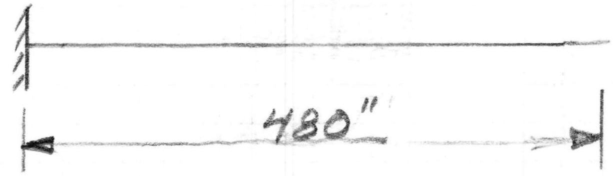
Lecture 3 - Formulation of EOM for MDOF Sys.

(3) Finite Element Approach

→ Assemble global M & K from element M & K...



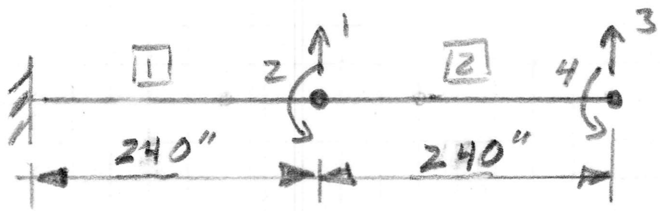
$E = 29{,}000\ ksi$

$I = 1{,}000\ in^4$

$\bar{m} = \dfrac{20 in^2 (490 lb/ft^3)}{386 in/sec^2}\left(\dfrac{1}{12}\right)^3$

$\bar{m} = 0.0146\ lbm/in.$

→ Use 2 elements.



→ Local to Global Mapping.

Element #	Element DOF 1	2	3	4
1	X	X	1	2
2	1	2	3	4

$$[K^G] = \begin{bmatrix} (k_{33}^{[1]} + k_{11}^{[2]}) & (k_{34}^{[1]} + k_{12}^{[2]}) & (k_{13}^{[2]}) & (k_{14}^{[2]}) \\ (k_{43}^{[1]} + k_{21}^{[2]}) & (k_{44}^{[1]} + k_{22}^{[2]}) & (k_{23}^{[2]}) & (k_{24}^{[2]}) \\ (k_{31}^{[2]}) & (k_{32}^{[2]}) & (k_{33}^{[2]}) & (k_{34}^{[2]}) \\ (k_{41}^{[2]}) & (k_{42}^{[2]}) & (k_{43}^{[2]}) & (k_{44}^{[2]}) \end{bmatrix}$$

Plugging in...

$$[K_G] = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

(Note $L = 240\,in$) ↑ e.g. element length.

Similarly, using consistent mass...

$$[M_G] = \frac{\bar{m}L}{420}\begin{bmatrix} 312 & 0 & 54 & -13L \\ 0 & 8L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

($L = 240\,in$)

→ Solve for frequencies and mode shapes.

$$\det\left|[K] - \omega_i^2[M]\right| = 0$$

→ this produces a 4th order equation that must be solved for 4 root (i.e. ω_i^2)

⇒ Use a computational tool like Matlab.

$$\omega = \begin{Bmatrix} 21.5 \\ 135.9 \\ 459.7 \\ 1334.4 \end{Bmatrix} \text{ RAD/SEC}$$

$$\phi = \begin{array}{cccc} \phi_1 & \phi & \phi_3 & \phi_4 \\ \end{array}$$

$$\phi = \begin{bmatrix} 0.3396 & -0.7219 & 0.1017 & 0.2532 \\ 0.0024 & 0.0009 & -0.0159 & 0.0108 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 0.0029 & 0.0101 & 0.0200 & 0.0403 \end{bmatrix}$$

Visual Analysis (240 elements)

$$\omega = \begin{Bmatrix} 21.45 \\ 134.3 \\ 375.8 \\ 736.0 \end{Bmatrix} \text{ rad/s}$$

$$\phi = \begin{array}{cccc} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \end{array}$$

$$\phi = \begin{bmatrix} 0.3395 & -0.7145 & 0.0221 & 0.7117 \\ 0.0024 & 0.0009 & -0.0116 & 0.0002 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 0.0029 & 0.0100 & 0.0164 & 0.0232 \end{bmatrix}$$

→ <u>Alternate Solution</u> ⇒ use <u>lumped mass</u>.

