ECE551 - Homework 3-4

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1 Deterministic Correlation Sequences

(a) It is obvious that σ is a delay operator, i.e. $\sigma = T_1$

$$\begin{aligned} \sigma^k x &= (\sigma(\sigma(...(\sigma x)))) \qquad (k \text{ times of } \sigma) \\ \Rightarrow \sigma^k x &= T_k x \\ \Rightarrow (\sigma^k x)[n] &= x[n-k]. \end{aligned}$$

Similarly, $\sigma^{-1}x = T_{-1}x \Rightarrow (\sigma^{-1}x)[n] = x[n+1]$.

(b) Prove or disprove:

i.
$$\begin{aligned} a_x[-k]^* = \left\langle x, \sigma^k x \right\rangle^* = \left\langle \sigma^k x, x \right\rangle &= \sum_{n\in\mathbb{Z}} x[n]^* x[n-k] \\ &= \sum_{n\in\mathbb{Z}} x[n+k]^* x[n-k+k] = \sum_{n\in\mathbb{Z}} x[n+k]^* x[n] = a_x[k] \end{aligned}$$

Hence, $a_x[k] = a_x[-k]^*$.

ii. We have $a_x[0] = \langle x, x\rangle = \|x\|^2$. By Cauchy-Schwarz inequality:

$$|a_x[k]| = \left\langle x, \sigma^{-k}x \right\rangle \leq \|x\| \left\| \sigma^{-k}x \right\|.$$

Since delay does not change the norm, $\left\|\sigma^{-k}x\right\| = \|x\|$. Therefore:

$$|a_x[k]| \leq \|x\|^2 = a_x[0].$$

iii.
$$\begin{aligned} c_{y,x}[-n]^* = \left(\sum_{i\in\mathbb{Z}} y[i]x[i+n]^*\right)^* &= \sum_{i\in\mathbb{Z}} x[i+n]y[i]^* \\ &= \sum_{i\in\mathbb{Z}} x[i+n-n]y[i-n]^* = \sum_{i\in\mathbb{Z}} x[i]y[i-n]^* = c_{x,y}[n] \end{aligned}$$

Hence, $c_{x,y}[n] = c_{y,x}[-n]^*$.

iv.

$$c_{x,y}[-n]^* = \left(\sum_{i\in\mathbb{Z}} x[i]y[i+n]^*\right)^* = \sum_{i\in\mathbb{Z}} x[i]^* y[i+n]$$
$$= \sum_{i\in\mathbb{Z}} x[i-n]^* y[i] = c_{y,x}[n] \neq c_{x,y}[n].$$

Hence, $c_{x,y}[n] \neq c_{x,y}[-n]^*$.

v. We have

$$\begin{aligned}
C_{x,y}[\omega] &= \sum_{n\in\mathbb{Z}} c_{x,y} e^{-j\omega n} \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} x[k]y[k-n]^*\right) e^{-j\omega n} \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x[k]y[k-n]^* e^{-j\omega(k-(k-n))} \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x[k]y[k-n]^* e^{-j\omega k} e^{j\omega(k-n)} \\
&= \sum_{k\in\mathbb{Z}} x[k]e^{-j\omega k} \sum_{n\in\mathbb{Z}} y[k-n]^* \left(e^{-j\omega(k-n)}\right)^* \\
&= X(\omega)Y(\omega)^*
\end{aligned}$$

(c) i. We have

$$\begin{aligned}
c_{x_1,x_2}[k] &= \sum_{n\in\mathbb{Z}} \alpha_1 x[n-n_1]\alpha_2 x[n-n_2-k] \\
&= \alpha_1\alpha_2 \sum_{n\in\mathbb{Z}} x[n-n_1]x[n-n_2-k].
\end{aligned}$$

Let $m = n - n_1 \Rightarrow n - n_2 - k = m + n_1 - n_2 - k = m - \Delta - k$. Then

$$c_{x_1,x_2}[k] = \alpha_1\alpha_2 \sum_{n\in\mathbb{Z}} x[m]x[m-\Delta-k] = \alpha_1\alpha_2 a_x[-\Delta-k].$$

We know that $|a_x[k]|$ is maximized at $k = 0$, therefore $a_x[-\Delta-k]$ is maximized when $-\Delta - k = 0 \Leftrightarrow \Delta = -k$. To determine the time delay Δ, we change the value of k until the crosscorelation between x_1 and x_2 is maximized; then the value of Δ is $-k$. After we have Δ we shift x_1 by Δ divide it with x_2 get $\rho = \frac{\alpha_1}{\alpha_2}$.

ii. We can shift x_1 and x_2 by the same amount and still get the same result for Δ. Therefore we cannot find n_1 and n_2 explicitly. The same thing apply for scaling α_1 and α_2.

2 Studying yet another Linear System

(a) The system is the linear combination of three states of x (i.e. $n-1, n, n+1$.) Therefore it is linear.

