Week 4 Solutions

1. Matrix Operations and Linear Systems

1.1 Compute the following using basic matrix operations.

$$\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -6 \\ -6 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Solution:

For the first expression:

$$\begin{aligned} \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1\cdot 2+0\cdot 1+2\cdot(-1) \\ 0\cdot 2+3\cdot 1+3\cdot(-1) \\ 1\cdot 2+2\cdot 1+1\cdot(-1) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \\ &= 2\cdot 0+1\cdot 0+(-1)\cdot 3 \\ &= -3 \quad \blacksquare \end{aligned}$$

For the second expression, we can sometimes save time if we multiply the vectors in a different order. Here, if we multiply the left vector with the matrix first, the answer is all zeros so our overall product is zero. This is an interesting property of some matrices, where they posses a set of vectors such that if we multiply the matrix by any vector in this set, the result is zero. (In linear algebra, this set of vectors is known as the *null space*, but this is beyond the scope of CME 100.)

$$\begin{aligned} \begin{bmatrix} 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -6 \\ -6 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} 0\cdot 2+3\cdot 4+2\cdot(-6) & 0\cdot 1+3\cdot 0+2\cdot 0 & 2\cdot 0+3\cdot(-6)+2\cdot 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= 0 \quad \blacksquare \end{aligned}$$

1.2 Prove the following:

(a) Assuming $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, show that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Solution: Much of linear algebra depends on doing rigorous mathematical proofs. Indeed, linear algebra is typically an engineering student's first exposure to mathematical proof techniques. A linear algebra proof generally follows that you are given facts A and B, you know definition C, and you use these to prove statement D. Here, we are given that $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible (something that is not always guaranteed for square matrices), and we know that definition of an inverse is a matrix such that

$$\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$$

So, to prove that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$, we simply need to show that $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ is an inverse for $\boldsymbol{AB}$. So trivially:

$$\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I} = (\boldsymbol{AB})^{-1}\boldsymbol{AB}$$

So, we must have $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

(b) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\vec{v} \in \mathbb{R}^n$. Show that

$$(\boldsymbol{A} + \vec{v}\,\vec{v}^T)^{-1} = \boldsymbol{A}^{-1} - \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}$$

Note: this is the famous Sherman-Morrison-Woodbury formula, which is an important result in numerical linear algebra.

Solution: This is actually nearly identical to the proof in the previous question, with just a slightly more complicated more matrix in question. Remember, the inverse of a matrix is unique, so if we want to prove that two inverses are equal, we just need to prove that they are both inverses for the same matrix.

The first expression is by definition the inverse of $\boldsymbol{A}+\vec{v}\,\vec{v}^T$. So, we need to prove that the second expression is also an inverse for this matrix:

$$\begin{aligned}
(\boldsymbol{A} + \vec{v}\,\vec{v}^T)\left(\boldsymbol{A}^{-1} - \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\right) &= \boldsymbol{A}\boldsymbol{A}^{-1} + \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} - \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\boldsymbol{A}\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} \\
&\quad - \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} \\
&= \boldsymbol{I} + \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\left(\frac{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\right) - \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\boldsymbol{A}\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} \\
&\quad - \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} \\
&= \boldsymbol{I} + \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\big(\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} + \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\vec{v}^T\boldsymbol{A}^{-1}\vec{v} - \\
&\quad \boldsymbol{A}\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} - \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\big)
\end{aligned}$$

Notice that some of the terms in the matrix multiplications actually evaluate to scalars. We can group these and rearrange, since scalar multiplication commutes with matrix multiplication.

$$\begin{aligned}
&= \boldsymbol{I} + \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\big(\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} + \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}(\vec{v}^T\boldsymbol{A}^{-1}\vec{v}) - \\
&\quad \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} - \vec{v}(\vec{v}^T\boldsymbol{A}^{-1}\vec{v})\vec{v}^T\boldsymbol{A}^{-1}\big) \\
&= \boldsymbol{I} + \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}\big(\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} + (\vec{v}^T\boldsymbol{A}^{-1}\vec{v})\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} - \\
&\quad \vec{v}\,\vec{v}^T\boldsymbol{A}^{-1} - (\vec{v}^T\boldsymbol{A}^{-1}\vec{v})\vec{v}\,\vec{v}^T\boldsymbol{A}^{-1}\big) \\
&= \boldsymbol{I} + \frac{1}{1+\vec{v}^T\boldsymbol{A}^{-1}\vec{v}}(0) \\
&= \boldsymbol{I} \quad \blacksquare
\end{aligned}$$

