ECE551 - Homework 6

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October 29, 2016

1 DTFT of Auto-correlation and Cross-correlation

$$\begin{aligned}
C_{x,y}(\omega) &= \sum_n c_{x,y}[n]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]y[n]\right]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n](x[n]+w[n])\right]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]x[n]\right]e^{-jn\omega} + \sum_n \mathbb{E}\left[x[n]w[n]\right]e^{-jn\omega} \\
&= \sum_n a_x[n]e^{-jn\omega} \qquad (\because x[n], w[n] \text{ are uncorrelated}) \\
&= A_x(\omega)
\end{aligned}$$

$$\begin{aligned}
A_y(\omega) &= \sum_n a_y[n]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[y[n]y[n]\right]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[(x[n]+w[n])(x[n]+w[n])\right]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]x[n]\right]e^{-jn\omega} + \sum_n \mathbb{E}\left[w[n]w[n]\right]e^{-jn\omega} + \sum_n 2\mathbb{E}\left[x[n]w[n]\right]e^{-jn\omega} \\
&= \sum_n \mathbb{E}\left[x[n]x[n]\right]e^{-jn\omega} + \sum_n \mathbb{E}\left[w[n]w[n]\right]e^{-jn\omega} \\
&= A_x(\omega) + A_w(\omega)
\end{aligned}$$

2 Higly Correlated Random Processes

(a)

$$x_1[n] = \begin{cases} A & \text{even } n \\ B & \text{odd } n \end{cases}$$

Half of the sequence is A and the other half is B, so $\mathbb{E}\left[x_1[n]\right] = \mathbb{E}\left[\frac{A+B}{2}\right] = 0$ is a constant.

$$a_{x_1}[n_1, n_2] = \mathbb{E}\left[x_1[n_1]x_1[n_2]\right] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & & n_1, n_2 \text{ even} \\ \mathbb{E}\left[B^2\right] = 1 & & n_1, n_2 \text{ odd} \\ \mathbb{E}\left[AB\right] = 0 & (A, B \text{ uncorrelated}) & \text{else} \end{cases}$$

We have $x_1[0] = A$, so

$$a_{x_1}[0, n1 - n_2] = \mathbb{E}\left[x_1[0]x_1[n_1 - n_2]\right] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & \text{both odd or even} \\ \mathbb{E}\left[AB\right] = 0 & \text{one odd, one even} \end{cases}$$

$a_{x_1}[n_1, n_2] = a_{x_1}[0, n_1 - n_2]$, so $x_1[n]$ is WSS. Since its values keep alternating between A and B, it is periodic.

$$x_2[n] = \begin{cases} A & n \geq 0 \\ B & n < 0 \end{cases}$$

Similarly, $\mathbb{E}\left[x_2[n]\right] = \mathbb{E}\left[\frac{A+B}{2}\right] = 0$. We have

$$a_{x_2}[n_1, n_2] = \mathbb{E}\left[x_2[n_1]x_1[n_2]\right] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & n_1, n_2 \geq 0 \\ \mathbb{E}\left[B^2\right] = 1 & n_1, n_2 < 0 \\ \mathbb{E}\left[AB\right] = 0 & \text{else} \end{cases}$$

and

$$a_{x_1}[0, n1 - n_2] = \mathbb{E}\left[x_1[0]x_1[n_1 - n_2]\right] = \begin{cases} \mathbb{E}\left[A^2\right] = 1 & n_1 \geq n_2 \\ \mathbb{E}\left[AB\right] = 0 & n_1 < n_2 \end{cases}$$

$a_{x_2}[n_1, n_2] \neq a_{x_2}[0, n_1 - n_2]$, so $x_2[n]$ is not WSS. $x_2 = B$ on the negative side and A on the positive side, so it is not periodic.

$$\begin{cases} x_3[n+1] = \frac{1}{2}x_3[n] + A \\ x_3[0] = A \end{cases}$$

We can see that

$$
\begin{aligned}
x_3[0] &= A \\
x_3[1] &= \frac{1}{2}A + A \\
x_3[2] &= \frac{1}{2}\left(\frac{1}{2}A + A\right) + A \\
x_3[3] &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}A + A\right) + A\right) + A \\
&\dots \\
\Rightarrow x_3[n] &= A\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i
\end{aligned}
$$

By geometric series

$$x_3[n] = A\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2A(1-2^{-n-1}) = A(2-2^{-n})$$

So

$$\mathbb{E}\left[x_3[n]\right] = (2-2^{-n-1})\mathbb{E}\left[A\right] = 0$$

We have

$$a_{x_3}[n_1, n_2] = \mathbb{E}\left[x_3[n_1]x_3[n_2]\right] = (2-2^{-n_1})(2-2^{-n_2})\mathbb{E}\left[A^2\right] = (2-2^{-n_1})(2-2^{-n_2})$$

and

$$a_{x_3}[0, n_1-n_2] = \mathbb{E}\left[x_3[0]x_3[n_1-n_2]\right] = (2-2^{-n_1+n_2})\mathbb{E}\left[A^2\right] = (2-2^{-n_1+n_2})$$

$a_{x_3}[n_1, n_2] \neq a_{x_3}[0, n_1-n_2]$, so $x_2[n]$ is not WSS. Since $x_3[n]$ is a geometric series, it is not periodic.