$x[n-1-k] + x[n+1-k] - 2x[n-k] = (Lx)[n-k]$. Therefore the system is shift invariant.

The system is defined from both previous and future state of x. Therefore it is not causal.

The system depends on the previous state of x, i.e. $x[n-1]$. Therefore it is not memoryless.

The impulse response of $x[n-1]+x[n+1]-2x$ is 3δ (each has impulse response of δ). Since δ is BIBO stable, the system is BIBO stable.

(b) The sketches of $(x_1, Lx_1), (x_2, Lx_2)$, and (x_3, Lx_3) are showed in Figure 1, 2, and 3, respectively.

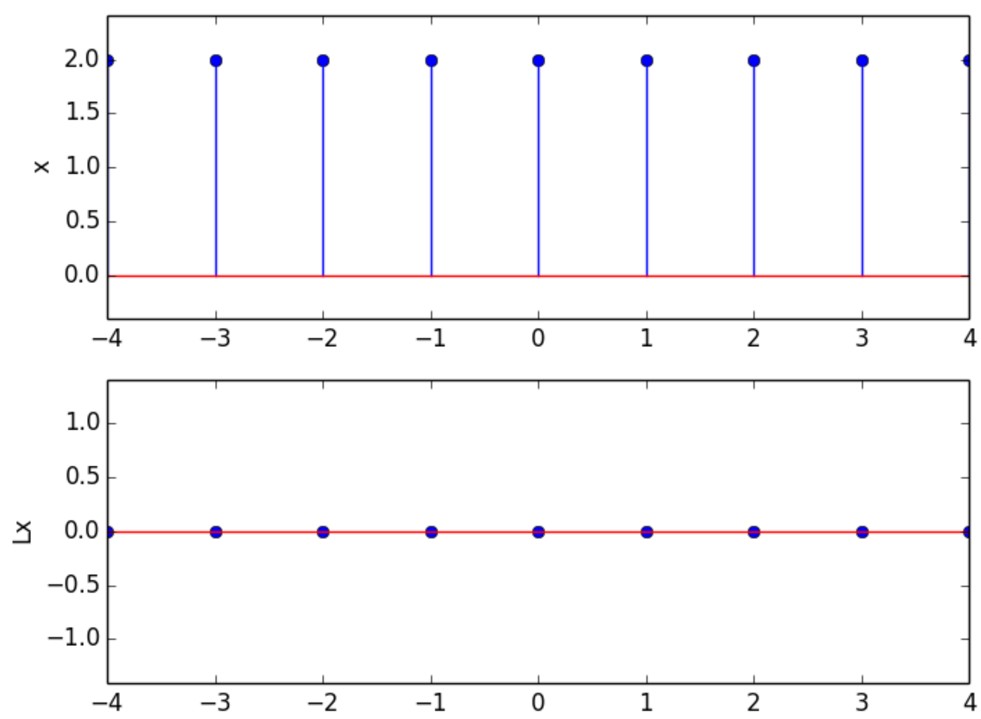

Figure 1: x_1 and Lx_1, where $x_1[n] = c, \forall n \in \mathbb{Z}$. Here $c = 2$, but choice of c does not affect Lx_1.

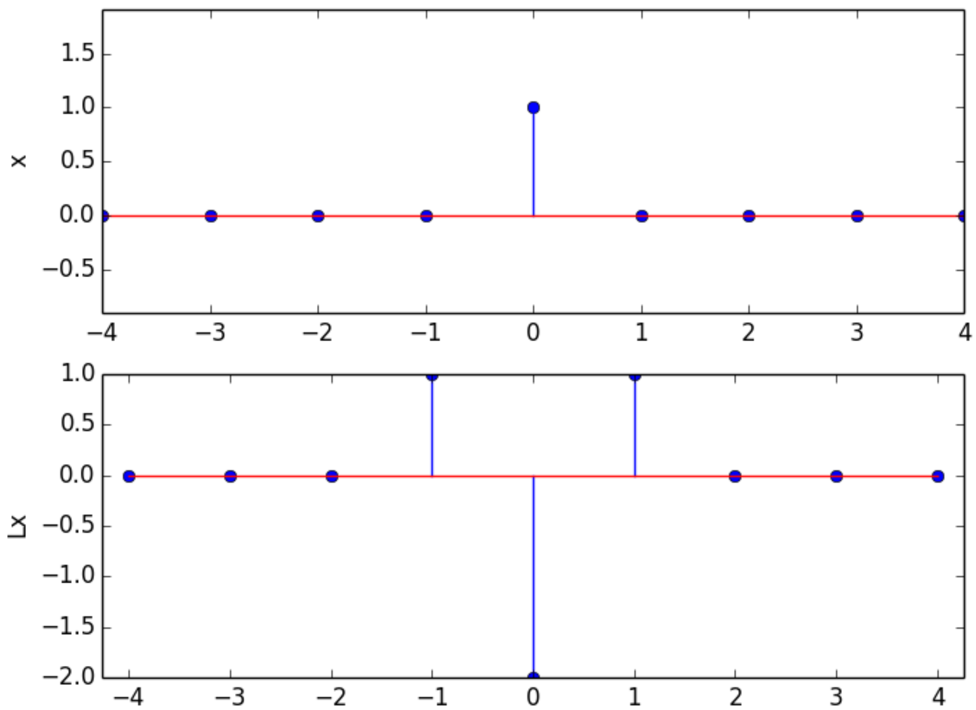

Figure 2: x_2 and Lx_2, where $x_2[n] = \delta[n]$.

