Week 4 Worksheet

1. Matrix Operations and Linear Systems

1.1 Compute the following using basic matrix operations.

$$\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -6 \\ -6 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

1.2 Prove the following:

(a) Assuming $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, show that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

(b) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\vec{v} \in \mathbb{R}^n$. Show that

$$(\boldsymbol{A} + \vec{v}\vec{v}^T)^{-1} = \boldsymbol{A} - \frac{1}{1 + \vec{v}^T \boldsymbol{A}^{-1}\vec{v}} \boldsymbol{A}^{-1}\vec{v}\vec{v}^T \boldsymbol{A}^{-1}$$

Note: this is the famous Sherman-Morrison-Woodbury formula, which is an important result in numerical linear algebra.

(c) In linear algebra, we often like to work with what are known as *canonical basis vectors*, which are vectors $\vec{e}_i$ such that the i^{th} element is 1 and all other elements are 0. Suppose we are given a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ such that for any vector $\vec{v} \in \mathbb{R}^n$, we have $\vec{v}^T \boldsymbol{A}\vec{v} > 0$ when $\vec{v} \neq \boldsymbol{0}$. Prove that all of the diagonal entries of $\boldsymbol{A}$ are positive and non-zero.

Note: these types of matrices are referred to as *positive definite*, and are extremely important in numerical linear algebra and optimization theory.

(d) Suppose I can decompose matrix $\boldsymbol{A}$ as $\boldsymbol{A} = \boldsymbol{QR}$ such that $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$. Show that $(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T = (\boldsymbol{R}^T\boldsymbol{R})^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T$.

1.3 Least-squares problems Suppose I have a set of data points $(x_1, y_2), (x_2, y_2), \ldots, (x_n, y_n)$ and I would like to find the n^{th}-degree polynomial

$$y(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

which exactly passes through these points. Set up a system of equations that you could solve for the c_i.

2. Matrix Inverse & Gaussian Elimination

Compute the inverse of the following matrices. If it is not invertible, state why. What is the rank of each matrix? Which matrices will have a unique solution for $\boldsymbol{A}x = b$?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 0 & 5 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & -6 \\ 1 & 0 & 0 \\ 0 & -6 & 9 \end{bmatrix}$$

3. Linear Systems & Gauss-Jordan Elimination

Solve the following linear systems using Gauss-Jordan elimination. If there is no solution, state why.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

4. Determinants

4.1 Compute the determinant of:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

4.2 Solve the following system using Cramer's rule.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

4.3 If the determinant of $\boldsymbol{A} \in \mathbb{R}^{5\times 5}$ is 3, what is the determinant of $2\mathbf{A}$?

4.4 Does the following system have a unique solution? Why or why not? (Use an answer based on determinants; do not simply restate the result form the previous problem.)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$