ECE551 - Homework 1

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1 Geometry of orthogonal transformations in Euclidean spaces

1.1

$U \in \mathbb{R}^{N \times N}$ is an orthogonal matrix, therefore:

$$U^\top U = UU^\top = I.$$

By definition,

$$\|x\|^2 := x^\top x = \langle x, x \rangle, \qquad x \in \mathbb{R}^N$$
$$\Rightarrow \|Ux\| = \langle Ux, Ux \rangle = (Ux)^\top Ux = x^\top U^\top Ux = x^\top x = \|x\|^2$$
$$\Rightarrow \|Ux\| = \|x\|, \qquad \because \text{ norms are non-negative.}$$

1.2

By definition,

$$\langle x, y \rangle := y^\top x = \sum_{i=0}^{N-1} x_i y_i, \qquad x, y \in \mathbb{R}^N$$
$$\Rightarrow \langle Ux, Uy \rangle = (Uy)^\top Ux = y^\top U^\top Ux = y^\top x = \langle x, y \rangle.$$

1.3

If $M > N$ and $U^\top U = I$, then $\text{rank}(U) = N \Rightarrow U^\top U = I_N$. Since $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$,

$$x^\top U^\top Ux = x^\top I_N x = x^\top x$$
$$y^\top U^\top Ux = y^\top I_N x = y^\top x.$$

Hence, 1.1 and 1.2 hold.

1.4

If $M < N \Rightarrow \text{rank}(U)$ is at most M.

$$U^\top U \neq I_N.$$

Hence, 1.1 and 1.2 do not hold.

2 Some basic properties of inner product spaces

Notice that:

$$\langle a, b+c \rangle = \overline{\langle b+c, a \rangle} = \overline{\langle b, a \rangle} + \overline{\langle c, a \rangle} = \langle a, b \rangle + \langle a, c \rangle.$$

2.1 The Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\|\|y\|$$

Proof: If $\langle x, y \rangle = 0$, then theorem holds because $\|x\|$ and $\|y\| \geq 0$.
Suppose that $x \neq 0$ and $y \neq 0$. Let $z \in \mathbb{C}$, such that:

$$z = \frac{\langle x, y \rangle}{\|y\|^2} = \frac{\overline{\langle y, x \rangle}}{\|y\|^2}.$$

We have

$$
\begin{aligned}
0 \leq \|x - zy\|^2 &= \langle x - zy, x - zy \rangle = \langle x, x - zy \rangle - \langle zy, x - zy \rangle \\
&= \langle x, x \rangle - \langle x, zy \rangle - \langle zy, x \rangle + \langle zy, zy \rangle \\
&= \|x\|^2 - \bar{z}\langle x, y \rangle - z\langle y, x \rangle + z\bar{z}\|y\|^2 \\
&= \|x\|^2 - \frac{\langle y, x \rangle\langle x, y \rangle}{\|y\|^2} - \frac{\langle x, y \rangle\langle y, x \rangle}{\|y\|^2} + \frac{\langle x, y \rangle}{\|y\|^2}\frac{\langle y, x \rangle}{\|y\|^2}\|y\|^2 \\
&= \|x\|^2 - \frac{\langle x, y \rangle\langle y, x \rangle}{\|y\|^2} \\
&= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}.
\end{aligned}
$$

Therefore,

$$
0 \leq \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}
$$

$$
\Leftrightarrow \|x\|^2 \geq \frac{|\langle x, y \rangle|^2}{\|y\|^2}
$$

$$
\Leftrightarrow \|x\|^2\|y\|^2 \geq |\langle x, y \rangle|^2
$$

$$
\Leftrightarrow \|x\|\|y\| \geq |\langle x, y \rangle|.
$$

$\square$

The equality occurs iff $x = \alpha y$, for some scalar α.

Proof: By substituting $x = \alpha y$, we have

$$|\langle x, y \rangle| = |\langle \alpha y, y \rangle| = |x||\langle y, y \rangle| = |\alpha|\|y\|^2 = |\alpha|\|y\|\|y\| = \|x\|\|y\|.$$

$\square$

2.2   The triangle inequality

$\|x + y\|^2 \le \|x\| + \|y\|$ with equality iff $y = \alpha x$

Proof:

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y \rangle \\
&= \langle x, x + y \rangle + \langle y, x + y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle
\end{aligned}$$

