ECE551 - Homework 5

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1 Sampling and Interpolation for Band-Limited Vectors

(a) We have the Fourier vector

$$w_k[n] = e^{j\frac{2\pi kn}{M}}$$

and

$$2\cos x = e^{jx} + e^{-jx}$$

So

$$\begin{aligned}
x_1[n] &= 1 + \cos\left(\frac{2\pi n}{M}\right) + \cos\left(\frac{8\pi n}{M}\right) \\
&= w_0[n] + \frac{1}{2}\left(e^{j\frac{2\pi n}{M}} + e^{-j\frac{2\pi n}{M}}\right) + \frac{1}{2}\left(e^{j\frac{8\pi n}{M}} + e^{-j\frac{8\pi n}{M}}\right) \\
&= w_0[n] + \frac{1}{2}\left(w_1[n] + w_{-1}[n] + w_4[n] + w_{-4}[n]\right)
\end{aligned}$$

Its DFT is

$$\begin{aligned}
X_1[k] &= \sum_{n=0}^{M-1} x_1[n] w_{-k}[n] \\
&= \frac{1}{2}\left(\sum_{n=0}^{M-1} 2w_{-k}[n] + w_{-k-1}[n] + w_{-k+1}[n] + w_{-k-4}[n] + w_{-k+4}[n]\right)
\end{aligned}$$

We can see that

$$\begin{aligned}
\sum_{n=0}^{M-1} w_k[n] &= \sum_{n=0}^{M-1} \exp\left(j\frac{2\pi k}{M}\right)^n \\
&= \frac{1-\exp\left(j\frac{2\pi k}{M}\right)^M}{1-\exp\left(j\frac{2\pi k}{M}\right)} \qquad (\because \text{geometric series}) \\
&= A
\end{aligned}$$

Whenever the numerator of A is 0 (k=0), its denominator is also 0. Therefore A has a peak at k. Hence, $X_1[k]$ has peaks at $k = 0, \pm 1, \pm 4$, so its bandwidth is $[-4, 4]$.

Similarly,

$$x_2[n] = \cos\left(\frac{3\pi n}{M}\right) = \frac{1}{2}\left(e^{j\frac{3\pi n}{M}} + e^{-j\frac{3\pi n}{M}}\right)$$

$$\Rightarrow X_2[n] = \sum_{n=0}^{M-1} x_2[n] e^{-j\frac{2\pi kn}{M}}$$

$$= \frac{1}{2}\sum_{n=0}^{M-1}\left(e^{j\frac{2\pi n}{(}3-2k)} + e^{j\frac{2\pi n}{(}-3-2k)}\right)$$

$$= \frac{1}{2}\left(\frac{1-\exp\left(j\frac{\pi n}{M}(3-2k)\right)^M}{1-\exp\left(j\frac{\pi n}{M}(3-2k)\right)} + \frac{1-\exp\left(j\frac{\pi n}{M}(-3-2k)\right)^M}{1-\exp\left(j\frac{\pi n}{M}(-3-2k)\right)}\right) \neq 0, \forall k \in \mathbb{Z}$$

Hence, $x_2[n]$ is full-band.

(b) We take

$$\Phi = \left[w_0, w_1, ..., w_{\frac{k_0+1}{2}-1}, w_{M-\frac{k_0+1}{2}+1}, ..., w_{M-1}\right]$$

Because x is band limited s.t. $X[k] = 0, \forall k \in \left[\frac{k_0+1}{2}, M - \frac{k_0+1}{2}\right]$, it means that we remove the part from $\frac{k_0+1}{2}$ to $M - \frac{k_0+1}{2}$ of the DFT.

2 Band Limited Space with Rational Sampling Rate Changes

(a) Since we only care about the effect of g, we consider only until $g[n]$ is apply (the first 5 steps).

Figure 1 shows the results for $M = 2, N = 3, K = 3$. After upsampling by 2 (second row), we need the cut-off frequency of g to be $\frac{\pi}{3} \leq w_c \leq \frac{\pi}{3}$. By applying the low-pass filter $g[-n]$ (third row), the gap between two copies is $\frac{5\pi}{3} - \frac{\pi}{3} = \frac{4\pi}{3}$. After downsampling by 3 (forth row), the gap is reduced to 0, so upsampling it by 3 (fifth row) also gives the same gap. So the cut-off frequency has to be $w_c = \frac{\pi}{3}$.

(b) Figure 2 shows the results for$M = 2, N = 3, K = 4$. After upsampling by 2 (second row), we need the cut-off frequency of g to be $\frac{\pi}{4} \leq w_c \leq \frac{3\pi}{4}$. After apply downsampling by 3 (forth row), the gap is $\frac{\pi}{2}$. Therefore the gap is reduced by