→ $[K_G]$ remains unchanged

→ $[M_G]$ becomes.

$$[M_G] = \frac{\bar{m}L}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Since only dof 1 & 3 have mass associated with them, they are the only two "dynamic" dofs.

→ Problem: Although the mass matrix is only a 2x2 (dofs 1 & 3), the stiffness matrix is a full 4x4.

→ Solution: "Condense" out dofs 2 & 4 in the stiffness matrix (e.g. rotational dofs) to produce a 2x2.

→ <u>Derivation</u> <u>of</u> <u>Static</u> <u>Condensation</u>

→ <u>Rule</u>: Dofs may be condensed out of the stiffness matrix if:

(1) Their mass influence coeffs are zero and

(2) They are not loaded by an external force.

→ Partition the stiffness and mass matricies based on "retained" dofs (u_r) & "condensed" dofs (u_c).

$$\left[\begin{array}{c|c} M_{rr} & 0 \\ \hline 0 & 0 \end{array}\right]\left\{\begin{array}{c} \ddot{u}_r \\ \hline \ddot{u}_c \end{array}\right\} + \left[\begin{array}{c|c} K_{rr} & K_{rc} \\ \hline K_{cr} & K_{cc} \end{array}\right]\left\{\begin{array}{c} u_r \\ \hline u_c \end{array}\right\} = \left\{\begin{array}{c} P_r \\ \hline 0 \end{array}\right\}$$

M_{rr} = mass matrix associate w/ u_r

K_{rr} = stiffness matrix assoc w/ u_r

K_{cr} & K_{rc} = Coupling stiffness matrix between u_r & u_c.

K_{cc} = Stiffness matrix associated w/ u_c

Multiplying out...

$$[M_{rr}]\{\ddot{u}_r\} + [K_{rr}]\{u_r\} + [K_{rc}]\{u_c\} = \{P_r\}$$

$$[K_{cr}]\{u_r\} + [K_{cc}]\{u_c\} = 0$$

$$\{u_c\} = \underbrace{-[K_{cc}]^{-1}[K_{cr}]\{u_r\}}_{\text{subst into 1st eqn for } \{u_c\}}$$

$$\underbrace{[M_{rr}]\{\ddot{u}_r\} + [K_{rr}]\{u_r\} - [K_{rc}][K_{cc}]^{-1}[K_{cr}]\{u_r\} = \{P_r\}}_{\text{No more } \{u_c\} \Rightarrow \text{ these degrees of freedom have been removed.}}$$

Re-writing

$$[M_{rr}]\{\ddot{u}_r\} + [K^*]\{u_r\} = \{P_r\}$$

where,

$[K^*]$ = the condensed stiffness matrix.

$$* \quad [K^*] = [K_{rr}] - [K_{rc}][K_{cc}]^{-1}[K_{cr}]$$

=> Back to Example

-> To partition => switch row 2 & 3 and switch column 2 & 3

=> this reorganizes the dofs to be $u_1, u_3, \theta_2, \theta_4$

$$\frac{\bar{m}L}{2}\left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left\{\begin{array}{c} \ddot{u}_1 \\ \ddot{u}_3 \\ \hline \ddot{\theta}_2 \\ \ddot{\theta}_4 \end{array}\right\} +$$

$$\frac{EI}{L^3}\left[\begin{array}{cc|cc} 24 & -12 & 0 & 6L \\ -12 & 12 & -6L & -6L \\ \hline 0 & -6L & 8L^2 & 2L^2 \\ 6L & -6L & 2L^2 & 4L^2 \end{array}\right]\left\{\begin{array}{c} u_1 \\ u_3 \\ \hline \theta_2 \\ \theta_4 \end{array}\right\} = \left\{\begin{array}{c} P_1 \\ P_2 \\ \hline 0 \\ 0 \end{array}\right\}$$

$$K_{rr} = \frac{EI}{L^3}\begin{bmatrix} 24 & -12 \\ -12 & 12 \end{bmatrix} \qquad K_{rc} = \frac{EI}{L^3}\begin{bmatrix} 0 & 6L \\ -6L & -6L \end{bmatrix}$$

$$K_{cr} = \frac{EI}{L^3}\begin{bmatrix} 0 & -6L \\ 6L & -6L \end{bmatrix} \qquad K_{cc} = \frac{EI}{L^3}\begin{bmatrix} 8L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix}$$

