Calculus Note - Sequences and Series

Goals to learn Series

We'll introduce several methods (tests) to solve the following problems,

1. Determine whether a series is convergent or divergent
2. Estimate the sum of a series if convergent

Basic Concepts

Sequences

A sequence is a infinite list of number: $a_1, a_2, a_3, \ldots$ also denoted by $\{a_n\}_{n=1}^{\infty}$

Definition A sequence $\{a_n\}$ as the limit L and $\lim_{n \to \infty} a_n = L$, we say the sequnce **converges**. Otherwise, we say the sequence **diverges**.

The Squeeze theorem If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = L$, then $\lim_{n \to \infty} b_n = L$.

Theorem If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all n > 1. It is called **decreasing** if $a_n > a_{n+1}$ for all n > 1. A sequence is **monotonic** if it is either increasing or decreasing.

Monotonic Sequence theorem Every bounded, monotonic sequence is convergent.

Series

Definition When we add the terms of an infinite sequence, we get a series.

Definition A partial sum $s_n = \sum_{i=1}^{n} a_i$ is the first n terms of the series.

Definition A series is called convergent if its partial sum has a limit of real number $\lim_{n \to \infty} s_n = s$. Otherwise it's divergent.

Theorem If the series $\sum a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Test for Divergence if $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} \neq 0$, then the series $\sum a_n$ is divergent.

Example

1. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$ and its sum is $\frac{a}{1-r}$. If $|r| \geq 1$ the series is divergent.
2. The p-series $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Integral Test

Convergence

For some series, we can extract the sum by getting the limit of it's partial sum, or canceling most of the terms in the series. However more often, it isn't easy to find a simple formula for partial sum, thus difficult to exact sum of a series.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is finite.

Example

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent since $\int_1^{\infty} \frac{1}{x^2} dx$ is finite.
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent since $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is infinite.
3. A general rule for 1 & 2, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **is convergent if $p > 1$ and divergent if $p \leq 1$.**
4. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent since $\int_1^{\infty} \frac{\ln x}{x} = \vert_1^{\infty} \frac{(\ln x)^2}{2} = \infty$

Estimating the Sum

When estimating the sum of a series with its partial sum of first n elements, we want to know how close the estimation is.

We introduce the *remainder* as the error of estimation, which is defined by $R_n = s - s_n = \sum_{i=n+1}^{\infty} a_i$. The boundaries of the remainder,

$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$
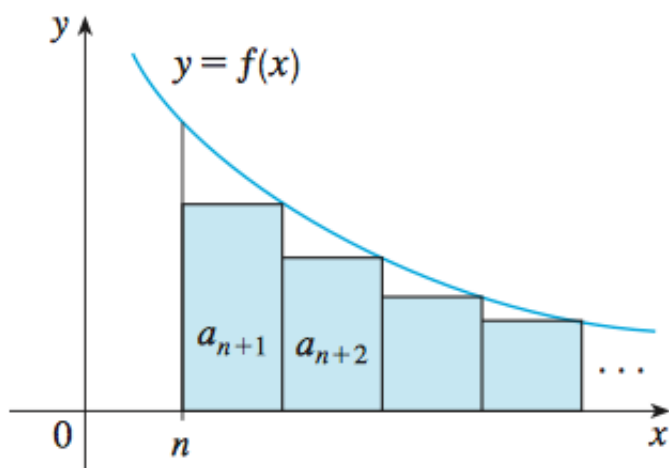
FIGURE 3
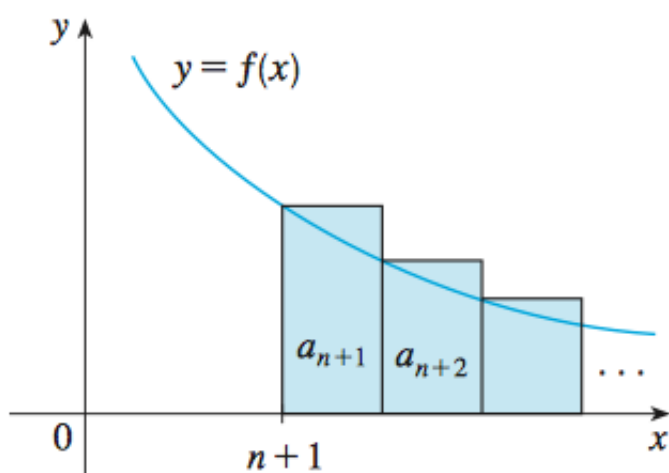
FIGURE 4

