Tax Analysis for Nickel Sales in Colombia: A Monte Carlo Simulation with Merton's Model using Historical Data.

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1 Introduction

All projects are subject to risk. Specifically, mining projects are exposed to high risk due to the variability of mineral and energy prices (Sauvageau & Kumral, 2018). In March 2022, nickel had a price of 33,924.17647 U.S. Dollars per Metric Ton on the international market, while in July 2022, it had a price of 21,481.89286 U.S. Dollars per Metric Ton (International Monetary Fund, 2023). Additionally, a new factor is influencing the risk of mining projects. In response to global warming, most countries' governments in recent years have been imposing high taxes on mining to incentivize the use of renewable energy sources. As a result, mining companies are affected and must analyze the volatility of prices to avoid losses.

The government proposed the tax reform in Colombia in 2022 with the aim of reducing the production of the mining sector. The goal is to raise approximately 48% of the resources, around $22 trillion by 2023, from the mining and energy sector (Morales, 2022). The tax rate has been set at 35%, with a surcharge based on the price of the resource compared to its historical prices. The historical prices of the last 10 years are taken into account, and in case the price in the international market exceeds certain percentile of these last 10 years, a surcharge is applied (Congreso de Colombia, 2022). In this way, a good approximation of future prices could shed light on the level of taxation to be paid to investors and entrepreneurs.

Sauvageau & Kumral (2018) study the valuation of cash flow in mining projects using Monte Carlo simulations with stochastic processes calibrated on historical data. They demonstrate that the probability of a significant loss can be extracted from the probability density function of the simulated prices at a given time in the future. On the other hand, Samis *et al.* (2007) proposed an approach that consists of cash flow estimation with stochastic processes, Monte Carlo simulation, and real options to analyze the effects of taxes on mining projects. The previous work used the Galore Creek project of NovaGold Resources Incorporated in Mongolia as a case study, a country with a tax rate of 68%. However, this solution is not so effective because it cannot be used at any stage of the mining project (Sauvageau & Kumral, 2018), and real options are currently not widely used in the real world Block (2007). Finally, Islam & Nguyen (2020) present a comparative study for stock price prediction using three different methods: autoregressive integrated moving average, artificial neural networks, and stochastic models.

This research paper determines the tax surcharge for the sale of nickel in Colombia. A database of monthly international nickel prices from 1990 to 2023 is used. The Merton's jump-diffusion model is used as the methodological approach through parameter estimation and calibration with data from the last 10 years. Finally, the Monte Carlo simulation is used to obtain an approximation of the future price of nickel at a specific time. Taking into account this future price, the percentage of surcharge can be determined and investors can make more efficient decisions.

The rest of the document is organized as follows. The next section provides a theoretical framework for various mathematical concepts used, such as the components of the Merton jump-diffusion model, the maximum likelihood method, numerical methods for parameter estimation, and the Monte Carlo method. The subsequent section presents the results, including the simulation of a Merton jump-diffusion model, the estimation of nickel price parameters from the data, and the determination of the percentage of tax to be paid. Finally, the most significant conclusions are presented.

2 Theoretical framework

2.1 Models

2.1.1 Brownian motion

Brownian motion, also known as Wiener process, is a stochastic process that models the random movement of particles or variables over time. It is named after the botanist Robert Brown, who observed the erratic motion of pollen particles suspended in water (Brown, 1828). In the context of finance, Brownian motion is used to describe the random and fluctuating behavior of the prices of financial assets. It is based on the idea that prices do not follow a smooth, predictable path, but fluctuate unpredictably in different directions.

