ECE551 - Homework 7

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1 Truncation as Filter Approximation

(a) Let $\psi = \{\varphi_k\}$ be the basis of $\mathbb{C}^{\mathbb{Z}}$

$$h_d \in \mathbb{C}^{\mathbb{Z}} \Rightarrow h_d = \sum_{\varphi_k \in \psi} \alpha_k \varphi_k$$

Since $I \subset \mathbb{Z}$, $\mathbb{C}^I \subset \mathbb{C}^{\mathbb{Z}}$, where $\mathbb{C}^I = \text{span}\{\phi^I\}, \phi^I \subset \phi$.

$$\begin{aligned} T_I h_d &= \sum_{\varphi_k \in \psi} w[k] \alpha_k \varphi_k \\ &= \sum_{\varphi_k \in \psi^I} 1 \cdot \alpha_k \varphi_k + \sum_{\varphi_k \in \psi / \psi^I} 0 \cdot \alpha_k \varphi_k \\ &= \sum_{\varphi_k \in \psi^I} \alpha_k \varphi_k \in \text{span}\{\psi^I\} = \mathbb{C}^I \\ \Rightarrow T_I h_d - h_d &= \sum_{\varphi_k \in \psi / \psi^I} \alpha_k \varphi_k \end{aligned}$$

$$\Rightarrow \langle T_I h_d - h_d, T_I h_d \rangle = 0 \Rightarrow T_I h_d - h_d \perp T_I h_d$$

By orthogonality principal, $T_I h_d$ is the least square approximation of h_d on $\ell_2(I)$.

(b) Let $h \in \mathbb{C}^{\mathbb{R}}$

$$\begin{aligned}
T_I h &= \sum_{\varphi_k \in \psi_I} 1 \cdot \alpha_k \phi_k + \sum_{\varphi_k \in \psi / \psi^I} 0 \cdot \alpha_k \phi_k \\
&= \sum_{\varphi_k \in \psi_I} \alpha_k \phi_k \\
&= \hat{h} \\
T_I(T_I h) &= \sum_{\varphi_k \in \psi_I} 1 \cdot \alpha_k \phi_k + \sum_{\varphi_k \in \psi^I / \psi^I} 0 \cdot \alpha_k \phi_k \\
&= \sum_{\varphi_k \in \psi_I} \alpha_k \phi_k \\
&= \hat{h} \\
\Rightarrow T_I &= T_I^2
\end{aligned}$$

Let $x, y \in \mathbb{C}^{\mathbb{R}}$

$$\begin{aligned}
\langle T_I x, y \rangle &= \left\langle \sum_{\varphi_k \in \psi_I} 1 \cdot \alpha_k \phi_k + \sum_{\varphi_k \in \psi / \psi^I} 0 \cdot \alpha_k \phi_k, \sum_{\varphi_k \in \psi_I} \beta_k \phi_k + \sum_{\varphi_k \in \psi / \psi^I} \beta_k \phi_k \right\rangle \\
&= \left\langle \sum_{\varphi_k \in \psi_I} \alpha_k \phi_k + \sum_{\varphi_k \in \psi / \psi^I} \alpha_k \phi_k, \sum_{\varphi_k \in \psi_I} 1 \cdot \beta_k \phi_k + \sum_{\varphi_k \in \psi / \psi^I} 0 \cdot \beta_k \phi_k \right\rangle \\
&= \langle x, T_I y \rangle \\
\Rightarrow T_I &= T_I^*
\end{aligned}$$

Hence, T_I is an orthogonal projection.

(c) For $I = \{0, \cdots, 4\}$,

$$T_I h_d = \begin{bmatrix} \cdots & 0 & \text{sinc}0 & \text{sinc}\frac{\pi}{3} & \text{sinc}\frac{2\pi}{3} & \text{sinc}1 & \text{sinc}\frac{4\pi}{3} & 0 & \cdots \end{bmatrix}^\top$$

(d) We can choose I as $\{-2, -1, 0, 1, 2\}$, so $T_I h_d$ is

$$T_I h_d = \begin{bmatrix} \cdots & 0 & -\text{sinc}\frac{2\pi}{3} & -\text{sinc}\frac{\pi}{3} & \text{sinc}0 & \text{sinc}\frac{\pi}{3} & \text{sinc}\frac{2\pi}{3} & 0 & \cdots \end{bmatrix}^\top$$

