Community Detection Methods for the Generalized Random Dot Product Graph Model

Dissertation Proposal

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Introduction

Community Detection for Networks

How can we cluster the nodes of a network?

Statistical inference (parametric approach):

1. Define a generative model $G \mid z_1, ..., z_n, \vec{\theta} \sim P(\vec{z}, \vec{\theta})$.
2. Develop a method for obtaining estimators $f(G) = \hat{z}_1, ..., \hat{z}_n$.
3. Identify asymptotic properties of estimators and prove consistency.

Overview

1. Probability Models for Networks
   - Block Models and Community Structure
   - (Generalized) Random Dot Product Graphs
   - Connecting Block Models to the (G)RDPG

2. Popularity Adjusted Block Model
   - Connecting the PABM to the GRDPG
   - Subspace Clustering for Community Detection

3. Community Detection for the (G)RDPG
   - Manifold Clustering
   - Manfolds as (G)RDPG Latent Configurations

Probability Models for Networks

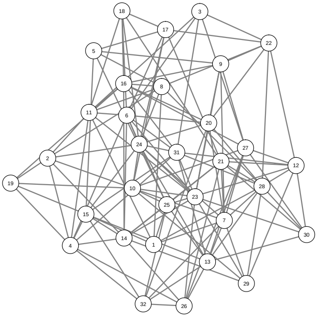
Bernoulli Graphs

Let $G = (V, E)$ be an undirected and unweighted graph with $|V| = n$.
G is described by adjacency matrix A such that $A_{ij} = \begin{cases} 1 & \exists \text{ edge between } i \text{ and } j \\ 0 & \text{else} \end{cases}$

$A \sim BernoulliGraph(P)$ iff

- $P \in [0, 1]^{n \times n}$ describes edge probabilities between pairs of vertices.
- $A_{ij} \overset{ind}{\sim} Bernoulli(P_{ij})$ for each $i < j$.
- $A_{ji} = A_{ij}$ and $A_{ii} = 0$.

Example: If G is an Erdos-Renyi graph, then $P_{ij} = \theta$.

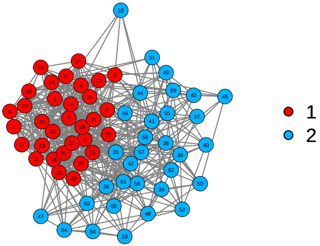


Block Models

Suppose each vertex $v_1, ..., v_n$ has hidden labels $z_1, ..., z_n \in [K]$, and each P_{ij} depends on labels z_i and z_j.
Then $A \sim BernoulliGraph(P)$ is a *block model*.

Example: Stochastic Block Model with two communities

- $z_1, ..., z_n \in \{1, 2\}$
- $P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$
- To make this an assortative SBM, set $pq > r^2$.
- In this example, $p = 1/2$, $q = 1/4$, and $r = 1/8$.



Block Models

Erdos-Renyi Model (1959)

- $P_{ij} = \theta$
- Not a block model

Stochastic Block Model (Lorrain and White, 1971)

- $P_{ij} = \theta_{z_i z_j}$
- $K(K+1)/2$ parameters θ_{kl}

Degree Corrected Block Model (Karrer and Newman, 2011)

- $P_{ij} = \theta_{z_i z_j} \omega_i \omega_j$
- $K(K+1)/2 + n$ parameters θ_{kl}, ω_i

Popularity Adjusted Block Model (Sengupta and Chen, 2017)

- $P_{ij} = \lambda_{i z_j} \lambda_{j z_i}$
- Kn parameters λ_{ik}

Hierarchy of Block Models

PABM $\to$ DCBM: $\lambda_{ik} = \sqrt{\theta_{z_i k}}\omega_i$

DCBM $\to$ SBM: $\omega_i = 1$

SBM $\to$ Erdos-Renyi: $\theta_{kl} = \theta$

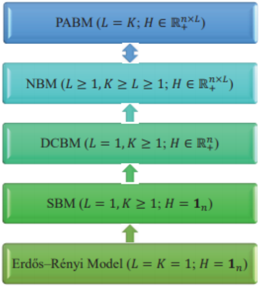


Figure 2: The hierarchy of block models

Majid Noroozi and Marianna Pensky, 2021

(Generalized) Random Dot Product Graph Model

Random Dot Product Graph $A \sim RDPG(X)$
(Young and Scheinerman, 2007)

