Block Models
and
(Generalized) Random Dot Product Graphs

STAT-S 675

Fall 2021

Recall From Last Time ...

- Let $G = (V, E)$ be an undirected and hollow graph with $|V| = n$ and adjacency matrix A
  - $A \in \mathbb{R}^{n \times n}$ is symmetric with zero diagonals
- Suppose $G \sim F(\theta)$
  - What kind of $F(\theta)$ make sense here?
  - Given F and observed G, how can we estimate θ?

Recall From Last Time . . .

$$A_{ij} = \begin{cases} 1 & \exists \text{ edge between } i \text{ and } j \\ 0 & \text{else} \end{cases}$$

$A_{ji} = A_{ij}$ and $A_{ii} = 0 \; \forall i, j \in [n]$.

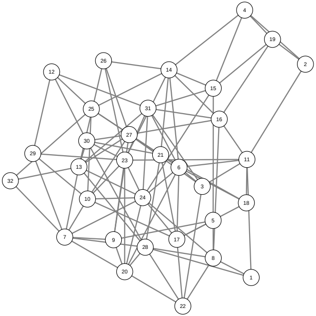
$A \sim \text{BernoulliGraph}(P)$ iff:

1. $P \in [0,1]^{n \times n}$ describes edge probabilities between pairs of vertices.
2. $A_{ij} \overset{\text{ind}}{\sim} \text{Bernoulli}(P_{ij})$ for each $i < j$.

For estimation, we need to impose some structure on P.

Example 1: If every entry $P_{ij} = \theta \in (0, 1)$, then $A \sim \text{BernoulliGraph}(P)$ is an Erdos-Renyi graph.

For this model, $A_{ij} \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$.

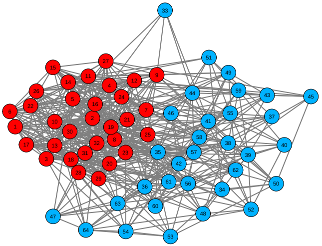


Recall From Last Time ...

Suppose each vertex $v_1, ..., v_n$ has hidden labels $z_1, ..., z_n \in [K]$, and each P_{ij} depends on labels z_i and z_j.
Then $A \sim \text{BernoulliGraph}(P)$ is a *block model*.

Example 2: Stochastic Block Model with two communities

- $z_1, ..., z_n \in \{1, 2\}$
- $P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$
- To make this an assortative SBM, set $pq > r^2$.
- In this example, $p = 1/2$, $q = 1/4$, and $r = 1/8$.



Recall From Last Time ...

Erdos-Renyi Model (1959)

- $P_{ij} = \theta$ (not a block model)
- 1 parameter θ

Stochastic Block Model (Lorrain and White, 1971)

- $P_{ij} = \theta_{z_i z_j}$
- $K(K+1)/2$ parameters θ_{kl}

Degree Corrected Block Model (Karrer and Newman, 2011)

- $P_{ij} = \theta_{z_i z_j} \omega_i \omega_j$
- $K(K+1)/2 + n$ parameters θ_{kl}, ω_i

Popularity Adjusted Block Model (Sengupta and Chen, 2017)

- $P_{ij} = \lambda_{i z_j} \lambda_{j z_i}$
- Kn parameters λ_{ik}

Recall From Last Time ...

Random Dot Product Graph $A \sim \text{RDPG}(X)$
(Young and Scheinerman, 2007)

- Latent vectors $x_1, ..., x_n \in \mathbb{R}^d$ such that $x_i^\top x_j \in [0, 1]$
- $A \sim \text{BernoulliGraph}(XX^\top)$ where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$

Generalized Random Dot Product Graph $A \sim \text{GRDPG}_{p,q}(X)$
(Rubin-Delanchy, Cape, Tang, Priebe, 2020)

- Latent vectors $x_1, ..., x_n \in \mathbb{R}^{p+q}$ such that $x_i^\top I_{p,q} x_j \in [0, 1]$ and $I_{p,q} = \text{blockdiag}(I_p, -I_q)$
- $A \sim \text{BernoulliGraph}(X I_{p,q} X^\top)$ where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$

Recall From Last Time . . .

Adjacency Spectral Embedding

To embed in $\mathbb{R}^d$,

1. Compute $A \approx \hat{V}\hat{\Lambda}\hat{V}^\top$ where $\hat{\Lambda} \in \mathbb{R}^{d\times d}$ and $\hat{V} \in \mathbb{R}^{n\times d}$.
   For RDPG, use d greatest eigenvalues; for GRDPG, use p most positive and q most negative eigenvalues.
2. For RDPG, let $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$; for GRDPG, let $\hat{X} = \hat{V}|\hat{\Lambda}|^{1/2}$.

