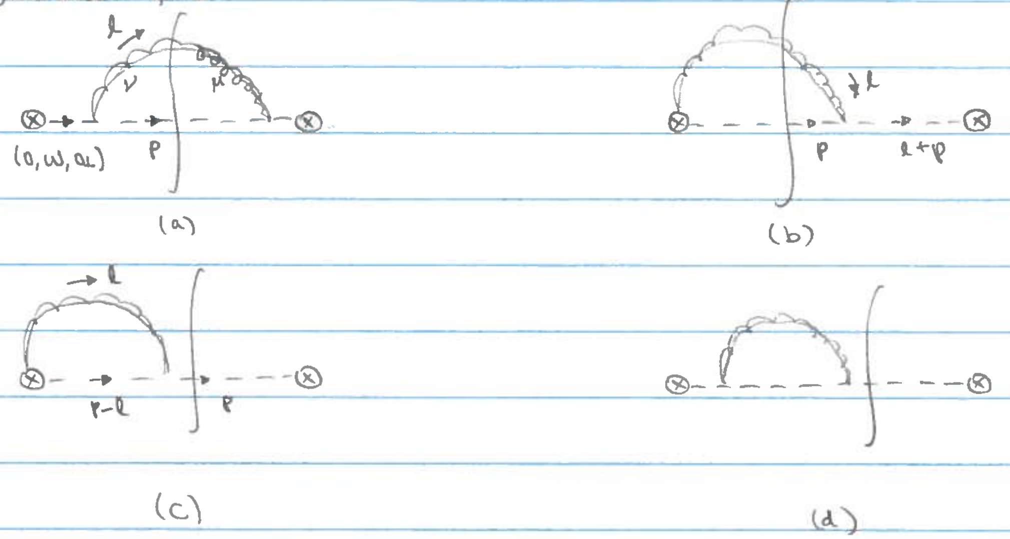
fragmentation function



(b) & (c) have mirror diagrams

(d) is the wave-function renormalization

$$\text{fig a} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c} \frac{1}{z} \int d^{d-2}p_\perp \int \frac{d^d l}{(2\pi)^d}\, 2\pi\delta(l^2)\, \delta(\omega - l^- - p^-)\, \delta^{d-2}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \mathrm{Tr}\left[\frac{\not{n}}{2} \frac{i(\not{k}+\not{l})}{(l+p)^2}\, ig\gamma^\mu T^a\, \not{p}\, ig\gamma^\nu T^a \frac{i(\not{k}+\not{l})}{(l+p)^2}\right] * (-g_{\mu\nu})$$

$$\mathrm{Tr}[T^a T^a] = N_c C_F$$

$$\int d^{d-2}p_\perp\, \delta^{d-2}(\vec{l}_\perp + \vec{p}_\perp) = 1$$ for integrated FFs

Now we want to have $D_q^h(z, \vec{p}_\perp^{\,2})$

Here $\vec{p}_\perp$ is the transverse momentum of the hadron with respect to the quark!!

Use the so-called $\boxed{\mathrm{CDR}_2}$

$$\int d^{d-2}p_\perp\, \delta^{d-2}(\vec{l}_\perp + \vec{p}_\perp) \Rightarrow \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

In other words, we consider $\vec{p}_\perp$ is the observed momentum, which should be only 2-dimensional in the physical 4-dimension space

$\Rightarrow \boxed{\mathrm{CDR}_2}$

fig a $= D_q^{q(a)}(z, \vec{P}_\perp^2)$

$$= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d\ell}{(2\pi)^d} 2\pi\delta(\ell^2)\, \delta(\omega - \ell^- - p^-)\, \delta^{(2)}(\vec{\ell}_\perp + \vec{P}_\perp)$$

$$* \mathrm{Tr}\left[\frac{\bar{n}\!\!\!/}{2} \frac{i(k\!\!\!/+\ell\!\!\!/)}{(\ell+p)^2} ig\gamma^\mu T^a \, p\!\!\!/ \, ig\gamma^\nu T^a \frac{i(k\!\!\!/+\ell\!\!\!/)}{(\ell+p)^2}\right] (-g_{\mu\nu})$$

$p^- = z\omega \qquad \ell^- = (1-z)\omega$

$0 = \ell^2 = \ell^+\ell^- - \vec{\ell}_\perp^{\,2} - \vec{\ell}_{\perp\epsilon}^{\,2} \Rightarrow \ell^+ = \frac{\vec{\ell}_\perp^{\,2} + \vec{\ell}_{\perp\epsilon}^{\,2}}{\ell^-} = \frac{\vec{\ell}_\perp^{\,2} + \vec{\ell}_{\perp\epsilon}^{\,2}}{(1-z)\omega}$

Note, we'll use $\vec{\ell}_\perp, \vec{p}_\perp \Rightarrow$ 2-dimensional vector

$\vec{\ell}_{\perp\epsilon}, \vec{P}_{\perp\epsilon} \Rightarrow$ (-2ϵ)-dimensional vector in $4-2\epsilon$ spa

in other words, the full

$$\ell^\mu = (\ell^+, \ell^-, \vec{\ell}_\perp, \vec{\ell}_{\perp\epsilon})$$

$$\boxed{\int d^d\ell = \frac{1}{2}\int d\ell^+ d\ell^- \, d^2\vec{\ell}_\perp \, d^{d-4}\ell_{\perp\epsilon}}$$

fig b = $D_q^{q(b)}(z, \vec{p}_\perp^{\,2})$ = (b) + complex conjugate

$$= (2) \times \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d l}{(2\pi)^d}\, 2\pi\, \delta(l^2)\, \delta(\omega - l^- - p^-)\, \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$\times \mathrm{Tr}\left[\frac{\bar{\not{n}}}{z} \frac{i(\not{k}+\not{p})}{(l+p)^2}\, ig\gamma^\mu T^a \not{p} \frac{g T^a \bar{n}^\nu}{l^-} \right] (-g_{\mu\nu})$$

fig c $= D_q^{q(c)}(z, \vec{\ell}_\perp^2) = (c) +$ complex conjugate

$$= (2) * \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d\ell}{(2\pi)^d} \delta(\omega - p^-) \, \delta^{(2)}(p_\perp)$$

$$* \mathrm{Tr}\left[\frac{\bar{n}\!\!\!/}{z} \, p\!\!\!/ \, i g \gamma^\mu T^a \frac{i(p\!\!\!/ - \ell\!\!\!/)}{(p-\ell)^2 + i\epsilon} \frac{g T^a \bar{n}^\nu}{\ell^-} \right] \frac{i}{\ell^2 + i\epsilon} (-g_{\mu\nu})$$

= a scaless integral for $d\vec{\ell}_\perp^2 \Rightarrow 0$

fig d = wave function diagram $= 0$ again scaless integral

only fig a + fig b contributes !

$$\text{fig a} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{2N_c}\frac{1}{z}\int \frac{d^d l}{(2\pi)^d}\, 2\pi\,\delta(l^2)\,\delta(\omega - l^- - p^-)\,\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \, \mathrm{Tr}\left[\frac{\bar{\not{n}}}{2}\frac{i(\not{k}+\not{l})}{(l+p)^2}\, ig\gamma^\mu T^a \not{p}\; ig\gamma^\nu T^a \frac{i(\not{k}+\not{l})}{(l+p)^2}\right](-g_{\mu\nu})$$

$$\text{fig b} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{2N_c}\frac{1}{z}\int \frac{d^d l}{(2\pi)^d}\, 2\pi\,\delta(l^2)\,\delta(\omega - l^- - p^-)\,\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \, 2\,\mathrm{Tr}\left[\frac{\bar{\not{n}}}{2}\frac{i(\not{k}+\not{l})}{(l+p)^2}\, ig\gamma^\mu T^a \not{p}\quad \frac{gT^a\bar{n}^\nu}{l^-}\right](-g_{\mu\nu})$$

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{2N_c}\frac{1}{z}\int \frac{d^d l}{(2\pi)^d}\, 2\pi\,\delta(l^2)\,\delta(\omega - l^- - p^-)\,\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \left\{ \underbrace{\frac{(d-2)\,\bar{n}\cdot l}{l\cdot p}}_{(a)} + \underbrace{\frac{4\,\bar{n}\cdot p\,\omega}{\bar{n}\cdot l\; l\cdot p}}_{(b)} \right\}$$

$$p^- = z\omega$$

$$l^- = (1-z)\,\omega$$

$$l\cdot p = \frac{1}{2}(l^+p^- + l^-p^+) - \underbrace{\vec{l}_{\perp d}\cdot\vec{p}_{\perp d}}_{\text{"d-2" dimensional}}$$

$$0 = l^+l^- - \vec{l}_{\perp d}^{\,2} \Rightarrow l^+ = \frac{\vec{l}_{\perp d}^{\,2}}{l^-} = \frac{\vec{l}_{\perp d}^{\,2}}{(1-z)\omega} = \frac{\vec{p}_{\perp d}^{\,2}}{(1-z)\omega}$$

$$0 = p^2 = p^+p^- - \vec{p}_{\perp d}^{\,2} \Rightarrow p^+ = \frac{\vec{p}_{\perp d}^{\,2}}{p^-} = \frac{\vec{p}_{\perp d}^{\,2}}{z\omega}$$

$$l\cdot p = \frac{1}{2}\frac{\vec{p}_{\perp d}^{\,2}}{(1-z)\omega}\, z\omega + \frac{1}{2}\frac{\vec{p}_{\perp d}^{\,2}}{z\omega}(1-z)\omega + \vec{p}_{\perp d}^{\,2}$$

$$= \left[\frac{1}{2}\frac{z}{1-z} + \frac{1}{2}\frac{1-z}{z} + 1\right]\vec{p}_{\perp d}^{\,2}$$

$$= \frac{\vec{p}_{\perp d}^{\,2}}{2z(1-z)} \quad \text{or} \quad \frac{\vec{l}_{\perp d}^{\,2}}{2z(1-z)}$$

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c}\frac{1}{z}\int \frac{d^d l}{(2\pi)^d}\, 2\pi\delta(l^2)\,\delta(\omega - l^- - p^-)\,\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$\underbrace{* \left\{(d-2)\,l^- + \frac{4p^-\omega}{l^-}\right\} * \frac{2z(1-z)}{\vec{l}_{\perp d}^{\,2}}} \quad *(\text{color} = N_c C_F) * g_s^2$$

$$(d-2)(1-z)\omega + \frac{4z\omega * \omega}{(1-z)\omega} = \left\{(d-2)(1-z) + \frac{4z}{1-z}\right\}\omega$$

$$\left(\frac{1}{N_c} N_c C_F\right) * \frac{1}{2z} * \frac{2z(1-z)\omega}{\vec{l}_{\perp d}^{\,2}} = \frac{(1-z)\omega}{\vec{l}_{\perp d}^{\,2}} = \frac{1}{l^+}$$

We thus have

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \int \frac{d^d l}{(2\pi)^d}\, 2\pi\,\delta(l^2)\,\underset{\delta[(1-z)\omega - l^-]}{\delta(\omega - l^- - p^-)}\,\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* (g_s^2 C_F) * \left[(d-2)(1-z) + \frac{4z}{1-z}\right]\left(\frac{1}{l^+}\right)$$

$$\int d^d l = \frac{1}{2}\int dl^+ dl^-\, d^2 l_\perp\, d^{d-4} l_{\perp\epsilon}$$

$$I = \int d^d l\; \delta(l^2)\;\delta[(1-z)\omega - l^-]\;\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$= \frac{1}{2}\int dl^+ dl^-\, d^2 l_\perp\, d^{d-4} l_{\perp\epsilon}\;\delta[l^+ l^- - \vec{l}_\perp^{\,2} - \vec{l}_{\perp\epsilon}^{\,2}]\;\delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \,\delta[(1-z)\omega - l^-]$$

$$\int d^{d-4} l_{\perp\epsilon} = \int d\Omega^{d-4}\, dl_{\perp\epsilon}\,(l_{\perp\epsilon})^{d-5}$$

$$= \frac{2\pi^{\frac{d-4}{2}}}{\Gamma\left(\frac{d-4}{2}\right)} * dl_{\perp\epsilon}\,(l_{\perp\epsilon})^{d-5}$$

$$I = \frac{1}{2}\int dl^+ dl^- \, d^2 l_\perp \, \underbrace{dl_{\perp\epsilon} \, (l_{\perp\epsilon})^{d-5}}_{\frac{1}{2} dl_{\perp\epsilon}^2 \, (l_{\perp\epsilon}^2)^{\frac{d-6}{2}}} * \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2})}$$

$$* \; \delta(l^+l^- - \vec{l}_\perp^{\,2} - \vec{l}_{\perp\epsilon}^{\,2}) \quad \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp) * \delta[(1-z)\omega - l^-]$$

$$= \left(\frac{1}{2}\right)^2 \int dl^+ \quad (l^+l^- - \vec{p}_\perp^{\,2})^{\frac{d-6}{2}} * \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2})}$$

$$\Downarrow \quad d = 4-2\epsilon$$

$$= \left(\frac{1}{2}\right)^2 \int dl^+ \quad (l^+l^- - \vec{p}_\perp^{\,2})^{-1-\epsilon} * \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}$$

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{(2\pi)^{d-1}} \quad (g_s^2 C_F)\left[(d-2)(1-z) + \frac{4z}{1-z}\right]\left(\frac{1}{l^+}\right)$$

$$* \left(\frac{1}{2}\right)^2 \int dl^+ \quad (\underset{\substack{\downarrow \\ (1-z)\omega}}{l^+l^-} - \vec{p}_\perp^{\,2})^{-1-\epsilon} * \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}$$

$$= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{(2\pi)^{d-1}} \; (g_s^2 C_F) \; \frac{\pi^{-\epsilon}}{\Gamma(-\epsilon)} \; [l^-]^{-1-\epsilon}$$

$$* \frac{1}{2}\left[(d-2)(1-z) + \frac{4z}{1-z}\right] * \int_0^\infty dl^+ \left[l^+ - \frac{\vec{p}_\perp^{\,2}}{l^-}\right]^{-1-\epsilon}\left(\frac{1}{l^+}\right)$$

$$l^- = (1-z)\omega$$

too quick!

the rapidity regulator is set "for" gauge link only!

See arXiv: 1202.0814 Eq.(4.8)

let's go back a bit

No change

$$\text{fig } a \overset{\downarrow}{=} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d l}{(2\pi)^d} \, 2\pi \, \delta(l^2) \, \delta(\omega - l^- - p^-) \, \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp) * g_s^2 \text{ color}$$

$$* \mathrm{Tr}\left[\frac{\bar{\not{n}}}{2} \frac{i(\not{k}+\not{p})}{(l+p)^2} \, ig\gamma^\mu T^a \, \not{p} \, ig\gamma^\nu T^a \frac{i(\not{k}+\not{l})}{(l+p)^2} \right] (-g_{\mu\nu})$$

$$\text{fig } b = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d l}{(2\pi)^d} \, 2\pi \, \delta(l^2) \, \delta(\omega - l^- - p^-) \, \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp) * g_s^2 \text{ color}$$

$$* 2\mathrm{Tr}\left[\frac{\bar{\not{n}}}{2} \frac{i(\not{k}+\not{l})}{(l+p)^2} \, ig\gamma^\mu T^a \, \not{p} \, \frac{g T^a \bar{n}^\nu}{l^-} \right] (-g_{\mu\nu})$$

$\frac{1}{l^-}$ comes from gauge link

$$\text{change} \quad \frac{1}{l^-} \to \frac{1}{l^-} * \left(\frac{l^-}{\nu}\right)^{-\eta}$$

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d l}{(2\pi)^{d-1}} \, \delta(l^2) \, \delta(\omega - l^- - p^-) \, \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* g_s^2 * (\text{color} = N_c C_F)$$

$$* \left\{ \underbrace{(d-2)\, l^-}_{(a)} + \underbrace{4\omega p^- * \left(\frac{1}{l^-}\right) * \left(\frac{l^-}{\nu}\right)^{-\eta}}_{(b)} \right\} * \frac{1}{l \cdot p}$$

NOTE:

$$\left.\begin{aligned} l \cdot p &= \frac{\vec{l}_{\perp d}^{\,2}}{2z(1-z)} \\ l^+ &= \frac{\vec{l}_{\perp d}^{\,2}}{(1-z)\omega} \end{aligned}\right\} \Rightarrow \frac{l \cdot p}{l^+} = \frac{\omega}{2z}$$

$$\Rightarrow \frac{1}{l \cdot p} = \left(\frac{2z}{\omega l^+}\right)$$

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{2N_c} \frac{1}{z} \int \frac{d^d l}{(2\pi)^{d-1}} \delta(l^2)\, \delta(\omega - l^- - p^-)\, \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \, g_s^2 * N_c C_F$$

$$* \left\{ (d-2)\, l^- + 4\omega p^- * \frac{\nu^\eta}{(l^-)^{1+\eta}} \right\} * \frac{2z}{\omega\, l^+}$$

$$\Downarrow \quad \frac{l^-}{\omega} = 1 - z$$

$$= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} * g_s^2 C_F \int \frac{d^d l}{(2\pi)^{d-1}} \delta(l^2)\, \delta(\omega - l^- - p^-)\, \delta^{(2)}(\vec{l}_\perp + \vec{p}_\perp)$$

$$* \underbrace{\left\{ (d-2)(1-z) + 4p^- * \frac{\nu^\eta}{(l^-)^{1+\eta}} \right\}}_{\Downarrow\ 4(z\omega) * \frac{\nu^\eta}{\{(1-z)\omega\}^{1+\eta}}} * \left(\frac{1}{l^+}\right)$$

$$\left\{ (d-2)(1-z) + \frac{4z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^\eta \right\} * \left(\frac{1}{l^+}\right)$$

go back, now

$$\text{fig}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{(2\pi)^{d-1}} (g_s^2 C_F) \frac{\pi^{-\epsilon}}{\Gamma(1-\epsilon)} \left[l^- \right]^{-1-\epsilon}$$

$$* \frac{1}{2} \left[(d-2)(1-z) + \frac{4z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^\eta \right]$$

$$* \int_0^\infty dl^+ \left[l^+ - \frac{\vec{p}_\perp^{\,2}}{l^-} \right]^{-1-\epsilon} \left(\frac{1}{l^+}\right)$$

something wrong!