(c) In linear algebra, we often like to work with what are known as *canonical basis vectors*, which are vectors $\vec{e}_i$ such that the i^{th} element is 1 and all other elements are 0. Suppose we are given a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ such that for any vector $\vec{v} \in \mathbb{R}^n$, we have $\vec{v}^T\boldsymbol{A}\vec{v} > 0$ when $\vec{v} \neq \boldsymbol{0}$. Prove that all of the diagonal entries of $\boldsymbol{A}$ are positive and non-zero.

Note: these types of matrices are referred to as *positive definite*, and are extremely important in numerical linear algebra and optimization theory.

Solution: Let $\vec{a}_i$ represent the i^{th} column of $\boldsymbol{A}$. First note that if we multiply an arbitrary vector $\vec{v}$ by a canonical basis vector $\vec{e}_i$, it will "pick out" the i^{th} element of $\vec{v}$. That is, $\vec{v}^T\vec{e}_i = v_i$. Now, consider multiplying $\boldsymbol{A}$ by $\vec{e}_i$. This will pick out the i^{th} element for every row of $\boldsymbol{A}$, giving us $\boldsymbol{A}\vec{e}_i = \vec{a}_i$, the i^{th} column. Now, consider $\vec{e}_i^T\boldsymbol{A}\vec{e}_i$. We know that for any vector, $\vec{v}^T\boldsymbol{A}\vec{v} > 0$, so

$$\vec{e}_i^T\boldsymbol{A}\vec{e}_i = \vec{e}_i^T\vec{a}_i = \vec{a}_{ii}$$

This will hold for all $i = 1, ..., n$, so all diagonal elements of $\boldsymbol{A}$ must be strictly positive.

(d) Suppose I can decompose matrix $\boldsymbol{A}$ as $\boldsymbol{A} = \boldsymbol{QR}$ such that $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{QQ}^T = \boldsymbol{I}$. Show that $(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T = (\boldsymbol{R}^T\boldsymbol{R})^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T$.

Solution: The first thing to note is our identity $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$. (Note: this does not always hold for inverses, but does always hold for transposes.) So, using this face, we can simply plug in the decomposition for $\boldsymbol{A}$ to show the statement:

$$(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T = ((\boldsymbol{QR})^T\boldsymbol{QR})^{-1}(\boldsymbol{QR})^T = (\boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{QR})^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T = (\boldsymbol{R}^T\boldsymbol{R})^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T \quad \blacksquare$$

1.3 Polynomial interpolation Suppose I have a set of data points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ and I would like to find the n^{th}-degree polynomial

$$y(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n$$

which exactly passes through these points. Set up a system of equations that you could solve for the c_i.

Solution: We know that at every x_k, we need to satisfy

$$y_k = c_0 + c_1 x_k + c_2 x_k^2 + ... + c_n x_k^n$$

So, we can set up this equation for all i, which will give us a system of equations we can solve for the c_i:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ & & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The matrix used in this setup is known as the *Gram matrix*, and is very important in linear algebra, especially when setting up least squares or interpolation problems.

2. Matrix Inverse & Gaussian Elimination

Compute the inverse of the following matrices. If it is not invertible, state why. What is the rank of each matrix? Which matrices will have a unique solution for $\boldsymbol{A}x = b$?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 0 & 5 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & -6 \\ 1 & 0 & 0 \\ 0 & -6 & 9 \end{bmatrix}$$

Solution: For all of the following, let $\boldsymbol{r}_i$ represent the i^{th} row of the matrix in question. Gaussian elimination is essentially an algorithm for computing the matrix inverse, solving a system of equations, or putting a given matrix into reduced row echelon form. (Recall that reduced row echelon form is where we do full row elimination on the matrix to determine the rank, and the rank of a matrix is the number of linearly independent rows remaining after we perform row elimination.)