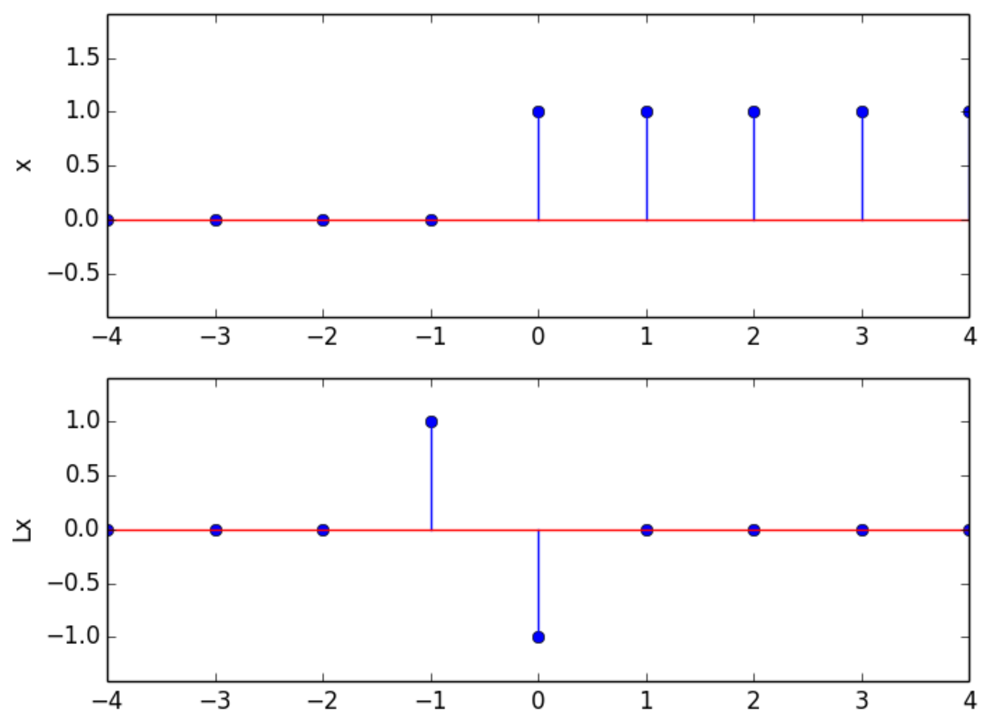

Figure 3: x_3 and Lx_3, where $x_3[n] = u[n]$.

3 DTFT Affairs

(a) Assume that $Z = \frac{1}{2\pi} X \circledast Y$. The inverse DTFT of $Z(\omega)$ is:

$$\begin{aligned}
z[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}(X \circledast Y)(\omega)\right)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\nu)Y(\omega-\nu)d\nu\right)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\nu)Y(\omega-\nu)d\nu\right)e^{j\nu n}e^{j(\omega-\nu)n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\nu)e^{j\nu n}d\nu \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega-\nu)e^{j(\omega-\nu)n}d(\omega-\nu) \\
&= x[n]y[n]
\end{aligned}$$

(b) i. We consider the low pass filter system:

$$G(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{else} \end{cases}$$

Then

$$\begin{aligned}
g[n] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_o} e^{j\omega n}d\omega \\
&= \frac{1}{2\pi jn}\, e^{j\omega n}\Big|_{-\omega_0}^{\omega_0} \\
&= \frac{1}{2\pi jn}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) \\
&= \frac{1}{2\pi jn} 2j\sin(\omega_0 n) \\
&= \begin{cases} \frac{1}{\pi n}\sin(\omega_0 n), & n \neq 0 \\ \frac{\omega_0}{\pi}, & n = 0 \end{cases}
\end{aligned}$$

Therefore, $h[n] = \sqrt{3}\frac{\sin\left(\frac{1}{3}\pi n\right)}{\pi n} = \sqrt{3}g[n]$, where $\omega_0 = \frac{\pi}{3}$. Hence, it is a low pass filter, whose DTFT is

$$H(\omega) = \begin{cases} \sqrt{3}, & |\omega| < \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

ii. We have $x[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$,

$$\Rightarrow X(\omega)$$

Since $y = h * x$, the DTFT of y is

$$Y(\omega) = H(\omega)X(\omega) = \begin{cases} \frac{\sqrt{3}}{2}(1 + e^{-j\omega}), & |\omega| < \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

