ECE551 - Homework 2

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1 Frames and Bases

(a) The synthesis operator associated with $\{\varphi_k\}_{k\in\mathcal{K}}$ in $\mathbb{R}^2$ is

$$\Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$$

Hence,

$$\Phi_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$
$$\Phi_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$\Phi_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$\Phi_4 = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

(Note that we reuse the notation Φ to represent the matrix representation.)

(b) For $\Phi \in \mathbb{R}^{M\times N}$, if $M = N$ then it is a basis, if $M > N$ then it is a frame.

Let A be the inverse of the Gram matrix of basis Φ, i.e. $A = (\Phi^*\Phi)^{-1}$. Then $\Phi = \Phi A = \Phi(\Phi^*\Phi)^{-1}$ forms a dual basis with Φ.

Let $B = (\Phi\Phi^*)^{-1}$, where Φ is a frame. Then $\tilde{\Phi} = B\Phi = (\Phi\Phi^*)^{-1}\Phi$ forms the canonical dual frame associated with frame Φ.

For Φ_1 (basis),

$$A_1 = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \Rightarrow A_1^{-1} = \begin{bmatrix} 4 & -2\sqrt{3} \\ -2\sqrt{3} & 4 \end{bmatrix}$$

$$\tilde{\Phi}_1 = \begin{bmatrix} 2 & -\sqrt{3} \\ 0 & 1 \end{bmatrix}$$

For Φ_2 (frame),

$$B_2 = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B_2^{-1}$$

$$\tilde{\Phi}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} = \Phi_2$$

For Φ_3 (basis),

$$A_3 = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_3^{-1}$$

$$\Rightarrow \tilde{\Phi}_3 = \Phi_3$$

For Φ_4 (basis),

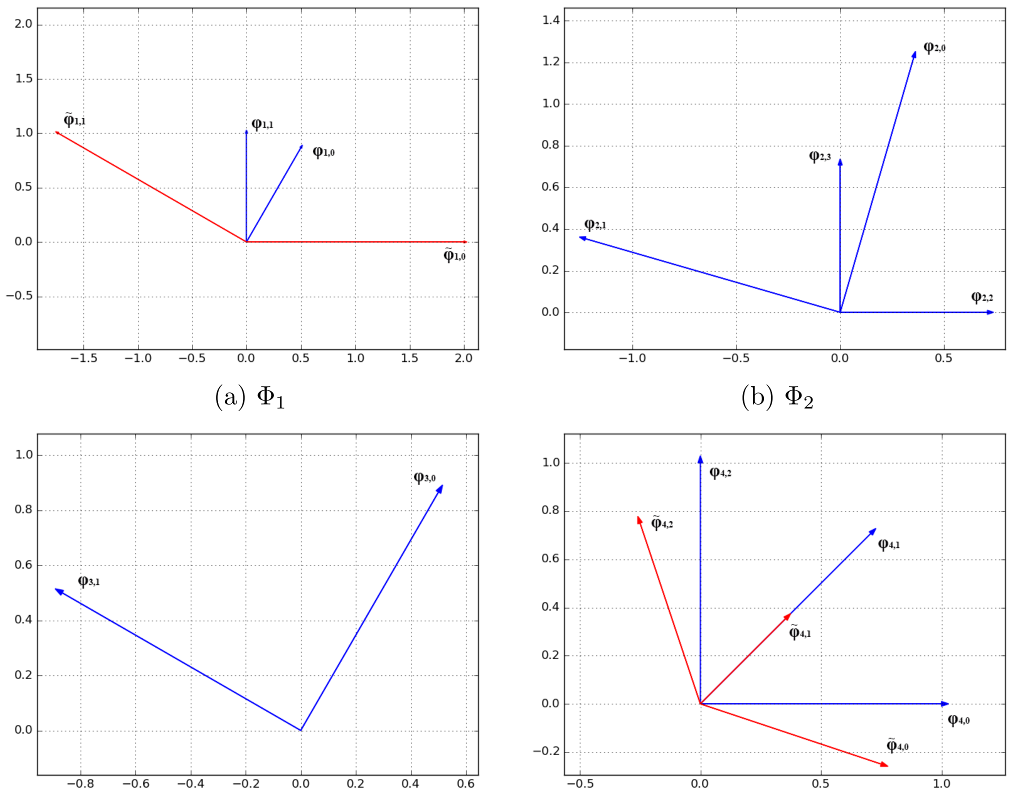
$$B_4 = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \Rightarrow B_4^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\tilde{\Phi}_4 = \begin{bmatrix} \frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{3}{4} \end{bmatrix}$$

Figure 1 shows the sketch of the sets and their duals.