When computing the inverse, remember that we first build the augmented matrix by placing the original matrix on the left, and the identity matrix on the right. From here, we do iterations of Gaussian elimination where on iteration i, we take a linear combination of row i with row j $(j > i)$ to zero-out the subdiagonal elements, then do the same process to zero out the super diagonal elements. You also must start this algorithm with column 1 and move to the right—you cannot do this with columns at random or the algorithm will not work. Please refer to the Wikipedia page for Gaussian elimination for a more formal statement of the algorithm.

For matrix one, notice that because the first entry in the second row is 0, so the only non-zero subdiagonal element is in the third row. Therefore, we only need to do row elimination on the third row during the first iteration of Gaussian elimination.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \overset{r_3 - r_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right]$$

Now that all of the subdiagonal elements in the first row are zero, we move on to the second column. Some statements of the algorithm would have you divide through the i^{th} row by the i^{th} diagonal element, but we will save this for the end in my statement here. Just understand that these statements of the algorithm are equivalent, and you should do this same problem both ways to convince yourself.

So, we perform row elimination to remove the subdiagonal elements in the second column. There are no super-diagonal elements, so we move on to the third column and eliminate the super diagonal elements.

$$\overset{r_3 - \frac{2}{3} r_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 & -\frac{2}{3} & 1 \end{array}\right] \overset{r_2 + r_3, r_1 + \frac{2}{3} r_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{4}{9} & \frac{2}{3} \\ 0 & 3 & 0 & -1 & \frac{1}{3} & 1 \\ 0 & 0 & -3 & -1 & -\frac{2}{3} & 1 \end{array}\right]$$

Finally, divide through each row by the diagonal entry in the reduced left matrix, to create the identity on the left and the matrix inverse on the right:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{4}{9} & \frac{2}{3} \\ 0 & 3 & 0 & -1 & \frac{1}{3} & 1 \\ 0 & 0 & -3 & -1 & -\frac{2}{3} & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{4}{9} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{9} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{9} & -\frac{1}{3} \end{array}\right] \quad \blacksquare$$

For the second matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 0 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \overset{r_2 - 3r_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \overset{r_3 + 5r_2, r_1 + r_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 5 & -15 & 5 & 1 \end{array}\right]$$

$$\overset{r_1 - \frac{1}{5} r_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{5} \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 5 & -15 & 5 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & -3 & 1 & \frac{1}{5} \end{array}\right] \quad \blacksquare$$

For the third matrix:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -6 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -6 & 9 & 0 & 0 & 1 \end{array}\right] \overset{r_2 - \frac{1}{2} r_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 2 & 4 & -6 & 1 & 0 & 0 \\ 0 & -2 & 3 & -\frac{1}{2} & 1 & 0 \\ 0 & -6 & 9 & 0 & 0 & 1 \end{array}\right] \overset{r_3 - 3r_2, r_1 + 2r_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{5}{2} & -3 & 0 \\ 0 & -2 & 3 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{array}\right]$$

At this point, notice that the left matrix is row-deficient since the third row is a linear combination of the first two. Therefore, the matrix is singular and the inverse does not exist.

3. Linear Systems & Gauss-Jordan Elimination

Solve the following linear systems using Gauss-Jordan elimination. If there is no solution, state why.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution: For the first system, we set up the augmented matrix by concatenating the $\boldsymbol{A}$ matrix with the right hand side vector, then do row elimination on this augmented matrix. Note though that when we are solving a linear system, we only need to eliminate the subdiagonal elements, since we can then do *backwards substitution* to solve for $\boldsymbol{x}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 1 & 2 & 1 & 1 \end{array}\right] \overset{r_3 - r_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 2 & -1 & 1 \end{array}\right] \overset{r_3 - \frac{2}{3}r_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & -3 & \frac{1}{3} \end{array}\right]$$

Now that we have done row elimination, we have the equivalent *upper triangular* system:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & -3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}$$

