

ECE551 - Homework 3-4

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1 Deterministic Correlation Sequences

(a) It is obvious that σ is a delay operator, i.e. $\sigma = T_1$

$$\begin{aligned}\sigma^k x &= (\sigma(\sigma(\dots(\sigma x)))) \quad (k \text{ times of } \sigma) \\ \Rightarrow \sigma^k x &= T_k x \\ \Rightarrow (\sigma^k x)[n] &= x[n - k].\end{aligned}$$

Similarly, $\sigma^{-1}x = T_{-1}x \Rightarrow (\sigma^{-1}x)[n] = x[n + 1]$.

(b) Prove or disprove:

i.

$$\begin{aligned}a_x[-k]^* &= \langle x, \sigma^k x \rangle^* = \langle \sigma^k x, x \rangle = \sum_{n \in \mathbb{Z}} x[n]^* x[n - k] \\ &= \sum_{n \in \mathbb{Z}} x[n + k]^* x[n - k + k] = \sum_{n \in \mathbb{Z}} x[n + k]^* x[n] = a_x[k]\end{aligned}$$

Hence, $a_x[k] = a_x[-k]^*$.

ii. We have $a_x[0] = \langle x, x \rangle = \|x\|^2$. By Cauchy-Schwarz inequality:

$$|a_x[k]| = \langle x, \sigma^{-k} x \rangle \leq \|x\| \|\sigma^{-k} x\|.$$

Since delay does not change the norm, $\|\sigma^{-k} x\| = \|x\|$. Therefore:

$$|a_x[k]| \leq \|x\|^2 = a_x[0].$$

iii.

$$\begin{aligned}c_{y,x}[-n]^* &= \left(\sum_{i \in \mathbb{Z}} y[i] x[i + n]^* \right)^* = \sum_{i \in \mathbb{Z}} x[i + n] y[i]^* \\ &= \sum_{i \in \mathbb{Z}} x[i + n - n] y[i - n]^* = \sum_{i \in \mathbb{Z}} x[i] y[i - n]^* = c_{x,y}[n]\end{aligned}$$

Hence, $c_{x,y}[n] = c_{y,x}[-n]^*$.

iv.

$$\begin{aligned} c_{x,y}[-n]^* &= \left(\sum_{i \in \mathbb{Z}} x[i]y[i+n]^* \right)^* = \sum_{i \in \mathbb{Z}} x[i]^*y[i+n] \\ &= \sum_{i \in \mathbb{Z}} x[i-n]^*y[i] = c_{y,x}[n] \neq c_{x,y}[n]. \end{aligned}$$

Hence, $c_{x,y}[n] \neq c_{x,y}[-n]^*$.

v.

(c) i. We have

$$\begin{aligned} c_{x_1,x_2}[k] &= \sum_{n \in \mathbb{Z}} \alpha_1 x[n-n_1] \alpha_2 x[n-n_2-k] \\ &= \alpha_1 \alpha_2 \sum_{n \in \mathbb{Z}} x[n-n_1] x[n-n_2-k]. \end{aligned}$$

Let $m = n - n_1 \Rightarrow n - n_2 - k = m + n_1 - n_2 - k = m - \Delta - k$. Then

$$c_{x_1,x_2}[k] = \alpha_1 \alpha_2 \sum_{n \in \mathbb{Z}} x[m] x[m - \Delta - k] = \alpha_1 \alpha_2 a_x[-\Delta - k].$$

We know that $|a_x[k]|$ is maximized at $k = 0$, therefore $a_x[-\Delta - k]$ is maximized when $-\Delta - k = 0 \Leftrightarrow \Delta = -k$. To determine the time delay Δ , we change the value of k until the crosscorrelation between x_1 and x_2 is maximized; then the value of Δ is $-k$. After we have Δ we delay one of the 2 signals x_1 or x_2 and divide them to get $\rho = \frac{\alpha_1}{\alpha_2}$.

ii. We can shift x_1 and x_2 by the same amount and still get the same result for Δ . Therefore we cannot find n_1 and n_2 explicitly. The same thing apply for scaling α_1 and α_2 .