(b)

$$x_1[n] = \begin{cases} A & \text{even } n \\ B & \text{odd } n \end{cases}$$

We can see that $x_1[n+1]$ only depends on $x_1[n-1]$ as the values alternate between A and B. Therefore, $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and prediction error is 0.

$$x_2[n] = \begin{cases} A & n \geq 0 \\ B & n < 0 \end{cases}$$

If $n \neq -1$ then $x_2[n+1] = x_2[n]$ and there is no prediction error. If $n = -1$ then the prediction error is $\mathbb{E}\left[x_2[0] \mid x_2[-1], x_2[-2]\right]$. Since $x_2[0] = A$, $x_2[-1] = x_2[-2] = B$, and A and B are independent, $\mathbb{E}\left[x_2[0] \mid x_2[-1], x_2[-2]\right] = \mathbb{E}\left[A\right] = 0$. Hence, $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and prediction error is 0.

$$\begin{cases} x_3[n+1] = \frac{1}{2}x_3[n] + A \\ x_3[0] = A \end{cases}$$

We have

$$\begin{aligned} x_3[n+1] - x_3[n] &= \frac{1}{2}x_3[n] - \frac{1}{2}x_3[n-1] \\ \Leftrightarrow x_3[n+1] &= \frac{3}{2}x_3[n] - \frac{1}{2}x_3[n-1] \end{aligned}$$

Hence, $w = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$ and prediction error is 0.

3 Adaptive Filter and LMS

(a) We are given the model $\mathbb{E}\left[x[0]x[m]\right] = 2^{-|m|} + 4^{-|m|} = a_x[m]$, therefore we can use probabilistic cost function for this problem.

(b) In general

$$\begin{aligned} R_x &= \mathbb{E}\left[X[n]X[n]^\top\right] \\ &= \begin{bmatrix} a_x[0] & a_x[1] & a_x[2] & \cdots & a_x[L-1] \\ a_x[1] & a_x[0] & a_x[1] & \cdots & a_x[L-2] \\ a_x[2] & a_x[1] & a_x[0] & \cdots & a_x[L-3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_x[L-1] & a_x[L-2] & a_x[L-3] & \cdots & a_x[0] \end{bmatrix} \end{aligned}$$

and the cost function

$$C(w) = \mathbb{E}\left[|e[n]|^2\right]$$

where $e[n] = y[n] - d[n]$ is the prediction error, $y[n]$ is the prediction, and $d[n]$ is the reference.

Let $X[n] = \begin{bmatrix} x[n] & x[n-1] & x[n-2] & x[n-3] & \cdots & x[n-L+1] \end{bmatrix}^\top$.

For $L \geq 3$,

$$\begin{aligned} & y[n] = w^\top X[n] \\ & d[n] = \alpha_1 x[n-1] + \alpha_2 x[n-2] = \begin{bmatrix} 0 & \alpha_1 & \alpha_2 & 0 & \cdots & 0 \end{bmatrix} X[n] = A^\top X[n] \\ \Rightarrow\ & e[n] = y[n] - d[n] = (w - A)^\top X[n] \end{aligned}$$

The cost function is

$$C(w) = \mathbb{E}\left[|e[n]|^2\right] = (w-A)^\top R_x (w-A)$$

Therefore, $\min C(w) = 0$ for $w = A$. Hence, $w_{opt} = \begin{bmatrix} 0 & \alpha_1 & \alpha_2 & 0 & \cdots & 0 \end{bmatrix}$
For $L = 2$

$$R_x = \begin{bmatrix} a_x[0] & a_x[1] \\ a_x[1] & a_x[0] \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{4} \\ \frac{3}{4} & 2 \end{bmatrix}$$

$$\begin{aligned} R_{xd} &= \mathbb{E}\left[X[n]d[n]\right] \\ &= \mathbb{E}\left[\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} (\alpha_1 x[n-1] + \alpha_2 x[n-2])\right] \\ &= \begin{bmatrix} \alpha_1 \mathbb{E}\left[x[n]x[n-1]\right] + \alpha_2 \mathbb{E}\left[x[n]x[n-2]\right] \\ \alpha_1 \mathbb{E}\left[x[n-1]x[n-1]\right] + \alpha_2 \mathbb{E}\left[x[n-1]x[n-2]\right] \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 a_x[1] + \alpha_2 a_x[2] \\ \alpha_1 a_x[0] + \alpha_2 a_x[1] \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{4}\alpha_1 + \frac{5}{16}\alpha_2 \\ 2\alpha_1 + \frac{3}{4}\alpha_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \gamma_d &= \mathbb{E}\left[d[n]^2\right] \\ &= \alpha_1^2 \mathbb{E}\left[x[n-1]x[n-1]\right] + \alpha_2^2 \mathbb{E}\left[x[n-2]x[n-2]\right] + 2\alpha_1\alpha_2 \mathbb{E}\left[x[n-1]x[n-2]\right] \\ &= \alpha_1^2 a_x[0] + \alpha_2^2 a_x[0] + 2\alpha_1\alpha_2 a_x[1] \\ &= 2(\alpha_1^2 + \alpha_2^2) + \frac{3}{2}\alpha_1\alpha_2 \end{aligned}$$