Brownian motion denoted by $\{W(t) \geq 0\}$ is a stochastic process (Brown, 1828) that satisfies the following properties:

- Starts at the origin with probability 1, i.e, P(W(0)=0)=1 or W(0)=0.
- The increment of Brownian motion, given by $W(t) - W(s)$ with $t \geq s$, are independent random variables.

  $$W(t_1) - W(t_0), W(t_2) - W(t_1), ..., W(t_{n-1}) - W(t_n)$$

  with $0 \leq t_1 \leq t_2 \leq ... \leq t_{n-1} \leq t_n$ are independent. This means that the movement at a given time does not depend on previous movements.
- It has stationary increments, i.e:

  $$W(t + \Delta t) - W(t) \stackrel{d}{=} W(s + \Delta t) - W(s), \quad \forall s, t : 0 \leq s \leq t$$

  Where the symbol $\stackrel{d}{=}$ denotes that the previous equality is in the sense of statistical distributions. This implying that statistical properties such as mean and variance remain constant over time.
- The increments of motion are normal with mean 0 and variance t-s, i.e

  $$W(t) - W(s) \sim N(0, t - s) \quad \forall s, t : 0 \leq s \leq t$$

Brownian motion has numerous applications in various fields, including physics, finance, biology, and engineering. It serves as a fundamental building block in the development of stochastic processes and provides a basis for modeling and analyzing random phenomena in a wide range of disciplines.

2.1.2 Poisson process

In this context the Poisson process is used to describe jump events, which are abrupt and unexpected changes in the price of a financial asset. Since asset prices at any time, without warning, experience a significant jump up or down. These jumps can be caused by important news, corporate announcements, changes in economic conditions, among other factors.

Thus, the Poisson process is used to model and quantify the occurrence of these jump events over time. It is based on the idea that jumps occur randomly, without following a regular pattern, and that their magnitude and frequency can be described by a Poisson distribution. The Poisson distribution describes the probability of a specific number of events occurring in a given time interval.

Poisson process denotes by $\{N(t) \geq 0\}$ is a stochastic process that satisfies the following properties:

- Starts at the origin with probability 1, i.e, N(0)=0.
- The increment of Poisson process, given by N(t)-N(s) with $t \geq s$, are independent random variables

  $$N(t_1) - N(t_0), N(t_2) - N(t_1), ..., N(t_{n-1}) - N(t_n)$$

  with $0 \leq t_1 \leq t_2 \leq ... \leq t_{n-1} \leq t_n$ are independent. This means that the occurrence of one event does not affect the occurrence of other events.
- The number of events or points in any interval of length t is a Poisson random variable with parameter λt, i.e., $N(t) - N(s) \sim Poisson(\lambda(t-s))$ with $t \geq s$. This implies that the probability of an event occurring in a given time interval is proportional to the duration of that interval.

2.1.3 Merton's jump-diffusion model

The Merton jump-diffusion model is a model used in finance to describe the price behavior of financial assets, such as stocks. This model combines two main components: diffusion, which represents the continuous and random movement of the price, and jumps, which represent sudden and significant changes in the price of the asset.

In simpler terms, Merton's jump-diffusion model suggests that the price of a financial asset can change continuously but can also experience abrupt changes at specific times. These abrupt changes are known as "jumps" and can be caused by unexpected events that affect the price of the asset.

Using the notation of Remillard (2013), the Merton's jump-diffusion model (MJD) can be represented with the following equation.

$$S(t) = S_0 e^{X(t)} \tag{1}$$

Where S_0 is the spot price at initiation, $S(t)$ is the expected price at any given point in time and $X(t)$ is represented by the following equation,

$$X(t) = (\mu - \lambda k - \frac{\sigma^2}{2})t + \sigma W(t) + \sum_{j=1}^{N(t)} \xi_j \tag{2}$$

Where μ is the drift parameter, λ is the intensity of the Poisson process, k is the expected value of the jump, σ is the e annualized standard deviation, W(t) is a standard Brownian motion, and $\sum_{j=1}^{N(t)} \xi_j$ is the compound Poisson process with $\xi_j \sim N(\gamma, \delta^2)$. Therefore, it follows that $X(t) \sim N(at + k\gamma, \sigma^2 t + k\delta^2)$, where $a = \mu - \lambda k - \frac{\sigma^2}{2}$ and the density of X(t) is given by:

$$f_{X(t)}(x) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \frac{e^{\frac{-1}{2}\frac{(x-at-k\gamma)^2}{\sigma^2 t + k\delta^2}}}{\sqrt{2\pi(\sigma^2 t + k\delta^2)}} \tag{3}$$

where $x \in \mathbb{R}$ (Remillard, 2013).

