ECE551 - Homework 1

Khoi-Nguyen Mac

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1 Geometry of orthogonal transformations in Euclidean spaces

1.1

$U \in \mathbb{R}^{N\times N}$ is an orthogonal matrix, therefore:

$$U^\top U = UU^\top = I.$$

By definition,

$$\begin{aligned} &\|x\|^2 := x^\top x = \langle x, x\rangle, \qquad x \in \mathbb{R}^N \\ \Rightarrow &\|Ux\| = \langle Ux, Ux\rangle = (Ux)^\top Ux = x^\top U^\top Ux = x^\top x = \|x\|^2 \\ \Rightarrow &\|Ux\| = \|x\|, \qquad \because \text{norms are non-negative.} \end{aligned}$$

1.2

By definition,

$$\begin{aligned} &\langle x, y\rangle := y^\top x = \sum_{i=0}^{N-1} x_i y_i, \qquad x, y \in \mathbb{R}^N \\ \Rightarrow &\langle Ux, Uy\rangle = (Uy)^\top Ux = y^\top U^\top Ux = y^\top x = \langle x, y\rangle. \end{aligned}$$

1.3

If $M > N$ and $U^\top U = I$, then $\text{rank}(U) = N \Rightarrow U^\top U = I_N$. Since $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$,

$$\begin{aligned} x^\top U^\top Ux = x^\top I_N x = x^\top x \\ y^\top U^\top Ux = y^\top I_N x = y^\top x. \end{aligned}$$

Hence, 1.1 and 1.2 hold.

1.4

If $M < N \Rightarrow \text{rank}(U)$ is at most M.

$$U^\top U \neq I_N.$$

Hence, 1.1 and 1.2 do not hold.

2 Some basic properties of inner product spaces

Notice that:

$$\langle a, b+c \rangle = \overline{\langle b+c, a \rangle} = \overline{\langle b, a \rangle} + \overline{\langle c, a \rangle} = \langle a, b \rangle + \langle a, c \rangle.$$

2.1 The Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Proof: If $\langle x, y \rangle = 0$, then theorem holds because $\|x\|$ and $\|y\| \geq 0$.

Suppose that $x \neq 0$ and $y \neq 0$. Let $z \in \mathbb{C}$, such that:

$$z = \frac{\langle x, y \rangle}{\|y\|^2} = \frac{\overline{\langle y, x \rangle}}{\|y\|^2}.$$

We have

$$\begin{aligned}
0 \leq \|x - zy\|^2 &= \langle x - zy, x - zy \rangle = \langle x, x - zy \rangle - \langle zy, x - zy \rangle \\
&= \langle x, x \rangle - \langle x, zy \rangle - \langle zy, x \rangle + \langle zy, zy \rangle \\
&= \|x\|^2 - \bar{z}\langle x, y \rangle - z\langle y, x \rangle + z\bar{z}\|y\|^2 \\
&= \|x\|^2 - \frac{\langle y, x \rangle \langle x, y \rangle}{\|y\|^2} - \frac{\langle x, y \rangle \langle y, x \rangle}{\|y\|^2} + \frac{\langle x, y \rangle}{\|y\|^2} \frac{\langle y, x \rangle}{\|y\|^2} \|y\|^2 \\
&= \|x\|^2 - \frac{\langle x, y \rangle \langle y, x \rangle}{\|y\|^2} \\
&= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&0 \leq \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\
\Leftrightarrow &\|x\|^2 \geq \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\
\Leftrightarrow &\|x\|^2 \|y\|^2 \geq |\langle x, y \rangle|^2 \\
\Leftrightarrow &\|x\| \|y\| \geq |\langle x, y \rangle|.
\end{aligned}$$

$\square$

The equality occurs iff $x = \alpha y$, for some scalar α.