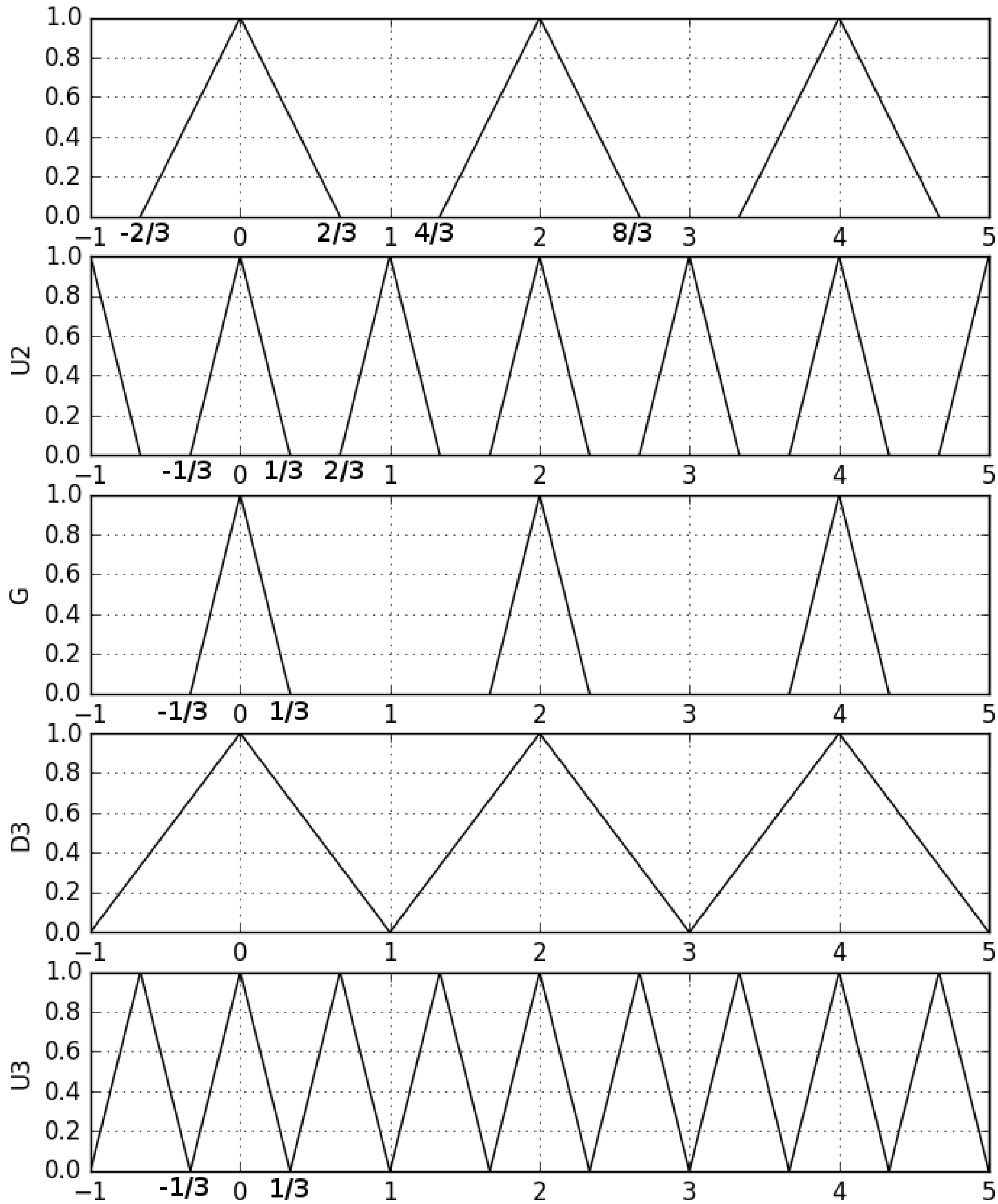

Figure 1: $M = 2, N = 3, K = 3$ (x scale is π)

a third after upsampling by 3 (fifth row). So the second condition for cut-off frequency is $\frac{\pi}{4} \le w_c \le \frac{5\pi}{12}$.

$$\frac{\pi}{4} \le w_c \le \frac{3\pi}{4} \text{ and } \frac{\pi}{4} \le w_c \le \frac{5\pi}{12} \Rightarrow \frac{\pi}{4} \le w_c \le \frac{5\pi}{12}$$

(c) If the signal in $[-\frac{2\pi}{K}, \frac{2\pi}{K}]$, the gap's width is

$$2\pi - \frac{2\pi}{K} - \frac{2\pi}{K} = 2\pi(1 - \frac{2}{K})$$

After upsampling by M, the range is $[-\frac{2\pi}{KM}, \frac{2\pi}{KM}]$ and the gap is $\frac{2\pi}{M}(1 - \frac{2}{K})$. Therefore the first condition of w_c is

$$\frac{2\pi}{KM} \le w_c \le \frac{2\pi}{KM} + \frac{2\pi}{M}(1 - \frac{2}{K}) \Leftrightarrow \frac{2\pi}{KM} \le w_c \le \frac{2\pi}{M}(1 - \frac{1}{K})$$

After downsampling by N, the $[-\frac{2\pi N}{KM}, \frac{2\pi N}{KM}]$ and the gap is $2\pi - \frac{2\pi N}{KM} - \frac{2\pi N}{KM} = 2\pi(1 - \frac{2}{KM})$.

Therefore, after upsampling by N, the lower bound of the first copy (after at frequency of 0) is

$$\frac{1}{N} \cdot \frac{2\pi N}{KM} + \frac{1}{N} \cdot 2\pi(1 - \frac{2N}{KM}) = \frac{2\pi}{KM} + \frac{2\pi}{N} - \frac{4\pi}{KM} = \frac{2\pi}{N} - \frac{2\pi}{KM} = 2\pi(\frac{1}{N} - \frac{1}{KM})$$

So the second condition of w_c is

$$\frac{2\pi}{KM} \le w_c \le 2\pi(\frac{1}{N} - \frac{1}{KM})$$

Combining the first and second condition gives

$$\frac{2\pi}{KM} \le w_c \le 2\pi(\frac{1}{N} - \frac{1}{KM}) \qquad (\because M < N \text{ so the second condition is tighter})$$