2 Studying yet another Linear System

(a) The system is the linear combination of three states of x (i.e. $n-1, n, n+1$.) Therefore it is linear.

$x[n-1-k] + x[n+1-k] - 2x[n-k] = (Lx)[n-k]$. Therefore the system is shift invariant.

The system is defined from both previous and future state of x . Therefore it is not causal.

The system depends on the previous state of x , i.e. $x[n-1]$. Therefore it is not memoryless.

The impulse response of $x[n-1] + x[n+1] - 2x$ is 3δ (each has impulse response of δ). Since δ is BIBO stable, the system is BIBO stable.

- (b) The sketches of (x_1, Lx_1) , (x_2, Lx_2) , and (x_3, Lx_3) are showed in Figure 1, 2, and 3, respectively.

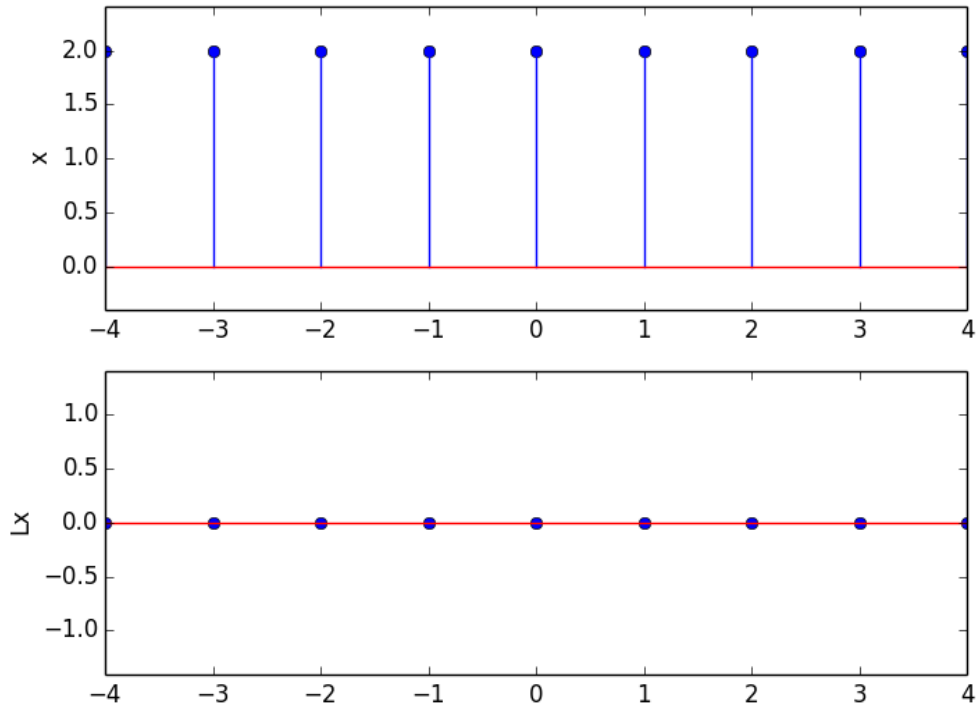


Figure 1: x_1 and Lx_1 , where $x_1[n] = c, \forall n \in \mathbb{Z}$. Here $c = 2$, but choice of c does not affect Lx_1 .

