ECE551 - Homework 5

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1 Sampling and Interpolation for Band-Limited Vectors

(a) We have the Fourier vector

$$w_k[n] = e^{j\frac{2\pi kn}{M}}$$

and

$$2\cos x = e^{jx} + e^{-jx}$$

So

$$\begin{aligned}
x_1[n] &= 1 + \cos\left(\frac{2\pi n}{M}\right) + \cos\left(\frac{8\pi n}{M}\right) \\
&= w_0[n] + \frac{1}{2}\left(e^{j\frac{2\pi n}{M}} + e^{-j\frac{2\pi n}{M}}\right) + \frac{1}{2}\left(e^{j\frac{8\pi n}{M}} + e^{-j\frac{8\pi n}{M}}\right) \\
&= w_0[n] + \frac{1}{2}\left(w_1[n] + w_{-1}[n] + w_4[n] + w_{-4}[n]\right)
\end{aligned}$$

Its DFT is

$$\begin{aligned}
X_1[k] &= \sum_{n=0}^{M-1} x_1[n] w_{-k}[n] \\
&= \frac{1}{2}\left(\sum_{n=0}^{M-1} 2w_{-k}[n] + w_{-k-1}[n] + w_{-k+1}[n] + w_{-k-4}[n] + w_{-k+4}[n]\right)
\end{aligned}$$

We can see that

$$\begin{aligned}
\sum_{n=0}^{M-1} w_k[n] &= \sum_{n=0}^{M-1} \exp\left(j\frac{2\pi k}{M}\right)^n \\
&= \frac{1 - \exp\left(j\frac{2\pi k}{M}\right)^M}{1 - \exp\left(j\frac{2\pi k}{M}\right)} \qquad (\because \text{geometric series}) \\
&= A
\end{aligned}$$

Whenever the numerator of A is 0 (k=0), its denominator is also 0. Therefore A has a peak at k. Hence, $X_1[k]$ has peaks at $k = 0, \pm 1, \pm 4$, so its bandwidth is $[-4, 4]$.

Similarly,

$$
\begin{aligned}
x_2[n] &= \cos\left(\frac{3\pi n}{M}\right) = \frac{1}{2}\left(e^{j\frac{3\pi n}{M}} + e^{-j\frac{3\pi n}{M}}\right) \\
\Rightarrow X_2[n] &= \sum_{n=0}^{M-1} x_2[n] e^{-j\frac{2\pi k n}{M}} \\
&= \frac{1}{2}\sum_{n=0}^{M-1}\left(e^{j\frac{2\pi n}{(}3-2k)} + e^{j\frac{2\pi n}{(}-3-2k)}\right) \\
&= \frac{1}{2}\left(\frac{1-\exp\left(j\frac{\pi n}{M}(3-2k)\right)^M}{1-\exp\left(j\frac{\pi n}{M}(3-2k)\right)} + \frac{1-\exp\left(j\frac{\pi n}{M}(-3-2k)\right)^M}{1-\exp\left(j\frac{\pi n}{M}(-3-2k)\right)}\right) \neq 0, \forall k \in \mathbb{Z}
\end{aligned}
$$

Hence, $x_2[n]$ is full-band.

(b) We take

$$
\Phi = \left[w_0, w_1, ..., w_{\frac{k_0+1}{2}-1}, w_{M-\frac{k_0+1}{2}+1}, ..., w_{M-1}\right]
$$

Because x is band limited s.t. $X[k] = 0, \forall k \in \left[\frac{k_0+1}{2}, M - \frac{k_0+1}{2}\right]$, it means that we remove the part from $\frac{k_0+1}{2}$ to $M - \frac{k_0+1}{2}$ of the DFT.

2 Band Limited Space with Rational Sampling Rate Changes

(a) Since we only care about the effect of g, we consider only until $g[n]$ is apply (the first 5 steps).

Figure 1 shows the results for $M = 2, N = 3, K = 3$. After upsampling by 2 (second row), we need the cut-off frequency of g to be $\frac{\pi}{3} \leq w_c \leq \frac{\pi}{3}$. By applying the low-pass filter $g[-n]$ (third row), the gap between two copies is $\frac{5\pi}{3} - \frac{\pi}{3} = \frac{4\pi}{3}$. After downsampling by 3 (forth row), the gap is reduced to 0, so upsampling it by 3 (fifth row) also gives the same gap. So the cut-off frequency has to be $w_c = \frac{\pi}{3}$.

(b) Figure 2 shows the results for$M = 2, N = 3, K = 4$. After upsampling by 2 (second row), we need the cut-off frequency of g to be $\frac{\pi}{4} \leq w_c \leq \frac{3\pi}{4}$. After apply downsampling by 3 (forth row), the gap is $\frac{\pi}{2}$. Therefore the gap is reduced by

