ECE551 - Homework 2

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1 Frames and Bases

(a) The synthesis operator associated with $\{\varphi_k\}_{k\in\mathcal{K}}$ in $\mathbb{R}^2$ is

$$\Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$$

Hence,

$$\Phi_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Phi_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Phi_4 = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

(Note that we reuse the notation Φ to represent the matrix representation.)

(b) For $\Phi \in \mathbb{R}^{M\times N}$, if $M = N$ then it is a basis, if $M > N$ then it is a frame.

Let A be the inverse of the Gram matrix of basis Φ, i.e. $A = (\Phi^*\Phi)^{-1}$. Then $\Phi = \Phi A = \Phi(\Phi^*\Phi)^{-1}$ forms a dual basis with Φ.

Let $B = (\Phi\Phi^*)^{-1}$, where Φ is a frame. Then $\tilde{\Phi} = B\Phi = (\Phi\Phi^*)^{-1}\Phi$ forms the canonical dual frame associated with frame Φ.

For Φ_1 (basis),

$$A_1 = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \Rightarrow A_1^{-1} = \begin{bmatrix} 4 & -2\sqrt{3} \\ -2\sqrt{3} & 4 \end{bmatrix}$$
$$\tilde{\Phi}_1 = \begin{bmatrix} 2 & -\sqrt{3} \\ 0 & 1 \end{bmatrix}$$

For Φ_2 (frame),

$$B_2 = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B_2^{-1}$$
$$\tilde{\Phi}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} = \Phi_2$$

For Φ_3 (basis),

$$A_3 = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_3^{-1}$$
$$\Rightarrow \tilde{\Phi}_3 = \Phi_3$$

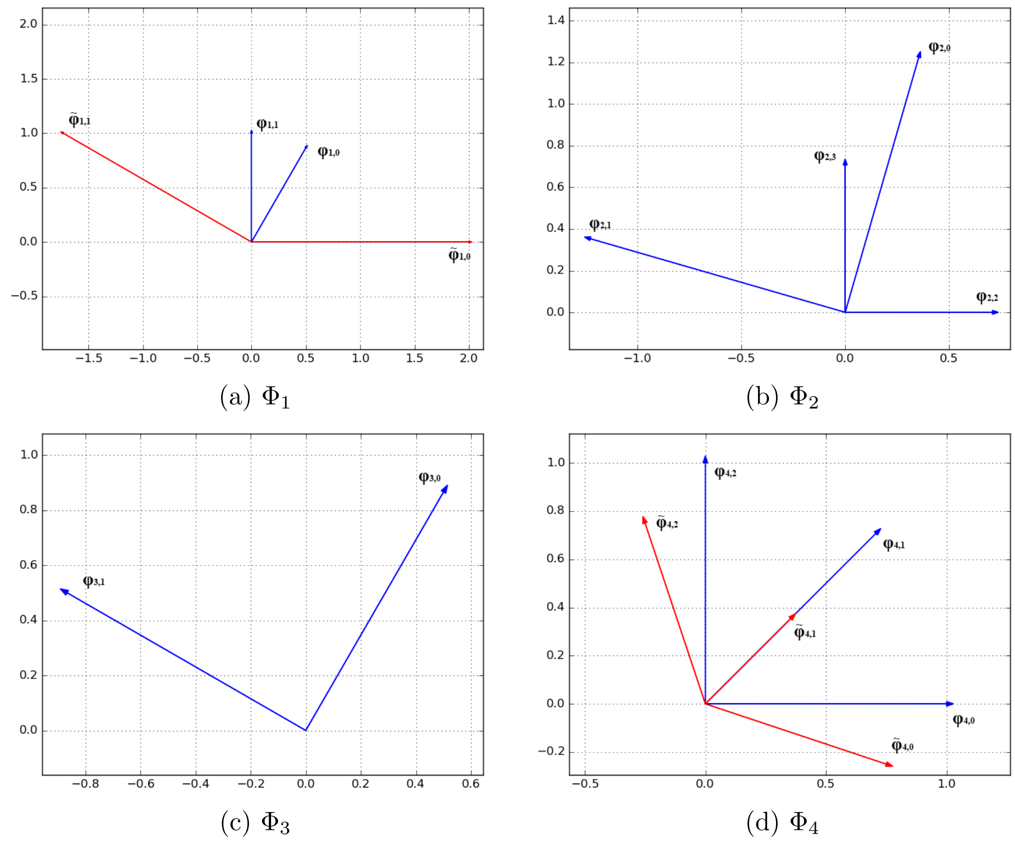
For Φ_4 (basis),

$$B_4 = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \Rightarrow B_4^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$
$$\tilde{\Phi}_4 = \begin{bmatrix} \frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{3}{4} \end{bmatrix}$$

Figure 1 shows the sketch of the sets and their duals.