Note we have $\delta_+(\ell^2)$ meaning $\Rightarrow$ $\delta(\ell^2)\,\theta(\ell^0)$

$\delta(\ell^+\ell^- - \vec{\ell}_\perp^{\,2} - \vec{\ell}_{\perp\epsilon}^{\,2})$ $\qquad$ $\ell^0 > 0 \Rightarrow$ real particle / on-shell

$\Rightarrow$ $\ell^0 =$ energy $> |\ell^2|$

$\ell^+\ell^- - \vec{\ell}_\perp^{\,2} = \vec{\ell}_{\perp\epsilon}^{\,2} > 0$ $\qquad$ both $\ell^+ > 0$ $\ell^- > 0$

$\ell^+\ell^- - \vec{\ell}_\perp^{\,2} > 0 \Rightarrow$ $\boxed{\ell^+ > \frac{\vec{\ell}_\perp^{\,2}}{\ell^-}} = \frac{\vec{p}_\perp^{\,2}}{\ell^-}$

Important for ℓ^+ - integral

Thus $\displaystyle\int_0^\infty d\ell^+ \left[\ell^+ - \frac{\vec{p}_\perp^{\,2}}{(1-z)W}\right]^{-1-\epsilon} (\ell^+)^{-1} \quad \theta\left(\ell^+ - \frac{\vec{p}_\perp^{\,2}}{\ell^-}\right)$

$$= \int_{\frac{p_\perp^2}{\ell^-}}^\infty d\ell^+ \left[\ell^+ - \frac{\vec{p}_\perp^{\,2}}{\ell^-}\right]^{-1-\epsilon} (\ell^+)^{-1}$$

define $\quad \ell^+ = \frac{\vec{p}_\perp^{\,2}}{\ell^-}\frac{1}{\alpha} \Rightarrow \frac{d\ell^+}{\ell^+} = *(-1)\frac{d\alpha}{\alpha}$

$$= \int_0^1 \frac{d\alpha}{\alpha} \left[\frac{\vec{p}_\perp^{\,2}}{\ell^-}\right]^{-1-\epsilon} \left(\frac{1}{\alpha} - 1\right)^{-1-\epsilon}$$

$$= \int_0^1 \frac{d\alpha}{\alpha} \left(\frac{\vec{p}_\perp^{\,2}}{\ell^-}\right)^{-1-\epsilon} (1-\alpha)^{-1-\epsilon}\,\alpha^{1+\epsilon}$$

$$= \underbrace{\int_0^1 d\alpha\,(1-\alpha)^{-1-\epsilon}\,\alpha^{\epsilon}}_{\frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1)}} \left(\frac{\vec{p}_\perp^{\,2}}{\ell^-}\right)^{-1-\epsilon}$$

$$= \Gamma(-\epsilon)\,\Gamma(1+\epsilon) \left(\frac{\vec{p}_\perp^{\,2}}{\ell^-}\right)^{-1-\epsilon}$$

Thus $\mathrm{frg}(a+b) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{(2\pi)^{d-1}} (g_s^2 C_F) \frac{\pi^{-\epsilon}}{\Gamma(-\epsilon)} (l^-)^{-1-\epsilon}$

$$* \frac{1}{2}\left[(d-2)(1-z) + \frac{4z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} \right]$$

$$* \left(\frac{\vec{p}_\perp^{\,2}}{l^-}\right)^{-1-\epsilon} \Gamma(-\epsilon)\Gamma(1+\epsilon)$$

$$= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{(2\pi)^{d-1}} (g_s^2 C_F) \frac{\pi^{-\epsilon}}{\cancel{\Gamma(-\epsilon)}} \frac{1}{(\vec{p}_\perp^{\,2})^{1+\epsilon}}$$

$$* \left[(1-\epsilon)(1-z) + \frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} \right]$$

$$* \Gamma(-\epsilon)\Gamma(1+\epsilon)$$

$$= \frac{\alpha_s}{2\pi^2} C_F \Gamma(1+\epsilon) (\mu^2 e^{\gamma_E})^{\epsilon} \frac{1}{(\vec{p}_\perp^{\,2})^{1+\epsilon}}$$

$$* \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-z) - \epsilon(1-z) \right]$$

We have

$$\boxed{\begin{aligned} D_q^q(z, \vec{p}_\perp^{\,2}) &= \frac{\alpha_s}{2\pi^2} C_F \Gamma(1+\epsilon) e^{\gamma_E \epsilon} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{p}_\perp^{\,2}}\right)^{1+\epsilon} \\ &* \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-z) - \epsilon(1-z) \right] \end{aligned}}$$

Fourier transform in b-space

$$D_q^q(z, b) \overset{?}{=} \int d^2\vec{p}_\perp \; e^{i\vec{p}_\perp \cdot \vec{b}} \; D_q^q(z, \vec{p}_\perp^{\,2})$$

This is not the convention !!

see arXiv: 1401.5078 Eq. (4)

$$D_q^q(z, b) = \frac{1}{z^2} \int d^2\vec{p}_\perp \; e^{-i\vec{p}_\perp \cdot \vec{b}/z} \; D_q^q(z, \vec{p}_\perp^{\,2})$$

$$= \frac{1}{z^2} \int d^2\vec{p}_\perp \; e^{-i\vec{p}_\perp \cdot \vec{b}/z} \; (\vec{p}_\perp^{\,2})^{-1-\epsilon} \; * \{ \cdots \}$$

identity : $\displaystyle \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2)^\beta} e^{ik\cdot b} = \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(\frac{n}{2}-\beta)}{\Gamma(\beta)} \left(\frac{b^2}{4}\right)^{\beta-\frac{n}{2}}$

our case $n = 2$ $\beta = 1+\epsilon$

$$\Rightarrow \int \frac{d^2 k}{(2\pi)^2} \frac{1}{(k^2)^{1+\epsilon}} e^{ik\cdot b} = \frac{1}{4\pi} \frac{\Gamma(1-1-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2}{4}\right)^{1+\epsilon-1}$$

$$= \frac{1}{4\pi} \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2}{4}\right)^{\epsilon}$$

$$D_q^q(z, b) = \int d^2\vec{k}_\perp \; e^{-i\vec{k}_\perp \cdot \vec{b}} \; (z^2 k_\perp^2)^{-1-\epsilon} * \{ \cdots \}$$

$$= (z^2)^{-1-\epsilon} \int d^2 k_\perp \; e^{-i\vec{k}_\perp \cdot \vec{b}} \; (k_\perp^2)^{-1-\epsilon} * \{ \cdots \}$$

$$= (z^2)^{-1-\epsilon} * \pi \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2}{4}\right)^{\epsilon} * \{ \cdots \}$$

Thus

$$D_q^q(z,b) = (z^2)^{-1-\epsilon} * \frac{\pi\Gamma(-\epsilon)}{\cancel{\Gamma(1+\epsilon)}} \left(\frac{b^2}{4}\right)^{\epsilon}$$

$$* \frac{\alpha_s}{2\pi^2} C_F \, \cancel{\Gamma(1+\epsilon)} \, e^{\gamma_E \epsilon} \, (\mu^2)^{\epsilon}$$

$$* \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-z) - \epsilon(1-z) \right]$$

$$= (z^2)^{-1-\epsilon} * \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2 e^{\gamma_E} b^2}{4}\right)^{\epsilon} \Gamma(-\epsilon)$$

$$* \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-z) - \epsilon(1-z) \right]$$

define $\mu_b = 2e^{-\gamma_E}/b$

$$D_q^q(z,b) = (z^2)^{-1} * \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{z^2\mu_b^2}\right)^{\epsilon} \Gamma(-\epsilon)\, e^{-\gamma_E \epsilon}$$

$$* \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-z) - \epsilon(1-z) \right]$$

first "η" expansion, then ϵ - expansion

$$(1-z)^{-1-\eta} = -\frac{1}{\eta}\delta(1-z) + \frac{1}{(1-z)_+} + O(\eta)$$

$$\left(\frac{\nu}{\omega}\right)^{\eta} = 1 + \eta \ln\left(\frac{\nu}{\omega}\right) + O(\eta^2)$$

$$(1-z)^{-1-\eta}\left(\frac{\nu}{\omega}\right)^{\eta} = -\frac{1}{\eta}\delta(1-z) - \delta(1-z)\ln\left(\frac{\nu}{\omega}\right) + \frac{1}{(1-z)_+} + O(\eta)$$

$$D_q^q(z,b) = \left(\frac{1}{z^2}\right) * \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{z^2\mu_b^2}\right)^{\epsilon} \Gamma(-\epsilon)\, e^{-\gamma_E \epsilon}$$

$$* \left[-\frac{1}{\eta}\delta(1-z) * 2z - 2z * \delta(1-z)\ln\left(\frac{\nu}{\omega}\right) + \underbrace{\frac{2z}{(1-z)_+} + (1-z) - \epsilon(1-z)}_{\frac{1+z^2}{(1-z)_+} - \epsilon(1-z)} \right]$$

$$= \left(\frac{1}{z^2}\right) * \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{z^2\mu_b^2}\right)^{\epsilon} \Gamma(-\epsilon)\, e^{-\gamma_E \epsilon}$$

$$* \left[-\frac{2}{\eta}\delta(1-z) - 2\,\delta(1-z)\ln\left(\frac{\nu}{\omega}\right) + \hat{P}_{qq}(z,\epsilon) \right]$$

where $\hat{P}_{qq}(z,\epsilon) = \frac{1+z^2}{(1-z)_+} - \epsilon(1-z)$

Now ϵ-expansion

$$\left(\frac{\mu^2}{z^2\mu_b^2}\right)^{\epsilon} \Gamma(-\epsilon)\, e^{-\gamma_E \epsilon} = -\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{z^2\mu_b^2}\right) + O(\epsilon)$$

$$D_q^q(z,b) = \left(\frac{1}{z^2}\right) * \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{z^2\mu_b^2}\right) + O(\epsilon) \right]$$

$$* \left[-\frac{2}{\eta}\delta(1-z) - 2\,\delta(1-z)\ln\frac{\nu}{\omega} + \frac{1+z^2}{(1-z)_+} - \epsilon(1-z) \right]$$

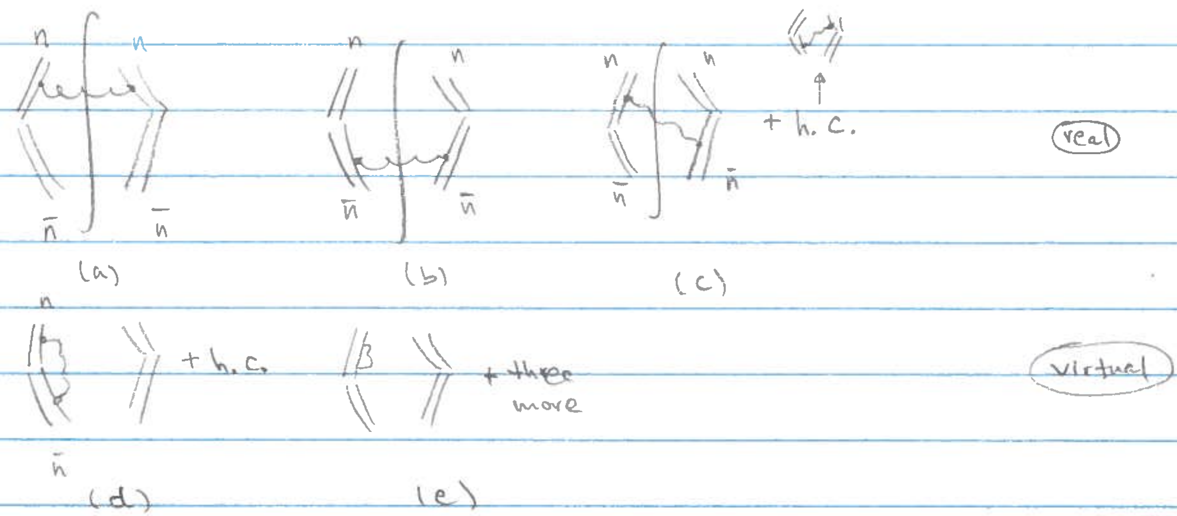
$$= \left(\frac{1}{z^2}\right) * \frac{\alpha_s}{2\pi} C_F * \left\{ \frac{2}{\eta\epsilon}\delta(1-z) + \frac{2}{\epsilon}\delta(1-z)\ln\left(\frac{\nu}{\omega}\right) \right.$$

$$+ \left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{z^2\mu_b^2}\right)\right) * \hat{P}_{qq}(z) + (1-z)$$

$\frac{1+z^2}{(1-z)_+}$

$$+ \frac{2}{\eta}\delta(1-z)\ln\left(\frac{\mu^2}{z^2\mu_b^2}\right) + 2\ln\left(\frac{\mu^2}{z^2\mu_b^2}\right)\delta(1-z)\ln\left(\frac{\nu}{\omega}\right)$$

Soft function



(a), (b) are zero, since $\propto \bar{n}^\mu \bar{n}^\nu (-g_{\mu\nu}) = \bar{n}^2 = 0$

or $n^\mu n^\nu (-g_{\mu\nu}) = n^2 = 0$

(e) should be zero, again due to same reason

So let's concentrate on diagrams (c) & (d)

(d) will involve a scaleless integral, and thus will be zero in dimensional regularization

$$\text{fig d} + \text{h.c.} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \delta^2(\lambda_\perp) \; (\text{color} = N_c C_F) \frac{1}{N_c}$$

$$* \int \frac{d^d l}{(2\pi)^d} \left[\frac{g_s n^\mu}{l^+} \quad \frac{g_s \bar{n}^\nu}{l^-} \right] \frac{i(-g_{\mu\nu})}{l^2 + i\epsilon} + \text{h.c.}$$

$\Downarrow$ $n \cdot \bar{n} = 2$

$$= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \delta^2(\lambda_\perp) \; C_F \; (+2i g_s^2) \; \delta^{(2)}(\lambda_\perp)$$

$$+ \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^+} \frac{1}{l^-} \frac{1}{l^2 + i\epsilon} + \text{h.c.}$$

Apparently "$d^d l$" integral will become scaleless integral
let's drop it for now

$$\text{fig c} + \text{h.c.} = 2\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \int \frac{d^d l}{(2\pi)^d} \, 2\pi \, \delta_+(l^2) \, * \, \delta^{(2)}(\vec{l}_\perp + \vec{\lambda}_\perp)$$

$$* \left[-\frac{g_s n^\mu}{l^+} \, \frac{g_s \bar{n}^\nu}{l^-}\right](-g_{\mu\nu}) \; * \; (\text{color} = C_F N_C) \frac{1}{N_c}$$

$\Downarrow$ $n \cdot \bar{n} = 2$

$$= +8\pi \, g_s^2 \, C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \int \frac{d^d l}{(2\pi)^d} \, \delta_+(l^2) \, \delta^{(2)}(\vec{l}_\perp + \vec{\lambda}_\perp)$$

$$* \frac{1}{l^+} \frac{1}{l^-}$$

it'll be good to see the sign & $i\epsilon$ convention from
arXiv: 1504.04006

There're two kinds of soft wilson line

$$(Y_n^+)^\dagger(x) = P\exp\left[ig\int_0^\infty ds \, n \cdot A_s(ns+x)\right]$$

$$Y_n^-(x) = P\exp\left[ig\int_{-\infty}^0 ds \, n \cdot A_s(ns+x)\right]$$

Y_n^+ : n = direction of outgoing jet
Y_n^- : n = direction of incoming jet

e.g. for e^+e^-, we have $(Y_n^+)^\dagger \, Y_{\bar{n}}^+$ both outgoing
n is quark jet

for pp→DY, we have $(Y_n^-)^\dagger \, Y_{\bar{n}}^-$ both are incoming
n is quark jet

for $e p \to e h X$, we have $(Y_n^+)^\dagger Y_{\bar{n}}^-$ "$\bar{n}$" is incoming quark
"n" is outgoing quark

$$\boxed{\begin{array}{ll} Y_n^+ : & -g \dfrac{n^\mu}{n\cdot k + i\epsilon} \\ Y_{\bar{n}}^- : & -g \dfrac{n^\mu}{n\cdot k - i\epsilon} \end{array}}$$

$(Y_{\bar{n}}^-)^\dagger \Rightarrow g \dfrac{\bar{n}^\mu}{\bar{n}\cdot k + i\epsilon}$

$(Y_{\bar{n}}^+)^\dagger \Rightarrow g \dfrac{\bar{n}^\mu}{\bar{n}\cdot k - i\epsilon}$

Thus

$$S(\lambda_\perp) = 8\pi g_s^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \int \frac{d^d l}{(2\pi)^d} \delta_+(l^2)\, \delta^{(2)}(\vec{l}_\perp + \vec{\lambda}_\perp) * \frac{1}{l^+}\frac{1}{l^-}$$

$\Downarrow$ rapidity regulator $\quad \frac{1}{l^+}\frac{1}{l^-} \Rightarrow \frac{1}{l^+ l^-}\left\{\left[\frac{2 l_z}{\nu}\right]^{-\eta/2}\right\}^2$

$$\Downarrow$$

$$\left|\frac{2 l_z}{\nu}\right|^{-\eta} \Leftarrow \left(\frac{2 l_z}{\nu}\right)^{-\eta}$$

here it's absolute value

$$= 8\pi g_s^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \int \frac{d^d l}{(2\pi)^d} \delta_+(l^2)\, \delta^{(2)}(\vec{l}_\perp + \vec{\lambda}_\perp)$$

$$* \frac{1}{l^+ l^-} * |2 l_z|^{-\eta} * \nu^\eta$$

See Eq. (4.? of arXiv: 1202.0814

Note:

$$I = \int d^d l\ \delta_+(l^2)\, \delta^{(2)}(\vec{l}_\perp + \vec{\lambda}_\perp)$$

$$= \frac{1}{2}\int dl^+ dl^- d^2 l_\perp d^{-2\epsilon} l_{\perp\epsilon}\ \delta^{(2)}(\vec{l}_\perp + \vec{\lambda}_\perp)\ \delta_+(l^2)$$

$$= \frac{1}{2}\int dl^+ dl^- * \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2})} \underbrace{dl_{\perp\epsilon}\ (l_{\perp\epsilon})^{d-5}}_{\frac{1}{2} dl_{\perp\epsilon}^2 * (l_{\perp\epsilon}^2)^{\frac{d-6}{2}}} \delta_+(l^+ l^- - \vec{l}_\perp^2 - \vec{l}_{\perp\epsilon}^2)$$

$$= \frac{1}{2}\int dl^+ dl^- \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)} \frac{1}{2}\left[l^+ l^- - \vec{l}_\perp^2\right]^{-1-\epsilon}$$

$$= \left(\frac{1}{2}\right)^2 \int dl^+ dl^- \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)} \left(l^+ l^- - \vec{\lambda}_\perp^2\right)^{-1-\epsilon}$$

Thus

$$S(\lambda_\perp) = 8\pi g_s^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon * \frac{1}{(2\pi)^d} \left(\frac{1}{2}\right)^2 \int d\ell^+ d\ell^- \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}$$

$$* \left(\ell^+\ell^- - \vec{\lambda}_\perp^2\right)^{-1-\epsilon} * \frac{1}{\ell^+\ell^-} |2\ell_z|^{-\eta} * \nu^\eta$$

$$0 = \ell^+\ell^- - \vec{\ell}_\perp^2 - \vec{\ell}_{\perp\epsilon}^2 = \ell^+\ell^- - \vec{\lambda}_\perp^2 - \vec{\ell}_{\perp\epsilon}^2$$

$$2\ell_z = \ell^- - \ell^+$$

$$\ell^- = \ell^0 + \ell z$$

$$\ell^+ = \ell^0 - \ell z$$

$$S(\lambda_\perp) = 8\pi g_s^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon * \frac{1}{(2\pi)^d} \left(\frac{1}{2}\right)^2 \nu^\eta \int d\ell^+ d\ell^- \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}$$

$$* \left(\ell^+\ell^- - \vec{\lambda}_\perp^2\right)^{-1-\epsilon} * \frac{1}{\ell^+\ell^-} \times |\ell^- - \ell^+|^{-\eta}$$

Note: $\ell^+\ell^- - \vec{\lambda}_\perp^2 > 0 \Rightarrow \ell^+ > \frac{\vec{\lambda}_\perp^2}{\ell^-}$

