# ECE551 - Homework 6

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## 1 DTFT of Auto-correlation and Cross-correlation

$$
\begin{aligned}
C_{x,y}(\omega) &= \sum_n c_{x,y}[n]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]y[n]\right] e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n](x[n]+w[n])\right] e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]x[n]\right] e^{-jn\omega} + \sum_n \mathbb{E}\left[x[n]w[n]\right] e^{-jn\omega} \\
&= \sum_n a_x[n]e^{-jn\omega} \qquad (\because x[n], w[n] \text{ are uncorrelated}) \\
&= A_x(\omega)
\end{aligned}
$$

$$
\begin{aligned}
A_y(\omega) &= \sum_n a_y[n]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[y[n]y[n]\right] e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[(x[n]+w[n])(x[n]+w[n])\right] e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]x[n]\right] e^{-jn\omega} + \sum_n \mathbb{E}\left[w[n]w[n]\right] e^{-jn\omega} + \sum_n 2\mathbb{E}\left[x[n]w[n]\right] e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]x[n]\right] e^{-jn\omega} + \sum_n \mathbb{E}\left[w[n]w[n]\right] e^{-jn\omega} \\
&= A_x(\omega) + A_w(\omega)
\end{aligned}
$$

## 2 Higly Correlated Random Processes

(a)

$$x_1[n] = \begin{cases} A & \text{even } n \\ B & \text{odd } n \end{cases}$$

Half of the sequence is $A$ and the other half is $B$, so $\mathbb{E}[x_1[n]] = \mathbb{E}\left[\frac{A+B}{2}\right] = 0$ is a constant.

$$a_{x_1}[n_1, n_2] = \mathbb{E}[x_1[n_1]x_1[n_2]] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & & n_1, n_2 \text{ even} \\ \mathbb{E}\left[B^2\right] = 1 & & n_1, n_2 \text{ odd} \\ \mathbb{E}[AB] = 0 & (A, B \text{ uncorrelated}) & \text{else} \end{cases}$$

We have $x_1[0] = A$, so

$$a_{x_1}[0, n1 - n_2] = \mathbb{E}[x_1[0]x_1[n_1 - n_2]] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & \text{both odd or even} \\ \mathbb{E}[AB] = 0 & \text{one odd, one even} \end{cases}$$

$a_{x_1}[n_1, n_2] = a_{x_1}[0, n_1 - n_2]$, so $x_1[n]$ is WSS. Since its values keep alternating between $A$ and $B$, it is periodic.

$$x_2[n] = \begin{cases} A & n \geq 0 \\ B & n < 0 \end{cases}$$

Similarly, $\mathbb{E}[x_2[n]] = \mathbb{E}\left[\frac{A+B}{2}\right] = 0$. We have

$$a_{x_2}[n_1, n_2] = \mathbb{E}[x_2[n_1]x_1[n_2]] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & n_1, n_2 \geq 0 \\ \mathbb{E}\left[B^2\right] = 1 & n_1, n_2 < 0 \\ \mathbb{E}[AB] = 0 & \text{else} \end{cases}$$

and

$$a_{x_1}[0, n1 - n_2] = \mathbb{E}[x_1[0]x_1[n_1 - n_2]] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & n_1 \geq n_2 \\ \mathbb{E}[AB] = 0 & n_1 < n_2 \end{cases}$$

$a_{x_2}[n_1, n_2] \neq a_{x_2}[0, n_1 - n_2]$, so $x_2[n]$ is not WSS. $x_2 = B$ on the negative side and $A$ on the positive side, so it is not periodic.

$$\begin{cases} x_3[n+1] = \frac{1}{2}x_3[n] + A \\ x_3[0] = A \end{cases}$$

We can see that

$$
\begin{aligned}
x_3[0] &= A \\
x_3[1] &= \frac{1}{2}A + A \\
x_3[2] &= \frac{1}{2}\left(\frac{1}{2}A + A\right) + A \\
x_3[3] &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}A + A\right) + A\right) + A \\
&\dots \\
\Rightarrow x_3[n] &= A\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i
\end{aligned}
$$

By geometric series

$$x_3[n] = A\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2A(1-2^{-n-1}) = A(2-2^{-n})$$

So

$$\mathbb{E}\left[x_3[n]\right] = (2-2^{-n-1})\mathbb{E}\left[A\right] = 0$$

We have

$$a_{x_3}[n_1, n_2] = \mathbb{E}\left[x_3[n_1]x_3[n_2]\right] = (2-2^{-n_1})(2-2^{-n_2})\mathbb{E}\left[A^2\right] = (2-2^{-n_1})(2-2^{-n_2})$$

and

$$a_{x_3}[0, n_1-n_2] = \mathbb{E}\left[x_3[0]x_3[n_1-n_2]\right] = (2-2^{-n_1+n_2})\mathbb{E}\left[A^2\right] = (2-2^{-n_1+n_2})$$

$a_{x_3}[n_1, n_2] \neq a_{x_3}[0, n_1-n_2]$, so $x_2[n]$ is not WSS. Since $x_3[n]$ is a geometric series, it is not periodic.

(b)

$$x_1[n] = \begin{cases} A & \text{even } n \\ B & \text{odd } n \end{cases}$$

We can see that $x_1[n+1]$ only depends on $x_1[n-1]$ as the values alternate between $A$ and $B$. Therefore, $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and prediction error is 0.

$$x_2[n] = \begin{cases} A & n \geq 0 \\ B & n < 0 \end{cases}$$

If $n \neq -1$ then $x_2[n+1] = x_2[n]$ and there is no prediction error. If $n = -1$ then the prediction error is $\mathbb{E}[x_2[0] \mid x_2[-1], x_2[-2]]$. Since $x_2[0] = A$, $x_2[-1] = x_2[-2] = B$, and $A$ and $B$ are independent, $\mathbb{E}[x_2[0] \mid x_2[-1], x_2[-2]] = \mathbb{E}[A] = 0$. Hence, $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and prediction error is 0.

$$\begin{cases} x_3[n+1] = \frac{1}{2}x_3[n] + A \\ x_3[0] = A \end{cases}$$

We have

$$\begin{aligned} x_3[n+1] - x_3[n] &= \frac{1}{2}x_3[n] - \frac{1}{2}x_3[n-1] \\ \Leftrightarrow x_3[n+1] &= \frac{3}{2}x_3[n] - \frac{1}{2}x_3[n-1] \end{aligned}$$

Hence, $w = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$ and prediction error is 0.

# 3 Adaptive Filter and LMS

(a) We are given the model $\mathbb{E}[x[0]x[m]] = 2^{-|m|} + 4^{-|m|} = a_x[m]$, therefore we can use probabilistic cost function for this problem.

$$C(w) = \gamma_d - 2w^\top R_{xd} + w^\top R_x w$$

(b)

$$\begin{aligned} R_x &= \mathbb{E}\left[X[n]X[n]^\top\right] \\ &= \begin{bmatrix} a_x[0] & a_x[1] & a_x[2] & \cdots & a_x[L-1] \\ a_x[1] & a_x[0] & a_x[1] & \cdots & a_x[L-2] \\ a_x[2] & a_x[1] & a_x[0] & \cdots & a_x[L-3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_x[L-1] & a_x[L-2] & a_x[L-3] & \cdots & a_x[0] \end{bmatrix} \end{aligned}$$

For $L \geq 3$,

For $L = 2$

For $L = 1$

(c)

(d)

# 4 Regularized Wiener Filter and Leaky LMS

# 5 Python Problem - Wiener's LMS

# 6 Python Problem - AR System Identification