ECE551, Fall 2016: Homework Problem Sets $\#\{3,4\}$
Rev 0. Due Sep. 27 2016 in class

1. **Deterministic Correlation sequences**

For a (countable) index set I we define the ℓ_p-signals space as

$$\ell_p(I) := \{x \in \mathbb{C}^I \mid \sum_{t\in I} |x[t]|^p < \infty\} \subset \mathbb{C}^I.$$

These (normed) spaces play an important role in analysis and probability theory, and particularly the space $\ell_2(I)$, which is a Hilbert space with $\langle u, v\rangle := \sum_{i\in I} u[i]\bar{v}[i]$. The theory focuses on $I = \mathbb{Z}$, as most results on $\mathbb{Z}$ can be easily generalized to any countable set.

(a) We define the *unit shift operator* $\sigma : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by $(\sigma x)[n] := x[n-1]$. Compute $(\sigma^k x)[n]$ and $(\sigma^{-1}x)[n]$ in terms of x, n and k.

Definition 1 (correlation sequences) *Let $x, y \in \ell_2(\mathbb{Z})$ be two sequences. The deterministic autocorrelation sequence is defined by*

$$a_x[k] := \left\langle x, \sigma^{-k}x \right\rangle = \sum_{n\in\mathbb{Z}} x[n]x[n+k]^* \tag{1}$$

and likewise the crosscorrelation sequence defined by

$$c_{x,y}[k] := \left\langle x, \sigma^{k}y \right\rangle = \sum_{n\in\mathbb{Z}} x[n]y[n-k]^* \tag{2}$$

(b) Prove or counter the following statements (note that we work on the *complex* field $\mathbb{C}$):

i. $a_x[k] = a_x[-k]^*$
ii. $|a_x[k]| \le a_x[0]$ for every $k \in \mathbb{Z}$
iii. $c_{x,y}[n] = c_{y,x}[-n]^*$
iv. $c_{x,y}[n] = c_{x,y}[-n]^*$
v. $C_{x,y}(\omega) = X(\omega)Y(\omega)^*$, where $X, Y, C_{x,y}$ are the DTFTs of x, y and $c_{x,y}$ respectively.

(c) **Simple Delay Detection**

A signal $x \in \mathbb{R}^{\mathbb{Z}}$ whose support is bounded to $[0 \dots N-1]$ is received by two antennas, each introduces a different gain and delay. The received signals x_1 and x_2 are given by

$$x_1[n] = \alpha_1 x[x - n_1], \qquad x_2[n] = \alpha_2 x[n - n_2]. \tag{3}$$

The constants $\alpha_1, \alpha_2 \in \mathbb{R}$ are gain coefficients and $n_1, n_2 \in \mathbb{Z}$ are delays, all unknown.

i. Based on the result from part 1(b)ii, derive an algorithm to determine the time delay $\Delta := n_2 - n_1$ and the gain ratio $\rho = \frac{\alpha_1}{\alpha_2}$ given inputs x_1 and x_2.
ii. Explain why the explicit delay values n_1 and n_2, cannot be determined, but only their difference Δ. Is the same true for the gains α_1, α_2 ?

2. **Studying yet another Linear System**
   Consider the system $L : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ defined by the relation

$$(Lx)[n] := x[n-1] + x[n+1] - 2x[n] \quad \text{for every } x \in \mathbb{C}^{\mathbb{Z}} \tag{4}$$

   Note that $(Lx)[n]$ is well defined being a finite sum for every n.

   (a) Is that system linear ? Shift invariant ? Causal ? memoryless ? BIBO stable ?

   (b) For each of the input signals given below, find the corresponding output $y_k = Lx_k$, sketch both input and output, and explain the effect of L.

$$x_1[n] = c \quad \text{for all } n \in \mathbb{Z} \qquad \text{(a constant sequence)}$$

$$x_2[n] = \delta[n] := \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases} \qquad \text{(the unit impulse sequence)}$$

$$x_3[n] = u[n] := \begin{cases} 1 & n \geq 0 \\ 0 & \text{else} \end{cases} \qquad \text{(the unit step sequence)}$$

3. **DTFT Affairs**

   (a) Let $x, y \in \ell_1(\mathbb{Z})$, $z[n] := x[n]y[n]$, and let $X, Y, Z : [-\pi, \pi] \to \mathbb{C}$ be their respective DTFTs. Show that $Z = \frac{1}{2\pi} X \circledast Y$, where

$$(X \circledast Y)(\omega) := \int_{-\pi}^{\pi} X(\nu) Y(\omega - \nu) d\nu$$

   (b) Some LTI system is characterized by the impulse response

$$h[n] = \sqrt{3} \frac{\sin(\frac{1}{3}\pi n)}{\pi n}$$

   i. Compute the DTFT $H(\omega)$. What kind of a filter is that ?

   ii. Let $x[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$ and $y = h * x$. Compute and sketch the DTFT $Y(\omega)$.

4. **The Z-transform of an Autocorrelation.**
   Let $x \in \ell_1(\mathbb{Z})$ with the deterministic autocorrelation sequence a_x, and let $X(z)$ and $A_x(z)$ be their Z-transforms.

   (a) Show that $A_x(z) = X(z)X(z^{-1})$, and determine the region of convergence for $A_x(z)$.

