

# ECE551 - Homework 7

Khôi-Nguyễn Mac

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## 1 Truncation as Filter Approximation

(a) Let  $\psi = \{\varphi_k\}$  be the basis of  $\mathbb{C}^{\mathbb{Z}}$

$$h_d \in \mathbb{C}^{\mathbb{Z}} \Rightarrow h_d = \sum_{\varphi_k \in \psi} \alpha_k \varphi_k$$

Since  $I \subset \mathbb{Z}$ ,  $\mathbb{C}^I \subset \mathbb{C}^{\mathbb{Z}}$ , where  $\mathbb{C}^I = \text{span}\{\phi^I\}$ ,  $\phi^I \subset \phi$ .

$$\begin{aligned} T_I h_d &= \sum_{\varphi_k \in \psi} w[k] \alpha_k \varphi_k \\ &= \sum_{\varphi_k \in \psi^I} 1 \cdot \alpha_k \varphi_k + \sum_{\varphi_k \in \psi / \psi^I} 0 \cdot \alpha_k \varphi_k \\ &= \sum_{\varphi_k \in \psi^I} \alpha_k \varphi_k \in \text{span}\{\psi^I\} = \mathbb{C}^I \\ \Rightarrow T_I h_d - h_d &= \sum_{\varphi_k \in \psi / \psi^I} \alpha_k \varphi_k \end{aligned}$$

$$\Rightarrow \langle T_I h_d - h_d, T_I h_d \rangle = 0 \Rightarrow T_I h_d - h_d \perp T_I h_d$$

By orthogonality principal,  $T_I h_d$  is the least square approximation of  $h_d$  on  $\ell_2(I)$ .

(b) Let  $h \in \mathbb{C}^{\mathbb{R}}$

$$\begin{aligned}
T_I h &= \sum_{\varphi_k \in \psi_I} 1 \cdot \alpha_k \phi_k + \sum_{\varphi_k \in \psi/\psi^I} 0 \cdot \alpha_k \phi_k \\
&= \sum_{\varphi_k \in \psi_I} \alpha_k \phi_k \\
&= \hat{h} \\
T_I(T_I h) &= \sum_{\varphi_k \in \psi_I} 1 \cdot \alpha_k \phi_k + \sum_{\varphi_k \in \psi^I/\psi^I} 0 \cdot \alpha_k \phi_k \\
&= \sum_{\varphi_k \in \psi_I} \alpha_k \phi_k \\
&= \hat{h} \\
&\Rightarrow T_I = T_I^2
\end{aligned}$$

Let  $x, y \in \mathbb{C}^{\mathbb{R}}$

$$\begin{aligned}
\langle T_I x, y \rangle &= \left\langle \sum_{\varphi_k \in \psi_I} 1 \cdot \alpha_k \phi_k + \sum_{\varphi_k \in \psi/\psi^I} 0 \cdot \alpha_k \phi_k, \sum_{\varphi_k \in \psi_I} \beta_k \phi_k + \sum_{\varphi_k \in \psi/\psi^I} \beta_k \phi_k \right\rangle \\
&= \left\langle \sum_{\varphi_k \in \psi_I} \alpha_k \phi_k + \sum_{\varphi_k \in \psi/\psi^I} \alpha_k \phi_k, \sum_{\varphi_k \in \psi_I} 1 \cdot \beta_k \phi_k + \sum_{\varphi_k \in \psi/\psi^I} 0 \cdot \beta_k \phi_k \right\rangle \\
&= \langle x, T_I y \rangle \\
&\Rightarrow T_I = T_I^*
\end{aligned}$$

Hence,  $T_I$  is an orthogonal projection.

