# SDP Notes

### The Ratio Cut Problem

Given a fully connected, undirected similarity graph $G = (V, E)$ represented by weight matrix $W$, and known number of clusters/communities $k$, ratio cut partitions the graph by minimizing the objective function:

$$\arg\min_{H} \operatorname{Tr}(H^\top L H)$$

where $L$ is the unnormalized combinatorial Laplacian matrix $L = D - W$, $D$ is the diagonal degree matrix $d_{ii} = \sum_j w_{ij}$, and $H \in \mathbb{R}^{n \times k}$ is restricted to the form $h_{ij} = \begin{cases} n_j^{-1/2} & \text{vertex } i \text{ is in cluster } j \\ 0 & \text{else} \end{cases}$

This problem is NP-hard, which brings us to ...

### Spectral Clustering

Note that $H^\top H = I$. So instead of restricting $H$ to a cluster membership matrix, we just restrict $H$ such that $H^\top H = I$. Then $H = \begin{bmatrix} v_0 & v_1 & \cdots & v_{k-1} \end{bmatrix}$, or the matrix of $k$ eigenvectors that correspond to the smallest $k$ eigenvalues. Since this does not provide cluster memberships, the first $k$ eigenvectors are then treated as an embedding of the graph $G$ and $k$-means clustering is then applied (e.g., using Lloyd's algorithm) to obtain cluster memberships.[1]

However, there isn't much theoretical justification for this approach, nor are there any guarantees that it will result in a clustering that minimizes the ratio cut objective, which brings us to ...

### Semidefininte Programming Approach

Ling and Strohmer[2] showed that under certain conditions, the ratio cut problem can be solved exactly using semidefinite programming.

First, we note that $\operatorname{Tr}(H^\top L H) = \operatorname{Tr}(L H H^\top)$. Let us denote $Z = HH^\top$. We can see that $z_{ij} = n_k^{-1}$ if vertices $i$ and $j$ both belong to the same cluster (where $n_k$ is the size of that cluster), otherwise it is 0. We can then note that $\operatorname{Tr}(Z) = k$, the number of clusters, and $Z$ is positive semidefinite of rank $k$. We also note that $Ze = e$ where $e$ is a constant vector.

The SDP approach relaxes these constraints on $Z$ but keeps more constraints than the spectral clustering approach. More formally, the SDP ratio cut algorithm is as follows:

1. Solve $\arg\min_Z \operatorname{Tr}(LZ)$ subject to
   - $Z$ is positive semidefinite
   - $Z \geq 0$ element-wise
   - $\operatorname{Tr}(Z) = k$
   - $Ze = e$
2. Use the solution to the above to obtain cluster memberships.

[1] This is almost (but not quite) the same as performing kernel $k$-means clustering using the Moore-Penrose pseudoinverse of $L$ as a kernel matrix.

[2] https://arxiv.org/abs/1806.11429

The paper does not make it clear how to actually solve the semidefinite programming problem or how to recover cluster memberships from $Z$, since we are not guaranteed to obtain the "correct" form of $Z$ using this method.

### Possibly Interesting Question

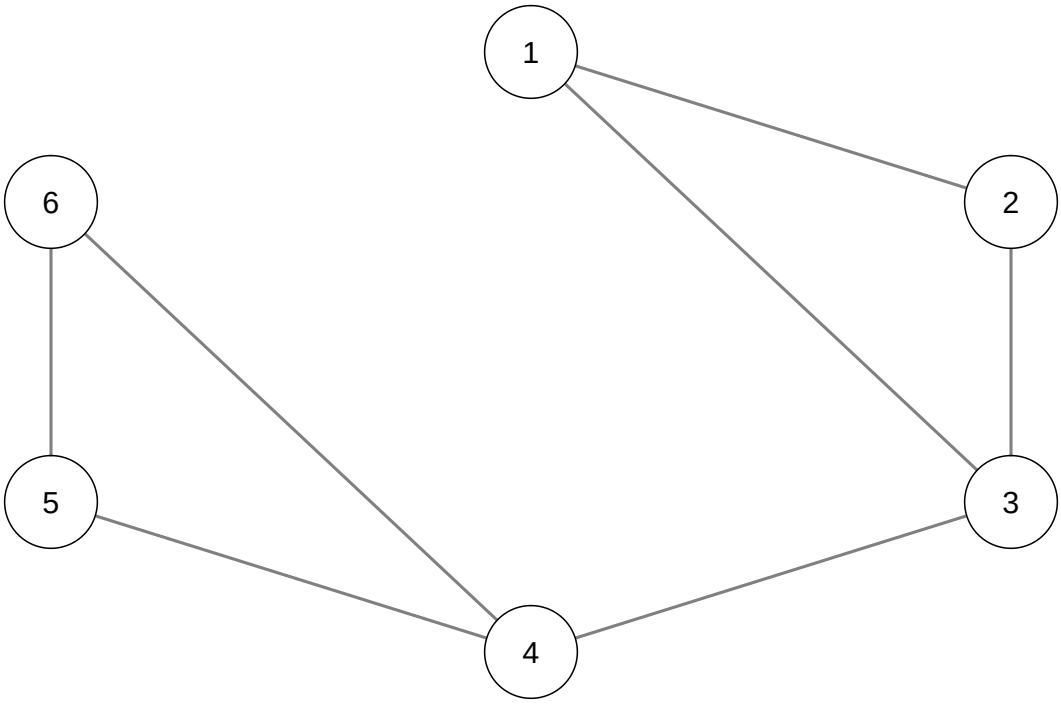
We previously noted that the spectral clustering approach to ratio cut is almost equivalent to kernel $k$-means using the pseudoinverse of $L$ as a kernel matrix. One problem that we have been trying to solve is under what conditions does kernel $k$-means provide the optimal ratio cut clustering. Similarly, we might be interested in a kernel $k$-means version of the semidefinite programming approach:

Let $L^\dagger$ be the pseudoinverse of $L$. Solve $\arg\max_Z \mathrm{Tr}(L^\dagger Z)$ subject to

- $Z$ is positive semidefinite
- $Z \geq 0$ element-wise
- $\mathrm{Tr}(Z) = k$
- $Ze = e$

### Some Examples

We will begin with a very simple graph with six vertices:




It is clear (and easy to verify) that cutting the $3-4$ edge is optimal under the ratio cut objective. Using the semidefinite programming method:

```
$Z
      [,1]  [,2]  [,3]  [,4]  [,5]  [,6]
[1,] 0.333 0.333 0.333 0.000 0.000 0.000
[2,] 0.333 0.333 0.333 0.000 0.000 0.000
```

```
[3,] 0.333 0.333 0.333 0.000 0.000 0.000
[4,] 0.000 0.000 0.000 0.333 0.333 0.333
[5,] 0.000 0.000 0.000 0.333 0.333 0.333
[6,] 0.000 0.000 0.000 0.333 0.333 0.333

$clustering
[1] 2 2 2 1 1 1
```

We in fact get the exact solution to $Z$. The clustering here is based on a spectral decomposition of $Z$, which can have up to $n-1$ nonzero eigenvectors (in this case we only have two). $k$-means is applied to this embedding.

We can also try the kernel $k$-means approach:

```
$Z
      [,1]  [,2]  [,3]  [,4]  [,5]  [,6]
[1,] 0.333 0.333 0.333 0.000 0.000 0.000
[2,] 0.333 0.333 0.333 0.000 0.000 0.000
[3,] 0.333 0.333 0.333 0.000 0.000 0.000
[4,] 0.000 0.000 0.000 0.333 0.333 0.333
[5,] 0.000 0.000 0.000 0.333 0.333 0.333
[6,] 0.000 0.000 0.000 0.333 0.333 0.333

$clustering
[1] 2 2 2 1 1 1
```

And in fact we get identical results as before for both $Z$ and the clustering.

Let's try the epsilon graph example from the text. Recall that the graph Laplacian is of the form $\begin{bmatrix} \epsilon & -\epsilon & 0 & 0 \\ -\epsilon & 1+\epsilon & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Ratio cut and kernel $k$-means gave different answers when $\epsilon \in (0.6, 0.75)$.

Plugging in $\epsilon = 0.6$, the SDP method for ratio cut gives:

```
$Z
      [,1]  [,2]  [,3]  [,4]
[1,] 0.777 0.223 0.000 0.000
[2,] 0.223 0.355 0.275 0.147
[3,] 0.000 0.275 0.370 0.355
[4,] 0.000 0.147 0.355 0.498

$clustering
[1] 1 2 2 2
```

While the SDP method for kernel $k$-means gives:

```
$Z
      [,1]  [,2]  [,3]  [,4]
[1,] 0.683 0.317 0.000 0.000
[2,] 0.317 0.394 0.211 0.077
[3,] 0.000 0.211 0.394 0.394
[4,] 0.000 0.077 0.394 0.528

$clustering
[1] 2 2 1 1
```

Plugging in $\epsilon = .7$:

```
$Z
      [,1]  [,2]  [,3]  [,4]
[1,] 0.708 0.292 0.000 0.000
[2,] 0.292 0.368 0.243 0.096
[3,] 0.000 0.243 0.389 0.368
[4,] 0.000 0.096 0.368 0.535

$clustering
[1] 1 1 2 2
```

While the SDP method for kernel $k$-means gives:

```
$Z
      [,1]  [,2]  [,3]  [,4]
[1,] 0.581 0.419 0.000 0.000
[2,] 0.419 0.439 0.122 0.020
[3,] 0.000 0.122 0.439 0.439
[4,] 0.000 0.020 0.439 0.541

$clustering
[1] 1 1 2 2
```

So while we can get SDP ratio cut and SDP kernel $k$-means to disagree, the conditions are not identical to when this happens for the spectral clustering methods.