Example

Approximate the same of series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error. How many terms are required to ensure the sum is accurate to within 0.0005?

$$\int_{11}^{\infty} \frac{1}{x^3} dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^3} dx$$

$$\frac{1}{2\times11^2} \le R_{10} \le \frac{1}{2\times10^2}$$

The error is within 0.005. To ensure the error is within 0.0005, we need $\frac{1}{2\times x^2} \le 0.0005$, which means $x \ge 32$.

Comparison Tests

Convergence

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

> We only need to verify $a_n \leq b_n$ or $a_n \geq b_n$ for $n \geq N$ where N is some fixed number, because the convergence is not affected by a finite number of terms.

Example

1. $\sum \frac{5}{2n^2+4n+3} \leq \sum \frac{5}{2} \frac{1}{n^2}$ for all n≥1 thus is convergent
2. $\sum \frac{\ln k}{k} > \sum \frac{1}{k}$ for k≥3 thus is divergent

The Limit Comparison Test Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example

1. $a_n = \sum \frac{1}{2^n-1}, b_n = \sum \frac{1}{2^n}, \lim_{n \to \infty} \frac{a_n}{b_n} = 1$, because b_n is a convergent geometric series, a_n is convergent
2. $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}, b = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}, \lim_{n \to \infty} \frac{a_n}{b_n} = 1$ because b_n is a divergent p-series, a_n is convergent

Estimating the Sum

Example

Use sum of first 100 terms the estimate the sum of series $\sum \frac{1}{n^3+1}$, estimate the error.

$R_n = R_{100} \leq \int_{100}^{\infty} \frac{1}{x^3+1} < \int_{100}^{\infty} \frac{1}{x^3} = 0.00005$

Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. For example
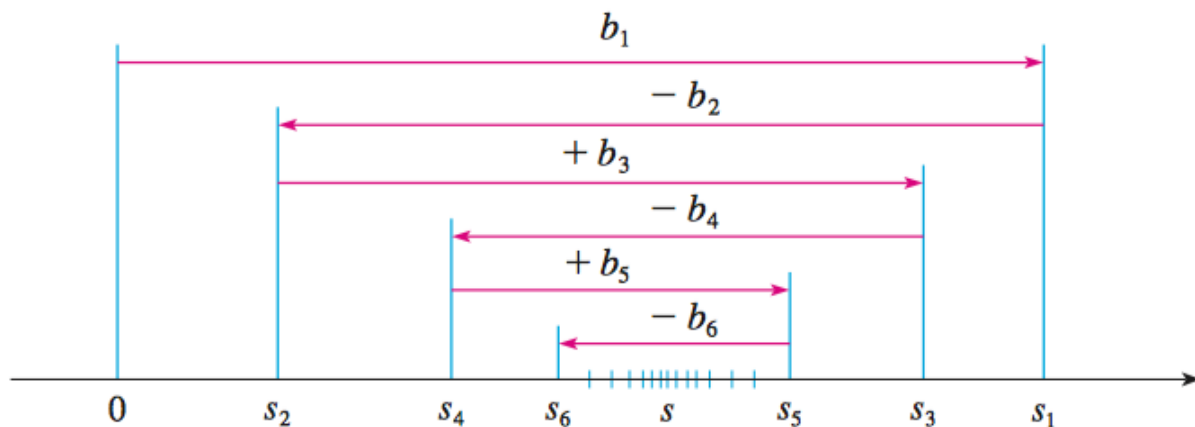
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum (-1)^{n+1} \frac{1}{n}$