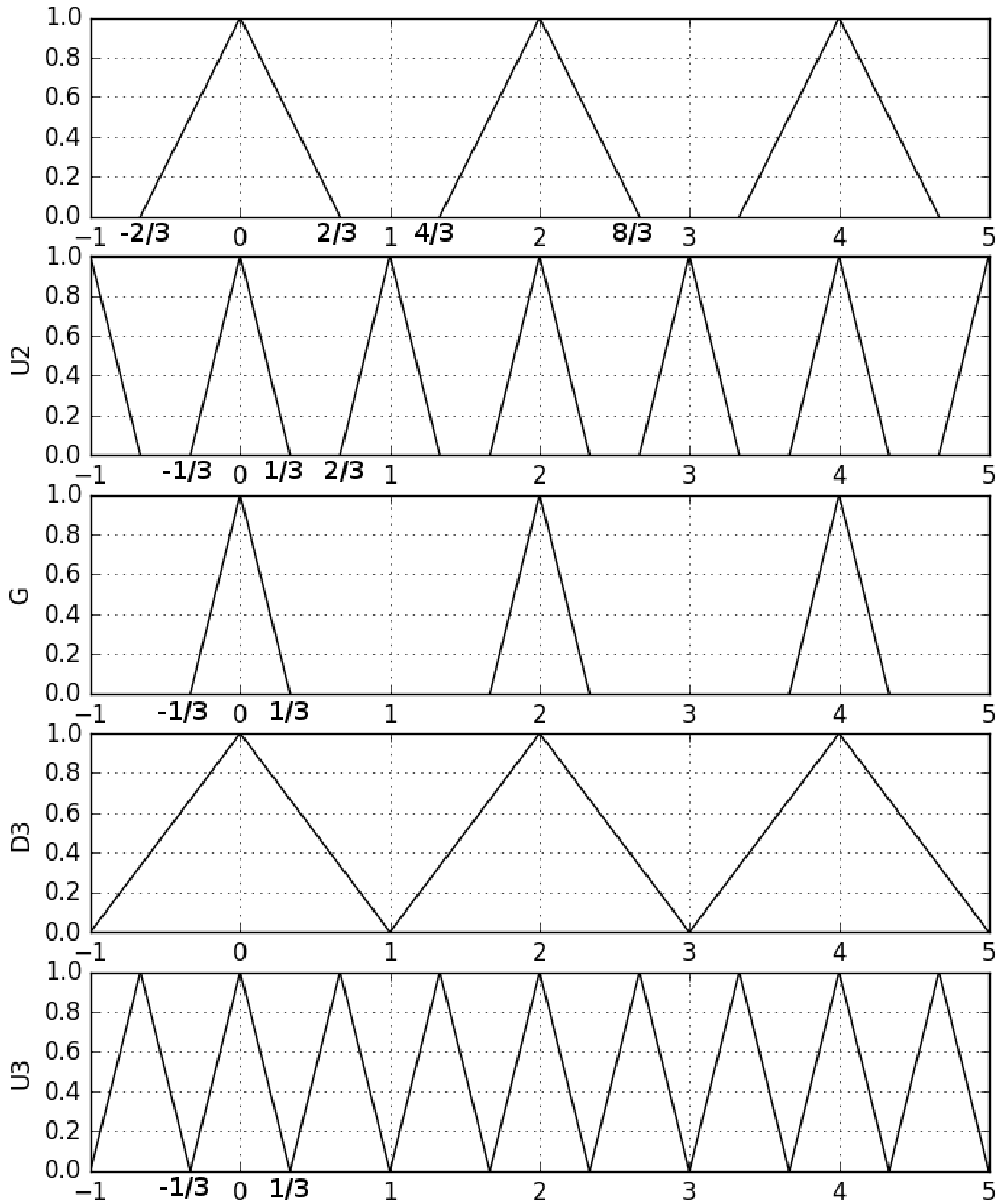

Figure 1: $M = 2, N = 3, K = 3$ (x scale is π)

a third after upsampling by 3 (fifth row). So the second condition for cut-off frequency is $\frac{\pi}{4} \leq w_c \leq \frac{5\pi}{12}$.

$$\frac{\pi}{4} \leq w_c \leq \frac{3\pi}{4} \text{ and } \frac{\pi}{4} \leq w_c \leq \frac{5\pi}{12} \Rightarrow \frac{\pi}{4} \leq w_c \leq \frac{5\pi}{12}$$

(c) If the signal in $[-\frac{2\pi}{K}, \frac{2\pi}{K}]$, the gap's width is

$$2\pi - \frac{2\pi}{K} - \frac{2\pi}{K} = 2\pi(1 - \frac{2}{K})$$

After upsampling by M, the range is $[-\frac{2\pi}{KM}, \frac{2\pi}{KM}]$ and the gap is $\frac{2\pi}{M}(1 - \frac{2}{K})$. Therefore the first condition of w_c is

$$\frac{2\pi}{KM} \leq w_c \leq \frac{2\pi}{KM} + \frac{2\pi}{M}(1 - \frac{2}{K}) \Leftrightarrow \frac{2\pi}{KM} \leq w_c \leq \frac{2\pi}{M}(1 - \frac{1}{K})$$

After downsampling by N, the $[-\frac{2\pi N}{KM}, \frac{2\pi N}{KM}]$ and the gap is $2\pi - \frac{2\pi N}{KM} - \frac{2\pi N}{KM} = 2\pi(1 - \frac{2}{KM})$.

Therefore, after upsampling by N, the lower bound of the first copy (after at frequency of 0) is

$$\frac{1}{N} \cdot \frac{2\pi N}{KM} + \frac{1}{N} \cdot 2\pi(1 - \frac{2N}{KM}) = \frac{2\pi}{KM} + \frac{2\pi}{N} - \frac{4\pi}{KM} = \frac{2\pi}{N} - \frac{2\pi}{KM} = 2\pi(\frac{1}{N} - \frac{1}{KM})$$

So the second condition of w_c is

$$\frac{2\pi}{KM} \leq w_c \leq 2\pi(\frac{1}{N} - \frac{1}{KM})$$

Combining the first and second condition gives

$$\frac{2\pi}{KM} \leq w_c \leq 2\pi(\frac{1}{N} - \frac{1}{KM}) \qquad (\because M < N \text{ so the second condition is tighter})$$