2.2 Parameter estimation

2.2.1 Maximum likelihood

Maximum likelihood is an approach used in statistics to estimate the unknown parameters of a model from observed data. In the context of the Merton's jump-diffusion model, maximum likelihood is used to estimate the model parameters from the observed prices of a financial asset. Maximum likelihood in the Merton's jump-diffusion model involves finding the values of the model parameters that maximize the probability of observing the asset prices as recorded in the data. In other words, we look for the parameters that make the prices simulated by the model best match the actual observed prices.

For this, the density given by equation (3) is calculated, then the log-likelihood is calculated as the sum of logarithms of the density and this result is multiplied by -1 since the objective is to maximize and *optim* function in R used, by default minimizes.

The likelihood function is given by:

$$L(\theta | x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i | \theta)$$

Where $\theta = (\mu, \sigma, \lambda, \gamma, \delta)$ and f is the density function of $X(t)$. In this case, log-likelihood is used, since the products are converted to sums, which facilitates numerical optimization. Optimization algorithms are usually designed to maximize or minimize objective functions by addition and subtraction, which makes working with the log-likelihood more computationally efficient. The log-likelihood function is given by:

$$lnL = \sum_{i=1}^{n} lnf(x_i|\theta) = \sum_{i=1}^{n} ln \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \frac{e^{\frac{-1}{2}\frac{(x_i - at - k\gamma)^2}{\sigma^2 t + k\delta^2}}}{\sqrt{2\pi(\sigma^2 t + k\delta^2)}} \tag{4}$$

The maximum likelihood method estimates θ_0 by finding the value of θ that maximizes the above equation (4).This is the maximum likelihood estimator of θ_0.

$$\hat{\theta}_{\text{mle}} = \underset{\theta \in \Theta}{\arg\max}\ lnL\left(\theta \mid x_1, \ldots, x_n\right).$$

Finally this likelihood function is maximized using numerical optimization techniques, specifically the BFGS method, to find the optimal values of the parameters.

2.2.2 Broyden-Fletcher-Goldfarb-Shanno (BFGS)

The Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm is a popular optimization method used in maximum likelihood estimation. It belongs to the class of quasi-Newton methods and is widely used for finding the maximum likelihood estimates of model parameters. The BFGS algorithm iteratively approximates the Hessian matrix of the likelihood function to determine the optimal parameter values. The Hessian matrix provides information about the curvature of the likelihood function, which is crucial for finding the maximum.

The BFGS algorithm starts with an initial guess for the parameter values and iteratively updates these values to improve the likelihood. At each iteration, it computes the gradient of the likelihood function, which represents the direction of steepest ascent, and uses this information to update the parameter values. One of the key advantages of the BFGS algorithm is that it avoids the direct computation of the Hessian matrix, which can be computationally expensive for large models. Instead, it approximates the inverse of the Hessian matrix using an iterative formula based on the gradient information from each iteration. This approximation is updated at each step to improve the estimate of the Hessian matrix (Nocedal & Wright, 2006).

