

Clustering of Distributions on 1-Dimensional Manifolds

Setup

Suppose we have two manifolds \mathcal{M}_1 and $\mathcal{M}_2 \in \mathbb{R}^d$, each of length 1, defined by $f_1(t)$ and $f_2(t)$ respectively ($f_i : [0, 1] \mapsto \mathbb{R}^d$). Define δ as the minimum distance between the two manifolds, i.e., $\delta = \min_t \|f_1(t) - f_2(t)\|$, and let $\delta > 0$. For now, restrict each f_i such that the distance along the manifold between $f_i(t)$ and $f_i(s)$ is equal to the difference between t and s (this also implies that each manifold is of length 1). We sample $T_1, \dots, T_n \stackrel{iid}{\sim} F$ for continuous F with support $[0, 1]$ and use f_1 to map the first n_1 points to \mathcal{M}_1 and f_2 to map the remaining $n_2 = n - n_1$ points to \mathcal{M}_2 . Let $X_i = f_1(T_i)$ and $Y_j = f_2(T_j)$. Without loss of generality, assume $n_1 \leq n_2$.

Preliminary Theory

Distributions of differences of order statistics

Let $D_i = X_{(i+1)} - X_{(i)}$. Then if $\max_i D_i < \delta$, we have sufficient separation of points in \mathcal{M}_1 . Then it is sufficient to quantify $P(\max_i D_i > \delta)$ as a function of n and δ and show that this converges to zero as n grows to ∞ .

We denote $f(x)$ as the density of each X_i , $g_i(x)$ as the density of $X_{(i)}$, $g_{ij}(x, y)$ as the joint density of $X_{(i)}, X_{(j)}$, and $h_i(d)$ as the density of D_i (with corresponding capital letters for the cumulative distribution functions).

The following are taken as given¹:

1. $g_i(x) = \frac{n!}{(n-i)!(i-1)!} (F(x))^{i-1} (1 - F(x))^{n-i} f(x)$.
2. $g_{ij}(x) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} (F(x))^{i-1} (F(y) - F(x))^{j-i-1} (1 - F(y))^{n-j} f(x) f(y)$.
3. By convolution, $h_i(d) = \int_0^1 g_{i,i+1}(x, x+d) dx$.

Lemma 1 (The probability density function of D_i).

$$h_i(d) = \int_0^{1-d} \frac{n!}{(i-1)!(n-i-1)!} (F(x))^{i-1} (1 - F(x+d))^{n-i-1} f(x) f(x+d) dx \quad (1)$$

Proof. This is just given by setting the integrand as $g_{i,i+1}(x, x+d)$. We briefly note that the integral can only be evaluated from 0 to $1-d$ in this case because of the $x+d$ terms. \square

Lemma 2 (The cumulative distribution function of D_i).

$$P(D_i < \delta) = H_i(\delta) = 1 - \int_0^{1-\delta} \frac{n!}{(n-i)!(i-1)!} (F(x))^{i-1} (1 - F(x+\delta))^{n-i} f(x) dx \quad (2)$$

¹https://en.wikipedia.org/wiki/Order_statistic

Proof.

$$\begin{aligned}
H_i(\delta) &= \int_x^{x+\delta} h_i(d) dd \\
&= \int_x^{x+\delta} \int_0^1 \frac{n!}{(i-1)!(n-i-1)!} ((F(x))^{i-1} (1-F(x+d))^{n-i-1} f(x) f(x+d) dx dd \\
&= \int_0^1 \frac{n!}{(i-1)!(n-i-1)!} (F(x))^{i-1} f(x) \int_x^{x+\delta} (1-F(x+d))^{n-i-1} f(x+d) dd dx \\
&= \int_0^1 \frac{n!}{(i-1)!(n-i-1)!} (F(x))^{i-1} f(x) \int_{F(x)}^{F(x+\delta)} (1-u)^{n-i-1} du dx \\
&= \int_0^1 \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} f(x) ((1-F(x))^{n-i} - (1-F(x+\delta))^{n-i}) dx \\
&= \int_0^1 g_i(x) dx - \int_0^1 \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1-F(x+\delta))^{n-i} f(x) dx \\
&= 1 - \int_0^1 \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1-F(x+\delta))^{n-i} f(x) dx
\end{aligned}$$

Because of the $x + \delta$ term, we can't actually evaluate this integral all the way up to 1, and so we are left with

$$= 1 - \int_0^{1-\delta} \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1-F(x+\delta))^{n-i} f(x) dx.$$

□

Thus if we are able to show that Eq. (2) converges to 1, we are done.