For $|\omega| < \frac{\pi}{3}$,

$$\begin{aligned} Y(\omega) &= \frac{\sqrt{3}}{2}(1 + e^{-j\omega}) \\ &= \frac{\sqrt{3}}{2} e^{-j\omega/2} e^{j\omega/2}(1 + e^{-j\omega}) \\ &= \frac{\sqrt{3}e^{-j\omega/2}}{2}(e^{j\omega/2} + e^{-j\omega/2}) \\ &= \frac{\sqrt{3}e^{-j\omega/2}}{2} \cos\left(\frac{\omega}{2}\right) \end{aligned}$$

4 The Z-Transform of Autocorrelation

(a) We have

$$\begin{aligned} A_x(z) &= \sum_{n\in\mathbb{Z}} a_x[n] z^{-n} \\ &= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x[k]x[k+n]z^{-n} \\ &= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x[k]x[k+n]z^{-n}z^{k}z^{-k} \\ &= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x[k]z^{k}x[k+n]z^{-(n+k)} \\ &= \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} x[k]z^{k}x[m]z^{-m} \\ &= \sum_{k\in\mathbb{Z}} x[k]z^{k} \sum_{m\in\mathbb{Z}} x[m]z^{-m} \\ &= X(z)X(z^{-1}) \end{aligned}$$

Consider $X(z) = \sum_{n\in\mathbb{Z}} x[n]z^{-n} = \sum_{n\in\mathbb{Z}} (x[n]^{1/n}z^{-1})^n = \frac{1}{1-x[n]^{1/n}z^{-1}}$. Then the $ROC_{X(z)}$ is

$$\left|x[n]^{1/n}z^{-1}\right| < 1 \Leftrightarrow |z| > \left|x[n]^{1/n}\right|.$$

Similarly, for $X(-z) = \frac{1}{1-x[n]^{1/n}z}$, the $ROC_{X(-z)}$ is

$$\left|x[n]^{1/n}z\right| < 1 \Leftrightarrow \left|\frac{1}{z}\right| > \left|x[n]^{1/n}\right| \Leftrightarrow |z| < \left|x[n]^{-1/n}\right|.$$

Hence, the ROC_A is

$$\left\{\left|x[n]^{1/n}\right| < |z| < \left|x[n]^{-1/n}\right|\right\}.$$

(b) i. $x_1[n] = \alpha^n u[n]$, therefore $ROC_{A_{x_1}}$ is

$$\left\{\left|(\alpha^n)^{1/n}\right| < |z| < \left|(\alpha^n)^{-1/n}\right|\right\} = \left\{|\alpha| < |z| < \left|\frac{1}{\alpha}\right|\right\}$$

ii. The z-transform of x_1 is

$$\begin{aligned}
X_1(z) &= \sum_{n\in\mathbb{Z}} x_1[n]z^{-n} \\
&= \sum_{n\in\mathbb{Z}} \alpha^n u[n]z^{-n} \\
&= \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=0}^{\infty} (\alpha z^{-1})^n \\
&= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \\
\Rightarrow A_{x_1}(z) &= X_1(z)X_1(z^{-1}) \\
&= \frac{z}{z-\alpha}\frac{z^{-1}}{z^{-1}-\alpha} \\
&= \frac{1}{(z-\alpha)(z^{-1}-\alpha)} \\
&= z\frac{1}{(z-\alpha)(1-\alpha z)}
\end{aligned}$$

Let $\frac{1}{(z-\alpha)(1-\alpha z)} = \frac{A}{z-\alpha} + \frac{B}{1-\alpha z} = \frac{A(1-\alpha z)+B(z-\alpha)}{(z-\alpha)(1-\alpha z)}$, then:

$$A(1-\alpha z) + B(z-\alpha) = 1 \Rightarrow (-A\alpha + B)z + A + \alpha B = 1$$

$$\Rightarrow \begin{cases} -A\alpha + B = 0 \\ A + \alpha B = 1 \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{1}{\alpha^2+1} \\ B = \frac{\alpha}{\alpha^2+1} \end{cases}$$

Therefore,

$$\begin{aligned}A_{x_1}(z) &= z\left(\frac{1}{(z-\alpha)(\alpha^2+1)} + \frac{1}{(1-\alpha z)(\alpha^2+1)}\right)\\ &= z\left(\frac{1}{z(\alpha^2+1)-\alpha^3-\alpha} + \frac{1}{z(-\alpha^3-\alpha)+\alpha^2+1}\right)\end{aligned}$$

Hence,

$$\begin{aligned}a_{x_1}[n] &= \frac{1}{2\pi j}\int_{ROC_{A_{x_1}}} A_{x_1}(z)z^{-1}dz\\ &= \frac{1}{2\pi j}\int_{ROC_{A_{x_1}}} \left(\frac{1}{z(\alpha^2+1)-\alpha^3-\alpha} + \frac{1}{z(-\alpha^3-\alpha)+\alpha^2+1}\right)dz\\ &= \frac{1}{2\pi j}\left(\frac{\log(\alpha-z)}{\alpha^2+1} + \frac{\log(-(\alpha^2+1)(\alpha z-1))}{\alpha^3+\alpha}\right)\Bigg|_{\alpha}^{1/\alpha}\end{aligned}$$

iii. Two other sequences that are not equal to x_1 and have the same deterministic autocorrelation sequence as that of x_1 are its time shifted versions (with different delays).