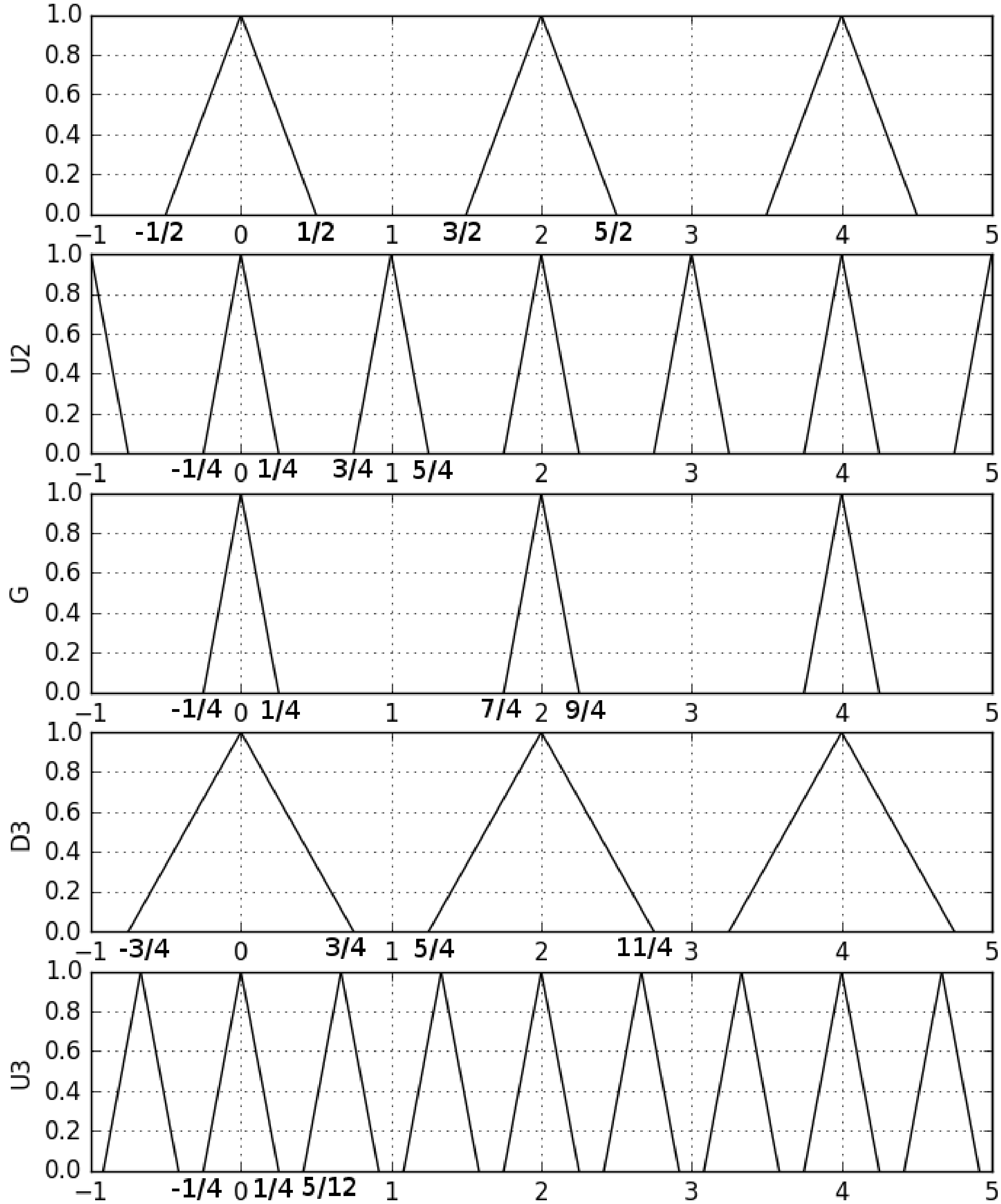

Figure 2: $M = 2, N = 3, K = 4$ (x scale is π)

3 Multirate Systems

(a) Let $u[n]$ be the output after downsampling and $v[n]$ be the output after convolving with g.

$$\begin{aligned}
U(z) &= \frac{1}{2}\sum_{k=0}^{1} X\left(e^{-j\frac{2\pi k}{2}} z^{1/2}\right) \\
&= \frac{1}{2}\left(X(z^{1/2} + X(e^{-j\pi} z^{1/2}))\right) \\
V(z) &= G(z)U(z) \\
Y(z) &= V(z^3) \\
&= G(z^3)U(z^3) \\
&= \frac{1}{2}\left(X(z^{3/2} + X(e^{-j\pi} z^{3/2}))\right)
\end{aligned}$$

(b) $x[n] = q(nT)$ is the same as downsampling by T, so $u[z]$ is obtained by downsampling $q(t)$ by $2T$. Therefore

$$\begin{aligned}
U(\omega) &= \frac{1}{2T}\sum_{k=0}^{2T-1} Q\left(\frac{\omega - 2\pi k}{2T}\right) \\
V(\omega) &= G(\omega)U(\omega) \\
Y(\omega) &= V(3\omega) \\
&= G(3\omega)U(3\omega) \\
&= \frac{G(3\omega)}{2T}\sum_{k=0}^{2T-1} Q\left(\frac{3\omega - 2\pi k}{2T}\right)
\end{aligned}$$

Since $q \in BL[-\frac{\pi}{T}, \frac{\pi}{T}]$ and the sampling rate $\frac{1}{T}$, we have the setting in Figure 3 To avoiud aliasing, we need $\frac{\pi}{T} < \frac{1}{2T} \Leftrightarrow \frac{2\pi - 1}{T} < 0$, which cannot satisfy. Therefore, q cannot avoid aliasing after sampling.

4 Pseudo-Inverse of Interpolation Filter: Single Channel Case

By orthogonality principle

$$x - \hat{x} \perp V = \mathrm{span}\{\sigma^{kN} g\}_{k\in\mathbb{Z}} \Leftrightarrow \left\langle x - \hat{x}, \sigma^{kN} g\right\rangle = 0$$

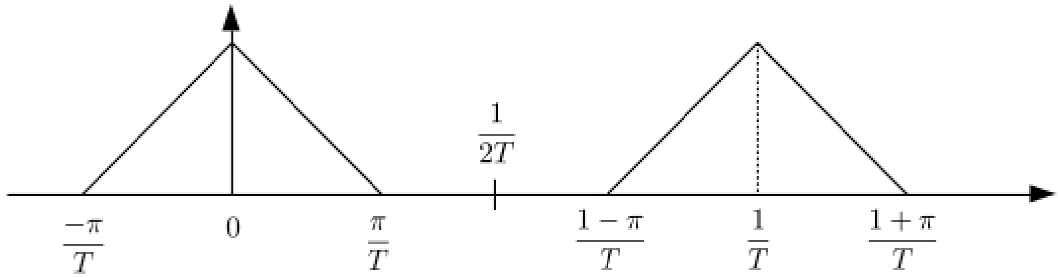

Figure 3: Sampling $q \in BL[-\frac{\pi}{T}, \frac{\pi}{T}]$ with sampling rate of $\frac{1}{T}$

Recall that convolution can be written as inner product, i.e. $(u * v)[n] = \langle u, \sigma^n \tilde{v} \rangle$. Therefore

$$\Leftrightarrow \left\langle x - \hat{x}, \sigma^{kN} g \right\rangle = ((x - \hat{x}) * \tilde{g})[kN] = 0$$