(c) $\langle \varphi_{1,0}, \varphi_{1,1} \rangle = \frac{\sqrt{3}}{2}$, so the basis Φ_1 is not orthogonal, thus not orthonormal. $B_2 = I$, so the frame Φ_2 is tight (a frame is tight if $\Phi\Phi^* = I$.) $\langle \varphi_{3,0}, \varphi_{3,1} \rangle = 0$ and $\|\varphi_{3,0}\| = \|\varphi_{3,1}\| = 1$, so the basis Φ_3 is orthonormal. $B_4 \neq I$, so the frame Φ_4 is not tight.

(d) $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\alpha_{i,k} = \langle x, \tilde{\varphi}_{i,k} \rangle$. Therefore,



(a) Φ_1

(b) Φ_2

(c) Φ_3

(d) Φ_4

Figure 1: Original sets and their duals.

For Φ_1,

$$\alpha_{1,0} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\rangle = 4 \qquad \alpha_{1,1} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\rangle = -2\sqrt{3}$$

For Φ_2,

$$\alpha_{2,0} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}} \qquad \alpha_{2,1} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle = -\sqrt{\frac{3}{2}}$$

$$\alpha_{2,2} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \frac{2}{\sqrt{2}} \qquad \alpha_{2,3} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = 0$$

For Φ_3,

$$\alpha_{3,0} = \frac{1}{2} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right\rangle = 1 \qquad \alpha_{3,1} = \frac{1}{2} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\rangle = -\sqrt{3}$$

For Φ_4,

$$\alpha_{4,0} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} \right\rangle = \frac{3}{2} \qquad \alpha_{4,1} = \frac{1}{2\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\alpha_{4,2} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \right\rangle = -\frac{1}{2}$$

(e) We check the values for $\alpha_{i,k}$ by verifying the expansion $x = \sum_k \alpha_{i,k}\varphi_{i,k}$.

For Φ_1,

$$\sum_k \alpha_{1,k}\varphi_{1,k} = 4 \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - 2\sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x$$

For Φ_2,

$$\sum_k \alpha_{2,k}\varphi_{2,k} = \frac{1}{2} \left(1 \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - \sqrt{3} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x$$

For Φ_3,

$$\sum_k \alpha_{3,k}\varphi_{3,k} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - \sqrt{3} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x$$

For Φ_4,

$$\sum_k \alpha_{4,k}\varphi_{4,k} = \frac{3}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$

(f) We verify that $\Phi\tilde{\Phi}^\top = I$.

$$\Phi_1 \tilde{\Phi}_1^\top = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -\sqrt{3} & 1 \end{bmatrix} = I$$

$$\Phi_2 \tilde{\Phi}_2^\top = \Phi_2 \Phi_2^\top = B_2 = I$$

$$\Phi_3 \tilde{\Phi}_3^\top = \Phi_3 \Phi_3^\top = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = I$$

$$\Phi_4 \tilde{\Phi}_4^\top = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} = I$$

(g) We check if $\|x\|^2 = \sum_k |\alpha_{i,k}|^2$.

$$\|x\|^2 = \left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\|^2 = 4^2 = 16$$

$$\sum_k |\alpha_{1,k}|^2 = 28$$

$$\sum_k |\alpha_{2,k}|^2 = \frac{13}{4}$$

$$\sum_k |\alpha_{3,k}|^2 = 4$$

$$\sum_k |\alpha_{4,k}|^2 = 3$$

Hence, the expansion does not preserve the norm.

(h) We see that since the frames have more column vectors than its dimension, so these column vectors are linearly dependent. Therefore we can have more than one way to represent a vector using these columns. Hence, the expansions of frames are redundant.

For the basis, since the columns are linearly independent, the expansions are not redundant.

2 Linear Least-Squares Approximation

(a) $y \in \text{colsp}(A) \Rightarrow y \in \mathcal{R}(A)$. Suppose that $A \in \mathbb{C}^{M\times N}$, by the definition of range space:

$$\mathcal{R}(A) = \{y \in \mathbb{C}^M \mid y = Ax, x \in \mathbb{C}^N\}$$

Hence, $\hat{y} = y$.

(b) $y \perp \text{colsp}(A) \Rightarrow y \perp \mathcal{R}(A) \Rightarrow y \in \mathcal{N}(A^*)$. Be definition of null space:

$$\mathcal{N}(A^*) = \{y \in \mathbb{C}^M \mid A^*y = 0\}$$

Hence, $\hat{y} = 0$.

