

# Manifold Clustering in the Setting of Generalized Random Dot Product Graphs

SDSS Lightning Presentation

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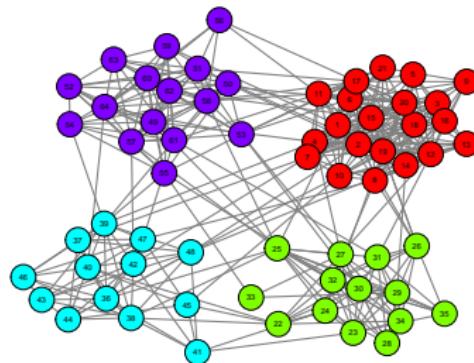


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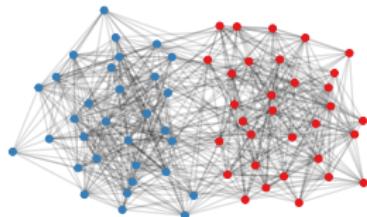
# Community Detection for Networks



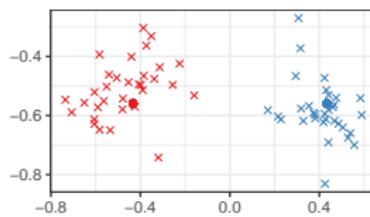
How might we cluster the nodes of a network?