Let $s = (x - \hat{x}) * \tilde{g}$, its z-transform is

$$S(z) = (X(z) - \hat{X}(x))G(z^{-1}) = 0$$

where $X(z), \hat{X}(z)$, and $G(z^{-1})$ are the z-transform of $x[n], \hat{x}[n]$, and $\tilde{g}[n]$. We have the polyphase decomposition as

$$\begin{aligned}
X(z) &= \pi(z) X_p(z^N) \\
G(z) &= G_p(z^N)^\top \pi(z)^\top \\
\hat{X}(z) &= \pi(z)\hat{X}_p(z^N) = \pi(z) G_p(z^N) \tilde{H}_p(z^N) X_p(z^N)
\end{aligned}$$

(because $z^N = \omega$ and $\hat{X}_p(\omega) = G_p(\omega)\tilde{H}_p(\omega)X_p(\omega)$). Substituting them to $S(z)$ gives:

$$\begin{aligned}
S(z) &= \left[\pi(z)X_p(z^N) - \pi(z)G_p(z^N)\tilde{H}_p(z^N)X_p(z^N)\right] G_p(z^{-N})^\top \pi(z^{-1})^\top \\
&= G_p(z^{-N})^\top \pi(z^{-1})^\top \pi(z) \left[X_p(z^N) - G_p(z^N)\tilde{H}_p(z^N)X_p(z^N)\right]
\end{aligned}$$

We know that $\pi(z^{-1})^\top \pi(z) = I - A(z)$, where $A(z)$ contains non-zero phases, thus will vanish $G_p(z^{-N})^\top A(z) \left[X_p(z^N) - G_p(z^N)\tilde{H}_p(z^N)X_p(z^N)\right]$. Therefore

$$\begin{aligned}
S(z) &= G_p(z^{-N})^\top \left[X_p(z^N) - G_p(z^N)\tilde{H}_p(z^N)X_p(z^N)\right] \\
&= G_p(z^{-N})^\top X_p(z^N) - G_p(z^{-N})^\top G_p(z^N)\tilde{H}_p(z^N)X_p(z^N) \\
&= \left[G_p(z^{-N})^\top - G_p(z^{-N})^\top G_p(z^N)\tilde{H}_p(z^N)\right] X_p(z^N) = 0, \forall X_p(z^N) \\
&\Rightarrow G_p(z^{-N})^\top - G_p(z^{-N})^\top G_p(z^N)\tilde{H}_p(z^N) = 0
\end{aligned}$$

Since $\omega = z^n$

$$\begin{aligned}
&G_p(\omega^{-1})^\top - G_p(\omega^{-1})^\top G_p(\omega)\tilde{H}_p(\omega) = 0 \\
\Leftrightarrow &G_p(\omega^{-1})^\top G_p(\omega)\tilde{H}_p(\omega) = G_p(\omega^{-1})^\top \\
\Leftrightarrow &\tilde{H}_p(\omega) = \left(G_p(\omega^{-1})^\top G_p(\omega)\right)^{-1} G_p(\omega^{-1})^\top
\end{aligned}$$

5 Ideal-Matched Sampling and Interpolation with Nonorthogonal Filters

(a) $N = 2 \Rightarrow \omega = z^N = z^2$

z-transform of $g[n]$ is

$$G(z) = 1 + \frac{1}{2}(z^{-1} + z)$$

Since $N = 2$, we need components z^0 and z^{-1}

$$\Rightarrow G(z) = 1 + z^{-1}\left(\frac{1}{2} + \frac{1}{2}z^2\right)$$

For type-I polyphase decomposition

$$G_p(\omega) = \begin{bmatrix} G_0(\omega) \\ G_1(\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\omega \end{bmatrix} \Rightarrow G_p(\omega^{-1})^\top = \begin{bmatrix} 1, & \frac{1}{2} + \frac{1}{2}\omega^{-1} \end{bmatrix}$$

From problem 4, we have

$$\tilde{H}_p(\omega) = \left(G_p(\omega^{-1})^\top G_p(\omega)\right)^{-1} G_p(\omega^{-1})^\top$$

$$\begin{aligned}
G_p(\omega^{-1})^\top G_p(\omega) &= \begin{bmatrix} 1, & \frac{1}{2} + \frac{1}{2}\omega^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\omega \end{bmatrix} \\
&= 1 + \frac{1}{4}(1 + \omega^{-1})(1 + \omega) \\
&= \frac{6 + \omega + \omega^{-1}}{4}
\end{aligned}$$

$$\Rightarrow \tilde{H}_p(\omega) = \left(\frac{6 + \omega + \omega^{-1}}{4}\right)^{-1} \begin{bmatrix} 1, & \frac{1}{2} + \frac{1}{2}\omega^{-1} \end{bmatrix}$$

We have

$$\begin{aligned}
H(z) &= \tilde{H}_p(z^2)\pi(z^{-1})^\top \qquad \text{(type-II, with } z^2=\omega\text{)}\\
&= \frac{4}{6+\omega+\omega^{-1}}\begin{bmatrix}1, & \frac{1}{2}+\frac{1}{2}\omega^{-1}\end{bmatrix}\begin{bmatrix}1\\ z\end{bmatrix}\\
&= \frac{4(1+\frac{z}{2}+\frac{z^{-1}}{2})}{6+z^2+z^{-2}}\\
&= \frac{2(2+z+z^{-1})}{6+z^2+z^{-2}}
\end{aligned}$$

For the numerator

$$\begin{aligned}
2(2+z+z^{-1}) &= 2z^{-1}(2z+z^2+1)\\
&= 2z^{-1}(z+1)(z+1)\\
&= 2(z^{-1}+1)(z+1)
\end{aligned}$$

For the denominator

$$\begin{aligned}
6+z^2+z^{-2} &= z^{-2}(6z^2+z^4+1)\\
&= z^{-2}(z^2+3+2\sqrt{2})(z^2+3-2\sqrt{2})\\
&= (1+(3+2\sqrt{2})z^{-2})(z^2+3-2\sqrt{2})\\
&= (1+(3+2\sqrt{2})z^{-2})\left(z^2+\frac{1}{3+2\sqrt{2}}\right)\\
&= (3+2\sqrt{2})\left(z^{-2}+\frac{1}{3+2\sqrt{2}}\right)\left(z^2+\frac{1}{3+2\sqrt{2}}\right)\\
&= \mathcal{C}^{-1}(z^{-2}+\mathcal{C})(z^2+\mathcal{C}), \qquad \text{where } \mathcal{C}=\frac{1}{3+2\sqrt{2}}
\end{aligned}$$

Therefore

$$\begin{aligned}
H(Z) &= \frac{2(z^{-1}+1)(z+1)}{\mathcal{C}^{-1}(z^{-2}+\mathcal{C})(z^2+\mathcal{C})}\\
&= 2\mathcal{C}\frac{z^{-1}+1}{z^{-2}+\mathcal{C}}\frac{z^1+1}{z^2+\mathcal{C}}\\
&= P(z^{-1})P(z), \qquad \text{where } P(z)=\sqrt{2\mathcal{C}}\frac{z+1}{z^2+\mathcal{C}}, \mathcal{C}=\frac{1}{3+2\sqrt{2}}
\end{aligned}$$

$h[n]$ is the inverse z-transform of $H(z)$.

