Popularity Adjusted Block Models are Generalized Random Dot Product Graphs

Future Leaders Summit

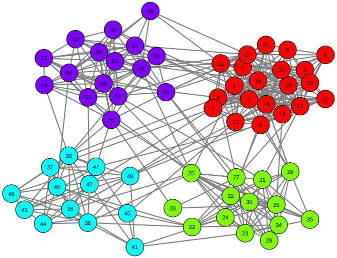
April 2022

John Koo,
PhD Student in
Statistical Science,
Indiana University

Minh Tang,
Assistant Professor of
Statistics,
NC State University

Michael Trosset,
Professor of Statistics,
Indiana University

Community Detection for Networks



How might we cluster the nodes of a network?

1. Define a probability model with communities that might generate the graph (e.g., popularity adjusted block model).
2. Develop estimators for the parameters of the probability model, including the community labels.
3. Describe the properties of the estimators (e.g., consistency).

Bernoulli Graphs

Let G be an undirected and unweighted graph with n vertices.

G is described by adjacency matrix A such that

$$A_{ij} = \begin{cases} 1 & \text{an edge connects vertices } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

$A_{ji} = A_{ij}$ and $A_{ii} = 0$.

$A \sim \text{BernoulliGraph}(P)$ iff:

1. P is a matrix of edge probabilities between pairs of vertices.
2. $A_{ij} \overset{\text{ind}}{\sim} \text{Bernoulli}(P_{ij})$ for each $i < j$.

Block Models

Suppose each vertex $v_1, ..., v_n$ has labels $z_1, ..., z_n \in \{1, ..., K\}$, and each P_{ij} depends on labels z_i and z_j.
Then $A \sim \text{BernoulliGraph}(P)$ is a *block model*.

Example 1: Stochastic Block Model with $K = 2$ communities.

$$P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$$

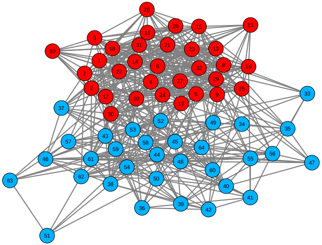


Figure 1: SBM with $p = 1/2$, $q = 1/4$, $r = 1/8$

Popularity Adjusted Block Model

Def Popularity Adjusted Block Model (Sengupta and Chen, 2017):

Let each vertex $i \in [n]$ have K popularity parameters $\lambda_{i1}, ..., \lambda_{iK} \in [0, 1]$. Then $A \sim$ PABM$(\{\lambda_{ik}\}_K)$ if each $P_{ij} = \lambda_{iz_j} \lambda_{jz_i}$.

Def (Noroozi, Rimal, and Pensky, 2020):

A is sampled from a PABM if P can be described as:

1. Let each $P^{(kl)}$ denote the $n_k \times n_l$ matrix of edge probabilities between communities k and l.
2. Organize popularity parameters as vectors $\lambda^{(kl)} \in \mathbb{R}^{n_k}$ such that $\lambda_i^{(kl)} = \lambda_{k_i l}$ is the popularity parameter of the i^{th} vertex of community k towards community l.
3. Each block can be decomposed as $P^{(kl)} = \lambda^{(kl)} (\lambda^{(lk)})^\top$.

Generalized Random Dot Product Graph

Def Generalized Random Dot Product Graph
(Rubin-Delanchy, Cape, Tang, Priebe, 2020)

Let $I_{p,q} = \text{blockdiag}(I_p, -I_q)$ and suppose that $x_1, \ldots, x_n \in \mathbb{R}^{p+q}$ are such that $x_i^\top I_{p,q} x_j \in [0, 1]$.
Then $A \sim \text{GRDPG}_{p,q}(X)$ iff $A \sim \text{BernoulliGraph}(X I_{p,q} X^\top)$, where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$.

Adjacency Spectral Embedding (Sussman et al., 2012) estimates $x_1, ..., x_n \in \mathbb{R}^{p+q}$ from A:

1. Let $\hat{\Lambda}$ be the diagonal matrix that contains the absolute values of the p most positive and the q most negative eigenvalues.
2. Let $\hat{V}$ be the matrix whose columns are the corresponding eigenvectors.
3. Compute $\hat{X} = \hat{V} \hat{\Lambda}^{1/2}$.

Theorem: $\max_i \|\hat{X}_i - Q_n X_i\| = O_P\left(\frac{(\log n)^c}{n^{1/2}}\right)$ as $n \to \infty$

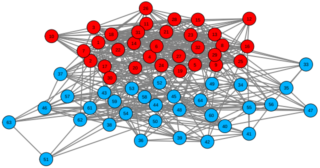
Connecting Block Models to the GRDPG Model

All Bernoulli Graphs are GRDPGs.

