt==1,print -f1 -deps v2_009b,end;
pause
mesh(abs(H),view)
%     title('Transfer Function (SDOF):  Magnitude')
if plt==1,print -f1 -deps v2_009c,end;
pause
mesh(angle(H),view)
%     title('Transfer Function (SDOF):  Phase')
if plt==1,print -f1 -deps v2_009d,end;
pause
mesh(log(abs(H)),view)
%     title('Transfer Function (SDOF):  Log Magnitude')
```

```
if plt==1,print -f1 -deps v2_009e,end;
```

```
% v2_011.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units.  The output includes
% poles, residues (modal coefficients) and both
% time and frequency domain plots of the impulse
% frequency response functions.
%
% SDOF System
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

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% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
mass=input('Mass value:  (10)'); if isempty(mass),mass=10;end
stiff=input('Stiffness value:  (4)'); if isempty(stiff),stiff=4;end
damp=input('Damping value:  (2)'); if isempty(damp),damp=2;end
mass,stiff,damp
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu=r(1);
lambda=p(1);
t=linspace(0,60,500);
xt=residu./2*exp(lambda.*t) + residu'./2*exp(lambda'.*t);
axis([0,60,-0.1,0.1])
plot(t,xt)
xlabel('Time (Sec.)'),ylabel ('Amplitude'),grid
if plt==1,print -f1 -deps v2_011a,end;
pause
clg
clear t,xt;
f=linspace(0,3,500);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
axis([0,3,-1.0,1.0])
plot(f,real(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_011b,end;
pause
clg
plot(f,imag(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_011c,end;
axis([1,2,3,4]);axis;
```

```
% v2_015.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units when the stiffness is varied.
% The output includes
% poles, residues (modal coefficients) and
% frequency domain plots of the
% frequency response functions.
%
% SDOF System, Change of Stiffness
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

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%*********************************************************************
%
clg,clear
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
pi=3.14159;
mass=input('Mass value: (10)');if isempty(mass),mass=10;end
istiff=input('Initial Stiffness value: (1000)');if isempty(istiff),istiff=1000;end
damp=input('Damping value: (10)');if isempty(damp),damp=10;end
f=linspace(0,50,300);
hold off
for i=1:10
stiff=istiff + (i-1).*200
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu= r(1);
lambda=p(1);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
semilogy(f,abs(xf))
hold on
end
xlabel('Frequency (Rad/Sec)'),ylabel('Log Magnitude'),grid
if plt==1,print -f1 -deps v2_015a,end;
pause
hold off
clg;
for i=1:10
stiff=istiff + (i-1).*200
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu= r(1);
lambda=p(1);
```

```
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
scale=360.0/(2.0*pi);
plot(f,scale.*angle(xf))
hold on
end
xlabel('Frequency (Rad/Sec)'),ylabel('Phase (Deg)'),grid
if plt==1,print -f1 -deps v2_015b,end;
hold off
```

```
% v2_016.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units when the damping is varied.
% The output includes
% poles, residues (modal coefficients) and
% frequency domain plots of the
% frequency response functions.
%
% SDOF System, Change of Damping
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

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%*********************************************************************
%
clg,clear
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
pi=3.14159;
mass=input('Mass value: (10)');if isempty(mass),mass=10;end
stiff=input('Stiffness value: (1600)');if isempty(stiff),stiff=1600;end
idamp=input('Initial Damping value: (2)');if isempty(idamp),idamp=2;end
f=linspace(0,50,300);
hold off;
for i=1:10
damp=idamp + (i-1).*2
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu= r(1);
lambda=p(1);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
semilogy(f,abs(xf))
hold on;
end
xlabel('Frequency (Rad/Sec)'),ylabel('Log Magnitude'),grid
title('Effect of Changing Damping')
if plt==1,print -f1 -deps v2_016a,end;
pause
hold off;
clg;
for i=1:10
damp=idamp + (i-1).*2
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu= r(1);
```

```
lambda=p(1);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
scale=360.0/(2.0*pi);
plot(f,scale.*angle(xf))
hold on;
end
xlabel('Frequency (Rad/Sec)'),ylabel('Phase (Deg)'),grid
if plt==1,print -f1 -deps v2_016b,end;
hold off;
```

