

Community Detection Methods for the Generalized Random Dot Product Graph Model

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1 Introduction

1.1 Research Goal

Graph and network data have become increasingly widespread in various fields including sociology, neuroscience, biostatistics, and computer science. This has resulted in various challenges for researchers who rely on traditional statistical and machine learning methods, many of which are incompatible with graph data and instead require the data to exist as feature vectors in Euclidean space. Such challenges often involve clustering and community detection, which are often the goal of graph data analysis. Common clustering methods typically involve calculating some central or representative point for each cluster around which the data belonging to that cluster lie (e.g., Lloyd’s algorithm for K -means clustering [4], Gaussian Mixture Models [3]). Because these methods involve computing summary statistics within each cluster, such as the sample average, they cannot be applied directly to graphs, necessitating methods for transforming the graph data into feature vectors.

One family of methods for unifying graph community detection with traditional clustering techniques is Spectral Clustering [9], which involves embedding the graph into Euclidean space, followed by applying a popular clustering algorithm such as K -means clustering. The Random Dot Product Graph (RDPG) [11] and Generalized Random Dot Product Graph (GRDPG) [7] models take this further by explicitly constructing generative models such that latent positions in Euclidean space are used to induce graphs. A community detection algorithm motivated by this may involve learning the latent positions given an observed graph and then learning the community labels given the latent positions.

The aim of our research is to develop consistent community detection techniques under the RDPG and GRDPG frameworks. First, we explore existing generative graph models with underlying community structures that can be inferred by connecting these models to the RDPG or GRDPG. Then we explore other latent structures or mixture distributions in the latent space that induce graphs for which consistent community detection is possible.

1.2 Notation

Let $G = (V, E)$ be an undirected, unweighted graph with no self-loops and n vertices. Denote $A \in \{0, 1\}^{n \times n}$ as the adjacency matrix for G such that $A_{ij} = 1$ if there exists an edge between vertices i and j and $A_{ij} = 0$ otherwise. Because G is symmetric and contains no self-loops, $A_{ij} = A_{ji}$ and $A_{ii} = 0$ for $i, j \in [n]$. We further restrict our analyses to independent Bernoulli graphs. Let $P \in [0, 1]^{n \times n}$ be a symmetric matrix of edge probabilities. Graph G is sampled from P by drawing $A_{ij} \stackrel{\text{indep}}{\sim} \text{Bernoulli}(P_{ij})$ for each $1 \leq i < j \leq n$ ($A_{ji} = A_{ij}$ and $A_{ii} = 0$). Finally, we denote $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^{n \times d}$ as the sample $x_1, \dots, x_n \in \mathbb{R}^d$ and $z_1, \dots, z_n \in [K]$ as their corresponding (hidden) labels.

2 Literature Review

2.1 Generative Graph Models and Community Detection

Generative models for independent Bernoulli graphs involve defining the edge probability matrix P whose ij^{th} entry is the probability of an edge between vertices i and j for each $i, j \in [n]$. In order to motivate community detection methods, we restrict the generative model such that for

each pair of vertices, the probability of an edge between the vertices is conditioned on their labels. One such model is the Stochastic Block Model (SBM) [5]: Given K communities and each vertex belonging to one community, the SBM restricts P to $K(K+1)/2$ unique entries such that $P_{ij} = p_{z_i z_j}$ where $p_{kl} = p_{lk}$ is the probability of an edge between each vertex in community k to each vertex in community l . The homogeneous SBM further restricts P to two unique entries: $P_{ij} = p$ if $z_i = z_j$ and $P_{ij} = q$ otherwise. Several generalizations of the SBM have been introduced since, including the Degree Corrected Block Model (DCBM) and the Popularity Adjusted Block Model (PABM). Like the SBM, these models involve imposing some restriction on P based on community labels (and possibly other factors).