(c) For  $I = \{0, \dots, 4\}$ ,

$$T_I h_d = [\dots \quad 0 \quad \text{sinc}0 \quad \text{sinc}\frac{\pi}{3} \quad \text{sinc}\frac{2\pi}{3} \quad \text{sinc}1 \quad \text{sinc}\frac{4\pi}{3} \quad 0 \quad \dots]^{\top}$$

(d) We can choose  $I$  as  $\{-2, -1, 0, 1, 2\}$ , so  $T_I h_d$  is

$$T_I h_d = [\dots \quad 0 \quad -\text{sinc}\frac{2\pi}{3} \quad -\text{sinc}\frac{\pi}{3} \quad \text{sinc}0 \quad \text{sinc}\frac{\pi}{3} \quad \text{sinc}\frac{2\pi}{3} \quad 0 \quad \dots]^{\top}$$

## 2 Lagrange Interpolation

(a) We have

$$p_{\bar{D}}(t) = p_D(t) + c(t - t_0)(t - t_1) \cdots (t - t_{N-1}) = p_D(t) = p_D(t) + c \prod_{j=0}^{N-1} (t - t_j)$$

We want  $p_{\tilde{D}}(t_N) = x_N$ , so

$$c \prod_{j=0}^{N-1} (t - t_j) = x_N \Leftrightarrow c = \frac{x_N}{\prod_{j=0}^{N-1} (t_N - t_j)}$$

(b) We already have

$$\begin{aligned} p_{\tilde{D}}(t) &= p_D(t) + \frac{x_N}{\prod_{j=0}^{N-1} (t_N - t_j)} \prod_{j=0}^{N-1} (t - t_j) \\ &= p_D(t) + x_N \prod_{j=0}^{N-1} \frac{t - t_j}{t_N - t_j} \\ &= p_{D^{(-1)}}(t) + x_{N-1} \prod_{j=0}^{N-2} \frac{t - t_j}{t_{N-1} - t_j} + x_N \prod_{j=0}^{N-1} \frac{t - t_j}{t_N - t_j} \end{aligned}$$

where  $D^{(-1)} = D / \{(t_{N-1} m x_{N-1})\} = \{(t_k, x_k)\}_{k=0}^{N-2}$ . Therefore, in general

$$\begin{aligned} p_{\tilde{D}}(t) &= \sum_{k=0}^N x_k \prod_{j \neq k} \frac{t - t_j}{t_k - t_j} \\ \Rightarrow p_D(t) &= \sum_{k=0}^{N-1} x_k \prod_{j \neq k} \frac{t - t_j}{t_k - t_j} \end{aligned}$$

### 3 Polynomial Spaces with Orthogonality

(a) Let  $v \in V_n$ , then

$$v = \sum_{j=0}^n \alpha_j v_j$$

$$\deg(v) = \max\{\deg(v_j)\}_{j=0}^n \leq n$$

Therefore  $v$  can be written as  $\sum_{j=0}^n \beta_j t^j$

$$\Rightarrow v \in W_n \Rightarrow V_n \subset W_n$$

We have

$$\begin{aligned} \dim(V_n) &= n & \because \langle v_k, v_j \rangle &= \delta[k - j] \\ \dim(W_n) &= n & \because \{1, t^1, t^2, \dots, t^n\} &\text{are independent} \end{aligned}$$

So  $\dim(V_n) = \dim(W_n)$ . Hence,  $V_n = W_n$ .

(b)  $p$  is a polynomial of degree  $m$ , so  $p \in V_n = W_n$ .

$$p = \sum_{j=0}^m \langle p, v_j \rangle v_j$$

For  $k > m$ ,

$$\begin{aligned} \langle p, v_k \rangle &= \left\langle \sum_{j=0}^m \langle p, v_j \rangle v_j, v_k \right\rangle \\ &= \sum_{j=0}^m \langle p, v_j \rangle \langle v_j, v_k \rangle \\ &= 0 \quad \because \langle v_k, v_k \rangle = 0 \end{aligned}$$

(c)  $v \in V_n = W_n \Rightarrow v(t) = \sum_{j=0}^n \alpha_j t^j$

$$\begin{aligned} \sum_{j=0}^n \alpha_j (t - t_0)^j &= \sum_{j=0}^n \alpha_j \left( \binom{j}{i} t^{j-i} (-t_0)^i \right) \\ &= \sum_{j=0}^n \alpha_j \binom{j}{i} t^j \frac{(-t_0)^i}{t^i} \\ &= \sum_{j=0}^n \left( \alpha_j \binom{j}{i} \frac{(-t_0)^i}{t^i} \right) t^j \end{aligned}$$

Since  $i \leq j$ ,  $\sum_{j=0}^n \left( \alpha_j \binom{j}{i} \frac{(-t_0)^i}{t^i} \right) t^j$  is a polynomial of degree up to  $n$ . So we can write it as

$$\sum_{j=0}^n \alpha_j (t - t_0)^j = \sum_{j=0}^n \beta_j t^j$$

Hence, it is shift-invariant.

