# Popularity Adjusted Block Models are Generalized Random Dot Product Graphs

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## Block Models

A *block model* is a random graph model in which each vertex $v_i$ has a community label $z_i \in [K]$ and the edge probability between each pair of vertices is determined in part by the pair of labels.

$A$ is an adjacency matrix of a block model if and only if each $A_{ij} \overset{\text{ind}}{\sim} \text{Bernoulli}(P_{ij})$ and $P_{ij} = f(z_i, z_j, \cdot)$.

### Stochastic Block Model

Each edge probability is fixed for each pair of communities, i.e., $P_{ij} = \theta_{z_i, z_j}$.

### Degree Corrected Block Model

Each edge probability is determined by the community edge probability as well as each vertex's degree factor, i.e., $P_{ij} = \theta_{z_i, z_j} \omega_i \omega_j$.

### Popularity Adjusted Block Model

Each vertex has a popularity parameter for each community that determines its affinity toward that community, i.e., $P_{ij} = \lambda_{i, z_j} \lambda_{j, z_i}$.

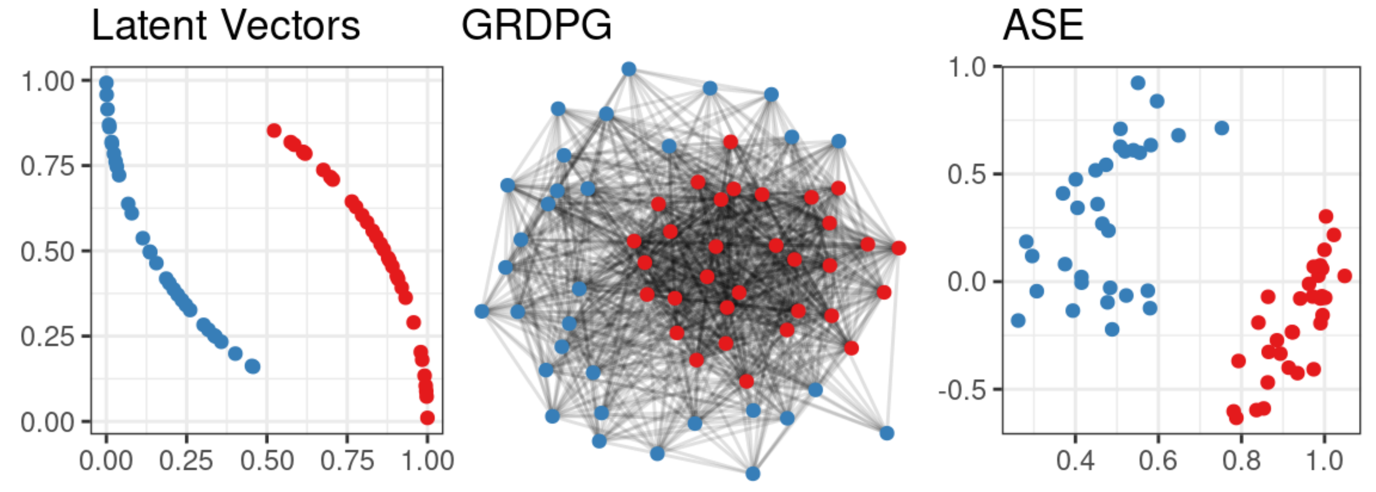
## Generalized Random Dot Product Graphs

The *generalized random dot product graph* is a random graph model in which each vertex $v_i$ has a corresponding hidden vector $x_i \in \mathbb{R}^{p+q}$ and each edge probability is the indefinite inner product of the corresponding pair of hidden vectors, i.e., $P_{ij} = x_i^\top I_{p,q} x_j$, $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$

### Adjacency Spectral Embedding

Approximate $A$ by spectral decomposition $A \approx V_{p,q} \Lambda_{p,q} V_{p,q}^\top$. The subscript $p, q$ denotes the $p$ most positive and $q$ most negative eigenvalues and corresponding eigenvectors. Each $\hat{x}_i$, the $i^{th}$ row of $\hat{X} = V_{p,q} |\Lambda_{p,q}|^{1/2}$, estimates the relative position of its corresponding latent vector $x_i$, up to an indefinite orthogonal transformation.