3 DTFT Affairs

- (a)
- (b) i. We consider the low pass filter system:

$$G(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{else} \end{cases}$$

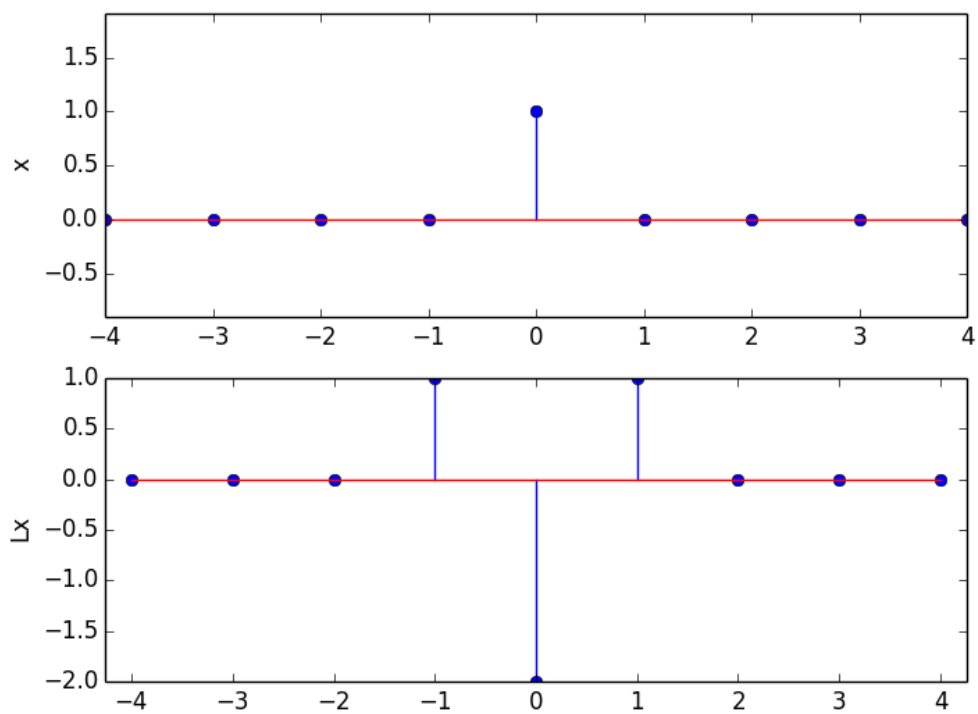


Figure 2: x_2 and Lx_2 , where $x_2[n] = \delta[n]$.

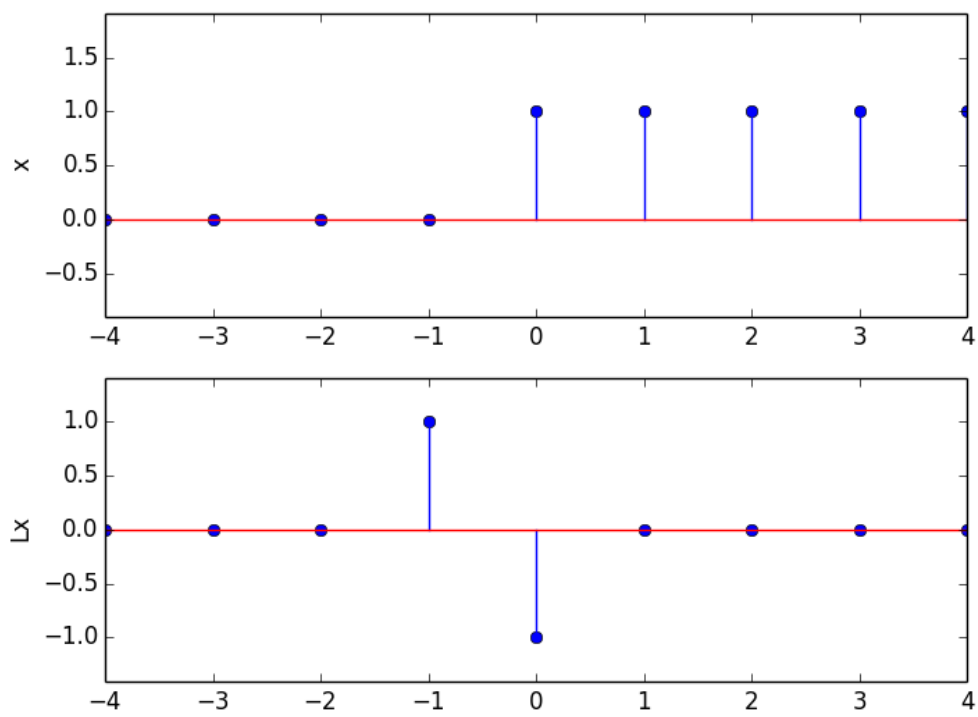


Figure 3: x_3 and Lx_3 , where $x_3[n] = u[n]$.