(c) $\langle \varphi_{1,0}, \varphi_{1,1} \rangle = \frac{\sqrt{3}}{2}$, so the basis Φ_1 is not orthogonal, thus not orthonormal. $B_2 = I$, so the frame Φ_2 is tight (a frame is tight if $\Phi\Phi^* = I$.) $\langle \varphi_{3,0}, \varphi_{3,1} \rangle = 0$ and $\|\varphi_{3,0}\| = \|\varphi_{3,1}\| = 1$, so the basis Φ_3 is orthonormal. $B_4 \neq I$, so the frame Φ_4 is not tight.

(d) $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\alpha_{i,k} = \langle x, \tilde{\varphi}_{i,k} \rangle$. Therefore,



(a) Φ_1 (b) Φ_2

(c) Φ_3 (d) Φ_4

Figure 1: Original sets and their duals.

For Φ_1,

$$\alpha_{1,0} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\rangle = 4 \qquad \alpha_{1,1} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\rangle = -2\sqrt{3}$$

For Φ_2,

$$\alpha_{2,0} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}} \qquad \alpha_{2,1} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle = -\sqrt{\frac{3}{2}}$$

$$\alpha_{2,2} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \frac{2}{\sqrt{2}} \qquad \alpha_{2,3} = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = 0$$

For Φ_3,

$$\alpha_{3,0} = \frac{1}{2} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right\rangle = 1 \qquad \alpha_{3,1} = \frac{1}{2} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\rangle = -\sqrt{3}$$

For Φ_4,

$$\alpha_{4,0} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} \right\rangle = \frac{3}{2} \qquad \alpha_{4,1} = \frac{1}{2\sqrt{2}} \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\alpha_{4,2} = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \right\rangle = -\frac{1}{2}$$

(e) We check the values for $\alpha_{i,k}$ by verifying the expansion $x = \sum_k \alpha_{i,k}\varphi_{i,k}$.
For Φ_1,

$$\sum_k \alpha_{1,k}\varphi_{1,k} = 4 \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - 2\sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x$$

For Φ_2,

$$\sum_k \alpha_{2,k}\varphi_{2,k} = \frac{1}{2} \left(1 \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - \sqrt{3} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x$$

For Φ_3,

$$\sum_k \alpha_{3,k}\varphi_{3,k} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - \sqrt{3} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x$$

For Φ_4,

$$\sum_k \alpha_{4,k}\varphi_{4,k} = \frac{3}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$

(f) We verify that $\Phi\tilde{\Phi}^\top = I$.

$$\Phi_1\tilde{\Phi}_1^\top = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -\sqrt{3} & 1 \end{bmatrix} = I$$

$$\Phi_2\tilde{\Phi}_2^\top = \Phi_2\Phi_2^\top = B_2 = I$$

$$\Phi_3\tilde{\Phi}_3^\top = \Phi_3\Phi_3^\top = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = I$$

$$\Phi_4\tilde{\Phi}_4^\top = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} = I$$

(g) We check if $\|x\|^2 = \sum_k |\alpha_{i,k}|^2$.

$$\|x\|^2 = \left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\|^2 = 4^2 = 16$$

$$\sum_k |\alpha_{1,k}|^2 = 28$$

$$\sum_k |\alpha_{2,k}|^2 = \frac{13}{4}$$

$$\sum_k |\alpha_{3,k}|^2 = 4$$

$$\sum_k |\alpha_{4,k}|^2 = 3$$

Hence, the expansion does not preserve the norm.

(h) Since all of the expansion $\sum_k \alpha_{i,k}\varphi_{i,k} = x$, the expansion is redundant.

2 Linear Least-Squares Approximation

3 Orthogonalization of a Projection

(a) If $P = P^*$,

$$\langle Px, y\rangle_0 = y^*(Px) = (y^*P)x = (y^*P^*)x = (Py)^*x = \langle x, Py\rangle_0 , \qquad \forall x, y \in V.$$

Hence, P is self-adjoint with respect to $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^N$ if $P = P^*$.