2 Lagrange Interpolation

(a) We have

$$p_{\tilde{D}}(t) = p_D(t) + c(t - t_0)(t - t_1) \cdots (t - t_{N-1}) = p_D(t) = p_D(t) + c \prod_{j=0}^{N-1} (t - t_j)$$

We want $p_{\tilde{D}}(t_N) = x_N$, so

$$c \prod_{j=0}^{N-1} (t - t_j) = x_N \Leftrightarrow c = \frac{x_N}{\prod_{j=0}^{N-1}(t_N - t_j)}$$

(b) We already have

$$\begin{aligned} p_{\tilde{D}}(t) &= p_D(t) + \frac{x_N}{\prod_{j=0}^{N-1}(t_N - t_j)} \prod_{j=0}^{N-1}(t - t_j) \\ &= p_D(t) + x_N \prod_{j=0}^{N-1} \frac{t - t_j}{t_N - t_j} \\ &= p_{D^{(-1)}}(t) + x_{N-1} \prod_{j=0}^{N-2} \frac{t - t_j}{t_{N-1} - t_j} + x_N \prod_{j=0}^{N-1} \frac{t - t_j}{t_N - t_j} \end{aligned}$$

where $D^{(-1)} = D/\{(t_{N-1} m x_{N-1})\} = \{(t_k, x_k)\}_{k=0}^{N-2}$. Therefore, in general

$$p_{\tilde{D}}(t) = \sum_{k=0}^{N} x_k \prod_{j \neq k} \frac{t - t_j}{t_k - t_j}$$

$$\Rightarrow p_D(t) = \sum_{k=0}^{N-1} x_k \prod_{j \neq k} \frac{t - t_j}{t_k - t_j}$$

3 Polynomial Spaces with Orthogonality

(a) Let $v \in V_n$, then

$$v = \sum_{j=0}^{n} \alpha_j v_j$$

$$\deg(v) = \max\{\deg(v_j)\}_{j=0}^{n} \leq n$$

Therefore v can be written as $\sum_{j=0}^{n} \beta_j t^j$

$$\Rightarrow v \in W_n \Rightarrow V_n \subset W_n$$

We have

$$\dim(V_n) = n \qquad \because \langle v_k, v_j \rangle = \delta[k - j]$$

$$\dim(W_n) = n \qquad \because \{1, t^1, t^2, \cdots t^n\} \text{ are independent}$$

So $\dim(V_n) = \dim(W_n)$. Hence, $v_n = W_n$.

(b) p is a polynomial of degree m, so $p \in V_n = W_n$.

$$p = \sum_{j=0}^{m} \langle p, v_j \rangle v_j$$

For $k > m$,

$$\begin{aligned}
\langle p, v_k \rangle &= \left\langle \sum_{j=0}^{m} \langle p, v_j \rangle v_j, v_k \right\rangle \\
&= \sum_{j=0}^{m} \langle p, v_j \rangle \langle v_j, v_k \rangle \\
&= 0 \qquad \because \langle v_k, v_k \rangle = 0
\end{aligned}$$

(c) $v \in V_n = W_n \Rightarrow v(t) = \sum_{j=0}^{n} \alpha_j t^j$

$$\begin{aligned}
\sum_{j=0}^{n} \alpha_j (t - t_0)^j &= \sum_{j=0}^{n} \alpha_j \left(\binom{j}{i} t^{j-i} (-t_0)^i \right) \\
&= \sum_{j=0}^{n} \alpha_j \binom{j}{i} t^j \frac{(-t_0)^i}{t^i} \\
&= \sum_{j=0}^{n} \left(\alpha_j \binom{j}{i} \frac{(-t_0)^i}{t^i} \right) t^j
\end{aligned}$$

Since $i \leq j$, $\sum_{j=0}^{n} \left(\alpha_j \binom{j}{i} \frac{(-t_0)^i}{t^i} \right) t^j$ is a polynomial of degree up to n. So we can write it as

$$\sum_{j=0}^{n} \alpha_j (t - t_0)^j = \sum_{j=0}^{n} \beta_j t^j$$

Hence, it is shift-invariant.