Solving this is extremely easy if we solve for the x_i in reverse order. Therefore:

$$x_3 = -\frac{1}{9} \implies x_2 = \frac{4}{9} \implies x_1 = \frac{2}{9} \quad \blacksquare$$

For the second matrix, we follow an identical procedure:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{array}\right] \overset{r_3 - r_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \overset{r_3 - \frac{1}{3}r_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & \frac{2}{3} \end{array}\right]$$

This gives the reduced system:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix}$$

The last row is all zero, and the last entry of the reduced right hand side is non-zero, so this system has no solution. (Were the third entry of the RHS vector 0, then the system would have infinitely many solution. What value would b_3 need to be in the original system to have infinitely many solutions?)

4. Determinants

4.1 Compute the determinant of:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution:

$$\begin{aligned} \det \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{vmatrix} &= 1\cdot 3\cdot 3 + 1\cdot 3\cdot 1 + 2\cdot 0\cdot 2 - 1\cdot 3\cdot 2 - 2\cdot 3\cdot 1 - 3\cdot 0\cdot 1 \\ &= 9 + 3 + 0 - 6 - 6 - 0 \\ &= 0 \quad \blacksquare \end{aligned}$$

4.2 Solve the following system using Cramer's rule.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution: For the matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$, let $\boldsymbol{A}_i$ be the matrix formed by replaces the i^{th} column of $\boldsymbol{A}$ with $\boldsymbol{b}$. Cramer's rule states that the solution of the system will satisfy:

$$x_i = \frac{\det \boldsymbol{A}_i}{\det \boldsymbol{A}}$$

Let's first compute the determinants:

$$\det \boldsymbol{A} = \det \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 1\cdot3\cdot1 + 0\cdot3\cdot1 + 2\cdot0\cdot2 - 1\cdot3\cdot2 - 2\cdot3\cdot1 - 1\cdot0\cdot0 = -9$$

$$\det \boldsymbol{A}_1 = \det \begin{vmatrix} 0 & 0 & 2 \\ 1 & 3 & 3 \\ 1 & 2 & 1 \end{vmatrix} = -2$$

$$\det \boldsymbol{A}_2 = \det \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -4$$

$$\det \boldsymbol{A}_3 = \det \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1$$

which gives us:

$$\boldsymbol{x} = \frac{1}{9} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

which is the same solution we obtained via Gaussian elimination.

Note: Cramer's rule is an interesting but ultimately incredibly laborious way of computing solutions to linear systems. While it will always work, in practice the actual algorithm we use to compute determinant requires us to first run the same algorithm that we end up using to solve the linear system. Cramer's rule therefore requires us to run this algorithm $n+1$ times, whereas computing the solution directly requires us to run it once. So, while it is a cool identity and has many useful theoretical uses, it is not terribly useful in practice.

4.3 If the determinant of $\boldsymbol{A} \in \mathbb{R}^{5\times5}$ is 3, what is the determinant of 2**A**?

Solution: Remember that the determinant is a function of the product of the elements along the diagonals. So, if I double every element, then each of the n elements along a diagonal contributes twice as much to the product. Therefore, the product of this term (and thus all the terms) increases by 2^n. To convince yourself, do this for the matrix in the following problem.

4.4 Does the following system have a unique solution? Why or why not? (Use an answer based on determinants; do not simply restate the result form the previous problem.)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Remember that a system of linear equations has a unique solution if and only if its matrix has an inverse. Then, remember that a matrix will not have an inverse if its determinant is non-zero. (The proof for this is fairly straightforward using Gaussian elimination, but such a proof is beyond the scope of CME 100.) So, since we are simply asked to argue whether or not this system has a unique solution and not actually determine the solution, we can simply take the determinant of the matrix for this problem and argue based on whether or not it is nonzero.

$$\det\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 1\cdot3\cdot3 + 1\cdot3\cdot1 + 2\cdot0\cdot2 - 1\cdot3\cdot2 - 2\cdot3\cdot1 - 3\cdot0\cdot1$$
$$= 9 + 3 + 0 - 6 - 6 - 0$$
$$= 0 \quad \blacksquare$$

Therefore, the system will not have a unique solution.