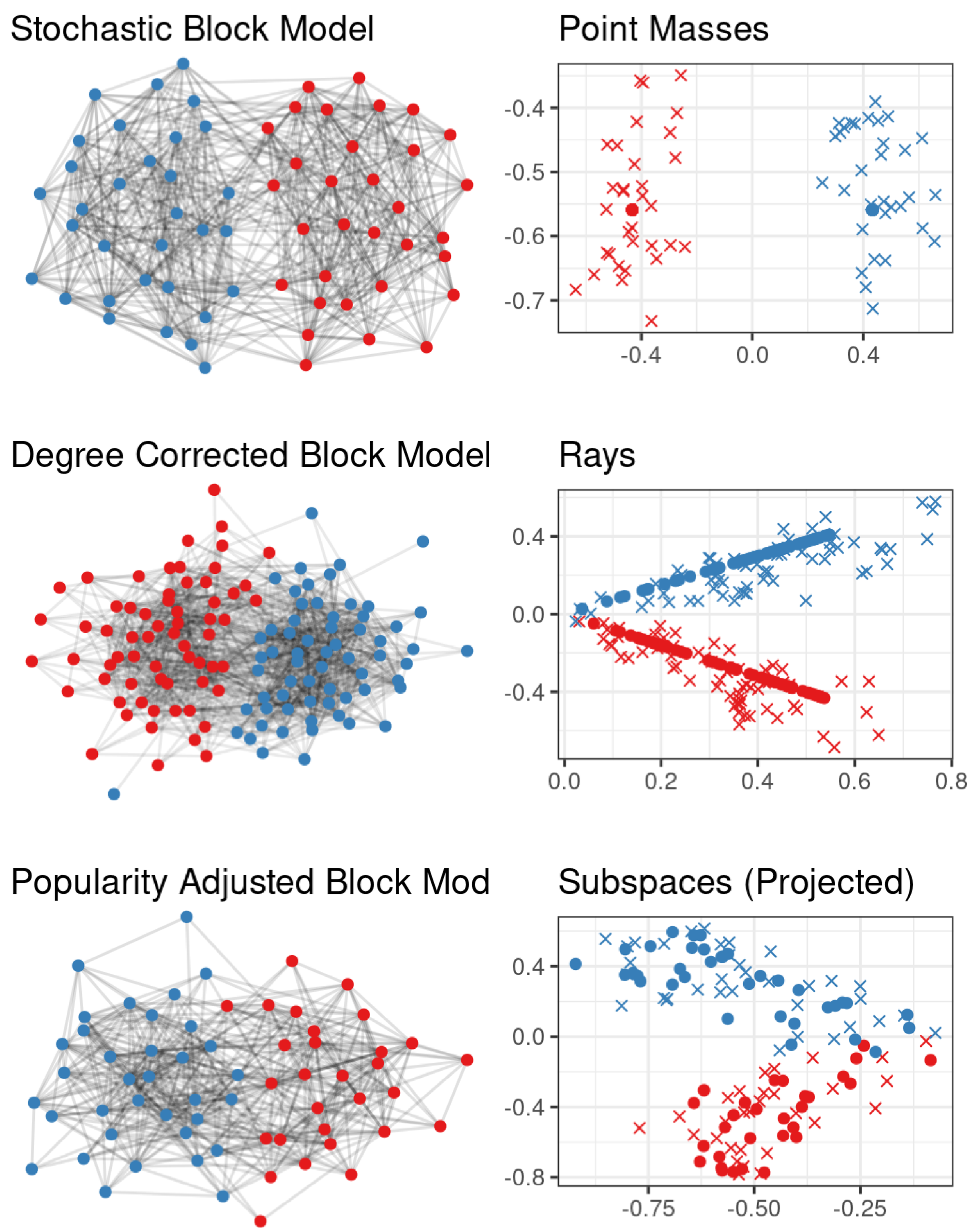
**Theorem**: $\max_i \|\hat{x}_i - Q_n x_i\| = O_P\left(\frac{\log^c n}{n^{1/2}}\right)$ for some $Q_n \in \mathbb{O}(p, q)$.