4 Polynomial Spaces vs. Spline Spaces

(a) Figure 1 shows the graph of $s_0, s_1 \in U$

(b) By inspecting Figure 1, if we increase N, s_0 will approach $p_0(t) = 1$ and s_1 will approach $p_1(t) = t$. Hence, $p_0(t)$ and $p_1(t)$ are contained (as limit) in U.

(c) u_0, u_1 belong to U as well as V_1. However, $V_1 = \text{span}\{v_0, v_1\} \Rightarrow \dim(V_1) = 2$ and $\dim(U) = \infty$. Hence $V_1 \neq U$.

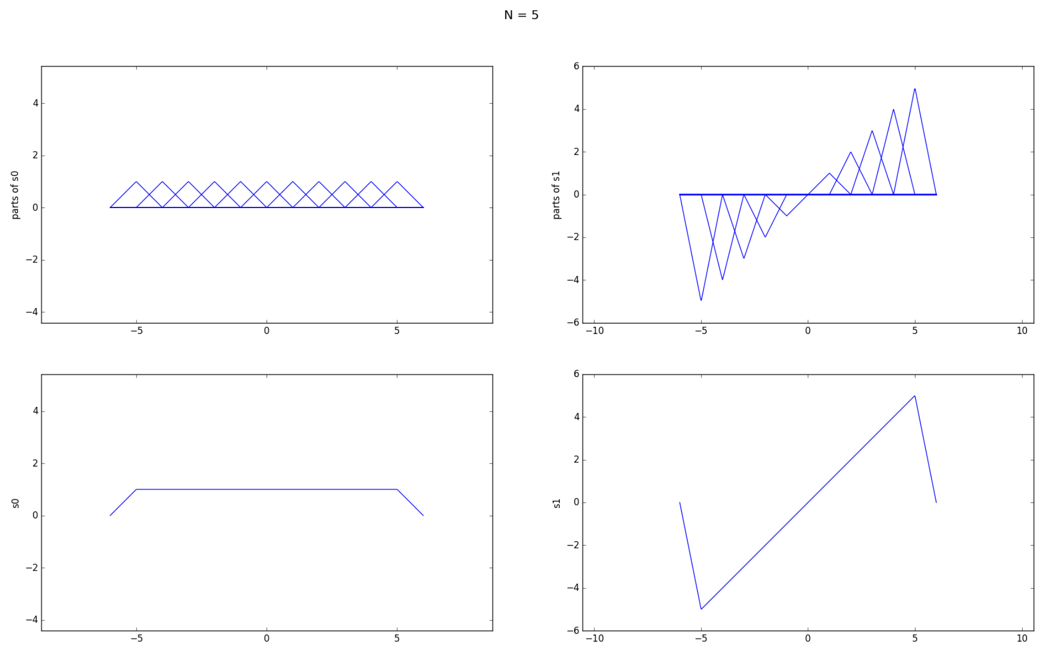

Figure 1: s_0 (left) and s_1 (right) with $N = 5$

5 Interpolation with Shifted Symmetric Functions

(a) We are given the coefficients $\{c[k]\}$, so

$$\begin{aligned} s(t) &= \sum_{k\in\mathbb{Z}} c[k]\phi(t-kT) \\ \Rightarrow s(nT) = x[n] &= \sum_{k\in\mathbb{Z}} c[k]\phi(nT-kT) \\ &= \sum_{k\in\mathbb{Z}} c[k]\phi((n-k)T) \\ &= \sum_{k\in\mathbb{Z}} c[k]b[n-k] \end{aligned}$$

where $b[m] = \phi(mT)$. So

$$x = c * b \Rightarrow X(z) = C(z)B(z) \Rightarrow C(z) = \frac{1}{B(z)}X(z) = H(z)X(z)$$

where $H(z) = \frac{1}{B(z)}$.

To enable this, we need $\phi(nT)$ to be band limited in the range of $[\frac{-\pi}{T}, \frac{\pi}{T}]$ to avoid aliasing.