2.2 (Generalized) Random Dot Product Graphs

Like the SBM, DCBM, and PABM, the RDPG and GRDPG are generative models for graphs involving an edge probability matrix P . The RDPG starts with points in latent space $X \in \mathcal{X} \subset \mathbb{R}^d$ such that $\forall x, y \in \mathcal{X}, x^\top y \in [0, 1]$. P is then constructed as $P = XX^\top$, and graph G with adjacency matrix A is drawn from P . We provide a more formal definition of the RDPG and GRDPG below.

Definition 1 ((Generalized) Random Dot Product Graph). *Let $X \in \mathbb{R}^{n \times d}$ be a collection of n points in $\mathcal{X} \subset \mathbb{R}^d$ such that $\forall x, y \in \mathcal{X}, x^\top y \in [0, 1]$. $G = (V, E)$ is a Random Dot Product Graph if its adjacency matrix A is drawn such that $A_{ij} \sim \text{Bernoulli}(x_i^\top x_j)$ for $i < j$, with $A_{ji} = A_{ij}$ and $A_{ii} = 0 \forall i, j \in [n]$. If on the other hand $A_{ij} \sim \text{Bernoulli}(x_i^\top I_{p,q} x_j)$ where $I_{p,q} = \text{blockdiag}(I_p, -I_q)$ and $p + q = d$, then A is the adjacency matrix of a Generalized Random Dot Product Graph. These are denoted by $A \sim \text{RDPG}(X)$ and $A \sim \text{GRDPG}_{p,q}(X)$ respectively.*

In addition, let F be a probability distribution with support \mathcal{X} , and $x_1, \dots, x_n \stackrel{iid}{\sim} F$ with $X = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top$. If A is drawn from X as before, then $(A, X) \sim \text{RDPG}(F, n)$ or $(A, X) \sim \text{GRDPG}_{p,q}(F, n)$.

The structure of the RDPG and GRDPG provides a straightforward method for recovery of the latent positions via spectral decomposition.

Definition 2 (Adjacency Spectral Embedding). *Let $A \sim \text{RDPG}(X)$ for $X \in \mathcal{X} \subset \mathbb{R}^{n \times d}$. Let $A = V\Lambda V^\top$ be the approximate spectral decomposition of A corresponding to the d largest eigenvalues and their corresponding eigenvectors. Then the rows of $V\Lambda^{1/2}$ are the scaled Adjacency Spectral Embedding (ASE) of A , and the rows of V are the unscaled ASE of A .*

If $A \sim \text{GRDPG}_{p,q}(X)$, then let $A = V\Lambda V^\top$ be the approximate spectral decomposition of A corresponding to the p most positive and q most negative eigenvalues of A and their corresponding eigenvectors. Then the rows of $V|\Lambda|^{1/2}$ and V are the scaled and unscaled ASE of A respectively.

Athreya et al. [1] showed that under mild conditions, if $(A_n, X_n) \sim \text{RDPG}(F, n)$ and \hat{X}_n is the scaled ASE of A_n , for some sequence of orthogonal matrices W_n ,

$$\max_i \|(\hat{X}_n)_i - W_n(X_n)_i\| \xrightarrow{a.s.} 0 \quad (1)$$

Similarly, Rubin-Delanchy et al. [7] showed that for $(A_n, X_n) \sim \text{GRDPG}_{p,q}(F, n)$,

$$\max_i \|(\hat{X}_n)_i - Q_n(X_n)_i\| \xrightarrow{a.s.} 0 \quad (2)$$

where Q_n is a sequence of matrices in $\mathbb{O}(p, q)$, the indefinite orthogonal group of order p, q .

It is straightforward to show that all Bernoulli graphs with positive semidefinite P are special cases of the RDPG, which is a special case of the GRDPG, and all graphs generated by P are special cases of the GRDPG. This includes the SBM, DCBM, and PABM. In the following example, we show that under the RDPG framework, the latent configuration that induces the SBM takes on a very particular form.