Convergence

Alternating Series Test If the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n, b_n > 0$ satisfies

- $b_{n+1} \leq b_n$ for all n
- $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Example

1. $\sum (-1)^{n+1} \frac{1}{n}$ satisfies both conditions thus is convergent
2. $\sum \frac{(-1)^n 3n}{4n-1}$ is alternating but the limit of the term is $\frac{3}{4}$ thus is divergent
3. $\sum (-1)^{n+1} \frac{n^2}{n^3+1}$, to verify condition 1, take the derivative of the term we have
   $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x > 2$, it's easy to verify condition 2 is satisfied, so the series is
   convergent

Estimating Sums

Alternating Series Estimation Theorem For a convergent alternating series, the remainder

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example

Find the sum of the series $\sum \frac{(-1)^n}{n!}$ correct to three decimal places.

From $|R_n| \leq \frac{1}{(n+1)!} < 0.001$ we have $n + 1 \geq 7$ so s_6 is correct to three decimal places.

Absolute Convergence and The Ratio and Root Tests

Definition A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent, and **conditionally convergent** if it's convergent but not absolutely convergent.

Theorem If a series is absolutely convergent $\Rightarrow$ it is convergent.

For example, $\sum \frac{\cos n}{n^2} \leq \sum |\frac{\cos n}{n^2}| \leq \sum \frac{1}{n^2}$ thus is convergent.

Convergence

The Ratio Test If $\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = L$, then $\sum a_n$ is aboslutely convergent if $L < 1$, divergent if $L > 1$, no conclusion if $L = 1$. The idea is to compare $\sum a_n$ with a geometric series.

The Root Test If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$, then $\sum a_n$ is aboslutely convergent if $L < 1$, divergent if $L > 1$, no conclusion if $L = 1$. The idea is to compare $\sum a_n$ with a geometric series.

Example

1. $\sum (-1)^n \frac{n^3}{3^n}$ is absolutely convergent since $|\frac{a_{n+1}}{a_n}| = \frac{1}{3}(1+1/n)^3 \to \frac{1}{3} < 1$
2. $\sum \frac{n^n}{n!}$ is divergent since $|\frac{a_{n+1}}{a_n}| = (1+1/n)^n \to e > 1$
3. For $\sum (\frac{2n+3}{3n+2})^n$, $\sqrt[n]{|a_n|} \to 2/3 < 1$, thus is absolutely convergent

Strategy for Testing Series

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \le 1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \ge 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered.
4. If you can see at a glance that $\lim_{n \to \infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

> See P746 in `Calculus 8th edition` for some exercises.

Power Series

A **power series** is a series of the form $\sum_{n=0}^\infty c_n x^n$ where the c_n are constants called the **coefficients** of the series. A power series may converge for some values of x and diverge for other values of x.

More generally, $\sum_{n=0}^\infty c_n (x-a)^n$ is called a **power series in (x-a) or a power series centered at a**.

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the convergence.

Example

Example 1 For what values of x is the series $\sum n! x^n$ convergent?

We use the Ratio Test. $|\frac{a_{n+1}}{a_n}| = (n+1)|x|$, so the series converge only if $x = 0$.

Example 2 For what values of x does the series $\sum \frac{(x-3)^n}{n}$ converge?

$|\frac{a_{n+1}}{a_n}| \to |x-3|$, when $2 < x < 4$ the series converge. *The Ratio Test gives no information when the ratio is 1, so we should consider them.* When $x = 4$ the series becomes a harmonic series which diverge. When $x = 2$ then series is an alternating series which converge. So when $2 \le x < 4$ the series converge.