## 4 Polynomial Spaces vs. Spline Spaces

(a) Figure 1 shows the graph of  $s_0, s_1 \in U$

(b) By inspecting Figure 1, if we increase  $N$ ,  $s_0$  will approach  $p_0(t) = 1$  and  $s_1$  will approach  $p_1(t) = t$ . Hence,  $p_0(t)$  and  $p_1(t)$  are contained (as limit) in  $U$ .

(c)  $u_0, u_1$  belong to  $U$  as well as  $V_1$ . However,  $V_1 = \text{span}\{v_0, v_1\} \Rightarrow \dim(V_1) = 2$  and  $\dim(U) = \infty$ . Hence  $V_1 \neq U$ .

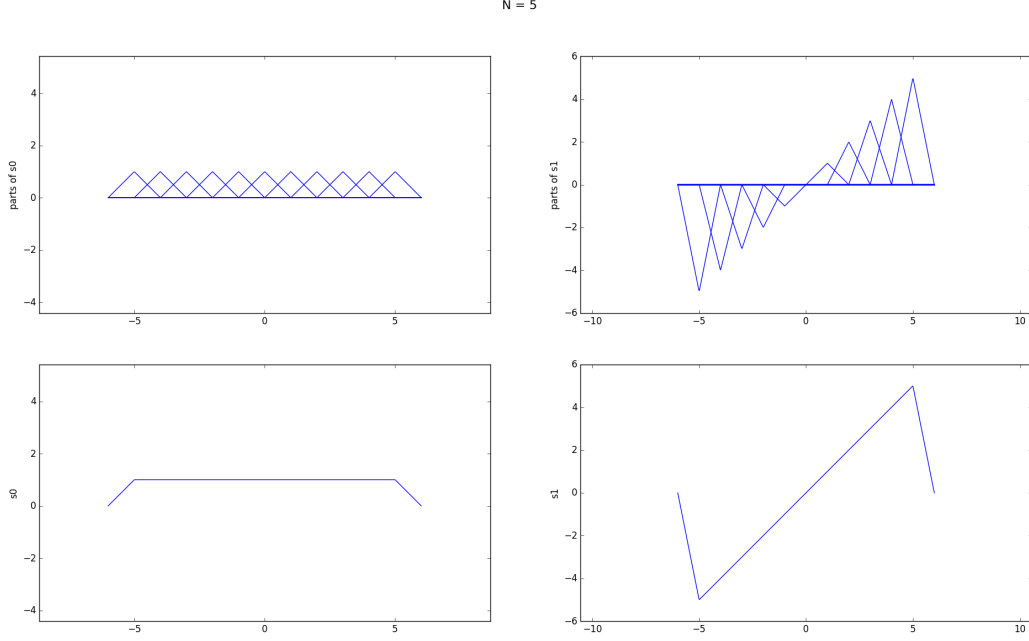


Figure 1:  $s_0$  (left) and  $s_1$  (right) with  $N = 5$

## 5 Interpolation with Shifted Symmetric Functions

(a) We are given the coefficients  $\{c[k]\}$ , so

$$\begin{aligned}
 s(t) &= \sum_{k \in \mathbb{Z}} c[k] \phi(t - kT) \\
 \Rightarrow s(nT) &= x[n] = \sum_{k \in \mathbb{Z}} c[k] \phi(nT - kT) \\
 &= \sum_{k \in \mathbb{Z}} c[k] \phi((n - k)T) \\
 &= \sum_{k \in \mathbb{Z}} c[k] b[n - k]
 \end{aligned}$$

where  $b[m] = \phi(mT)$ . So

$$x = c * b \Rightarrow X(z) = C(z)B(z) \Rightarrow C(z) = \frac{1}{B(z)}X(z) = H(z)X(z)$$

where  $H(z) = \frac{1}{B(z)}$ .

