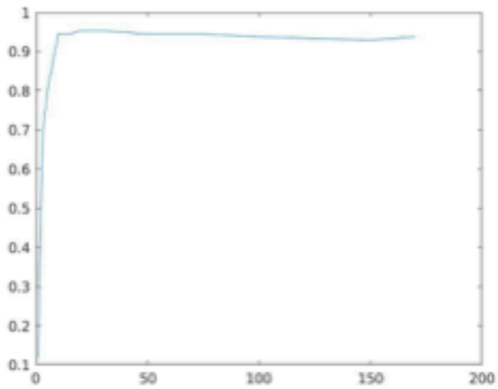
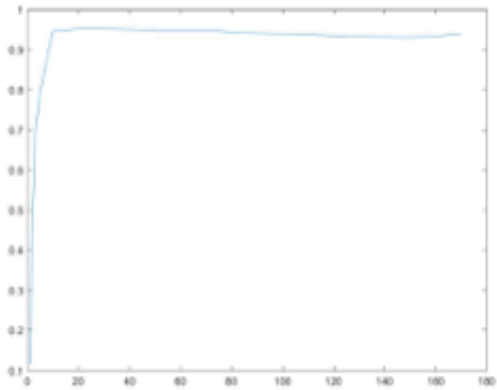
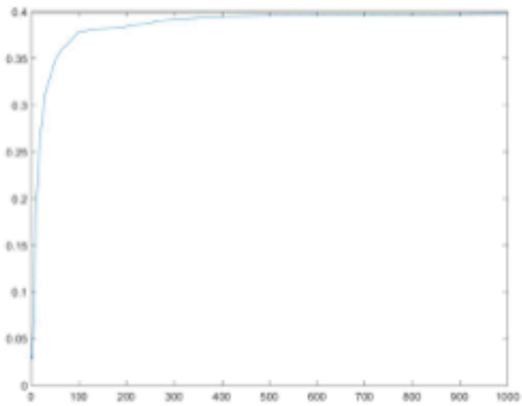
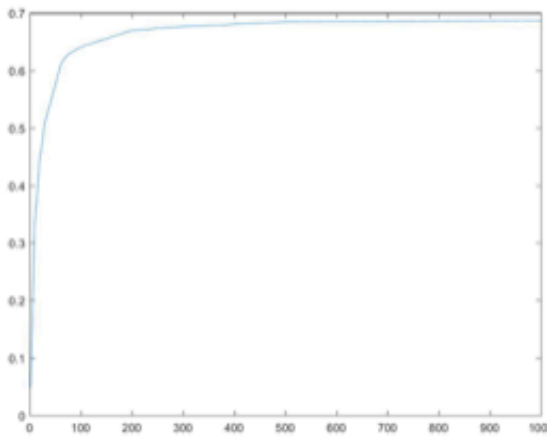
1. Just run the file ‘myMainScript1.m’ with all other files and folders of the given databases in appropriate positions. It will produce all the graphs and respective images.









2.

Run the code p2.m , you will obtain these Eigen faces.

4.

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To Evaluate this function , make a separate
script and call the function mySVD with a
single argument : A matrix.

The return values of the function are three
matrices U , S and V that satisfy the
relation : U * S * V' = A.
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5.

-: Solution - 5 :-

Using Lagrange's Multiplier Method & subjected to constraints $f^t f = 1$ and $f^t e = 0$; we have

$$E(f) = f^t C f - \lambda(f^t f - 1) - \mu(f^t e)$$

Taking derivative w.r.t. f :-

$$Cf - \lambda f - \mu e = 0$$

Premultiplying both sides by e^t :-

$$e^t C f - e^t(\lambda f) - \mu e^t e = 0 \qquad -①$$

$\because$ e is an eigen-vector of C & with λ as it's eigen-value :-

$$Ce = \lambda e$$

$$(Ce)^t = \lambda e^t$$

$$e^t C = \lambda e^t \qquad \{\because C \text{ is symmetric}\}$$

$$\therefore, \; e^t C f = \lambda e^t f$$

Using this result in eq. ① :-

$$\mu e^t e = 0$$

$$\therefore, \; \mu = 0$$

Putting this value of μ in eq. ① :-

$$e^t C f - e^t(\lambda f) = 0$$

$$\therefore, \; Cf = \lambda f$$

It proves that f is an eigenvector of C.

