Week 3 Solutions

1. MATLAB Review

2. Integration of Vector-valued Functions

Integrate the following vector-valued functions

(a)
$$\int_0^1 \left[\frac{1}{\sqrt{1-t^2}}\vec{i}+\frac{\sqrt{3}}{1+t^2}\vec{k}\right]dt$$

Solution: When integrating vector-valued functions, remember that we can just integrate each component individually. Note that on this problem, you need to use some less-common integrals (specifically inverse trig function integrals). While these don't show up very often, you should have these memorized or easily accessible on your cheatsheet.

$$\int_0^1 \left[\frac{1}{\sqrt{1-t^2}}\vec{i}+\frac{\sqrt{3}}{1+t^2}\vec{k}\right]dt$$
$$\sin^{-1}(t)\vec{i}+\sqrt{3}\tan^{-1}(t)\vec{k}\Big|_0^1$$
$$\frac{\pi}{2}\vec{i}+\frac{\sqrt{3}\pi}{4}\vec{k} \quad \blacksquare$$

(b)
$$\int_0^{\pi/3}\left[(\sec t\tan t)\vec{i}+\tan t\,\vec{j}+2\sin t\cos t\,\vec{k}\right]dt$$

Solution:

$$\int_0^{\pi/3}\left[(\sec t\tan t)\vec{i}+\tan t\,\vec{j}+2\sin t\cos t\,\vec{k}\right]dt$$
$$\sec(t)\vec{i}+-\ln(\cos(t))\vec{j}+\sin^2(t)\vec{k}\Big|_0^{\pi/3}$$
$$\vec{i}+\ln(2)\vec{j}+\frac{3}{4}\vec{k} \quad \blacksquare$$

3. Arc Length

Find the arc length of the following vector-valued functions over the given intervals.

(a) $r(t)=(6\sin 2t)\vec{i}+(6\cos 2t)\vec{j}+5t\vec{k}$, $0\le t\le\pi$

Solution: Recall that the arc length for

$$r(t)=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{k}$$

with $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\frac{df}{dt}^2 + \frac{dg}{dt}^2 + \frac{dh}{dt}^2}\, dt$$

This comes from the more fundamental definition of arc length:

$$L = \int_a^b ds$$

where ds represents a change in length. We also know that the speed at any given point will be:

$$\frac{ds}{dt} = \sqrt{\frac{df}{dt}^2 + \frac{dg}{dt}^2 + \frac{dh}{dt}^2}$$

so using the chain rule, we are able to derive the formula for arc length that we know and use.

A vector valued function is essentially a function that maps a single parameter (in this case t) to a vector quantity. (Vector valued functions and parametric equations are very closely related: in parametric equations, we wanted to parameterize x and y by a single parameter; here, we are doing the same thing, except now we are representing x and y (and sometime z) as the components of a vector.)

Superficially this may seem to be extra work, but it is quite actually very useful to link all of the components of our vector to a single parameter since if we want to integrate anything (e.g. arc length, work, etc.), we can integrate against this single parameter instead of having lots of different variables and integrals floating around.

For this problem, the arc length is then:

$$L = \int_0^\pi \sqrt{\frac{df}{dt}^2 + \frac{dg}{dt}^2 + \frac{dh}{dt}^2}\, dt$$

$$L = \int_0^\pi \sqrt{\left(\frac{d}{dt}(6\sin(2t))\right)^2 + \left(\frac{d}{dt}(6\cos(2t))\right)^2 + \left(\frac{d}{dt}(5t)\right)^2}\, dt$$

$$L = \int_0^\pi \sqrt{(12\cos(2t))^2 + (-12\sin(2t))^2 + (5)^2}\, dt$$

$$L = \int_0^\pi \sqrt{144\cos^2(2t) + 144\sin^2(2t) + 25}\, dt$$

$$L = \int_0^\pi \sqrt{144 + 25}\, dt$$

$$L = 13\int_0^\pi dt$$

$$L = 13\pi \quad \blacksquare$$

(b) $r(t) = (2+t)\vec{i} - (t+1)\vec{j} + t\vec{k},\ 0 \le t \le 3$

Solution: Using the formula:

$$L = \int_0^3 \sqrt{1+1+1}\, dt$$

$$= \sqrt{3}\int_0^3 dt$$
$$= 3\sqrt{3} \quad \blacksquare$$

(c) $r(t) = 6t^3\vec{i} - 2t^3\vec{j} - 3t^3\vec{k}$, $1 \le t \le 2$

Solution: using the formula:

$$\begin{aligned} L &= \int_1^2 \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2}\,dt \\ &= \int_1^2 \sqrt{441t^4}\,dt \\ &= \int_1^2 21t^2\,dt \\ &= 7\left(t^3\Big|_1^2\right) \\ &= 49 \quad \blacksquare \end{aligned}$$

4. "Special Unit Vectors"

Find $\boldsymbol{T}$, $\boldsymbol{N}$, κ, $\boldsymbol{B}$, and τ for

(a) $\boldsymbol{r}(t) = \cosh(t)\vec{i} - \sinh(t)\vec{j} + t\vec{k}$

Solution: Recall the formula for the unit tangent vector:

$$\boldsymbol{T} = \frac{\boldsymbol{v}(t)}{|\boldsymbol{v}(t)|}$$

$\boldsymbol{v}(t)$ is just the component-wise derivative of the original vector:

$$\boldsymbol{v}(t) = \sinh(t)\vec{i} - \cosh(t)\vec{j} + \vec{k}$$

Using the identities for hyperbolic trig functions:

$$|\boldsymbol{v}(t)| = \sqrt{\sinh(t)^2 + \cosh(t)^2 + 1} = \sqrt{2}\cosh(t)$$

so

$$\boldsymbol{T} = \frac{1}{\sqrt{2}\cosh(t)}\left(\sinh(t)\vec{i} - \cosh(t)\vec{j} + \vec{k}\right) = \frac{\sqrt{2}}{2}\tanh(t)\vec{i} - \frac{\sqrt{2}}{2}\vec{j} + \frac{\sqrt{2}}{2}\operatorname{sech}(t)\vec{k} \quad \blacksquare$$