Then

$$\begin{aligned}
g[n] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi j n} e^{j\omega n} \Big|_{-\omega_0}^{\omega_0} \\
&= \frac{1}{2\pi j n} (e^{j\omega_0 n} - e^{-j\omega_0 n}) \\
&= \frac{1}{2\pi j n} 2j \sin(\omega_0 n) \\
&= \begin{cases} \frac{1}{\pi n} \sin(\omega_0 n), & n \neq 0 \\ \frac{\omega_0}{\pi}, & n = 0 \end{cases}
\end{aligned}$$

Therefore, $h[n] = \sqrt{3} \frac{\sin(\frac{1}{3}\pi n)}{\pi n} = \sqrt{3}g[n]$, where $\omega_0 = \frac{\pi}{3}$. Hence, it is a low pass filter, whose DTFT is

$$H(\omega) = \begin{cases} \sqrt{3}, & |\omega| < \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

ii. We have $x[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$,

$$\Rightarrow X(\omega)$$

Since $y = h * x$, the DTFT of y is

$$Y(\omega) = H(\omega)X(\omega) = \begin{cases} \frac{\sqrt{3}}{2}(1 + e^{-j\omega}), & |\omega| < \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

For $|\omega| < \frac{\pi}{3}$,

$$\begin{aligned}
Y(\omega) &= \frac{\sqrt{3}}{2}(1 + e^{-j\omega}) \\
&= \frac{\sqrt{3}}{2}e^{-j\omega/2}e^{j\omega/2}(1 + e^{-j\omega}) \\
&= \frac{\sqrt{3}e^{-j\omega/2}}{2}(e^{j\omega/2} + e^{-j\omega/2}) \\
&= \frac{\sqrt{3}e^{-j\omega/2}}{2} \cos\left(\frac{\omega}{2}\right)
\end{aligned}$$

4 The Z -Transform of Autocorrelation

(a) We have

$$\begin{aligned}
 A_x(z) &= \sum_{n \in \mathbb{Z}} a_x[n] z^{-n} \\
 &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[k] x[k+n] z^{-n} \\
 &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[k] x[k+n] z^{-n} z^k z^{-k} \\
 &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[k] z^k x[k+n] z^{-(n+k)} \\
 &= \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} x[k] z^k x[m] z^{-m} \\
 &= \sum_{k \in \mathbb{Z}} x[k] z^k \sum_{m \in \mathbb{Z}} x[m] z^{-m} \\
 &= X(z) X(z^{-1})
 \end{aligned}$$

Consider $X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n} = \sum_{n \in \mathbb{Z}} (x[n]^{1/n} z^{-1})^n = \frac{1}{1 - x[n]^{1/n} z^{-1}}$. Then the $ROC_{X(z)}$ is

$$|x[n]^{1/n} z^{-1}| < 1 \Leftrightarrow |z| > |x[n]^{1/n}|.$$

Similarly, for $X(-z) = \frac{1}{1 - x[n]^{1/n} z}$, the $ROC_{X(-z)}$ is

$$|x[n]^{1/n} z| < 1 \Leftrightarrow \left| \frac{1}{z} \right| > |x[n]^{1/n}| \Leftrightarrow |z| < |x[n]^{-1/n}|.$$