- Latent vectors $x_1, ..., x_n \in \mathbb{R}^d$ such that $x_i^\top x_j \in [0, 1]$
- $A \sim BernoulliGraph(XX^\top)$ where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$

Generalized Random Dot Product Graph $A \sim GRDPG_{p,q}(X)$
(Rubin-Delanchy, Cape, Tang, Priebe, 2020)

- Latent vectors $x_1, ..., x_n \in \mathbb{R}^{p+q}$ such that $x_i^\top I_{p,q} x_j \in [0, 1]$ and $I_{p,q} = blockdiag(I_p, -I_q)$
- $A \sim BernoulliGraph(XI_{p,q}X^\top)$ where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$

If latent vectors $X_1, ..., X_n \overset{iid}{\sim} F$, then we write
$(A, X) \sim RDPG(F, n)$ or $(A, X) \sim GRDPG_{p,q}(F, n)$.

(Generalized) Random Dot Product Graph Model

Recovery/Estimation

Want to estimate X given A, or alternatively, interpoint distances, inner products, or angles.

Adjacency Spectral Embedding

To embed in $\mathbb{R}^d$,

1. Compute $A \approx \hat{V}\hat{\Lambda}\hat{V}^\top$ where $\hat{\Lambda} \in \mathbb{R}^{d \times d}$ and $\hat{V} \in \mathbb{R}^{n \times d}$.
   For RDPG, use d greatest eigenvalues;
   for GRDPG, use p most positive and q most negative eigenvalues.
2. For RDPG, let $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$;
   for GRDPG, let $\hat{X} = \hat{V}|\hat{\Lambda}|^{1/2}$.

RDPG: $\max_i \|\hat{X}_i - W_n X_i\| \stackrel{a.s.}{\to} 0$ (Athreya et al., 2018)

GRDPG: $\max_i \|\hat{X}_i - Q_n X_i\| \stackrel{a.s.}{\to} 0$ (Rubin-Delanchy et al., 2020)

Connecting Block Models to the (G)RDPG Model

All G with $A \sim BernoulliGraph(P)$ are RDPG (if P is positive semidefinite) or GRDPG (includes all block models).

Example: Assortative SBM

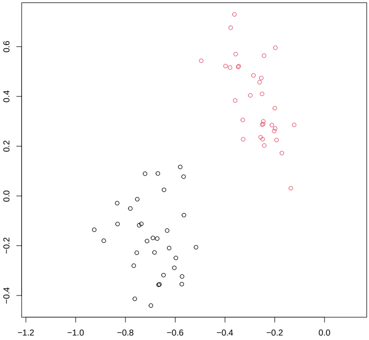
$$X = \begin{bmatrix} \sqrt{p} & 0 \\ \vdots & \vdots \\ \sqrt{p} & 0 \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \\ \vdots & \vdots \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \end{bmatrix}$$

$$P = XX^\top$$

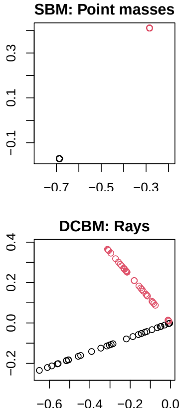
Connecting Block Models to the (G)RDPG Model

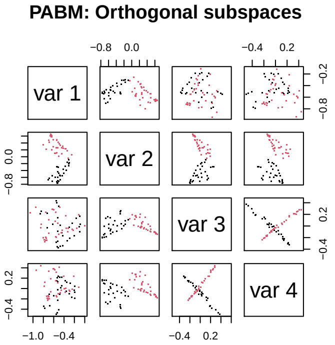
Example: SBM (cont'd)

- $A \sim BernoulliGraph(XX^\top)$
- $A \approx \hat{X}\hat{X}^\top$
  - $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$
- Apply clustering algorithm (e.g., K-means) on $\hat{X}$



Connecting Block Models to the (G)RDPG Model





Popularity Adjusted Block Model

Popularity Adjusted Block Model

Definition based on Noroozi, Rimal, and Pensky (2020).

$A \sim PABM(\{\lambda^{(kl)}\}_K)$ iff

1. w.l.o.g., organize P such that each block $P^{(kl)} \in [0, 1]^{n_k \times n_l}$ contains edge probabilities between communities k and l.
2. Organize parameters as vectors such that $\lambda^{(kl)} \in \mathbb{R}^{n_k}$ are the popularity parameters of members of community k to community l.
   $\{\lambda^{(kl)}\}_K$ is the set of K^2 popularity vectors.
3. Then we can write each block of P as $P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top$.
4. Sample $A \sim BernoulliGraph(P)$.