The formula for the normal vector is given by:

$$\boldsymbol{N} = \frac{\boldsymbol{T}'}{|\boldsymbol{T}'|}$$

which we can compute as:

$$\boldsymbol{T}' = \frac{\sqrt{2}}{2}\operatorname{sech}^2(x)\vec{i} - \frac{\sqrt{2}}{2}\operatorname{sech}(t)\tanh(t)\vec{k}$$

$$
\begin{aligned}
|\boldsymbol{T}'| &= \sqrt{\left(\frac{\sqrt{2}}{2}\operatorname{sech}^2(t)\right)^2 + \left(\frac{\sqrt{2}}{2}\operatorname{sech}(t)\tanh(t)\right)^2} \\
&= \frac{\sqrt{2}}{2}\sqrt{\operatorname{sech}^2(t)\big(\operatorname{sech}^2(t)+\tanh^2(t)\big)} \\
&= \frac{\sqrt{2}}{2}\operatorname{sech}(t)\sqrt{1} \\
&= \frac{\sqrt{2}}{2}\operatorname{sech}(t) \\
\boldsymbol{N} &= \operatorname{sech}(t)\vec{i} - \tanh(t)\vec{k} \quad \blacksquare
\end{aligned}
$$

The formula for curvature is given by:

$$\kappa = \frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right|$$

so we can calculate curvature as

$$\kappa = \frac{1}{\sqrt{2}\cosh(t)}\frac{\sqrt{2}}{2} = \frac{1}{2}\operatorname{sech}(t) \quad \blacksquare$$

The formula for the binormal vector is:

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$$

which we can solve as:

$$
\begin{aligned}
\boldsymbol{B} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\sqrt{2}}{2}\tanh(t) & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\operatorname{sech}(t) \\ \operatorname{sech}(t) & 0 & -\tanh(t) \end{vmatrix} \\
&= \frac{\sqrt{2}}{2}\tanh(t)\vec{i} + \frac{\sqrt{2}}{2}\vec{j} + \frac{\sqrt{2}}{2}\operatorname{sech}(t) \quad \blacksquare
\end{aligned}
$$

Finally, for the torsion, recall the formula:

$$\tau = -\frac{d\boldsymbol{B}}{ds}\cdot\boldsymbol{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\boldsymbol{v}\times\boldsymbol{a}|}$$

The acceleration vector is given by:

$$\boldsymbol{a} = \cosh(t)\vec{i} - \sinh(t)\vec{j}$$

so

$$\boldsymbol{v}\times\boldsymbol{a} = \sinh(t)\vec{i} + \cosh(t)\vec{j} + (\cosh^2(t) - \sinh^2(t))\vec{k} = \sinh(t)\vec{i} + \cosh(t)\vec{j} + \vec{k}$$

and therefore:

$$|\boldsymbol{v}\times\boldsymbol{a}| = \sinh^2(t) + \cosh^2(t) + 1 = 2\cosh^2(t)$$

Plugging into the formula for τ:

$$
\begin{aligned}
\tau &= \frac{\begin{vmatrix} \sinh(t) & -\cosh(t) & 1 \\ \cosh(t) & -\sinh(t) & 0 \\ \sinh(t) & -\cosh(t) & 0 \end{vmatrix}}{2\cosh^2(t)} \\
&= \frac{1}{2\cosh^2(t)}\left(-\cosh^2(t)+\sinh^2(t)\right) \\
&= -\frac{1}{2}\operatorname{sech}^2(t) \quad \blacksquare
\end{aligned}
$$

(b) $\boldsymbol{r}(t) = (\ln \sec t)\boldsymbol{i} + t\boldsymbol{j}, -\pi/2 < t < \pi/2$

Solution: Tangent vector:

$$
\begin{aligned}
\boldsymbol{v} &= \tan(t)\vec{i} + \vec{j} \\
|\boldsymbol{v}| &= \sqrt{\tan^2(t)+1} = \sqrt{\sec^2(t)} = \sec(t) \\
\boldsymbol{T} &= \frac{1}{\sec(t)}\left(\tan(t)\vec{i} + \vec{j}\right) = \sin(t)\vec{i} + \cos(t)\vec{j} \quad \blacksquare
\end{aligned}
$$

For the normal vector:

$$
\begin{aligned}
\boldsymbol{T}' &= \cos(t)\vec{i} - \sin(t)\vec{j} \\
|\boldsymbol{T}'| &= \sqrt{\cos^2(t) + (-\sin(t))^2} = 1 \\
\boldsymbol{N} &= \cos(t)\vec{i} - \sin(t)\vec{j} \quad \blacksquare
\end{aligned}
$$

The curvature:

$$\kappa = \frac{1}{\sec(t)}(1) = \cos(t) \quad \blacksquare$$

Observe that both $\boldsymbol{T}$ and $\boldsymbol{N}$ lie in the $x - y$ plane only, so we know that their cross product will only have a component in the z direction (since the cross of two vectors is mutually orthogonal to either). Therefore,

$$\boldsymbol{B} = (-\sin^2(t) - \cos^2(t))\vec{k} = -\vec{k} \quad \blacksquare$$

Finally, the binormal vector is time-invariant, so we have simply:

$$\tau = 0 \quad \blacksquare$$