Now, since all the eigen-values are distinct (given) and

$$Cf = \lambda f$$

$$\Rightarrow f^t C f = \lambda$$

$\therefore$, It must correspond to the second highest eigen-value.

6.

6. (a) Given $P = A^T \cdot A$ and $Q = A \cdot A^T$

To Prove: For any vectors y and z;

(i) $y^t \cdot P \cdot y \geq 0$ and $z^t \cdot Q \cdot z \geq 0$

(ii) Why are the eigen values of P and Q non-negative?

Proof:

(i) $y^t * P * y = y^t (A^T * A) * y$

$= (y^t * A^t) * (A * y)$

$= (A * y)^t * (A * y) = \underline{\underline{\|(A * y)\|^2 \geq 0}}$

$z^t * Q * z = z^t * A * A^t * z$

$= (A^t * z)^t * (A^t * z) = \underline{\underline{\|(A^t * z)\|^2 \geq 0}}$

(ii) Let λ be the eigenvalue of $A^T \cdot A$ i.e. P, then for some nonzero x,

$\lambda \|x\|^2 = \langle \lambda x, x \rangle = \langle A^T A x, x \rangle$

$= \underline{\langle Ax, Ax \rangle \geq 0.}$

Similarly, for $A \cdot A^T$ i.e. Q

$\lambda \|x\|^2 = \langle \lambda x, x \rangle = \langle A \cdot A^T x, x \rangle$

$= \underline{\langle Ax, Ax \rangle \geq 0.}$

6.(b) To prove : (i) If u is an eigenvector of P of eigen value λ, to show that Au is an eigenvector of Q, with eigenvalue λ.

(ii) If v is an ~~eigenvalue~~ eigenvector of Q, with eigenvalue μ, show $A^T v$ is an eigenvector of P with eigenvalue μ.

Proof : (i) $P\vec{u} = \lambda\vec{u}$

$$A^T A\vec{u} = \lambda\vec{u}$$

$$AA^T A\vec{u} = A(\lambda\vec{u})$$

$$AA^T(A\vec{u}) = \lambda(A\vec{u})$$

$$Q(A\vec{u}) = \lambda(A\vec{u})$$

$\therefore$ $A\vec{u}$ is an eigenvector of Q with eigenvalue λ.

(ii) $Q\vec{v} = \mu\vec{v}$

$$AA^T\vec{v} = \mu\vec{v}$$

$$A^T AA^T\vec{v} = A^T(\mu\vec{v})$$

$$A^T A(A^T\vec{v}) = \mu(A^T\vec{v})$$

$\therefore$ $A^T\vec{v}$ is an eigenvector of $A^T A$ with eigenvalue μ

No. of elements in $\vec{u}$: $n \times 1$

No. of elements in $\vec{v}$: $m \times 1$

6 (c) Given: v_i eigenvector of Q

$$u_i \triangleq \frac{A^T v_i}{\|A^T * v_i\|^2}$$

To prove: There exists some real, non-negative γ_i, such that

$$A u_i = \gamma_i v_i$$

Proof:

Assume, $A \mu_i = \gamma_i v_i$

Claim: γ_i is some real, non-negative value.

then, $$A * \left(\frac{A^T v_i}{\|A^T * v_i\|^2}\right) = \gamma_i v_i$$

$$\left(\frac{(A * A^T)}{\|A^T * v_i\|^2}\right) v_i = \gamma_i v_i$$

$$\gamma_i = \frac{A A^T}{\|A^T v_i\|^2} \quad - (1)$$

Since, $Q \vec{v}_i = \lambda \vec{v}_i$

$$(A . A^T) . \vec{v}_i = \lambda \vec{v}_i$$

$$\frac{(A * A^T}{\| A^T * \ldots}$$

$$\frac{(A*A^T)*\vec{v}_i}{\|A^T*v_i\|^2} = \frac{\lambda}{\|A^T*v_i\|^2}\cdot\vec{v}_i = \gamma_i v_i$$

$$\Rightarrow \gamma_i = \frac{\lambda}{\|A^T*v_i\|^2} > 0.$$

$\therefore \lambda$ is positive

6.