   (b) Let $x_1[n] := \alpha^n u[n]$ where $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ is the Heaviside step sequence.

   i. Determine the region of convergence for $A_{x_1}(z)$.

   ii. Find a_{x_1} by evaluating the inverse z-transform of $A_{x_1}(z)$.

   iii. Specify two other sequences that are not equal to x_1 and have the same deterministic autocorrelation sequence as that of x_1.

5. **Some DFT properties.**
Let $x, y \in \mathbb{C}^N$ be signals and let $X, Y \in \mathbb{C}^N$ be their corresponding DFTs. Prove the following:

(a) DFT pair property for time reversal, that is

$$x[-n \ mod \ N] \quad \overset{DFT}{\longleftrightarrow} \quad X[-k \ mod \ N] \tag{5}$$

(b) DFT pair property for circular convolution in time, that is

$$(x \circledast y)[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[k]Y[k] \tag{6}$$

(c) DFT pair property for circular convolution in frequency

$$x[n]y[n] \quad \overset{DFT}{\longleftrightarrow} \quad \frac{1}{N}(X \circledast Y)[k] \tag{7}$$

(d) If x is a real symmetric signal, namely $x[n] = x[-n \ mod \ N]$, then X is real.

(e) If x is a real antisymmetric signal, $x[n] = -x[-n \ mod \ N]$, then X is imaginary.

6. **Z-transform of Downsampled signals**
Let $y[n] = x[Nn]$. Show that the Z-transform of this downsampling satisfies

$$y[n] = x[Nn] \quad \overset{ZT}{\longleftrightarrow} \quad Y(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(W^k z^{1/N})$$

here $X(z)$ is the Z-transform of x, and W is a primitive N^{th} root of unity, namely $W^N = 1$ and $W^k \neq 1$ for all $0 < k < N$.

Show that the DTFT transform pair given by

$$y[n] = x[Nn] \quad \overset{DTFT}{\longleftrightarrow} \quad Y(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{\omega - 2\pi k}{N})$$

where $X(\omega)$ is the DTFT of x.

7. **Interchange of Multirate Operations and LTI Filtering**.
Consider the system given by the input-output relation

$$y = D_2 A D_2 A D_2 A x \tag{8}$$

where A is a convolution filter.

(a) Use the *multirate noble identities* to find the simplest equivalent system of the form $y = D_N H x$, where D_N is downsampling by N and H is a convolution filter. Specify the downsampling factor N , and write H in the z-transform and Fourier domains.

(b) If A is an ideal half-band lowpass filter, draw the DTFT $H(\omega)$, clearly specifying the cutoff frequencies.

(c) If A is an ideal half-band highpass filter, draw the DTFT $H(\omega)$, clearly specifying the cutoff frequencies.
Is this transfer function capturing the highest frequency content in the sequence x? Explain.

8. **Python Exercise: Two-Channel Delay Recovery**
Implement the algorithm you suggested in problem (1c). Use the code below to generate two samples $x_1, x_2 \in \mathbb{R}^{100}$ of some randomly shifted/scaled waveform based on your UIN.

   Plot the input signals, their crosscorrelation, and estimated values. Comment on the results.

```python
import numpy as np
from numpy.random import randint, randn
from scipy.interpolate import splev, splrep

def gen_wave(UIN,n0 = 0):
    T = np.arange(100)
    Ts = 1+3*np.arange(len(UIN))
    sp = splrep(Ts, UIN, k=3)
    return  splev(T-n0,sp,ext=1)

def estimate_delay_and_gain(x1,x2):
    # This is where you need to get some work done
    return delta, rho

UIN = np.array([ Fill In your UIN here, comma separated ])
n1,n2 = randint(1,40,size = 2)
alpha1,alpha2 = randn(2)
x1, x2= gen_wave(alpha1*UIN, n1), gen_wave(alpha2*UIN, n2)

delta, rho = estimate_delay_and_gain(x1,x2)
# Then compare with n1-n2, alpha1/alpha2
```

9. **Python Exercise: Multirate Systems**

   (a) Write two functions that implement downsampling and upsampling operators:

```python
def downsample(x,N):
    # Do your thing here. x is a 1D NumPy array

def upsample(x,N):
    # Just add water, eggs, butter, and mix...
```

   Use array slicing (`x[::N]`), and avoid loops.

   (b) Generate an FIR lowpass of length 30 and cutoff $\omega_c = \pi/3$ using `scipy.signal.firwin()`. Then, using `scipy.signal.freqz()`, obtain and plot the frequency response of the FIR, as well of its downsampled and upsampled (at rate 2) version. Verify that these plots comply with the theory of spectrum change by downsampling and upsampling.

   (c) Let the signal $x \in \mathbb{R}^{8192}$ be a sum of three sinusoids with frequencies 800_{Hz}, 1600_{Hz} and 2400_{Hz}, and amplitudes 1, 0.2, and 0.4 respectively, sampled at 8192_{Hz}. Let

$$y_1 = D_3 x \qquad\qquad y_2 = D_3 L x$$

   where L is the FIR lowpass that you created in part (9b). Plot the (logarithmic) DTFT of the input, y_1 and y_2, and clearly identify any aliased components.