# Connecting Block Models to the GRDPG

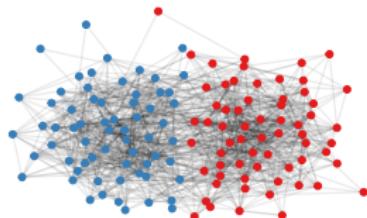
SBM



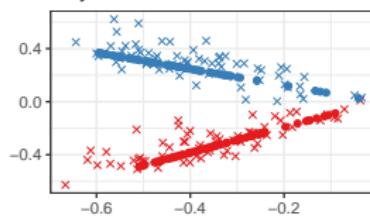
Point Masses



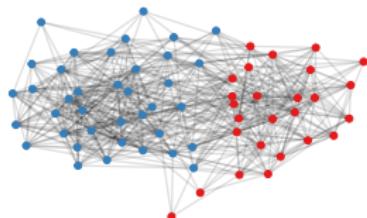
DCBM



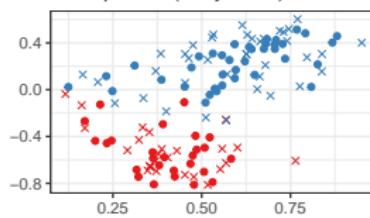
Rays



PABM

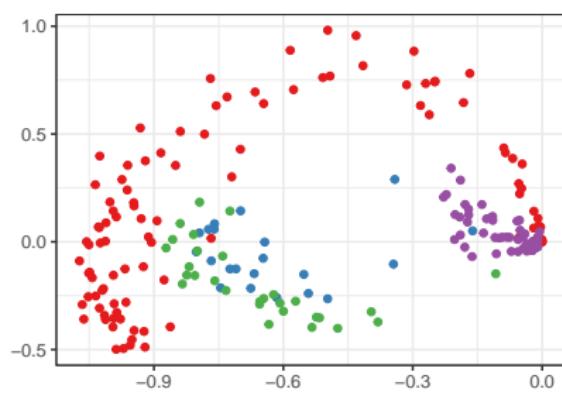
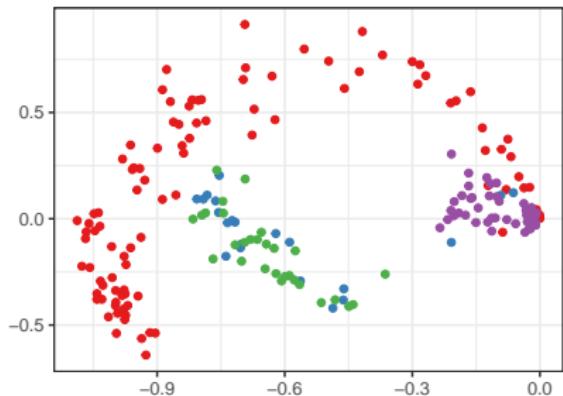
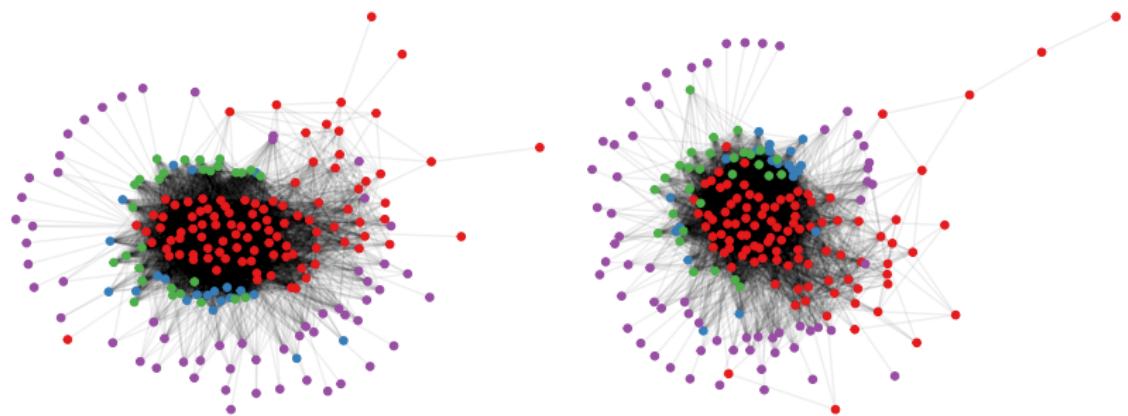


Subspaces (Projected)



- K-means clustering
- Gaussian mixture models
- K-means with cosine similarity
- GMM on angles
- Orthogonal Spectral Clustering
- Sparse Subspace Clustering

# Nonlinear Community Structure



## Manifold Block Model

Let  $p, q \geq 0$ ,  $d = p + q \geq 1$ ,  $1 \leq r < d$ ,  $K \geq 2$ , and  $n > K$  be integers. Define manifolds  $\mathcal{M}_1, \dots, \mathcal{M}_K \subset \mathcal{X}$  for

$\mathcal{X} = \{x, y \in \mathbb{R}^d : x^\top I_{p,q} y \in [0, 1]\}$  each by continuous function  $g_k : [0, 1]^r \rightarrow \mathcal{X}$ . Define probability distribution  $F$  with support  $[0, 1]^r$ . Then the following mixture model is a *manifold block model*:

1. Draw labels  $z_1, \dots, z_n \stackrel{\text{iid}}{\sim} \text{Categorical}(\alpha_1, \dots, \alpha_K)$ .
2. Draw latent vectors by first taking  $t_1, \dots, t_n \stackrel{\text{iid}}{\sim} F$  and then computing each  $x_i = g_{z_i}(t_i)$ .
3. Compile the latent vectors into data matrix  $X = [x_1 \mid \dots \mid x_n]^\top$  and define the adjacency matrix as  $A \sim \text{GRDPG}_{p,q}(X)$ .

# Manifold Block Model

1.  $z_1, \dots, z_n \stackrel{\text{iid}}{\sim} \text{Categorical}(1/2, 1/2)$
2.  $t_1, \dots, t_n \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$
3.  $x_i = g_{z_i}(t_i)$ 
  - $g_1(t) = [t^2, 2t(1-t)]^\top$
  - $g_2(t) = [2t(1-t), (1-t)^2]^\top$
4.  $A \sim \text{GRDPG}_{2,0}(X)$

