LECTURE 5: Undamped Free Vibration & Intro to Forced Vibration

→ Systems w/ Viscous Damping ⇒ $c \neq 0$

$$m\ddot{u} + c\dot{u} + ku = 0 \quad \text{(EOM}$$

$$\ddot{u} + \left(\frac{c}{m}\right)\dot{u} + (\omega_n)^2 u = 0$$

natural frequency (undamped)

define $\frac{c}{m} = 2\xi\omega_n$

damping ratio

$\xi < 1.0$ under-damped (oscillation)
$\xi = 1.0$ critically damped
$\xi > 1.0$ over damped

$$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0$$

⇒ Assume $u = e^{rt}$

⇒ substituting in

$$(r^2 + 2\xi\omega_n r + \omega_n^2)e^{rt} = 0$$

this can't be zero.

$$r^2 + 2\xi\omega_n r + \omega_n^2 = 0 \Leftarrow \text{characteristic equation}$$

$$r_{1,2} = \omega_n\left[-\xi \pm \sqrt{\xi^2 - 1}\right]$$

⇒ Critically Damped Systems ($\xi = 1.0$)

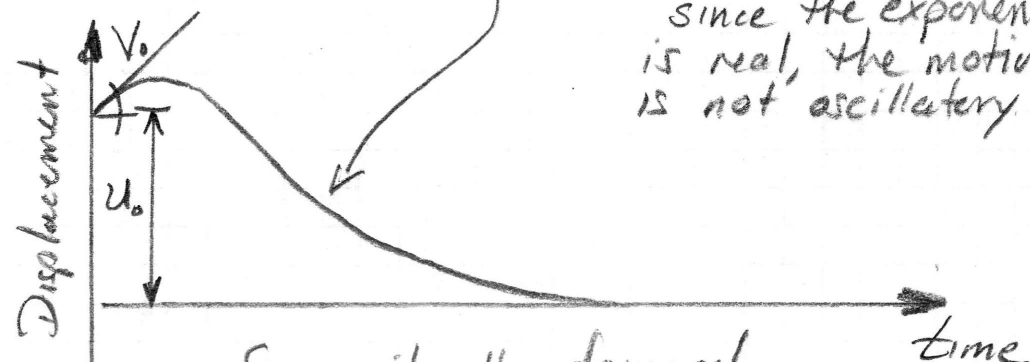
$r_{1,2} = -\omega_n$ (repeated real roots)

$u = (G_1 + G_2 t)e^{-\omega t}$ ← solution

constants determined by initial conditions

$$u = \left[u_0 + (V_0 + \omega_n u_0)t\right]\underbrace{e^{-\omega t}}$$

since the exponent is real, the motion is not oscillatory



For critically damped structures the response slowly decays to zero → no oscillation.

recall $\frac{c}{m} = 2\xi\omega_n$

for $\xi = 1$ → $c = c_{cr}$ ← critical damping

$$\Rightarrow c_{cr} = 2m\omega_n$$

$\Rightarrow$ <u>Underdamped Systems</u> $(\zeta < 1.0)$

$$r_{1,2} = \omega_n\left[-\zeta \pm \underbrace{\sqrt{\zeta^2 - 1}}_{}\right]$$

this is negative for $\zeta < 1.0$

$$r_{1,2} = -\omega_n\zeta \pm i\,\omega_n\sqrt{1-\zeta^2}$$

imaginary number $= \sqrt{-1}$

$\Rightarrow$ imaginary roots $\Rightarrow$ oscillations

$$u = e^{-\omega_n\zeta t}\left(G_1 e^{i\omega_d t} + G_2 e^{-i\omega_d t}\right)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

damped natural frequency.

Alternative (and more intuitive) form

$\Rightarrow$ de Moivre's theorm. gives.

$$u = e^{-\omega_n\zeta t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right)$$

$\Rightarrow$ Solve for A & B using initial cond gives...

$$\Rightarrow \quad u = e^{-\omega_n\zeta t}\left(u_0\cos(\omega_d t) + \left(\frac{V_0 + \omega_n\zeta u_0}{\omega_d}\right)\sin(\omega_d t)\right)$$