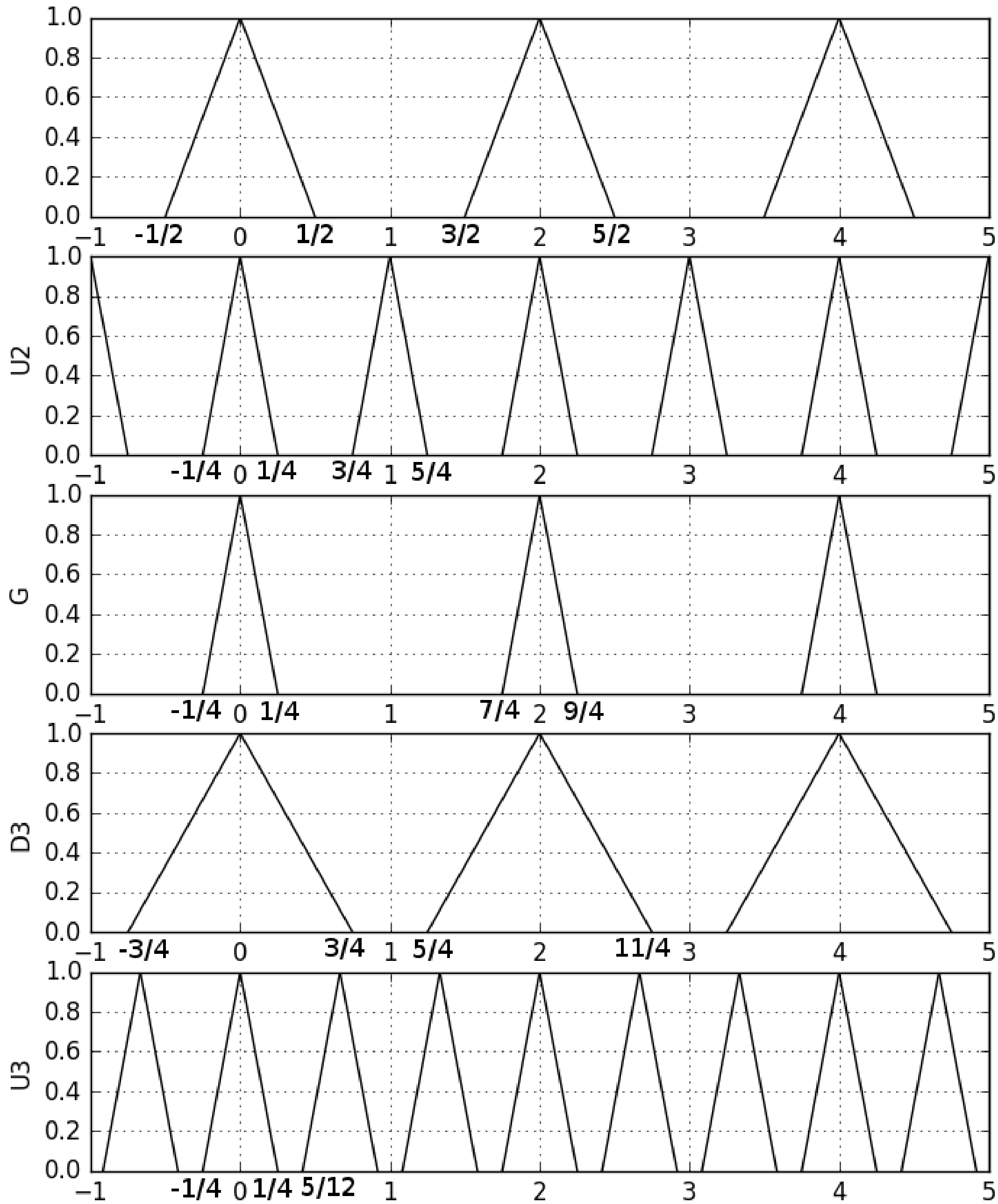

Figure 2: $M = 2, N = 3, K = 4$ (x scale is π)

3 Multirate Systems

(a) Let $u[n]$ be the output after downsampling and $v[n]$ be the output after convolving with g.

$$
\begin{aligned}
U(z) &= \frac{1}{2}\sum_{k=0}^{1} X\left(e^{-j\frac{2\pi k}{2}} z^{1/2}\right) \\
&= \frac{1}{2}\left(X(z^{1/2} + X(e^{-j\pi} z^{1/2}))\right) \\
V(z) &= G(z)U(z) \\
Y(z) &= V(z^3) \\
&= G(z^3)U(z^3) \\
&= \frac{1}{2}\left(X(z^{3/2} + X(e^{-j\pi} z^{3/2}))\right)
\end{aligned}
$$

(b) $x[n] = q(nT)$ is the same as downsampling by T, so $u[z]$ is obtained by downsampling $q(t)$ by $2T$. Therefore

$$
\begin{aligned}
U(\omega) &= \frac{1}{2T}\sum_{k=0}^{2T-1} Q\left(\frac{\omega - 2\pi k}{2T}\right) \\
V(\omega) &= G(\omega)U(\omega) \\
Y(\omega) &= V(3\omega) \\
&= G(3\omega)U(3\omega) \\
&= \frac{G(3\omega)}{2T}\sum_{k=0}^{2T-1} Q\left(\frac{3\omega - 2\pi k}{2T}\right)
\end{aligned}
$$

Since $q \in BL[-\frac{\pi}{T}, \frac{\pi}{T}]$ and the sampling rate $\frac{1}{T}$, we have the setting in Figure 3 To avoiud aliasing, we need $\frac{\pi}{T} < \frac{1}{2T} \Leftrightarrow \frac{2\pi - 1}{T} < 0$, which cannot satisfy. Therefore, q cannot avoid aliasing after sampling.