Example (Connecting the SBM to the RDPG). *Let $G = (V, E)$ with adjacency matrix A be sampled from the SBM with two communities for which the within-edge probabilities are p and q for communities 1 and 2 respective, and the between-community edge probability is r , with $pq > r^2$. Let community 1 have n_1 vertices and community 2 have n_2 vertices such that $n_1 + n_2 = n$. Without loss of generality, organize P and A such that the kl^{th} block represents edges between communities k and l . Then $P = \begin{bmatrix} P^{(11)} & P^{(12)} \\ P^{(21)} & P^{(22)} \end{bmatrix}$ where each block is a constant value, e.g., $P_{ij}^{(11)} = p$. One RDPG representation of this SBM is:*

$$X = \begin{bmatrix} \sqrt{p} & 0 \\ \vdots & \vdots \\ \sqrt{p} & 0 \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \\ \vdots & \vdots \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \end{bmatrix} \in \mathbb{R}^{n \times 2}$$

where the first n_1 rows are $\begin{bmatrix} \sqrt{p} & 0 \end{bmatrix}$ and the next n_2 rows are $\begin{bmatrix} \sqrt{r^2/p} & \sqrt{q - r^2/p} \end{bmatrix}$. Then it can be shown that

$$P = XX^\top$$

The ASE of the assortative SBM consists of points in \mathbb{R}^K that lie near one of K centers, depending on the community label, leading to ASE followed by K -means clustering (or similar, e.g., GMM) as a consistent community detection algorithm [6]. A similar result can be shown for the DCBM and PABM but with different structures in the ASE.

2.3 Manifold Learning

Trosset et al. [8] showed that the ASE of a RDPG can be used to recover one-dimensional manifolds. Suppose $f : [0, 1] \mapsto \mathcal{X}$ such that f is smooth and \mathcal{X} represents a curve or one-dimensional manifold in \mathbb{R}^d . If $t_1, \dots, t_n \stackrel{\text{iid}}{\sim} F$ such that F has support $[0, 1]$, the latent positions are $x_i = f(t_i)$ with y_i is its corresponding point in the scaled ASE, and $d_\epsilon(\cdot, \cdot)$ is the shortest path distance of an ϵ -neighborhood graph. Under certain mild conditions, the shortest path distances of the ϵ -neighborhood graph of the ASE approaches the arc lengths along f :

$$d_\epsilon(y_i, y_j) \xrightarrow{p} \int_{t_i}^{t_j} \sqrt{\sum_r^d \left(\frac{df_r}{dt} \right)^2} dt \quad (3)$$

Athreya et al. [2] extended this further by generating a RDGP from a mixture of distributions on a curve. In their example, points were sampled from a mixture of two Beta distributions on the Hardy-Weinberg curve to construct the latent positions of a RDGP, with the goal of recovering the hidden mixture distribution from an observed graph.

3 Proposed Research

3.1 Connecting Existing Generative Graph Models to the RDGP and GRDGP

As discussed in §2.1, all Bernoulli graph models can be expressed as a RDGP or GRDGP model, and in particular, the (associative) SBM is a RDGP with a very specific structure in the latent space. A similar result has been shown for the DCBM [6]. We have also extended this to the PABM. In §3.3.1, we show that the PABM is equivalent to the GRDGP model for which the communities correspond to subspaces in the latent configuration. We further show that while the latent configuration is not unique, there exists one in particular for which the subspaces are orthogonal. This leads to two straightforward methods for community detection, one involving computing inner products in the ASE and another which is a slight modification of an existing algorithm for subspace clustering. Finally, we will briefly explore alternative, non-spectral-based methods for community detection in the ASE involving Expectation Maximization, MCMC sampling, and Mean Field Variational Inference.

3.1.1 Popularity Adjusted Block Model

We first define the PABM.