Hence, the ROC_A is

$$\left\{ |x[n]^{1/n}| < |z| < |x[n]^{-1/n}| \right\}.$$

(b) i. $x_1[n] = \alpha^n u[n]$, therefore $ROC_{A_{x_1}}$ is

$$\left\{ |(\alpha^n)^{1/n}| < |z| < |(\alpha^n)^{-1/n}| \right\} = \left\{ |\alpha| < |z| < \left| \frac{1}{\alpha} \right| \right\}$$

ii. The z -transform of x_1 is

$$\begin{aligned}
X_1(z) &= \sum_{n \in \mathbb{Z}} x_1[n] z^{-n} \\
&= \sum_{n \in \mathbb{Z}} \alpha^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=0}^{\infty} (\alpha z^{-1})^n \\
&= \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow A_{x_1}(z) &= X_1(z) X_1(z^{-1}) \\
&= \frac{z}{z - \alpha} \frac{z^{-1}}{z^{-1} - \alpha} \\
&= \frac{1}{(z - \alpha)(z^{-1} - \alpha)} \\
&= z \frac{1}{(z - \alpha)(1 - \alpha z)}
\end{aligned}$$

Let $\frac{1}{(z-\alpha)(1-\alpha z)} = \frac{A}{z-\alpha} + \frac{B}{1-\alpha z} = \frac{A(1-\alpha z) + B(z-\alpha)}{(z-\alpha)(1-\alpha z)}$, then:

$$\begin{aligned}
A(1 - \alpha z) + B(z - \alpha) &= 1 \Rightarrow (-A\alpha + B)z + A + \alpha B = 1 \\
\Rightarrow \begin{cases} -A\alpha + B = 0 \\ A + \alpha B = 1 \end{cases} \\
\Rightarrow \begin{cases} A = \frac{1}{\alpha^2 + 1} \\ B = \frac{\alpha}{\alpha^2 + 1} \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_{x_1}(z) &= z \left(\frac{1}{(z - \alpha)(\alpha^2 + 1)} + \frac{1}{(1 - \alpha z)(\alpha^2 + 1)} \right) \\
&= z \left(\frac{1}{z(\alpha^2 + 1) - \alpha^3 - \alpha} + \frac{1}{z(-\alpha^3 - \alpha) + \alpha^2 + 1} \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
a_{x_1}[n] &= \frac{1}{2\pi j} \int_{ROC_{A_{x_1}}} A_{x_1}(z) z^{-1} dz \\
&= \frac{1}{2\pi j} \int_{ROC_{A_{x_1}}} \left(\frac{1}{z(\alpha^2 + 1) - \alpha^3 - \alpha} + \frac{1}{z(-\alpha^3 - \alpha) + \alpha^2 + 1} \right) dz \\
&= \frac{1}{2\pi j} \left(\frac{\log(\alpha - z)}{\alpha^2 + 1} + \frac{\log(-(\alpha^2 + 1)(\alpha z - 1))}{\alpha^3 + \alpha} \right) \Big|_{\alpha}^{1/\alpha}
\end{aligned}$$

- iii. Two other sequences that are not equal to x_1 and have the same deterministic autocorrelation sequence as that of x_1 are its time shifted versions (with different delays).

5 Some DFT Properties

- (a) Define $k \bmod N$ as $\langle k \rangle_N$, i.e.

$$\langle k + N \rangle_N = \langle k \rangle_N.$$

The DFT of $x[n]$ is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-kn}$$

where

$$W_N = e^{j\frac{2\pi}{N}} = \cos(2\pi/N) + j \sin(2\pi/N).$$

Assume that $k = lN + r \Leftrightarrow \langle k \rangle_N = r$, then

$$\begin{aligned}
W_N^k &= \exp(j\frac{2\pi}{N}(lN + r)) \\
&= \exp(j\frac{2\pi}{N}lN) \exp(j\frac{2\pi}{N}r) \\
&= 1 \exp(j\frac{2\pi}{N}r) \\
&= W_N^r = W_N^{\langle k \rangle_N}.
\end{aligned}$$

Similarly, $W_N^{mk} = W_N^{m\langle k \rangle_N}$.