5 Some DFT Properties

(a) Define k mod N as $\langle k\rangle_N$, i.e.

$$\langle k+N\rangle_N = \langle k\rangle_N.$$

The DFT of $x[n]$ is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{-kn}$$

where

$$W_N = e^{j\frac{2\pi}{N}} = \cos(2\pi/N) + j\sin(2\pi/N).$$

Assume that $k = lN + r \Leftrightarrow \langle k\rangle_N = r$, then

$$\begin{aligned}W_N^k &= \exp(j\frac{2\pi}{N}(lN+r))\\ &= \exp(j\frac{2\pi}{N}lN)\exp(j\frac{2\pi}{N}r)\\ &= 1\exp(j\frac{2\pi}{N}r)\\ &= W_N^r = W_N^{\langle k\rangle_N}.\end{aligned}$$

Similarly, $W_N^{mk} = W_N^{m\langle k\rangle_N}$.

Furthermore,

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{-n(lN+r)} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{-nlN} W_N^{-nr} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{-nr} \\
&= X[r] = X[\langle k\rangle_N].
\end{aligned}$$

Therefore,

$$\begin{aligned}
DFT(x[\langle -n\rangle_N]) &= \sum_{n=0}^{N-1} x[\langle -n\rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{-\langle -m\rangle_N k} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{mk} \\
&= X[-k] \\
&= X[\langle -k\rangle_N].
\end{aligned}$$

(b) The circular convolution between x and y can be defined as

$$(x \circledast y)[n] = \sum_{m=0}^{N-1} x[m] y[\langle n-m\rangle_N].$$

Therefore

$$
\begin{aligned}
DFT((x \circledast y)[n]) &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m] y[\langle n-m \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} y[\langle n-m \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{-mk} Y[k] \\
&= Y[k] \sum_{m=0}^{N-1} x[m] W_N^{-nk} \\
&= Y[k] X[k]
\end{aligned}
$$

(c) The inverse DFT is defined as

$$
x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{kn}
$$

Therefore

$$
\begin{aligned}
IDFT((X \circledast Y)[n]) &= \frac{1}{N} \sum_{k=0}^{N-1} (X \circledast Y)[k] W_N^{kn} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \left(X[m] Y[\langle k-m \rangle_N] \right) W_N^{kn} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \left(X[m] Y[\langle k-m \rangle_N] \right) W_N^{mn} W_N^{(k-m)n} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_N^{mn} \sum_{k=0}^{N-1} Y[\langle k-m \rangle_N] W_N^{(k-m)n} \\
&= x[n] y[n].
\end{aligned}
$$

(d) If $x[n]$ is real,

$$
\begin{aligned}
&\Rightarrow x[n] = x^*[n] \\
&\Rightarrow DFT(x[n]) = DFT(x^*[n]) \\
&\Rightarrow X[k] = X^*[\langle -k \rangle_N].
\end{aligned}
$$

If $x[n]$ is also symmetric,

$$\begin{aligned}
&\Rightarrow x[n] = x[\langle -n \rangle_N] \\
&\Rightarrow DFT(x[n]) = DFT(x[\langle -n \rangle_N]) \\
&\Rightarrow X[k] = X[\langle -k \rangle_N].
\end{aligned}$$

Therefore

$$X[k] = X^*[\langle -k \rangle_N] = X[\langle -k \rangle_N].$$

Hence, $X[k]$ is real (and also symmetric.)

(e) If $x[n]$ is symmetric (and real),

$$\begin{aligned}
&\Rightarrow x[n] = -x[\langle -n \rangle_N] \\
&\Rightarrow DFT(x[n]) = -DFT(x[\langle -n \rangle_N]) \\
&\Rightarrow X[k] = -X[\langle -k \rangle_N].
\end{aligned}$$

Therefore

$$X[k] = X^*[\langle -k \rangle_N] = -X[\langle -k \rangle_N].$$

Hence, $X[k]$ is imaginary.