Definition 3 (Popularity Adjusted Block Model). *Let $P \in [0, 1]^{n \times n}$ be a symmetric edge probability matrix for a set of n vertices, V . Each vertex has a community label $1, \dots, K$, and the rows and columns of P are arranged by community label such that $n_k \times n_l$ block $P^{(kl)}$ describes the edge probabilities between vertices in communities k and l ($P^{(lk)} = (P^{(kl)})^\top$). Let graph $G = (V, E)$ be an undirected, unweighted graph such that its corresponding adjacency matrix $A \in \{0, 1\}^{n \times n}$ is a realization of $\text{Bernoulli}(P)$, i.e., $A_{ij} \stackrel{\text{indep}}{\sim} \text{Bernoulli}(P_{ij})$ for $i > j$ ($A_{ij} = A_{ji}$ and $A_{ii} = 0$).*

If each block $P^{(kl)}$ can be written as the outer product of two vectors:

$$P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top \quad (4)$$

for a set of K^2 fixed vectors $\{\lambda^{(st)}\}_{s,t=1}^K$ where each $\lambda^{(st)}$ is a column vector of dimension n_s , then graph G and its corresponding adjacency matrix A is a realization of a popularity adjusted block model with parameters $\{\lambda^{(st)}\}_{s,t=1}^K$.

We will use the notation $A \sim \text{PABM}(\{\lambda^{(kl)}\}_K)$ to denote a random adjacency matrix A drawn from a PABM with parameters $\lambda^{(kl)}$ consisting of K underlying communities.

It is straightforward to show that the PABM, as well as all graphs such that the adjacency matrix is drawn such that $A_{ij} \stackrel{\text{indep}}{\sim} \text{Bernoulli}(P_{ij})$, is a special case of the GRDGP. It can also be shown that the latent positions of the PABM under the GRDGP framework consists of K K -dimensional subspaces in \mathbb{R}^{K^2} . While there is no unique latent configuration X such that $XX^\top = P$, the edge probability P for the PABM, they all have this subspace structure, and one in particular consists of *orthogonal* subspaces.

Theorem 1 (Connecting the PABM to the GRDPG for $K = 2$). *Let*

$$X = \begin{bmatrix} \lambda^{(11)} & \lambda^{(12)} & 0 & 0 \\ 0 & 0 & \lambda^{(21)} & \lambda^{(22)} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where each $\lambda^{(kl)}$ is a vector as in Definition 1. Then $A \sim \text{GRDPG}_{3,1}(XU)$ and $A \sim \text{PABM}(\{(\lambda^{(kl)})_2\})$ are equivalent.

Theorem 2 (Generalization to $K > 2$). *There exists a block diagonal matrix $X \in \mathbb{R}^{n \times K^2}$ defined by PABM parameters $\{\lambda^{(kl)}\}_K$ and orthonormal matrix $U \in \mathbb{R}^{K^2 \times K^2}$ that is fixed for each K such that $A \sim \text{GRDPG}_{K(K+1)/2, K(K-1)/2}(XU)$ and $A \sim \text{PABM}(\{(\lambda^{(kl)})_K\})$ are equivalent.*

Proof. Define the following matrices from $\{\lambda^{(kl)}\}_K$:

$$\Lambda^{(k)} = \begin{bmatrix} \lambda^{(k,1)} & \dots & \lambda^{(k,K)} \end{bmatrix} \in \mathbb{R}^{n_k \times K}$$

$$X = \text{blockdiag}(\Lambda^{(1)}, \dots, \Lambda^{(K)}) \in \mathbb{R}^{n \times K^2} \quad (5)$$

$$L^{(k)} = \text{blockdiag}(\lambda^{(1k)}, \dots, \lambda^{(Kk)}) \in \mathbb{R}^{n \times K}$$

$$Y = \begin{bmatrix} L^{(1)} & \dots & L^{(K)} \end{bmatrix} \in \mathbb{R}^{n \times K^2}$$

Then $P = XY^\top$.

Similar to the $K = 2$ case, we have $Y = X\Pi$ for a permutation matrix Π , resulting in $P = X\Pi X^\top$. The permutation described by Π has K fixed points, which correspond to K eigenvalues equal to 1 with corresponding eigenvectors e_k where $k = r(K+1) + 1$ for $r = 0, \dots, K-1$. It also has $\binom{K}{2} = K(K-1)/2$ cycles of order 2. Each cycle corresponds to a pair of eigenvalues $+1$ and -1 and a pair of eigenvectors $(e_s + e_t)/\sqrt{2}$ and $(e_s - e_t)/\sqrt{2}$.