(b)

$$\tilde{H}_p(\omega)G_p(\omega) = 1 \Leftrightarrow \tilde{H}_p(\omega)\begin{bmatrix} 1 \\ \frac{1}{2}+\frac{1}{2}\omega \end{bmatrix} = 1$$

Therefore, the shortest $\tilde{H}_p(\omega)$ is $\begin{bmatrix} 0 & 1 \end{bmatrix}$

$$H(z) = \tilde{H}_p(z^2)\pi(z^{-1})^\top = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix} = 1$$

Hence, $h = \delta$

6 Python Exercise: DTFT Approximation using DFT

Figure 4-8 show the output with $N = 4096, M = 200$. The runtime for `dtft_approx` is 0.1s and `eq_dtft_approx` is 0.0005s. Figure 9-13 show the output with $N = 4096, M = 5000$. The runtime for `dtft_approx` is 2.7s and `eq_dtft_approx` is still 0.0005s. `dtft_approx`'s runtime significantly increases with $M = 5000$; however, M does not affect `eq_dtft_approx` because it is implemented using `fft`. Figure 14-18 show the output with $N = 400, M = 200$. The runtime of `eq_dtft_approx` is around 0.0001s and `dtft_approx`'s is around 0.02s.

Overall, the results of `dtft_approx` is similar to `eq_dtft_qpprox` but with significantly longer runtime. Increasing M or N makes the results finer and decreasing N or M makes the them rougher. Changing M only affects the frequency responses while changing N affect the original signals directly.