To enable this, we require  $B(z) \neq 0 \Leftrightarrow \phi(jT) \neq 0, \forall j$ .

(b) If  $\phi(t) = \phi(-t)$ , then

$$\begin{aligned}\phi(jT) &= \phi(-jT) \\ \Leftrightarrow b[j] &= b[-j]\end{aligned}$$

If  $\lambda$  is a pole/root of  $H(z)$  then  $B(\lambda) = 0$ . We have

$$\begin{aligned}B(z^{-1}) &= \sum_{n=-N}^N b[n]z^n \\ &= \sum_{n=-N}^N b[-n]z^n \\ &= \sum_{m=-N}^N b[m]z^{-m} \quad \because m = -n \\ &= B(z)\end{aligned}$$

Therefore  $\lambda^{-1}$  is also a root of  $H(z)$ .

(c) Assume that  $\lambda_j$  is a pole of  $H(z)$ , then  $z = \lambda_j \Rightarrow 1 - \lambda_j z^{-1} = 0$ . Since  $\lambda_j^{-1}$  is also a pole,  $z = \lambda_j^{-1} \Rightarrow 1 - \lambda_j z = 0$ . Therefore, we can write  $H(z)$  as

$$H(z) = \frac{1}{\prod_j (1 - \lambda_j z^{-1})} \cdot \frac{1}{\prod_j (1 - \lambda_j z)}$$

Let  $G(z) = \frac{1}{\prod_j (1 - \lambda_j z^{-1})}$  (causal), then  $G(z^{-1}) = \frac{1}{\prod_j (1 - \lambda_j z)}$ . Hence,

$$H(z) = G(z)G(z^{-1}), \quad \text{with } G(z) = \frac{1}{\prod_j (1 - \lambda_j z^{-1})}$$

(d) We notice that  $G(z)$  is an IIR filter, so we can use direct form I to sketch  $G(z)$  and  $G(z^{-1})$  is sketched by replacing  $z^{-1}$  blocks in  $G(z)$  by  $z$  blocks (Figure 2).

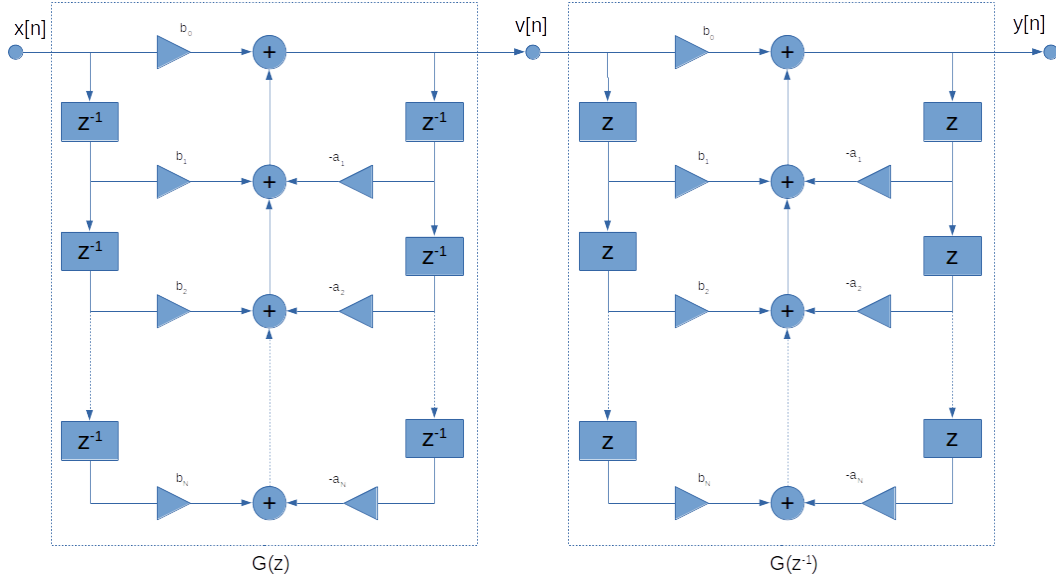


Figure 2:  $H(z)$  as a cascade of causal  $G(z)$  and anti-causal  $G(z^{-1})$

## 6 Python: Interpolation Games

For  $\phi_2$ , we consider the equation  $1/(z/8 + 3z/4 + z^{-1}/8)$ :