Then Π has $K(K+1)/2$ eigenvalues equal to 1 and $K(K-1)/2$ eigenvalues equal to -1 . Π has the decomposed form

$$\Pi = UI_{K(K+1)/2, K(K-1)/2}U^\top \quad (6)$$

The edge probability matrix then can be written as:

$$P = XU I_{p,q}(XU)^\top \quad (7)$$

$$p = K(K + 1)/2 \quad (8)$$

$$q = K(K - 1)/2 \quad (9)$$

and we can describe the PABM with K communities as a GRDPG with latent positions XU with signature $(K(K + 1)/2, K(K - 1)/2)$. \square

Example ($K = 3$). *Using the same notation as in Theorem 2:*

$$X = \begin{bmatrix} \lambda^{(11)} & \lambda^{(12)} & \lambda^{(13)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{(21)} & \lambda^{(22)} & \lambda^{(23)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{(31)} & \lambda^{(32)} & \lambda^{(33)} \end{bmatrix}$$

$$Y = \begin{bmatrix} \lambda^{(11)} & 0 & 0 & \lambda^{(12)} & 0 & 0 & \lambda^{(13)} & 0 & 0 \\ 0 & \lambda^{(21)} & 0 & 0 & \lambda^{(22)} & 0 & 0 & \lambda^{(23)} & 0 \\ 0 & 0 & \lambda^{(31)} & 0 & 0 & \lambda^{(32)} & 0 & 0 & \lambda^{(33)} \end{bmatrix}$$

Then $P = XY^\top$ and $Y = X\Pi$ where Π is a permutation matrix consisting of 3 fixed points and 3 cycles of order 2:

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

* Positions 1, 5, 9 are fixed.

* The cycles of order 2 are (2, 4), (3, 7), and (6, 8).

Therefore, we can decompose $\Pi = UI_{6,3}U^\top$ where the first three columns of U consist of e_1 , e_5 , and e_9 corresponding to the fixed positions 1, 5, and 9, the next three columns consist of eigenvectors $(e_k + e_l)/\sqrt{2}$, and the last three columns consist of eigenvectors $(e_k - e_l)/\sqrt{2}$, where pairs (k, l) correspond to the cycles of order 2 described above.

The latent positions are the rows of

$$XU = \begin{bmatrix} \lambda^{(11)} & 0 & 0 & \lambda^{(12)}/\sqrt{2} & \lambda^{(13)}/\sqrt{2} & 0 & \lambda^{(12)}/\sqrt{2} & \lambda^{(13)}/\sqrt{2} & 0 \\ 0 & \lambda^{(22)} & 0 & \lambda^{(21)}/\sqrt{2} & 0 & \lambda^{(23)}/\sqrt{2} & -\lambda^{(21)}/\sqrt{2} & 0 & \lambda^{(23)}/\sqrt{2} \\ 0 & 0 & \lambda^{(33)} & 0 & \lambda^{(31)}/\sqrt{2} & \lambda^{(32)}/\sqrt{2} & 0 & -\lambda^{(31)}/\sqrt{2} & -\lambda^{(32)}/\sqrt{2} \end{bmatrix}$$

Algorithm 1: Orthogonal Spectral Clustering.

Data: Adjacency matrix A , number of communities K

Result: Community assignments $1, \dots, K$

- 1 Compute the eigenvectors of A that correspond to the $K(K+1)/2$ most positive eigenvalues and $K(K-1)/2$ most negative eigenvalues. Construct V using these eigenvectors as its columns.
 - 2 Compute $B = |nVV^\top|$, applying $|\cdot|$ entry-wise.
 - 3 Construct graph G using B as its similarity matrix.
 - 4 Partition G into K disconnected subgraphs (e.g., using edge thresholding or spectral clustering).
 - 5 Map each partition to the community labels $1, \dots, K$.
-

In Theorem 2, we showed that a possible latent configuration for a K -community PABM under the GRDPG framework consists of vectors in \mathbb{R}^{K^2} such that each community corresponds to a K -dimensional subspace and each subspace is orthogonal to the others.