(b) If $\phi(t) = \phi(-t)$, then

$$\begin{aligned} &\phi(jT) = \phi(-jT) \\ \Leftrightarrow\ & b[j] = b[-j] \end{aligned}$$

If λ is a pole/root of $H(z)$ then $B(\lambda) = 0$. We have

$$\begin{aligned} B(z^{-1}) &= \sum_{n=-N}^{N} b[n] z^{n} \\ &= \sum_{n=-N}^{N} b[-n] z^{n} \\ &= \sum_{m=-N}^{N} b[m] z^{-m} \qquad \because m = -n \\ &= B(z) \end{aligned}$$

Therefore λ^{-1} is also a root of $H(z)$.

(c) Assume that λ_j is a pole of $H(z)$, then $z = \lambda_j \Rightarrow 1 - \lambda_j z^{-1} = 0$. Since λ_j^{-1} is also a pole, $z = \lambda_j^{-1} \Rightarrow 1 - \lambda_j z = 0$. Therefore, we can write $H(z)$ as

$$H(z) = \frac{1}{\prod_j (1 - \lambda_j z^{-1})} \cdot \frac{1}{\prod_j (1 - \lambda_j z)}$$

Let $G(z) = \frac{1}{\prod_j (1-\lambda_j z^{-1})}$ (causal), then $G(z^{-1}) = \frac{1}{\prod_j (1-\lambda_j z)}$. Hence,

$$H(z) = G(z) G(z^{-1}), \qquad \text{with } G(z) = \frac{1}{\prod_j (1 - \lambda_j z^{-1})}$$

(d) We can see that

$$\begin{aligned} &(1 - \lambda_1 z^{-1}) = 1 - \lambda_1 z^{-1} \\ &(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) \\ &\qquad\qquad = 1 - (\lambda_1 + \lambda_2) z^{-1} + \lambda_1 \lambda_2 z^{-2} \\ &(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1})(1 - \lambda_3 z^{-1}) \\ &\qquad\qquad = (1 - (\lambda_1 + \lambda_2) z^{-1} + \lambda_1 \lambda_2 z^{-2})(1 - \lambda_3 z^{-1}) \\ &\qquad\qquad = 1 - (\lambda_1 + \lambda_2 + \lambda_3) z^{-1} + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) z^{-2} - \lambda_1\lambda_2\lambda_3 z^{-3} \\ &\qquad\quad \cdots \end{aligned}$$

Therefore, in general

$$\prod_{j=1}^{M}(1-\lambda_j z^{-1}) = \sum_{j=0}^{M}\xi_j z^{-j}$$

where $\xi_j = (-1)^j\zeta_j$ and ζ_j is the sum of products of j elements from the set $\{\lambda_1, \cdots, \lambda_M\}$ (Vieta's formula), i.e.

$$\begin{cases}\zeta_1 = \lambda_1 + \lambda_2 + ...\lambda_M \\ \zeta_2 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + ... + \lambda_1\lambda_M) + (\lambda_2\lambda_3 + \lambda_2\lambda_4 + ... + \lambda_2\lambda_M) + ... + \lambda_{M-1}\lambda_M \\ \vdots \\ \zeta_M = \lambda_1\lambda_2...\lambda_M\end{cases}$$