Representations of Functions as Power Series

> You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials.

Start with a simple equation *(all following representations are based on it)*

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum x^n, |x| < 1$

Example

Example 1 Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum(-x^2)^n = \sum(-1)^n x^{2n}$ the interval of convergence is $(-1, 1)$

Example 2 Find a power series representation for $1/(x+2)$.

$\frac{1}{2+x} = \frac{1}{2[1-(-\frac{x}{2})]} = \frac{1}{2}\sum(-\frac{x}{2})^n$ The interval of convergence is $(-2, 2)$

Example 3 Find a power series representation for $x^3/(x+2)$.

$\frac{x^3}{2+x} = x^3 \frac{1}{2+x} = \sum \frac{(-1)^n}{2^{n+1}} x^{n+3}$

Differentiation and Integration of Power Series

Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 = \sum n_c n(x-a)^{n-1}$
2. $\int f(x)dx = C + \sum c_n \frac{(x-a)^{n+1}}{n+1}$

The radius of convergence of the power series in above equations are both R.

> We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. The theorem above assert that the same is true for infinite sums, provided we are dealing with power series.

Example

Example 1 Differentiate **Bessel function** $J_0(x) = \sum \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

By Ratio Test we get $|\frac{a_{n+1}}{a_n}| = \frac{x^2}{2^2(n+1)^2} \to 0$ so the series converge on $(-\infty, \infty)$. So the derivative is found by term-by-term differentiation: $J_0'(x) = \sum \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$

Example 2 Express $1/(1-x)^2$ as a power series. Differentiating each side of $\frac{1}{1-x} = \sum x^n$, we get $\frac{1}{(1-x)^2} = \sum n x^{n-1}$. The radius of convergence stays the same as original, namely, $R = 1$.

Example 3 Find a power series representation for $\ln(1+x)$ and its radius of convergence. It's easy to find a power series representation of its derivative. Integrating both sides of it to get the representation of the original.

Example 4 Find a power series representation for $f(x) = \tan^{-1}x$. The derivative is $\frac{1}{1+x^2}$, find a power series representation and integrate both sides.

Taylor and Maclaurin Series

Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

Theorem If f has a power series representation (expansion) at a, that is, if $f(x) = \sum c_n(x-a)^n, |x-a| < R$, then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

Substituting c_n back into the series, we have $f(x) = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$, which is called the **Taylor series of the function f at a**. For the special case $a = 0$ the Taylor series becomes $f(x) = \sum \frac{f^{(n)}(0)}{n!}x^n$, which is usually called **Maclaurin series**.

> The theorem above is based on **if f has a power series representation**.

Definition nth-degree Taylor polynomial of f at a,
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Definition remainder of Taylor series $R_n(x) = f(x) - T_n(x)$

Theorem WHEN does a function equals to its Taylor series. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem HOW to determine the remainder approach to zero. If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$

A useful fact $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x.

Example

Using above theorems, we can prove that

> Practice with the below examples.

1. $\frac{1}{1-x} = \sum x^n = 1 + x + x^2 + x^3 + \ldots$ with $R = 1$
2. $e^x = \sum \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ with $R = \infty$
3. $\sin x = \sum (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ with $R = \infty$
4. $\cos x = \sum (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ with $R = \infty$
5. $\tan x = \sum (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ with $R = 1$

6. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ with $R = 1$

7. $(1+x)^k = \sum C_n^k x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \ldots$ with $R = 1$

Error of Estimationg using nth-degree Taylor polynomial

It's common to use first n terms of Taylor expansion as an estimation of f. Actutally this is how some of the calculator or computer programs do while calculating e^x, $\sin x$ ... Here're possible methods for estimating the size of error:

1. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
2. In all cases we can use Taylor's Inequality, which says that if $|f^{(n+1)}(x)| \le M$, then $|R_n(x)| \le \frac{M}{(n+1)!} |x - a|^{n+1}$