$$\begin{aligned}
 \frac{1}{\frac{1}{8}z + \frac{3}{4} + \frac{1}{8}z^{-1}} &= \frac{8}{z + 6 + z^{-1}} \\
 &= \frac{8}{z^{-1}(z^2 + 6z + 1)} \\
 &= \frac{8}{z^{-1}(z + 3 + 2\sqrt{2})(z + 3 - 2\sqrt{2})} \\
 &= \frac{8}{(1 + (3 + 2\sqrt{2})z^{-1})(z + 3 - 2\sqrt{2})} \\
 &= \frac{\frac{8}{3+2\sqrt{2}}}{(z^{-1} + 3 - 2\sqrt{2})(z + 3 - 2\sqrt{2})} \\
 &= \frac{\sqrt{\frac{8}{3+2\sqrt{2}}}}{z^{-1} + 3 - 2\sqrt{2}} \cdot \frac{\sqrt{\frac{8}{3+2\sqrt{2}}}}{z + 3 - 2\sqrt{2}}
 \end{aligned}$$

Therefore, we can choose  $\mu = \sqrt{\frac{8}{3+2\sqrt{2}}}$  and  $\gamma = 2\sqrt{2} - 3$ .

Figure 3 shows the interpolation results on the UIN and Figure 3 shows the results on 5 points inputted by mouse. The results of  $\phi_0, \phi_1$ , and  $\phi_3$  are the same

for the second figure.