$\ell^- > \frac{\lambda_\perp^2}{\ell^+}$

We need to perform

$$\boxed{I = \int d\ell^+ d\ell^- \left(\ell^+\ell^- - \vec{\lambda}_\perp^2\right)^{-1-\epsilon} \frac{1}{\ell^+\ell^-} |\ell^- - \ell^+|^{-\eta}}$$

change variable

$$x = \ell^+ \ell^- > \lambda_\perp^2 \qquad xy = \ell^{+2} \qquad \frac{x}{y} = \ell^{-2}$$

$$y = \frac{\ell^+}{\ell^-}$$

$$dx\,dy = \begin{vmatrix} \ell^- & \ell^+ \\ \frac{1}{\ell^-} & -\frac{\ell^+}{\ell^{-2}} \end{vmatrix} d\ell^+ d\ell^-$$

$$= -\frac{2\ell^+}{\ell^-} d\ell^+ d\ell^-$$

$$\Rightarrow \quad d\ell^+ d\ell^- = -\frac{1}{2} dx\,dy * \left(\frac{\ell^-}{\ell^+}\right)$$

$$= -\frac{1}{2y} dx\,dy$$

we should have

$$I = \int dx\,dy \; \frac{1}{2y} * (x - \vec{\lambda}_\perp^2)^{-1-\epsilon} \; \frac{1}{x} \; \underbrace{[\ell^-]^{-\eta}}_{\left(\frac{x}{y}\right)^{-\frac{\eta}{2}}} \; \underbrace{\left|1 - \frac{\ell^+}{\ell^-}\right|^{-\eta}}_{|1-y|^{-\eta}}$$

$$= \int dx\,dy \; \frac{1}{2y} \; (x - \lambda_\perp^2)^{-1-\epsilon} \; x^{-1-\eta/2} \; y^{\frac{\eta}{2}} \; |1-y|^{-\eta}$$

$$= \frac{1}{2} \int_{\lambda_\perp^2}^{\infty} dx \; (x - \lambda_\perp^2)^{-1-\epsilon} \; x^{-1-\eta/2}$$

$$* \int_0^{\infty} dy \; y^{\frac{\eta}{2}-1} \; |1-y|^{-\eta}$$

define $\quad x = \dfrac{\lambda_\perp^2}{\alpha} \qquad \dfrac{dx}{x} = \dfrac{d\alpha}{\alpha}$

$$I = \frac{1}{2}\int_0^1 \frac{d\alpha}{\alpha}\left(\frac{\lambda_\perp^2}{\alpha} - \lambda_\perp^2\right)^{-1-\epsilon}\left(\frac{\lambda_\perp^2}{\alpha}\right)^{-\eta/2}$$

$$* \; I_y$$

$$= \frac{1}{2}\int_0^1 d\alpha \, (\lambda_\perp^2)^{-1-\epsilon} * (1-\alpha)^{-1-\epsilon} \;\; \alpha^{1+\epsilon} * \alpha^{-1+\eta/2} * (\lambda_\perp^2)^{-\eta/2}$$

$$* \; I_y$$

$$= \frac{1}{2}(\lambda_\perp^2)^{-1-\epsilon-\eta/2} * \underbrace{\int_0^1 d\alpha \, (1-\alpha)^{-1-\epsilon} \;\; \alpha^{\epsilon+\eta/2}}_{\frac{\Gamma(-\epsilon)\,\Gamma(1+\epsilon+\eta/2)}{\Gamma(1+\eta/2)}} * I_y$$

$$= \frac{1}{2}(\lambda_\perp^2)^{-1-\epsilon-\eta/2} \;\; \frac{\Gamma(-\epsilon)\,\Gamma(1+\epsilon+\eta/2)}{\Gamma(1+\eta/2)} * I_y$$

$$I_y = \int_0^\infty dy \;\; y^{\frac{\eta}{2}-1} \;\; |1-y|^{-\eta}$$

$$I_y = \int_0^1 dy \;\; y^{\frac{\eta}{2}-1}(1-y)^{-\eta} + \int_1^\infty dy \;\; y^{\frac{\eta}{2}-1} \; (y-1)^{-\eta}$$

$$= \;\; I_{y_1} + I_{y_2}$$

$$I_{y_1} = \int_0^1 dy \;\; y^{\frac{\eta}{2}-1} \; (1-y)^{-\eta} = \frac{\Gamma(\frac{\eta}{2}) \;\; \Gamma(1-\eta)}{\Gamma(1-\frac{\eta}{2})}$$

$$I_{y2} = \int_1^\infty dy \quad y^{\frac{\eta}{2}-1} \quad (y-1)^{-\eta}$$

$$\Downarrow \quad y = \frac{1}{\alpha} \qquad \frac{dy}{y} = \frac{d\alpha}{\alpha}$$

$$= \int_0^1 \frac{d\alpha}{\alpha} \quad \left(\frac{1}{\alpha}\right)^{\frac{\eta}{2}} \quad \left(\frac{1}{\alpha}-1\right)^{-\eta}$$

$$= \int_0^1 \frac{d\alpha}{\alpha} \quad \alpha^{-\frac{\eta}{2}} \quad \alpha^{\eta} \quad (1-\alpha)^{-\eta}$$

$$= \int_0^1 d\alpha \quad \alpha^{\frac{\eta}{2}-1} \quad (1-\alpha)^{-\eta}$$

$$= \frac{\Gamma\left(\frac{\eta}{2}\right)\Gamma(1-\eta)}{\Gamma\left(1-\frac{\eta}{2}\right)}$$

We have

$$I_y = I_{y1} + I_{y2} = \frac{2\,\Gamma\left(\frac{\eta}{2}\right)\Gamma(1-\eta)}{\Gamma\left(1-\frac{\eta}{2}\right)}$$

$$\Downarrow \quad \Gamma(z)\,\Gamma\left(\tfrac{1}{2}+z\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

$$\Downarrow \quad z = \tfrac{1}{2} - \tfrac{\eta}{2}$$

$$\Gamma\left(\tfrac{1}{2}-\tfrac{\eta}{2}\right)\Gamma\left(1-\tfrac{\eta}{2}\right) = 2^{\eta}\sqrt{\pi}\,\Gamma(1-\eta)$$

$$\Rightarrow \quad \frac{\Gamma(1-\eta)}{\Gamma\left(1-\frac{\eta}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}-\frac{\eta}{2}\right)}{2^{\eta}\sqrt{\pi}}$$

$$I_y = \frac{2\,\Gamma\left(\frac{\eta}{2}\right) \times \Gamma\left(\frac{1}{2}-\frac{\eta}{2}\right)}{2^{\eta}\sqrt{\pi}}$$

finally

$$I = \frac{1}{2} \frac{1}{(\vec{\lambda}_\perp^2)^{1+\epsilon+\eta/2}} \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\eta/2)} * \frac{2\Gamma(\frac{\eta}{2})\ \Gamma(\frac{1}{2}-\frac{\eta}{2})}{2^\eta \sqrt{\pi}}$$

$$= \frac{1}{(\vec{\lambda}_\perp^2)^{1+\epsilon+\eta/2}} \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\eta/2)} \frac{2^{-\eta}\Gamma(\frac{1}{2}-\frac{\eta}{2})\Gamma(\frac{\eta}{2})}{\sqrt{\pi}}$$

finally

$$S(\vec{\lambda}_\perp) = 8\pi g_s^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{(2\pi)^{4-2\epsilon}} \left(\frac{1}{2}\right)^2 \nu^\eta * \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}$$

$$* (I)$$

$$= \frac{2 g_s^2 C_F}{(2\pi)^3} \frac{e^{\gamma_E \epsilon} \mu^{2\epsilon} \nu^\eta}{(\vec{\lambda}_\perp^2)^{1+\epsilon+\eta/2}} \frac{\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\eta/2)} \frac{2^{-\eta}\ \Gamma(\frac{1}{2}-\frac{\eta}{2})\Gamma(\frac{\eta}{2})}{\sqrt{\pi}}$$

Same as Eq. (5.61) of arxiv: 1202.0814

except for $C_F \to C_A$

$$\frac{2 g_s^2}{(2\pi)^3} = \frac{2 * 4\pi \alpha_s}{(2\pi)^3} = \frac{\alpha_s}{\pi^2}$$

- quark TMD

lowest order (LO)

$$f^{(0)}_{q/q}(x, \vec{k}_\perp^2) = \delta(1-x)\,\delta^2(\vec{k}_\perp)$$

Next-to-leading order (NLO)

$$f^{(1)}_{q/q}(x, \vec{k}_\perp^2) = \frac{\alpha_s}{2\pi^2} C_F\, \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon}$$

$$* \left[\frac{2x}{(1-x)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-x) - \epsilon(1-x) \right]$$

converted to Fourier space

$$f(x, \vec{b}) = \int d^2\vec{k}_\perp\, e^{-i\vec{k}_\perp\cdot\vec{b}}\, f(x, \vec{k}_\perp)$$

$$f(x, \vec{k}_\perp) = \int \frac{d^2\vec{b}}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot\vec{b}}\, f(x, \vec{b})$$

Now

$$f^{(0)}_{q/q}(x, b) = \int d^2\vec{k}_\perp\, e^{-i\vec{k}_\perp\cdot\vec{b}}\, \delta(1-x)\,\delta^2(\vec{k}_\perp) = \delta(1-x)$$

$$f^{(1)}_{q/q}(x, b) = \int d^2\vec{k}_\perp\, e^{-i\vec{k}_\perp\cdot\vec{b}}\, \frac{\alpha_s}{2\pi^2} C_F\, \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon}$$

$$* \left[\frac{2x}{(1-x)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-x) - \epsilon(1-x) \right]$$

integration formula

$$\int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{(\vec{k}_\perp^{\,2})^{1+\epsilon}} e^{-i\vec{k}_\perp \cdot \vec{b}} = \frac{1}{4\pi} \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2}{4}\right)^{\epsilon}$$

$$f_{q/q}^{(1)}(x,b) = \frac{\alpha_s}{2\pi^2} C_F \,\Gamma\cancel{(1+\epsilon)}\, e^{\epsilon\gamma_E} * \pi \frac{\Gamma(-\epsilon)}{\Gamma\cancel{(1+\epsilon)}} \left(\frac{b^2\mu^2}{4}\right)^{\epsilon}$$
$$* \left[\frac{2x}{(1-x)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-x) - \epsilon(1-x) \right]$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{\gamma_E}}{4}\right)^{\epsilon} \Gamma(-\epsilon) \left[\frac{2x}{(1-x)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^{\eta} + (1-x) - \epsilon(1-x) \right]$$

$$\frac{1}{(1-x)^{1+\eta}} = -\frac{1}{\eta}\delta(1-x) + \frac{1}{(1-x)_+}$$

$$\left(\frac{\nu}{\omega}\right)^{\eta} = 1 + \eta \ln\left(\frac{\nu}{\omega}\right)$$

Thus

$$f_{q/q}^{(1)}(x,b) = \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{\gamma_E}}{4}\right)^{\epsilon} \Gamma(-\epsilon) \left[\frac{2x}{(1-x)_+} - 2\ln\left(\frac{\nu}{\omega}\right)\delta(1-x) - \frac{2}{\eta}\delta(1-x) + (1-x) - \epsilon(1-x) \right]$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{\gamma_E}}{4}\right)^{\epsilon} \Gamma(-\epsilon) \left[\frac{1+x^2}{(1-x)_+} - \epsilon(1-x) - \frac{2}{\eta}\delta(1-x) - 2\ln\left(\frac{\nu}{\omega}\right)\delta(1-x) \right]$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{\epsilon} e^{-\gamma_E\epsilon}\, \Gamma(-\epsilon)$$
$$* \left[-\frac{2}{\eta}\delta(1-x) - \ln\left(\frac{\nu^2}{\omega^2}\right)\delta(1-x) + \frac{1+x^2}{(1-x)_+} - \epsilon(1-x) \right]$$

- Soft function

lowest order $\quad S^{(0)}(\vec{\lambda}_\perp) = \delta^2(\lambda_\perp) \quad \Rightarrow \quad S(b) = \int d^2\lambda_\perp \; e^{-i\vec{\lambda}_\perp\cdot\vec{b}} \, \delta^2(\lambda_\perp)$

$= 1$

NLO

$$S^{(1)}(\lambda_\perp) = \frac{\alpha_s}{2\pi^2} C_F \frac{e^{\gamma_E \epsilon} \mu^{2\epsilon} \nu^\eta}{(\vec{\lambda}_\perp^2)^{1+\epsilon+\eta/2}} \frac{\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} \frac{2^{1-\eta}\Gamma(\frac{1}{2}-\frac{\eta}{2})\Gamma(\frac{\eta}{2})}{\sqrt{\pi}}$$

$$S^{(1)}(b) = \int d^2\lambda_\perp \; e^{-i\vec{\lambda}_\perp\cdot\vec{b}} \; S^{(1)}(\lambda_\perp)$$

$$= \frac{\alpha_s}{2\pi^2} C_F \; e^{\gamma_E\epsilon} \mu^{2\epsilon} \nu^\eta * \pi \frac{\Gamma(1-1-\epsilon-\frac{\eta}{2})}{\Gamma(1+\epsilon+\frac{\eta}{2})} \left(\frac{b^2}{4}\right)^{1+\epsilon+\frac{\eta}{2}-1}$$

$$* \frac{\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} \frac{2^{1-\eta}\,\Gamma(\frac{1}{2}-\frac{\eta}{2})\Gamma(\frac{\eta}{2})}{\sqrt{\pi}}$$

$$= \frac{\alpha_s}{\pi} C_F \; 2^{-2\epsilon-\eta} \; e^{\gamma_E\epsilon} \mu^{2\epsilon} \nu^\eta \; b^{2\epsilon+\eta} \frac{\Gamma(-\epsilon-\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} \frac{2\Gamma(-\eta)\Gamma(\frac{\eta}{2})}{\Gamma(-\frac{\eta}{2})}$$

Perform expansion. first in $\eta \to 0$, then in $\epsilon \to 0$

$$S^{(1)}(b) = \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^\epsilon e^{-\gamma_E\epsilon}\,\Gamma(-\epsilon)$$

$$* \left[\frac{4}{\eta} - \frac{2}{\epsilon} + 2\ln\left(\frac{b^2\nu^2 e^{2\gamma_E}}{4}\right) + \frac{\pi^2}{3}\epsilon\right]$$

NOW

$$f_{q/q}(x,b) = \delta(1-x) \qquad \leftarrow f_{q/q}^{(0)}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{\epsilon} e^{-\gamma_E \epsilon} \Gamma(-\epsilon)$$

$$* \left[-\frac{2}{\eta}\delta(1-x) - \ln\left(\frac{\nu^2}{w^2}\right)\delta(1-x) + \frac{1+x^2}{(1-x)_+} - \epsilon(1-x) \right] \qquad \leftarrow f_{q/q}^{(1)}$$

$$S(b) = 1 \qquad \leftarrow S^{(0)}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{\epsilon} e^{-\gamma_E \epsilon} \Gamma(-\epsilon) \qquad \leftarrow S^{(1)}$$

$$* \left[\frac{4}{\eta} - \frac{2}{\epsilon} + 2\ln\left(\frac{b^2\nu^2 e^{2\gamma_E}}{4}\right) + \frac{\pi^2}{3}\epsilon \right]$$

Now the "correct" definition is

$$\sqrt{1 + S^{(1)}(b)} \approx 1 + \frac{1}{2} S^{(1)}(b)$$

$$S^{(0)}(b)$$

$$\hat{f}_{q/q}(x,b) = f_{q/q}(x,b) * \sqrt{S(b)}$$

$$= \left[f_{q/q}^{(0)} + f_{q/q}^{(1)} \right] * \left[S^{(0)}(b) + \frac{1}{2} S^{(1)}(b) \right]$$

$$= f_{q/q}^{(0)} S^{(0)}(b) + \frac{1}{2} f_{q/q}^{(0)} S^{(1)}(b) + f_{q/q}^{(1)} S^{(0)}(b) + O(\alpha_s^2)$$

$$= \delta(1-x)$$

$$+ \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{\epsilon} e^{-\gamma_E \epsilon} \Gamma(-\epsilon)$$

$$* \left\{ \left[\frac{2}{\eta} - \frac{1}{\epsilon} + \ln\left(\frac{b^2\nu^2 e^{2\gamma_E}}{4}\right) + \frac{\pi^2}{6}\epsilon \right] \delta(1-x) \right.$$

$$\left. + \left[-\frac{2}{\eta}\delta(1-x) - \ln\left(\frac{\nu^2}{w^2}\right)\delta(1-x) + \frac{1+x^2}{(1-x)_+} - \epsilon(1-x) \right] \right\}$$

$$\hat{f}_{q/q}(x,b) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{\epsilon} e^{-\gamma_E \epsilon}\, \Gamma(1-\epsilon)$$

$$* \left\{ \left[-\frac{1}{\epsilon} + \ln\left(\frac{b^2\omega^2 e^{2\gamma_E}}{4}\right) + \frac{\pi^2}{6}\epsilon \right] \delta(1-x) + \frac{1+x^2}{(1-x)_+} - \epsilon(1-x) \right\}$$

NOTE: No rapidity divergence any more

i.e. rapidity divergences cancel between $f_{q/q}(x,b)$ and $\sqrt{S(b)}$

further expansion, we obtain

$$\hat{f}_{q/q}(x,b) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right) + \epsilon\left(-\frac{1}{2}\ln^2\left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right) - \frac{\pi^2}{12}\right) \right]$$

$$* \left\{ \left[-\frac{1}{\epsilon} + \ln\left(\frac{b^2\omega^2 e^{2\gamma_E}}{4}\right) + \frac{\pi^2}{6}\epsilon \right] \delta(1-x) + \frac{1+x^2}{(1-x)_+} - \epsilon(1-x) \right\}$$

$$= \delta(1-x) + \frac{\alpha_s}{2\pi} C_F\, \delta(1-x) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\left(\frac{\mu^2}{\omega^2}\right) + \frac{3}{2}\right) \right]$$

$$+ \frac{\alpha_s}{2\pi} C_F \left\{ \left[-\frac{1}{\epsilon} - L_T \right] P_{qq}(x) + (1-x) - \delta(1-x)\left[-\frac{1}{2}L_T^2 - \frac{3}{2}L_T - L_T \ln\left(\frac{\mu^2}{\omega^2}\right) + \frac{\pi^2}{12} \right] \right\}$$

Finally

quark TMD $\qquad P_{qq}(x) = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)$

$$\hat{f}^{(0)}_{q/q}(x,b) = \delta(1-x)$$

$$\hat{f}^{(1)}_{q/q}(x,b) = \frac{\alpha_s}{2\pi} C_F \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\left(\frac{\mu^2}{\omega^2}\right) + \frac{3}{2} \right) \right] \delta(1-x) \quad \leftarrow \text{UV}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - L_T \right] P_{qq}(x) \quad \leftarrow \text{IR}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left\{ (1-x) - \delta(1-x) \left[-\frac{1}{2} L_T^2 - \frac{3}{2} L_T - L_T \ln\left(\frac{\mu^2}{\omega^2}\right) + \frac{\pi^2}{12} \right] \right\}$$

- There's NO rapidity divergence any more

- the natural scale from $L_T = \ln\left(\frac{b^2 \mu^2 e^{2\gamma_E}}{4} \right)$

define $\mu_b = 2e^{-\gamma_E}/b$

then $L_T = \ln\left(\mu^2/\mu_b^2 \right)$

See also the discussion in arXiv: 1111.4996

Since the term $\underbrace{\boxed{\ln\left(\frac{\mu^2}{\omega^2}\right) L_T}}_{\text{mixing}}$

It's instructive to realize the first line ⟹ UV divergence

Renormalization

$$\hat{f}_i^{\,bare}(x,b;\mu) = Z_i(\mu)\ \underset{\substack{\uparrow\\ \text{renormalized}}}{\hat{f}_i(x,b;\mu)}$$

NOTE "i" is not summed over !

$$\Rightarrow \qquad Z_q^{(0)} = 1$$

$$Z_q^{(1)} = \frac{\alpha_s}{2\pi} C_F \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\left(\frac{\mu^2}{w^2}\right) + \frac{3}{2} \right) \right]$$

$$\mu \frac{d}{d\mu} \hat{f}_i(x,b;\mu) = \gamma_i(\mu)\ \hat{f}_i(x,b;\mu)$$

$$\gamma_i(\mu) = - Z_i^{-1}\ \mu \frac{d}{d\mu} Z_i$$

$\mu \frac{d}{d\mu}\alpha_s = -2\epsilon\,\alpha_s + \cdots$

$$\mu \frac{d}{d\mu} Z_q^{(1)} = \frac{\alpha_s}{2\pi} C_F (-2\epsilon) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{\mu^2}{w^2} + \frac{3}{2} \right) \right]$$
$$+ \frac{\alpha_s}{2\pi} C_F \frac{1}{\epsilon}\ \mu \frac{d}{d\mu} \ln\left(\frac{\mu^2}{w^2}\right)$$

$$= \frac{\alpha_s}{2\pi} C_F \left[-2\ln\left(\frac{\mu^2}{w^2}\right) - 3 \right]$$

$$\gamma_q(\mu) = \frac{\alpha_s}{2\pi} C_F \left[2\ln\left(\frac{\mu^2}{w^2}\right) + 3 \right]$$