4 Pseudo-Inverse of Interpolation Filter: Single Channel Case

By orthogonality principle

$$
x - \hat{x} \perp V = \operatorname{span}\{\sigma^{kN} g\}_{k\in\mathbb{Z}} \Leftrightarrow \left\langle x - \hat{x}, \sigma^{kN} g\right\rangle = 0
$$

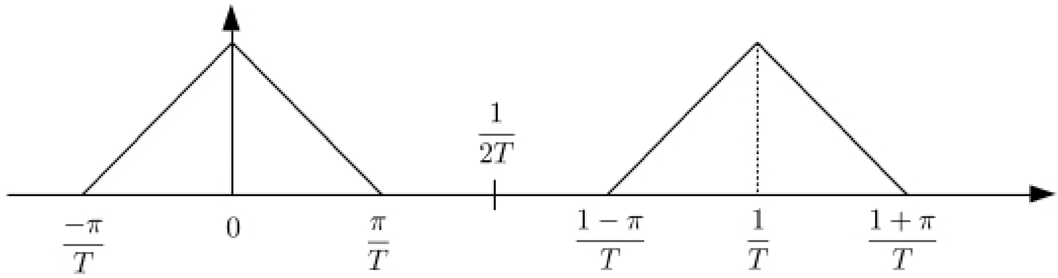

Figure 3: Sampling $q \in BL[-\frac{\pi}{T}, \frac{\pi}{T}]$ with sampling rate of $\frac{1}{T}$

Recall that convolution can be written as inner product, i.e. $(u * v)[n] = \langle u, \sigma^n \tilde{v} \rangle$. Therefore

$$\Leftrightarrow \left\langle x - \hat{x}, \sigma^{kN} g \right\rangle = ((x - \hat{x}) * \tilde{g})[kN] = 0$$

Let $s = (x - \hat{x}) * \tilde{g}$, its z-transform is

$$S(z) = (X(z) - \hat{X}(x))G(z^{-1}) = 0$$

where $X(z), \hat{X}(z)$, and $G(z^{-1})$ are the z-transform of $x[n], \hat{x}[n]$, and $\tilde{g}[n]$. We have the polyphase decomposition as

$$\begin{aligned}
X(z) &= \pi(z) X_p(z^N) \\
G(z) &= G_p(z^N)^\top \pi(z)^\top \\
\hat{X}(z) &= \pi(z) \hat{X}_p(z^N) = \pi(z) G_p(z^N) \tilde{H}_p(z^N) X_p(z^N)
\end{aligned}$$

(because $z^N = \omega$ and $\hat{X}_p(\omega) = G_p(\omega)\tilde{H}_p(\omega)X_p(\omega)$). Substituting them to $S(z)$ gives:

$$\begin{aligned}
S(z) &= \left[\pi(z) X_p(z^N) - \pi(z) G_p(z^N) \tilde{H}_p(z^N) X_p(z^N)\right] G_p(z^{-N})^\top \pi(z^{-1})^\top \\
&= G_p(z^{-N})^\top \pi(z^{-1})^\top \pi(z) \left[X_p(z^N) - G_p(z^N) \tilde{H}_p(z^N) X_p(z^N)\right]
\end{aligned}$$

We know that $\pi(z^{-1})^\top \pi(z) = I - A(z)$, where $A(z)$ contains non-zero phases, thus will vanish $G_p(z^{-N})^\top A(z) \left[X_p(z^N) - G_p(z^N)\tilde{H}_p(z^N)X_p(z^N)\right]$. Therefore

$$\begin{aligned}
S(z) &= G_p(z^{-N})^\top \left[X_p(z^N) - G_p(z^N) \tilde{H}_p(z^N) X_p(z^N)\right] \\
&= G_p(z^{-N})^\top X_p(z^N) - G_p(z^{-N})^\top G_p(z^N) \tilde{H}_p(z^N) X_p(z^N) \\
&= \left[G_p(z^{-N})^\top - G_p(z^{-N})^\top G_p(z^N) \tilde{H}_p(z^N)\right] X_p(z^N) = 0, \forall X_p(z^N) \\
&\Rightarrow G_p(z^{-N})^\top - G_p(z^{-N})^\top G_p(z^N) \tilde{H}_p(z^N) = 0
\end{aligned}$$

Since $\omega = z^n$

$$
\begin{aligned}
&G_p(\omega^{-1})^\top - G_p(\omega^{-1})^\top G_p(\omega)\tilde{H}_p(\omega) = 0 \\
\Leftrightarrow &G_p(\omega^{-1})^\top G_p(\omega)\tilde{H}_p(\omega) = G_p(\omega^{-1})^\top \\
\Leftrightarrow &\tilde{H}_p(\omega) = \left(G_p(\omega^{-1})^\top G_p(\omega)\right)^{-1} G_p(\omega^{-1})^\top
\end{aligned}
$$

5 Ideal-Matched Sampling and Interpolation with Nonorthogonal Filters

(a) $N = 2 \Rightarrow \omega = z^N = z^2$

z-transform of $g[n]$ is

$$G(z) = 1 + \frac{1}{2}(z^{-1} + z)$$

Since $N = 2$, we need components z^0 and z^{-1}

$$\Rightarrow G(z) = 1 + z^{-1}\left(\frac{1}{2} + \frac{1}{2}z^2\right)$$

For type-I polyphase decomposition

$$G_p(\omega) = \begin{bmatrix} G_0(\omega) \\ G_1(\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\omega \end{bmatrix} \Rightarrow G_p(\omega^{-1})^\top = \begin{bmatrix} 1, & \frac{1}{2} + \frac{1}{2}\omega^{-1} \end{bmatrix}$$