```
% v2_017.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units when the mass is varied.
% The output includes
% poles, residues (modal coefficients) and
% frequency domain plots of the
% frequency response functions.
%
% SDOF System, Change of Mass
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

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%*********************************************************************
%
clg,clear
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
pi=3.14159;
imass=input('Initial Mass value: (2)');if isempty(imass),imass=2;end
stiff=input('Stiffness value: (1600)');if isempty(stiff),stiff=1600;end
damp=input('Damping value: (10)');if isempty(damp),damp=10;end
f=linspace(0,50,300);
hold off
for i=1:10
mass=2 + (i-1).*1
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu= r(1);
lambda=p(1);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
semilogy(f,abs(xf))
hold on
end
xlabel('Frequency (Rad/Sec)'),ylabel('Log Magnitude'),grid
if plt==1,print -f1 -deps v2_017a,end;
pause
hold off
for i=1:10
mass=2 + (i-1).*1
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu= r(1);
lambda=p(1);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
```

```
scale=360.0/(2.0*pi);
plot(f,scale.*angle(xf))
hold on
end
xlabel('Frequency (Rad/Sec)'),ylabel('Log Magnitude'),grid
if plt==1,print -f1 -deps v2_017b,end;
hold off
```

```
% v2_020.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units.  The output is a three
% dimensional plot in the s domain (complex independent
% variable.
%
% SDOF System, Laplace Domain (3D) plot
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

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%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end
mass=10;
a=input('Real Part of Pole: (-0.0625)'); if isempty(a),a=-0.0625;end
b=input('Imaginary Part of Pole: (0.51)'); if isempty(b),b=0.51;end
rot=input('Rotation View Angle: (60)'); if isempty(rot),rot=60;end
lambda(1)=a+j*b;
lambda(2)=a-j*b;
%
%        Set up mesh for only quadrants two and three
%
[sigma,omega]=meshdom(-0.2:0.005:0,-1:0.02:1);
s=sigma+j*omega;
H=(1.0./mass).*(1.0./((s-lambda(1)).*(s-lambda(2))));
view=[rot,30];
mesh(real(H),view)
%     title('Transfer Function (SDOF):  Real Part')
if plt==1,print -f1 -deps v2_020a,end;
pause
mesh(imag(H),view)
%     title('Transfer Function (SDOF):  Imaginary Part')
if plt==1,print -f1 -deps v2_020b,end;
pause
mesh(abs(H),view)
%     title('Transfer Function (SDOF):  Magnitude')
if plt==1,print -f1 -deps v2_020c,end;
pause
mesh(angle(H),view)
%     title('Transfer Function (SDOF):  Phase')
if plt==1,print -f1 -deps v2_020d,end;
pause
mesh(log(abs(H)),view)
%     title('Transfer Function (SDOF):  Log Magnitude')
```

```
if plt==1,print -f1 -deps v2_020e,end;
```

```
%  v2_052.m
%
% This is a script file to solve a sdof system
% given the mass, damping and stiffness terms
% in dimensionless units.  The output includes
% poles, residues (modal coefficients) and both
% time and frequency domain plots of the impulse
% frequency response functions.
%
% SDOF Damped System
% Figures for UC-SDRL-CN-20-263-662, Chapter 2

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%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
mass=input('Mass value: (10)'); if isempty(mass), mass=10;end;
stiff=input('Stiffness value: (4)'); if isempty(stiff),stiff=4;end;
damp=input('Damping value: (2)'); if isempty(damp),damp=2;end;
a=[mass,damp,stiff];
b=[0,0,1];
[r,p,k]=residue(b,a);
residu = r(1)
lambda=p(1)
t=linspace(0,60,500);
xt=residu./2*exp(lambda.*t) + residu'./2*exp(lambda'.*t);
axis([0,60,-0.1,0.1])
plot(t,xt)
xlabel('Time (Sec.)'),ylabel ('Amplitude'),grid
if plt==1,print -f1 -deps v2_052a,end;
pause
clg; clear t; clear xt;
f=linspace(0,3,500);
xf=residu./(j.*f-lambda) + residu'./(j.*f-lambda');
axis([0,3,-1.0,1.0])
plot(f,real(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_052b,end;
pause
clg
plot(f,imag(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_052c,end;
axis([1,2,3,4]);axis;
```

