# Popularity Adjusted Block Models are Generalized Random Dot Product Graphs



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#### Block Models

A block model is a random graph model in which each vertex  $v_i$  has a community label  $z_i \in [K]$  and the edge probability between each pair of vertices is determined in part by the pair of labels.

A is an adjacency matrix of a block model if and only if each  $A_{ij} \stackrel{\mathrm{ind}}{\sim} \mathrm{Bernoulli}(P_{ij})$  and  $P_{ij} = f(z_i, z_j, \cdot).$ 

#### Stochastic Block Model

Each edge probability is fixed for each pair of communities, i.e.,  $P_{ij} = \theta_{z_i,z_j}$ .

#### Degree Corrected Block Model

Each edge probability is determined by the community edge probability as well as each vertex's degree factor, i.e.,  $P_{ij} = \theta_{z_i,z_j}\omega_i\omega_j$ .

# Popularity Adjusted Block Model (Sengupta and Chen 2018)

Each vertex has a popularity parameter for each community that determines its affinity toward that community, i.e.,  $P_{ij} = \lambda_{i,z_j} \lambda_{j,z_i}$ .

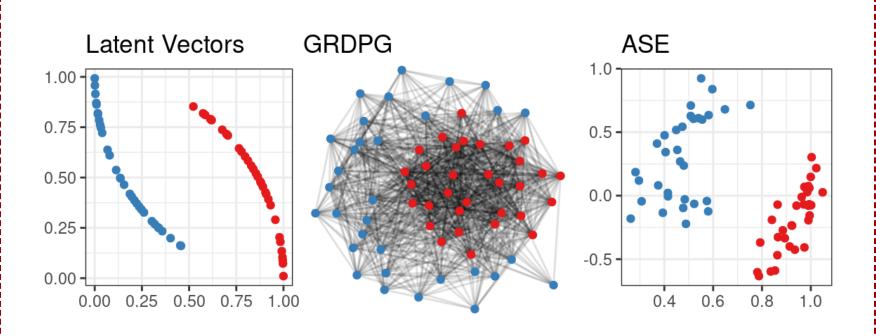
### Generalized Random Dot Product Graphs

The generalized random dot product graph is a random graph model in which each vertex  $v_i$  has a corresponding hidden vector  $x_i \in \mathbb{R}^{p+q}$  and each edge probability is the indefinite inner product of the corresponding pair of hidden vectors, i.e.,  $P_{ij} = x_i^{\top} I_{p,q} x_j$ ,  $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ 

#### Adjacency Spectral Embedding

Approximate A by spectral decomposition  $A \approx V_{p,q} \Lambda_{p,q} V_{p,q}^{\top}$ . The subscript p,q denotes the p most positive and q most negative eigenvalues and corresponding eigenvectors. Each  $\hat{x}_i$ , the  $i^{th}$  row of  $\hat{X} = V_{p,q} |\Lambda_{p,q}|^{1/2}$ , estimates the relative position of its corresponding latent vector  $x_i$ , up to an indefinite orthogonal transformation.