Connecting the PABM to the GRDPG ($K = 2$)

Theorem (KTT): $A \sim PABM(\{\lambda^{(kl)}\}_2)$ is equivalent to $A \sim GRDPG_{3,1}(XU)$ for X constructed from $\{\lambda^{(kl)}\}_2$ and $U \in \mathbb{O}(4)$

Proof:

$$X = \begin{bmatrix} \lambda^{(11)} & \lambda^{(12)} & 0 & 0 \\ 0 & 0 & \lambda^{(21)} & \lambda^{(22)} \end{bmatrix}$$

$$Y = \begin{bmatrix} \lambda^{(11)} & 0 & \lambda^{(12)} & 0 \\ 0 & \lambda^{(21)} & 0 & \lambda^{(22)} \end{bmatrix}$$

$$P = XY^\top$$

Connecting the PABM to the GRDPG ($K = 2$)

Proof (cont'd):

$$Y = X\Pi$$

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U I_{3,1} U^\top$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P = (XU) I_{3,1} (XU)^\top$$

Connecting the PABM to the GRDPG

Theorem (KTT): $A \sim PABM(\{\lambda^{(kl)}\}_K)$ is equivalent to $A \sim GRDPG_{p,q}(XU)$ such that

- $p = K(K+1)/2$
- $q = K(K-1)/2$
- U is orthogonal and predetermined for each K
- X is block diagonal and composed of $\{\lambda^{(kl)}\}_K$ $\implies$ if $x_i^\top$ and $x_j^\top$ are two rows of XU corresponding to different communities, $x_i^\top x_j = 0$.

Remark: Non-uniqueness of the latent configuration $A \sim GRDPG_{p,q}(XU) \implies A \sim GRDPG_{p,q}(XUQ)$ $\forall Q \in \mathbb{O}(p,q)$

Orthogonal Spectral Clustering

Theorem (KTT): If $P = V\Lambda V^\top$ and $B = nVV^\top$, then $B_{ij} = 0$ $\forall i, j$ in different communities.

Orthogonal Spectral Clustering algorithm:

1. Compute the eigenvectors of A that correspond to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues to construct V.
2. Compute $B = |nVV^\top|$ applying $|\cdot|$ entry-wise.
3. Construct graph G using B as its similarity matrix.
4. Partition G into K disconnected subgraphs (e.g., using edge thresholding or spectral clustering). Map each partition to the community labels $1, ..., K$.

Theorem (KTT): Let $\hat{B}_n$ with entries $\hat{B}_n^{(ij)}$ be the affinity matrix from OSC. Then $\forall$ pairs (i, j) belonging to different communities and sparsity factor satisfying $n\rho_n = \omega\{(\log n)^{4c}\}$,

$$\hat{B}_n^{(ij)} = O\left(\frac{(\log n)^c}{\dots}\right)$$

Sparse Subspace Clustering

- X is block diagonal and U is orthogonal $\implies$ each community corresponds to a subspace in $\mathbb{R}^d$.
- Subspace property holds even with linear transformation $Q \in \mathbb{O}(p, q)$.
- If $P = V\Lambda V^\top$, then V consists of *orthogonal* subspaces.

Sparse Subspace Clustering algorithm:

1. Solve n optimization problems $c_i = \arg\min_c \|c\|_1$ subject to $x_i = X_{-i}c$ and $c_i^{(i)} = 0$.
2. Compile solutions $C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$
3. Construct affinity matrix $B = |C| + |C^\top|$

- If X obeys the Subspace Detection Property, then B is sparse such that $B_{ij} = 0$ if i and j belong to different communities and $\|c_i\| > 0$.
- Step (1) of SSC typically performed via LASSO: $c_i = \arg\min \frac{1}{2}\|x_i - X_{-i}c\|_2^2 + \lambda\|c\|_1$

Sparse Subspace Clustering

Theorem (KTT):

Let

- P_n describe the edge probability matrix of the PABM with n vertices
- $A_n \sim BernoulliGraph(P_n)$
- $\hat{V}_n$ be the matrix of eigenvectors of A_n corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues.

