Clustering of Distributions on 1-Dimensional Manifolds

Setup

Suppose we have two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2 \in \mathbb{R}^d$, each of length 1, defined by $f_1(t)$ and $f_2(t)$ respectively ($f_i : [0,1] \mapsto \mathbb{R}^d$). Define δ as the minimum distance between the two manifolds, i.e., $\delta = \min_t \|f_1(t) - f_2(t)\|$, and let $\delta > 0$. For now, restrict each f_i such that the distance along the manifold between $f_i(t)$ and $f_i(s)$ is equal to the difference between t and s (this also implies that each manifold is of length 1). We sample $T_1, ..., T_n \overset{iid}{\sim} F$ for continuous F with support $[0,1]$ and use f_1 to map the first n_1 points to $\mathcal{M}_1$ and f_2 to map the remaining $n_2 = n - n_1$ points to $\mathcal{M}_2$. Let $X_i = f_1(T_i)$ and $Y_j = f_2(T_j)$. Without loss of generality, assume $n_1 \leq n_2$.

Theory

Let $D_i = X_{(i)} - X_{(i-1)}$. Then if $\max_i D_i < \delta$, we have sufficient separation of points in $\mathcal{M}_1$.

Uniform case

It can be shown that if $X_i \overset{iid}{\sim} Uniform(0,1)$, then $D_i \sim Beta(1,n)$. Therefore, $P(\max_i D_i < \delta) \geq (P(D_i < \delta))^n = (1-(1-\delta)^n)^n$, which is a decreasing function for sufficiently large n. This gives us the result $\max_i D_i \overset{a.s.}{\to} 0$.

Beta case

General case

If F is absolutely continuous, then it can be shown that

$$P(D_i \leq \delta) = 1 - \int_0^{1-\delta} \frac{n!}{(n-i+1)!(i-2)!} (F(x))^{i-2}(1-F(x+\delta))^{n-i} f(x)dx$$

Making the approximation $F(x+\delta) \approx F(x) + \delta f(x)$ and bounding $f(x) \geq a > 0$, we get:

$$P(D_i \leq \delta) \geq 1 - \int_0^{1-\delta} \frac{n!}{(n-i+1)!(i-2)!} (F(x))^{i-2}(1-F(x)-a\delta)^{n-i} f(x)dx$$

Then making the substitution $u = F(x) \implies du = f(x)dx$, we get

$$1 - \int_0^{F(1-\delta)} \frac{n!}{(n-i+1)!(i-2)!} u^{i-2}(1-u-a\delta)^{n-i} du$$

Which becomes $1 - CP(n)e^{-n}$ for some $C > 0$ and $P(n)$ is a polynomial of n. This goes to 0 as $n \to \infty$, giving us the result $D_i \overset{p}{\to} 0$.

Computational Results

TBD