5. Polar Coordinates and Kepler's Laws

5.1 Find the velocity and acceleration in terms of the radial and angular unit vectors.

(a) $r = a(1 - \cos\theta)$, $\dot{\theta} = 3$

Solution: recall that in 2D, we have the relations:

$$\boldsymbol{r} = r(t)\boldsymbol{u}_r$$
$$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$

This and the following problems are a matter of using these relations to derive the velocity and acceleration vectors. However, please review Thomas Calculus section 13.6 so you thoroughly understand where these come from. The main point to keep in mind when working with vector polar coordinates is that the $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ vectors are also functions of time, so chain rule will apply when taking derivatives. If you feel bogged down by the math, it may be good to restate the problem in Cartesian coordinates, take derivatives, and convert back to polar coordinates (even just as a sanity check).

Using the above relations, we find:

$$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$$
$$= 3a\sin\theta\boldsymbol{u}_r + 3a(1 - \cos\theta)\boldsymbol{u}_\theta \quad \blacksquare$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$
$$= ((3)^2 a\cos\theta - a(1 - \cos\theta)(3)^2)\boldsymbol{u}_r + (a(1 - \cos\theta)(0) + 2(a\cos\theta)(3)^2)\boldsymbol{u}_\theta$$
$$= (18a\cos\theta - 9a)\boldsymbol{u}_r + 18(a\cos\theta))\boldsymbol{u}_\theta \quad \blacksquare$$

(b) $r = a\sin 2\theta$, $\dot{\theta} = 2t$

Solution: Using the above formulas:

$$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$$
$$= 4ta\cos 2\theta\boldsymbol{u}_r + 2ta\sin 2\theta\boldsymbol{u}_\theta \quad \blacksquare$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$
$$= (-16t^2 a\sin 2\theta + 4a\cos 2\theta - a\sin 2\theta(4t^2))\boldsymbol{u}_r + (a\sin 2\theta(2) + 2(2t)(2a\cos 2\theta)(2t))\boldsymbol{u}_\theta$$
$$= (4a\cos 2\theta - 20t^2 a\sin 2\theta)\boldsymbol{u}_r + (2a\sin 2\theta + 16t^2 a\cos 2\theta)\boldsymbol{u}_\theta$$

(c) $r = a(1 + \sin t)$, $\theta = 1 - e^{-t}$

Solution: Using the above formulas:

$$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$$

$$
\begin{aligned}
&= a\cos t\,\boldsymbol{u}_r + a(1+\sin t)e^{-t}\boldsymbol{u}_\theta \\
\boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta \\
&= (-a\sin t - a(1+\sin t)e^{-2t})\boldsymbol{u}_r + (-a(1+\sin t)e^{-t} + 2a\cos t\, e^{-t})\boldsymbol{u}_\theta \quad \blacksquare
\end{aligned}
$$

5.2 Show that a planet in a circular orbit moves with a constant speed.

Solution: When given a problem like this, the most important (but also most difficult) part is setting up the problems we will plug into the equations that we know. Here, we are given that the orbit is circular, which translates to the radius being constant. We state this as:

$$\boldsymbol{r} = R\boldsymbol{u}_r$$

Now, recall Kepler's first law of Planetary motion:

$$e = \frac{r_0 v_0^2}{GM} - 1$$

The eccentricity is a metric for how oblong an ellipse is, and is defined as:

$$e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}$$

For a circle, we have $e = 0$. Therefore, the first law becomes:

$$0 = \frac{r_0 v_0^2}{GM} - 1$$

which we rewrite as

$$r_0 = \frac{GM}{v_0^2}$$

Then, using the second part of Kepler's first law:

$$r = \frac{(1+e)r_0}{1+e\cos\theta} = r_0 = \frac{GM}{v^2} \implies v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R}} = C$$

Note: while the derivation for Kepler's first law is omitted in Thomas Calculus, the proof is available on Wikipedia and worth looking over. It would also be worth to work through and complete the derivation of Kepler's third law, since they give only the first half of the proof in your textbook.