Thus $\quad \mu \frac{d}{d\mu} f_q(x,b;\mu) = \gamma_q(\mu)\ f_q(x,b;\mu)$

OR. $\quad \mu \frac{d}{d\mu} \ln f_q(x,b;\mu) = \gamma_q(\mu)$

After renormalization, we obtain the renormalized TMD

$$\hat{f}^{(1)}_{q/q}(x,b,\mu) = \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - L_T \right] P_{qq}(x)$$

$$+ \frac{\alpha_s}{2\pi} C_F \left\{ (1-x) - \delta(1-x) \left[-\frac{1}{2} L_T^2 - \frac{3}{2} L_T - L_T \ln\left(\frac{\mu^2}{\nu^2}\right) + \frac{\pi^2}{12} \right] \right\}$$

Notice: $-\frac{1}{\epsilon}$ here is IR pole

now can be matched onto collinear PDF

$$\hat{f}^{(0)}_{q/q}(x,\mu) = \delta(1-x)$$

$$\hat{f}^{(1)}_{q/q}(x,\mu) = \frac{\alpha_s}{2\pi} C_F \left(-\frac{1}{\epsilon}\right) P_{qq}(x)$$

$$f_i(x,b,\mu) = \int_x^1 \frac{dx'}{x'} C_{i\leftarrow j}\left(\frac{x}{x'}, b, \mu\right) f_j(x',\mu)$$

$$\Rightarrow \quad C^{(0)}_{i\leftarrow j} = \delta_{ij}$$

$$\Rightarrow \quad C^{(1)}_{i\leftarrow j} = -\frac{\alpha_s}{2\pi} C_F L_T P_{qq}(x)$$

$$+ \frac{\alpha_s}{2\pi} C_F \left\{ (1-x) - \delta(1-x) \left[-\frac{1}{2} L_T^2 - \frac{3}{2} L_T - L_T \ln\left(\frac{\mu^2}{\nu^2}\right) + \frac{\pi^2}{12} \right] \right\}$$

$$= \frac{\alpha_s}{2\pi} C_F \Big[-P_{qq}(x) L_T + (1-x)$$
$$- \delta(1-x) \left(-\frac{1}{2} L_T^2 - \frac{3}{2} L_T - L_T \ln\left(\frac{\mu^2}{\nu^2}\right) + \frac{\pi^2}{12} \right) \Big]$$

Soft function

$$S^{(0)}(b) = 1$$

$$S^{(1)}(b) = \frac{\alpha_s}{\pi} C_F \, 2^{-2\epsilon-\eta} \, e^{\epsilon\gamma_E} \mu^{2\epsilon} \nu^{\eta} \, b^{2\epsilon+\eta} \frac{\Gamma(-\epsilon-\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} \frac{2\Gamma(-\eta)\Gamma(\frac{\eta}{2})}{\Gamma(-\frac{\eta}{2})} * \underline{\underline{w^2}}$$

w is a book-keeping parameter that tracks the number of eikonal vertices, basically $\nu^{\eta} \Leftrightarrow w^2$ accompanying ∴ eventually take $w \to 1$

see Eq. (4.9) of 1202.0814

Thus

$$S^{(1)}(b) = w^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{\mu_b^2}\right)^{\epsilon} e^{-\gamma_E \epsilon} \Gamma(-\epsilon) \left[\frac{4}{\eta} - \frac{2}{\epsilon} + 2\ln\left(\frac{\nu^2}{\mu_b^2}\right) + \frac{\pi^2}{3}\epsilon\right]$$

where $\mu_b = 2e^{-\gamma_E}/b$

$$S^{(1)}(b) = w^2 \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right) + \epsilon\left(-\frac{1}{2}\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{12}\right)\right]$$

$$* \left[\frac{4}{\eta} - \frac{2}{\epsilon} + 2\ln\left(\frac{\nu^2}{\mu_b^2}\right) + \frac{\pi^2}{3}\epsilon\right]$$

$$= \frac{\alpha_s}{2\pi} C_F w^2 \left[\frac{4}{\eta}\left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right) + \frac{2}{\epsilon^2} + \left(-\frac{1}{\epsilon}\right) 2\ln\left(\frac{\nu^2}{\mu^2}\right)\right.$$

$$\left. - 2\ln\left(\frac{\mu^2}{\mu_b^2}\right)\ln\left(\frac{\nu^2}{\mu_b^2}\right) + \ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6}\right]$$

Renormalization

$$S_{bare}(b) = Z_S(b,\mu,\nu) \, S_{Ren}(b,\mu,\nu)$$

$$Z_s^{(0)}(b,\mu,\nu) = 1$$

$$Z_s^{(1)}(b,\mu,\nu) = \frac{\alpha_s}{2\pi} C_F\, w^2 \left[\frac{4}{\eta}\left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right) + \frac{2}{\epsilon^2} + \left(-\frac{1}{\epsilon}\right) 2\ln\left(\frac{\nu^2}{\mu^2}\right) \right]$$

Renormalization Group equation from

$$\frac{d}{d\ln\mu} S_{bare} = \frac{d}{d\ln\nu} S_{bare} = 0$$

Renormalized

$$\Rightarrow \quad \frac{d}{d\ln\mu} S(b,\mu,\nu) = \gamma_\mu^S\, S(b,\mu,\nu)$$

$$\frac{d}{d\ln\nu} S(b,\mu,\nu) = \gamma_\nu^S\, S(b,\mu,\nu)$$

where
$$\gamma_\mu^S = -Z_s^{-1} \frac{d}{d\ln\mu} Z_s$$

$$\gamma_\nu^S = -Z_s^{-1} \frac{d}{d\ln\nu} Z_s$$

Now we can easily obtain

- $$\gamma_\mu^S = -Z_s^{-1} \frac{d}{d\ln\mu} Z_s$$

$\mu\frac{\partial}{\partial\mu}\alpha_s = -2\epsilon\,\alpha_s$

$$= -1 * \left\{ \frac{\alpha_s}{2\pi} C_F (-2\epsilon) \left[\frac{4}{\eta}\left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right) + \frac{2}{\epsilon^2} + \left(-\frac{1}{\epsilon}\right) 2\ln\left(\frac{\nu^2}{\mu^2}\right)\right] \right.$$

$$\left. + \frac{\alpha_s}{2\pi} C_F \left[\frac{4}{\eta}(-2) + \left(-\frac{1}{\epsilon}\right) 2 * (-2) \right] \right\}$$

$$\Downarrow \epsilon \to 0$$

$$= -1 * \frac{\alpha_s}{2\pi} C_F * 4\ln\left(\frac{\nu^2}{\mu^2}\right)$$

$$= -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2}{\mu^2}\right)$$

- $$\gamma_\nu^S = -Z_S^{-1} \frac{d}{d\ln\nu} Z_S$$

$\Downarrow$ Note $\nu \frac{\partial}{\partial \nu} w = -\frac{\eta}{2} w$

$$= -1 * \left\{ \frac{\alpha_s}{2\pi} C_F w^2 (-\eta) \left[\frac{4}{\eta} \left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right) \right) + \frac{2}{\epsilon^2} + \left(-\frac{1}{\epsilon}\right) 2\ln\left(\frac{\nu^2}{\mu^2}\right) \right] \right.$$

$$\left. + \frac{\alpha_s}{2\pi} C_F w^2 \left[\left(-\frac{1}{\epsilon}\right) 4 \right] \right\}$$

$\Downarrow$ $\eta \to 0$ first

$$= -\frac{\alpha_s}{2\pi} C_F \; 4\ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

$$= -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

After the renormalization (subtract the counter term), we have

$$S(b,\mu,\nu) = 1 + \frac{\alpha_s}{2\pi} C_F \left[-2\ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu^2}{\mu_b^2}\right) + \ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6} \right]$$

follows

μ-RG equation: $\mu \frac{d}{d\mu} S(b,\mu,\nu) = \gamma_\mu^S \, S(b,\mu,\nu)$

ν-RG equation: $\nu \frac{d}{d\nu} S(b,\mu,\nu) = \gamma_\nu^S \, S(b,\mu,\nu)$

with $\gamma_\mu^S = -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2}{\mu^2}\right)$

$$\gamma_\nu^S = -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

independence of μ and ν leads to

$$\left[\frac{d}{d\ln\mu}, \frac{d}{d\ln\nu}\right] = 0$$

from which, we can have

$$\left[\frac{d}{d\ln\mu}, \frac{d}{d\ln\nu}\right] \ln Z_s = 0$$

$$\Rightarrow \frac{d}{d\ln\mu}\Big(\underbrace{\frac{d}{d\ln\nu}\ln Z_s}_{\gamma_\nu^s}\Big) - \frac{d}{d\ln\nu}\Big(\underbrace{\frac{d}{d\ln\mu}\ln Z_s}_{\gamma_\mu^s}\Big) = 0$$

$$\Rightarrow \frac{d}{d\ln\mu}\gamma_\nu^s = \frac{d}{d\ln\nu}\gamma_\mu^s$$

let's check the above equation using our one-loop result

$$\frac{d}{d\ln\mu}\gamma_\nu^s = \frac{d}{d\ln\mu}\left[-2\frac{\alpha_s}{\pi}C_F \ln\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$

$$= -2\frac{\alpha_s}{\pi}C_F * 2 = -4\frac{\alpha_s}{\pi}C_F$$

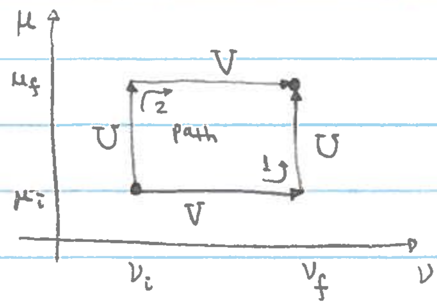
$$\frac{d}{d\ln\nu}\gamma_\mu^s = \frac{d}{d\ln\nu}\left[-2\frac{\alpha_s}{\pi}C_F \ln\left(\frac{\nu^2}{\mu^2}\right)\right]$$

$$= -2\frac{\alpha_s}{\pi}C_F * 2 = -4\frac{\alpha_s}{\pi}C_F$$

$\Rightarrow$ We verified

$$\boxed{\frac{d}{d\ln\mu}\gamma_\nu^s = \frac{d}{d\ln\nu}\gamma_\mu^s}$$

How to solve the evolution equation



There're two possible paths

(1) first evolve from ν_i to ν_f (at μ_i) : $V(\nu_f, \nu_i; \mu_i)$

then evolve from μ_i to μ_f (at ν_f) : $U(\mu_f, \mu_i; \nu_f)$

path 1 = $U(\mu_f, \mu_i; \nu_f)\, V(\nu_f, \nu_i; \mu_i)$

(2) first evolve from μ_i to μ_f (at ν_i) : $U(\mu_f, \mu_i; \nu_i)$

then evolve from ν_i to ν_f (at μ_f) : $V(\nu_f, \nu_i; \mu_f)$

path 2 = $V(\nu_f, \nu_i; \mu_f)\, U(\mu_f, \mu_i; \nu_i)$

from the consistent relation

$$\frac{d}{d\ln\mu} \gamma_V^S = \frac{d}{d\ln\nu} \gamma_\mu^S$$

$$\Rightarrow \quad \gamma_V^S = \int^{\ln\mu} d\ln\mu' \underbrace{\frac{d}{d\ln\nu} \gamma_\mu^S}_{-4\frac{\alpha_s}{\pi}C_F}(\mu') + \text{constant}$$

$\Downarrow$ compare with fixed-order $\gamma_V^S = -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_s^2}\right)$

if we choose lower integration limit to be μ_b

then constant $= 0$

$$= -4\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu}{\mu_b}\right) = -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

quark TMD

$$f^{(0)}_{q/q}(x, \vec{k}_\perp^2) = \delta(1-x)\,\delta^2(\vec{k}_\perp)$$

$$f^{(1)}_{q/q}(x, \vec{k}_\perp^2) = \frac{\alpha_s}{2\pi^2} C_F \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon} * \left[\frac{2x}{(1-x)^{1+\eta}}\left(\frac{\nu}{\omega}\right)^\eta + (1-x) - \epsilon(1-x)\right]$$

Note: for rapidity regulator, there's a book-keeping index "ω"

to be consistent with that, let's change the above proton

light-cone momentum "ω" → "Q"

book-keeping $\quad w^2 \left|\frac{\nu}{\bar{n}\cdot p_g}\right|^\eta$

Thus

$$f^{(1)}_{q/q}(x, \vec{k}_\perp^2) = \frac{\alpha_s}{2\pi^2} C_F \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon}$$

$$* \left[\frac{2x}{(1-x)^{1+\eta}}\, \underset{\uparrow}{w^2} \left(\frac{\nu}{Q}\right)^\eta + (1-x) - \epsilon(1-x)\right]$$

book-keeping

Note: $\boxed{\nu \frac{\partial}{\partial \nu} w = -\frac{\eta}{2} w}$

after obtain rapidity anomalous dimension and etc, we

take "$w \to 1$" to obtain the final result

$$f^{(1)}_{q/q}(x, b) = \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^\epsilon e^{-\epsilon\gamma_E}\,\Gamma(-\epsilon)$$

$$* \left[\frac{2x}{(1-x)^{1+\eta}}\, w^2 \left(\frac{\nu}{Q}\right)^\eta + (1-x) - \epsilon(1-x)\right]$$

$$f^{(1)}_{q/q}(x,b) = \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{\mu_b^2}\right)^{\epsilon} e^{-\epsilon\gamma_E} \Gamma(-\epsilon)\left[\frac{2x}{(1-x)^{1+\eta}}\left(\frac{\nu}{Q}\right)^{\eta} w^2 + (1-x) - \epsilon(1-x)\right]$$

$$= \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right]$$

$$* \left\{\left[\frac{2}{\eta}\delta(1-x) - 2\ln\left(\frac{\nu}{Q}\right)\delta(1-x) + \frac{2x}{(1-x)_+}\right] w^2 + (1-x) - \epsilon(1-x)\right\}$$

where $Q = p^- =$ proton energy

$w =$ book-keeping index

NOTE: since we did calculation in pure DR

we don't know/seperate UV and IR

However, $f^{(1)}_{q/q}(x,b)$ should also contain IR divergence,

such IR divergence should be exactly reproducing those

from standard collinear PDF $f^{(1)}_{q/q}(x,\mu)$

We have to isolate these IR, then the rest of $\frac{1}{\epsilon}$

pole will be related to UV, and thus renormalization !!

$$f^{(1)}_{q/q}(x,b) = \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right]$$

$$* \left\{ w^2\left[-\frac{2}{\eta}\delta(1-x) - 2\ln\left(\frac{\nu}{Q}\right)\delta(1-x)\right] + \frac{1+x^2}{(1-x)_+} - \epsilon(1-x)\right\}$$

Note: w is for book-keeping only, it's only relevant for

"$\frac{1}{\eta}$" divergent term as in " $\nu\frac{\partial}{\partial\nu}w = -\frac{\eta}{2}w$ "

for finite-terms, we can choose $w \to 1$

$$f^{(1)}_{q/q}(x,b) = \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right] * \left\{ w^2\left[-\frac{2}{\eta}\delta(1-x) - 2\ln\left(\frac{\nu}{Q}\right)\delta(1-x)\right] - \epsilon(1-x) \right\}$$

$$+ \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right] * \left[\underbrace{\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)}_{P_{qq}(x)} - \frac{3}{2}\delta(1-x)\right]$$

$$= \frac{\alpha_s}{2\pi} C_F \left\{ w^2 \frac{2}{\eta}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right]\delta(1-x) + \frac{2}{\epsilon}\ln\left(\frac{\nu}{Q}\right)\delta(1-x) + \frac{3}{2}\frac{1}{\epsilon}\delta(1-x) \right. \quad \leftarrow \text{UV}$$

$$+ 2\ln\left(\frac{\mu^2}{\mu_b^2}\right)\ln\left(\frac{\nu}{Q}\right)\delta(1-x) + (1-x) + \frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right)\delta(1-x) \quad \leftarrow \text{finite}$$

$$\left. + \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right] P_{qq}(x) \right\} \quad \leftarrow \text{IR}$$

Thus

$$f^{\text{bare}}_{q/p}(x,b) = Z_f(b,\mu,\nu)\, f^{\text{ren}}_{q/p}(x,b,\mu,\nu)$$

$$f^{(0)\,\text{bare}}_{q/q}(x,b) = \delta(1-x)$$

$$\Rightarrow \quad f^{(1)\,\text{bare}}_{q/q}(x,b) = Z_f^{(0)} f^{(1)}_{q/q} + Z_f^{(1)} f^{(0)}_{q/q}$$

$$\Rightarrow \quad \boxed{\begin{aligned} Z_f^{(0)} &= 1 \\ Z_f^{(1)} &= \frac{\alpha_s}{2\pi} C_F \left\{ w^2 \frac{2}{\eta}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right] + \frac{2}{\epsilon}\ln\left(\frac{\nu}{Q}\right) + \frac{3}{2}\frac{1}{\epsilon} \right\} \end{aligned}}$$

then anomalous dimension

- $$\gamma_\mu^f = -Z_f^{-1} \frac{d}{d\ln\mu} Z_f$$

$$= -1 * \left[\frac{\alpha_s}{2\pi} C_F (-2\epsilon) \left\{ \frac{2}{\eta}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{M_b^2}\right)\right) + \frac{2}{\epsilon}\ln\left(\frac{\nu}{Q}\right) + \frac{3}{2}\frac{1}{\epsilon} \right\} \right.$$

$$\left. + \frac{\alpha_s}{2\pi} C_F \frac{2}{\eta} * 2 \right]$$

$$\Downarrow \quad \epsilon \to 0$$

$$= \left\{ 4\ln\left(\frac{\nu}{Q}\right) + 3 \right\} \frac{\alpha_s}{2\pi} C_F$$

- $$\gamma_\nu^f = -Z_f^{-1} \frac{d}{d\ln\nu} Z_f$$

$$= -1 * \frac{\alpha_s}{2\pi} C_F \left\{ (-\eta w^2) \frac{2}{\eta} \left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{M_b^2}\right) \right] \right.$$

$$\left. + \frac{2}{\epsilon} * 1 \right\}$$

$$\Downarrow \quad w \to 1 \quad \eta \to 0$$

$$= \frac{\alpha_s}{2\pi} C_F * 2\ln\left(\frac{\mu^2}{M_b^2}\right)$$

$$= \frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{M_b^2}\right)$$

for a proton with light-cone momentum $p^- = p_0 + p_z$

μ-RG: $$\mu \frac{d}{d\mu} \ln f_{q/p}(x, b, \mu, \nu) = \gamma_\mu^f$$

ν-RG: $$\nu \frac{d}{d\nu} \ln f_{q/p}(x, b, \mu, \nu) = \gamma_\nu^f$$

$$\gamma_\mu^f = \left[4 \ln\left(\frac{\nu}{p^-}\right) + 3\right] \frac{\alpha_s}{2\pi} C_F$$

$$\gamma_\nu^f = \frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right)$$