From problem 4, we have

$$\tilde{H}_p(\omega) = \left(G_p(\omega^{-1})^\top G_p(\omega)\right)^{-1} G_p(\omega^{-1})^\top$$

$$
\begin{aligned}
G_p(\omega^{-1})^\top G_p(\omega) &= \begin{bmatrix} 1, & \frac{1}{2} + \frac{1}{2}\omega^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\omega \end{bmatrix} \\
&= 1 + \frac{1}{4}(1 + \omega^{-1})(1 + \omega) \\
&= \frac{6 + \omega + \omega^{-1}}{4}
\end{aligned}
$$

$$\Rightarrow \tilde{H}_p(\omega) = \left(\frac{6 + \omega + \omega^{-1}}{4}\right)^{-1} \begin{bmatrix} 1, & \frac{1}{2} + \frac{1}{2}\omega^{-1} \end{bmatrix}$$

We have

$$\begin{aligned}
H(z) &= \tilde{H}_p(z^2)\pi(z^{-1})^\top \qquad \text{(type-II, with } z^2 = \omega) \\
&= \frac{4}{6+\omega+\omega^{-1}} \begin{bmatrix} 1, & \frac{1}{2}+\frac{1}{2}\omega^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} \\
&= \frac{4(1+\frac{z}{2}+\frac{z^{-1}}{2})}{6+z^2+z^{-2}} \\
&= \frac{2(2+z+z^{-1})}{6+z^2+z^{-2}}
\end{aligned}$$

For the numerator

$$\begin{aligned}
2(2+z+z^{-1}) &= 2z^{-1}(2z+z^2+1) \\
&= 2z^{-1}(z+1)(z+1) \\
&= 2(z^{-1}+1)(z+1)
\end{aligned}$$

For the denominator

$$\begin{aligned}
6+z^2+z^{-2} &= z^{-2}(6z^2+z^4+1) \\
&= z^{-2}(z^2+3+2\sqrt{2})(z^2+3-2\sqrt{2}) \\
&= (1+(3+2\sqrt{2})z^{-2})(z^2+3-2\sqrt{2}) \\
&= (1+(3+2\sqrt{2})z^{-2})\left(z^2+\frac{1}{3+2\sqrt{2}}\right) \\
&= (3+2\sqrt{2})\left(z^{-2}+\frac{1}{3+2\sqrt{2}}\right)\left(z^2+\frac{1}{3+2\sqrt{2}}\right) \\
&= \mathcal{C}^{-1}(z^{-2}+\mathcal{C})(z^2+\mathcal{C}), \qquad \text{where } \mathcal{C} = \frac{1}{3+2\sqrt{2}}
\end{aligned}$$

Therefore

$$\begin{aligned}
H(Z) &= \frac{2(z^{-1}+1)(z+1)}{\mathcal{C}^{-1}(z^{-2}+\mathcal{C})(z^2+\mathcal{C})} \\
&= 2\mathcal{C}\frac{z^{-1}+1}{z^{-2}+\mathcal{C}}\frac{z^1+1}{z^2+\mathcal{C}} \\
&= P(z^{-1})P(z), \qquad \text{where } P(z) = \sqrt{2\mathcal{C}}\frac{z+1}{z^2+\mathcal{C}}, \mathcal{C} = \frac{1}{3+2\sqrt{2}}
\end{aligned}$$

$h[n]$ is the inverse z-transform of $H(z)$.

(b)

$$\tilde{H}_p(\omega)G_p(\omega) = 1 \Leftrightarrow \tilde{H}_p(\omega)\begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2}\omega \end{bmatrix} = 1$$

Therefore, the shortest $\tilde{H}_p(\omega)$ is $\begin{bmatrix} 0 & 1 \end{bmatrix}$

$$H(z) = \tilde{H}_p(z^2)\pi(z^{-1})^\top = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix} = 1$$

Hence, $h = \delta$

6 Python Exercise: DTFT Approximation using DFT

Figure 4-8 show the output with $M = 200$. The runtime for `dtft_approx` is 0.1s and `eq_dtft_approx` is 0.0005s. Figure 9-13 show the output with $M = 5000$. The runtime for `dtft_approx` is 2.7s and `eq_dtft_approx` is still 0.0005s. `dtft_approx`'s runtime significantly increases with $M = 5000$; however, M does not affect `eq_dtft_approx` because it is implemented using `fft`.