(d) We know that for an arbitrary $m \times n$ matrix, the matrices $A^T A$ and AA^T are symmetric.

$A^T A$ then has n pairwise orthogonal eigenvectors. Let $v_1, \ldots, v_r$ be those normalised such eigenvectors pertaining to the non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A^T A$. We know that because by definition, eigenvectors pertaining to an eigenvalue zero are mapped to zero; r is simply the rank of $A^T A$, which is also the rank of A.

We then observed that $AA^T A v_i = A\lambda_i v_i = \lambda_i A v_i$, i.e., that v_i is an eigenvector of $A^T A$ implies that Av_i is an eigenvector of AA^T with the same eigenvalue λ_i. We computed the norm of Av_i via

$$(Av_i)^T(Av_i) = v_i^T A^T A v_i = v_i^T \lambda v_i = \lambda v_i^T v_i.$$

Since for any vector x, the squared norm $\|x\|_2^2 := \sum_i x_i^2$ is just the scalar product with itself: $x^T \cdot x$, we have that the norm of Av_i is the square root of λ_i, which we denoted by σ_i

Now, define $u_i = Av_i/\sigma_i$; by the above $u_1, \ldots, u_r$ are then unit vectors just like the $v_1, \ldots, v_r$. Then, by the same argument as used for the previous displayed formula

$$u_i^T A v_j = (Av_i/\sigma_i)^T A v_j = v_i^T A^T A v_j/\sigma_i$$

i.e $u_i^T A v_j$ is zero for $i \neq j$ (because $v_1, \ldots v_r$ are pairwise orthogonal) and σ_i for $i = j$ (because the $v_1, \ldots, v_r$ are unit vectors $d_i = \sigma_i^2$).

Writing all the equations for $i, j = 1, \ldots, r$, in matrix form gives:

$$\begin{bmatrix} u_1^T \\ \vdots \\ u_r^T \end{bmatrix} A \cdot [v_1 \cdots v_r] = \mathrm{diag}(\sigma_1, \ldots, \sigma_r),$$

where the right hand side denotes an $r \times r$ matrix with $\sigma_1, \ldots, \sigma_r$ on the diagonal and all zeros elsewhere.

The u_i are vectors ~~for~~ from an m-dimensional space, and we only have r of them, so we can pick unit vectors $u_{r+1}, \cdots u_m$ that are pairwise orthogonal, and orthogonal to $u_1 \cdots u_r$. Similarly we can find $v_{r+1}, \cdots, v_n$ such that $v_1 \cdots v_n$ are pairwise orthogonal unit vectors. We then still have that $u_i^T A v_j$ is σ_i when $i = j \leq r$ and by a calculation analagous to the above it is zero for all other cases. In matrix form, this gives us

$$\begin{bmatrix} u_1^T \\ \vdots \\ u_m^T \end{bmatrix} \cdot A \cdot [v_1 \cdots v_n] = \Sigma,$$

where Σ is now an $m \times n$ matrix with it's first r diagonal entries being the $\sigma_1, \cdots, \sigma_r$ and zeros everywhere else.

Defining $U = [u_1 \cdots u_m]$ and $V = [v_1 \cdots v_n]$, that is, writing the $u_1, \cdots, u_m$ resp the $v_1, \cdots, v_n$ column by column, we now have $U^T A V = \Sigma$. By the remains following the eigenvector decomposition theorem stated in the beginning, $UU^T = U^TU = I$ and also $VV^T = V^TV = I$ and thus, we get $A = U \Sigma V^T$.

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