$$C(w) = \gamma_d - 2w^\top R_{xd} + w^\top R_x w$$

$$\begin{aligned} w_{opt} = R_x^{-1} R_{xd} &= \frac{4}{55}\begin{bmatrix} 8 & -3 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} \frac{3}{4}\alpha_1 + \frac{5}{16}\alpha_2 \\ 2\alpha_1 + \frac{3}{4}\alpha_2 \end{bmatrix} \\ &= \frac{4}{55}\begin{bmatrix} \frac{1}{4}\alpha_2 \\ \frac{55}{4}\alpha_1 + \frac{81}{16}\alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{55}\alpha_1 \\ \alpha_1 + \frac{81}{220}\alpha_2 \end{bmatrix} \end{aligned}$$

For $L = 1$

$$
\begin{aligned}
R_x &= a_x[0] = 2 \\
R_{xd} &= \alpha_1 a_x[1] + \alpha_2 a_x[2] = \frac{3}{4}\alpha_1 + \frac{5}{16}\alpha_2 \\
\gamma_d &= 2(\alpha_1^2 + \alpha_2^2) + \frac{3}{2}\alpha_1\alpha_2 \\
C(w) &= \gamma_d - 2w^\top R_{xd} + w^\top R_x w \\
&= 2(\alpha_1^2 + \alpha_2^2) + \frac{3}{2}\alpha_1\alpha_2 - \left(\frac{3}{2}\alpha_1 + \frac{5}{8}\alpha_2\right) w + 2w^2 \\
w_{opt} &= R_x^{-1} R_{xd} = \frac{3}{8}\alpha_1 + \frac{5}{32}\alpha_2
\end{aligned}
$$

(c) The gradient of selected cost

$$
\begin{aligned}
\nabla_w C(\hat{w}) &= \nabla_w \mathbb{E}\left[|e[n]|^2\right] \\
&= \mathbb{E}\left[2e[n]\nabla_w e[n]\right] && \text{(chain rule)} \\
&= -2\mathbb{E}\left[e[n]X[n]\right] && (\nabla_w e[n] = -X[n])
\end{aligned}
$$

The gradient descent update equation (with μ as the learning rate)

$$
\hat{w}[n+1] = \hat{w}[n] - \frac{1}{2}\mu\nabla_w C(\hat{w}[n]) = \hat{w}[n] + \mu\mathbb{E}\left[e[n]X[n]\right]
$$

converges to a local minimum if $C(w)$ is strictly convex (R_x is invertible) and differentiable. Indeed, if $\hat{w}[n] \to \hat{w}$ converges then $\hat{w}[n+1]$ to the same limit, the gradient equation becomes

$$
\hat{w} = \hat{w} - \frac{1}{2}\mu\nabla_w C(\hat{w}) \Rightarrow \nabla_W C(\hat{w}) = 0
$$

which is a characterization of a local minimum of $C(w)$.

(d) In LMD, we assume that $\mathbb{E}\left[e[n]X[n]\right] \approx e[n]X[n]$. Therefore, the LMS update equations are

$$
\begin{aligned}
e[n] &= d[n] - \hat{w}[n]^\top X[n] \\
\hat{w}[n+1] &= \hat{w}[n] + \mu X[n]e[n]
\end{aligned}
$$

We have

$$
\nabla_w C(\hat{w}) = -2\mathbb{E}\left[X[n](d[n] - \hat{X}[n]^\top \hat{w})\right] = -w(R_{xd} - R_x\hat{w})
$$

so that the ideal iterations are

$$\hat{w}[n+1] = (I - \mu R_x)\hat{w}[n] + \mu R_{xd}$$

This is a linear difference equation in the vector $\hat{w}[n]$. Such difference equation has a convergent solution iff the eigenvalues of $I - \mu R_x$ are contained in the unit circle. The eigenvalues of $I - \mu R_x$ are given by

$$\lambda_k = 1 - \mu\psi_k, \qquad k = 1, \cdots, L$$

where $\psi_1 < \psi_2 \leq \cdots \leq \psi_L$ are the eigenvalues of R_x, sorted by increasing order. We want

$$\begin{aligned}
&-1 < \lambda_k < 1 \\
\Leftrightarrow &-1 < 1 - \mu\psi_k < 1 \\
\Leftrightarrow &1 > \mu\psi_k - 1 > -1 \\
\Leftrightarrow &0 < \mu\psi_k < 2 \\
\Rightarrow &0 < \mu < \frac{2}{\psi_L}
\end{aligned}$$

where ψ_L is the largest eigenvalue of R_x. Since $tr(R_x) = \sum_{k=1}^{L} \psi_k \geq \psi_L$

$$0 < \mu < \frac{2}{tr(R_x)}$$

4 Regularized Wiener Filter and Leaky LMS

5 Python Problem - Wiener's LMS

6 Python Problem - AR System Identification