LECTURE 6: Modal Analysis for Forced Vibration

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{P\}$$

$\{P\}$ → force vector that varies w/ time.

→ Using mass normalized mode shapes, the forcing function can be brought into normal coordinates as:

$$P_i = \{\phi_{N_i}\}^T \{P\}$$

P_i → forcing function associated with mode i in mass normal coord.

→ Writing the EOM for mode i.

$$\ddot{z}_i + 2\xi_i \omega_i \dot{z}_i + \omega_i^2 z_i = P_i$$

$M_i^* = 1$ $\quad C_i^* = \xi_i \omega_i$ $\quad K_i^* = \omega_i^2$ $\quad P_i^* = P_i$

→ The solution for mode i can be obtained for any loading using the solutions outlined in CIVE 801

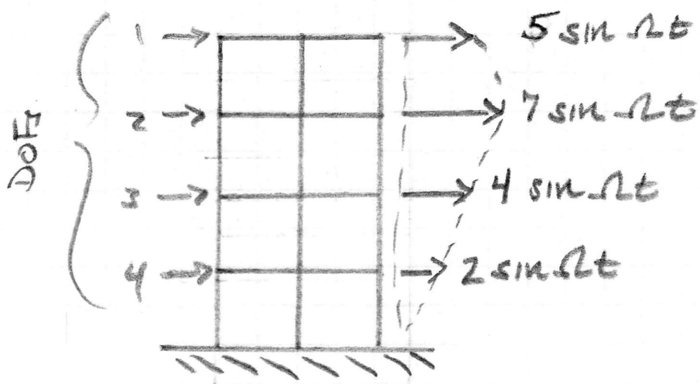
→ <u>Free</u> <u>Vibration</u> <u>Responses</u>.

$$z_i = e^{\xi_i \omega_i t}\left(z_{0_i} \cos(\omega_i t) + \frac{\dot{z}_{0_i} + z_{0_i}\omega_i \xi_i}{\omega_i} \sin(\omega_i t)\right)$$

→ It is important to recognize that not all modes need to be included within an analysis.

↳ Typically only a few modes are required...

⇒ Qualitative Example.



→ It is common to write $\{P\}$ as ...

$$\{P\} = \underbrace{\{f\}}_{\text{spatial distribution of amplitude.}} \; \underbrace{g(t)}_{\text{time varying component.}}$$

$$\{P\} = \underbrace{\begin{Bmatrix} 5 \\ 7 \\ 4 \\ 2 \end{Bmatrix}}_{\{f\}} \overbrace{\sin(\Omega t)}^{g(t)}$$

→ To bring this forcing function into normal coordinates, pre-multiply by ϕ_N^T

$$P_1 = \underbrace{\{\phi_{N,1}\}^T \{f\}}_{\text{known as the participation factor} \Rightarrow \text{related to how much mode 1 is excited by } \{f\}.} \sin(\Omega t)$$

$$\gamma_i = \{\phi_{N_i}\}^T \{f\}$$

participation factor for mode i

=> looking @ steady-state response of mode i

$$z_{i_{ss}} = \frac{\gamma_i}{\omega_i^2} \frac{1}{(1-\beta_i^2)^2 + (2\xi_i \beta_i)^2} \left[(1-\beta^2)\sin(\Omega t) - 2\xi_i \beta_i \cos(\Omega t) \right]$$

=> 2 primary parameters that impact the response of mode i

(1) γ_i => participation factor

=> If the spatial distribution of the load is similar to the mode shape, the response of the mode will be relatively large

=> Conversely, if the spatial dist. of load is orthogonal to the mode shape then.

$$\gamma_i = \{\phi_{N_i}\}^T \{f\} = 0 \Rightarrow z_i = 0$$

No response.