A step-by-step outline of the BFGS algorithm for maximum likelihood estimation is presented below (Encyclopedia of Mathematics, 2011):

1. Initialize the parameter vector θ with an initial guess.

2. Compute the log-likelihood function $L(\theta)$ and its gradient $\nabla L(\theta)$ with respect to θ.

3. Initialize an approximation of the inverse Hessian matrix, denoted as B_0.

4. Iterate until convergence:
   - Compute the search direction by multiplying the negative inverse Hessian approximation with the gradient: $p = B_k \nabla L(\theta_k)$.
   - Use a line search method, to determine an appropriate step size α along the search direction p.
   - Update the parameter vector: $\theta_{k+1} = \theta_k + \alpha p$.
   - Compute the new gradient: $\nabla L(\theta_{k+1})$.
   - Compute the difference between the new and previous gradients: $\Delta \nabla L = \nabla L(\theta_{k+1}) - \nabla L(\theta_k)$.
   - Update the inverse Hessian approximation: $B_{k+1} = B_k + \frac{\Delta \nabla (\Delta \theta)^T \Delta \theta}{\Delta \theta^T B_k \theta \Delta \theta} + R$
     - R is a correction term that accounts for the curvature information lost during the line search.
5. If the change in the log- likelihood or the parameter values is below a predefined threshold, stop iterating. Otherwise, update θ_k to θ_{k+1} and go back to step 4.

2.2.3 Fisher information

Fisher information tells us how much information about an unknown parameter we can obtain from a sample. In other words, it tells us how well we can measure a parameter, given a certain amount of data. More formally, it measures the expected amount of information provided by a random variable (X) for a parameter (θ) of interest (Ly *et al.*, 2017).

More formally, in the context of maximum likelihood estimation, the Fisher information matrix is defined as the negative expected second derivative of the log-likelihood function with respect to the parameters. It captures the local curvature of the log-likelihood function and provides insights into the amount of information the data carries about the parameters. The Fisher information matrix is a symmetric and positive definite matrix. Its diagonal elements represent the variances of the individual parameters, while the off-diagonal elements represent the covariances between parameter estimates.

Additionally, the Fisher information matrix has implications for hypothesis testing and the asymptotic behavior of maximum likelihood estimators. It can be used to derive the standard errors of the parameter estimates, construct confidence intervals, perform likelihood ratio tests, and assess the overall fit of the model. In this case is used to calculate the standard errors and confidence intervals (Benites, 2022).

2.3 Monte Carlo simulation

The Monte Carlo method is a simple computer technique based on performing numerous fictitious experiments with random numbers (Hammersley & Handscomb, 1964). The only information one needs is the relationship between the output and input quantities, and the knowledge of probability distributions of the input variables. The method repeats multiple trials with random numbers, in each trial the input variables are assigned random values, taking into account their probability distribution. With these values the output is calculated and with the results it is possible to observe the behavior of the output distribution, determine its mean value, percentiles or certain probabilities (Menčík, 2016).

The basic steps of the Monte Carlo method are as follows:

1. Identify the system or process to be analyzed and the variables involved.
2. Determine the probability distributions that represent the uncertainty or variability of the input variables. These distributions may be based on historical data.
3. Generate a large number of random samples or scenarios by sampling the input distributions.
4. For each sample, simulate the system or process using the sampled input values.
5. Collect and analyze the output or result of the simulations.

In this way the Monte Carlo method allows analysts to understand the range of possible outcomes, assess risks, make informed decisions and optimize strategies. In this paper the Monte Carlo method is used to generalize multiple trajectories of the forward price of nickel.

2.4 Tax reform in Colombia

According to Law No. 2277 approved on December 13, 2022, by the Congress of Colombia, it was decreed in Chapter II, Article 10, which modifies Article 240 of the Tax Statute, that the general income tax rate applicable to national corporations and their related entities, permanent establishments of foreign entities, and foreign legal entities with or without residence in the country, obligated to file the annual income tax and complementary tax return, will be 35%.

Additionally, Paragraph 3 establishes that national corporations and their related entities, permanent establishments of foreign entities, and foreign legal entities with or without residence in the country, must add additional points to the general income tax rate when engaging in economic activities related to the extraction of oil, coal, nickel, and other non-renewable resources.

