ECE551 - Homework 7

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1 Truncation as Filter Approximation

(a) Let $\psi = \{\varphi_k\}$ be the basis of $\mathbb{C}^{\mathbb{Z}}$

$$h_d \in \mathbb{C}^{\mathbb{Z}} \Rightarrow h_d = \sum_{\varphi_k \in \psi} \alpha_k \varphi_k$$

Since $I \subset \mathbb{Z}$, $\mathbb{C}^I \subset \mathbb{C}^{\mathbb{Z}}$, where $\mathbb{C}^I = \text{span}\{\phi^I\}, \phi^I \subset \phi$.

$$\begin{aligned} T_I h_d &= \sum_{\varphi_k \in \psi} w[k] \alpha_k \varphi_k \\ &= \sum_{\varphi_k \in \psi^I} 1 \cdot \alpha_k \varphi_k + \sum_{\varphi_k \in \psi / \psi^I} 0 \cdot \alpha_k \varphi_k \\ &= \sum_{\varphi_k \in \psi^I} \alpha_k \varphi_k \in \text{span}\{\psi^I\} = \mathbb{C}^I \\ \Rightarrow T_I h_d - h_d &= \sum_{\varphi_k \in \psi / \psi^I} \alpha_k \varphi_k \end{aligned}$$

$$\Rightarrow \langle T_I h_d - h_d, T_I h_d \rangle = 0 \Rightarrow T_I h_d - h_d \perp T_I h_d$$

By orthogonality principal, $T_I h_d$ is the least square approximation of h_d on $\ell_2(I)$.

(b) $\forall z \in \mathbb{C}^{\mathbb{Z}}$, we have

$$T_I z = \sum_{\varphi_k \in \psi} \beta_k \varphi_k \perp T_I h_d - h_d = \sum_{\varphi_k \in \psi / \psi^I} \alpha_k \varphi_k$$

$$\Rightarrow \langle T_I z, T_I h_d - h_d \rangle = 0, \forall z \in \mathbb{C}^{\mathbb{Z}}$$

Hence, T_I is an orthogonal projection.

(c) For $I = \{0, \cdots, 4\}$,

$$T_I h_d = \begin{bmatrix} \cdots & 0 & \text{sinc}0 & \text{sinc}\frac{\pi}{3} & \text{sinc}\frac{2\pi}{3} & \text{sinc}1 & \text{sinc}\frac{4\pi}{3} & 0 & \cdots \end{bmatrix}^\top$$

(d) We can choose I as $\{-2, -1, 0, 1, 2\}$, so $T_I h_d$ is

$$T_I h_d = \begin{bmatrix} \cdots & 0 & -\text{sinc}\frac{2\pi}{3} & -\text{sinc}\frac{\pi}{3} & \text{sinc}0 & \text{sinc}\frac{\pi}{3} & \text{sinc}\frac{2\pi}{3} & 0 & \cdots \end{bmatrix}^\top$$

2 Lagrange Interpolation

(a)

(b)

3 Polynomial Spaces with Orthogonality

(a) Let $v \in V_n$, then

$$v = \sum_{j=0}^{n} \alpha_j v_j$$
$$\deg(v) = \max\{\deg(v_j)\}_{j=0}^{n} \leq n$$

Therefore v can be written as $\sum_{j=0}^{n} \beta_j t^j$

$$\Rightarrow v \in W_n \Rightarrow V_n \subset W_n$$

We have

$$\dim(V_n) = n \qquad \because \langle v_k, v_j \rangle = \delta[k-j]$$
$$\dim(W_n) = n \qquad \because \{1, t^1, t^2, \cdots t^n\} \text{ are independent}$$

So $\dim(V_n) = \dim(W_n)$. Hence, $v_n = W_n$.