(c) In general, consider $\|y - Ax\|$,

$$\begin{aligned}
\frac{\partial \|y - Ax\|}{\partial x_i} &= \frac{\partial (y - Ax)^\top (y - Ax)}{\partial x_i} \\
&= \frac{\partial (y - Ax)^\top}{\partial x_i}(y - Ax) + (y - Ax)^\top \frac{\partial (y - Ax)}{\partial x_i} \\
&= 2\frac{\partial (y - Ax)^\top}{\partial x_i}(y - Ax)
\end{aligned}$$

Since we can express x as $\sum_{i=1}^N x_i e_i$, where e_i is a elementary basis, then

$$\begin{aligned}
\frac{\partial (y - Ax)}{\partial x_i} &= \frac{\partial y}{\partial x_i} - A\frac{\partial x}{\partial x_i} \\
&= -A\frac{\partial x}{\partial x_i} \\
&= -Ae_i = -a_i
\end{aligned}$$

where a_i is a column vector of A. Therefore

$$\frac{\partial \|y - Ax\|}{\partial x_i} = -2a_i^\top (y - Ax)$$

Since $\hat{x} = \arg\min \|y - Ax\|^2$, $\hat{y} = A\hat{x}$ is the projection of y on space spanned by columns of A. Therefore $y - \hat{y} = y - A\hat{x} \perp \text{span}\{a_i\} \Rightarrow y - A\hat{x} \perp a_i \Rightarrow a_i^\top (y - A\hat{x}) = 0$. Hence, the partial derivatives vanish.

3 Orthogonalization of a Projection

(a) If $P = P^*$,

$$\langle Px, y \rangle_0 = y^*(Px) = (y^*P)x = (y^*P^*)x = (Py)^*x = \langle x, Py \rangle_0 , \qquad \forall x, y \in V.$$

Hence, P is self-adjoint with respect to $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^N$ if $P = P^*$.

(b) Remind that a non-zero vector $v \in \mathbb{C}^N$ is an eigenvector of square matrix $P \in \mathbb{C}^{N \times N}$ if $Pv = \lambda v$, where λ is the eigenvalue associated with v.

If $P = P^2$,

$$\lambda v = Pv = P^2 v = \lambda^2 v$$

Since $v \neq 0$, $\lambda = \lambda^2 \Leftrightarrow \lambda = 0$ or $\lambda = 1$. Hence the eigenvalues of an oblique projection is 0 or 1.

(c) Since $P = T^{-1}DT$, where D is a diagonal matrix with eigenvalues found in part (b), D's diagonal is formed by 1 and 0. Therefore, $D^* = D$

Since $\langle x, y \rangle_T \triangleq y^*T^*Tx$, we have

$$\begin{aligned}
\langle Px, y \rangle_T &= y^*T^*TPx \\
&= y^*T^*T(T^{-1}DT)x \\
&= y^*T^*DTx \\
&= y^*T^*DI^*Tx \\
&= y^*T^*D(TT^{-1})^*Tx \\
&= y^*T^*D(T^{-1})^*T^*Tx \\
&= y^*T^*D^*(T^{-1})^*T^*Tx \\
&= y^*(T^{-1}DT)^*T^*Tx \\
&= y^*P^*T^*Tx \\
&= (Py)^*T^*Tx \\
&= \langle x, Py \rangle_T
\end{aligned}$$

Bonus A matrix U is diagonalizable if $\exists V$, s.t. $V^{-1}UV$ is a diagonal matrix.

$$P = T^{-1}DT \Rightarrow TPT^{-1} = TT^{-1}DTT^{-1} = D.$$

Since D is a diagonal matrix, P is diagonalizable.

(d) We have $P = T^{-1}DT \Rightarrow P^* = T^*D^*(T^{-1})^*$.

$$\begin{aligned}\langle x - Px, Px\rangle_T &= x^*P^*T^*T(x - Px) = x^*P^*T^*Tx - x^*P^*T^*TPx \\ &= x^*T^*D^*(T^{-1})^*T^*Tx - x^*T^*D^*(T^{-1})^*T^*TT^{-1}DTx \\ &= x^*T^*D^*((T^{-1})^*T^*)Tx - x^*T^*D^*((T^{-1})^*T^*)(TT^{-1})DTx \\ &= x^*T^*D^*Tx - x^*T^*D^*DTx\end{aligned}$$

Since D is a diagonal matrix with only 1 and 0, $DD^* = D^*D = D = D^*$. Therefore,

$$\begin{aligned}\langle x - Px, Px\rangle_T &= x^*T^*D^*Tx - x^*T^*D^*DTx \\ &= x^*T^*D^*Tx - x^*T^*D^*Tx = 0\end{aligned}$$

Hence, $x - Px \perp Px$.