Furthermore,

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{-n(lN+r)} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{-nlN} W_N^{-nr} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{-nr} \\
&= X[r] = X[\langle k \rangle_N].
\end{aligned}$$

Therefore,

$$\begin{aligned}
DFT(x[\langle -n \rangle_N]) &= \sum_{n=0}^{N-1} x[\langle -n \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{-\langle -m \rangle_N k} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{mk} \\
&= X[-k] \\
&= X[\langle -k \rangle_N].
\end{aligned}$$

(b) The circular convolution between x and y can be defined as

$$(x \circledast y)[n] = \sum_{m=0}^{N-1} x[m] y[\langle n - m \rangle_N].$$

Therefore

$$\begin{aligned}
DFT((x \otimes y)[n]) &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m] y[\langle n - m \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} y[\langle n - m \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{-mk} Y[k] \\
&= Y[k] \sum_{m=0}^{N-1} x[m] W_N^{-mk} \\
&= Y[k] X[k]
\end{aligned}$$

(c) The inverse DFT is defined as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{kn}$$

Therefore

$$\begin{aligned}
IDFT((X \otimes Y)[n]) &= \frac{1}{N} \sum_{k=0}^{N-1} (X \otimes Y)[k] W_N^{kn} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} (X[m] Y[\langle k - m \rangle_N]) W_N^{kn} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} (X[m] Y[\langle k - m \rangle_N]) W_N^{mn} W_N^{(k-m)n} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_N^{mn} \sum_{k=0}^{N-1} Y[\langle k - m \rangle_N] W_N^{(k-m)n} \\
&= x[n] y[n].
\end{aligned}$$

(d) If $x[n]$ is real,

$$\begin{aligned}
&\Rightarrow x[n] = x^*[n] \\
&\Rightarrow DFT(x[n]) = DFT(x^*[n]) \\
&\Rightarrow X[k] = X^*[-k]_N.
\end{aligned}$$

If $x[n]$ is also symmetric,

$$\begin{aligned}\Rightarrow x[n] &= x[\langle -n \rangle_N] \\ \Rightarrow DFT(x[n]) &= DFT(x[\langle -n \rangle_N]) \\ \Rightarrow X[k] &= X[\langle -k \rangle_N].\end{aligned}$$

Therefore

$$X[k] = X^*[\langle -k \rangle_N] = X[\langle -k \rangle_N].$$

Hence, $X[k]$ is real (and also symmetric.)

(e) If $x[n]$ is symmetric (and real),

$$\begin{aligned}\Rightarrow x[n] &= -x[\langle -n \rangle_N] \\ \Rightarrow DFT(x[n]) &= -DFT(x[\langle -n \rangle_N]) \\ \Rightarrow X[k] &= -X[\langle -k \rangle_N].\end{aligned}$$

Therefore

$$X[k] = X^*[\langle -k \rangle_N] = -X[\langle -k \rangle_N].$$

Hence, $X[k]$ is imaginary.

6 Z-Transform of Downsampled Signals

Let

$$p_N[n] = \begin{cases} 1, & n = mN \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow P_N(z) = \sum_{n \in \mathbb{Z}} p_N[n] z^{-n} = \sum_{n \in \mathbb{Z}} z^{-Nn} = \sum_{n \in \mathbb{Z}} W^{-n}$$

We have

$$\frac{1}{N} \sum_{k=0}^{N-1} W^{nk} = \begin{cases} \frac{1}{N} \times N = 1, & n = mN \\ 0, & \text{else} \end{cases} = p_N[n]$$

Therefore

$$\begin{aligned}
Y(z) &= \sum_{n \in \mathbb{Z}} y[n] z^{-n} \\
&= \sum_{n \in \mathbb{Z}} x[Nn] z^{-n} \\
&= \sum_{k=Nd, n \in \mathbb{Z}} x[k] z^{-k/N} \\
&= \sum_{k \in \mathbb{Z}} x[k] z^{-k/N} p_N[k] \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k \in \mathbb{Z}} W^{mk} z^{-k/N} x[k] \\
&= \frac{1}{N} \sum_{m=0}^{N-1} X(W^m z^{1/N})
\end{aligned}$$