Renormalized result

$$f_{q/q}^{(1)}(x, b, \mu, \nu) = \frac{\alpha_s}{2\pi} C_F \Big\{ \left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right) P_{qq}(x)$$
$$+ \left[2 \ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu}{p^-}\right) + \frac{3}{2} \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right] \delta(1-x)$$
$$+ (1-x) \Big\}$$

a couple of consistent check

renormalized

$$\hat{f}^{(1)}_{q/q}(x,b,\mu) = f_{q/q}(x,b,\mu,\nu)\sqrt{S(b,\mu,\nu)}$$

$$= f^{(0)}_{q/q}\sqrt{S^{(1)}} + f^{(1)}_{q/q}\sqrt{S^{(0)}}$$

$$\qquad\qquad\qquad\qquad\qquad\quad \nwarrow 1$$

$$= \delta(1-x)\,\frac{1}{2}\frac{\alpha_s}{2\pi}C_F\left[-2\ln\left(\frac{\mu^2}{\mu_b^2}\right)\ln\left(\frac{\nu^2}{\mu_b^2}\right)+\ln^2\left(\frac{\mu^2}{\mu_b^2}\right)-\frac{\pi^2}{6}\right]$$

$$+\frac{\alpha_s}{2\pi}C_F\left(-\frac{1}{\epsilon}-\ln\left(\frac{\mu^2}{\mu_b^2}\right)\right)P_{qq}(x)$$

$$+\frac{\alpha_s}{2\pi}C_F\left[2\ln\left(\frac{\mu^2}{\mu_b^2}\right)\ln\left(\frac{\nu}{p^-}\right)+\frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right)\right]\delta(1-x)$$

$$+\frac{\alpha_s}{2\pi}C_F\left[(1-x)\right]$$

$$= \frac{\alpha_s}{2\pi}C_F\left(-\frac{1}{\epsilon}-L_T\right)P_{qq}(x)$$

$$+\frac{\alpha_s}{2\pi}C_F\left\{(1-x)-\delta(1-x)\left[-\frac{1}{2}L_T^2-\frac{3}{2}L_T+L_T\ln\left(\frac{p^{-2}}{\mu_b^2}\right)+\frac{\pi^2}{12}\right.\right.$$

$L_T = \ln(\mu^2/\mu_b^2)$

through slightly different RG, the renormalized $\hat{f}_{q/q}$ is also different from the original one where we simply multiply un-renormalized $f_{q/q}$ and $\sqrt{S}$

one thing: Now at $\mu=\mu_b$, all large logs disappear!!

another check

$$\frac{d}{d\ln\mu}\gamma_\nu^f = \frac{d}{d\ln\nu}\gamma_\mu^f$$

$$LHS = \frac{d}{d\ln\mu}\left[\frac{\alpha_s}{\pi}C_F\ln\left(\frac{\mu^2}{\mu_0^2}\right)\right] = 2\frac{\alpha_s}{\pi}C_F$$

$$RHS = \frac{d}{d\ln\nu}\left[\frac{\alpha_s}{2\pi}C_F\left(4\ln\left(\frac{\nu}{p^-}\right)+3\right)\right] = \frac{\alpha_s}{2\pi}C_F * 4 = 2\frac{\alpha_s}{\pi}C_F$$

LHS = RHS

Q. E. D.

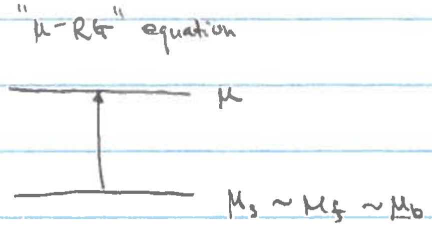
if like DY case, we have two quark TMDs, then since hard-function doesn't have rapidity divergence, we should then have

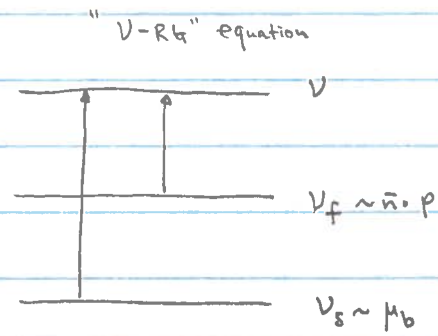
$$\gamma_\nu^{f_n} + \gamma_\nu^{f_{\bar{n}}} + \gamma_\nu^s = 0$$

$$\Rightarrow \quad \frac{\alpha_s}{\pi}C_F\ln\left(\frac{\mu^2}{\mu_0^2}\right) + \frac{\alpha_s}{\pi}C_F\ln\left(\frac{\mu^2}{\mu_0^2}\right) + \left[-2\frac{\alpha_s}{\pi}C_F\ln\left(\frac{\mu^2}{\mu_0^2}\right)\right] = 0$$

Consistent

the idea will be to evolve both of them from their "characteristic" scales up to a common scale





Thus the starting point of our evolution can be

$$\hat{f}^{\boxed{\text{sub}}}_{q/q}(x, b, \mu, \nu_0) = f_{q/q}(x, b, \mu, \nu_f)\, \sqrt{S(b, \nu_s)}$$

where we choose $\nu_f \sim \bar{n}\cdot p$

$\nu_s \sim \mu_b$

use the fixed-order result from Eqs. (1) & (2), we obtain

$$\begin{aligned}
\hat{f}^{\text{sub}}(x, b, \mu, \nu_0) = {} & \delta(1-x) \\
& + \frac{\alpha_s}{2\pi} C_F \left[-\ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu_s^2}{\mu_b^2}\right) + \frac{1}{2}\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{12} \right] \delta(1-x) \\
& + \frac{\alpha_s}{2\pi} C_F \Bigg\{ \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right) \right] P_{qq}(x) \\
& \qquad + (1-x) + \left[\ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu_f^2}{(\bar{n}\cdot p)^2}\right) + \frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right) \right] \delta(1-x) \Bigg\}
\end{aligned}$$

$\Downarrow$ choose $\nu_f = \bar{n}\cdot p \qquad \nu_s = \mu_b$

$$\begin{aligned}
= {} & \delta(1-x) + \frac{\alpha_s}{2\pi} C_F \Bigg\{ \delta(1-x)\left[\frac{1}{2}L_T^2 - \frac{\pi^2}{12} + \frac{3}{2}L_T \right] + (1-x) \\
& + \left[-\frac{1}{\epsilon} - L_T \right] P_{qq}(x) \Bigg\}
\end{aligned}$$

where $L_T = \ln\left(\frac{\mu^2}{\mu_b^2}\right)$

link to collinear PDFs : $f_{q/q}(x,\mu) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_F \left(-\frac{1}{\epsilon}\right) P_{qq}(x)$

thus $\tilde{f}^{sub}_{q/q}(x, b, \mu, \nu_0) = C_{q\leftarrow q} \otimes f_{q/q}(x,\mu)$

$$C_{q\leftarrow q}(x,b,\mu) = \frac{\alpha_s}{2\pi} C_F \Big\{ [-L_T] P_{qq}(x) + (1-x) - \left[-\frac{1}{2}L_T^2 - \frac{3}{2}L_T + \frac{\pi^2}{12}\right]\delta(1-x) \Big\}$$

Consistent with 1111.4996 & 1101.5057

Another interesting thing is that :

we have $\nu_f \sim \bar{n}\cdot p$ $\nu_s \sim \mu_b$ at this order to eliminate the large logarithm !

if we think $\nu_f \sim e^{y_f}$ rapidity for $f(x,b)$

$\nu_s \sim e^{y_s}$ ... for $S(b)$

then $\hat{f}^{sub}$ should have rapidity $\frac{\nu_f}{\nu_s} \sim e^{(y_f - y_s)}$

Similar to the new Collins convention, see 1509.04766

This is $\underline{\underline{\text{not}}}$ very relevant to us though !!

let's now solve the evolution equation !

Finally what is the evolution of TMD ?

whats the connection to the Standard CSS formalism ?

- fixed-order "renormalized" results

$$S(b,\mu,\nu) = 1 + \frac{\alpha_s}{2\pi} C_F \left[-2\ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu^2}{\mu_b^2}\right) + \ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6} \right] \quad (1)$$

$$\frac{d}{d\ln\mu} \ln S(b,\mu,\nu) = \gamma_\mu^S = -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2}{\mu_b^2}\right) \quad (1a)$$

$$\frac{d}{d\ln\nu} \ln S(b,\mu,\nu) = \gamma_\nu^S = -2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right) \quad (1b)$$

$$f_{q/q}(x,b,\mu,\nu) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_F \Big\{ \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right] P_{qq}(x) \quad (2)$$
$$+ (1-x) + \left[\ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu^2}{(\bar{n}\cdot p)^2}\right) + \frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right) \right] \delta(1-x) \Big\}$$

$$\frac{d}{d\ln\mu} \ln f_{q/q}(x,b,\mu,\nu) = \gamma_\mu^f = \frac{\alpha_s}{\pi} C_F \left[\ln\left(\frac{\nu^2}{(\bar{n}\cdot p)^2}\right) + \frac{3}{2} \right] \quad (2a)$$

$$\frac{d}{d\ln\nu} \ln f_{q/q}(x,b,\mu,\nu) = \gamma_\nu^f = \frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right) \quad (2b)$$

From the fixed-order result, we find the characteristic scales

$S(b,\mu,\nu) \Rightarrow \quad \mu_s \sim \mu_b \quad \nu_s \sim \mu_b$

$f(x,b,\mu,\nu) \Rightarrow \quad \mu_f \sim \mu_b \quad \nu_f \sim \bar{n}\cdot p$

In other words: μ-scale "S and f" the same

but they have different ν-scale !!

- from Eq. (2b)

$$\ln \frac{f_q(x,b,\mu_0,\nu)}{f_q(x,b,\mu_0,\nu_0)} = \ln\left(\frac{\nu}{\nu_0}\right) * \gamma_\nu^f(\mu_0)$$

from Eq. (2a)

$$\ln \frac{f_q(x,b,\mu,\nu)}{f_q(x,b,\mu_0,\nu)} = \int_{\mu_0}^{\mu} d\ln\mu' \ \frac{\alpha_s}{\pi} C_F\left[\ln\left(\frac{\nu^2}{(\bar{n}\cdot p)^2}\right) + \frac{3}{2}\right]$$

add together

$$\text{LHS} = \ln \frac{f_q(x,\cancel{b,\mu_0},\nu)}{f_q(x,b,\mu_0,\nu_0)} * \frac{f_q(x,b,\mu,\nu)}{f_q(\cancel{x},b,\mu_0,\nu)} = \ln \frac{f_q(x,b,\mu,\nu)}{f_q(x,b,\mu_0,\nu_0)}$$

$$\text{RHS} = \int_{\mu_0}^{\mu} d\ln\mu' \ \frac{\alpha_s}{\pi} C_F\left[\ln\left(\frac{\nu^2}{(\bar{n}\cdot p)^2}\right) + \frac{3}{2}\right] + \ln\left(\frac{\nu}{\nu_0}\right) * \gamma_\nu^f(\mu_0)$$

$$\Rightarrow \quad \frac{f_q(x,b,\mu,\nu)}{f_q(x,b,\mu_0,\nu_0)} = \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \ \frac{\alpha_s}{\pi} C_F\left(\ln\frac{\nu^2}{(\bar{n}\cdot p)^2} + \frac{3}{2}\right)\right] * \left(\frac{\nu}{\nu_0}\right)^{\gamma_\nu^f(\mu_0)} \qquad (3)$$

- Do the same thing for soft factor

from Eq. (1b)

$$\ln \frac{S(b,\mu_0,\nu)}{S(b,\mu_0,\nu_0)} = \ln\left(\frac{\nu}{\nu_0}\right) * \gamma_\nu^S(\mu_0)$$

from Eq. (1a)

$$\ln \frac{S(b,\mu,\nu)}{S(b,\mu_0,\nu)} = \int_{\mu_0}^{\mu} d\ln\mu' \ \left[-2\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2}{\mu'^2}\right)\right]$$

again add together

$$\sqrt{\frac{S(b,\mu,\nu)}{S(b,\mu_0,\nu_0)}} = \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \left(-\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2}{\mu'^2}\right)\right)\right] * \left(\frac{\nu}{\nu_0}\right)^{\frac{1}{2}\gamma_\nu^s(\mu_0)} \quad (4)$$

Now in Eq. (3), choose $\nu_0 = \bar{n}\cdot p$

in Eq. (4), choose $\nu_0 = \mu_b$

then we obtain

$$I = \frac{f_q(x,b,\mu,\nu)\sqrt{S(b,\mu,\nu)}}{f_q(x,b,\mu_0,\underset{\uparrow\, \bar{n}\cdot p}{\nu_f})\sqrt{S(b,\mu_0,\underset{\uparrow\, \mu_b}{\nu_s})}} \equiv \frac{\hat{f}_q^{\,sub}(x,b,\mu,\nu)}{\hat{f}_q^{\,sub}(x,b,\mu_0,\nu_0)}$$

$$= \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \frac{\alpha_s}{\pi} C_F \left(\ln\frac{\nu^2}{(\bar{n}\cdot p)^2} + \frac{3}{2} - \ln\left(\frac{\nu^2}{\mu'^2}\right)\right)\right]$$

$$* \left(\frac{\nu}{\underset{\uparrow\, \bar{n}\cdot p}{\nu_f}}\right)^{\gamma_\nu^f(\mu_0)} * \left(\frac{\nu}{\underset{\uparrow\, \mu_b}{\nu_s}}\right)^{\frac{1}{2}\gamma_\nu^s(\mu_0)}$$

Note: $\gamma_\nu^f(\mu_0) = -\frac{1}{2}\gamma_\nu^s(\mu_0) = \int_{\mu_b}^{\mu_0} d\ln\mu' \left(\frac{2\alpha_s}{\pi} C_F\right)$

use integral form
not fixed-order
see 1202.0814

$$\left(\frac{\nu}{\bar{n}\cdot p}\right)^{\gamma_\nu^f(\mu_0)} * \left(\frac{\nu}{\mu_b}\right)^{-\gamma_\nu^f(\mu_0)} = \left(\frac{\mu_b}{\bar{n}\cdot p}\right)^{\gamma_\nu^f(\mu_0)}$$

Finally

$$I = \frac{\tilde{f}_q^{sub}(x, b, \mu, \nu)}{\tilde{f}_q^{sub}(x, b, \mu_0, \nu_0)}$$

$$= \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \frac{\alpha_s}{\pi} C_F \left(\ln \frac{\mu'^2}{(\bar{n}\cdot p)^2} + \frac{3}{2} \right) \right] * \left(\frac{\mu_b}{\bar{n}\cdot p} \right)^{\gamma_\nu^f(\mu_0)}$$

$$= \exp\left[- \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left(A \ln \frac{(\bar{n}\cdot p)^2}{\mu'^2} + B \right) \right] * \left(\frac{(\bar{n}\cdot p)^2}{\mu_b^2} \right)^{-\frac{1}{2}\gamma_\nu^f(\mu_0)}$$

$$\frac{1}{2}\gamma_\nu^f(\mu_0) = \int_{\mu_b}^{\mu_0} \frac{d\mu'}{\mu'} \left(\frac{\alpha_s}{\pi} C_F \right)$$

we have $A = \frac{\alpha_s}{\pi} C_F$

$B = -\frac{\alpha_s}{\pi} \frac{3}{2} C_F$

This is exactly the usual Collins-Soper evolution formalism for TMD, in those formula we usually write

$$\bar{n}\cdot p = Q$$

Soft function

- standard one

$$S(\vec{\lambda}_\perp) = 8\pi g_s^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{1}{(2\pi)^{4-2\epsilon}} \left(\frac{1}{2}\right)^2 \nu^\eta * \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)} * (I)$$

where $I = \int dl^+ dl^- \, (l^+l^- - \vec{\lambda}_\perp^2)^{-1-\epsilon} \frac{1}{l^+l^-} |l^- - l^+|^{-\eta}$

- soft function for our case

we need to further require soft radiation should be happening only inside the jet cone!

$\Rightarrow$ ange between soft radation l and jet direction should be smaller than R

$\Rightarrow$ $\theta_{lJ} < R$

$\Rightarrow$ for the case where jet has only "z" momentum, No transverse component, we should then have

$$\Theta\left(\frac{l^+}{l^-} < \tan^2\frac{R}{2}\right)$$

Thus our soft function should be given by above, but

$I \rightarrow I_R$

$$I_R = \int dl^+ dl^- \, (l^+l^- - \vec{\lambda}_\perp^2)^{-1-\epsilon} \frac{1}{l^+l^-} |l^- - l^+|^{-\eta} \, \theta\left(\frac{l^+}{l^-} < \tan^2\frac{R}{2}\right)$$

also $l^+l^- > \vec{\lambda}_\perp^2$

It's a good idea to make variable exchange, we thus define

$$x = \ell^+ \ell^-$$
$$y = \frac{\ell^+}{\ell^-}$$

$$dx\,dy = \begin{vmatrix} \ell^- & \ell^+ \\ \frac{1}{\ell^-} & -\frac{\ell^+}{\ell^{-2}} \end{vmatrix} d\ell^+ d\ell^-$$

$$= 2\frac{\ell^+}{\ell^-} d\ell^+ d\ell^-$$
$$= 2y\, d\ell^+ d\ell^-$$

$$\Rightarrow \quad d\ell^+ d\ell^- = \left(\frac{1}{2y}\right) dx\,dy$$

$$\ell^- = \left(\frac{x}{y}\right)^{1/2}$$

$$I_R = \int dx\,dy\, \frac{1}{2y} \left(x - \vec{\lambda}_\perp^2\right)^{-1-\epsilon} \frac{1}{x} |1-y|^{-\eta} \left(\frac{x}{y}\right)^{-\eta/2} \theta\left(y < \tan^2\frac{R}{2}\right)$$

NOTE: $\infty > x > \vec{\lambda}_\perp^2 \qquad 0 < y < \tan^2\frac{R}{2}$

$$I_R = \underbrace{\int_{\vec{\lambda}_\perp^2}^{\infty} dx\, (x - \vec{\lambda}_\perp^2)^{-1-\epsilon}\, x^{-1-\eta/2}}_{I_{Rx}} * \frac{1}{2} \underbrace{\int_0^{\tan^2\frac{R}{2}} dy\, y^{-1+\eta/2} |1-y|^{-\eta}}_{I_{Ry}}$$

$$= \frac{1}{2} I_{Rx} * I_{Ry}$$

$$I_{Rx} = \int_{\vec{\lambda}_\perp^2}^{\infty} dx \quad (x - \vec{\lambda}_\perp^2)^{-1-\epsilon} \quad x^{-1-\eta/2}$$

$$\text{define} \quad x = \frac{\vec{\lambda}_\perp^2}{\alpha} \qquad \frac{dx}{x} = \frac{d\alpha}{\alpha}$$

$$= \int_0^1 \frac{d\alpha}{\alpha} \quad \left(\frac{\vec{\lambda}_\perp^2}{\alpha} - \vec{\lambda}_\perp^2 \right)^{-1-\epsilon} \quad \left(\frac{\vec{\lambda}_\perp^2}{\alpha} \right)^{-\eta/2}$$

$$= (\vec{\lambda}_\perp^2)^{-1-\epsilon-\eta/2} \int_0^1 d\alpha \quad (1-\alpha)^{-1-\epsilon} \quad \alpha^{\epsilon+\eta/2}$$

$$= (\vec{\lambda}_\perp^2)^{-1-\epsilon-\eta/2} \quad \frac{\Gamma(-\epsilon)\,\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})}$$

$$I_{Ry} = \int_0^{\tan^2\frac{R}{2}} dy \quad y^{-1+\eta/2} \quad |1-y|^{-\eta}$$

generally $R<1$, we consider small jet radius $R \ll 1$

then $\tan^2\frac{R}{2} < 1$ thus integration limit $\in [0, 1)$

$1-y > 0 \qquad y > 0$

$$I_{Ry} = \int_0^{\tan^2\frac{R}{2}} dy \quad y^{-1+\eta/2} \quad (1-y)^{-\eta}$$

$$= \frac{2}{\eta} \left(\tan\frac{R}{2}\right)^{\eta} \quad \underbrace{{}_2F_1\left(\frac{\eta}{2}, \eta; 1+\frac{\eta}{2}; \tan^2\frac{R}{2}\right)}_{\text{expand}}$$

$$= 1 + \frac{\eta^2}{2} \mathrm{Li}_2\left(\tan^2\frac{R}{2}\right) + O(\eta^3)$$

$$\Downarrow \quad \text{for } x \ll 1 \qquad \mathrm{Li}_2(x) = x + \frac{x^2}{4} + \cdots$$

$$= 1 + \frac{\eta^2}{2}\left(\tan^2\frac{R}{2} + \cdots\right) + O(\eta^3)$$

$$= 1 + O(\eta^2 R^2)$$

$$= \frac{2}{\eta} \left(\tan\frac{R}{2}\right)^{\eta} \quad \left[1 + O(R^2)\right]$$