Since $H(z) = G(z)G(z^{-1})$

$$\begin{aligned}Y(z) &= X(z)H(z) \\ &= X(z)G(z)G(z^{-1})\end{aligned}$$

Let $V(z) = X(z)G(z)$, we can sketch a diagram of the system as in Figure 2

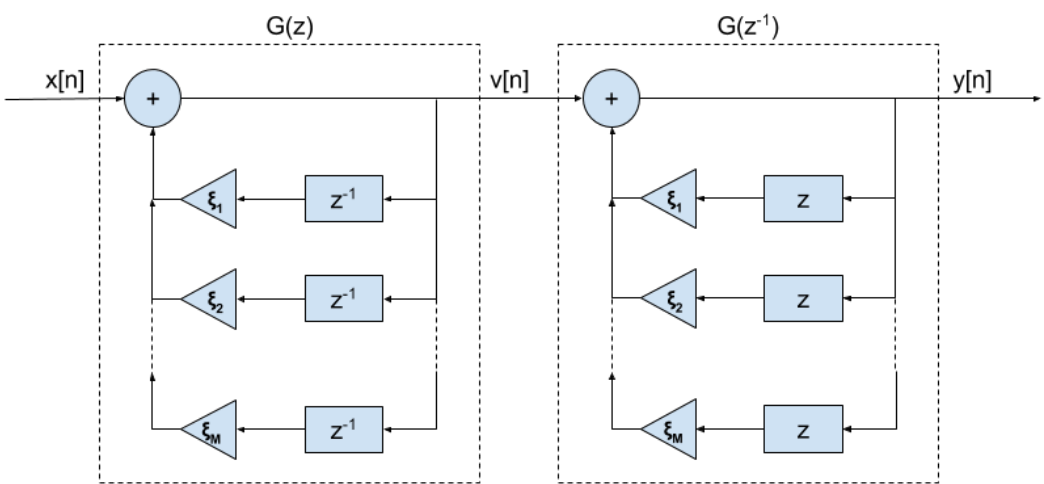

Figure 2: $H(z)$ as a cascade of causal $G(z)$ and anti-causal $G(z^{-1})$

6 Python: Interpolation Games

For ϕ_2, we consider the equation $1/(z/8+3z/4+z^{-1}/8)$:

$$\begin{aligned}
\frac{1}{\frac{1}{8}z+\frac{3}{4}+\frac{1}{8}z^{-1}} &= \frac{8}{z+6+z^-1} \\
&= \frac{8}{z^{-1}(z^2+6z+1)} \\
&= \frac{8}{z^{-1}(z+3+2\sqrt{2})(z+3-2\sqrt{2})} \\
&= \frac{8}{(1+(3+2\sqrt{2})z^{-1})(z+3-2\sqrt{2})} \\
&= \frac{\frac{8}{3+2\sqrt{2}}}{(z^{-1}+3-2\sqrt{2})(z+3-2\sqrt{2})} \\
&= \frac{\sqrt{\frac{8}{3+2\sqrt{2}}}}{z^{-1}+3-2\sqrt{2}} \cdot \frac{\sqrt{\frac{8}{3+2\sqrt{2}}}}{z+3-2\sqrt{2}}
\end{aligned}$$

Therefore, we can choose $\mu=\sqrt{\frac{8}{3+2\sqrt{2}}}$ and $\gamma=2\sqrt{2}-3$.

Figure 3 shows the interpolation results on the UIN and Figure 3 shows the results on 5 points inputted by mouse. The results of ϕ_0, ϕ_1, and ϕ_3 are the same for the second figure.