Theorem 3. Let $P = V\Lambda V^\top$ be the spectral decomposition of the edge probability matrix of a PABM. Define $B = nVV^\top$. Then $B_{ij} = 0 \forall i, j$ in different communities.

If \hat{V} is the *unscaled* ASE of A , then results from Rubin-Delanchy et al. [7] imply $n\hat{V}\hat{V}^\top \xrightarrow{a.s.} nVV^\top$. Then for vertices i and j belonging to separate communities, the ij^{th} entry of $n\hat{V}\hat{V}^\top$ approaches 0 with probability 1:

Theorem 4. Let \hat{B}_n with entries $\hat{B}_n^{(ij)}$ be the affinity matrix from OSC (Alg. 1). Then \forall pairs (i, j) belonging to different communities and sparsity factor satisfying $n\rho_n = \omega\{(\log n)^{4c}\}$,

$$\max_{i,j} |n(\hat{v}_n^{(i)})^\top \hat{v}_n^{(j)}| = O_P\left(\frac{(\log n)^c}{\sqrt{n\rho_n}}\right) \quad (10)$$

This provides the result that $\forall i, j$ in different communities, $\hat{B}_n^{(ij)} \xrightarrow{a.s.} 0$.

Since the communities are subspaces in the latent space, we can also use existing methods for subspace clustering on the ASE. Of particular interest is Sparse Subspace Clustering (SSC), which is performed by solving an optimization problem for each observed point in a sample. Given $X \in \mathbb{R}^{n \times d}$ with vectors $x_i^\top \in \mathbb{R}^d$ as rows of X , the optimization problem $c_i = \min_c \|c\|_1$ subject to $x_i = X_{-i}c$ and $c^{(i)} = 0$ is solved for each $i \in [n]$. The solutions are collected into matrix $C = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}^\top$ to construct affinity matrix $B = |C| + |C^\top|$. If each x_i lie perfectly on one of K subspaces, B is sparse such that $B_{ij} = 0 \forall x_i, x_j$ belonging to different subspaces. Then B can describe a graph with at least K disjoint subgraphs, and if the number of subgraphs is exactly K , each subgraph maps onto a subspace.

In practice, SSC is performed by solving the LASSO problems:

$$c_i = \arg \min_c \frac{1}{2} \|x_i - X_{-i}c\|_2^2 + \lambda \|c\|_1 \quad (11)$$

for some sparsity parameter $\lambda > 0$. The c_i vectors are then collected into C and B as described before. If X is noisy such that each x_i does not lie exactly on one of K subspaces but near it, the choice of λ becomes important in guaranteeing the Subspace Detection Property (SDP) [10].

Definition 4 (Subspace Detection Property). Let $X = [x_1 \ \cdots \ x_n]^\top$ be noisy points sampled from K subspaces. Let C and B be constructed from the solutions of LASSO problems as described in (11). If each column of C has nonzero norm and $B_{ij} = 0 \ \forall \ x_i$ and x_j sampled from different subspaces, then X obeys the Subspace Detection Property.

Remark. In practice, a noisy sample X often does not obey SDP. In such cases, B is treated as an affinity matrix for a graph which is then partitioned into K subgraphs to obtain the clustering. On the other hand, if X does obey the SDP, B describes a graph with at least K disconnected subgraphs. Ideally, when SDP holds, there are exactly K subgraphs which map to each subspace, but it could be the case that some of the subspaces are represented by multiple disconnected subgraphs.

Since every ASE of the PABM consists of subspaces and as $n \rightarrow \infty$ each vector of the ASE approaches its subspace uniformly [7], SSC is also able to perform community detection for the PABM. It has been shown by Wang and Xu [10] that if the points lie sufficiently close to their respective subspaces and the cosine of the angles between subspaces is sufficiently small, SDP will hold. We now state that the unscaled ASE of the PABM exhibits exactly these conditions for sufficiently large n .