Thus

$$I_R = \frac{1}{2}\underbrace{\int_{\vec{\lambda}_\perp^2}^{\infty} dx\, (x-\vec{\lambda}_\perp^2)^{-1-\epsilon}\, x^{-1-\eta/2}}_{I_{Rx}} * \underbrace{\int_0^{\tan^2\frac{R}{2}} dy\;\; y^{-1+\frac{\eta}{2}}\, |1-y|^{-\eta}}_{I_{Ry}}$$

$$= \frac{1}{2} * \frac{1}{(\vec{\lambda}_\perp^2)^{1+\epsilon+\frac{\eta}{2}}} \frac{\Gamma(-\epsilon)\,\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} * \frac{2}{\eta}\left(\tan\frac{R}{2}\right)^{\eta} \left[1+O(R^2)\right]$$

Then we have

$$S(\vec{\lambda}_\perp, R) = 8\pi g_s^2\, C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{1}{(2\pi)^{4-2\epsilon}} \left(\frac{1}{2}\right)^2 \nu^{\eta} * \frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}$$

$$* \frac{1}{2} \frac{1}{(\vec{\lambda}_\perp^2)^{1+\epsilon+\frac{\eta}{2}}} \frac{\Gamma(-\epsilon)\,\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} * \frac{2}{\eta}\left(\tan\frac{R}{2}\right)^{\eta} \left[1+O(R^2)\right]$$

$$= \frac{\alpha_s}{2\pi^2} C_F\, e^{\gamma_E \epsilon} \frac{\Gamma(1+\epsilon+\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})} \frac{1}{\mu^2}\left[\frac{\mu^2}{\vec{\lambda}_\perp^2}\right]^{1+\epsilon+\frac{\eta}{2}} \frac{2}{\eta}\left(\frac{\nu \tan\frac{R}{2}}{\mu}\right)^{\eta} \left[1+O(R^2)\right]$$

Fourier transform to b-space

$$\int \frac{d^2\vec{k}_\perp}{(2\pi)^2}\, e^{-i\vec{k}_\perp\cdot\vec{b}}\, \frac{1}{(\vec{k}_\perp^{\,2})^{1+\alpha}} = \frac{1}{4\pi}\,\frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}\left(\frac{\vec{b}^2}{4}\right)^{\alpha}$$

Thus $$\int \frac{d^2\vec{k}_\perp}{(2\pi)^2}\, e^{-i\vec{k}_\perp\cdot\vec{b}}\, \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^{\,2}}\right)^{1+\alpha} = \frac{1}{4\pi}\,\frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}\left(\frac{b^2\mu^2}{4}\right)^{\alpha}$$

$$S(b,R) = \int d^2\lambda_\perp\, e^{-i\vec{\lambda}_\perp\cdot\vec{b}}\, S(\lambda_\perp, R)$$

$$= \frac{\alpha_s}{2\pi^2} C_F\, e^{\gamma_E \epsilon}\, \frac{\cancel{\Gamma(1+\epsilon+\frac{\eta}{2})}}{\Gamma(1+\frac{\eta}{2})} * \pi\, \frac{\Gamma(-\epsilon-\frac{\eta}{2})}{\cancel{\Gamma(1+\epsilon+\frac{\eta}{2})}}\left(\frac{b^2\mu^2}{4}\right)^{\epsilon+\frac{\eta}{2}}$$

$$* \frac{2}{\eta}\left(\frac{\nu\tan\frac{R}{2}}{\mu}\right)^{\eta}$$

$$= \frac{\alpha_s}{2\pi} C_F\, e^{\gamma_E\epsilon}\, \frac{\Gamma(-\epsilon-\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})}\left(\frac{b^2\mu^2}{4}\right)^{\epsilon+\frac{\eta}{2}} \frac{2}{\eta}\left(\frac{\nu\tan\frac{R}{2}}{\mu}\right)^{\eta}$$

Now perform the expansion, first $\eta\to 0$, then $\epsilon\to 0$

$$I = \frac{\Gamma(-\epsilon-\frac{\eta}{2})}{\Gamma(1+\frac{\eta}{2})}\left(\frac{b^2\mu^2}{4} * \frac{\nu^2\tan^2\frac{R}{2}}{\mu^2}\right)^{\eta/2} * \frac{1}{\eta}$$

$\Downarrow$ expand at $\eta \to 0$

$$S(b,R) = \frac{\alpha_s}{2\pi} C_F * \left[\frac{1}{\epsilon}\left(-\frac{2}{\eta} + \ln\frac{\mu^2}{\nu^2\tan^2\frac{R}{2}}\right) - \ln\left(\frac{\nu^2\tan^2\frac{R}{2}}{\mu_b^2}\right)\ln\left(\frac{\mu^2}{\mu_b^2}\right)\right.$$

$$\left. - \frac{2}{\eta}\ln\left(\frac{\mu^2}{\mu_b^2}\right) + \frac{1}{2}\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) + \frac{1}{\epsilon^2} - \frac{\pi^2}{12}\right]$$

$$= \frac{\alpha_s}{2\pi} C_F \left[\frac{2}{\eta}\left(-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\right) + \frac{1}{\epsilon^2} + \left(-\frac{1}{\epsilon}\right)\ln\left(\frac{\nu^2\tan^2\frac{R}{2}}{\mu^2}\right)\right.$$

$$\left. - \ln\left(\frac{\mu^2}{\mu_b^2}\right)\ln\left(\frac{\nu^2\tan^2\frac{R}{2}}{\mu_b^2}\right) + \frac{1}{2}\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{12}\right]$$

Summary :

• quark TMD

$$f_{q/q}(x,b) = \delta(1-x)$$

book-keeping for rapidity divergence

$$+ \frac{\alpha_s}{2\pi} C_F \Big\{ w^2 \frac{2}{\eta} \Big[\frac{1}{\epsilon} + \ln\Big(\frac{\mu^2}{\mu_b^2}\Big) \Big] \delta(1-x) + \frac{2}{\epsilon} \ln\Big(\frac{\nu}{p^-}\Big) \delta(1-x) + \frac{3}{2}\frac{1}{\epsilon}\delta(1-x)$$

$$+ \Big[-\frac{1}{\epsilon} - \ln\Big(\frac{\mu^2}{\mu_b^2}\Big) \Big] P_{qq}(x)$$

$$+ 2\ln\Big(\frac{\mu^2}{\mu_b^2}\Big) \ln\Big(\frac{\nu}{p^-}\Big) \delta(1-x) + \frac{3}{2}\ln\Big(\frac{\mu^2}{\mu_b^2}\Big) \delta(1-x) + (1-x) \Big\}$$

• Soft function

$$S(b,R) = 1 + \frac{\alpha_s}{2\pi} C_F w^2 \Big[\frac{2}{\eta}\Big(-\frac{1}{\epsilon} - \ln\Big(\frac{\mu^2}{\mu_b^2}\Big)\Big) + \frac{2}{\epsilon^2} + \Big(-\frac{1}{\epsilon}\Big) \ln\Big(\frac{\nu^2 \tan^2\frac{R}{2}}{\mu^2}\Big)$$

$$- \ln\Big(\frac{\mu^2}{\mu_b^2}\Big) \ln\Big(\frac{\nu^2 \tan^2\frac{R}{2}}{\mu_b^2}\Big) + \frac{1}{2}\ln^2\Big(\frac{\mu^2}{\mu_b^2}\Big) - \frac{\pi^2}{12} \Big]$$

• only quark is inside

$\vec{k}_\perp$ is the relative momentum of hadron w.r.t. jet

$$G_{q,\mathrm{bare}}^{q}(z, \omega_J, z_h, \vec{k}_\perp) = \delta(1-z_h)\,\delta^2(\vec{k}_\perp)$$
$$\quad * \frac{\alpha_s}{2\pi} \frac{(e^{\gamma_E}\mu^2)^\epsilon}{\Gamma(1-\epsilon)} P_{qq}(z,\epsilon) \int_{[(1-z)\omega_J \tan\frac{R}{2}]^2}^{\infty} \frac{dq_\perp^2}{(q_\perp^2)^{1+\epsilon}}$$

Note since we have only one parton inside the jet, this parton will become the hadron, at partonic level, there's no relative transverse momentum at this order, i.e., $\vec{k}_\perp = 0$

This is why we have $\delta^2(\vec{k}_\perp)$ factor

on the other hand, $\vec{q}_\perp$ is the partonic transverse momentum of the radiated gluon with respect to the parent quark

We have to integrate over as long as gluon is outside the jet cone, this is why we have

$$\int_{[(1-z)\omega_J \tan\frac{R}{2}]^2}^{\infty} dq_\perp^2 \quad \cdots$$

$$G_{q,\mathrm{bare}}^{q}(z, \omega_J, z_h, \vec{k}_\perp) = \delta(1-z_h)\,\delta^2(\vec{k}_\perp)$$
$$\quad * \frac{\alpha_s}{2\pi} \frac{(e^{\gamma_E}\mu^2)^\epsilon}{\Gamma(1-\epsilon)} \left(\frac{1}{\epsilon}\right) \left(\omega_J^2 \tan^2\frac{R}{2}\right)^{-\epsilon}$$
$$\quad * P_{qq}(z,\epsilon)\,(1-z)^{-2\epsilon}$$

using $$P_{qq}(z,\epsilon)(1-z)^{-2\epsilon} = C_F\left[-\frac{1}{\epsilon}\delta(1-z) + \frac{1+z^2}{(1-z)_+} - \epsilon z(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \epsilon(1-z)\right]$$

$$\frac{(\mu^2 e^{\gamma_E})^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon}\right)\left(\omega_J^2 \tan^2\frac{R}{2}\right)^{-\epsilon} = \frac{1}{\epsilon} + L + \epsilon\left(\frac{L^2}{2} - \frac{\pi^2}{12}\right)$$

where $L = \ln\left(\frac{\mu^2}{\omega_J^2 \tan^2\frac{R}{2}}\right)$

$$G^q_{q,\text{bare}}(z,\omega_J,z_h,\vec{k}_\perp) \overset{\text{"}q(g)\text{"}}{=} \delta(1-z_h)\,\delta^2(\vec{k}_\perp)\,\frac{\alpha_s}{2\pi}C_F$$
$$* \left[\frac{1}{\epsilon} + L + \epsilon\left(\frac{L^2}{2} - \frac{\pi^2}{12}\right)\right]$$
$$* \left[-\frac{1}{\epsilon}\delta(1-z) + \frac{1+z^2}{(1-z)_+} - \epsilon z(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \epsilon(1-z)\right]$$

$$= \delta(1-z_h)\,\delta^2(\vec{k}_\perp)\,\frac{\alpha_s}{2\pi}C_F$$
$$* \left[-\frac{1}{\epsilon^2}\delta(1-z) + \frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+} - \frac{1}{\epsilon}L\,\delta(1-z) - \left(\frac{L^2}{2} - \frac{\pi^2}{12}\right)\delta(1-z)\right.$$
$$\left. + L\frac{1+z^2}{(1-z)_+} - z(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - (1-z)\right]$$

if we Fourier transform to b-space, we have

$$G^{\text{bare}}_{q\to q(g)}(z,\omega_J,z_h,b) = \int d^2k_\perp\, e^{-i\vec{k}_\perp\cdot\vec{b}}\, G^{\text{bare}}_{q\to q(g)}(z,\omega_J,z_h,k_\perp)$$
$$\Downarrow \quad \int d^2\vec{k}_\perp\, e^{-i\vec{k}_\perp\cdot\vec{b}}\,\delta^2(\vec{k}_\perp) = 1$$
$$= \delta(1-z_h)\,\frac{\alpha_s}{2\pi}C_F * \left[-\frac{1}{\epsilon^2}\delta(1-z) + \frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+} - \frac{1}{\epsilon}\delta(1-z)L\right.$$
$$\left. - \left(\frac{L^2}{2} - \frac{\pi^2}{12}\right)\delta(1-z) + L\frac{1+z^2}{(1-z)_+} - z(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - (1-z)\right]$$

$$= \delta(1-z_h)\,\frac{\alpha_s}{2\pi}C_F * \left[-\frac{1}{\epsilon^2}\delta(1-z) - \frac{3}{2}\frac{1}{\epsilon}\delta(1-z) - \frac{1}{\epsilon}\delta(1-z)L\right.$$
$$+ \left(\frac{1}{\epsilon} + L\right)P_{qq}(z)$$
$$\left. - \left(\frac{L^2}{2} + \frac{3}{2}L - \frac{\pi^2}{12}\right)\delta(1-z) - z(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - (1-z)\right]$$

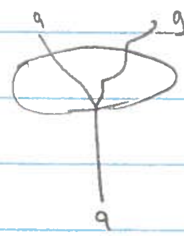
- Now study both q and g inside the jet cone

$$G^{bare}_{q\to qg}(z, \omega_J, z_h, k_\perp) = \delta(1-z)\, \frac{\alpha_s}{2\pi} \frac{(e^{\gamma_E}\mu^2)^\epsilon}{\Gamma(1-\epsilon)} P_{qq}(z_h, \epsilon)$$

$$* \int \frac{d^2 q_\perp}{(q_\perp^2)^{1+\epsilon}} \frac{1}{\pi} \, \Theta\left(q_\perp < z_h(1-z_h)\,\omega_J \tan\frac{R}{2}\right)$$

used $d^2 q_\perp = \pi\, dq_\perp^2$

$$* \; \delta^2(\vec{q}_\perp - \vec{k}_\perp)$$



in this case, if $q \to h$, then $k_\perp$ is the transverse momentum with respect to the jet direction, which is exactly the same as $\vec{q}_\perp$

Thus we have

$$G^{bare}_{q\to qg}(z, \omega_J, z_h, k_\perp) = \delta(1-z)\, \frac{\alpha_s}{2\pi^2} \frac{(\mu^2 e^{\gamma_E})^\epsilon}{\Gamma(1-\epsilon)} P_{qq}(z_h, \epsilon) * \frac{1}{(\vec{k}_\perp^2)^{1+\epsilon}}$$

we assume $|k_\perp| < z_h(1-z_h)\,\omega_J \tan\frac{R}{2}$

since we're interested in small $|\vec{k}_\perp| \lesssim \omega_J R$ region, this is exactly well take

$$P_{qq}(z_h, \epsilon) = C_F\left[\frac{1+z_h^2}{1-z_h} - \epsilon(1-z_h)\right]$$

Apparently the above expression contains a "rapidity" divergence at $z_h \to 1$

We should use the same method to regularize the rapidity divergence, just like for quark TMD $D_q^q(z_h, k_\perp)$

$$G^{bare}_{q\to qg}(z, w_J, z_h, k_\perp) = \delta(1-z) \underbrace{\frac{\alpha_s}{2\pi^2} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} P_{qq}(z_h, \epsilon) \frac{1}{\mu^2}}_{\text{just like a quark TMD}} \left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon}$$

Note $\int \frac{d^2\vec{k}_\perp}{(2\pi)^2} e^{-i\vec{k}_\perp\cdot\vec{b}} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon} = \frac{1}{4\pi}\frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)}\left(\frac{b^2\mu^2}{4}\right)^{\epsilon}$

thus there will be a "$\frac{1}{\epsilon}$" pole from there (at most No $\frac{1}{\epsilon^2}$ pole)

In other words, we have

$$\boxed{G^{bare}_{q\to qg}(z, w_J, z_h, k_\perp) = \delta(1-z)\, D^q_{q,bare}(z_h, k_\perp)}$$

Thus the rapidity-regularized expression should be the same!

$$G^{bare}_{q\to qg}(z, w_J, z_h, k_\perp) = \delta(1-z) \frac{\alpha_s}{2\pi^2} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{k}_\perp^2}\right)^{1+\epsilon}$$

$$* \left[\frac{2z_h}{(1-z_h)^{1+\eta}}\left(\frac{\nu}{p^-}\right)^{\eta} + (1-z_h) - \epsilon(1-z_h)\right]$$

What should be p^- here?

p^- should be the parent quark light-cone energy

for our case, this is basically / exactly the jet energy ω_J

so we have

$$G^{bare}_{q\to qg}(z, \omega_J, z_h, k_\perp) = \delta(1-z) \frac{\alpha_s}{2\pi^2} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \frac{1}{\mu^2} \left(\frac{\mu^2}{k_\perp^2}\right)^{1+\epsilon}$$

$$* \left[\frac{2 z_h}{(1-z_h)^{1+\eta}} \left(\frac{\nu}{\omega_J}\right)^\eta + (1-z_h) - \epsilon(1-z_h)\right]$$

Fourier transform into b-space

$$G^{bare}_{q\to qg}(z, \omega_J, z_h, b) = \int d^2\vec{k}_\perp \, e^{-i\vec{k}_\perp \cdot \vec{b}} \, G^{bare}_{q\to qg}(z, \omega_J, z_h, \vec{k}_\perp)$$

$$= \delta(1-z) \frac{\alpha_s}{2\pi^2} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} * \pi \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2\mu^2}{4}\right)^\epsilon$$

$$* \left[\frac{2 z_h}{(1-z_h)^{1+\eta}} \left(\frac{\nu}{\omega_J}\right)^\eta + (1-z_h) - \epsilon(1-z_h)\right]$$

$$= \delta(1-z) \frac{\alpha_s}{2\pi} C_F \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2\mu^2}{4}\right)^\epsilon$$

$$* \left[\frac{2 z_h}{(1-z_h)^{1+\eta}} \left(\frac{\nu}{\omega_J}\right)^\eta + (1-z_h) - \epsilon(1-z_h)\right]$$

Then

$$G^{bare}_{q\to qg}(z, \omega_J, z_h, b) = \frac{\alpha_s}{2\pi} C_F \Big\{ \frac{2}{\eta}\Big[\frac{1}{\epsilon} + \ln\Big(\frac{\mu^2}{\mu_b^2}\Big)\Big] \delta(1-z_h) + \frac{2}{\epsilon}\ln\Big(\frac{\nu}{\omega_J}\Big)\delta(1-z_h) + \frac{3}{2}\frac{1}{\epsilon}\delta(1-z_h)$$

$$+\Big[-\frac{1}{\epsilon} - \ln\Big(\frac{\mu^2}{\mu_b^2}\Big)\Big] P_{qq}(z_h)$$

$$+ 2\ln\Big(\frac{\mu^2}{\mu_b^2}\Big)\ln\Big(\frac{\nu}{\omega_J}\Big)\delta(1-z_h) + (1-z_h) + \frac{3}{2}\ln\Big(\frac{\mu^2}{\mu_b^2}\Big)\delta(1-z_h)\Big\}$$

$$* \, \delta(1-z)$$

on the other hand

$$\mathcal{J}^{bare}_{q\to q(g)}(z, \omega_J, z_h, b) = \frac{\alpha_s}{2\pi} C_F \Big[-\frac{1}{\epsilon^2}\delta(1-z) - \frac{3}{2}\frac{1}{\epsilon}\delta(1-z) - \frac{1}{\epsilon}\delta(1-z)\ln\Big(\frac{\mu^2}{\omega_J^2\tan^2 R/2}\Big)$$

$$+\Big(\frac{1}{\epsilon} + \ln\Big(\frac{\mu^2}{\omega_J^2\tan^2 R/2}\Big)\Big) P_{qq}(z)$$

$$-\Big(\frac{L^2}{2} + \frac{3}{2}L - \frac{\pi^2}{12}\Big)\delta(1-z) - 2(1+z^2)\Big(\frac{\ln(1-z)}{1-z}\Big)_+ - (1-z)\Big]$$

$$* \, \delta(1-z_h)$$

define $\quad L = \ln\Big(\frac{\mu^2}{\omega_J^2\tan^2 R/2}\Big) \qquad L_b = \ln\Big(\frac{\mu^2}{\mu_b^2}\Big) \quad$ with $\mu_b = 2e^{-\gamma_E}/b$

soft function

$$S(b, R) = 1 + \frac{\alpha_s}{2\pi} C_F \Big[\frac{2}{\eta}\Big(-\frac{1}{\epsilon} - L_b\Big) + \frac{2}{\epsilon^2} + \Big(-\frac{1}{\epsilon}\Big)\ln\Big(\frac{\nu^2\tan^2 R/2}{\mu^2}\Big)$$

$$- L_b \ln\Big(\frac{\nu^2\tan^2 R/2}{\mu_b^2}\Big) + \frac{1}{2}L_b^2 - \frac{\pi^2}{12}\Big]$$

