Midterm 2 Review Problems

1. Gaussian & Gauss-Jordan Elimination

Solve the following systems using Gaussian elimination.

1.

$$\begin{aligned} x+y+z&=5\\ 2x+3y+5z&=8\\ 4x+5z&=2 \end{aligned}$$

2.

$$\begin{aligned} x+2y-3z&=2\\ 6x+3y-9z&=6\\ 7x+14y-21z&=13 \end{aligned}$$

3.

$$\begin{aligned} A+B+2C&=1\\ 2A-B+D&=-2\\ A-B-C-2D&=4\\ 2A-B+2C-D&=0 \end{aligned}$$

2. Partial Derivatives

1. For exercises 1(a) and (b), verify that $w_{xy}=w_{yx}$.
   (a) $w=\ln(2x+3y)$
   (b) $w=x\sin y+y\sin x+xy$
2. Now let's talk about something with meaning. If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the one-dimensional wave equation:

$$\frac{\partial^2 w}{\partial t^2}=c^2\frac{\partial^2 w}{\partial x^2}$$

   Show that the functions below satisfy the wave equation.

   (a) $w=\cos(2x+2ct)$
   (b) $w=5\cos(3x+3ct)+e^{x+ct}$

3. Chain Rule

1. Assuming that in the following equations y is a differentiable function of x, find dy/dx at the given point.
   (a) $xe^y + \sin(xy) + y - \ln 2 = 0, \quad (0, \ln 2)$
   (b) $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$
2. For the following function, find the values of dz/dx and dz/dy at the point specified.

$$xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, \quad (1, \ln 2, \ln 3)$$

4. Directional Derivatives

1. (a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $(1, 1, 0)$ in the direction of $\boldsymbol{v} = 2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.
   (b) In what directions does f change most rapidly at $(1, 1, 0)$? What are the rates of change in these directions?
2. Find the direction in which $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z$ increases and decreases most rapidly at $(1, 1, 0)$, then find the derivative in this direction.
3. In what directions is the derivative of $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ at $(1, 1)$ equal to zero?

5. Gradients

Compute the gradient of the following functions at the given point.

1. $f(x, y, z) = 2z^3 - 2(x^2 + y^2)z + \tan^{-1}(xz), \quad (1, 1, 1)$
2. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -1)$
3. $f(x, y, z) = e^{x+y}\cos(z) + (y + 1)\sin^{-1}(x), \quad (0, 0, \pi/6)$

6. Double Integrals

1. Sketch the region of integration and reverse the order of integration. Then, evaluate the integral.
   (a) $\int_0^{\pi/6} \int_{\sin(x)}^{1/2} xy^2 \, dy \, dx$
   (b) $\int_0^8 \int_{x^{1/3}}^2 \frac{1}{y^4 + 1} \, dy \, dx$
2. Convert to polar coordinates, then evaluate the integral.
   (a) $\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \sqrt{25 - x^2 - y^2} \, dx \, dy$
   (b) $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) \, dx \, dy$

7. Unconstrained Optimization

1. Consider the function $f(x, y) = x^2 + kxy + y^2$. for what values of k will $f(x, y)$ have a saddle? A minimum? A maximum? Mathematically justify your answers.
2. Find the maxima and minima of $f(x, y) = 48xy - 32x^3 - 24y^2$ on $0 \le x \le 1$ and $0 \le y \le 1$.
3. Find local max, min, and saddle points of $f(x, y) = e^x(x^2 - y^2)$.

8. Constrained Optimization

1. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.
2. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.
3. Let $a_1, a_2, ..., a_n$ be positive numbers. Find the maximum of $\sum_{i=1}^n a_i x_i$ subject to $\sum_{i=1}^n x_i^2 = 1$.

9. Linear Regression

1. A herpetologist has been tasked with studying the local salamander population. To do so, they have been collecting data on the average winter temperature (in $°C$) in a region and how many salamanders per square kilometer they find on a given patch of land the following spring. The data points are $(8,2)$, $(6,1)$, $(11,3)$, and $(3,1)$ (with salamander population give in hundreds of salamanders). They would like to model the relationship between the average temperature and salamander population, and believe a linear model may work well.

   Solve for the least-squares best fit line for these four data points. What are the squared errors for each data point? Do you think a linear fit is appropriate?

2. In some cases, the linear fit may not be very good. An alternative to fitting a line through data is to fit a higher-order polynomial such as a quadratic. That is, we fit a quadratic of the form:

   $$y(x) = m_1 x^2 + m_2 x + b$$

   to our data, which requires us to solve out for m_1, m_2, and b.

   Reformulate the least-squares problem to fit a quadratic, and write out the system we should solve for m_1, m_2, and b.

3. Sometimes an outlier can have a huge effect on our solution, or if we are fitting a high-degree polynomial, the coefficients may become very large and our model will *overfit* the data. (Indeed, this is one of the biggest problems that machine learning researchers face when formulating models.)

   To combat this, we *regularize* the regression coefficients. Regularization involves adding a penalty term to our squared error to penalize large coefficients:

   $$E = \sum_{i=1}^{n} (mx_i + b - y_i)^2 + \lambda(m^2 + b^2)$$

   Rewrite the linear regression equations and include the regularization terms.

10. MATLAB

1. For the function $f(x, y) = \tan^{-1}(x) + y^2$, write MATLAB to compute $\int_1^{10}\int_2^4 f(x, y)\,dy\,dx$ using a right Riemann sum. Use 1001 grid points in each dimension.

2. The steepest descent method is one of the cornerstones of optimization theory. In fact, steepest descent is the main workhorse for most machine learning algorithms in industry, and forms the backbone of most systems used in other fields. The steepest descent algorithm is given as follows:

$$x_{k+1} = x_k - \gamma_k \nabla f(x_k)$$

   where x_k is your current point, γ_k is your step size at step k, and $\nabla f(x_k)$ is the gradient calculated at your current point.

   (a) Based on your knowledge of gradient vectors, what is the intuition behind this algorithm? Will it always converge to a local minimum? A global minimum? No need to mathematically justify your answers here—we are just looking for a qualitative explanation.

   (b) Consider the function

$$f(x, y) = 2e^{-y}(x-2)^2 + e^{-x}(y-4)^4$$

   Write the gradient descent update equations.

   (c) For the function in the previous part, write MATLAB code that implements gradient descent with an initial guess of $(0, 0)$ and runs for 5,000 iterations. Use an initial step size of 10^{-2} and reduce the step size by 1/2 ever 1000 iterations.