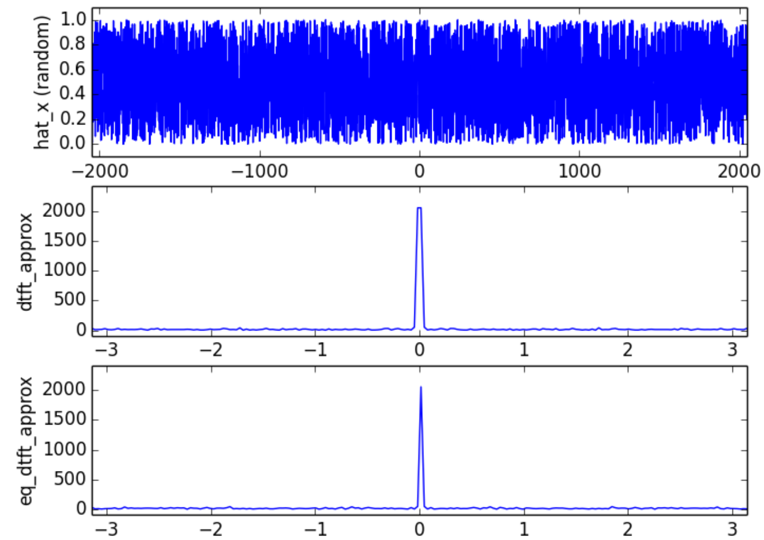

Figure 4: Random signal with $N = 4096, M = 200$

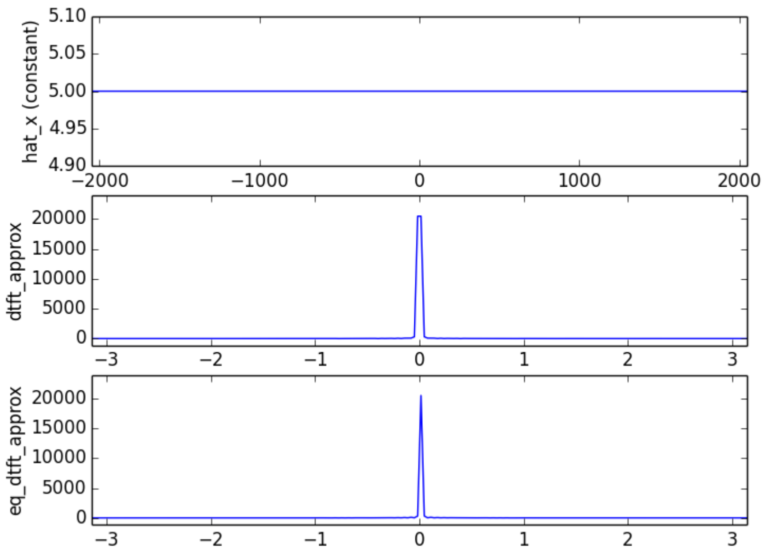

Figure 5: Constant signal with $N = 4096, M = 200$

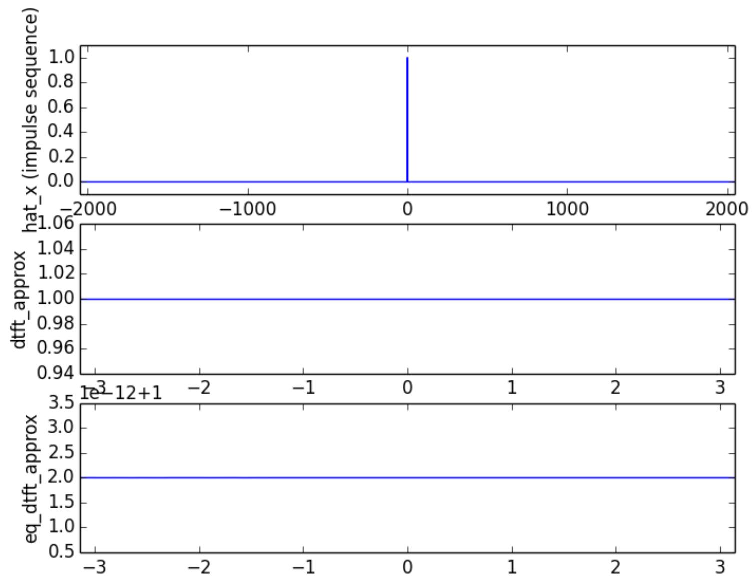

Figure 6: Impulse signal ($x[n] = \delta[n]$) with $N = 4096, M = 200$

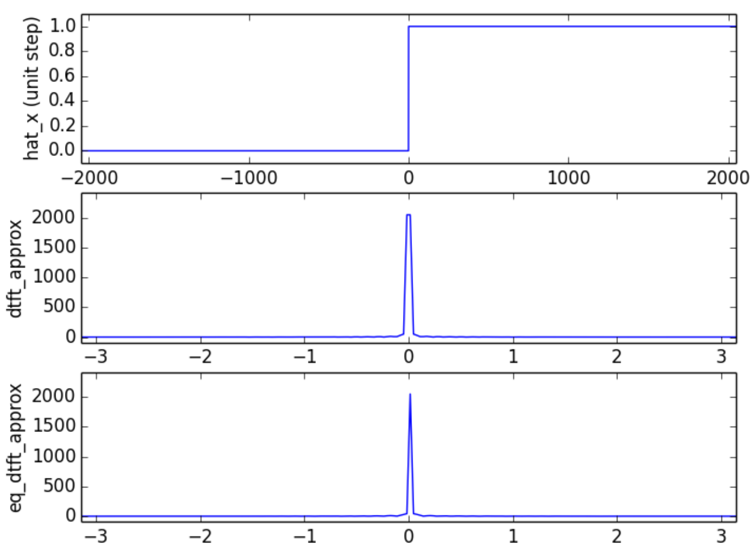

Figure 7: Unit step signal ($x[n] = u[n]$) with $N = 4096, M = 200$

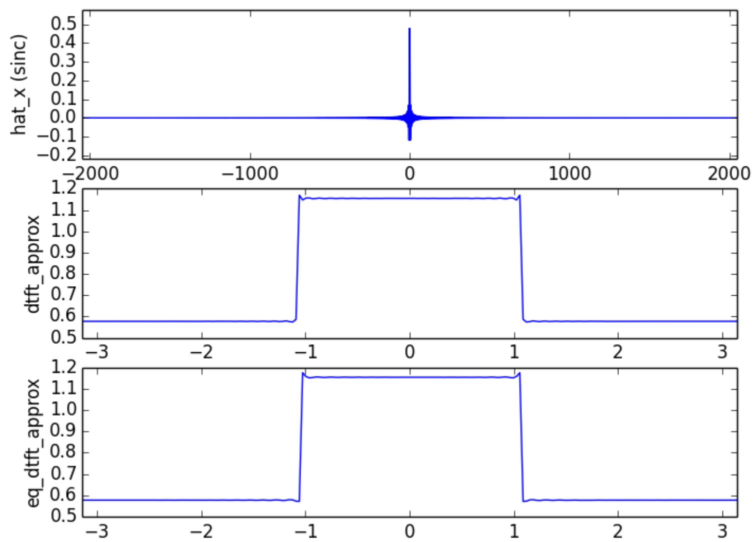

Figure 8: $x[n] = \sqrt{3}\frac{\sin(\frac{\pi n}{3})}{\pi n}$ with $N = 4096, M = 200$