Proof: By substituting $x = \alpha y$, we have

$$|\langle x, y\rangle| = |\langle \alpha y, y\rangle| = |x||\langle y, y\rangle| = |\alpha|\|y\|^2 = |\alpha|\|y\|\|y\| = \|x\|\|y\|.$$

$\square$

2.2 The triangle inequality

$\|x + y\|^2 \leq \|x\| + \|y\|$ with equality iff $y = \alpha x$

Proof:

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x + y\rangle + \langle y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle
\end{aligned}$$

By Cauchy-Schwarz inequality:

$$\begin{aligned}
&\langle x, y\rangle \leq \|x\|\|y\| \\
\Rightarrow &\langle y, x\rangle \leq \|x\|\|y\| \\
\Rightarrow &\langle x, y\rangle + \langle y, x\rangle \leq 2\|x\|\|y\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|x + y\|^2 \leq \|x\| + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \\
\Rightarrow &\|x + y\| \leq \|x\| + \|y\| \qquad \because \text{norms are non-negative.}
\end{aligned}$$

If $y = \alpha x$ then the equality of Cauchy-Schwarz theorem occurs. Therefore,

$$\begin{aligned}
&\langle x, y\rangle + \langle y, x\rangle = 2\|x\|\|y\| \\
\Rightarrow &\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \\
\Rightarrow &\|x + y\| = \|x\| + \|y\|.
\end{aligned}$$

$\square$

2.3 Parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Proof:

$$\begin{aligned} LHS &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= (\|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle) + (\|x\|^2 + \|y\|^2 - \langle x,y\rangle - \langle y,x\rangle) \\ &= 2(\|x\|^2 + \|y\|^2) = RHS. \end{aligned}$$

$\square$

3 Least-squares approximation with orthonormal bases

3.1

We notice that $\langle x, \varphi_i\rangle\varphi_i$ is the orthogonal projection of x onto φ_i. We want to show that $\sum_{\varphi_i\in\hat{\mathcal{B}}}\langle x,\varphi_i\rangle\varphi_i$ is the orthogonal projection of x onto the subspace $\text{span}(\hat{\mathcal{B}}) = \hat{V}$:

$$\langle \hat{x}, \varphi_i\rangle = \langle\langle x,\varphi_i\rangle\varphi_i, \varphi_i\rangle = \langle x,\varphi_i\rangle\langle\varphi_i,\varphi_i\rangle = \langle x,\varphi_i\rangle, \forall\varphi_i \in \hat{\mathcal{B}}.$$

$$\begin{aligned} &\langle x,\varphi_i\rangle = \langle\hat{x},\varphi_i\rangle, \forall\varphi_i\in\hat{\mathcal{B}} \\ \Leftrightarrow &\langle x,\varphi_i\rangle - \langle\hat{x},\varphi_i\rangle = 0, \forall\varphi_i\in\hat{\mathcal{B}} \\ \Leftrightarrow &\langle x-\hat{x},\varphi_i\rangle, \forall\varphi_i\in\hat{\mathcal{B}} \\ \Leftrightarrow &x-\hat{x}\perp\hat{V} \end{aligned}$$

Let $z = x - \hat{x}$, where $\hat{x} = \sum_{\varphi_i\in\hat{\mathcal{B}}}\langle x,\varphi_i\rangle\varphi_i$. Thus, $z\perp\hat{V}$.

Since any vectors in $\hat{V}$ can be written as $f(\alpha) = \sum_{\varphi_i\in\hat{\mathcal{B}}}\alpha_i\varphi_i$ (α is a vector and α_i's are scalars,) we have:

$$\begin{aligned} \|x - f(\alpha)\|^2 &= \|x - \hat{x} + \hat{x} - f(\alpha)\|^2 = \|z + \hat{x} - f(\alpha)\|^2 \\ &= \langle z + \hat{x} - f(\alpha), z + \hat{x} - f(\alpha)\rangle \\ &= \|z\|^2 + \|\hat{x} - f(\alpha)\|^2 + \langle z, \hat{x} - f(\alpha)\rangle + \langle\hat{x} - f(\alpha), z\rangle \end{aligned}$$

We can see that $\hat{x}\in\hat{V}$ and $f(\alpha)\in\hat{V}$, therefore $\hat{x} - f(\alpha)\in\hat{V}$. Since $z\perp\hat{V}\Rightarrow z\perp\hat{x} - f(\alpha)\Rightarrow\langle z,\hat{x} - f(\alpha)\rangle = \langle\hat{x} - f(\alpha), z\rangle = 0$.

