Week 2 Solutions

1. Unit Vectors

1.1 Describe conceptually what a unit vector is.

Solution: A unit vector is a vector with unit magnitude. On a conceptual level, this is the vector equivalent of the number 1, since we can scale a unit vector by a scalar amount to get any vector in that direction. The simplest unit vectors are the $\vec{i}$, $\vec{j}$, and $\vec{k}$ vectors we work with in Cartesian space, but in fact we could use any three *orthogonal* unit vectors in cartesian space to use to decompose a 3-dimensional vector into separate components.

As an aside, it is obvious that we can decompose any 3-dimensional vector into the $\vec{i}$, $\vec{j}$, and $\vec{k}$ vectors. This means that these three vectors form a *basis* for our space. However, this is not the only basis. Indeed, any three mutually orthogonal unit vectors could be used as a basis for 3-dimensional cartesian space. (You can think of any set of vectors like this being a rotation of the three we typically work with.) Because the $\vec{i}$, $\vec{j}$, and $\vec{k}$ vectors only have one non-zero component in each direction, we refer to these as our *canonical basis* vectors.

1.2 Compute the unit vectors for the following vectors.

(a) $\vec{v} = \langle 1, 1, 1\rangle$

Solution: Remember that the unit vector is simply the original vector *normalized* by its magnitude. Therefore:

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1^2+1^2+1^2}}\vec{v} = \frac{\sqrt{3}}{3}\vec{v} \quad \blacksquare$$

(b) $\vec{w} = \langle 0, -1, -1\rangle$

Solution:

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{0^2+(-1)^2+(-1)^2}}\vec{w} = \frac{\sqrt{2}}{2}\vec{w}$$

(c) $\vec{u} = \langle 10, 8, -7\rangle$

Solution:

$$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{10^2+8^2+(-7)^2}}\vec{u} = \frac{\sqrt{213}}{213}\vec{u}$$

1.3 The unit vector from point $A = (0, 2, 3)$ to $B = (1, 6, -2)$.

Solution: First find the vector between the two points, calculate the magnitude, then find the unit vector:

$$B - A = \langle 1-0, 6-2, -2-3\rangle = \langle 1, 4, -5\rangle$$
$$|B - A| = \sqrt{1^2+4^2+(-5)^2} = \sqrt{42}$$
$$\frac{B-A}{|B-A|} = \frac{\sqrt{42}}{42}\langle 1, 4, -5\rangle \quad \blacksquare$$

1.4 If $\vec{v}$ and $\vec{w}$ are orthogonal, will their unit vectors also be orthogonal? Why?

Solution: Yes, their associated unit vectors are just $\vec{v}$ and $\vec{w}$ scaled by their magnitudes. So the dot product of the unit vector will still be zero:

$$\vec{v}\cdot\vec{w} = 0 \implies \left(\frac{\vec{v}}{|\vec{v}|}\right)\cdot\left(\frac{\vec{w}}{|\vec{w}|}\right) = \frac{1}{|\vec{v}||\vec{w}|}(\vec{v}\cdot\vec{w}) = 0 \quad \blacksquare$$

2. Vector Operations

Compute the following.

$$\vec{v} = \langle 1, 2, 5 \rangle, \quad \vec{w} = \langle 3, -4, 2 \rangle$$

(a) $2\vec{v} - \vec{w}$

Solution:

$$2\vec{v} - \vec{w} = 2\langle 1,2,5\rangle - \langle 3,-4,2\rangle = \langle 2-3, 4-(-4), 10-2\rangle = \langle -1, 8, 8\rangle \quad \blacksquare$$

(b) $(2\vec{v}) \cdot \vec{w}$

Solution:

$$(2\vec{v}) \cdot \vec{w} = (2\langle 1,2,5\rangle) \cdot (\langle 3,-4,2\rangle) = 2\cdot 3 + 4\cdot(-4) + 10\cdot 2 = 42 \quad \blacksquare$$