(2) $$A_{d_i} = \frac{1}{(1-\beta_i^2)^2 + (2\xi_i \beta_i)^2}$$ = dynamic amplification factor

=> $A_d = 1.0$ implies static response

=> $\Delta = \frac{P}{K}$

=> This considers the influence of the loading frequency

$\Rightarrow$ For small β_i (i.e. $\omega_i >> \Omega$)

$A_{d_i} \approx 1.0$

$\therefore$ the amplitude of $z_i = \dfrac{\gamma_i}{\omega_i^2} = \underbrace{\dfrac{P_i}{K_i}}_{\text{modal force \& stiffness of mode } i}$

$\Rightarrow$ Stiffness controlled.

$\Rightarrow$ For $\beta = 1.0$ (i.e. $\omega_i = \Omega$)

$A_{d_i} = \left(\dfrac{1}{2\xi_i}\right)$

$\therefore$ the amplitude of $z_i = \dfrac{\gamma_i}{\omega_i^2}\left(\dfrac{1}{2\xi_i}\right)$

$\Rightarrow$ Damping controlled

$\Rightarrow$ For large β (i.e. $\omega_i << \Omega$)

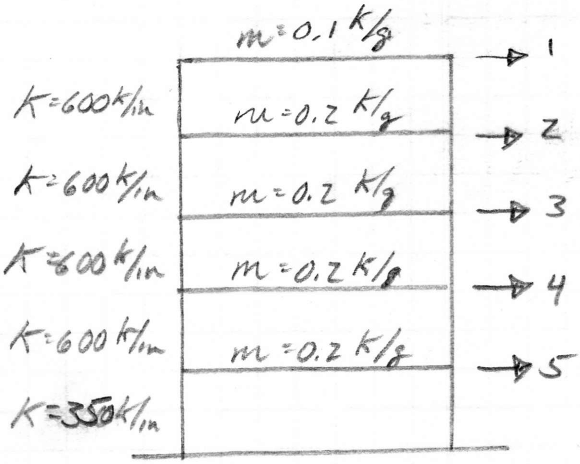
$A_{d_i} \approx \dfrac{1}{\beta^2}$

$\therefore$ the amplitude of $z_i = \dfrac{\gamma_i}{\omega_i^2 \beta_i^2} = \dfrac{\gamma_i}{(1)\Omega^2}$

modal mass of mode i.

$\Rightarrow$ Mass controlled.

Example: Compute the steady-state response of the following structure to loads (1) & (2)



(1) $P(t)=\begin{Bmatrix}30\\30\\30\\30\\15\end{Bmatrix}\sin(25t)$ ↑ $\Omega=25\ \text{rad/s}$

(2) $P(t)=\begin{Bmatrix}0\\0\\0\\30\\30\\30\end{Bmatrix}\sin(45t)$ ↑ $\Omega=45\ \text{rad/s}$

$$[K]=\begin{bmatrix}600 & -600 & 0 & 0 & 0\\ -600 & 1200 & -600 & 0 & 0\\ 0 & -600 & 1200 & -600 & 0\\ 0 & 0 & -600 & 1200 & -600\\ 0 & 0 & 0 & -600 & 950\end{bmatrix}\ \text{kip/in}$$

$$[M]=\begin{bmatrix}0.1 & 0 & 0 & 0 & 0\\ 0 & 0.2 & 0 & 0 & 0\\ 0 & 0 & 0.2 & 0 & 0\\ 0 & 0 & 0 & 0.2 & 0\\ 0 & 0 & 0 & 0 & 0.2\end{bmatrix}\ \text{kip/g}$$

→ From Matlab.

$$\omega=\begin{Bmatrix}15.103\\45.697\\74.252\\96.183\\108.023\end{Bmatrix}\ \frac{\text{rad}}{\text{s}}\qquad [\phi_N]=\begin{bmatrix}1.335 & 1.393 & 1.432 & 1.450 & -1.457\\ 1.284 & 0.908 & 0.116 & -0.786 & 1.377\\ 1.136 & -0.209 & -1.414 & -0.579 & -1.144\\ 0.901 & -1.180 & -0.346 & 1.435 & 0.786\\ 0.598 & -1.330 & 1.358 & -0.956 & -0.341\end{bmatrix}$$

↑ Mass normalized.

