Connecting the Popularity Adjusted Block Model to the Generalized Random Dot Product Graph

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The Popularity Adjusted Block Model

Def Popularity Adjusted Block Model (Sengupta and Chen, 2017):

Let each vertex $i \in [n]$ have K popularity parameters $\lambda_{i1}, ..., \lambda_{iK} \in [0, 1]$. Then $A \sim \text{PABM}(P)$ if each $P_{ij} = \lambda_{i z_j} \lambda_{j z_i}$, e.g., if $z_i = k$ and $z_j = l$, $P_{ij} = \lambda_{il} \lambda_{jk}$.

Lemma (Noroozi, Rimal, and Pensky, 2020):

A is sampled from a PABM if P can be described as:

1. Let each $P^{(kl)}$ denote the $n_k \times n_l$ matrix of edge probabilities between communities k and l.
2. Organize popularity parameters as vectors $\lambda^{(kl)} \in \mathbb{R}^{n_k}$ such that $\lambda_i^{(kl)} = \lambda_{k_i l}$ is the popularity parameter of the i^{th} vertex of community k towards community l.
3. Each block can be decomposed as $P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top$.

Notation: $A \sim \text{PABM}(\{\lambda^{(kl)}\}_K)$.

The Generalized Random Dot Product Graph

Generalized Random Dot Product Graph $A \sim GRDPG_{p,q}(X)$ (Rubin-Delanchy, Cape, Tang, Priebe, 2020)

- Latent vectors $x_1, ..., x_n \in \mathbb{R}^{p+q}$ such that $x_i^\top I_{p,q} x_j \in [0, 1]$ and $I_{p,q} = blockdiag(I_p, -I_q)$
- $A \sim BernoulliGraph(X I_{p,q} X^\top)$ where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$

If latent vectors $X_1, ..., X_n \overset{iid}{\sim} F$, then we write $(A, X) \sim GRDPG_{p,q}(F, n)$.

(Generalized) Random Dot Product Graph Model

Recovery/Estimation

Want to estimate X given A.

Adjacency Spectral Embedding

To embed in $\mathbb{R}^{p+q}$,

1. Compute $A \approx \hat{V}\hat{\Lambda}\hat{V}^\top$ where $\hat{\Lambda} \in \mathbb{R}^{(p+q)\times(p+q)}$ and $\hat{V} \in \mathbb{R}^{n\times(p+q)}$ by using p most positive and q most negative eigenvalues.
2. Let $\hat{X} = \hat{V}|\hat{\Lambda}|^{1/2}$.

Rubin-Delanchy et al., 2020:

$$\max_i \|\hat{X}_i - Q_n X_i\| \overset{a.s.}{\to} 0$$

$$Q_n \in \mathbb{O}(p,q)$$

Connecting the PABM to the GRDPG

Theorem (KTT): $A \sim PABM(\{\lambda^{(kl)}\}_K)$ is equivalent to $A \sim GRDPG_{p,q}(XU)$ such that

- $p = K(K+1)/2$
- $q = K(K-1)/2$
- U is orthogonal and predetermined for each K
- X is block diagonal and composed of $\{\lambda^{(kl)}\}_K$ $\implies$ if $x_i^\top$ and $x_j^\top$ are two rows of XU corresponding to different communities, then $x_i^\top x_j = 0$.

Remark (non-uniqueness of the latent configuration): $A \sim GRDPG_{p,q}(XU) \implies A \sim GRDPG_{p,q}(XUQ)$ $\forall Q \in \mathbb{O}(p,q)$

Corollary: X is block diagonal by community and U is orthogonal $\implies$ each community corresponds to a subspace in $\mathbb{R}^{K^2}$.

Subspace property holds even with linear transformation $Q \in \mathbb{O}(p,q)$.

Connecting the PABM ot the GRDPG

Theorem (KTT): If $P = V \Lambda V^\top$ is the spectral decomposition of P for the PABM and V has rows $v_i^\top$, then $v_i^\top v_j = 0 \; \forall z_i \neq z_j$.

Theorem (KTT): If $A \approx \hat{V} \hat{\Lambda} \hat{V}^\top$ is the spectral decomposition of A for the PABM using the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues, and we denote $\hat{v}_i^\top$ as the rows of $\hat{V}$, then

$$\max_{i,j : z_i \neq z_j} n \hat{v}_i^\top \hat{v}_j = O_P\Big(\frac{(\log n)^c}{\sqrt{n \rho_n}} \Big)$$

Orthogonal Spectral Clustering (KTT):

1. Let V be the eigenvectors of A corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues.
2. Compute $B = |n V V^\top|$ applying $| \cdot |$ entry-wise.
3. Construct graph $\hat{G}$ using B as its similarity matrix.
4. Partition $\hat{G}$ into K disconnected subgraphs.

Simulation Study

Simulation setup:

1. $Z_1, ..., Z_n \overset{iid}{\sim} Categorical(1/K, ..., 1/K)$
2. $\lambda_{ik} \overset{iid}{\sim} Beta(a_{ik}, b_{ik})$
$$a_{ik} = \begin{cases} 2 & z_i = k \\ 1 & z_i \neq k \end{cases}, \; b_{ik} = \begin{cases} 1 & z_i = k \\ 2 & z_i \neq k \end{cases}$$
3. $P_{ij} = \lambda_{iz_j} \lambda_{jz_i}$
4. $A \sim BernoulliGraph(P)$