Then

- $\exists \lambda > 0$ and $N < \infty$ such that when $n > N$, $\sqrt{n}\hat{V}_n$ obeys the Subspace Detection Property with probability 1.

Remarks:

- For large n, we can identify λ for SDP (Wang and Xu, 2016).
- SDP does not guarantee community detection.

General Community Detection for the (G)RDPG

Generative Model

Let $(A, X) \sim RDPG(F, n)$ such that

1. Define functions $f_1, ..., f_K$ such that $f_k : [0, 1] \mapsto \mathcal{X}$ and $f_k(t) \neq f_l(t)\ \forall k, l \in [K]$.
2. Sample labels $Z_1, ..., Z_n \overset{iid}{\sim} Categorical(\pi_1, ..., \pi_K)$.
3. Sample $T_1, ..., T_n \overset{iid}{\sim} D$ with support $[0, 1]$.
4. Set latent positions $X_i = f_{Z_i}(T_i)$ and $X = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}^\top$.
5. $A \sim BernoulliGraph(XX^\top)$

Community Detection

- Athreya et al. and Rubin-Delanchy et al.: we can approximate properties of the latent configurations via ASE.
- General community detection method: Given A, K, and d (or p and q),
  1. Use ASE to approximate the latent configuration.
  2. Use the appropriate clustering algorithm for the form of the latent configuration (manifolds).

Parallel Segments

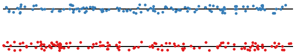
Example: Let $U_1, ..., U_{n_1}, U_{n_1+1}, ..., U_n \overset{iid}{\sim} Uniform(0, \cos \frac{\pi}{2} a)$, $f_1(t) = (t, 0)$, and $f_2(t) = (t, a)$. $X_i = f_1(U_i)$ for $i \leq n_1$ and $X_j = f_2(U_j)$ for $n_1 + 1 \leq j \leq n$. If we observe $X_1, ..., X_{n_1}, X_{n_1+1}, ..., X_n$, what approach will allow us to group the observations by segment?

$\forall a \in (0, 1)$, $\delta \in (0, 1)$, and $K \geq 2$, $\exists N(a, \delta, K) < \infty$ such that when $\min_k n_k \geq N$, with probability at least $1 - \delta$,

1. Single linkage clustering will produce perfect community detection.
2. Any ϵ-neighborhood graph with $\epsilon \leq a$ will consist of at least K disjoint subgraphs such that no subgraph contains members of two different communities.

Noisy Parallel Segments and One-Dimensional Manifolds

Example: Starting with the parallel segments as before, suppose instead of observing $X_1, ..., X_n$, we have noisy observations $X_1 + \xi_1, ..., X_n + \xi_n$ such that $\max_i \|\xi_i\| = \xi \leq a/3$.

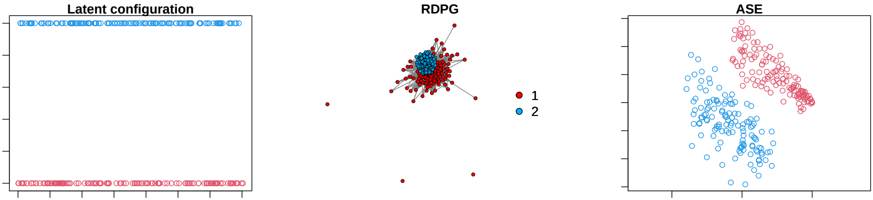


Then $\forall a \in (0, 1)$, $\delta \in (0, 1)$, $K \geq 2$, $\xi \leq a/3$, $\exists N(a, \delta, K, \xi) < \infty$ such that when $\min_k n_k \geq N$, with probability at least $1 - \delta$,

1. Single linkage will produce perfect community detection.
2. Any ϵ-neighborhood graph will consist of at least K sub-graphs with no subgraph containing vertices from multiple communities.

This also holds for noisy points sampled from one-dimensional manifolds such that the manifolds are distance at least a apart.

Recovery from the Adjacency Matrix



Future Work

1. Show that the ASE of a random graph generated by these latent vectors produces the correct conditions for sufficiently large n.
2. Extend results to non-uniform distributions.
3. Extend results to multidimensional manifolds.
4. Relax condition for the minimum distance between manifolds.
5. Explore more robust clustering techniques for these latent configurations.
6. Extend results to the GRDPG.