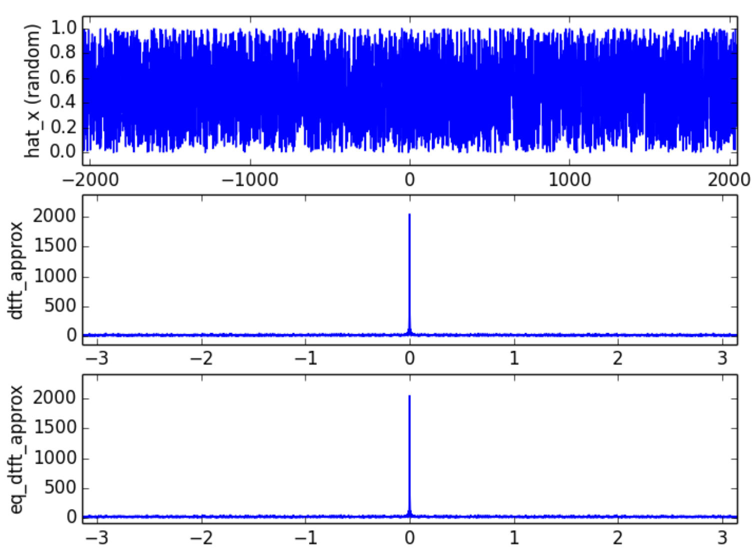

Figure 9: Random signal with $N = 4096, M = 5000$

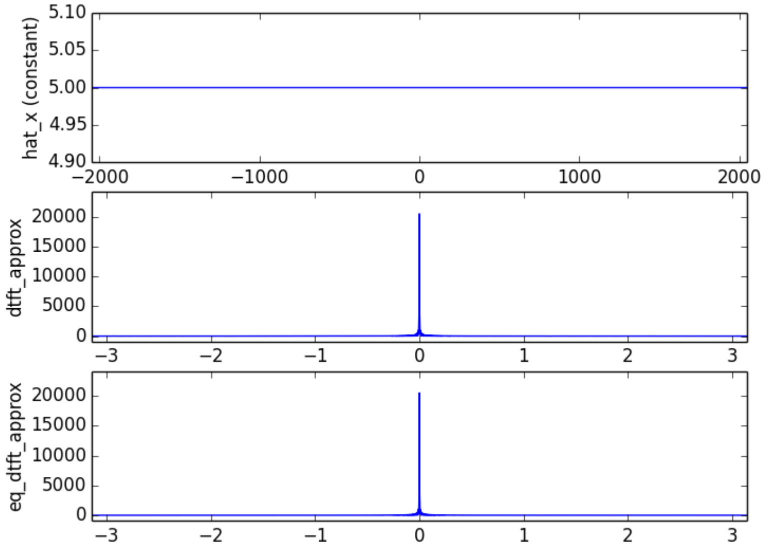

Figure 10: Constant signal with $N = 4096, M = 5000$

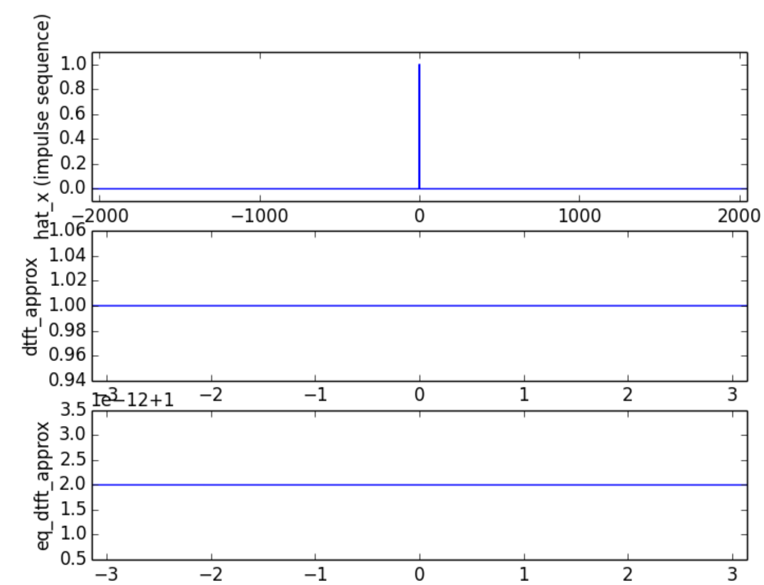

Figure 11: Impulse signal ($x[n] = \delta[n]$) with $N = 4096, M = 5000$

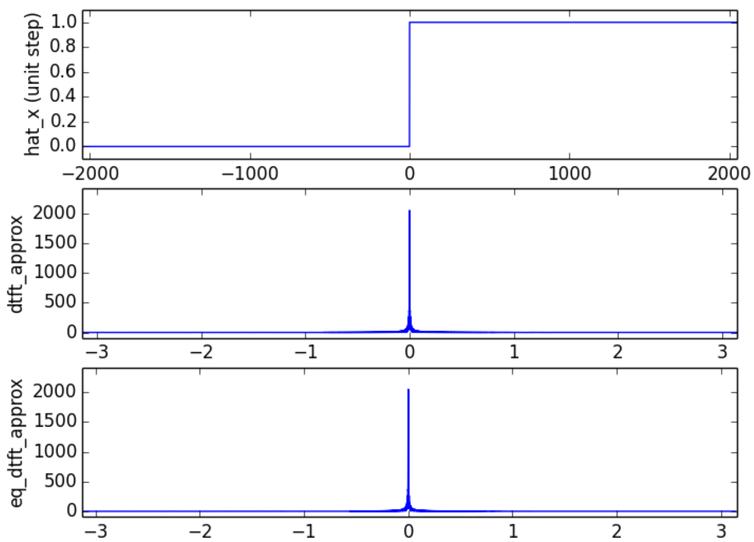

Figure 12: Unit step signal ($x[n] = u[n]$) with $N = 4096, M = 5000$

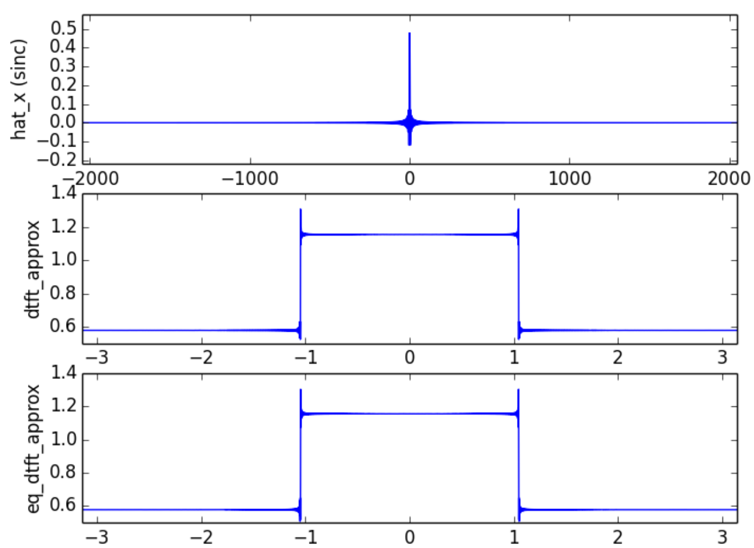

Figure 13: $x[n] = \sqrt{3}\frac{\sin(\frac{\pi n}{3})}{\pi n}$ with $N = 4096, M = 5000$