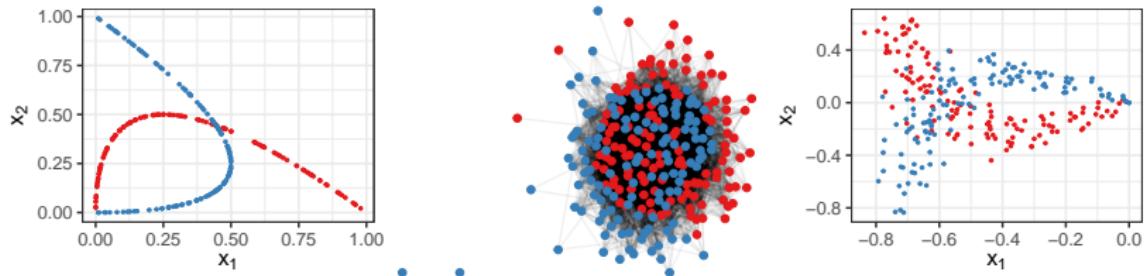


Figure 1: Latent vectors on intersecting curves (left), along with an RDGP drawn from this configuration (center) and its ASE (right).

# *K*-Curves Clustering

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## Algorithm 1: *K*-curves clustering.

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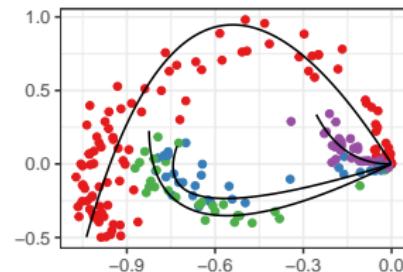
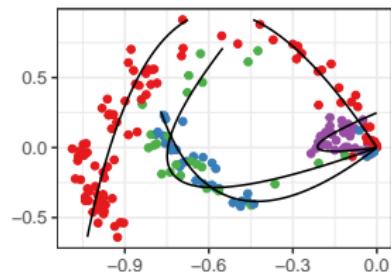
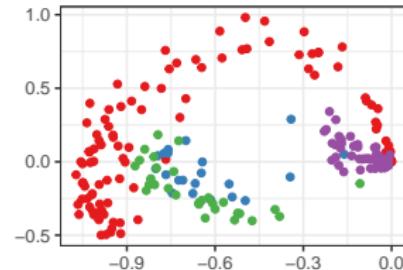
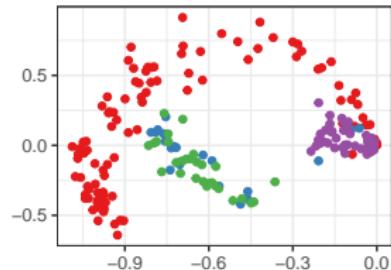
**Data:** Adjacency matrix  $A$ , number of communities  $K$ , embedding dimensions  $p$ ,  $q$ , stopping criterion  $\epsilon$

**Result:** Community assignments  $1, \dots, K$ , curves  $g_1, \dots, g_K$

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1 Compute  $X$ , the ASE of  $A$  using the  $p$  most positive and  $q$  most negative
   eigenvalues and their corresponding eigenvectors.
2 Initialize community labels  $z_1, \dots, z_n$ .
3 repeat
4   for  $k = 1, \dots, K$  do
5     Define  $X_k$  as the rows of  $X$  for which  $z_i = k$ .
6     Fit curve  $g_k$  and positions  $t_{k_i}$  to  $X_k$  by minimizing
          $\sum_{k_i} \|x_{k_i} - g_k(t_{k_i})\|^2$ .
7   end
8   for  $k = 1, \dots, K$  do
9     Assign  $z_i \leftarrow \arg \min_\ell \|x_i - g_\ell(t_i)\|^2$ .
10  end
11 until the change in  $\sum_k \sum_{i \in C_k} \|x_i - g_k(t_i)\|^2$  is less than  $\epsilon$ 
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# Example