RDPG: $\max_i \|\hat{x}_i - W x_i\| \stackrel{a.s.}{\to} 0$ (Athreya et al., 2018)

GRDPG: $\max_i \|\hat{x}_i - Q x_i\| \stackrel{a.s.}{\to} 0$ (Rubin-Delanchy et al., 2020)

Two Types of Bernoulli Graphs

Block Models

- $P_{ij} = f(z_i, z_j)$
- Community structure
- Estimation via EM
  - May require *ad hoc* approximations
  - Can be sensitive to initial guess

(G)RDPGs

- $P_{ij} = x_i^\top I_{p,q} x_j$
- Latent space structure
- Estimation via ASE
  - Consistency theorem
  - Straightforward algorithm
  - Non-identifiable linear transformation

All Bernoulli Graph models are (G)RDPGs

- Can we use this for inference on Block Models?

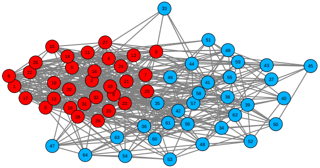
Connecting Block Models to (G)RDPGs

All Bernoulli Graphs are RDPG (if P is positive semidefinite) or GRDPG (in general).

Example 2 (cont'd): Assortative SBM ($pq > r^2$) with $K = 2$

$$P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$$

$$P = \begin{bmatrix} P^{(11)} & P^{(12)} \\ P^{(21)} & P^{(22)} \end{bmatrix} = XX^\top$$

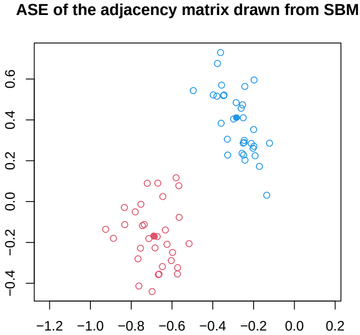


$$X = \begin{bmatrix} \sqrt{p} & 0 \\ \vdots & \vdots \\ \sqrt{p} & 0 \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \\ \vdots & \vdots \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \end{bmatrix}$$

Connecting the SBM to the RDPG

Example 2 (cont'd): If we want to perform community detection,

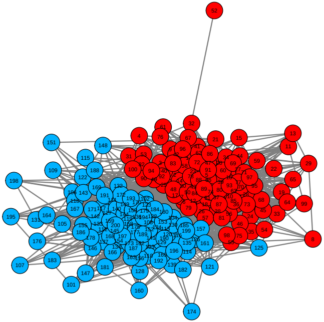
1. Note that A is a RDPG because $P = XX^\top$.
2. Compute the ASE $A \approx \hat{X}\hat{X}^\top$ with $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$.
3. Apply clustering algorithm (e.g., K-means) to $\hat{X}$, noting that as $n \to \infty$, the ASE approaches point masses.



Connecting the DCBM to the RDPG

Example 3: DCBM with two communities

- $z_1, ..., z_n \in \{1, 2\}$
- $P_{ij} = \begin{cases} p\omega_i\omega_j & z_i, z_j = 1 \\ q\omega_i\omega_j & z_i, z_j = 2 \\ r\omega_i\omega_j & z_i \neq z_j \end{cases}$
- To make this an assortative DCBM, set $pq > r^2$.
- In this example, $p = q = 1/2$, $r = 1/8$, and $\omega_i \in (0, 1)$.

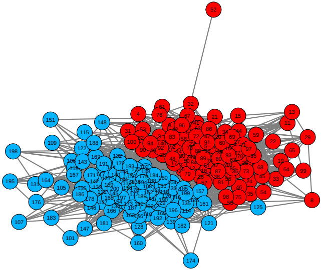


Connecting the DCBM to the RDPG

Example 3 (cont'd): Assortative DCBM ($pq > r^2$) with $K = 2$

$$P_{ij} = \begin{cases} p\omega_i\omega_j & z_i, z_j = 1 \\ q\omega_i\omega_j & z_i, z_j = 2 \\ r\omega_i\omega_j & z_i \neq z_j \end{cases}$$

$$P = \begin{bmatrix} P^{(11)} & P^{(12)} \\ P^{(21)} & P^{(22)} \end{bmatrix} = XX^\top$$

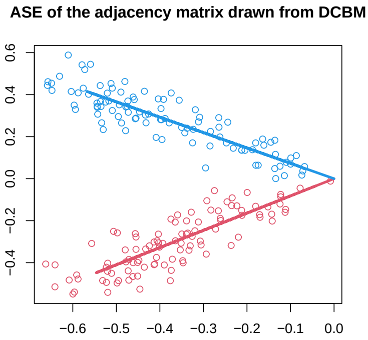


$$X = \begin{bmatrix} \sqrt{p}\omega_1 & 0 \\ \vdots & \vdots \\ \sqrt{p}\omega_{n_1} & 0 \\ \sqrt{\frac{r^2}{p}}\omega_{n_1+1} & \sqrt{q - \frac{r^2}{p}}\omega_{n_1+1} \\ \vdots & \vdots \\ \sqrt{\frac{r^2}{p}}\omega_n & \sqrt{q - \frac{r^2}{p}}\omega_n \end{bmatrix}$$

Connecting the SBM to the RDPG

Example 3 (cont'd): If we want to perform community detection,

1. Note that A is a RDPG because $P = XX^\top$.
2. Compute the ASE $A \approx \hat{X}\hat{X}^\top$ with $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$.
3. Apply clustering algorithm on $\hat{X}$,
   noting that as $n \to \infty$, the ASE approaches line segments.