6 Z-Transform of Downsampled Signals

Let

$$p_N[n] = \begin{cases} 1, & n = mN \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow P_N(z) = \sum_{n \in \mathbb{Z}} p_N[n] z^{-n} = \sum_{n \in \mathbb{Z}} z^{-Nn} = \sum_{n \in \mathbb{Z}} W^{-n}$$

We have

$$\frac{1}{N} \sum_{k=0}^{N-1} W^{nk} = \begin{cases} \frac{1}{N} \times N = 1, & n = mN \\ 0, & \text{else} \end{cases} = p_N[n]$$

Therefore

$$
\begin{aligned}
Y(z) &= \sum_{n\in\mathbb{Z}} y[n]z^{-n} \\
&= \sum_{n\in\mathbb{Z}} x[Nn]z^{-n} \\
&= \sum_{k=Nn, n\in\mathbb{Z}} x[k]z^{-k/N} \\
&= \sum_{k\in\mathbb{Z}} x[k]z^{-k/N} p_N[k] \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k\in\mathbb{Z}} W^{mk} z^{-k/N} x[k] \\
&= \frac{1}{N}\sum_{m=0}^{N-1} X(W^m z^{1/N})
\end{aligned}
$$

Replacing z with $e^{j\omega}$, we have

7 Interchange of Multirate Operations and LTI Filtering

(a) We have

$$
\begin{aligned}
y &= D_2AD_2AD_2Ax \\
&= D_2(AD_2)(AD_2)Ax \\
&= D_2(D_2A(z^2))(D_2A(z^2))Ax \\
&= D_2D_2(A(z^2)D_2)A(z^2)Ax \\
&= D_2D_2D_2A(z^4)A(z^2)Ax \\
&= D_8A(z^4)A(z^2)A(z)x
\end{aligned}
$$

Hence, the downsampling factor $N = 8$ and $H = A(z^4)A(z^2)A(z)$.

(b) Figure 4 shows the combination $H(\omega)$ if A is an ideal half-band lowpass filter. The cut-off frequency is $\pm\pi/8$.

(c) Figure 5 shows the combination $H(\omega)$ if A is an ideal half-band highpass filter. The cut-off frequency is $\pm\pi/2$. The transfer function captures the highest frequency because lower ones are removed by $A(z)$.

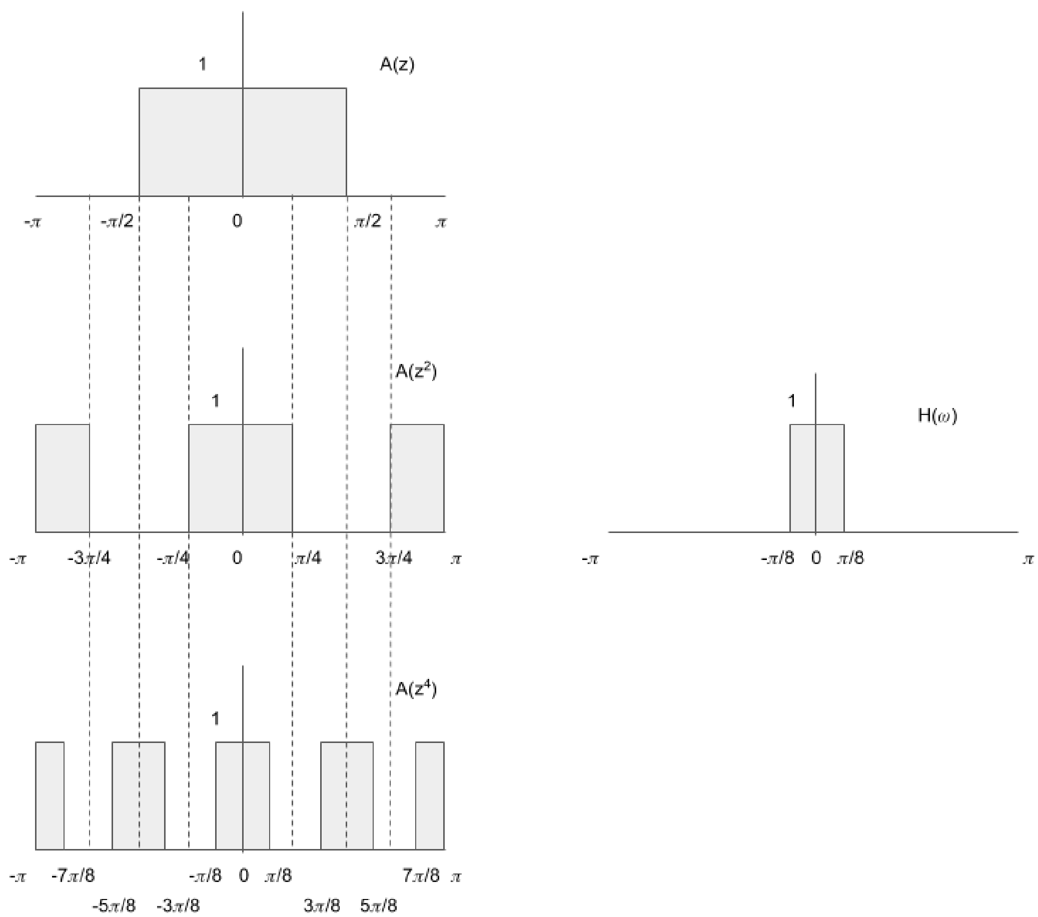

Figure 4: Sketch of the low pass filters and their combination.