Therefore, $\|x - f(\alpha)\|^2 = \|z\|^2 + \|\hat{x} - f(\alpha)\|^2$. Since $\|\hat{x} - f(\alpha)\|^2 \geq 0$,

$$\begin{aligned} &\|x - f(\alpha)\|^2 \geq \|z\|^2 \\ \Leftrightarrow &\|x - f(\alpha)\|^2 \geq \|x - \hat{x}\|^2 \\ \Leftrightarrow &\|x - f(\alpha)\| \geq \|x - \hat{x}\| \end{aligned}$$

3.2

We know that any inner products can define the valid norm

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Therefore $\|x - f(\alpha)\|^2 = \langle z + \hat{x} - f(\alpha), z + \hat{x} - f(\alpha)\rangle$, with other inner products. The expansion

$$\langle z + \hat{x} - f(\alpha), z + \hat{x} - f(\alpha)\rangle = \|z\|^2 + \|\hat{x} - f(\alpha)\|^2 + \langle z, \hat{x} - f(\alpha)\rangle + \langle \hat{x} - f(\alpha), z\rangle$$

also uses the properties of general inner products. Hence, the proof holds for other kinds of inner products, instead of only standard Euclidean one.

4 Signal Sets and Spaces

4.1

For $S = \mathbb{R}$, we have

$$\mathbb{R}^I = \{v \mid v : I \to \mathbb{R}\}.$$

We know that $\mathbb{R}$ is closed under addition and scalar multiplication ($\mathbb{R}$ is a vector space.) Therefore, for $t \in I$ and $u, v \in \mathbb{R}^I$,

$$\begin{aligned} &u[t], v[t] \in \mathbb{R} \\ \Rightarrow &u[t] + v[t] \in \mathbb{R} \\ \Rightarrow &(u + v)[t] := u[t] + v[t] \in \mathbb{R}. \end{aligned}$$

For scalar $\alpha \in \mathbb{R}$,

$$\begin{aligned} &\alpha u[t] \in \mathbb{R} \\ \Rightarrow &(\alpha u)[t] := \alpha u[t] \in \mathbb{R} \end{aligned}$$

4.2

(i) For complex-valued sequences indexed by the integers, signal values live in complex space and indices live in integer space, i.e.

$$\mathbb{C}^I = \{v \mid v : \mathbb{Z} \to \mathbb{C}\}.$$

Since $\mathbb{C}$ is also a vector space, this signal set is linear. The proof is similar as in 4.1.

Zero vector in $\mathbb{R}^I$ is defined as $\{a_i\}_{i \in I}$, where $a_i = 0, \forall i \in I$.

(ii) For 8-bit RGB color (three channels) digital photos of dimension $W \times H$, we can choose the value set to be $\mathcal{B}_8 = \{\overline{x_0 x_1 ... x_7}\}, x_i = \{0, 1\}, i = \{0, 7\}$, denoting the set of all binary sequences with length of 8. The indices are chosen as set of 3 natural numbers, corresponding to the width, height, and channel, i.e.

$$\mathcal{B}_8^I = \left\{ v \mid v : \mathbb{N}^{W \times H \times 3} \to \mathcal{B}_8 \right\}.$$

This set is not linear because it is not closed under addition, e.g. $(11111111)_2 + (00000001)_2 = (100000000)_2 \notin \mathcal{B}_8$. We can also see that , in general, $\mathcal{B}_k$ with finite k is not a linear space.