When the average price of the respective taxable year, subject to declaration, is below the 65th percentile of the average prices of the last 120 months, excluding the price of the months elapsed in the year of the declaration, the additional percentage of the income tax rate will be 0%. On the other hand, when the average falls between the 65th and 75th percentiles of the average prices of the last 120 months, the additional percentage of the income tax rate will be 5%. Lastly, when the average is above the 75th percentile of the average prices of the last 120 months, the additional percentage of the income tax rate will be 10%. Consequently, companies and legal entities engaged in nickel extraction could end up paying up to 45% in income tax (Congreso de Colombia, 2022).

3 Results

3.1 Simulation of Merton's jump-diffusion model

First, a simulation of Merton's jump-diffusion model is performed using the equation (3) as a starting point. The simulation can be found in the file *sim.jump.diff.merton.R*. The results of the simulation are shown below, with the parameters $\mu = 0.08$, $\sigma = 0.22$, $\lambda = 100$, $\gamma = 0.05$, $\delta = 0.005$ y $S_0 = 100$.

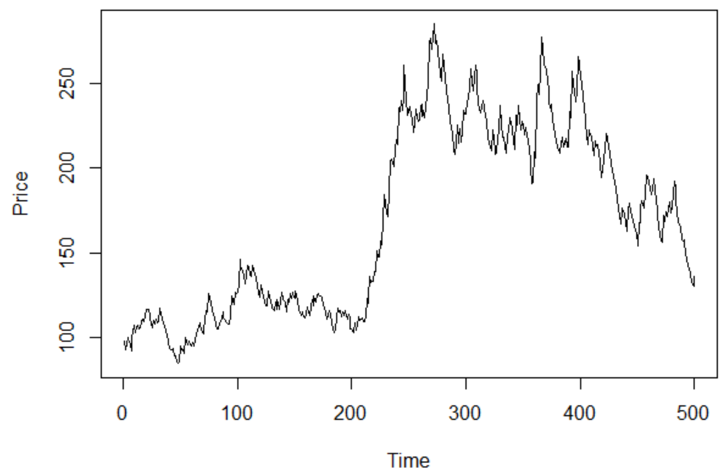

Figure 1: Simulation of the Merton's jump-diffusion model

The above figure depicts a price path generated by a Merton jump-diffusion model, showing the simulated evolution of prices over time. As the Merton model includes random jumps in prices, sharp fluctuations can be observed in the prices at times when jumps occur, particularly around time 210.

3.2 Estimation of parameters

Using the maximum likelihood approach, the parameter estimation of Merton's jump-diffusion model given in equation (2) is performed based on the methodology described in Remillard (2013). The likelihood function is optimized with the BFGS method, and Fisher's information is used to estimate the precision of the parameters. The resulting output is a list of the estimated parameters and their 95% confidence interval.

To perform the estimation, log-returns are calculated from the input data, and a vector of initial parameter values is defined. The likelihood function is then optimized with the BFGS method, and Fisher's information is calculated using the Hessian matrix or the numerical gradient. Corrections are made to ensure that the estimated parameters are positive, and the covariance matrix of the parameters is estimated. Finally, confidence intervals are calculated. This code can be found in the file *est.merton1.R*.

The development is first tested with a price simulation of Merton's jump-diffusion model, which can be found in the *DataMerton.R* file, to check that the estimated parameters give good results since the real parameters are known. The following table shows the estimate of each of the parameters along with their actual value.

Parameter	**True value**	**Confidence interval (95%)**
μ	0.08	0.1579 ± 0.7688
σ	0.22	0.2187 ± 0.0214
λ	100	98.7610 ± 17.2674
γ	0.05	0.0493 ± 0.0037
δ	0.005	0.0066 ± 0.0074

Table 1: 95% confidence intervals for the parameters of the Merton model using the maximum likelihood method, applied to the simulated data set *DataMerton*.

From the results in the table, all the parameters are estimated satisfactorily, except for mu, however, this is within the confidence intervals. Additionally, the following graph presents a comparison between the nonparametric density estimation using a Gaussian kernel, the true density of the data, and the density evaluated on the estimated parameters.

