Popularity Adjusted Block Models are Generalized Random Dot Product Graphs

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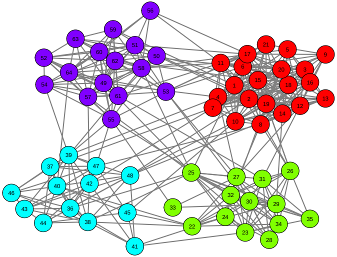
Community Detection for Networks



How might we cluster the nodes of a network?

Bernoulli Graphs

Let $G = (V, E)$ be an undirected and unweighted graph with $|V| = n$.

G is described by adjacency matrix A such that

$$A_{ij} = \begin{cases} 1 & \exists \text{ edge between } i \text{ and } j \\ 0 & \text{else} \end{cases}$$

$A_{ji} = A_{ij}$ and $A_{ii} = 0 \ \forall i, j \in [n]$.

$A \sim \text{BernoulliGraph}(P)$ iff:

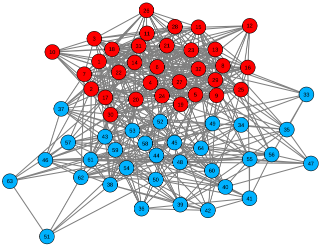
1. $P \in [0, 1]^{n \times n}$ describes edge probabilities between pairs of vertices.
2. $A_{ij} \overset{\text{ind}}{\sim} \text{Bernoulli}(P_{ij})$ for each $i < j$.

Block Models

Suppose each vertex $v_1, ..., v_n$ has labels $z_1, ..., z_n \in \{1, ..., K\}$, and each P_{ij} depends on labels z_i and z_j.
Then $A \sim \text{BernoulliGraph}(P)$ is a *block model*.

Example 1: Stochastic Block Model with $K = 2$ communities

- $z_1, ..., z_n \in \{1, 2\}$
- $P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$



Popularity Adjusted Block Model

Def Popularity Adjusted Block Model (Sengupta and Chen, 2017):

Let each vertex $i \in [n]$ have K popularity parameters $\lambda_{i1}, ..., \lambda_{iK} \in [0, 1]$. Then $A \sim$ PABM($\{\lambda_{ik}\}_K$) if each $P_{ij} = \lambda_{i z_j} \lambda_{j z_i}$.

Lemma (Noroozi, Rimal, and Pensky, 2020):

A is sampled from a PABM if P can be described as:

1. Let each $P^{(kl)}$ denote the $n_k \times n_l$ matrix of edge probabilities between communities k and l.
2. Organize popularity parameters as vectors $\lambda^{(kl)} \in \mathbb{R}^{n_k}$ such that $\lambda_i^{(kl)} = \lambda_{k_i l}$ is the popularity parameter of the i^{th} vertex of community k towards community l.
3. Each block can be decomposed as $P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top$.

Generalized Random Dot Product Graph

Def Generalized Random Dot Product Graph
(Rubin-Delanchy, Cape, Tang, Priebe, 2020)

Let $I_{p,q} = \text{blockdiag}(I_p, -I_q)$ and suppose that $x_1, \ldots, x_n \in \mathbb{R}^{p+q}$ are such that $x_i^\top I_{p,q} x_j \in [0, 1]$.
Then $A \sim \text{GRDPG}_{p,q}(X)$ iff $A \sim \text{BernoulliGraph}(X I_{p,q} X^\top)$, where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$

Adjacency Spectral Embedding (Sussman et al., 2012) estimates $x_1, ..., x_n \in \mathbb{R}^{p+q}$ from A:

1. Let $\hat{\Lambda}$ be the diagonal matrix that contains the absolute values of the p most positive and the q most negative eigenvalues.
2. Let $\hat{V}$ be the matrix whose columns are the corresponding eigenvectors.
3. Compute $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$.

Theorem: $\max_i \|\hat{X}_i - Q_n X_i\| = O_P\left(\frac{(\log n)^c}{n^{1/2}}\right)$

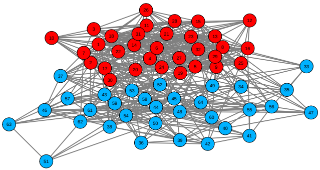
Connecting Block Models to the (G)RDPG Model

All Bernoulli Graphs are RDPG (if P is positive semidefinite) or GRDPG (in general).