## Asymptotic Results

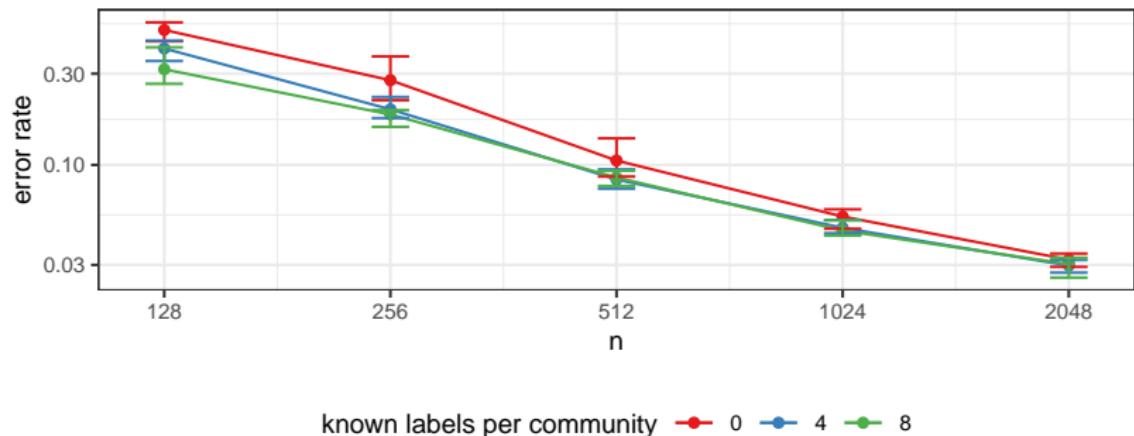
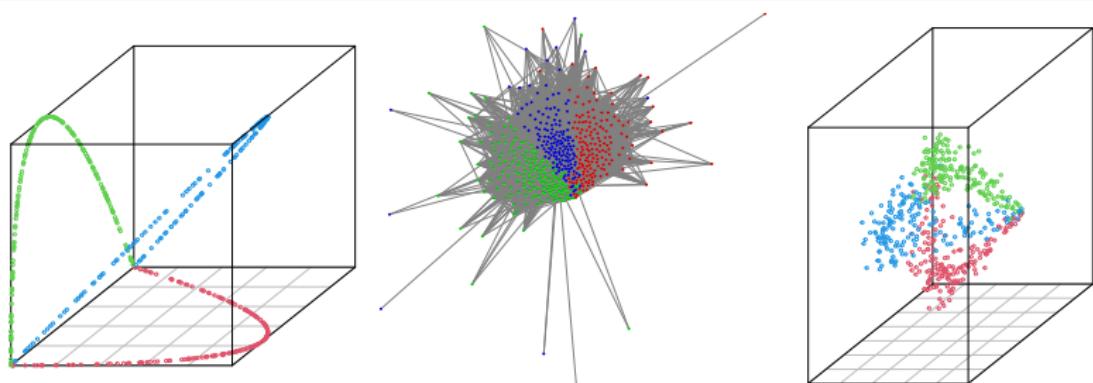
**Theorem.** Let an MBM be such that the manifolds are described by functions  $g_1(t), \dots, g_K(t)$  which are polynomial curves of order  $R$ . Define the loss of  $K$ -curves clustering as:

$$L(\hat{z}_1, \dots, \hat{z}_n, \hat{g}_1, \dots, \hat{g}_K; A) = \sum_k \sum_{i: \hat{z}_i=k} \|\hat{x}_i - \hat{g}_k(t_i)\|^2,$$

where  $\hat{x}_i$  are the embedding vectors of  $A$ . Suppose that for each community  $k$ , we have labels for at least  $R + 1$  vertices. Then as  $n \rightarrow \infty$ ,  $K$ -curves clustering outputs estimators such that

$$L(\hat{z}_1, \dots, \hat{z}_n, \hat{g}_1, \dots, \hat{g}_K; A) \xrightarrow{p} 0.$$

# Simulation



Thank you!

Code and drafts available at

<https://github.com/johneverettkoo/manifold-block-models>