→ Assume 5% damping in each mode.

$$\xi=\begin{Bmatrix}0.05\\0.05\\0.05\\0.05\\0.05\\0.05\end{Bmatrix}$$

=> Compute the participation factors for (1)

$$\{f\} = \begin{Bmatrix} 30 \\ 30 \\ 30 \\ 30 \\ 30 \\ 15 \end{Bmatrix}$$

$$\gamma_1 = \{\phi_{N,1}\}^T \{f\}$$

$$\gamma_1 = \begin{bmatrix} 1.335 & 1.284 & 1.136 & 0.901 & 0.598 \end{bmatrix} \begin{Bmatrix} 30 \\ 30 \\ 30 \\ 30 \\ 30 \\ 15 \end{Bmatrix}$$

$$\gamma_1 = 148.65$$

$$\gamma_2 = \{\phi_{N,2}\}^T \{f\}$$

$$\gamma_2 = \begin{bmatrix} 1.393 & 0.908 & -0.209 & -1.180 & -1.330 \end{bmatrix} \begin{Bmatrix} 30 \\ 30 \\ 30 \\ 30 \\ 15 \end{Bmatrix}$$

$$\gamma_2 = 7.41$$

Similarly,

$$\gamma_3 = 14.05$$
$$\gamma_4 = 30.67$$
$$\gamma_5 = -18.28$$

=> Note, the participation factors bring the forcing function into modal space.

e.g. $P_1^* = \gamma_1 \sin(\Omega t)$

=> Compute the steady-state response for each mode.

for $P = \gamma \sin(\Omega t)$

$$z_i = \frac{\gamma_i}{\omega_i^2} \underbrace{\frac{1}{(1-\beta_i^2)^2 - (2\zeta_i\beta_i)^2}}_{A_d} \left[(1-\beta_i^2)\sin(\Omega t) - 2\zeta_i\beta_i\cos(\Omega t)\right]$$

Mode 1:

$$\beta_1 = \frac{25}{15.103} = 1.655$$

$$A_{d_1} = \frac{1}{(1-1.655^2)^2 - (2(0.05)1.655)^2} = 0.334$$

$$z_1 = \frac{148.65}{(15.103)^2}(0.334)\left[(1-1.655^2)\sin(25t) - 2(0.05)(1.655)\cos(25t)\right]$$

* $z_1 = -0.378\sin(25t) - 0.037\cos(25t)$

Mode 2:

$$\beta_2 = \frac{25}{45.7} = 0.547$$

$$A_{d_2} = \frac{1}{(1-0.547^2)^2 + (2(0.05)0.547)^2} = 2.05$$

$$z_2 = \frac{7.41}{(45.7)^2}(2.05)\left[(1-0.547^2)\sin(25t) - 2(0.05)(0.547)\cos(25t)\right]$$

* $z_2 = 0.00509\sin(25t) - 0.000398\cos(25t)$

Similarly,

$$Ad_3 = 1.274$$
$$Ad_4 = 1.151$$
$$Ad_5 = 1.117$$

$$Z_3 = 2.88\times10^{-3}\sin(25t) + 1.09\times10^{-4}\cos(25t)$$
$$Z_4 = 3.558\times10^{-3}\sin(25t) + 9.92\times10^{-5}\cos(25t)$$
$$Z_5 = -1.66\times10^{-3}\sin(25t) - 4.05\times10^{-5}\cos(25t)$$

⇒ Total Response

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \{\phi_{N_1}\} Z_1 + \{\phi_{N_2}\} Z_2 + \{\phi_{N_3}\} Z_3 + \{\phi_{N_4}\} Z_4 + \{\phi_{N_5}\} Z_5$$

Question: What if we only include one or two modes in the response?
What type of errors occur?
Is this also true for load case (2)?
Why or why not?

→ In class example: solve for the response due to load case (2)