(e) From part (c), we know that P is oblique and self-adjoint, i.e. $P = P^2 = P^*$. Since $P \in \mathbb{C}^{N\times N}$, $I = I_N$ and $PI = IP = P$. Therefore,

$$(I-P)^2 = (I-P)(I-P) = I^2-IP-PI+P^2 = I-2P+P^2 = I-2P+P = I-P$$

and

$$(I-P)^* = I^* - P^* = I - P$$

Hence, $I - P$ is oblique and self-adjoint.

We know that for any matrix A, $\mathcal{R}(A) \perp \mathcal{N}(A^\top)$. Therefore,

$$\mathcal{R}(I-P) \perp \mathcal{N}((I-P)^\top)$$

Since $I - P$ is proven to be self-adjoint, $(I-P)^\top = I - P$

$$\Rightarrow \mathcal{R}(I-P) \perp \mathcal{N}(I-P)$$

From the definition of null space:

$$\begin{aligned}\mathcal{N}(P) &= \{x \mid Px = 0\} \\ \mathcal{N}(I-P) &= \{x \mid (I-P)x = 0\}\end{aligned}$$

We proved that $x - Px \perp Px$ wrt $\langle\cdot,\cdot\rangle_T$, so

$$(I-P)x \perp Px \Rightarrow \mathcal{N}(I-P) \perp \mathcal{N}(P)$$

We already have $\mathcal{R}(I-P) \perp \mathcal{N}(I-P)$, thus $\mathcal{R}(I-P) = \mathcal{N}(P)$.

(f) We have

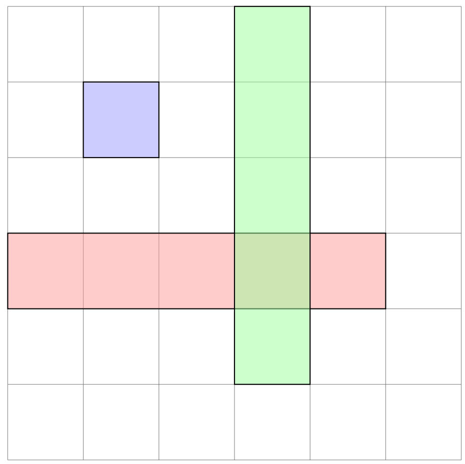
$$x - Py = x - Px + Px - Py = (x - Px) + P(x-y)$$

From part (d), $x - Px \perp P(x-y)$. By Pythagorean,

$$\|x-Py\|_T^2 = \|x-Px\|_T^2 + \|P(x-y)\|_T^2 \geq \|x-Px\|_T^2\,.$$

4 Approximation by Orthogonal Indicator Tiles

(a) A sample of tiles on I, where tiles with different colors correspond to E_a, E_b, E_c.



(b) Since $\dim(\text{span}\{\phi_a\}_{a\in A}) \leq |A| < |I| = \dim(\mathbb{R}^I)$, $\{\phi_a\}_{a\in A}$ cannot span I. Hence, it is not a basis or a frame.

(c) For $u, v \in A$ s.t. $u \neq v$. If $\phi_u \perp \phi_v$

$$\langle \phi_u, phi_v \rangle = \sum_{i\in I} \phi_u[i]\phi_v[i] = \left| E_u \bigcap E_v \right| = \lambda \delta_{u,v} = \begin{cases} 1, & u = v \\ 0, & \text{else} \end{cases} = 0 \qquad (u \neq v)$$

It means that $E_u \bigcap E_v = \emptyset$. For example, in the sample figure in part (a), the blue tile is orthogonal with the red and the green tiles, but the red and green are not orthogonal.

(d) Since $\{\phi_a\}_{a\in A}$ are orthogonal, the best approximation of x is its projection on the space spanned by normalized $\{\phi_a\}_{a\in A}$, i.e.

$$\hat{x} = \sum_{a\in A} \frac{\langle x, \phi_a \rangle}{\|\Phi_a\|^2} \phi_a.$$

(e) If $\{\phi_a\}_{a\in A}$ are not orthogonal, we can construct the corresponding set orthogonal bases using Gram-Schmidt algorithm and project x on it.

5 Python Problem

The script is written with Python 2.7. It uses cv2 to read images instead of scipy. Figure 2 shows the projection of x on different spaces S_1, S_2, S_3, S_4, and S_5. Figure 3 and 4 show the projection of an image with resolution of 1024x768 (grayscale and color, respectively) on non-orthogonal tiles with $R = 20$.

