Reformulation of the Ratio Cut Problem

The ratio cut objective can be written as:

$$\arg\min_H \operatorname{Tr}(H^\top L H)$$

where $L \in \mathbb{R}^{n\times n}$ is the unnormalized combinatorial graph Laplacian and $H \in \mathbb{R}^{n\times k}$ is a cluster assignment matrix. Since this problem is NP-hard, this is often "solved" via a continuous relaxation which yields $\hat{H} = \begin{bmatrix} u_1 \cdots u_k \end{bmatrix}$, the first k eigenvectors of L, followed by a "rounding step" in which $\hat{H}$ is treated as an embedding of the graph and k-means is applied to the embedding (via Lloyd's algorithm).

Based on this, we can rewrite $L = U\Sigma^{-2}U^\top$ where U is a matrix of eigenvectors and Σ^{-2} is a diagonal matrix of eigenvalues. Then we can write:

$$\begin{aligned}
\operatorname{Tr}(H^\top L H) &= \operatorname{Tr}(H^\top U \Sigma^{-2} U^\top H) \\
&= \operatorname{Tr}\big((H^\top U \Sigma^{-1})(\Sigma^{-1} U^\top H)\big) \\
&= \operatorname{Tr}\big((\Sigma^{-1} U^\top H)^\top (\Sigma^{-1} U^\top H)\big) \\
&= ||\Sigma^{-1} U^\top H||_F^2
\end{aligned}$$

Note that we can rewrite the k-means objective (in the Laplacian eigenmap) very similarly:

$$\arg\max_H ||\Sigma U^\top H||_F^2$$