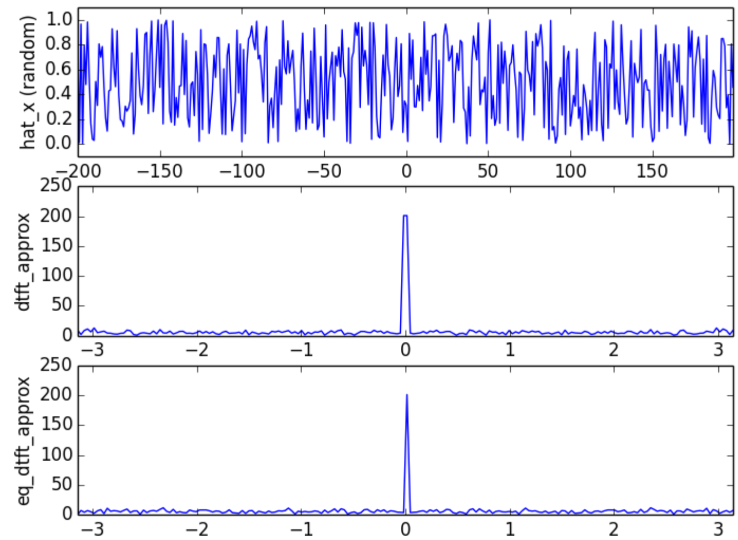

Figure 14: Random signal with $N = 400, M = 200$

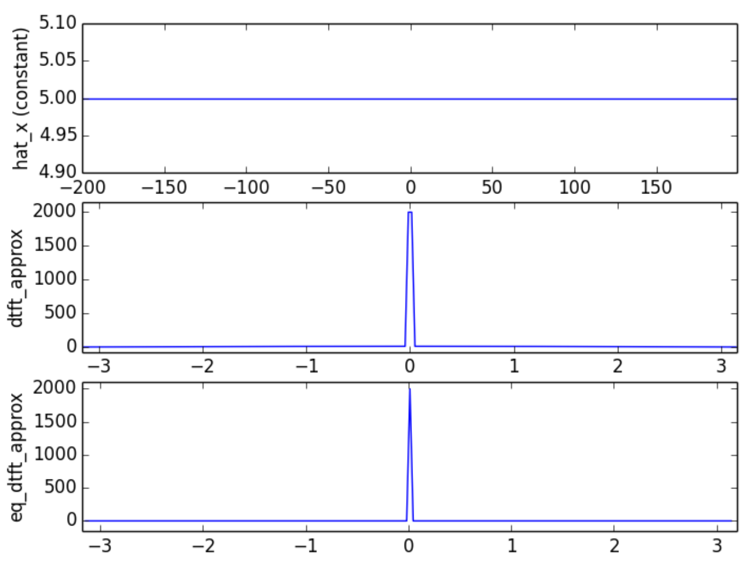

Figure 15: Constant signal with $N = 400, M = 200$

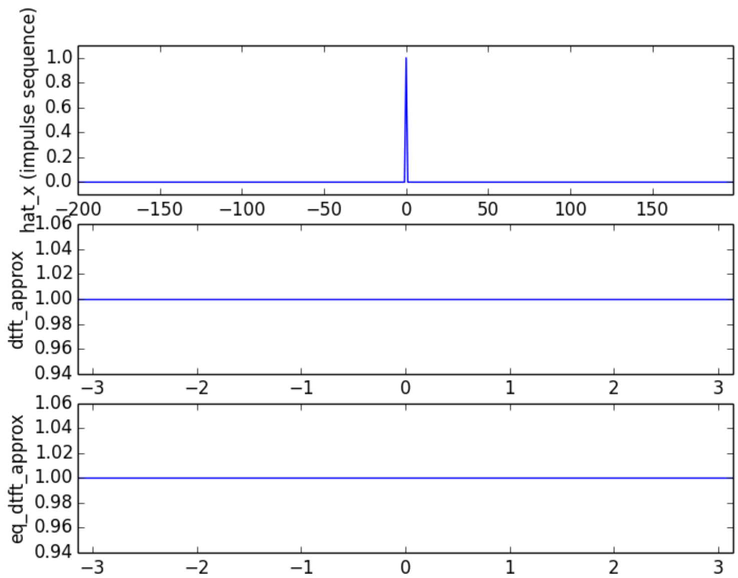

Figure 16: Impulse signal ($x[n] = \delta[n]$) with $N = 400, M = 200$

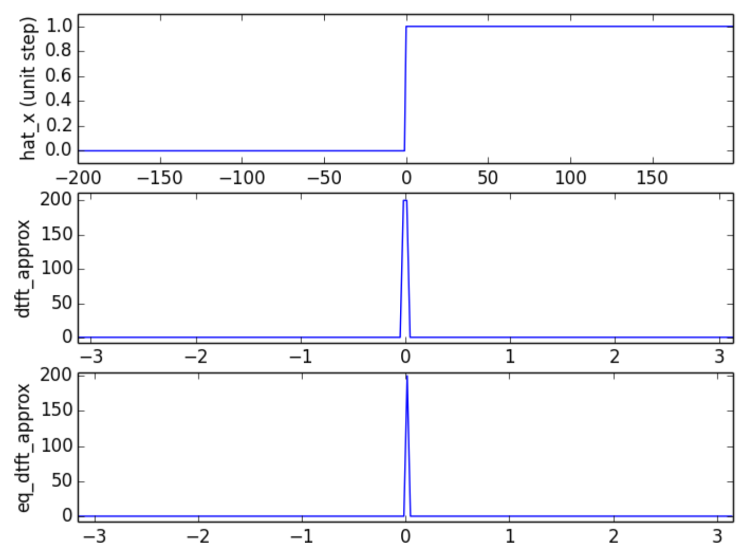

Figure 17: Unit step signal ($x[n] = u[n]$) with $N = 400, M = 200$

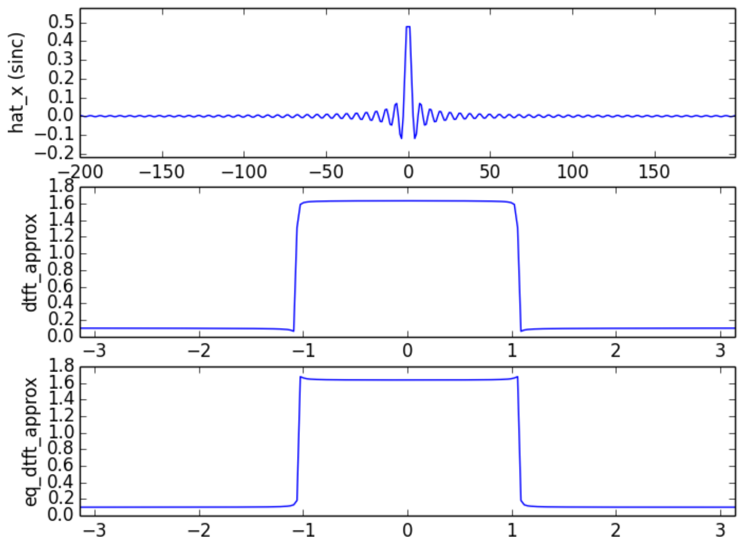

Figure 18: $x[n] = \sqrt{3}\frac{\sin(\frac{\pi n}{3})}{\pi n}$ with $N = 400, M = 200$

7 Python Exercise: Image Scaling with Separable Filters

Figure 19 shows the results of the four pre-filters. Since the visual difference is subtle, their corresponding MSEs are included. Method (a) and (c) are actually the same since $h = \delta$ so their MSEs are the same (MSE = 209.9389). The pattern (around leg and scarf area) of method (b) is overly smoothed while the alias effect of (a) and (c) is clearly visible; therefore its MSE is much higher (MSE = 147068.8310). Method (d) is uses the optimal filter so it has the lowest MSE (MSE = 178.7584).

Figure 19: Original image and four pre-filters (a-d) with corresponding MSE