Replacing z with $e^{j\omega}$, we have

7 Interchange of Multirate Operations and LTI Filtering

(a) We have

$$\begin{aligned}
y &= D_2 A D_2 A D_2 A x \\
&= D_2 (A D_2) (A D_2) A x \\
&= D_2 (D_2 A(z^2)) (D_2 A(z^2)) A x \\
&= D_2 D_2 (A(z^2) D_2) A(z^2) A x \\
&= D_2 D_2 D_2 A(z^4) A(z^2) A x \\
&= D_8 A(z^4) A(z^2) A(z) x
\end{aligned}$$

Hence, the downsampling factor $N = 8$ and $H = A(z^4)A(z^2)A(z)$.

- (b) Figure 4 shows the combination $H(\omega)$ if A is an ideal half-band lowpass filter. The cut-off frequency is $\pm\pi/8$.
- (c) Figure 5 shows the combination $H(\omega)$ if A is an ideal half-band highpass filter. The cut-off frequency is $\pm\pi/2$. The transfer function captures the highest frequency because lower ones are removed by $A(z)$.

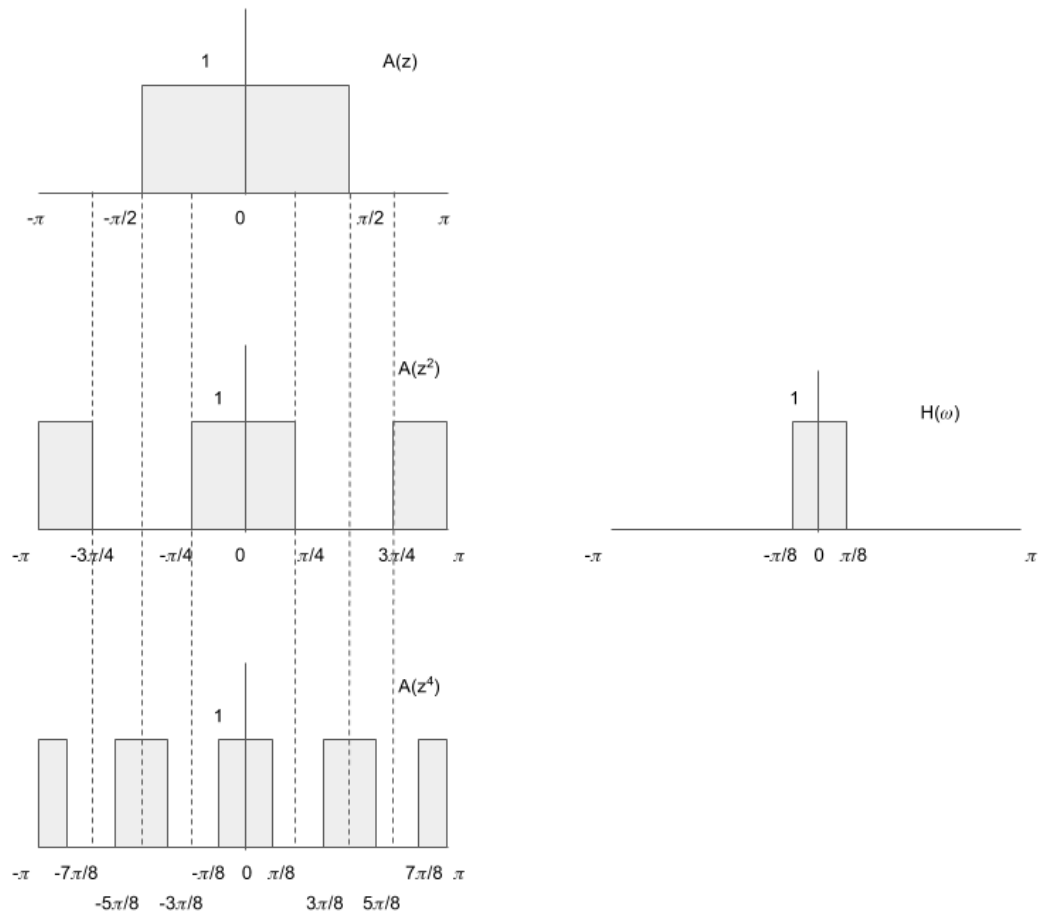


Figure 4: Sketch of the low pass filters and their combination.

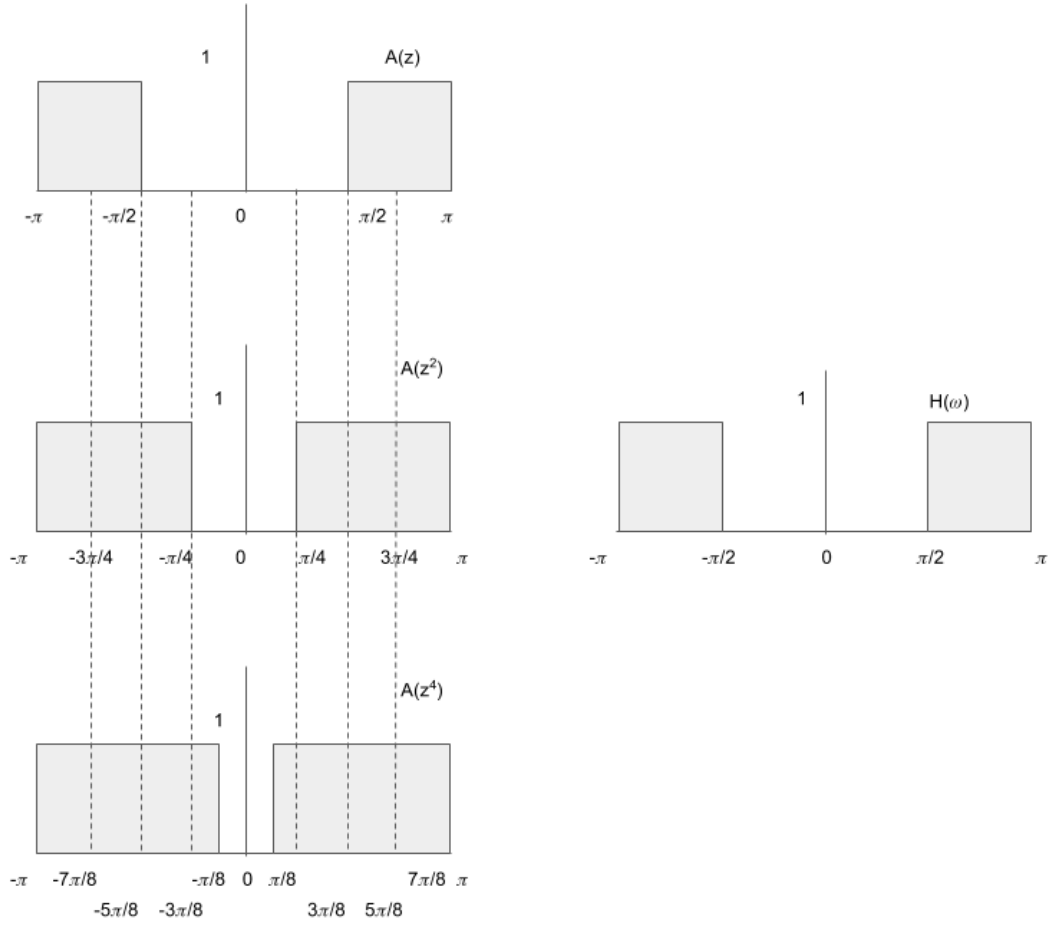


Figure 5: Sketch of the high pass filters and their combination.