Using Excel...

$$[K^*] = \begin{bmatrix} 28770 & -8991 \\ -8991 & 3596 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 3.504 & 0 \\ 0 & 1.752 \end{bmatrix}$$

$a = 6.139$

$b = -63006$

$c = 2.263 \times 10^7$

$\omega_i^2 = 372.75$

$\omega_i^2 = 9890.46$

$\omega_1 = 19.31$ rad/s

$\omega_2 = 99.45$ rad/s.

Visual Analysis

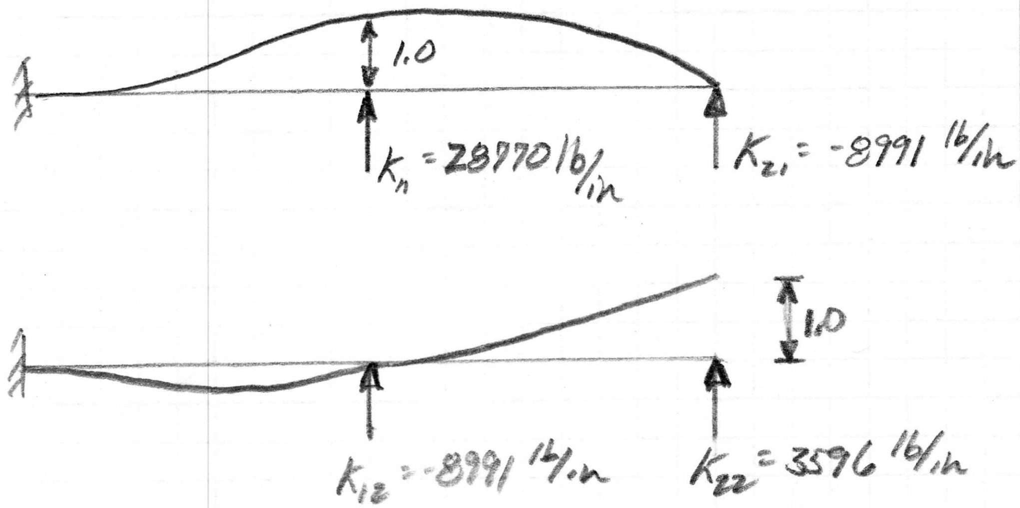
$\omega_1 = 19.26$ rad/s

$\omega_2 = 99.19$ rad/s

→ What does static condensation actually do?

→ It eliminates dofs as if the original stiffness matrix was formulated w/o them.

Show this is true using VA..



⇒ What if we want to condense out degrees of freedom that have mass?

⇒ <u>Guyan's</u> <u>Reduction</u>.

$$\underbrace{\left[\begin{array}{c|c} M_{rr} & M_{rc} \\ \hline M_{cr} & M_{cc} \end{array}\right]}_{\text{full mass matrix}}\left\{\begin{array}{c} \ddot{u}_r \\ \hline \ddot{u}_c \end{array}\right\} + \left[\begin{array}{c|c} K_{rr} & K_{rc} \\ \hline K_{cr} & K_{cc} \end{array}\right]\left\{\begin{array}{c} u_r \\ \hline u_c \end{array}\right\} = \underbrace{\left\{\begin{array}{c} P_r \\ \hline 0 \end{array}\right\}}_{\text{still can only deal w/ loads @ } u_r}$$

→ Multiplying out the bottom row...

$$[M_{cr}]\{\ddot{u}_r\} + [M_{cc}]\{\ddot{u}_c\} + [K_{cr}]\{u_r\} + [K_{cc}]\{u_c\} = 0$$

⇒ Assumption ⇒ M_{cr} & M_{cc} are negligible in the previous equation.