```
% v2_055.m
%
% This is a script file to solve a 2 DOF system
% given the mass, damping and stiffness matrices
% in dimensionless units.  The output includes poles,
% residues(modal coefficients) and time and frequency
% domain plots of impulse and frequency response
% functions.
%
% 2 DOF Proportionally Damped System
% Figures for UC-SDRL-CN-20-263-662, Chapter 5

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% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
pi=3.14159265;
alpha=0.0
beta=0.5
mass=[5,0;0,10];
stiff=[4,-2;-2,6];
damp=alpha*mass + beta*stiff;
null=[0,0;0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(-inv(a)*b);
% Sort Modal Frequencies
orig_lambda=diag(d);
[Y,I]=sort(imag(orig_lambda));
lambda=orig_lambda(I);
xx=x(:,I);
%     Normalize x matrix
for ii=1:4
xx(1:4,ii)=xx(1:4,ii)./xx(3,ii);
end
%     Extract modal vectors from state-space formulation
psi(1:2,1)=xx(3:4,1);
psi(1:2,2)=xx(3:4,2);
%      Calculate modal mass matrix
mm=psi.'*mass*psi;
%      Calculate modal scaling value (Q)
Q(1)=1./(2*j*imag(lambda(1))*mm(1,1));
Q(2)=1./(2*j*imag(lambda(2))*mm(2,2));
%      Calculate residue matrices
A1=Q(1).*psi(1:2,1)*psi(1:2,1).';
```

```
A2=Q(2).*psi(1:2,2)*psi(1:2,2).';
%      Formulate H(1,2) FRF as Default
resp=input('Enter response (row) DOF: (1)');if isempty(resp),resp=1;end
inp=input('Enter input (column) DOF: (2)');if isempty(inp),inp=2;end
residu(1) = A1(resp,inp);
residu(2) = A2(resp,inp);
A1,A2
pause
magA1=abs(A1);
phaseA1=angle(A1).*360.0./(2.0*pi);
magA2=abs(A2);
phaseA2=angle(A2).*360.0./(2.0*pi);
magA1,phaseA1
magA2,phaseA2
pause
lambda,residu
pause
%      Calculate desired plots
t=linspace(0,30,500);
xt1=residu(1).*exp(lambda(1).*t) + residu(1)'.*exp(lambda(1)'.*t);
xt2=residu(2).*exp(lambda(2).*t) + residu(2)'.*exp(lambda(2)'.*t);
xt=xt1+xt2;
axis([0,30,-0.10,0.10])
plot(t,xt)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -f1 -deps v2_055c,end;
pause
clg
plot(t,xt1)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -f1 -deps v2_055f,end;
pause
clg
plot(t,xt2)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -f1 -deps v2_055i,end;
pause
clg
clear t; clear xt; clear xt1; clear xt2;
f=linspace(0,2.5,500);
xf1=residu(1)./(j.*f-lambda(1)) + residu(1)'./(j.*f-lambda(1)');
xf2=residu(2)./(j.*f-lambda(2)) + residu(2)'./(j.*f-lambda(2)');
xf=xf1+xf2;
axis([0,2.5,-0.6,0.6])
plot(f,real(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_055a,end;
pause
clg
plot(f,imag(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_055b,end;
pause
clg
plot(f,real(xf1))
```

```
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_055d,end;
pause
clg
plot(f,imag(xf1))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_055e,end;
pause
clg
axis([0,2.5,-0.2,0.2])
plot(f,real(xf2))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_055g,end;
pause
clg
plot(f,imag(xf2))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_055h,end;
axis([1 2 3 4]);axis;
```

```
% v2_056.m
%
% This is a script file to solve a 2 DOF system
% given the mass, damping and stiffness matrices
% in dimensionless units.  The output includes poles,
% residues(modal coefficients) and time and frequency
% domain plots of impulse and frequency response
% functions.
%
% 2 DOF Non-Proportionally Damped System
% Figures for UC-SDRL-CN-20-263-662, Chapter 5