- normal soft function

$$S(b) = 1 + \frac{\alpha_s}{2\pi} C_F \Big[\frac{4}{\eta}\Big(-\frac{1}{\epsilon} - L_b\Big) + \frac{2}{\epsilon^2} - \frac{1}{\epsilon} 2L_{\nu\mu}$$

$$- 2 L_b L_{\nu\mu} + L_b^2 - \frac{\pi^2}{6} \Big]$$

where $L_b = \ln\Big(\frac{\mu^2}{\mu_b^2}\Big)$ $\qquad \mu_b = \frac{2e^{-\gamma_E}}{b}$

$$L_{\nu\mu} = \ln\Big(\frac{\nu^2}{\mu^2}\Big)$$

- Now soft function with jet algorithm constraint

$$S(b, R) = 1 + \frac{\alpha_s}{2\pi} C_F \Big[\frac{2}{\eta}\Big(-\frac{1}{\epsilon} - L_b\Big) + \frac{1}{\epsilon^2} - \frac{1}{\epsilon} L_{\nu\mu}^R$$

$$- L_b L_{\nu\mu}^R + \frac{1}{2} L_b^2 - \frac{\pi^2}{12} \Big]$$

where $$L_{\nu\mu}^R = \ln\Big(\frac{\nu^2 \tan^2\frac{R}{2}}{\mu^2} \Big)$$

- Renormalization

$$S_{bare}(b, R) = Z_S(b, \mu, \nu)\, S_{Ren}(b, R, \mu, \nu)$$

$$Z_S^{(0)}(b, \mu, \nu) = 1$$

$$Z_S^{(1)}(b, \mu, \nu) = \frac{\alpha_s}{2\pi} C_F\, w^2 \Big[\frac{2}{\eta}\Big(-\frac{1}{\epsilon} - L_b\Big) + \frac{1}{\epsilon^2} - \frac{1}{\epsilon} L_{\nu\mu}^R \Big]$$

Here "w" is a book-keeping parameter

Renormalization group equations from

$$\frac{d}{d\ln\mu} S_{bare} = \frac{d}{d\ln\nu} S_{bare} = 0$$

$$\Rightarrow \quad \frac{d}{d\ln\mu} S(b, R, \mu, \nu) = \gamma_\mu^S S(b, R, \mu, \nu)$$

$$\frac{d}{d\ln\nu} S(b, R, \mu, \nu) = \gamma_\nu^S S(b, R, \mu, \nu)$$

where

$$\gamma_\mu^S = - Z_S^{-1} \frac{d}{d\ln\mu} Z_S$$

$$\gamma_\nu^S = - Z_S^{-1} \frac{d}{d\ln\nu} Z_S$$

$$\Rightarrow \quad \gamma_\mu^S = - Z_S^{-1} \frac{d}{d\ln\mu} Z_S \qquad \mu \frac{d}{d\mu} \alpha_s = -2\epsilon \alpha_s + \cdots$$

$$= -1 * \left\{ \frac{\alpha_s}{2\pi} C_F (-2\epsilon) \left[\frac{2}{\eta}\left(-\frac{1}{\epsilon} - L_b\right) + \frac{1}{\epsilon^2} - \frac{1}{\epsilon} L_{\nu\mu}^R \right] + \frac{\alpha_s}{2\pi} C_F \left[\frac{2}{\eta}(-2) + \frac{1}{\epsilon} * 2 \right] \right\}$$

$$= - \frac{\alpha_s}{2\pi} C_F * 2 L_{\nu\mu}^R$$

$$\boxed{\gamma_\mu^S = - \frac{\alpha_s}{\pi} C_F L_{\nu\mu}^R}$$

$$\Rightarrow \quad \gamma_\nu^S = - Z_S^{-1} \frac{d}{d\ln\nu} Z_S \qquad \nu \frac{\partial}{\partial\nu} w = -\frac{\eta}{2} w$$

$$= -1 * \left\{ \frac{\alpha_s}{2\pi} C_F w^2 (-\eta) \left[\frac{2}{\eta}\left(-\frac{1}{\epsilon} - L_b\right) + \frac{1}{\epsilon^2} - \frac{1}{\epsilon} L_{\nu\mu}^R \right] + \frac{\alpha_s}{2\pi} C_F w^2 \left[-\frac{1}{\epsilon} * 2 \right] \right\}$$

$$\Downarrow \quad w \to 1 \qquad \eta \to 0 \text{ first}$$

$$= - \frac{\alpha_s}{2\pi} C_F 2 L_b$$

$$\boxed{\gamma_\nu^S = - \frac{\alpha_s}{\pi} C_F L_b}$$

check the consistency

$$\frac{d}{d\ln\mu} \gamma_\nu^s = \frac{d}{d\ln\nu} \gamma_\mu^s$$

$$\text{LHS} = \frac{d}{d\ln\mu}\left[-\frac{\alpha_s}{\pi} C_F L_b\right] = -\frac{\alpha_s}{\pi} C_F * 2$$

$$\text{RHS} = \frac{d}{d\ln\nu}\left[-\frac{\alpha_s}{\pi} C_F L_{\nu\mu}^R\right] = -\frac{\alpha_s}{\pi} C_F * 2$$

Q.E.D.

- solving RG evolution equation

$$\frac{d}{d\ln\mu} \ln S(b,R,\mu,\nu) = \gamma_\mu^S = -\frac{\alpha_s}{\pi} C_F L_{\nu\mu}^R = -\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2 \tan^2\frac{R}{2}}{\mu^2}\right) \quad (1)$$

$$\frac{d}{d\ln\nu} \ln S(b,R,\mu,\nu) = \gamma_\nu^S = -\frac{\alpha_s}{\pi} C_F L_b = -\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2}\right) \quad (2)$$

from Eq.(2), one obtains

$$\ln \frac{S(b,R,\mu_0,\nu)}{S(b,R,\mu_0,\nu_0)} = \int_{\nu_0}^{\nu} d\ln\nu' \, \gamma_\nu^s = \ln\left(\frac{\nu}{\nu_0}\right) * \gamma_\nu^S(\mu_0)$$

from Eq.(1), we have

$$\ln \frac{S(b,R,\mu,\nu)}{S(b,R,\mu_0,\nu)} = \int_{\mu_0}^{\mu} d\ln\mu' \, \gamma_\mu^S = \int_{\mu_0}^{\mu} d\ln\mu' \left[-\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2 \tan^2\frac{R}{2}}{\mu'^2}\right)\right]$$

add the above two equations together, we'll have

$$\ln \frac{S(b,R,\mu,\nu)}{S(b,R,\mu_0,\nu_0)} = \int_{\mu_0}^{\mu} d\ln\mu' \left[-\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2 \tan^2\frac{R}{2}}{\mu'^2}\right)\right] + \ln\left(\frac{\nu}{\nu_0}\right)^{\gamma_\nu^S(\mu_0)}$$

$$\frac{S(b,R,\mu,\nu)}{S(b,R,\mu_0,\nu_0)} = \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \left(-\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2\tan^2\frac{R}{2}}{\mu'^2}\right)\right)\right] * \left(\frac{\nu}{\nu_0}\right)^{\gamma_\nu^S(\mu_0)}$$

where $\gamma_\nu^S(\mu_0) = -\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu_0^2}{\mu_b^2}\right)$

we might still choose $\mu_0 = \mu_b$ at this order, then we'll have

$$\left.\frac{S(b,R,\mu,\nu)}{S(b,R,\mu_0,\nu_0)}\right|_{\mu_0=\mu_b} = \exp\left[\int_{\mu_b}^{\mu} d\ln\mu' \left(-\frac{\alpha_s}{\pi} C_F \ln\left(\frac{\nu^2\tan^2\frac{R}{2}}{\mu'^2}\right)\right)\right]$$

The usual TMD is still running the same

In other words, we have

$$\frac{f_q(x,b,\mu,\nu)}{f_q(x,b,\mu_0,\nu_0)} = \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \, \frac{\alpha_s}{\pi} C_F \left(\ln\frac{\nu^2}{(\bar{n}\cdot p)^2} + \frac{3}{2}\right)\right] * \left(\frac{\nu}{\nu_0}\right)^{\gamma_\nu^f(\mu_0)}$$

combine, we obtain

$$I = \frac{f_q(x,b,\mu,\nu)\, S(b,R,\mu,\nu)}{f_q(x,b,\mu_0,\nu_0^f)\, S(b,R,\mu_0,\nu_0^s)} = \frac{\hat{f}_q^{\,sub}(x,b,\mu,\nu)}{\hat{f}_q^{\,sub}(x,b,\mu_0,\nu_0)}$$

$$= \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \, \frac{\alpha_s}{\pi} C_F \left(\ln\frac{\nu^2}{(\bar{n}\cdot p)^2} + \frac{3}{2} - \ln\frac{\nu^2 \tan^2\frac{R}{2}}{\mu'^2}\right)\right]$$

$$* \left(\frac{\nu}{\nu_0^f}\right)^{\gamma_\nu^f(\mu_0)} * \left(\frac{\nu}{\nu_0^s}\right)^{\gamma_\nu^s(\mu_0)}$$

$$= \exp\left[\int_{\mu_0}^{\mu} d\ln\mu' \, \frac{\alpha_s}{\pi} C_F \left(\ln\frac{\mu'^2}{(\bar{n}\cdot p)^2 \tan^2\frac{R}{2}} + \frac{3}{2}\right)\right]$$

$$* \left(\frac{\nu}{\nu_0^f}\right)^{\gamma_\nu^f(\mu_0)} \left(\frac{\nu}{\nu_0^s}\right)^{\gamma_\nu^s(\mu_0)}$$

$$\gamma_\nu^f(\mu_0) = -\gamma_\nu^s(\mu_0) = \int_{\mu_b}^{\mu_0} d\ln\mu' \left(\frac{2\alpha_s}{\pi} C_F\right)$$

if we choose initial rapidity scale

$$\nu_0^f = \bar{n}\cdot p$$

$$\nu_0^s = \left(\frac{\mu_b}{\tan\frac{R}{2}}\right)$$

$$\left(\frac{\nu}{\nu_0^f}\right)^{\gamma_\nu^f(\mu_0)} \left(\frac{\nu}{\nu_0^s}\right)^{\gamma_\nu^s(\mu_0)}$$

$$= \left(\frac{\nu_0^s}{\nu} * \frac{\nu}{\nu_0^f}\right)^{\gamma_\nu^f(\mu_0)} \qquad \text{since } \gamma_\nu^f = -\gamma_\nu^s$$

$$= \left(\frac{\nu_0^s}{\nu_0^f}\right)^{\int_{\mu_b}^{\mu_0} d\ln\mu' \left(2\frac{\alpha_s}{\pi} C_F\right)}$$

$$= \left(\frac{\mu_b/\tan\frac{R}{2}}{\bar{n}\cdot p}\right)^{\int_{\mu_b}^{\mu_0} d\ln\mu' \left(2\frac{\alpha_s}{\pi} C_F\right)}$$

$$= \left[\frac{(\bar{n}\cdot p\tan\frac{R}{2})^2}{\mu_b^2}\right]^{-\int_{\mu_b}^{\mu_0} d\ln\mu' \left(\frac{\alpha_s}{\pi} C_F\right)}$$

Thus eventally

$$I = \exp\left[-\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\left(A\ln\frac{(\bar{n}\cdot p\tan\frac{R}{2})^2}{\mu'^2} + B\right)\right] * \left[\frac{(\bar{n}\cdot p\tan\frac{R}{2})^2}{\mu_b^2}\right]^{-\int_{\mu_b}^{\mu_0}\frac{d\mu'}{\mu'}\left(\frac{\alpha_s}{\pi}C_F\right)}$$

$$A = \frac{\alpha_s}{\pi} C_F \qquad B = -\frac{\alpha_s}{\pi}\frac{3}{2} C_F$$

one can choose $\mu_0 = \mu_b$ then the second part $= 1$

$$I = \exp\left[-\int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'}\left(A\ln\frac{(\bar{n}\cdot p\tan\frac{R}{2})^2}{\mu'^2} + B\right)\right]$$

evolution from μ_b to μ

this suggests that we should evolve the "TMD" from its natural scale μ_b up to the characteristic jet scale $\omega_J \tan\frac{R}{2}$

$$f(x, b, R; \mu) = f(x, b, R; \mu_b) \times \exp\left[- \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left(A \ln\left(\frac{(\bar{n} \cdot p \tan\frac{R}{2})^2}{\mu'^2} \right) + B \right) \right]$$

$$\hat{C}_{q' \leftarrow q}(z, \mu_b) = \delta_{qq'} \left[\delta(1-z) + \frac{\alpha_s}{\pi} \left(\frac{C_F}{2}(1-z) + P_{q\leftarrow q}(z) \ln z \right) \right]$$

$$\hat{C}_{g \leftarrow q}(z, \mu_b) = \frac{\alpha_s}{\pi} \left(\frac{C_F}{2} z + P_{g\leftarrow q}(z) \ln z \right)$$

where $\quad P_{q\leftarrow q}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right]$

$$P_{g\leftarrow q}(z) = C_F \frac{1+(1-z)^2}{z}$$

$$D_{h/q}(z, p_\perp^2; Q) = \frac{1}{z^2} \int_0^\infty \frac{db\, b}{2\pi} J_0(p_\perp b/z)\, \hat{C}_{i\leftarrow q} \otimes D_{h/i}(z, \mu_b)$$
$$* \exp\left[-\frac{1}{2} S_{pert}(Q, b_*) - S_{NP}(Q, b) \right]$$

where $\quad S_{pert}(Q, b) = \int_{\mu_b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A \ln \frac{Q^2}{\mu^2} + B \right]$

with $\quad A = \sum_{n=1} A^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n$

$$B = \sum_{n=1} B^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n$$

$$A^{(1)} = C_F$$

$$A^{(2)} = \frac{C_F}{2} \left[C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{10}{9} T_R n_f \right]$$

$$B^{(1)} = -\frac{3}{2} C_F$$

$$S_{NP}(Q, b) = \frac{g_2}{2} \ln\left(\frac{b}{b_*}\right) \ln\left(\frac{Q}{Q_0}\right) + \frac{g_h}{z^2} b^2 .$$

TMD fragmentation function

- $$D^{(1)}_{q/q}(z, \vec{p}_\perp) = \frac{\alpha_s}{2\pi^2} C_F \, \Gamma(1+\epsilon) \, e^{\epsilon\gamma_E} \, \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{p}_\perp^{\,2}}\right)^{1+\epsilon} * \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^\eta + (1-z) - \epsilon(1-z) \right]$$

$$D^{(0)}_{q/q}(z, \vec{p}_\perp) = \delta(1-z)\, \delta^2(\vec{p}_\perp)$$

Now study them in Fourier-transformed b-space

$$D_{q/q}(z, b) = \frac{1}{z^2} \int d^2\vec{p}_\perp \; e^{-i\vec{p}_\perp \cdot \vec{b}/z} \; D_{q/q}(z, \vec{p}_\perp)$$

$$D^{(0)}_{q/q}(z,b) = \frac{1}{z^2} \int d^2\vec{p}_\perp \; e^{-i\vec{p}_\perp\cdot\vec{b}/z} \; \delta(1-z)\, \delta^2(\vec{p}_\perp)$$

$$= \frac{1}{z^2}\, \delta(1-z)$$

$$D^{(1)}_{q/q}(z,b) = \frac{1}{z^2} \frac{\alpha_s}{2\pi^2} C_F \, \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \; \frac{1}{\mu^2} \int d^2\vec{p}_\perp \, e^{-i\vec{p}_\perp\cdot\vec{b}/z} \left(\frac{\mu^2}{\vec{p}_\perp^{\,2}}\right)^{1+\epsilon} * \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^\eta + (1-\epsilon)(1-z) \right]$$

using $$\int \frac{d^2k_\perp}{(2\pi)^2} \, e^{-i\vec{k}_\perp\cdot\vec{b}} \, \frac{1}{(\vec{k}_\perp^{\,2})^{1+\epsilon}} = \frac{1}{4\pi} \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2}{4}\right)^\epsilon$$

$$\Rightarrow D^{(1)}_{q/q}(z,b) = \frac{1}{z^2} \frac{\alpha_s}{2\pi^2} C_F \, \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \; \frac{1}{\mu^2} * \pi * \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2}{4z^2}\right)^\epsilon * (\mu^2)^{1+\epsilon} * [\ldots]$$

$$= \frac{1}{z^2} \frac{\alpha_s}{2\pi} C_F \, \Gamma(1+\epsilon)\, e^{\epsilon\gamma_E} \; \frac{\Gamma(-\epsilon)}{\Gamma(1+\epsilon)} \left(\frac{b^2\mu^2}{4z^2}\right)^\epsilon * [\ldots]$$

$$= \frac{1}{z^2} \frac{\alpha_s}{2\pi} C_F \left(\frac{b^2\mu^2 e^{\gamma_E}}{4z^2}\right)^\epsilon \Gamma(-\epsilon) * \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{\omega}\right)^\eta + (1-\epsilon)(1-z) \right]$$

define $\mu_b = 2e^{-\gamma_E}/b$

then $D^{(1)}_{q/q}(z, b) = \frac{1}{z^2} \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{\mu_b^2 z^2}\right)^{\epsilon} e^{-\gamma_E \epsilon} \Gamma(-\epsilon)$

$$* \left[\frac{2z}{(1-z)^{1+\eta}} \left(\frac{\nu}{p^-}\right)^{\eta} + (1-\epsilon)(1-z) \right]$$

only difference is that $\left(\frac{1}{z^2}\right)$ as an overall normalization,

and "$\mu_b \to \mu_b z$" (compared with PDF side)

expand we have

$$z^2 * D^{(1)}_{q/q}(z, b) = \frac{\alpha_s}{2\pi} C_F \left[\frac{2}{\eta}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right) + \frac{1}{\epsilon}\left(2\ln\left(\frac{\nu}{p^-}\right) + \frac{3}{2}\right)\right] \delta(1-z) \quad \swarrow \text{UV}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right] P_{qq}(z) \quad \leftarrow \text{IR}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left\{ \left[2\ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right) \ln\left(\frac{\nu}{p^-}\right) + \frac{3}{2} \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right] \delta(1-z) \quad \swarrow \text{finite}$$

$$+ (1-z) \right\}$$

$$D^{\text{bare}}_{q/q}(z, b) = Z_f(b, \mu, \nu) \, D^{\text{ren}}_{q/q}(z, b, \mu, \nu)$$

$$D^{(0)\,\text{bare}}_{q/q} = Z_f^{(0)} \, D^{(0)}_{q/q}$$

$$D^{(1)\,\text{bare}}_{q/q} = \underbrace{Z_f^{(0)}}_{1} D^{(1)}_{q/q} + Z_f^{(1)} D^{(0)}_{q/q}$$

$$Z_f^{(1)} = \frac{\alpha_s}{2\pi} C_F \left\{ \frac{2}{\eta}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right) + \frac{1}{\epsilon}\left(2\ln\left(\frac{\nu}{p^-}\right) + \frac{3}{2}\right)\right\}$$

$$\gamma_\mu^D = -Z_f^{-1} \frac{d}{d\ln\mu} Z_f = \frac{\alpha_s}{2\pi} C_F \left[4\ln\left(\frac{\nu}{p^-}\right) + 3\right]$$

$$\gamma_\nu^D = -Z_f^{-1} \frac{d}{d\ln\nu} Z_f = \frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)$$