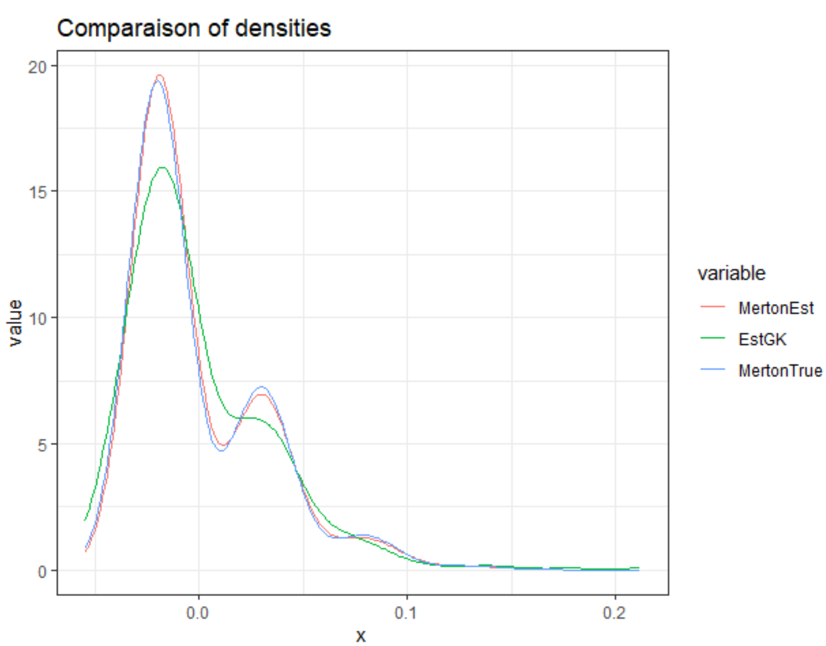

Figure 2: Comparison of densities

From the above figure, it is possible to see that the estimation of the parameters is good since the density curve evaluated in the estimated parameters compared to the actual density is very similar. Once the effectiveness of the estimation is verified, it is applied with monthly nickel price data in the global market from January 1990 to April 2023, obtained from database of International Monetary Fund (2023), these prices are on file *DataNiquel.R* and the behavior of these prices is shown below.

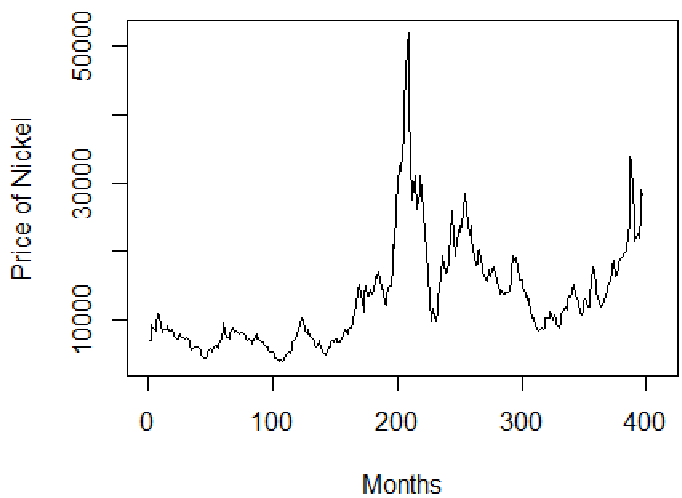

Figure 3: Nickel price in the global market from January 1990 to April 2023 (International Monetary Fund, 2023).