By Cauchy-Schwarz inequality:

$$\begin{aligned}
\langle x, y \rangle &\le \|x\|\|y\| \\
\Rightarrow \langle y, x \rangle &\le \|x\|\|y\| \\
\Rightarrow \langle x, y \rangle + \langle y, x \rangle &\le 2\|x\|\|y\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x + y\|^2 &\le \|x\| + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \\
\Rightarrow \|x + y\| &\le \|x\| + \|y\| \qquad \because \text{norms are non-negative.}
\end{aligned}$$

If $y = \alpha x$ then the equality of Cauchy-Schwarz theorem occurs. Therefore,

$$\begin{aligned}
\langle x, y \rangle + \langle y, x \rangle &= 2\|x\|\|y\| \\
\Rightarrow \|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \\
\Rightarrow \|x + y\| &= \|x\| + \|y\|.
\end{aligned}$$

$\square$

2.3   Parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Proof:

$$
\begin{aligned}
LHS &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\
&= (\|x\|^2 + \|y\|^2 + \langle x,y \rangle + \langle y,x \rangle) + (\|x\|^2 + \|y\|^2 - \langle x,y \rangle - \langle y,x \rangle) \\
&= 2(\|x\|^2 + \|y\|^2) = RHS.
\end{aligned}
$$

$\square$

3 Least-squares approximation with orthonormal bases

3.1

We notice that $\langle x, \varphi_i \rangle \varphi_i$ is the orthogonal projection of x onto φ_i. Since $\varphi_i \in \hat{\mathcal{B}}$ are orthonormal, $\sum_{\varphi_i \in \hat{\mathcal{B}}} \langle x, \varphi_i \rangle \varphi_i$ is the orthogonal projection of x onto the subspace $\text{span}(\hat{\mathcal{B}}) = \hat{V}$.

We need to prove that the Euclidean distance between x and its orthogonal projection on $\hat{V}$ is the shortest among Euclidean distances between x and other vectors in $\hat{V}$.

Let $z = x - \hat{x}$, where $\hat{x} = \sum_{\varphi_i \in \hat{\mathcal{B}}} \langle x, \varphi_i \rangle \varphi_i$. Thus, $z \perp \hat{V}$.

Since any vectors in $\hat{V}$ can be written as $f(\alpha) = \sum_{\varphi_i \in \hat{\mathcal{B}}} \alpha_i \varphi_i$ (α is a vector and α_i's are scalars,) we have:

$$
\begin{aligned}
\|x - f(\alpha)\|^2 &= \|x - \hat{x} + \hat{x} - f(\alpha)\|^2 = \|z + \hat{x} - f(\alpha)\|^2 \\
&= \langle z + \hat{x} - f(\alpha), z + \hat{x} - f(\alpha) \rangle \\
&= \|z\|^2 + \|\hat{x} - f(\alpha)\|^2 + \langle z, \hat{x} - f(\alpha) \rangle + \langle \hat{x} - f(\alpha), z \rangle
\end{aligned}
$$

We can see that $\hat{x} \in \hat{V}$ and $f(\alpha) \in \hat{V}$, therefore $\hat{x} - f(\alpha) \in \hat{V}$. Since $z \perp \hat{V} \Rightarrow z \perp \hat{x} - f(\alpha) \Rightarrow \langle z, \hat{x} - f(\alpha) \rangle = \langle \hat{x} - f(\alpha), z \rangle = 0$.

Therefore, $\|x - f(\alpha)\|^2 = \|z\|^2 + \|\hat{x} - f(\alpha)\|^2$. Since $\|\hat{x} - f(\alpha)\|^2 \geq 0$,

$$
\begin{aligned}
\|x - f(\alpha)\|^2 &\geq \|z\|^2 \\
\Leftrightarrow \|x - f(\alpha)\|^2 &\geq \|x - \hat{x}\|^2 \\
\Leftrightarrow \|x - f(\alpha)\| &\geq \|x - \hat{x}\|
\end{aligned}
$$

3.2

We know that any inner products can define the valid norm

$$
\|x\| = \sqrt{\langle x, x \rangle}.
$$

4

Therefore $\|x - f(\alpha)\|^2 = \langle z + \hat{x} - f(\alpha), z + \hat{x} - f(\alpha)\rangle$, with other inner products. The expansion

$$\langle z + \hat{x} - f(\alpha), z + \hat{x} - f(\alpha)\rangle = \|z\|^2 + \|\hat{x} - f(\alpha)\|^2 + \langle z, \hat{x} - f(\alpha)\rangle + \langle \hat{x} - f(\alpha), z\rangle$$

also uses the properties of general inner products. Hence, the proof holds for other kinds of inner products, instead of only standard Euclidean one.

4 Signal Sets and Spaces