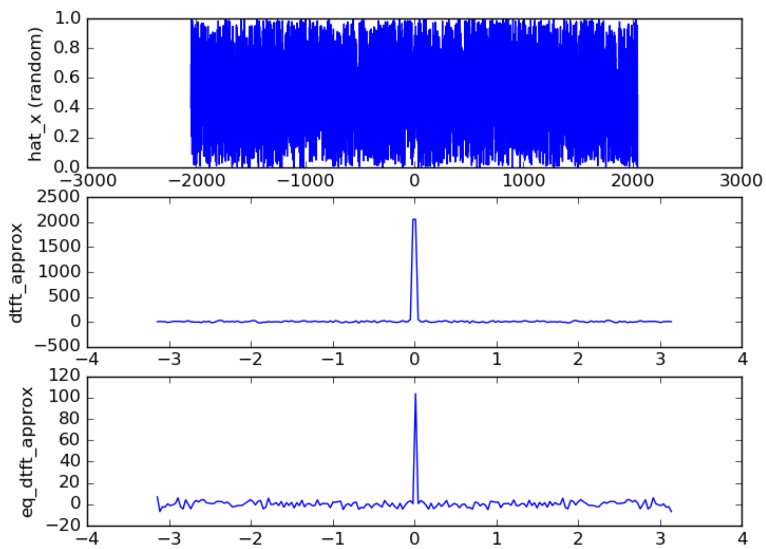

Figure 4: Random signal with $M = 200$

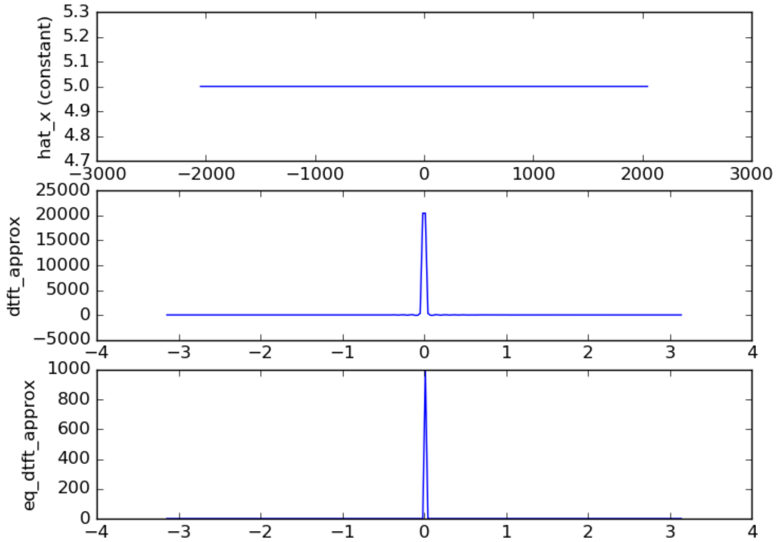

Figure 5: Constant signal with $M = 200$

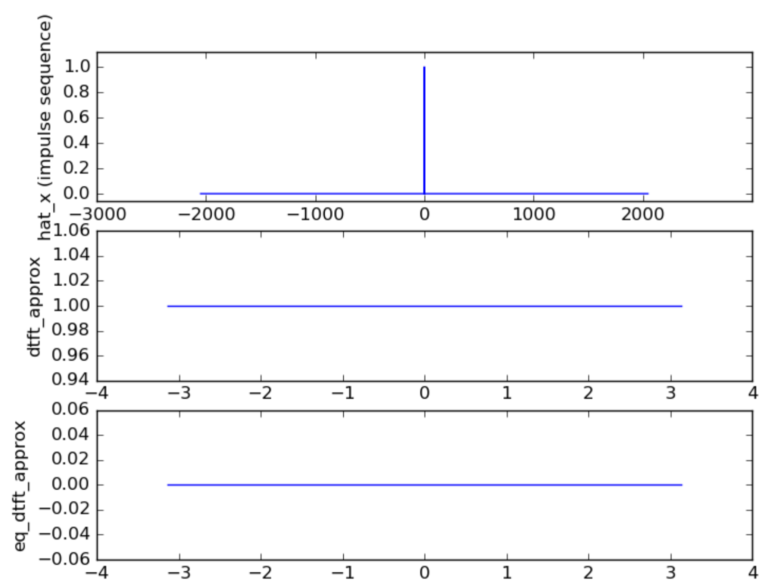

Figure 6: Impulse signal ($x[n] = \delta[n]$) with $M = 200$

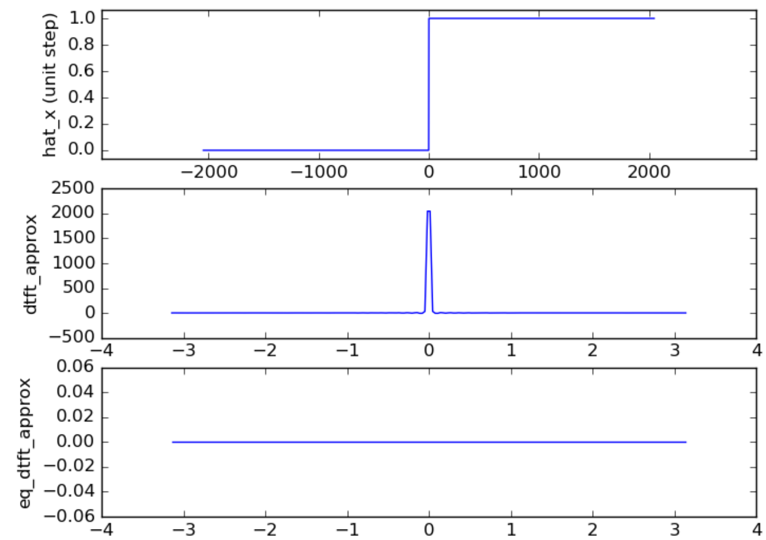

Figure 7: Unit step signal ($x[n] = u[n]$) with $M = 200$

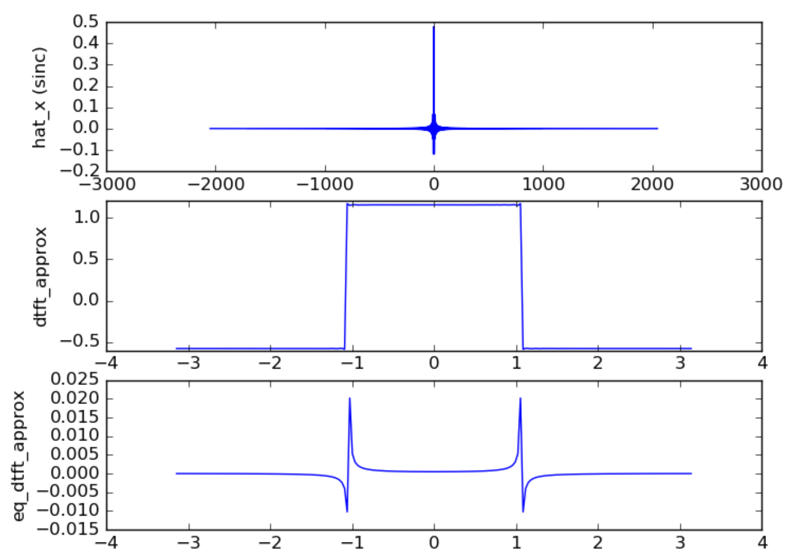

Figure 8: $x[n] = \sqrt{3}\frac{\sin(\frac{\pi n}{3})}{\pi n}$ with $M = 200$