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% FAX:  513-556-3390
% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
pi=3.14159265;
mass=[5,0;0,10];
stiff=[4,-2;-2,6];
damp=[6,-4;-4,5];
null=[0,0;0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(-inv(a)*b);
% Sort Modal Frequencies
orig_lambda=diag(d);
[Y,I]=sort(imag(orig_lambda));
lambda=orig_lambda(I);
xx=x(:,I);
xxx=input('Hit any key to continue');
%     Compute 'modal a' and 'modal b' matrix
ma=xx.'*a*xx;
mb=xx.'*b*xx;
%     Extract modal vectors from state-space formulation
psi(1:2,1)=xx(3:4,1);
psi(1:2,2)=xx(3:4,2);
psi(1:2,3)=xx(3:4,3);
psi(1:2,4)=xx(3:4,4);
%      Calculate residue matrices
A1=psi(1:2,1)*psi(1:2,1).'./ma(1,1);
A2=psi(1:2,2)*psi(1:2,2).'./ma(2,2);
A3=psi(1:2,3)*psi(1:2,3).'./ma(3,3);
A4=psi(1:2,4)*psi(1:2,4).'./ma(4,4);
%      Formulate H(1,2) FRF as Default
resp=input('Enter response (row) DOF: (1)');if isempty(resp),resp=1;end;
```

```
inp=input('Enter input (column) DOF: (2)');if isempty(inp),inp=2;end;
residu(1) = A1(resp,inp);
residu(2) = A2(resp,inp);
residu(3) = A3(resp,inp);
residu(4) = A4(resp,inp);
A1,A2,A3,A4
xxx=input('Hit any key to continue');
lambda,residu
xxx=input('Hit any key to continue');
%      Calculate desired plots
t=linspace(0,30,500);
xt1=residu(1).*exp(lambda(1).*t) + residu(2).*exp(lambda(2).*t);
xt2=residu(3).*exp(lambda(3).*t) + residu(4).*exp(lambda(4).*t);
xt=xt1+xt2;
axis([0,30,-0.10,0.10])
plot(t,xt)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -f1 -deps v2_056c,end;
pause
clg
plot(t,xt1)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -f1 -deps v2_056i,end;
pause
clg
plot(t,xt2)
xlabel('Time (Sec.)'),ylabel('Amplitude'),grid
if plt==1,print -f1 -deps v2_056f,end;
pause
clg
clear t; clear xt; clear xt1; clear xt2;
f=linspace(0,2.5,500);
xf1=residu(1)./(j.*f-lambda(1)) + residu(2)./(j.*f-lambda(2));
xf2=residu(3)./(j.*f-lambda(3)) + residu(4)./(j.*f-lambda(4));
xf=xf1+xf2;
axis([0,2.5,-0.6,0.6])
plot(f,real(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_056a,end;
pause
clg
plot(f,imag(xf))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_056b,end;
pause
clg
plot(f,real(xf1))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_056g,end;
pause
clg
plot(f,imag(xf1))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_056h,end;
pause
```

```
clg
axis([0,2.5,-0.6,0.6])
plot(f,real(xf2))
xlabel('Frequency (Rad/Sec)'),ylabel('Real'),grid
if plt==1,print -f1 -deps v2_056d,end;
pause
clg
plot(f,imag(xf2))
xlabel('Frequency (Rad/Sec)'),ylabel('Imaginary'),grid
if plt==1,print -f1 -deps v2_056e,end;
axis([1 2 3 4]);axis;
```

```
% v2_057.m
%
% This is a script file to solve a 2 DOF system
% given the mass, damping and stiffness matrices
% in dimensionless units and plot any desired
% frequency response function.
%
% Plot Formats for FRF and IRF Measurements
% Figures for UC-SDRL-CN-20-263-662, Appendix A