(c) The unit vector of $\vec{v} \times \vec{w}$

Solution: first find the cross product, then the magnitude, and finally the unit vector:

$$\vec{v} \times \vec{w} = (2\cdot 2 - (-4)\cdot 5)\vec{i} - (1\cdot 2 - 3\cdot 5)\vec{j} + (1\cdot(-4) - 3\cdot 2)\vec{k} = 24\vec{i} + 13\vec{j} - 10\vec{k}$$
$$|\vec{v} \times \vec{w}| = \sqrt{24^2 + 13^2 + (-10)^2} = \sqrt{845} = 13\sqrt{5}$$
$$\frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|} = \frac{\sqrt{5}}{65}\langle 24, 13, -10\rangle \quad \blacksquare$$

$$\vec{v} = \langle 1, 0, 0 \rangle, \quad \vec{w} = \langle \sqrt{3}, \sqrt{3}, \sqrt{3} \rangle$$

(d) $\vec{v} \cdot \left(\frac{1}{\sqrt{3}}\vec{w}\right)$

Solution:

$$\vec{v} \cdot \left(\frac{1}{\sqrt{3}}\vec{w}\right) = \langle 1,0,0\rangle \cdot \left(\frac{1}{\sqrt{3}}\langle \sqrt{3}, \sqrt{3}, \sqrt{3}\rangle\right) = 1\cdot 1 + 0\cdot 1 + 0\cdot 1 = 1 \quad \blacksquare$$

(e) $\vec{w} \cdot \vec{w}$

Solution:

$$\vec{w} \cdot \vec{w} = \langle \sqrt{3}, \sqrt{3}, \sqrt{3}\rangle \cdot \langle \sqrt{3}, \sqrt{3}, \sqrt{3}\rangle = 3(\sqrt{3}^2) = 27 \quad \blacksquare$$

(f) The angle between $\vec{v}$ and $\vec{w}$.

Solution: The easiest formula is use is $\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right)$.

$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{|1||3\sqrt{3}|}\right)| \approx 54.74^\circ$$

3. Projections

Compute the following.

$$\vec{v} = \langle 1, 2, 5 \rangle, \quad \vec{w} = \langle 3, -4, 2 \rangle$$

(a) $\text{proj}_{\vec{v}}\,\vec{w}$

Solution:

$$\operatorname*{proj}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} = \frac{3-8+10}{\sqrt{1^2+2^2+5^2}}\frac{1}{\sqrt{1^2+2^2+5^2}}\langle 1,2,5\rangle = \frac{5}{\sqrt{30}}\frac{1}{\sqrt{30}}\langle 1,2,5\rangle = \frac{1}{6}\langle 1,2,5\rangle \quad \blacksquare$$

(b) $\text{proj}_{\vec{w}}\,\vec{v}$

Solution:

$$\underset{\vec{w}}{\text{proj}}\,\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}\frac{\vec{w}}{|\vec{w}|} = \frac{3-8+10}{\sqrt{3^2+(-4)^2+2^2}}\frac{1}{\sqrt{3^2+(-4)^2+2^2}}\langle 3,-4,2\rangle = \frac{5}{\sqrt{29}}\frac{1}{\sqrt{29}}\langle 3,-4,2\rangle = \frac{5}{29}\langle 3,-4,2\rangle \quad \blacksquare$$

$$\vec{v} = \langle 1,0,0\rangle, \quad \vec{w} = \langle\sqrt{3},\sqrt{3},\sqrt{3}\rangle$$

(c) $\text{proj}_{\vec{v}}\,\vec{w}$

Solution:

$$\underset{\vec{v}}{\text{proj}}\,\vec{w} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} = \frac{\sqrt{3}+0+0}{\sqrt{1^2+0^2+0^2}}\frac{1}{\sqrt{1^2+0^2+0^2}}\langle 1,0,0\rangle = \frac{\sqrt{3}}{1}\langle 1,0,0\rangle = \langle\sqrt{3},0,0\rangle \quad \blacksquare$$

(d) $\text{proj}_{\vec{w}}\,\vec{v}$

Solution:

$$\underset{\vec{w}}{\text{proj}}\,\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}\frac{\vec{w}}{|\vec{w}|} = \frac{\sqrt{3}+0+0}{\sqrt{\sqrt{3}^2+\sqrt{3}^2+\sqrt{3}^2}}\frac{1}{\sqrt{\sqrt{3}^2+\sqrt{3}^2+\sqrt{3}^2}}\langle\sqrt{3},\sqrt{3},\sqrt{3}\rangle = \frac{\sqrt{3}}{3}\frac{1}{3}\langle\sqrt{3},\sqrt{3},\sqrt{3}\rangle = \frac{1}{3}\langle 1,1,1\rangle \quad \blacksquare$$