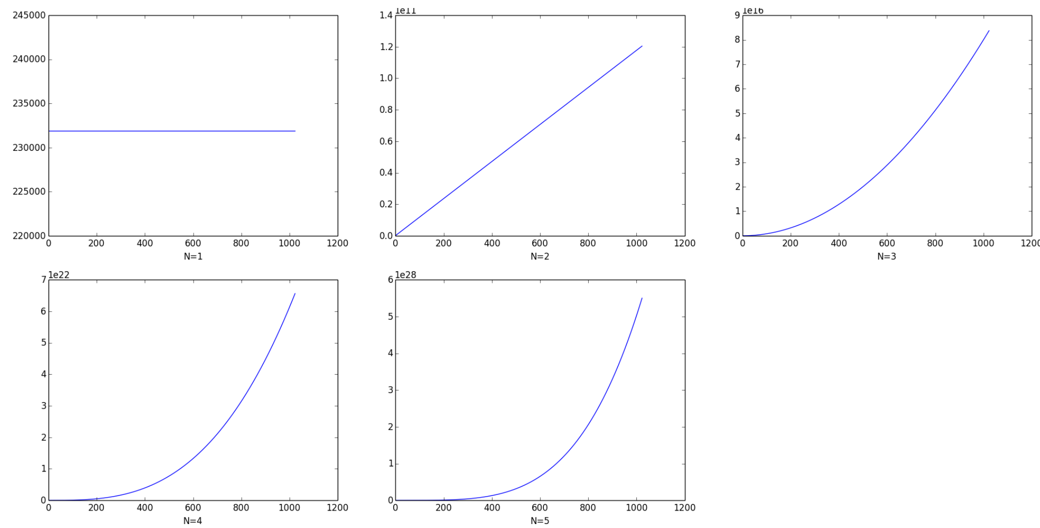


Figure 2: Projecting x on S_1, S_2, S_3, S_4, and S_5.

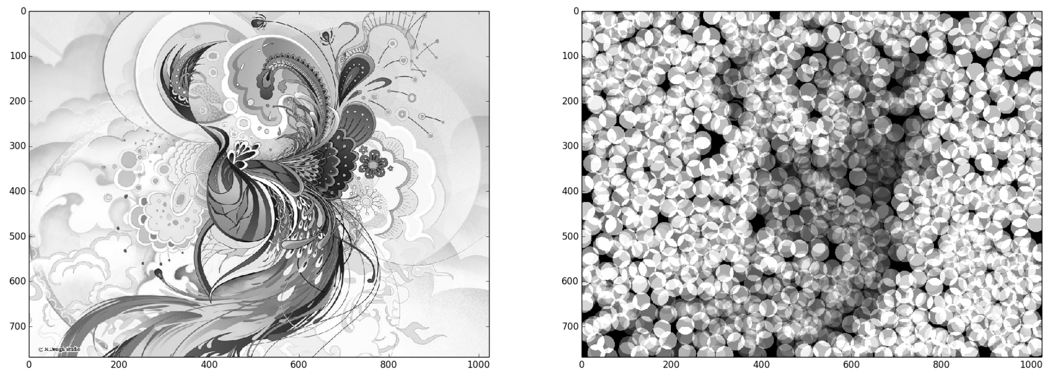


Figure 3: Projecting grayscale image on non-orthogonal tiles.

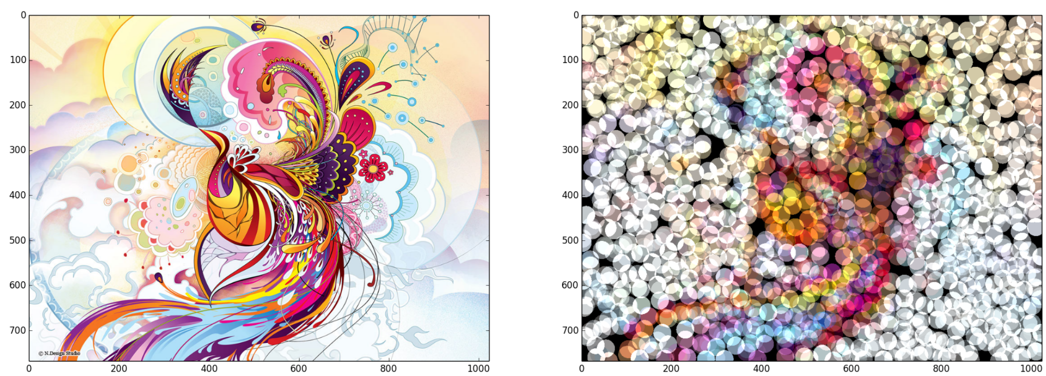


Figure 4: Projecting color image on non-orthogonal tiles.