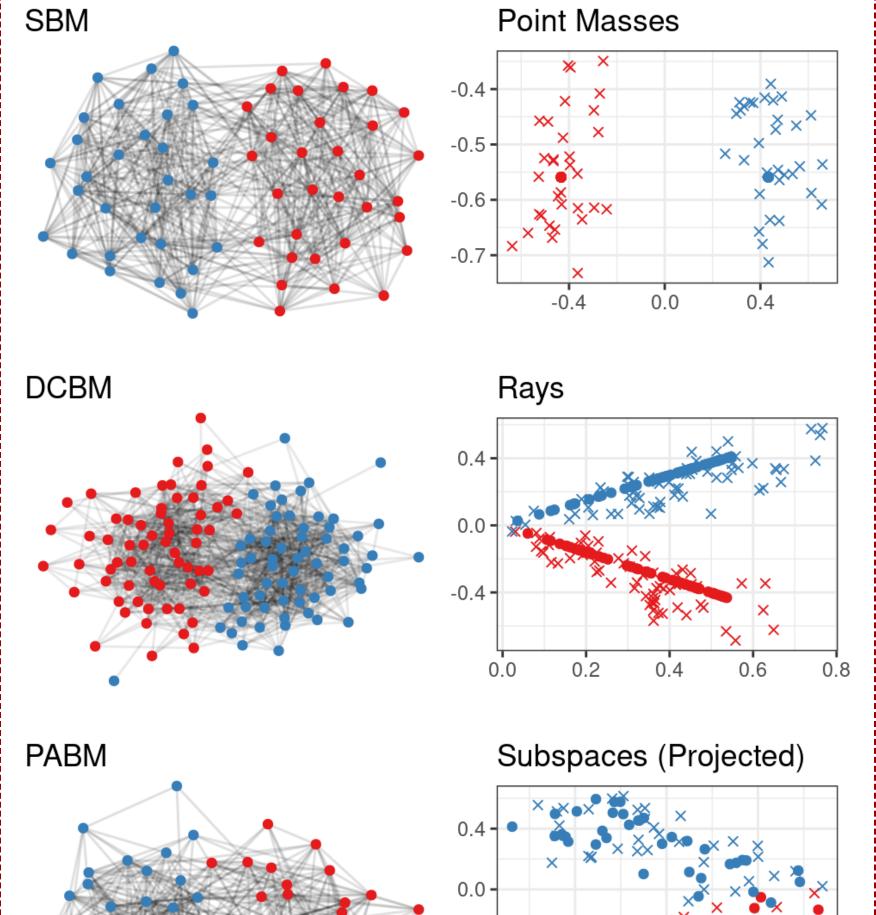
**Theorem** (Rubin-Delanchy et al. 2022):  $\max_i \|\hat{x}_i - Q_n x_i\| = O_P \left(\frac{\log^c n}{n^{1/2}}\right)$  for some  $Q_n \in \mathbb{O}(p,q)$ .



#### Connecting Block Models to the GRDPG

It has been previously shown that the SBM and DCBM are GRDPGs in which the communities lie on point masses and line segments respectively.

**Theorem** (KTT): The PABM is GRDPG in which the communities lie on mutually orthogonal K-dimensional subspaces.



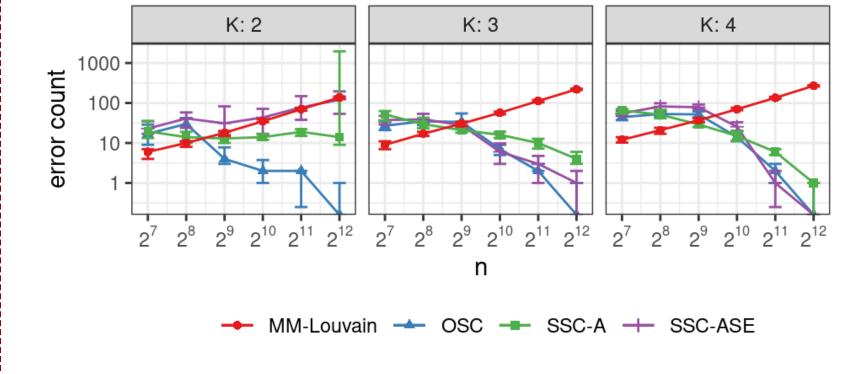
#### Orthogonal Spectral Clustering

- 1. Set  $p=rac{K(K+1)}{2}$ ,  $q=rac{K(K-1)}{2}$  and decompose  $Approx V_{p,q}\Lambda_{p,q}V_{p,q}^ op.$
- 2. Construct graph G' from  $B = |nV_{p,q}V_{p,q}^{\top}|$ .
- 3. Partition G' into K disconnected subgraphs.

**Theorem** (KTT):  $\max_{i,j} B_{ij} = O_P \Big( \frac{\log^c n}{\sqrt{n\rho_n}} \Big)$  for each (i,j) in different communities.

**Corollary**: OSC results in zero community detection error with probability 1 as  $n \to \infty$ .

## Simulation Study



#### Conclusion

Exploiting the geometry of PABMs (and other block models) results in intuitive and consistent community detection algorithms.