4. Lines and Planes

4.1 Compute the area enclosed by the parallelogram defined by:

$$A(0,0) \quad B(7,3) \quad C(9,8) \quad D(2,5)$$

Solution: Recall that the area of a parallelogram is just the magnitude of the cross product of two of its sides. Here we will pick AD and AB as the sides and find $|AB\times AD|$.

$$AB = \langle 7,3\rangle, \quad AD = \langle 2,5\rangle$$
$$AB\times AD = \langle 0,0,35-6\rangle = \langle 0,0,29\rangle$$
$$|AB\times AD| = 29 \quad \blacksquare$$

4.2 Compute the area enclosed by the triangle defined by:

$$A(0,0) \quad B(-2,3) \quad C(3,1)$$

Solution: Similarly, recall that the area of a triangle is half the magnitude of the cross product of two of its sides. Here we will pick AB and AC as the sides and find $|AC\times AB|$.

$$AC = \langle 3,1\rangle, \quad AB = \langle -2,3\rangle$$
$$AB\times AD = \langle 0,0,9-(-2)\rangle = \langle 0,0,11\rangle$$
$$\frac{1}{2}|AB\times AD| = \frac{11}{2} \quad \blacksquare$$

4.3 Find the equation for the line through $(1,2,1)$ in the direction of $\vec{v} = \langle 0,1,0\rangle$

Solution: we parameterize the line by $t\in\mathbb{R}$ and construct the line $L = P_0 + t\,\vec{v}$.

$$L = P_0 + t\,\vec{v} = \langle 1,2,1\rangle + t\langle 0,1,0\rangle = \langle 1,2+t,1\rangle \quad \blacksquare$$

4.4 Find the equation for the plane through $(1,2,1)$ with normal $\vec{n} = \langle -1, 0, 1\rangle$

Solution: a plane is just a set of points, and we are looking for an equation that defines the set of points (x, y, z) such that $\vec{n}$ is always orthogonal to a vector formed between these points and our points $P_0 = (1,2,1)$. From this, we can easily re-derive the plane equation:

$$0 = \vec{n} \cdot (\langle x, y, z\rangle - \langle 1,2,1\rangle) = \langle -1,0,1\rangle \cdot \langle x-1, y-2, z-1\rangle = -(x-1)+(z-1) = 0 \quad \blacksquare$$

5. Vector-valued functions

Compute the velocity and acceleration vectors of the following. In (c), also compute the tangent vector at the given point.

(a) $\vec{r}(t) = (1+t)\vec{i} + \frac{t^2}{\sqrt{2}}\vec{j} + \frac{t^3}{3}\vec{k}$

Solution:

$$\vec{v}(t) = \vec{i} + \sqrt{2}t\vec{j} + t^2\vec{k} \quad \blacksquare$$
$$\vec{a}(t) = \sqrt{2}\vec{j} + 2t\vec{k} \quad \blacksquare$$

(b) $\vec{r}(t) = \sec(t)\vec{i} + \tan(t)\vec{j} + t\vec{k}$

Solution:

$$\vec{v}(t) = \tan(x)\sec(x)\vec{i} + \sec(x)^2\vec{j} + \vec{k} \quad \blacksquare$$
$$\vec{a}(t) = (\sec(x)\tan(x)^2 + \sec(x)^3)\vec{i} + 2\tan(x)\sec(x)^2\vec{j} \quad \blacksquare$$

(c) $\vec{r}(t) = \ln(t)\vec{i} + \frac{t-1}{t+2}\vec{j} + t\ln(t)\vec{k}$, and $t_0 = 1$

Solution:

$$\vec{v}(t) = \frac{1}{t}\vec{i} + \frac{3}{(t+2)^2}\vec{j} + (\ln(t)+1)\vec{k} \quad \blacksquare$$
$$\vec{a}(t) = \frac{-1}{t^2}\vec{i} - \frac{6}{(t+2)^3}\vec{j} + \frac{1}{t}\vec{k} \quad \blacksquare$$