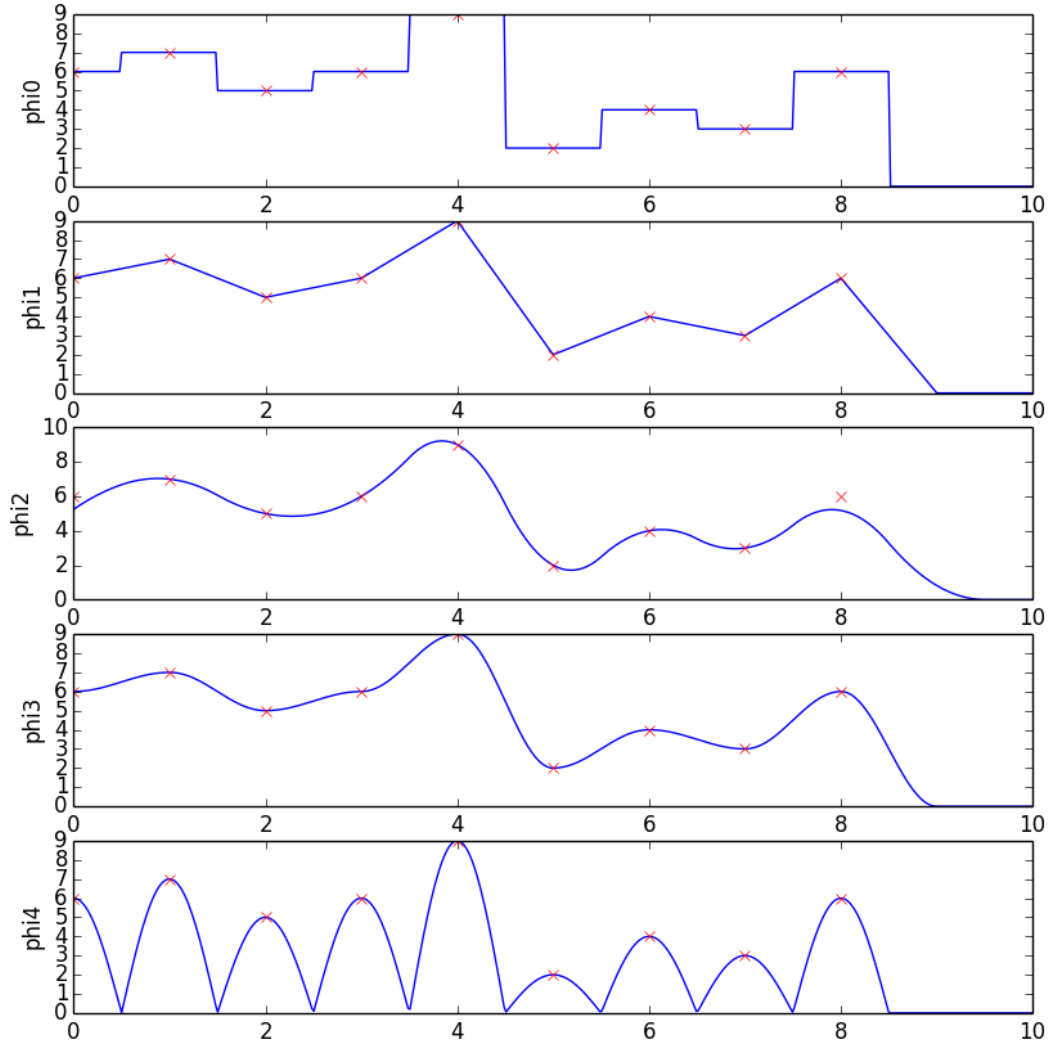


Figure 3: Interpolation results on the UIN



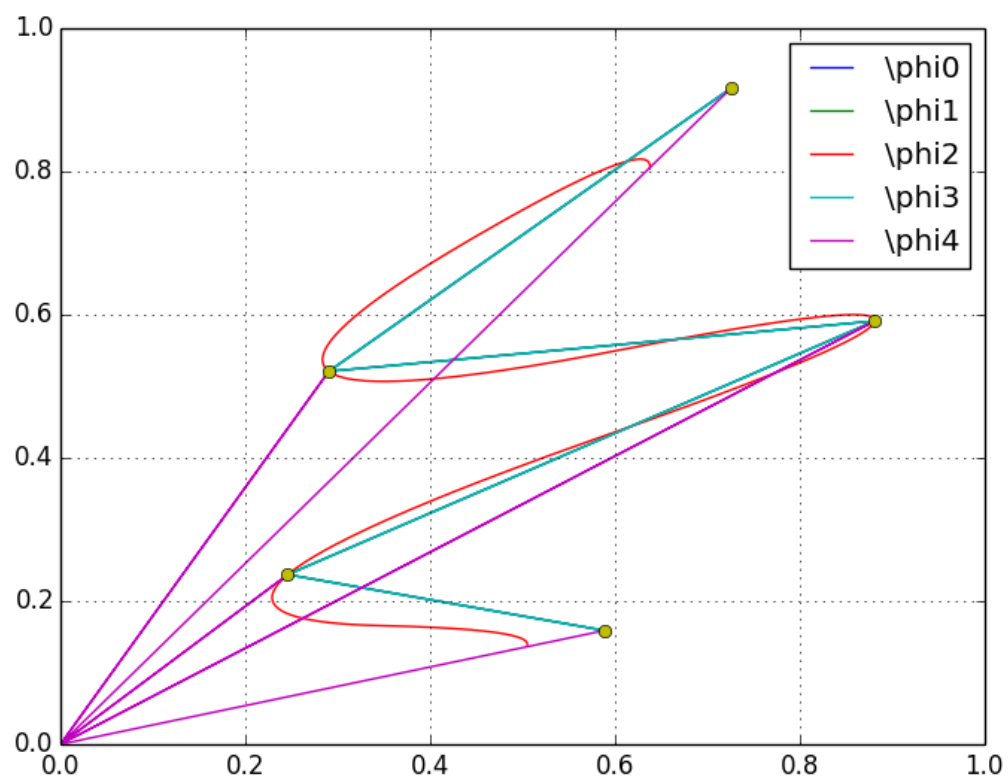


Figure 4: Interpolation results on 5 inputted points