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% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
pi=3.14159265;
axis('normal');
axis([1 2 3 4]),axis;
pi=3.14159265;
alpha=5;
mass=[8,0;0,10];
stiff=[600000,-200000;-200000,200000];
damp=alpha.*mass;
null=[0,0;0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(b,-a);
% Sort Modal Frequencies
orig_lambda=diag(d);
[Y,I]=sort(imag(orig_lambda));
lambda=orig_lambda(I);
xx=x(:,I);
xx,lambda
xxx=input('Hit any key to continue');
%     Compute 'modal a' and 'modal b' matrix
ma=xx.'*a*xx;
mb=xx.'*b*xx;
%     Extract modal vectors from state-space formulation
psi(1:2,1)=xx(3:4,1);
psi(1:2,2)=xx(3:4,2);
psi(1:2,3)=xx(3:4,3);
psi(1:2,4)=xx(3:4,4);
%      Calculate residue matrices
A1=psi(1:2,1)*psi(1:2,1).'./ma(1,1);
A2=psi(1:2,2)*psi(1:2,2).'./ma(2,2);
A3=psi(1:2,3)*psi(1:2,3).'./ma(3,3);
```

```
A4=psi(1:2,4)*psi(1:2,4).'./ma(4,4);
resp=input('Enter response (row) DOF: (1)');if isempty(resp),resp=1;end
inp=input('Enter input (column) DOF: (2)');if isempty(inp),inp=2;end
residu(1) = A1(resp,inp);
residu(2) = A2(resp,inp);
residu(3) = A3(resp,inp);
residu(4) = A4(resp,inp);
xxx=input('Hit any key to continue');
lambda,residu
xxx=input('Hit any key to continue');
%      Calculate desired plots
t=linspace(0,1,501);
xt1=residu(1)./2*exp(lambda(1).*t);
xt2=residu(2)./2*exp(lambda(2).*t);
xt3=residu(3)./2*exp(lambda(3).*t);
xt4=residu(4)./2*exp(lambda(4).*t);
xt=xt1+xt2+xt3+xt4;
plot(t,xt)
%       Reset plot axis for same + and - limits
V=axis;
ymax=max([abs(V(3)),abs(V(4))]);V(3)=-ymax;V(4)=+ymax;axis(V);
%
plot(t,xt)
xlabel('Time (Sec.)'),ylabel ('Amplitude'),grid
if plt==1,meta v2_057a,end;
pause
axis([1 2 3 4]),axis;
clear t; clear xt;
f=linspace(0,100,501);
om=f.*2.*pi;
H1=residu(1)./(j.*om-lambda(1));
H2=residu(2)./(j.*om-lambda(2));
H3=residu(3)./(j.*om-lambda(3));
H4=residu(4)./(j.*om-lambda(4));
H=H1+H2+H3+H4;
plot(f,real(H))
V=axis;
ymax=max([abs(V(3)),abs(V(4))]);V(3)=-ymax;V(4)=+ymax;axis(V);
plot(f,real(H))
xlabel('Frequency (Hertz)'),ylabel('Amplitude (Real)'),grid
if plt==1,meta v2_057b,end;
pause
axis([1 2 3 4]),axis;
plot(f,imag(H))
V=axis;
ymax=max([abs(V(3)),abs(V(4))]);V(3)=-ymax;V(4)=+ymax;axis(V);
plot(f,imag(H))
xlabel('Frequency (Hertz)'),ylabel('Amplitude (Imag)'),grid
if plt==1,meta v2_057c,end;
pause
axis([1 2 3 4]),axis;
plot(f,abs(H))
xlabel('Frequency (Hertz)'),ylabel('Magnitude'),grid
if plt==1,meta v2_057d,end;
pause
```

```
axis([1 2 3 4]),axis;
plot(f,20.*log10(abs(H)))
xlabel('Frequency (Hertz)'),ylabel('Log Magnitude (dB)'),grid
if plt==1,meta v2_057e,end;
pause
plot(f,360./(2.*pi).*angle(H))
V=axis;
ymax=max([abs(V(3)),abs(V(4))]);V(3)=-ymax;V(4)=+ymax;axis(V);
plot(f,360./(2.*pi).*angle(H))
xlabel('Frequency (Hertz)'),ylabel('Phase (Degrees)'),grid
if plt==1,meta v2_057f,end;
pause
axis([1 2 3 4]),axis;
axis('square')
polar(angle(H),abs(H))
xlabel('Amplitude (Real)'),ylabel('Amplitude (Imag)'),grid
if plt==1,meta v2_057g,end;
```

```
% v2_090.m
%
% This is a script file to solve a 3 DOF system
% given the mass, damping and stiffness matrices
% in dimensionless units and plot any desired
% frequency response function.