Example 1 (cont'd): SBM with $K = 2$.

$$P_{ij} = \begin{cases} p & z_i = z_j = 1 \\ q & z_i = z_j = 2 \\ r & z_i \neq z_j \end{cases}$$

$$P = \begin{bmatrix} P^{(11)} & P^{(12)} \\ P^{(21)} & P^{(22)} \end{bmatrix} = X I_{2,0} X^\top$$

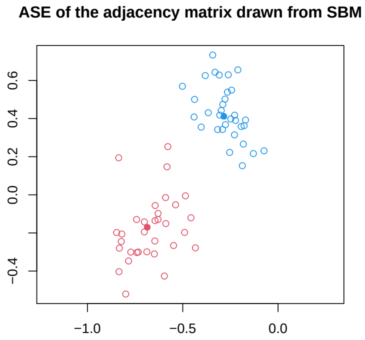


$$X = \begin{bmatrix} \sqrt{p} & 0 \\ \vdots & \vdots \\ \sqrt{p} & 0 \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \\ \vdots & \vdots \\ \sqrt{r^2/p} & \sqrt{q - r^2/p} \end{bmatrix}$$

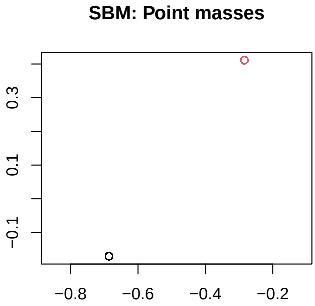
Connecting Block Models to the GRDPG Model

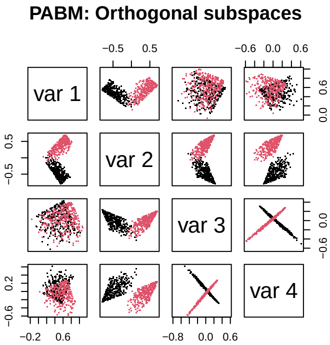
Example 1 (cont'd): To perform community detection,

1. Note that A is a RDPG because $P = XX^\top$.
2. Compute the ASE $A \approx \hat{X}\hat{X}^\top$ with $\hat{X} = \hat{V}\hat{\Lambda}^{1/2}$.
3. Apply a clustering algorithm (e.g., K-means) to $\hat{X}$, noting that $\hat{X}$ approaches point masses as $n \to \infty$.



Connecting Block Models to the GRDPG Model





Connecting the PABM to the GRDPG

Theorem (KTT): $A \sim \text{PABM}(\{\lambda_{ik}\}_K)$ is equivalent to $A \sim \text{GRDPG}_{p,q}(XU)$ with

- $p = K(K+1)/2$, $q = K(K-1)/2$;
- U is an orthogonal matrix;
- $X \in \mathbb{R}^{n \times K^2}$ is a block diagonal matrix composed of popularity vectors with each block corresponding to a community.

$$X = \begin{bmatrix} \Lambda^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \Lambda^{(K)} \end{bmatrix} \in \mathbb{R}^{n \times K^2}$$

$$\Lambda^{(k)} = \begin{bmatrix} \lambda^{(k1)} & \cdots & \lambda^{(kK)} \end{bmatrix} \in \mathbb{R}^{n_k \times K}$$

$$A \sim \text{PABM}(\{\lambda_{ik}\}_K) \text{ iff } A \sim \text{GRDPG}_{p,q}(XU)$$

Orthogonal Spectral Clustering

Theorem (KTT): If $P = V\Lambda V^\top$ and $B = nVV^\top$, then $B_{ij} = 0$ if $z_i \neq z_j$.

Algorithm: Orthogonal Spectral Clustering:

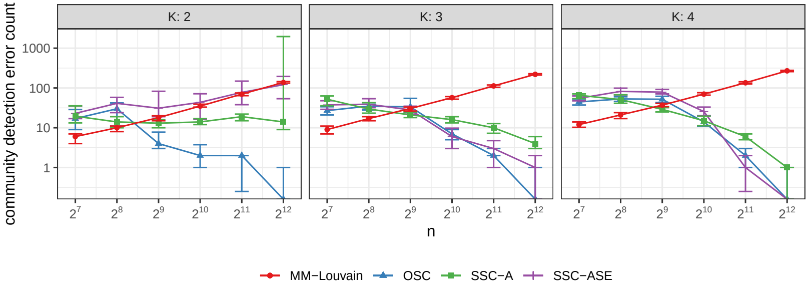
1. Let V be the eigenvectors of A corresponding to the $K(K+1)/2$ most positive and $K(K-1)/2$ most negative eigenvalues.
2. Compute $B = |nVV^\top|$ applying $|\cdot|$ entry-wise.
3. Construct graph G using B as its similarity matrix.
4. Partition G into K disconnected subgraphs.

Theorem (KTT): Let $\hat{B}$ with entries $\hat{B}_{ij}$ be the affinity matrix from OSC. Then $\forall$ pairs (i, j) belonging to different communities and sparsity factor satisfying $n\rho_n = \omega((\log n)^{4c})$, $\max_{i,j} \hat{B}_{ij} = O_P\left(\frac{(\log n)^c}{\sqrt{n\rho_n}}\right)$ as $n \to \infty$.

Simulation Results

We compare four algorithms for community detection on randomly generated PABMs:

- Modularity Maximization (Sengupta and Chen) using the Louvain algorithm;
- Orthogonal Spectral Clustering (KTT);
- Sparse Subspace Clustering on the columns of A (Noorozi, Rimal, Pensky);
- Sparse Subspace Clustering on the ASE (KTT).



Additional Slides

Simulation Setup

1. $z_1, ..., z_n \stackrel{\text{iid}}{\sim} \text{Categorical}(1/K, ..., 1/K)$
2. $\lambda_{ik} \stackrel{\text{iid}}{\sim} \text{Beta}(a_{ik}, b_{ik})$
    - $a_{ik} = \begin{cases} 2 & z_i = k \\ 1 & z_i \neq k \end{cases}$
    - $b_{ik} = \begin{cases} 1 & z_i = k \\ 2 & z_i \neq k \end{cases}$
3. $P_{ij} = \lambda_{iz_j} \lambda_{jz_i}$
4. $A \sim \text{BernoulliGraph}(P)$

Future Work