Uniform case

Lemma 3 (Differences between order statistics of a uniform distribution). *If $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$, then each $D_i \sim \text{Beta}(1, n)$.*

Proof. We begin with Eq. (1), plugging in $f(x) = 1$ and $F(x) = x$:

$$h_i(d) = \int_0^{1-d} \frac{n!}{(i-1)!(n-i-1)!} x^{i-1} (1-x-d)^{n-i-1} dx$$

Then we proceed with integration by parts, setting $u = x^{i-1} \implies du = (i-1)x^{i-2}$ and $dv = (1-x-d)^{n-i-1} dx \implies v = -\frac{1}{n-i}(1-x-d)^{n-i}$. Note that $uv|_0^{1-d} = 0$ in this case. This yields

$$= \frac{n!}{(i-1)!(n-i-1)!} \int \frac{i-1}{n-i} x^{i-2} (1-x-d)^{n-i} dx$$

Then applying integration by parts again until the x^p term disappears, we get:

$$\begin{aligned}
&= \frac{n!}{(i-1)!(n-i-1)!(n-i)\cdots(n-2)} \int_0^{1-d} (1-x-d)^{n-2} dx \\
&= -\frac{n(n-1)}{n-1} (1-x-d)^{n-1} \Big|_0^{1-d} \\
&= n(1-d)^{n-1}
\end{aligned}$$

This the density function for $\text{Beta}(1, n)$, completing the proof. \square

Theorem 1. Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$. Then for any $\delta > 0$, $P(\max_i X_{(i+1)} - X_{(i)} < \delta) \rightarrow 1$ as $n \rightarrow \infty$.

Proof (sketch). Since $X_{(i+1)} - X_{(i)} = D_i \sim \text{Beta}(1, n)$, $P(X_{(i+1)} - X_{(i)} < \delta) = 1 - (1 - \delta)^n$. This yields

$$\begin{aligned}
P(\max_i D_i < \delta) &\geq (P(D_i < \delta))^{n-1} \\
&= (1 - (1 - \delta)^n)^{n-1} \\
&\approx e^{-n \exp(-n\delta)}.
\end{aligned}$$

In the limit $n \rightarrow \infty$, this goes to 1. \square

General case

Theorem 2. Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F$ with support $[0, 1]$, and suppose $f(x)$ is continuous and $f(x) \geq a > 0$ everywhere on the support. Let $D_i = X_{(i+1)} - X_{(i)}$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $P(\max_i D_i < \delta) > 1 - \epsilon$ when $n > N$.

Proof (sketch). We start with Eq. (2):

$$P(D_i \leq \delta) = 1 - \int_0^{1-\delta} \frac{n!}{(n-i)!(i-1)!} (F(x))^{i-1} (1 - F(x + \delta))^{n-i} f(x) dx.$$

Making the approximation $F(x + \delta) \approx F(x) + \delta f(x)$ and bounding $f(x) \geq a$, we get:

$$P(D_i \leq \delta) \geq 1 - \int_0^{1-\delta} \frac{n!}{(n-i)!(i-1)!} (F(x))^{i-1} (1 - F(x) - a\delta)^{n-i} f(x) dx.$$

Then making the substitution $u = F(x) \implies du = f(x) dx$, we obtain

$$1 - \int_0^{F(1-\delta)} \frac{n!}{(n-i)!(i-1)!} u^{i-1} (1 - u - a\delta)^{n-i} du$$

Evaluating the integral yields

$$P(D_i < \delta) = 1 - (1 - a\delta)^n + (1 - F(1 - \delta) - a\delta)^n.$$

Then as before,

$$\begin{aligned} P(\max_i D_i < \delta) &\geq (P(D_i < \delta))^{n-1} \\ &\approx (1 - (1 - a\delta)^n + (1 - F(1 - \delta) - a\delta)^n)^{n-1} \\ &\rightarrow 1 \end{aligned}$$

□

Algorithms

TBD

Computational Results

TBD