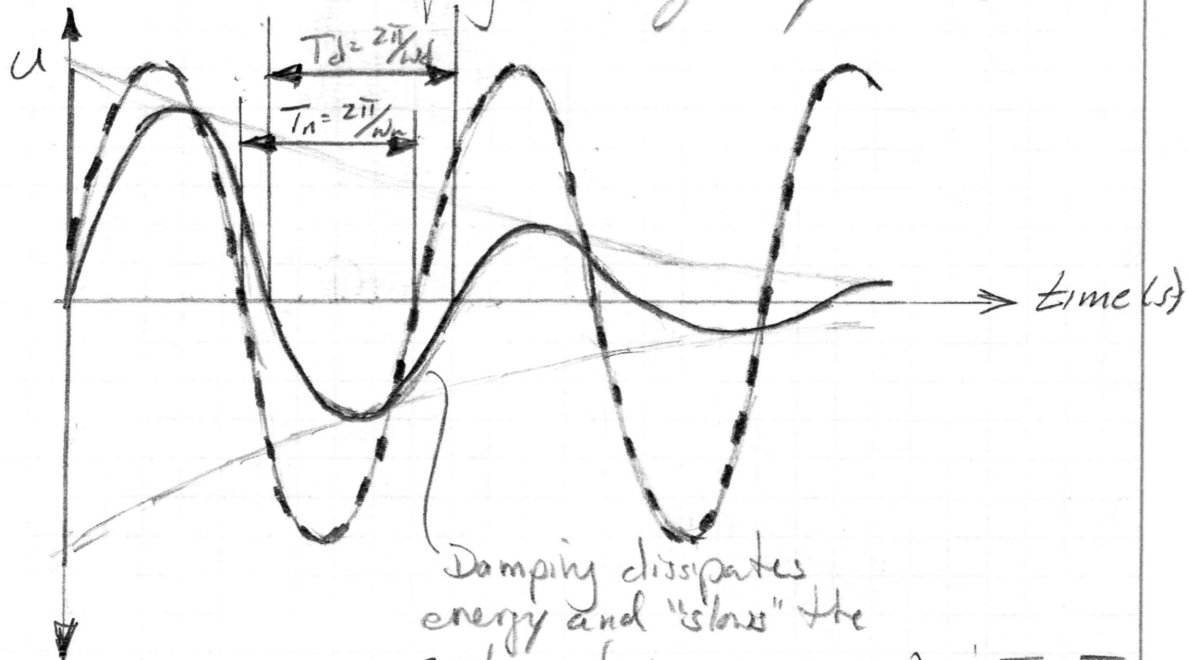
or

$$\Rightarrow \quad u = \rho e^{-\omega_n\zeta t}\sin(\omega_d t + \phi)$$

$$\rho = \sqrt{u_0^2 + \left(\frac{V_0 + \omega_n\zeta_n u_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{u_0\omega_d}{V_0 + u_0\omega_n\zeta}\right)$$

→ Effect of Damping on Frequency of Vibration



Damping dissipates energy and "slows" the system down ⇒ $\omega_d < \omega_n$ & $T_d > T_n$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

	ξ	ω_n (rad/s)	ω_d (rad/s)	% diff
	0.01	50	49.998	0.005
	0.02	50	49.990	0.020
Most common assumption →	0.05	50	49.937	0.125
	0.10	50	49.749	0.504
Upper bound →	0.15	50	49.434	1.144

↳ structure ⇒ rubble.

→ Phase-Plane Diagram for Damped System

$$u = \rho e^{-\omega_n \zeta t} \sin(\omega_d t + \phi)$$

$$\dot{u} = \frac{du}{dt} = \underbrace{-\omega_n \zeta \rho e^{-\omega_n \zeta t} \sin(\omega_d t + \phi)}$$
$$+ \omega_d \rho e^{-\omega_n \zeta t} \cos(\omega_d t + \phi)$$

- Problem #1 since the amplitude is a function of 't', the velocity has two terms.

- Simplification #1 ⇒ For small ζ (~5%), neglect the first term.

$$\dot{u} \approx \omega_d \rho e^{-\omega_n \zeta t} \cos(\omega_d t + \phi)$$

- Problem #2 → this expression doesn't satisfy the initial velocity

$$\dot{u}\Big|_{t=0} \neq V_0$$

- Simplification #2 ⇒ for small ζ (~5%), define ρ & ϕ as if the system is undamped.

$$\rho = \sqrt{(u_0)^2 + \left(\frac{V_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{u_0 \omega_d}{V_0}\right)$$

- Simplification #3 ⇒ for small ζ

$$\omega_d \approx \omega_n \leftarrow \text{just for convience.}$$

→ Using Excel, show...