Renormalized result

$$D_{q/q}^{(1)}(z, b, \mu, \nu) = \frac{1}{z^2}\left\{ \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right] P_{qq}(z) \right.$$
$$+ \frac{\alpha_s}{2\pi} C_F \left[\left(2\ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\ln\left(\frac{\nu}{p^-}\right) + \frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right)\delta(1-z)\right.$$
$$\left.\left. + (1-z)\right]\right\}$$

$$D_{q/q}^{(0)}(z, b, \mu, \nu) = \frac{1}{z^2}\delta(1-z)$$

$$\gamma_\mu^D = \frac{\alpha_s}{\pi} C_F \left[2\ln\left(\frac{\nu}{p^-}\right) + \frac{3}{2}\right]$$

$$\gamma_\nu^D = \frac{\alpha_s}{\pi} C_F \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)$$

- Soft function

$$S(b,\mu,\nu) = 1 + \frac{\alpha_s}{2\pi} C_F \left[-2\ln\left(\frac{\mu^2}{\mu_b^2}\right) \ln\left(\frac{\nu^2}{\mu_b^2}\right) + \ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6} \right]$$

Now study its running

natural scale for

$S(b,\mu,\nu)$ $\quad \mu_S \sim \mu_b$ $\quad \nu_S \sim \mu_b$

$D(z,b,\mu,\nu)$ $\quad \mu_D \sim \mu_b$ $\quad \nu_D \sim p^-$

choose the natural scale for both S and D

Note. $\quad D_{q/q}^{(1)}(z) = \frac{\alpha_s}{2\pi} C_F \left(-\frac{1}{\epsilon}\right) P_{qq}(z)$

$\quad D_{q/q}^{(0)}(z) = \delta(1-z)$

- Now we have at $\mu = \mu_b$ and their natural "ν" scales

$$D_{q/q}^{(0)}(z,b) = \frac{1}{z^2}\delta(1-z)$$

$$D_{q/q}^{(1)}(z,b) = \frac{1}{z^2}\left\{ \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} + 2\ln z\right] P_{qq}(z) - \frac{\alpha_s}{2\pi} C_F * \frac{3}{2} * \underbrace{2\ln z\ \delta(1-z)}_{\substack{\ln z \to 0 \\ \text{when } z\to 1}} + \frac{\alpha_s}{2\pi} C_F (1-z) \right\}$$

$$= \frac{1}{z^2}\left\{ \frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} + 2\ln z\right] P_{qq}(z) + \frac{\alpha_s}{2\pi} C_F (1-z) \right\}$$

$$= \frac{1}{z^2} \frac{\alpha_s}{2\pi} C_F \left\{ \left(-\frac{1}{\epsilon} + 2\ln z\right) P_{qq}(z) + (1-z) \right\}$$

$$D_{q/q}(z,b) = \frac{1}{z^2}\int_z^1 \frac{dz'}{z'}\, C_{q/q}(z',b)\, D_{q/q}\left(\frac{z}{z'}\right)$$

- LO

$$D^{(0)}_{q/q}\left(\frac{z}{z'}\right) = \delta\left(1-\frac{z}{z'}\right)$$

$$D^{(0)}_{q/q}(z,b) = \frac{1}{z^2}\delta(1-z) = \delta(1-z)$$

$$\Rightarrow \quad C_{q/q}(z',b) = \delta(1-z')$$

- NLO

$$D^{(1)}_{q/q}(z,b) = \frac{1}{z^2}\left[C^{(0)}_{q/q} \otimes D^{(1)}_{q/q} + C^{(1)}_{q/q} \otimes D^{(0)}_{q/q}\right]$$

$$D^{(1)}_{q/q}(z,b) = \frac{1}{z^2} D^{(1)}_{q/q}(z) + \frac{1}{z^2}\int_z^1 \frac{dz'}{z'}\, C^{(1)}_{q/q}(z')\,\delta\left(1-\frac{z}{z'}\right)$$

$$= \frac{1}{z^2} D^{(1)}_{q/q}(z) + \frac{1}{z^2} C^{(1)}_{q/q}(z)$$

$$\Rightarrow \quad D^{(1)}_{q/q}(z,b) * z^2 - D^{(1)}_{q/q}(z) = C^{(1)}_{q/q}(z)$$

$$\Rightarrow \quad C^{(1)}_{q/q}(z) = \frac{\alpha_s}{2\pi} C_F \left\{ \left[-\frac{1}{\epsilon} + 2\ln z\right] P_{qq}(z) + (1-z) \right\} - \frac{\alpha_s}{2\pi} C_F \left(-\frac{1}{\epsilon}\right) P_{qq}(z)$$

$$= \frac{\alpha_s}{2\pi} C_F \left[2\ln z\, P_{qq}(z) + (1-z) \right]$$

$$= \frac{\alpha_s}{\pi}\left[\frac{C_F}{2}(1-z) + C_F \ln z\, P_{qq}(z)\right]$$

here $\quad P_{qq}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$

- if we choose an arbitrary μ-scale but at their "ν" scale

$$D^{com}(z,b) = D(z,b) * \sqrt{S(b)} \Rightarrow$$

$$D^{com(0)}(z,b) = \delta(1-z)$$

$$D^{com(1)}(z,b) = \frac{1}{z^2}\left\{\frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right] P_{qq}(z) + \frac{\alpha_s}{2\pi} C_F \left[\frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\delta(1-z) + (1-z)\right]\right\}$$

$$+ \delta(1-z)\frac{\alpha_s}{2\pi} C_F \left[\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6}\right] * \frac{1}{2}$$

$$= \delta(1-z)\frac{\alpha_s}{2\pi} C_F \left[\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6}\right] * \frac{1}{2}$$

$$+ \frac{1}{z^2}\frac{\alpha_s}{2\pi} C_F \left[-\frac{1}{\epsilon} - \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right] P_{qq}(z)$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[\frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right)\delta(1-z) + (1-z)\right]\frac{1}{z^2}$$

$$D^{(1)}(z) = \frac{\alpha_s}{2\pi} C_F \left(-\frac{1}{\epsilon}\right) P_{qq}(z)$$

$$z^2 * D^{com(1)}(z,b) = \underbrace{C^{(0)}_{q/q} \otimes D^{(1)}}_{D^{(1)}_{q/q}(z)} + \underbrace{C^{(1)}_{q/q} \otimes D^{(0)}}_{C^{(1)}_{q/q}(z)}$$

$$\Rightarrow C^{(1)}_{q/q}(z) = \delta(1-z)\frac{\alpha_s}{2\pi} C_F \left[\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{6}\right] * \frac{1}{2}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[-\ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right)\right] P_{qq}(z)$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[\frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right)\delta(1-z) + (1-z)\right]$$

$$C^{(1)}_{q/q}(z,\mu) = \frac{\alpha_s}{2\pi} C_F \left\{ \left[\frac{1}{2}\ln^2\left(\frac{\mu^2}{\mu_b^2}\right) + \frac{3}{2}\ln\left(\frac{\mu^2}{\mu_b^2}\right) - \frac{\pi^2}{12}\right] \delta(1-z) + [(1-z)] - \ln\left(\frac{\mu^2}{\mu_b^2 z^2}\right) \underbrace{P_{qq}(z)}_{\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)} \right\}$$

$$C^{(1)}_{q/q}(z,\mu_b) = \frac{\alpha_s}{2\pi} C_F \left\{ \left(-\frac{\pi^2}{12}\right) \delta(1-z) + (1-z) + 2 P_{qq}(z) \ln(z) \right\}$$

in the standard $\overline{MS}$.

but for collins-11, one does not have this term!

Thus this term will be in the <u>hard-part function</u>

Thus our result is consistent now!

one simply needs the inversion, which is

$$D_{q/q}(z, p_\perp) = \int \frac{d^2b}{(2\pi)^2} e^{i\vec{p}_\perp \cdot \vec{b}/z} D_{q/q}(z, b)$$

for hadron level, we have

$$D_{h/q}(z, p_\perp) = \int \frac{d^2b}{(2\pi)^2} e^{i\vec{p}_\perp \cdot \vec{b}/z} D_{h/q}(z, b)$$

Similar for

$$D_{h/g}(z, p_\perp) = \int \frac{d^2b}{(2\pi)^2} e^{i\vec{p}_\perp \cdot \vec{b}/z} D_{h/g}(z, b)$$

- definition and convention for TMD fragmentation function

$$D_{h/i}(z, b; Q) = \frac{1}{z^2}\int d^2\vec{p}_\perp \; e^{-i\vec{p}_\perp\cdot\vec{b}/z} \; D_{h/i}(z, p_\perp; Q)$$

$$D_{h/i}(z, p_\perp; Q) = \int \frac{d^2\vec{b}}{(2\pi)^2} \; e^{i\vec{p}_\perp\cdot\vec{b}/z} \; D_{h/i}(z, b; Q)$$

$$= \int \frac{b\,db\,d\phi}{(2\pi)^2} \; e^{i\vec{p}_\perp\cdot\vec{b}/z} \; D_{h/i}(z, b; Q)$$

$$\Downarrow \qquad J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\phi}\, d\phi$$

$$= \int \frac{b\,db}{2\pi} \; J_0(p_\perp b/z) \; D_{h/i}(z, b; Q)$$

At the same time

in perturbative region, i.e. when $\mu_b = \frac{c}{b} \gg \Lambda_{QCD}$ with $c = 2e^{-\gamma_E}$, we have

$$D_{h/i}(z, b; Q) = D_{h/i}(z, b; \mu_b) * \exp\Big[-\underbrace{\int_{\mu_b}^{Q} \frac{d\mu}{\mu}\Big(A\ln\frac{Q^2}{\mu^2} + B\Big)}_{S_{pert}(b,\,Q)}\Big]$$

define $\quad S_{pert}(b, Q) = \int_{\mu_b}^{Q} \frac{d\mu}{\mu}\left[A\ln\frac{Q^2}{\mu^2} + B\right]$

note, since S_{pert} depends on whether it's a gluon or quark TMD we might put such an index in, we can write

$\Rightarrow$ when $\mu_b = \frac{c}{b} \gg \Lambda_{QCD}$, we have

$$D_{h/i}(z, b; Q) = D_{h/i}(z, b; \mu_b)\, \exp\left[-S^i_{pert}(b, Q)\right]$$

where $\quad S^i_{pert}(b, Q) = \int_{\mu_b}^{Q} \frac{d\mu}{\mu}\left[A_i \ln\left(\frac{Q^2}{\mu^2}\right) + B_i\right]$

Here we have $\qquad A_i = \sum_{n=1} A_i^{(n)}\left(\frac{\alpha_s}{\pi}\right)^n \qquad B_i = \sum_{n=1} B_i^{(n)}\left(\frac{\alpha_s}{\pi}\right)^n$

$$A_q^{(1)} = C_F \qquad A_q^{(2)} = \frac{C_F}{2}\left[C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9}T_R n_f\right]$$

$$B_q^{(1)} = -\frac{3}{2}C_F$$

$A_g^{(1)} = C_A \qquad A_g^{(2)} = \frac{C_A}{2}\left[C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9} T_R n_f \right]$

in other words, $\quad A_g = \frac{C_A}{C_F} A_q$

$B_g^{(1)} = -\frac{\beta_0}{2} \qquad$ with $\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_R n_f$

At the same time $D_{h/i}(z, b; \mu_b)$ can be expanded in terms of the collinear fragmentation function as follows

$$D_{h/i}(z, b; \mu_b) = \frac{1}{z^2} \int_z^1 \frac{dz'}{z'} C_{j\leftarrow i}(z', b) D_{h/j}\left(\frac{z}{z'}, \mu_b\right)$$

The coefficient functions are given by in standard $\overline{MS}$ at $\mu = \mu_b$

$$C_{q'\leftarrow q}(z) = \delta_{qq'}\left[\delta(1-z) + \frac{\alpha_s}{\pi}\left(\underset{\sim}{-C_F \frac{\pi^2}{24}} \delta(1-z) + \frac{C_F}{2}(1-z) + P_{qq}(z) \ln z \right) \right]$$

$$C_{g\leftarrow q}(z) = \frac{\alpha_s}{\pi}\left(\frac{C_F}{2} z + P_{gq}(z) \ln z \right)$$

where $\quad P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]$

$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z}$$

$$C_{g\leftarrow g}(z) = \delta(1-z) + \frac{\alpha_s}{\pi}\left[\underset{\sim}{-C_A \frac{\pi^2}{24}} \delta(1-z) + \ln z * P_{gg}(z) \right]$$

$$C_{q\leftarrow g}(z) = \frac{\alpha_s}{\pi}\left[T_R z(1-z) + \ln z * P_{qg}(z) \right]$$

where $\quad P_{qg}(z) = T_R\left[z^2 + (1-z)^2 \right]$

- computation of the convolution part

perturbative expansion part

$$D_{h/i}(z_h, b; \mu_b) = \frac{1}{z_h^2} \int_{z_h}^{1} \frac{dz'}{z'} \, C_{j\leftarrow i}(z', b) \, D_{h/j}(z_h/z', \mu_b)$$

$$C_{q\leftarrow q'}(z', \mu_b) = \delta_{qq'} \left[\delta(1-z') + \frac{\alpha_s}{\pi} \left(-C_F \frac{\pi^2}{24} \delta(1-z') + \frac{C_F}{2}(1-z') + P_{qq}(z') \ln z' \right) \right.$$

$$P_{qq}(z') = C_F \left[\frac{1+z'^2}{(1-z')_+} + \frac{3}{2} \delta(1-z') \right]$$

most-part are okay, now ( )+ function part

now

$$\int_x^1 dz \, \frac{f(z)}{(1-z)_+} = \int_x^1 dz \, \frac{f(z) - f(1)}{1-z} + f(1) \ln(1-x)$$

so

$$\frac{1+z'^2}{(1-z')_+} = \frac{-1 + z'^2 + 2}{(1-z')_+} = -(1+z') + \frac{2}{(1-z')_+}$$

$$\int_{z_h}^1 \frac{dz'}{z'} \, \frac{\alpha_s}{\pi} (\ln z') * C_F \left[\frac{1+z'^2}{(1-z')_+} \right] D\left(\frac{z_h}{z'}\right)$$

$$= \frac{\alpha_s}{\pi} C_F \int_{z_h}^1 \frac{dz'}{z'} \ln z' \left[-(1+z') + \frac{2}{(1-z')_+} \right] D\left(\frac{z_h}{z'}\right)$$

$$= \frac{\alpha_s}{\pi} C_F \int_{z_h}^1 \frac{dz'}{z'} \ln z' \left[-(1+z') \right] D\left(\frac{z_h}{z'}\right)$$

$$+ \frac{\alpha_s}{\pi} C_F \int_{z_h}^1 dz' \, \frac{\ln z'}{z'} * \frac{2}{(1-z')_+} D\left(\frac{z_h}{z'}\right)$$

$\Downarrow$ $\quad f(z') = \frac{\ln z'}{z'} * 2 D\left(\frac{z_h}{z'}\right)$ note $\Rightarrow$ $f(1) = 0$

no singularity problem

similar $\quad \delta(1-z') (\ln z') * D\left(\frac{z_h}{z'}\right) \to 0$

Thus we simply have

$$\bullet \quad \boxed{z_h^2} * D_{h/q}(z_h, b; \mu_b) = D_{q/h}(z_h) + \frac{\alpha_s}{\pi} * \Big\{ -C_F \frac{\pi^2}{24} D_{h/q}(z_h)$$

$$+ \int_{z_h}^{1} \frac{dz'}{z'} \Big[\frac{C_F}{2}(1-z') + C_F \frac{1+z'^2}{(1-z')} \ln z' \Big] D_{h/q}\Big(\frac{z_h}{z'}\Big) \Big\}$$

we simply drop $\delta(1-z')\ln z'$
as they go to zero!

$$= D_{h/q}(z_h) - \frac{\alpha_s}{\pi} C_F \frac{\pi^2}{24} D_{h/q}(z_h)$$

$$+ \frac{\alpha_s}{\pi} C_F \int_{z_h}^{1} \frac{dz'}{z'} \Big[\frac{1}{2}(1-z') + \frac{1+z'^2}{(1-z')} \ln z' \Big] \underset{\Delta}{D_{h/q}}\Big(\frac{z_h}{z'}\Big)$$

also $\quad C_{g \leftarrow q}(z', \mu_b) = \frac{\alpha_s}{\pi}\Big(\frac{C_F}{2} z' + P_{gq}(z') \ln z' \Big)$

$$P_{gq}(z') = C_F \frac{1+(1-z')^2}{z'}$$

$$+ \frac{\alpha_s}{\pi} C_F \int_{z_h}^{1} \frac{dz'}{z'} \Big[\frac{1}{2} z' + \frac{1+(1-z')^2}{z'} \ln z' \Big] \underset{\Delta}{D_{h/g}}\Big(\frac{z_h}{z'}\Big)$$

Now

$$D_{g\to h}(z_h, b; \mu_b) = \frac{1}{z_h^2}\left[C_{g\leftarrow g} \otimes D_{h/g}(z_h, \mu_b) + C_{q\leftarrow g} \otimes D_{h/q}(z_h, \mu_b)\right]$$

$$C_{g\leftarrow g}(z', \mu_b) = \delta(1-z') + \frac{\alpha_s}{\pi}\left[-C_A \frac{\pi^2}{24}\delta(1-z') + \ln z' * P_{gg}(z')\right]$$

where $P_{gg}(z') = 2C_A\left[\frac{z'}{(1-z')_+} + \frac{(1-z')}{z'} + z'(1-z')\right] + \frac{\beta_0}{2}\delta(1-z')$

with $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_R n_f$

again there should be no issue for $\frac{z'}{(1-z')_+}\ln z'$ at $z'\to 1$

$$C_{q\leftarrow g}(z', \mu_b) = \frac{\alpha_s}{\pi}\left[T_R z'(1-z') + \ln z' * P_{qg}(z')\right]$$

where $P_{qg}(z') = T_R\left[z'^2 + (1-z')^2\right]$

Thus we'll have

- $\boxed{z_h^2} * D_{h/g}(z_h, b; \mu_b) = D_{h/g}(z_h, \mu_b) - \frac{\alpha_s}{\pi} C_A \frac{\pi^2}{24} D_{h/g}(z_h, \mu_b)$

$$+ \frac{\alpha_s}{\pi}\int_{z_h}^1 \frac{dz'}{z'}\left\{2C_A\left[\frac{z'}{(1-z')_+} + \frac{1-z'}{z'} + z'(1-z')\right]\ln z' \, D_{h/g}(z_h/z', \mu_b)\right.$$

$$\left. + \left(T_R z'(1-z') + \ln z' * T_R\left[z'^2 + (1-z')^2\right]\right) D_{h/q}\left(\frac{z_h}{z'}, \mu_b\right)\right.$$

note. again $\frac{\beta_0}{2}\delta(1-z')\ln z'$ do not contribute