# Popularity Adjusted Block Models are Generalized Random Dot Product Graphs

JSM Speed Presentation

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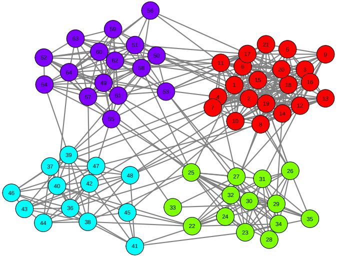
# Community Detection for Networks




**Def** Popularity Adjusted Block Model (Sengupta and Chen, 2017):

Let each vertex $i \in [n]$ have $K$ popularity parameters $\lambda_{i1}, ..., \lambda_{iK} \in [0, 1]$.
Then $A \sim \text{BernoulliGraph}(P)$ is a PABM if each $P_{ij} = \lambda_{iz_j} \lambda_{jz_i}$

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**Def** (Noroozi, Rimal, and Pensky, 2020):

$A$ is sampled from a PABM if $P$ can be described as:

1. Let each $P^{(kl)}$ denote the $n_k \times n_l$ matrix of edge probabilities between communities $k$ and $l$.
2. Organize popularity parameters as vectors $\lambda^{(kl)} \in \mathbb{R}^{n_k}$ such that $\lambda^{(kl)}_i = \lambda_{k_i l}$ is the popularity parameter of the $i^{\text{th}}$ vertex of community $k$ towards community $l$.
3. Each block can be decomposed as $P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top$.

# Generalized Random Dot Product Graph

**Def** Generalized Random Dot Product Graph
(Rubin-Delanchy, Cape, Tang, Priebe, 2020)

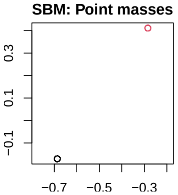
Let $I_{p,q} = \text{blockdiag}(I_p, -I_q)$ and suppose that $x_1, \ldots, x_n \in \mathbb{R}^{p+q}$ are such that $x_i^\top I_{p,q} x_j \in [0, 1]$.
Then $A \sim \text{GRDPG}_{p,q}(X)$ iff $A \sim \text{BernoulliGraph}(X I_{p,q} X^\top)$, where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$.

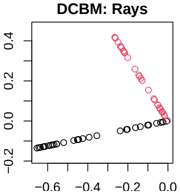
Adjacency Spectral Embedding (Sussman et al., 2012) estimates $x_1, ..., x_n \in \mathbb{R}^{p+q}$ from $A$:

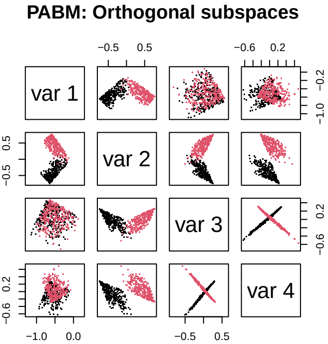
1. Let $\hat{\Lambda}$ be the diagonal matrix that contains the absolute values of the $p$ most positive and the $q$ most negative eigenvalues.
2. Let $\hat{V}$ be the matrix whose columns are the corresponding eigenvectors.
3. Compute $\hat{X} = \hat{V} \hat{\Lambda}^{1/2}$.

**Theorem**: $\max_i \|\hat{X}_i - Q_n X_i\| = O_P\left(\frac{(\log n)^c}{n^{1/2}}\right)$ as $n \to \infty$

# Connecting Block Models to the (G)RDPG Model



# Connecting the PABM to the GRDPG

**Theorem** (KTT): $A \sim \text{PABM}(\{\lambda_{ik}\}_K)$ is equivalent to $A \sim \text{GRDPG}_{p,q}(XU)$ with

- $p = K(K+1)/2$, $q = K(K-1)/2$;
- $U$ is an orthogonal matrix;
- $X \in \mathbb{R}^{n \times K^2}$ is a block diagonal matrix composed of popularity vectors with each block corresponding to a community.

$$X = \begin{bmatrix} \Lambda^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \Lambda^{(K)} \end{bmatrix} \in \mathbb{R}^{n \times K^2}$$

$$\Lambda^{(k)} = \begin{bmatrix} \lambda^{(k1)} & \cdots & \lambda^{(kK)} \end{bmatrix} \in \mathbb{R}^{n_k \times K}$$

$$A \sim \text{PABM}(\{\lambda_{ik}\}_K) \text{ iff } A \sim \text{GRDPG}_{p,q}(XU)$$

# Orthogonal Spectral Clustering

**Theorem** (KTT): If $P = V\Lambda V^\top$ and $B = nVV^\top$, then $B_{ij} = 0$ if $z_i \neq z_j$.

**Algorithm**: Orthogonal Spectral Clustering:

1. Let $V$ be the eigenvectors of $A$ corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues.
2. Compute $B = |nVV^\top|$ applying $|\cdot|$ entry-wise.
3. Construct graph $G$ using $B$ as its similarity matrix.
4. Partition $G$ into $K$ disconnected subgraphs.

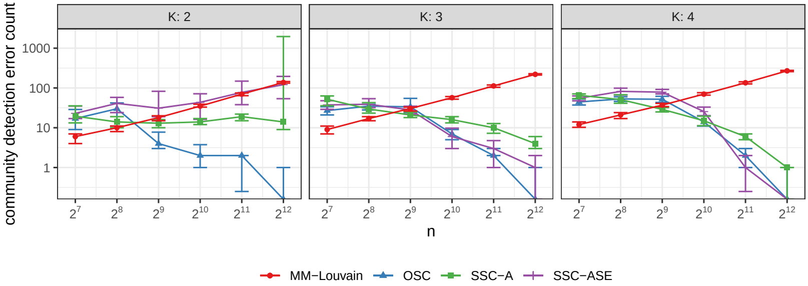
**Theorem** (KTT): Let $\hat{B}$ with entries $\hat{B}_{ij}$ be the affinity matrix from OSC. Then $\forall$ pairs $(i,j)$ belonging to different communities and sparsity factor satisfying $n\rho_n = \omega((\log n)^{4c})$, $\max_{i,j} \hat{B}_{ij} = O_P\left(\frac{(\log n)^c}{\sqrt{n\rho_n}}\right)$.

**Corollary**: OSC results in zero clustering error as $n \to \infty$, with probability 1.

# Simulation Study

We compare four algorithms for community detection on randomly generated PABMs:

- Modularity Maximization (Sengupta and Chen) using the Louvain algorithm;
- Orthogonal Spectral Clustering (KTT);
- Sparse Subspace Clustering on the columns of $A$ (Noorozi, Rimal, Pensky);
- Sparse Subspace Clustering on the ASE (KTT).



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arXiv preprint: https://arxiv.org/abs/2109.04010

GitHub repository: https://github.com/johneverettkoo/pabm-grdpg

R package: https://github.com/johneverettkoo/osc