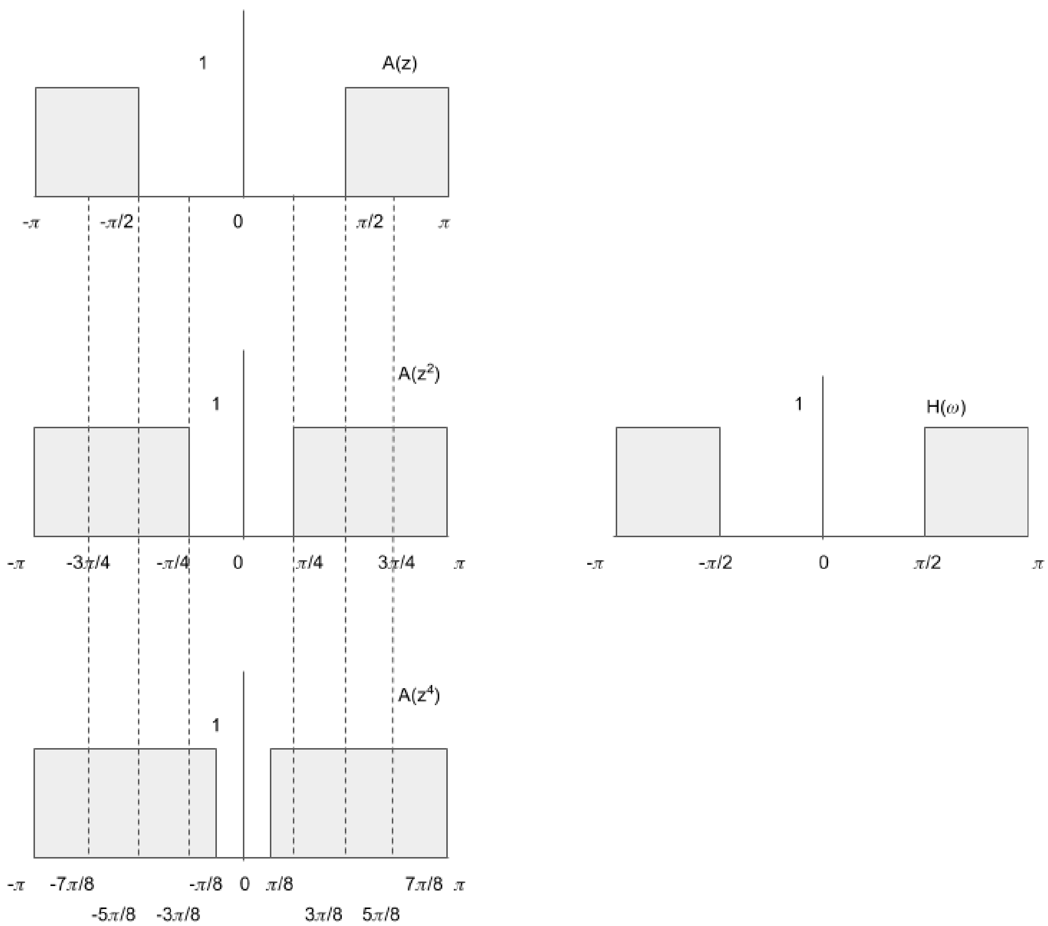

Figure 5: Sketch of the high pass filters and their combination.

8 Python Exercise: Two-Channel Delay Recovery

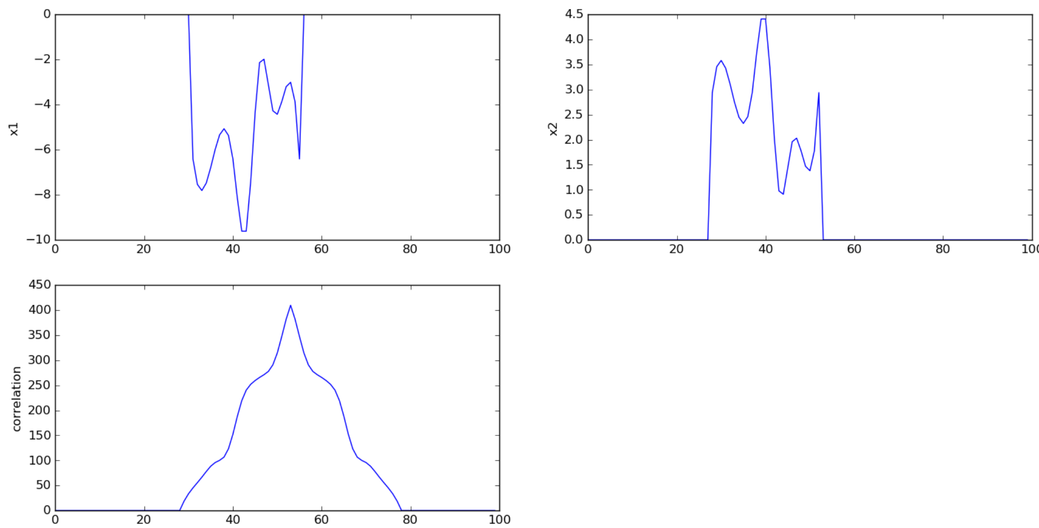

Figure 6: An output of two-channel delay recovery. The top 2 images are x_1 and x_2, where the third one is their cross-correlation.