(b) Remind that a non-zero vector $v \in \mathbb{C}^N$ is an eigenvector of square matrix $P \in \mathbb{C}^{N\times N}$ if $Pv = \lambda v$, where λ is the eigenvalue associated with v.

If $P = P^2$,

$$\lambda v = Pv = P^2 v = \lambda^2 v$$

Since $v \neq 0$, $\lambda = \lambda^2 \Leftrightarrow \lambda = 0$ or $\lambda = 1$. Hence the eigenvalues of an oblique projection is 0 or 1.

(c) Since $P = T^{-1}DT$, where D is a diagonal matrix with eigenvalues found in part (b), D's diagonal is formed by 1 and 0. Therefore, $D^* = D$

Since $\langle x, y\rangle_T \triangleq y^*T^*Tx$, we have

$$\begin{aligned}
\langle Px, y\rangle_T &= y^*T^*TPx \\
&= y^*T^*T(T^{-1}DT)x \\
&= y^*T^*DTx \\
&= y^*T^*DI^*Tx \\
&= y^*T^*D(TT^{-1})^*Tx \\
&= y^*T^*D(T^{-1})^*T^*Tx \\
&= y^*T^*D^*(T^{-1})^*T^*Tx \\
&= y^*(T^{-1}DT)^*T^*Tx \\
&= y^*P^*T^*Tx \\
&= (Py)^*T^*Tx \\
&= \langle x, Py\rangle_T
\end{aligned}$$

(d) We have $P = T^{-1}DT \Rightarrow P^* = T^*D^*(T^{-1})^*$.

$$\begin{aligned}
\langle x - Px, Px\rangle_T &= x^*P^*T^*T(x - Px) = x^*P^*T^*Tx - x^*P^*T^*TPx \\
&= x^*T^*D^*(T^{-1})^*T^*Tx - x^*T^*D^*(T^{-1})^*T^*TT^{-1}DTx \\
&= x^*T^*D^*((T^{-1})^*T^*)Tx - x^*T^*D^*((T^{-1})^*T^*)(TT^{-1})DTx \\
&= x^*T^*D^*Tx - x^*T^*D^*DTx
\end{aligned}$$

Since D is a diagonal matrix with only 1 and 0, $DD^* = D^*D = D = D^*$. Therefore,

$$\begin{aligned}\langle x - Px, Px\rangle_T &= x^*T^*D^*Tx - x^*T^*D^*DTx \\ &= x^*T^*D^*Tx - x^*T^*D^*Tx = 0\end{aligned}$$

Hence, $x - Px \perp Px$.

(e) From part (c), we know that P is oblique and self-adjoint, i.e. $P = P^2 = P^*$. Since $P \in \mathbb{C}^{N\times N}$, $I = I_N$ and $PI = IP = P$. Therefore,

$$(I-P)^2 = (I-P)(I-P) = I^2 - IP - PI + P^2 = I - 2P + P^2 = I - 2P + P = I - P$$

and

$$(I-P)^* = I^* - P^* = I - P$$

Hence, $I - P$ is oblique and self-adjoint.

We know that for any matrix A, $\mathcal{R}(A) \perp \mathcal{N}(A^\top)$. Therefore,

$$\mathcal{R}(I-P) \perp \mathcal{N}((I-P)^\top)$$

Since $I - P$ is proven to be self-adjoint, $(I-P)^\top = I - P$

$$\Rightarrow \mathcal{R}(I-P) \perp \mathcal{N}(I-P)$$

From the definition of null space:

$$\begin{aligned}\mathcal{N}(P) &= \{x \mid Px = 0\} \\ \mathcal{N}(I-P) &= \{x \mid (I-P)x = 0\}\end{aligned}$$

We proved that $x - Px \perp Px$ wrt $\langle\cdot,\cdot\rangle_T$, so

$$(I-P)x \perp Px \Rightarrow \mathcal{N}(I-P) \perp \mathcal{N}(P)$$

We already have $\mathcal{R}(I-P) \perp \mathcal{N}(I-P)$, thus $\mathcal{R}(I-P) = \mathcal{N}(P)$.

(f)

4 Approximation by Orthogonal Indicator Tiles