Theorem 5. Let P_n describe the edge probability matrix of the PABM with n vertices, and let $A_n \sim \text{Bernoulli}(P_n)$. Let \hat{V}_n be the matrix of eigenvectors of A_n corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues. Then $\exists \lambda > 0$ and $N < \infty$ such that when $n > N$, $\sqrt{n}\hat{V}_n$ obeys the subspace detection property with probability 1.

3.1.1.1 Alternative Approaches to Community Detection So far, we connected the PABM to the GRDPG and used this to motivate community detection methods based on the ASE of the adjacency matrix sampled from the PABM. In this section, we will explore non-spectral methods for community detection.

Denoting $\{\lambda_{ik}\}$ as the scalar popularity parameters for the PABM such that $\lambda^{(kl)}$ is the vector of λ_{jl} terms for which vertex j is in community k , we can write the likelihood for the PABM:

$$\begin{aligned} p(A|z, \{\lambda_{ik}\}) &= \prod_{i < j} (\lambda_{iz_j} \lambda_{jz_i})^{A_{ij}} (1 - \lambda_{iz_j} \lambda_{jz_i})^{1-A_{ij}} \\ &= \prod_{i < j} \prod_k \prod_l (\lambda_{il} \lambda_{jk})^{A_{ij} z_{ik} z_{jl}} (1 - \lambda_{il} \lambda_{jk})^{(1-A_{ij}) z_{ik} z_{jl}} \end{aligned} \quad (12)$$

where $z_{ik} = I(Z_i = k)$.

The log-likelihood is

$$\log p = \sum_{i < j} \sum_k \sum_l z_{ik} z_{jl} (A_{ij} \log \lambda_{il} \lambda_{jk} + (1 - A_{ij}) \log(1 - \lambda_{il} \lambda_{jk})) \quad (13)$$

While direct maximization of the likelihood or log-likelihood is NP-complete, it is possible to apply an Expectation Maximization (EM) algorithm to this problem.

3.2 Proposed Generative Graph Models

In the previous sections, we connected well-known and highly restricted generative graph models to the RDPG and GRDPG to show that highly structured latent configurations generate graphs consistent with these models: The latent space for the SBM consists of K point masses, the latent space for the DCBM consists of K rays emanating from the origin, and the latent space for the PABM consists of K K -dimensional subspaces. In the following sections, we explore additional structured latent configurations corresponding to community structure and develop methods for community detection based on the consistency of the ASE and the structural forms of the latent configurations.

The general structure of interest can be described as follows: Suppose that in the latent space $\mathcal{X} \subset \mathbb{R}^d$, sample X of n points lie on a union of K disjoint manifolds with each manifold corresponding to a community. If $A \sim \text{RDPG}(X)$, we wish to recover the community labels (up to permutation) from A .

Equivalently, suppose that probability distribution F is described as follows:

1. Define functions f_1, \dots, f_K such that $f_k : [0, 1] \mapsto \mathcal{X}$ and $f_k(t) \neq f_l(t) \forall k, l \in [K]$ and $t \in [0, 1]$.
2. Sample labels $z_1, \dots, z_n \stackrel{\text{iid}}{\sim} \text{Categorical}(\pi_1, \dots, \pi_K)$.
3. Sample $t_1, \dots, t_n \stackrel{\text{iid}}{\sim} D$ where D has support $[0, 1]$.
4. Set latent positions $x_i = f_{z_i}(t_i)$ and $X = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top$.

Then if $(A, X) \sim \text{RDPG}(F, n)$ and we observe A , we wish to recover hidden labels z_1, \dots, z_n .

3.2.1 Affine Segments

In this section, we will consider latent positions sampled uniformly from parallel unit length segments.

Example. Let $U_1, \dots, U_{n_1}, V_1, \dots, V_{n_2} \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$, $f_1(t) = (t, 0)$, and $f_2(t) = (t, a)$. $X_i = f_1(U_i)$ and $Y_j = f_2(V_j) \forall i \in [n_1]$ and $j \in [n_2]$. If we observe $X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}$, what approach will allow us to group the observations by segment?