Figure 6 shows an the generated signals and their cross-correlation. The output of the code is:

```
delta=-3, rho=-2.1807
n2=27, n1=30, n2-n1=-3
alpha1=-1.0682, alpha2=0.4899, alpha1/alpha2=-2.1807
```

9 Python Exercise: Multirate Systems

Figure 7 shows the frequency response of FIR lowpass filter (with sample rate of 100). Although the plots look similar due to scaling, the number of samples differs, i.e. downsampled version has half the amount and upsampled version has twice the amount of samples as the original one.

Figure 8 shows the frequency response of y_1 and y_2 in log-scale. It is obvious that y_1 suffers the aliasing effects (2 spikes towards the end) while that effect on y_2 is not significant.

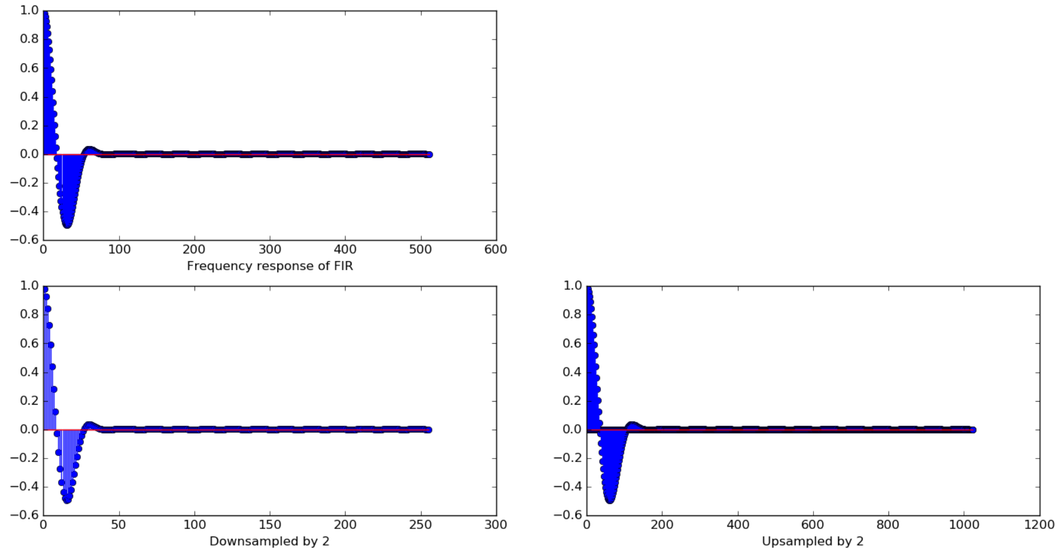

Figure 7: An output of generated FIR filter (frequency response).

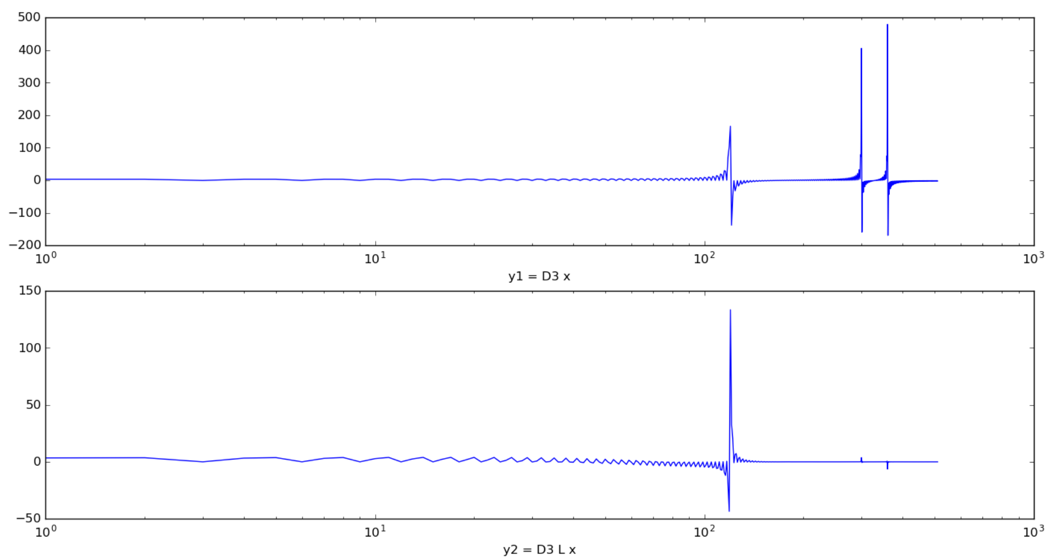

Figure 8: An output of $y_1 = D_3x$ (top) and $y_2 = D_3Lx$ (bottom) (frequency response, in log-scale).