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% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clg
pi=3.14159265;
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
mass=[10,0,0;0,14,0;0,0,12];
stiff=[5000,-3000,0;-3000,5500,-2500;0,-2500,2500];
damp=[50,-30,0;-30,55,-25;0,-25,25];
null=[0,0,0;0,0,0;0,0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(b,-a);
% Sort Modal Frequencies
orig_lambda=diag(d);
[Y,I]=sort(imag(orig_lambda));
lambda=orig_lambda(I);
xx=x(:,I);
%     Normalize x matrix to real vectors if possible
for ii=1:6
xx(1:6,ii)=xx(1:6,ii)./xx(4,ii);
end
%     Compute 'modal a' and 'modal b' matrix
ma=xx.'*a*xx;
mb=xx.'*b*xx;
%     Extract modal vectors from state-space formulation
psi(1:3,1)=xx(4:6,1);
psi(1:3,2)=xx(4:6,2);
psi(1:3,3)=xx(4:6,3);
psi(1:3,4)=xx(4:6,4);
psi(1:3,5)=xx(4:6,5);
psi(1:3,6)=xx(4:6,6);
lambda
xxx=input('Hit any key to continue');
psi
xxx=input('Hit any key to continue');
ma
xxx=input('Hit any key to continue');
%      Calculate residue matrices
```

```
A1=psi(1:3,1)*psi(1:3,1).'./ma(1,1);
A2=psi(1:3,2)*psi(1:3,2).'./ma(2,2);
A3=psi(1:3,3)*psi(1:3,3).'./ma(3,3);
A4=psi(1:3,4)*psi(1:3,4).'./ma(4,4);
A5=psi(1:3,5)*psi(1:3,5).'./ma(5,5);
A6=psi(1:3,6)*psi(1:3,6).'./ma(6,6);
resp=input('Enter response (row) DOF: (1)');if isempty(resp),resp=1;end
inp=input('Enter input (column) DOF: (1)');if isempty(inp),inp=1;end
residu(1) = A1(resp,inp);
residu(2) = A2(resp,inp);
residu(3) = A3(resp,inp);
residu(4) = A4(resp,inp);
residu(5) = A5(resp,inp);
residu(6) = A6(resp,inp);
%      Calculate desired plots
om=linspace(0,50,501);
H1=residu(1)./(j.*om-lambda(1));
H2=residu(2)./(j.*om-lambda(2));
H3=residu(3)./(j.*om-lambda(3));
H4=residu(4)./(j.*om-lambda(4));
H5=residu(5)./(j.*om-lambda(5));
H6=residu(6)./(j.*om-lambda(6));
H=H1+H2+H3+H4+H5+H6;
om(151),H(151)
om(251),H(251)
xxx=input('Hit any key to continue');
subplot(211),semilogy(om,abs(H),om(151),abs(H(151)),'*',om(251),abs(H(251)),'*')
xlabel('Frequency (Rad/Sec)'),ylabel('Magnitude'),grid
subplot(212),plot(om,angle(H),om(151),angle(H(151)),'*',om(251),angle(H(251)),'*')
xlabel('Frequency (Rad/Sec)'),ylabel('Phase'),grid
title(['FRF   Input: ',num2str(inp),'   Response: ',num2str(resp)])
pause
if plt==1,print -f1 -deps v2_090a,end;
```

```
% v2_095.m
%
% This is a script file to solve a 2 DOF system
% given the mass, damping and stiffness matrices
% in dimensionless units and plot any desired
% frequency response function.  A third DOF is
% then added to act as a spring-mass-damper to
% reduce the magnitude of the first mode of the
% original system.  The 3 DOF system is then
% solved and any frequency response functions
% can be plotted.