(1) When the simplifications cause large errors

(2) That the proper ρ & ϕ do not satisfy the initial cond. when the sin term is neglected (in $\frac{\dot{u}}{\omega_d}$ eqn).

(3) The spiral nature of the phase-plane diagram → i.e. a continual loss of energy

→ If the damping is large, the phase-plane representation is too cumbersome and should not be used

↳ use the direct u & $\dot{u}$ expressions.

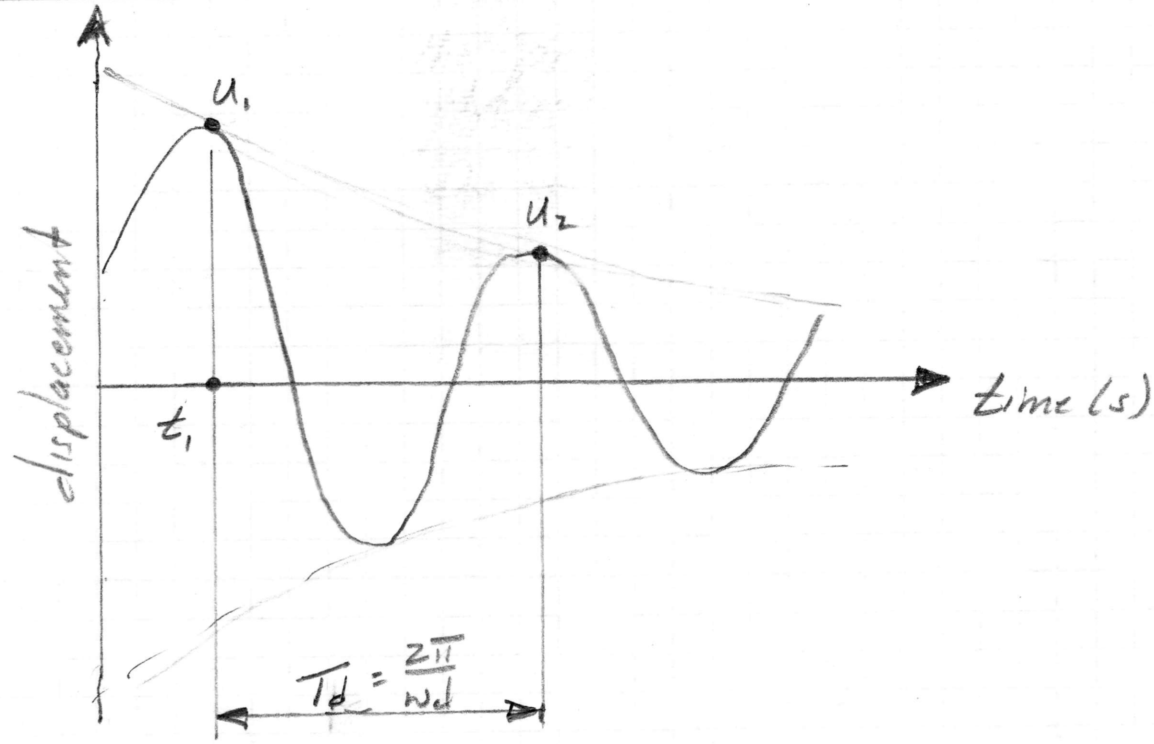
⇒ 'Log' Decrement

(1) Provides a physical understanding of the damp ratio ζ

(2) Provides an easy means to experimentally estimate damping

↳ this is important because it is very difficult to estimate damping in an apriori sense

→ key question → While it is obvious that an increase in ζ results in a faster 'decay' of vibrations, what is the exact relationships between ζ and vibration decay.



$$u_1 = u(t_1) = \rho e^{-\zeta\omega_n t_1} \sin(\omega_d t_1 + \phi)$$

causes 1 full rotation

$$u_2 = u\left(t_1 + \frac{2\pi}{\omega_d}\right) = \rho e^{-\zeta\omega_n\left(t_1 + \frac{2\pi}{\omega_d}\right)} \underbrace{\sin\left(\omega_d\left(t_1 + \frac{2\pi}{\omega_d}\right) + \phi\right)}_{\sin(\omega_d t_1 + \phi)}$$

$$\frac{u_1}{u_2} = \frac{\cancel{\rho} e^{-\zeta\omega_n t_1}}{\cancel{\rho} e^{-\zeta\omega_n\left(t_1 + \frac{2\pi}{\omega_d}\right)}}$$

define log dec: $\delta = \ln\left[\frac{u_1}{u_2}\right] = \ln\left[\frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n\left(t_1 + \frac{2\pi}{\omega_d}\right)}}\right]$

$$\delta = -\zeta\omega_n t_1 - \left(-\zeta\omega_n t_1 - \zeta\omega_n \frac{2\pi}{\omega_d}\right)$$

$\delta = \zeta \frac{\omega_n 2\pi}{\omega_d}$ ← δ is a constant!