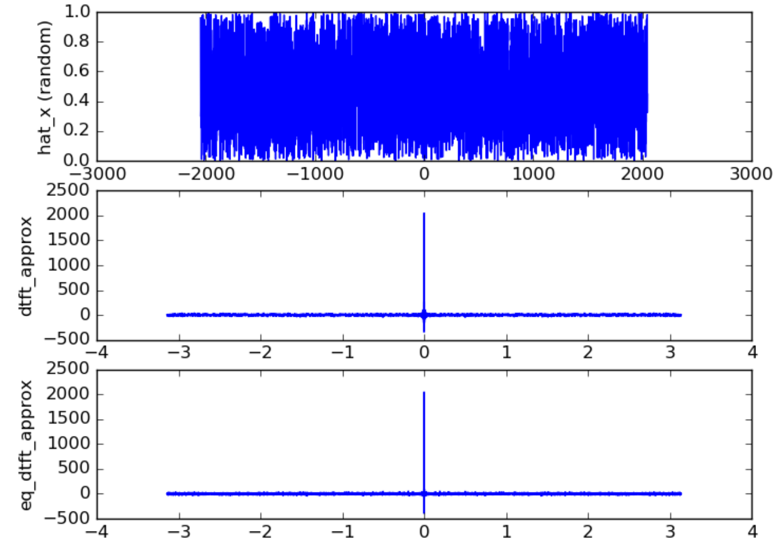

Figure 9: Random signal with $M = 5000$

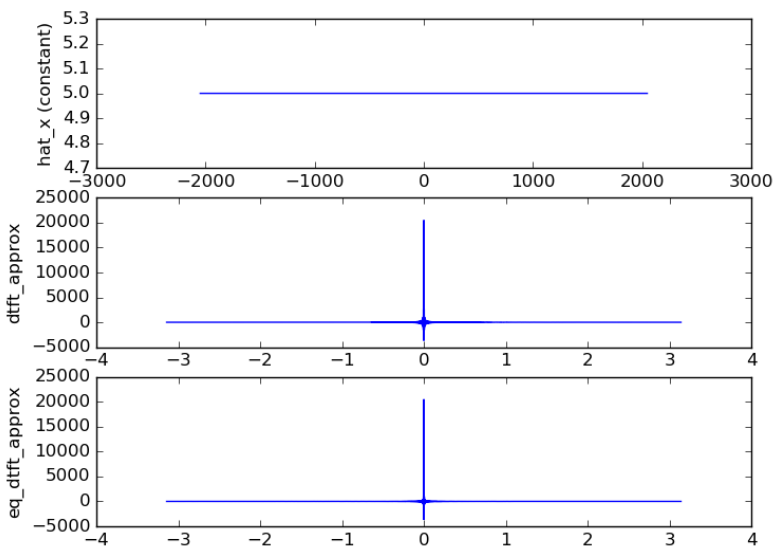

Figure 10: Constant signal with $M = 5000$

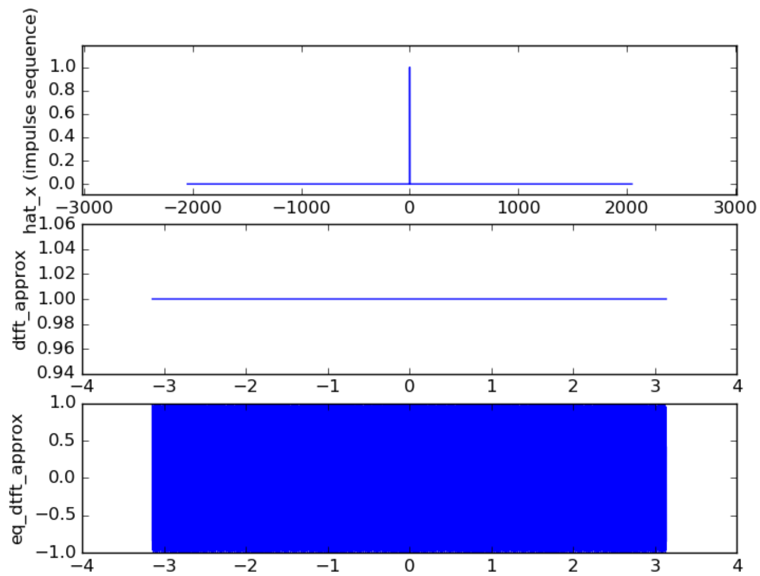

Figure 11: Impulse signal ($x[n] = \delta[n]$) with $M = 5000$

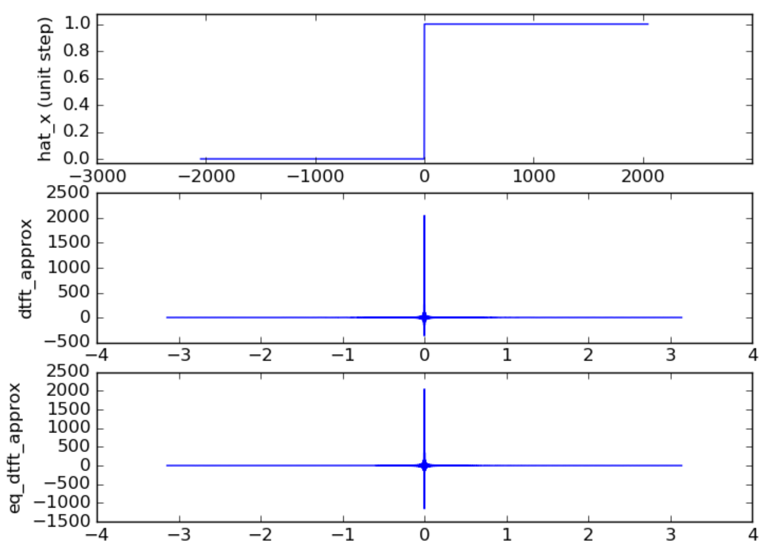

Figure 12: Unit step signal ($x[n] = u[n]$) with $M = 5000$

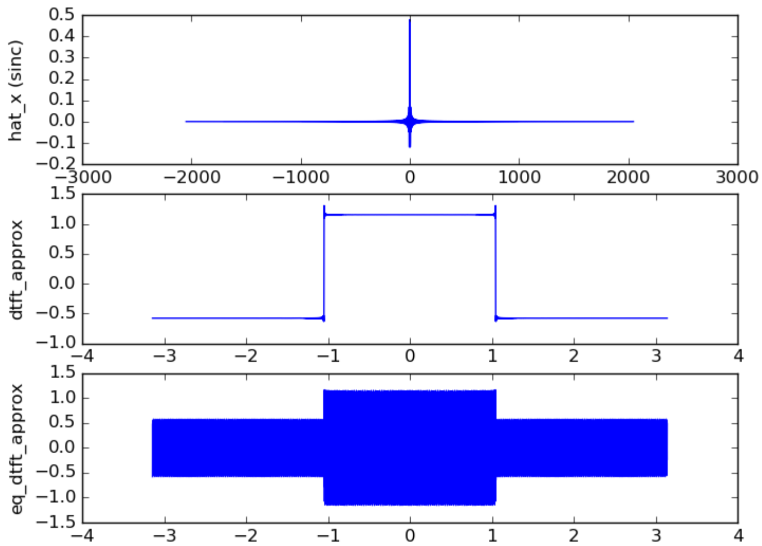

Figure 13: $x[n] = \sqrt{3}\frac{\sin(\frac{\pi n}{3})}{\pi n}$ with $M = 5000$