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% FAX:  513-556-3390
% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear, clg
pi=3.14159265;
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
alpha=0.001
mass=[8,0;0,10];
stiff=[6000,-2000;-2000,2000];
damp=alpha.*stiff;
null=[0,0;0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(b,-a);
% Sort Modal Frequencies
orig_lambda=diag(d);
[Y,I]=sort(imag(orig_lambda));
lambda=orig_lambda(I);
xx=x(:,I);
%     Normalize x matrix to real vectors if possible
for ii=1:4
xx(1:4,ii)=xx(1:4,ii)./xx(3,ii);
end
%     Compute 'modal a' and 'modal b' matrix
ma=xx.'*a*xx;
mb=xx.'*b*xx;
%     Extract modal vectors from state-space formulation
psi(1:2,1)=xx(3:4,1);
psi(1:2,2)=xx(3:4,2);
psi(1:2,3)=xx(3:4,3);
psi(1:2,4)=xx(3:4,4);
lambda
xxx=input('Hit any key to continue');
psi
```

```
xxx=input('Hit any key to continue');
ma
xxx=input('Hit any key to continue');
%      Calculate residue matrices
A1=psi(1:2,1)*psi(1:2,1).'./ma(1,1);
A2=psi(1:2,2)*psi(1:2,2).'./ma(2,2);
A3=psi(1:2,3)*psi(1:2,3).'./ma(3,3);
A4=psi(1:2,4)*psi(1:2,4).'./ma(4,4);
resp=input('Enter response (row) DOF: (1)');if isempty(resp),resp=1;end
inp=input('Enter input (column) DOF: (1)');if isempty(inp),inp=1;end
residu(1) = A1(resp,inp);
residu(2) = A2(resp,inp);
residu(3) = A3(resp,inp);
residu(4) = A4(resp,inp);
A1,A2,A3,A4
xxx=input('Hit any key to continue');
lambda,residu
xxx=input('Hit any key to continue');
%      Calculate desired plots
om=linspace(0,50,500);
H1=residu(1)./(j.*om-lambda(1));
H2=residu(2)./(j.*om-lambda(2));
H3=residu(3)./(j.*om-lambda(3));
H4=residu(4)./(j.*om-lambda(4));
HH=H1+H2+H3+H4;
fig1=figure(1);
subplot(211),semilogy(om,abs(HH))
xlabel('Frequency (Rad/Sec)'),ylabel('Magnitude'),grid
title('Frequency Response Function')
subplot(212),plot(om,angle(HH))
xlabel('Frequency (Rad/Sec)'),ylabel('Phase'),grid
pause
if plt==1,print -f1 -deps v2_095a,end;
%
% now solve 3 dof system (with spring-mass-damper added)
%
pi=3.14159265;
alpha=0.001
mass=[8,0,0;0,10,0;0,0,2.50];
stiff=[6000,-2000,0;-2000,2300,-300;0,-300,300];
damp=alpha.*stiff;
null=[0,0,0;0,0,0;0,0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(b,-a);
x,d
xxx=input('Hit any key to continue');
%     Pick frequency roots
lambda=diag(d);
%     Normalize x matrix to real vectors if possible
for ii=1:6
x(1:6,ii)=x(1:6,ii)./x(4,ii);
end
%     Compute 'modal a' and 'modal b' matrix
```

```
ma=x.'*a*x;
mb=x.'*b*x;
%     Extract modal vectors from state-space formulation
psi(1:3,1)=x(4:6,1);
psi(1:3,2)=x(4:6,2);
psi(1:3,3)=x(4:6,3);
psi(1:3,4)=x(4:6,4);
psi(1:3,5)=x(4:6,5);
psi(1:3,6)=x(4:6,6);
%      Calculate residue matrices
A1=psi(1:3,1)*psi(1:3,1).'./ma(1,1);
A2=psi(1:3,2)*psi(1:3,2).'./ma(2,2);
A3=psi(1:3,3)*psi(1:3,3).'./ma(3,3);
A4=psi(1:3,4)*psi(1:3,4).'./ma(4,4);
A5=psi(1:3,5)*psi(1:3,5).'./ma(5,5);
A6=psi(1:3,6)*psi(1:3,6).'./ma(6,6);
residu(1) = A1(resp,inp);
residu(2) = A2(resp,inp);
residu(3) = A3(resp,inp);
residu(4) = A4(resp,inp);
residu(5) = A5(resp,inp);
residu(6) = A6(resp,inp);
A1,A2,A3,A4,A5,A6
xxx=input('Hit any key to continue');
lambda,residu
xxx=input('Hit any key to continue');
%      Calculate desired plots
om=linspace(0,50,500);
H1=residu(1)./(j.*om-lambda(1));
H2=residu(2)./(j.*om-lambda(2));
H3=residu(3)./(j.*om-lambda(3));
H4=residu(4)./(j.*om-lambda(4));
H5=residu(5)./(j.*om-lambda(5));
H6=residu(6)./(j.*om-lambda(6));
H=H1+H2+H3+H4+H5+H6;
fig2=figure(2);
subplot(211),semilogy(om,abs(H),om,abs(HH))
xlabel('Frequency (Rad/Sec)'),ylabel('Magnitude'),grid
title('Frequency Response Function')
subplot(212),plot(om,angle(H),om,angle(HH))
xlabel('Frequency (Rad/Sec)'),ylabel('Phase'),grid
pause
if plt==1,print -f2 -deps v2_095b,end;
% Now compute final H22 using impedance method
M3=2.5;
C3=.3;
K3=300;
H33=-M3.*om.*om+j.*C3.*om+K3;
H33=H33./((-M3.*om.*om).*(K3+j.*om.*C3));
HHH=HH.*H33./(HH+H33);
fig3=figure(3);
subplot(211),semilogy(om,abs(H),om,abs(HHH))
xlabel('Frequency (Rad/Sec)'),ylabel('Magnitude'),grid
title('Frequency Response Function')
subplot(212),plot(om,angle(H),om,angle(HHH))
```