(b) p is a polynomial of degree m, so $p \in V_n = W_n$.

$$p = \sum_{j=0}^{m} \langle p, v_j \rangle v_j$$

For $k > m$,

$$\begin{aligned} \langle p, v_k \rangle &= \left\langle \sum_{j=0}^{m} \langle p, v_j \rangle v_j, v_k \right\rangle \\ &= \sum_{j=0}^{m} \langle p, v_j \rangle \langle v_j, v_k \rangle \\ &= 0 \qquad \because \langle v_k, v_k \rangle = 0 \end{aligned}$$

(c) $v \in V_n = W_n \Rightarrow v(t) = \sum_{j=0}^{n} \alpha_j t^j$

$$\begin{aligned}\sum_{j=0}^{n} \alpha_j (t-t_0)^j &= \sum_{j=0}^{n} \alpha_j \left(\binom{j}{i} t^{j-i} (-t_0)^i \right) \\ &= \sum_{j=0}^{n} \alpha_j \binom{j}{i} t^j \frac{(-t_0)^i}{t^i} \\ &= \sum_{j=0}^{n} \left(\alpha_j \binom{j}{i} \frac{(-t_0)^i}{t^i} \right) t^j\end{aligned}$$

Since $i \leq j$, $\sum_{j=0}^{n} \left(\alpha_j \binom{j}{i} \frac{(-t_0)^i}{t^i} \right) t^j$ is a polynomial of degree up to n. So we can write it as

$$\sum_{j=0}^{n} \alpha_j (t-t_0)^j = \sum_{j=0}^{n} \beta_j t^j$$

Hence, it is shift-invariant.

4 Polynomial Spaces vs. Spline Spaces

(a) Figure 1 shows the graph of $s_0, s_1 \in U$

(b)

(c)

5 Interpolation with Shifted Symmetric Functions

(a)

(b)

(c)

6 Python: Interpolation Games

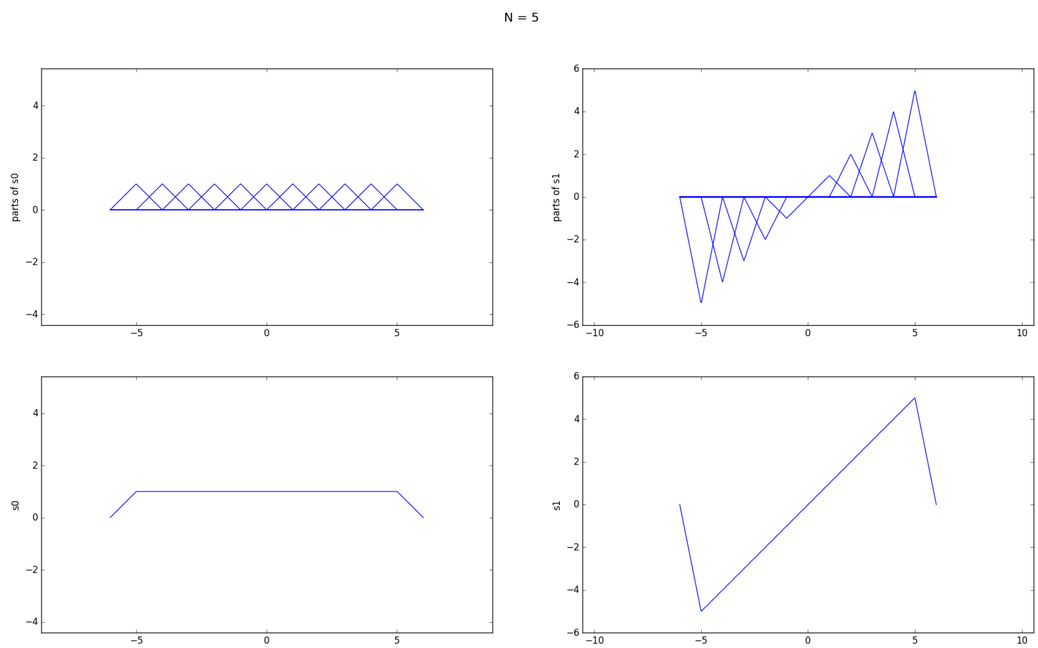

Figure 1: s_0 and s_1 with $N = 5$