Example 1 (cont'd): Assortative SBM ($pq > r^2$) with $K = 2$

$$P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$$

$$P = \begin{bmatrix} P^{(11)} & P^{(12)} \\ P^{(21)} & P^{(22)} \end{bmatrix} = XX^\top$$

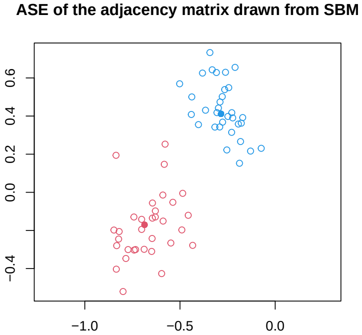


$$X = \begin{bmatrix} \sqrt{p} & 0 \\ \vdots & \vdots \\ \sqrt{p} & 0 \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \\ \vdots & \vdots \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \end{bmatrix}$$

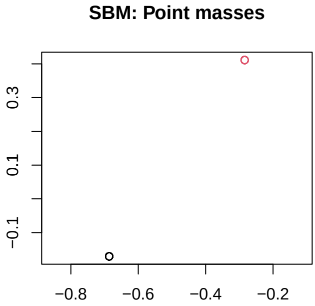
Connecting Block Models to the (G)RDPG Model

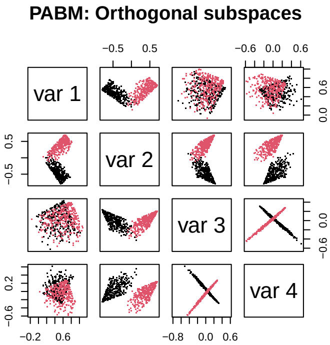
Example 1 (cont'd): If we want to perform community detection,

1. Note that A is a RDPG because $P = XX^\top$.
2. Compute the ASE $A \approx \hat{X}\hat{X}^\top$ with $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$.
3. Apply clustering algorithm (e.g., K-means) to $\hat{X}$,
   noting that as $n \to \infty$, the ASE approaches point masses.



Connecting Block Models to the (G)RDPG Model





Connecting the PABM to the GRDPG

Theorem (KTT): $A \sim \text{PABM}(\{\lambda^{(kl)}\}_K)$ is equivalent to $A \sim \text{GRDPG}_{p,q}(XU)$ with

- $p = K(K+1)/2$, $q = K(K-1)/2$
- $U \in \mathbb{O}(K^2)$
- $X \in \mathbb{R}^{n \times K^2}$ is block diagonal and composed of $\{\lambda^{(kl)}\}_K$ with each block corresponding to a community.

$$X = \begin{bmatrix} \Lambda^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \Lambda^{(K)} \end{bmatrix} \in \mathbb{R}^{n \times K^2}$$

$$\Lambda^{(k)} = \begin{bmatrix} \lambda^{(k1)} & \cdots & \lambda^{(kK)} \end{bmatrix} \in \mathbb{R}^{n_k \times K}$$

$$A \sim \text{PABM}(\{\lambda_{ik}\}_K) \iff A \sim \text{GRDPG}_{p,q}(XU)$$

Orthogonal Spectral Clustering

Theorem (KTT): If $P = V \Lambda V^\top$ and $B = nVV^\top$, then $B_{ij} = 0$ if $z_i \neq z_j$.

Algorithm: Orthogonal Spectral Clustering:

1. Let V be the eigenvectors of A corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues.
2. Compute $B = |nVV^\top|$ applying $|\cdot|$ entry-wise.
3. Construct graph G using B as its similarity matrix.
4. Partition G into K disconnected subgraphs.

Theorem (KTT): Let $\hat{B}_n$ with entries $\hat{B}_n^{(ij)}$ be the affinity matrix from OSC. Then $\forall$ pairs (i, j) belonging to different communities and sparsity factor satisfying $n\rho_n = \omega((\log n)^{4c})$,

$$\max_{i,j} \hat{B}_n^{(ij)} = O_P\Big(\frac{(\log n)^c}{\sqrt{n\rho_n}}\Big)$$

Simulation Results

Simulation setup:

1. $z_1, ..., z_n \overset{\text{iid}}{\sim} \text{Categorical}(1/K, ..., 1/K)$
2. $\lambda_{ik} \overset{\text{iid}}{\sim} \text{Beta}(a_{ik}, b_{ik})$
   $$a_{ik} = \begin{cases} 2 & z_i = k \\ 1 & z_i \neq k \end{cases} \quad b_{ik} = \begin{cases} 1 & z_i = k \\ 2 & z_i \neq k \end{cases}$$
3. $P_{ij} = \lambda_{iz_j} \lambda_{jz_i}$
4. $A \sim \text{BernoulliGraph}(P)$