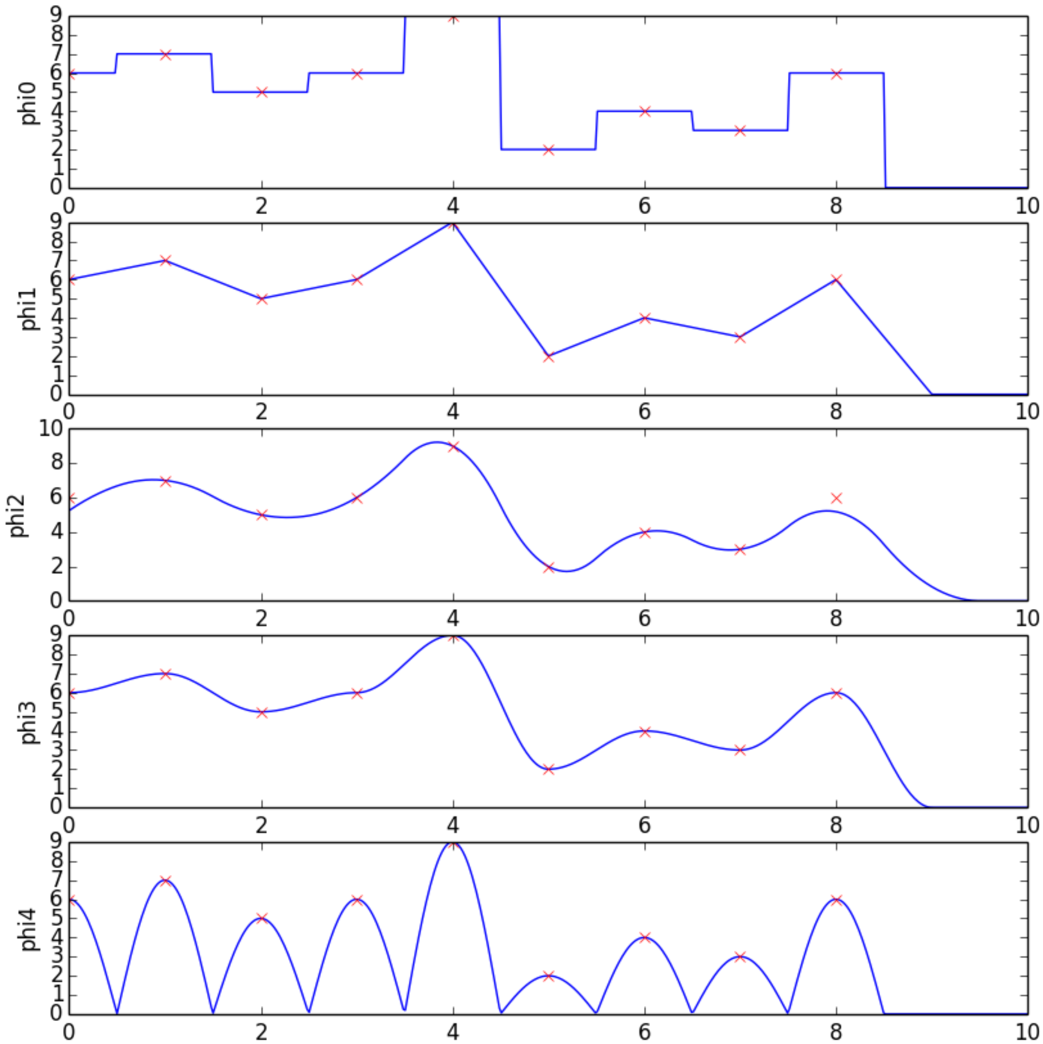

Figure 3: Interpolation results on the UIN

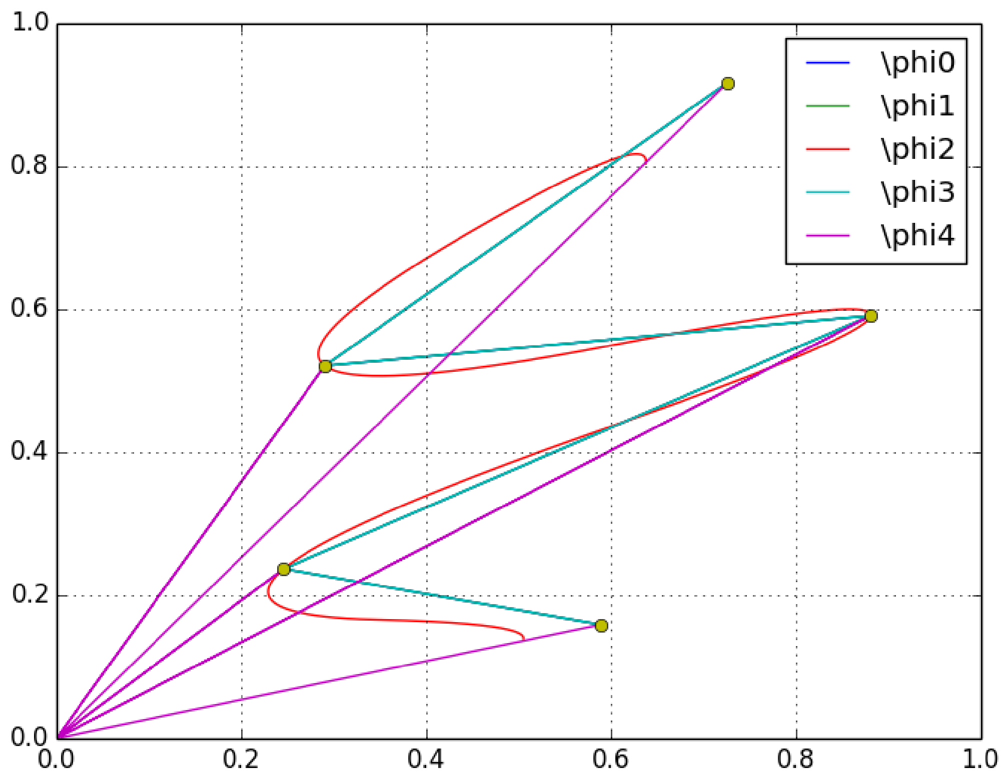

Figure 4: Interpolation results on 5 inputted points