7 Python Exercise: Image Scaling with Separable Filters

Figure 14 shows the results of the four pre-filters. Since the visual difference is subtle, their corresponding MSEs are included. Method (a) and (c) are actually the same since $h = \delta$ so their MSEs are the same. The pattern (around leg and scarf area) of method (b) is overly smoothed while the alias effect of (a) and (c) is clearly visible.

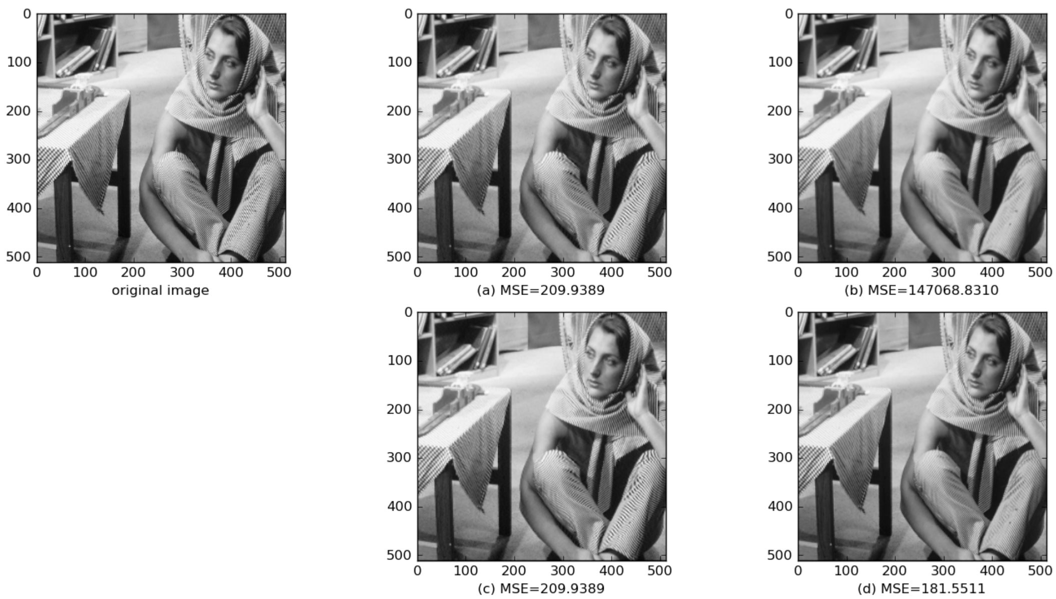

Figure 14: Original image and four pre-filters (a-d) with corresponding MSE