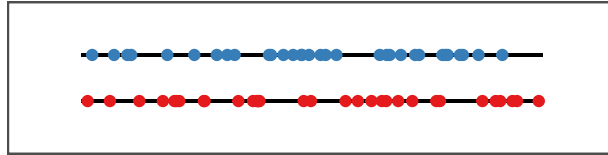


Figure 1: Points sampled uniformly on two parallel unit length line segments

We will motivate an approach by the following example.

Example. Let $U_1, \dots, U_n \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$ with order statistics $U_{(1)}, \dots, U_{(n)}$. Then $\forall a \in (0, 1)$ and $\delta \in (0, 1)$, $\exists N = N(\delta, a) < \infty$ such that $\forall n \geq N$,

$$P(\max_i U_{(i+1)} - U_{(i)} \leq a) \geq 1 - \delta/2 \quad (14)$$

Where $N(\delta, a)$ is monotone increasing with respect to δ and a . To prove this, we start with the fact that $U_{(i+1)} - U_{(i)} \sim \text{Beta}(1, n)$. Then

$$P(U_{(i+1)} - U_{(i)} \leq a) = 1 - (1 - a)^n \quad (15)$$

and

$$P(\max_i U_{(i+1)} - U_{(i)} \leq a) \geq (P(U_{(i+1)} - U_{(i)} \leq a))^n = (1 - (1 - a)^n)^n \quad (16)$$

This expression is monotone increasing $\forall n \geq N_1$ for some $N_1 < \infty$. Setting $(1 - (1 - a)^{N_2})^{N_2} \geq 1 - \delta/2$, we can solve for a finite N_2 . Then $N = \max(N_1, N_2)$.

If we extend this example such that n_1 points are sampled uniformly from the segment $f_1(t) = (t, 0)$ and n_2 points are sampled uniformly from the segment $f_2(t) = (t, a)$ for $t \in [0, 1]$, then a sample of size $\min(n_1, n_2) \geq N(\delta, a)$ is sufficient to satisfy:

$$P(\max_i X_{(i+1)} - X_{(i)} \leq a \leq \min_{i,j} \|X_i - Y_j\|) \geq 1 - \delta/2 \quad (17)$$

and similar for $P(\max_j Y_{(j+1)} - Y_{(j)} \leq \min_{i,j} \|X_i - Y_j\|)$, for X_i in the first segment and Y_j in the second segment and $X_{(i)}, Y_{(j)}$ are order statistics in the first coordinate. If each segment corresponds to a community, this leads to the following two results:

1. Single linkage clustering with $K = 2$ will perform perfect community detection with probability at least $1 - \delta$.
2. An ϵ -neighborhood graph with $\epsilon \in (0, a)$ will consists of at least 2 disjoint subgraphs such that no subgraph consists of members of two different communities (analogous to the SDP), with probability at least $1 - \delta$.

We can then further extend this to the case where points are drawn from unit segments with noise.

3.2.2 Manifold Learning

If instead of sampling uniformly from line segments of unit length, we sample uniformly from a 1 dimensional manifolds of unit length, the above property still holds. Let $U_1, \dots, U_n \stackrel{iid}{\sim} \text{Uniform}(0, 1)$ and $f : [0, 1] \mapsto \mathbb{R}^d$ be a smooth function such that $\int_u^v \sqrt{\sum_k (df_k/dt)^2} dt = \|u - v\|$. Then $U_{(i+1)} - U_{(i)} \geq \|f(U_{(i+1)}) - f(U_{(i)})\|$, so $P(U_{(i+1)} - U_{(i)} \leq \alpha) \leq P(\|f(U_{(i+1)}) - f(U_{(i)})\| \leq \alpha)$. If the shortest distance between the two manifolds defined by f_1 and f_2 with the same restriction is a , then the same N as before is sufficient, although perhaps a more lenient lower bound can be derived based on the shape of $f_k(\cdot)$.

4 Summary

5 Estimated Timeline of Completion

Literature review: August 2021

Complete proofs of main theorems: January 2022

Simulations and real data analyses: March 2022

Dissertation completion: April 2022

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