## Connecting Block Models to the GRDPG

It has been previously shown that the SBM and DCBM are GRDPGs in which the communities lie on point masses and line segments respectively.

**Theorem** (KTT): The PABM is GRDPG in which the communities lie on mutually orthogonal $K$-dimensional subspaces.

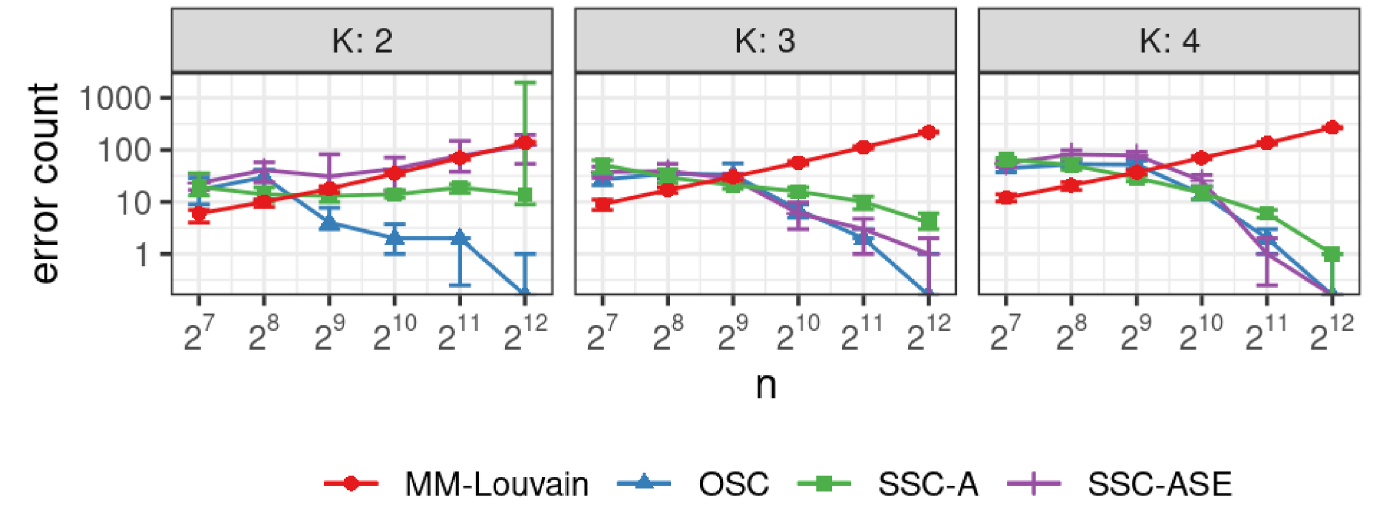



## Orthogonal Spectral Clustering

1. Decompose $A \approx V_{p,q} \Lambda_{p,q} V_{p,q}^\top$, $p = \frac{K(K+1)}{2}$ and $q = \frac{K(K-1)}{2}$.
2. Compute $B = |n V_{p,q} V_{p,q}^\top|$.
3. Construct graph $G'$ from $B$.
4. Partition $G'$ into $K$ disconnected subgraphs.

**Theorem** (KTT): Let $B$ be the affinity matrix from OSC. Then $\forall$ pairs $(i, j)$ belonging to different communities and sparsity factor satisfying $n \rho_n = \omega(\log^{4c} n)$, $\max_{i,j} B_{ij} = O_P\left(\frac{\log^c n}{\sqrt{n \rho_n}}\right)$.

### Simulation Study




## Conclusion

Exploiting the geometry of PABMs (and other block models) results in consistent community detection algorithms.