```
xlabel('Frequency (Rad/Sec)'),ylabel('Phase'),grid
pause
if plt==1,print -f3 -deps v2_095c,end;
```

```
% v2_099.m
%
% This is a script file to solve a 3 DOF system
% given the mass, damping and stiffness matrices
% in dimensionless units and plot any desired
% frequency response function.
%
% Solution for FRF by Inverse of [B]

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% E-MAIL: randy.allemang@uc.edu
%*********************************************************************
%
clear,clf,close all;
pi=3.14159265;
plt=input('Store plots to file (Yes=1): (0)');if isempty(plt),plt=0;end;
%
%  solve 3 dof system
%
mass=[10,0,0;0,14,0;0,0,12];
stiff=[5000,-3000,0;-3000,5500,-2500;0,-2500,2500];
damp=[50,-30,0;-30,55,-25;0,-25,25];
null=[0,0,0;0,0,0;0,0,0];
%    Form 2N x 2N state space equation.
a=[null,mass;mass,damp];
b=[-mass,null;null,stiff];
[x,d]=eig(b,-a);
% Sort Modal Frequencies
orig_lambda=diag(d);
[Y,I]=sort(imag(orig_lambda));
lambda=orig_lambda(I);
xx=x(:,I);
%     Normalize x matrix to real vectors if possible
for ii=1:6
xx(1:6,ii)=xx(1:6,ii)./xx(4,ii);
end
%     Compute 'modal a' and 'modal b' matrix
ma=xx.'*a*xx;
mb=xx.'*b*xx;
%     Extract modal vectors from state-space formulation
psi(1:3,1)=xx(4:6,1);
psi(1:3,2)=xx(4:6,2);
psi(1:3,3)=xx(4:6,3);
psi(1:3,4)=xx(4:6,4);
psi(1:3,5)=xx(4:6,5);
psi(1:3,6)=xx(4:6,6);
lambda
xxx=input('Hit any key to continue');
```

```
psi
xxx=input('Hit any key to continue');
ma
xxx=input('Hit any key to continue');
clear a,clear b,clear x,clear d;

omega=linspace(0,100,501);
for ii=1:501;
FRF(:,:,ii)=inv(-mass.*omega(ii).*omega(ii) + damp.*j.*omega(ii) + stiff);
end;

for Ni=1:3;
for No=1:3;
H=squeeze(FRF(No,Ni,:));
figure;
subplot(211),semilogy(omega,abs(H))
xlabel('Frequency (Rad/Sec)'),ylabel('Magnitude'),grid
scale=360.0/(2.0*pi);
subplot(212),plot(omega,scale.*angle(H))
xlabel('Frequency (Rad/Sec)'),ylabel('Phase'),grid
title('Frequency Response Function: H(x,x)');
pause
end
end
if plt==1,print -f1 -deps v2_099a,end;
pause
```