Thus, it is possible to observe that the behavior of prices can be in agreement with Merton's jump-diffusion model, since its three main characteristics can be identified. First, stochasticity is evidenced through Brownian motion, which is reflected in the curve that does not present a smooth trajectory. In addition, the jumps caused by the Poisson Process can be observed, for example around month 200, which corresponds to April 2007. Finally, its trend behavior is highlighted. Therefore, the results of the estimation of parameters applied to nickel price are shown in the following table:

Parameter	**Confidence interval (95%)**
μ	0.0816 ± 0.0968
σ	0.2535 ± 0.0249
λ	0.5436 ± 1.0213
γ	0.0158 ± 0.1056
δ	0.1625 ± 0.1241

Table 2: 95% confidence interval for the parameters of the Merton model using the maximum likelihood method, applied to monthly nickel prices.

3.3 Estimated income tax payable

Once the parameters of the previous database (Table 2) have been obtained. Monte Carlo simulation is used to perform 10000 possible trajectories of nickel prices for the next year, i.e from April 2023 to April 2024, using the parameters obtained previously together with the simulation of Merton's jump-diffusion model (*sim.jum.diff.merton.R*). The purpose is to determine the income tax payable and its respective probabilities, taking into account the Colombian tax reform established in 2022. In this way, the following graphical results are obtained. This results are on file *Montecarlo.R*.

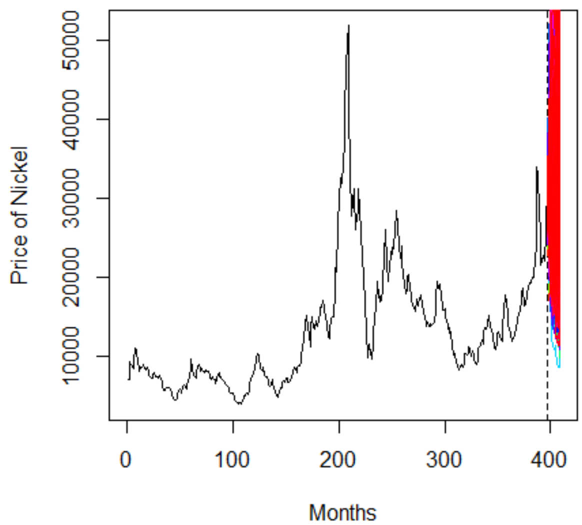

Figure 4: Monte Carlo simulation for 10000 possible trajectories of nickel prices from April 2023 to April 2024.

To show the visual result more clearly, the previous graph is modified by trimming the x-axis.

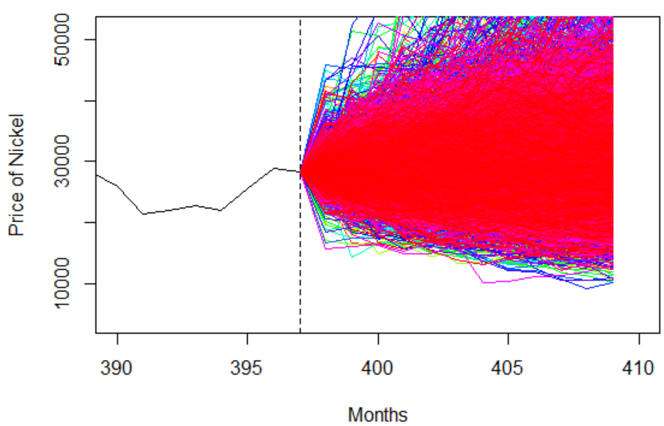

Figure 5: 10000 possible trajectories of nickel prices from April 2023 to April 2024.

The average price during the year is then calculated for each of the trajectories, as shown in the histogram below.