Additional Slides

Parameter Estimation

Indefinite Orthogonal Group

$$\mathbb{O}(p, q) = \{Q : QI_{p,q}Q^\top = I_{p,q}\}$$

- $Q^\top Q \neq I$
- If $A \sim GRDPG_{p,q}(X)$, then $A \sim GRDPG_{p,q}(XQ)$
- $(Qx)^\top (Qy) = x^\top Q^\top Qy \neq x^\top y$
- $\|Q\| \neq 1 \implies \|Qx - Qy\| \neq \|x - y\|$

Community Detection in Block Models

Likelihood

$$L = \prod_{i<j} \prod_{k,l}^{K} \left(p_{k,l,i,j}^{A_{ij}} (1 - p_{k,l,i,j})^{1-A_{ij}}\right)^{z_{ik} z_{jl}}$$

Example: DCBM ($p_{k,l,i,j} = \theta_{kl}\omega_i\omega_j$)

$$L = \prod_{i<j} \prod_{k,l}^{K} \left((\theta_{kl}\omega_i\omega_j)^{A_{ij}} (1 - \theta_{kl}\omega_i\omega_j)^{1-A_{ij}}\right)^{z_{ik} z_{jl}}$$

- ML method for community detection: $\hat{\vec{z}} = \arg\max_{\vec{z}} L$
- NP-complete
  - Expectation-Maximization
  - Bayesian methods
  - Spectral methods

Expectation Maximization for the PABM

Full data log-likelihood

$$\begin{aligned}\log L = &\sum_{i<j} \sum_{k,l} z_{ik} z_{jl} (A_{ij} \log \lambda_{il} \lambda_{jk} + (1 - A_{ij}) \log(1 - \lambda_{il} \lambda_{jk})) \\ &+ \sum_i \sum_k z_{ik} \log \pi_k\end{aligned}$$

E-step

- $\gamma_{ik} = P(Z_i = k \mid \{\pi_l\}, \{\lambda_{jl}\})$
- $\log \gamma_{ik} \propto$
  $\log \pi_k + \sum_{j \neq i} \sum_l \pi_{jl} (A_{ij} \log \lambda_{il} \lambda_{jk} + (1 - A_{ij}) \log(1 - \lambda_{il} \lambda_{jk}))$

M-step

- $\pi_k = \frac{1}{n} \sum_i \gamma_{ik}$
- $\{\lambda_{ik}\} = \arg\max_{\{\lambda_{ik}\}} E_Z[\log L]$

MCMC Sampling for the PABM

Priors:

- $Z_i \overset{iid}{\sim} Categorical(\pi_1, ..., \pi_K)$
- $\lambda_{ik} \overset{ind}{\sim} Beta(a_{ik}, b_{ik})$

Full joint distribution:

$$
\begin{aligned}
\log p = &\, constant \\
&+ \sum_{i<j} \sum_k \sum_l z_{ik} z_{jl} (A_{ij} \log \lambda_{il} \lambda_{jk} + (1 - A_{ij}) \log(1 - \lambda_{il} \lambda_{jk})) \\
&+ \sum_k \sum_i z_{ik} \log \pi_k \\
&+ \sum_i \sum_k (a_{ik} - 1) \log \lambda_{ik} + (b_{ik} - 1) \log(1 - \lambda_{ik})
\end{aligned}
$$

Variational Inference for the PABM

Mean Field Variational Inference

- Minimize $d_{KL}(p||q)$
  - p is the joint distribution
  - q is a density of some form
- Restrict $q(\vec{z}, \{\lambda_{ik}\}) = \Big(\prod_i q_{z_i}(z_i)\Big)\Big(\prod_{i,k} q_{\lambda_{ik}}(\lambda_{ik})\Big)$
- Iterative solution: $q_{\theta_i}^{(t+1)} \propto \exp(E_{\theta_{-i}^{(t)}}[\log p])$
- Approximate solution for the PABM
  - $Z_i \mid \{a'_{ik}\}, \{b'_{ik}\} \sim Categorical(\pi'_1, ..., \pi'_K)$
  - $\lambda_{ik} \mid \{a'_{-i,-k}\}, \{b'_{-i,-k}\}, \{\pi'_k\} \sim Beta(a'_{ik}, b'_{ik})$
  - Iteratively update $\{\pi'_K\}, \{a'_{ik}\}, \{b'_{ik}\}$ until convergence