(iii) For 32-bit floating point buffers containing 1 second of stereo audio at 48KHz, we can choose the value space as $\mathcal{B}_{32}$, with similar definition as $\mathcal{B}_8$. Since the audio is 1-second long with 48KHz, there are 48k samples. Therefore, the signal set can be described as:

$$\mathcal{B}_{32}^I = \left\{ v \mid v : \mathbb{N}^{48K \times 2} \to \mathcal{B}_{32} \right\}.$$

We can see that $\mathcal{B}_{32}$ is not a linear, therefore the signal set is not linear.

4.3

We can consider signals with indices $I_1 \subset I_2$ are truncated version of signals with indices I_2, where the signals in $I_d = I_2 / I_1$ are reduced to 0. So $\mathbb{R}^{I_1}$ is a subset of $\mathbb{R}^{I_2}$. Since $\mathbb{R}$ is a subspace, $\mathbb{R}^{I_1}$ is closed under addition and scalar multiplication. Hence, $\mathbb{R}^{I_1}$ is a subspace of $\mathbb{R}^{I_2}$.

4.4

Linearity Let $u, v \in \mathbb{R}^I$ and α, β be scalars

$$T(\alpha u + \beta v)_k = T(\alpha u)_k + T(\beta v)_k = \alpha (Tu)_k + \beta (T_v)_k.$$

Hence, T is linear.

Invertibility Let $Tu = v$. T can be seen as a permutation matrix that maps the k entry of v to i_k entry of u. Therefore T has full rank and $\text{rank}(T) = N$ (there are N indices). Hence, T is invertible (full rank matrices are invertible.)

4.5

We see that $u[i], v[i] \in \mathbb{R}, \forall i \in I$. We need to prove that the defined inner product satisfies the three axioms:

Conjugate symmetry:

$$\overline{\langle v, u \rangle}_I = \overline{\sum_{i \in I} u[i]v[i]} = \sum_{i \in I} \overline{u[i]v[i]} = \sum_{i \in I} u[i]v[i] = \langle u, v \rangle_I.$$

Linearity in the first argument:

$$\langle \alpha u, v \rangle = \sum_{i \in I} \alpha u[i]v[i] = \alpha \sum_{i \in I} u[i]v[i] = \alpha \langle u, v \rangle.$$

Let $u_1, u_2 \in \mathbb{R}^I$,

$$\langle u_1 + u_2, v \rangle = \sum_{i \in I} (u_1[i] + u_2[i])v[i] = \sum_{i \in I} u_1[i]v[i] + \sum_{i \in I} u_2[i]v[i] = \langle u_1, v \rangle + \langle u_2, v \rangle.$$

Positive-definiteness:

$$\langle u, u \rangle = \sum_{i \in I} u[i]u[i] \sum_{i \in I} (u[i])^2 \geq 0$$

Let u be a zero vector, i.e. $u[i] = 0, \forall i \in I$,

$$\langle u, u \rangle = \sum_{i \in I} (u[i])^2 = \sum_{i \in I} 0 = 0.$$

Hence, $\langle u, v \rangle_I := \sum_{i \in I} v[i]u[i]$ is an inner-product in $\mathbb{R}^I$.

4.6

For an arbitrary $t \in I$, let $e_t = \{\epsilon_\tau\}_{\tau \in I}$, s.t.

$$\epsilon_\tau = \begin{cases} 1, & \tau = t \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $\{e_\tau\}$ are linearly independent, as $\sum_\tau \alpha_\tau e_\tau = 0 \Leftrightarrow \alpha_\tau = 0, \forall \tau$. Therefore $\{e_\tau\}$ is a basis. Since $e_t \subset \mathbb{R}^I$, we have

$$\langle u, e_t \rangle_I = \sum_{\tau \ inI} u[\tau]\epsilon_\tau = u[t].$$

which satisfies the criteria. Hence, the defined e_t is the standard basis (or reproducing kernel).