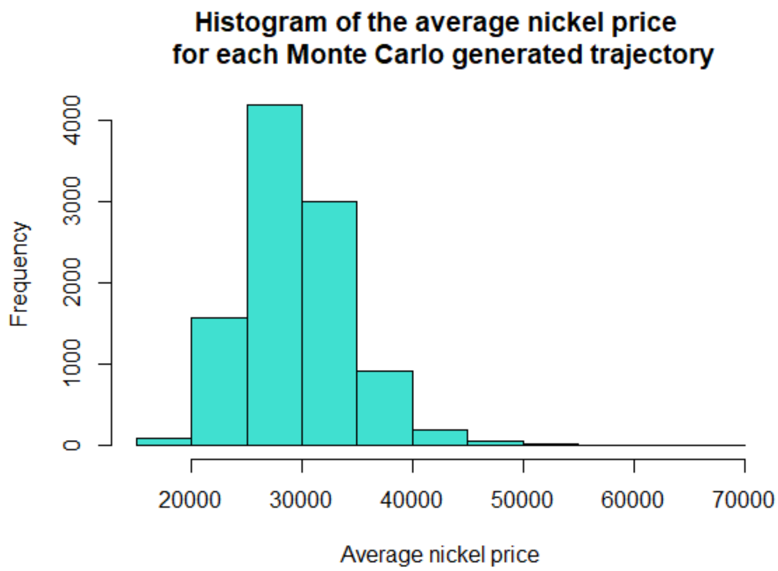

Figure 6: Histogram of the average nickel price for each Monte Carlo generated trajectory.

Thus, it is possible to see that most nickel price averages from April 2023 to April 2024 are between 20000 and 40000 U.S. Dollars per Metric Ton. Finally, according to the Colombian tax reform in the context of nickel extraction (Sección 2.4), the 65th and 75th percentile of the last 10 years of the international nickel price is calculated, excluding the year being simulated, i.e. from April 2013 to April 2023.

Two approaches are used to determine the percentage of income tax payable. The first one is based on calculating the average of the average nickel prices from April 2023 to April 2024 and then using the 65th and 75th percentiles previously calculated, it is determined in which percentile the average is located to establish the income tax percentage payable. The second approach is based on probabilities, i.e., from the averages of each of the trajectories determine what percentage of them is in each of the percentiles to determine the probability of paying each of the income taxes.

The first approach results in an income tax payable of 45%, i.e. a 10% surtax was applied since the average of the average price in each of the trajectories is above the 75th percentile of the international price of nickel during the last 10 years. On the other hand, the second approach yields as results that an income tax of 35% with a probability of 0, 40% with a probability of 9×10^{-4} and 45% with a probability of 0.9991 will be paid. This development can be found in the *PercentToPay.R* file.

4 Conclusions

This document presents a detailed analysis of the tax burden associated with the sale of nickel in Colombia. A database of monthly international nickel prices from 1990 to 2023 is utilized, and the Merton jump-diffusion model is applied to estimate parameters and calibrate the data for the last 10 years. Finally, Monte Carlo simulation is used to obtain an approximation of the future price of nickel at a specific point in time.

The results demonstrate that a high proportion of taxes will be paid in the upcoming year for nickel extraction, which can significantly impact the profitability of the mining project. Therefore, companies must carefully calculate their earnings, taking into account this tax percentage, to determine whether it is truly viable to continue with this type of work or halt it. Additionally, this analysis also highlights the importance of using advanced mathematical models to predict future prices of nickel. Monte Carlo simulation provides a valuable tool for estimating the future price of nickel and, therefore, determining the appropriate percentage of taxes to be paid. This approach enables investors to make more informed decisions and mitigate risks associated with market volatility.

Regarding potential future work, it would be interesting to explore how this methodology could be applied to other mining projects in different countries and regions. Furthermore, it could also be valuable to investigate how fluctuations in international nickel prices impact other economic and social aspects in Colombia and other producer countries. Overall, this analysis provides valuable insights for investors and decision-makers seeking to maximize profitability while minimizing risks associated with mining projects.

5 Intellectual property

According to the internal regulation on intellectual property within Universidad EAFIT, the results of this research practice are product of John Esteban Castro Ramirez and Nicolas Alberto Moreno Reyes.

In case further products, beside academic articles, that could be generated from this work, the intellectual property distribution related to them will be directed under the current regulation of this matter determined by Universidad EAFIT (2017).

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