$\Rightarrow$ for small ζ $\Rightarrow$ $\omega_n \approx \omega_d$

$\delta = \zeta(2\pi)$ or $\zeta = \frac{\delta}{2\pi}$ *

⇒ From Excel...

$$u_1 = -8.479 \text{ in}$$

$$u_2 = -6.231 \text{ in}$$

$$\delta = \ln\left[\frac{-8.479}{-6.231}\right] = 0.3081$$

$$\zeta = \frac{0.3081}{2\pi} = 0.0490 \sim 0.05$$

↳ actual ζ

⇒ Example: Inverse problem

→ You are hire to test a cantilever structure w/ to determine its mass, damping, & stiffness properties.

⇒ You elect to perform an impact test using a device that imparts a impulse of 20 kips over a 0.01 sec duration.

⇒ Using disp sensors, you capture the resulting free vibration response.

⇒ Goal: Determine m, c, & k...

⇒ Method 1: Model Updating.

⇒ Construct a model and modify m, c, k parameters until is matches the measured response.

⇒ Which perameters should we chage? why?

⇒ try to build a physical feel.

⇒ <u>Method 2</u>: Analytical.

<u>Step 1</u>: Determine ζ

⇒ log dec

→ from Excel.

$$\delta = \ln\left(\frac{u_1}{u_2}\right) = \ln\left(\frac{1.924''}{1.319''}\right) = 0.3775$$

$$\zeta = \frac{\delta}{2\pi} = \frac{0.3775}{2\pi}$$

$$\underline{\underline{\zeta = 0.0601}} \sim 0.06$$

<u>Step 2</u>: Determine 'm'

$$\Delta V = \frac{F\Delta t}{m}$$

} every thing is known except 'm'

1st two measurmt points.

$$\Delta V = \underbrace{\frac{\overbrace{(0.2204 - 0)''}}{0.005\,s}} = 44.08\ ^{m}/_{s}$$

← time step.

this assumes no noise or other measurement error.

$$44.08\ ^{m}/_{s} = \frac{(20{,}000\,lb)(0.01\,s)}{m}$$

$$\underline{\underline{m = 4.537\ lbm.}} \sim 4.5$$

Step 3: Determine 'k'

$$\omega_d = \sqrt{1-\zeta^2}\sqrt{\frac{k}{m}}$$

$$\omega_d = \frac{2\pi}{T_d}$$

from Excel.

$$T_d = -0.15 + 0.44 = 0.29 \text{ sec.}$$

$$\omega_d = 21.67 \text{ rad/s.}$$

$$21.67 = \sqrt{1-0.06^2}\sqrt{\frac{k}{4.537}}$$

$$21.70 = \sqrt{\frac{k}{4.537}}$$

$$\underline{\underline{k = 2137 \text{ kip/in.}}} \neq 2000 \text{ kip/in.}$$

→ Where did the error come from?

⇒ using $\zeta = 0.06$ & $m = 4.5$

$k = 2102$ kip/in

↳ still 5% error.

⇒ Use more precise T_d estimate.

$$t_1 = 0.145 + 0.1584\left(\frac{0.005}{0.1584 + 0.02618}\right) = 0.14929 \text{ sec}$$

$$t_2 = 0.445 + 0.07244\left(\frac{0.005}{0.07244 + 0.05323}\right)$$

$$t_2 = 0.44787 \text{ sec.}$$

$T_d = 0.44787 - 0.14929 = 0.29858$ sec

$\omega_d = 21.044$ rad/sec. (2.98% diff from before)

$\omega_n = 21.082$ rad/sec

why??

$k = 2000.025$ kip/in (5.10% diff from before)

=> ω_d is squared to solve for 'k' => all errors are magnified.

=> Sample HW Problems.

-> Code the damped free vibration response of an SDOF in Excel and examine the accuracy of the simplified phase-plane representation.

-> What is an upper bound for ξ?

-> Does any other parameter affect the accuracy?

-> Using the above coding try to solve the inverse problem using model updating and the analytical approach.

-> Which is more efficient?

-> Which is more realistic?

-> Why?