⇒ Guyan's reduction is approximate
→ Be careful which dofs you decide to condense out...

→ Given this assumption

$$\{U_c\} = -[K_{cc}]^{-1}[K_{cr}]\{U_r\}$$

or

Identity matrix

$$\left\{\begin{array}{c} U_r \\ \hline U_c \end{array}\right\} = \underbrace{\left[\begin{array}{c} I \\ \hline -[K_{cc}]^{-1}[K_{cr}] \end{array}\right]}_{=[T]}\{U_r\}$$

transformation matrix.

→ Reduction of stiffness dofs using static condensation can be written as:

$$\underbrace{[K^*] = [T]^T[K][T]}_{\text{this is identical to}}$$

$$[K^*] = [K_{rr}] - [K_{rc}][K_{cc}]^{-1}[K_{cr}]$$

→ Guyan's Reduction applies the same transformation to the mass matrix

$$[M^*] = [T]^T [M][T]$$

⇒ Again, this is approximate as the mass terms were neglected in computing $[T]$.

<u>Example</u>: Revisit the cantilever w/ consistent mass and reduce it to a 2DOF system.

$$[K_G] = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ & 8L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm} & & & 4L^2 \end{bmatrix}$$

$$[M_G] = \frac{\bar{m}L}{420}\begin{bmatrix} 312 & 0 & 54 & -13L \\ & 8L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{symm} & & & 4L^2 \end{bmatrix}$$

→ Reorder

→ Identify $[K_{rr}]$, $[K_{cc}]$, $[K_{rc}]$, $[K_{cr}]$

→ Perform the matrix operations

→ See Excel...

Comparison of Frequencies

Mode # (ω rad/s, % error)

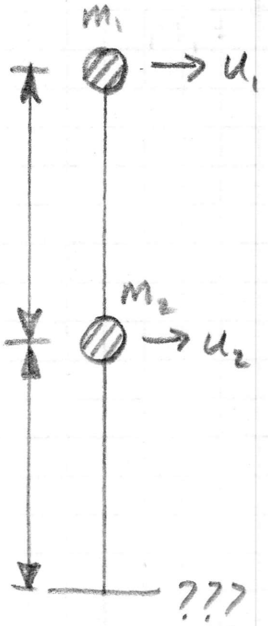
	1	2	3	4
VA Highly discretized	21.45	134.3	375.8	736.0
4DOF Consis. Mass	21.50 (0.2%)	135.9 (1.2%)	459.7 (22.3%)	1334.4 (81.3%)
2DOF Lumped Mass + Static Cond.	19.31 (9.9%)	99.45 (25.9%)	—	—
2DOF Guyan's (eliminate rotation)	21.54 (0.4%)	136.3 (1.5%)	—	—
2DOF Guyan's (eliminat translation)	22.6 (5.4%)	230.0 (71.3%)	—	—

→ What trends are apparent? Why?

→ In what cases is Static Condensation better than Guyan's reduction?

→ If Static Condensation is exact, why does it under perform Guyan's Reduction?

→ In class example:
Compute and validate the modal parameters for a simple 2DOF system



$m_1 = m_2 = 2.25\,lb/g$

$E = 29{,}000{,}000$ psi

$I = \frac{\pi (1/8)^4}{4} = 1.9175 \times 10^{-4}\ in^4$

→ Homework 6 - Solve for the frequencies & mode shapes of the following structure using:

(1) 4DOF Consistent Mass

(2) 2DOF Lumped Mass w/ Static Condensation

(3) 2DOF w/ Guyan's Reduction to eliminate rotational dofs.

(4) 2DOF w/ Guyan's Reduction to eliminate translation dofs.

(5) Highly discretized lumped mass (e.g. SAP)

→ Compare the results and comment on the trade-offs between accuracy & efficiency