Assortative SBM and DCBM are RDPGs

- P is positive semidefinite $\implies$ RDPG
- Number of embedding dimensions = number of communities
- Latent structure for the SBM are point masses, latent structure for the DCBM are line segments
- If the SBM or DCBM is not assortative, then P is not positive semidefinite
  - GRDPG instead of RDPG
  - Number of embedding dimensions = number of communities, except split by positive and negative eigenvalues
  - ASE still converges to true latent positions up to multiplication by unidentifiable $Q \in \mathbb{O}(p, q)$
  - Multiplication by Q doesn't change overall latent structure

Demo

Connecting the PABM to the GRDPG

Def Popularity Adjusted Block Model (Sengupta and Chen, 2017):

Let each vertex $i \in [n]$ have K popularity parameters $\lambda_{i1}, ..., \lambda_{iK} \in [0, 1]$. Then $A \sim$ PABM(P) if each $P_{ij} = \lambda_{i z_j} \lambda_{j z_i}$, e.g., if $z_i = k$ and $z_j = l$, $P_{ij} = \lambda_{il} \lambda_{jk}$.

Lemma (Noroozi, Rimal, and Pensky, 2020):

A is sampled from a PABM if P can be described as:

1. Let each $P^{(kl)}$ denote the $n_k \times n_l$ matrix of edge probabilities between communities k and l.
2. Organize popularity parameters as vectors $\lambda^{(kl)} \in \mathbb{R}^{n_k}$ such that $\lambda_i^{(kl)} = \lambda_{k_i l}$ is the popularity parameter of the i^{th} vertex of community k towards community l.
3. Each block can be decomposed as $P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top$.

Notation: $A \sim$ PABM($\{\lambda^{(kl)}\}_K$).

Connecting the PABM to the GRDPG

Theorem: $A \sim \text{PABM}(\{\lambda^{(kl)}\}_K)$ is equivalent to $A \sim \text{GRDPG}_{p,q}(XU)$ with

- $p = K(K+1)/2$, $q = K(K-1)/2$
- $U \in \mathbb{O}(K^2)$
- $X \in \mathbb{R}^{n \times K^2}$ is block diagonal and composed of $\{\lambda^{(kl)}\}_K$ with each block corresponding to a community.

$$X = \begin{bmatrix} \Lambda^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \Lambda^{(K)} \end{bmatrix} \in \mathbb{R}^{n \times K^2}$$

$$\Lambda^{(k)} = \begin{bmatrix} \lambda^{(k1)} & \cdots & \lambda^{(kK)} \end{bmatrix} \in \mathbb{R}^{n_k \times K}$$

Connecting the PABM to the GRDPG

$$X = \begin{bmatrix} \Lambda^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \Lambda^{(K)} \end{bmatrix} \qquad U \in \mathbb{O}(K^2)$$

$$A \sim \text{PABM}(\{\lambda^{(kl)}\}_K) \iff A \sim \text{GRDPG}_{p,q}(XU)$$

Remark 1 (orthogonality of subspaces): If $y_i^\top$ and $y_j^\top$ are two rows of XU corresponding to different communities, then $y_i^\top y_j = 0$.

Remark 2 (non-uniqueness of the latent configuration):
If $A \sim \text{GRDPG}_{p,q}(Y)$, then $A \sim \text{GRDPG}_{p,q}(YQ)$ for any Q in the indefinite orthogonal group with signature p, q.

Remark 3: Communities correspond to subspaces even with linear transformation $Q \in \mathbb{O}(p, q)$, but this may break the orthogonality property.

Orthogonal Spectral Clustering

Theorem (KTT): If $P = V\Lambda V^\top$ and $B = nVV^\top$, then $B_{ij} = 0$ if $z_i \neq z_j$.

Algorithm: Orthogonal Spectral Clustering:

1. Let V be the eigenvectors of A corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues.
2. Compute $B = |nVV^\top|$ applying $|\cdot|$ entry-wise.
3. Construct graph G using B as its similarity matrix.
4. Partition G into K disconnected subgraphs.

Theorem (KTT): Let $\hat{B}_n$ with entries $\hat{B}_n^{(ij)}$ be the affinity matrix from OSC. Then $\forall$ pairs (i,j) belonging to different communities and sparsity factor satisfying $n\rho_n = \omega((\log n)^{4c})$,

$$\max_{i,j} \hat{B}_n^{(ij)} = O_P\Big(\frac{(\log n)^c}{\sqrt{n\rho_n}}\Big)$$