# S722 HW3

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To save on typing, I will denote $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$.

## Part 1

### 8.17

#### a

$L(\mu, \theta) = f(x, y | \mu, \theta) = \Big(\prod x_i^{\mu - 1} \mu\Big)\Big(\prod y_i^{\theta - 1} \theta\Big)$
$= \mu^n \theta^m (\prod x_i)^{\mu - 1} (\prod y_i)^{\theta - 1}$

Under $H_0$, we can just use the MLE for $\mu = \theta$, so we get $\hat{\mu}_0 = \hat{\theta}_0 = -\frac{n+m}{\sum \log x_i + \sum \log y_i}$.

For the unrestricted MLEs, we can take the log likelihood:

$\ell(\mu, \theta) = n \log \mu + m \log \theta + (\mu - 1) \sum \log x_i + (\theta - 1) \sum \log y_i$

Then take the derivative and set to zero:

$0 = n/\mu + \sum \log x_i$
$0 = m/\theta + \sum \log y_i$

And we obtain:

$\hat{\mu} = -\frac{n}{\sum \log x_i}$
$\hat{\theta} = -\frac{m}{\sum \log y_i}$

The LRT statistic is $\frac{L(\hat{\mu}_0 = \hat{\theta}_0, \hat{\theta}_0)}{L(\hat{\mu}, \hat{\theta})}$

$= \frac{(-\frac{n+m}{\sum \log x_i + \sum \log y_i})^{n+m} (\prod x_i)^{-\frac{n+m}{\sum \log x_i + \sum \log y_i} - 1} (\prod y_i)^{-\frac{n+m}{\sum \log x_i + \sum \log y_i} - 1}}{(-\frac{n}{\sum \log x_i})^n (-\frac{m}{\sum \log y_i})^m (\prod x_i)^{-\frac{n}{\sum \log x_i} - 1} (\prod y_i)^{-\frac{m}{\sum \log y_i} - 1}}$

$= -(\frac{n+m}{m})^n (\frac{n+m}{m})^m (\frac{\sum \log x_i}{\sum \log x_i + \sum \log y_i})^n (\frac{\sum \log y_i}{\sum \log x_i + \sum \log y_i})^m (\prod x_i \prod y_i)^{-\frac{n+m}{\log \prod x_i \prod y_i}} (\prod x_i)^{\frac{n}{\log \prod x_i}} (\prod y_i)^{\frac{m}{\log \prod y_i}}$

Then using the fact that $t^{1/\log t} = e$, we get

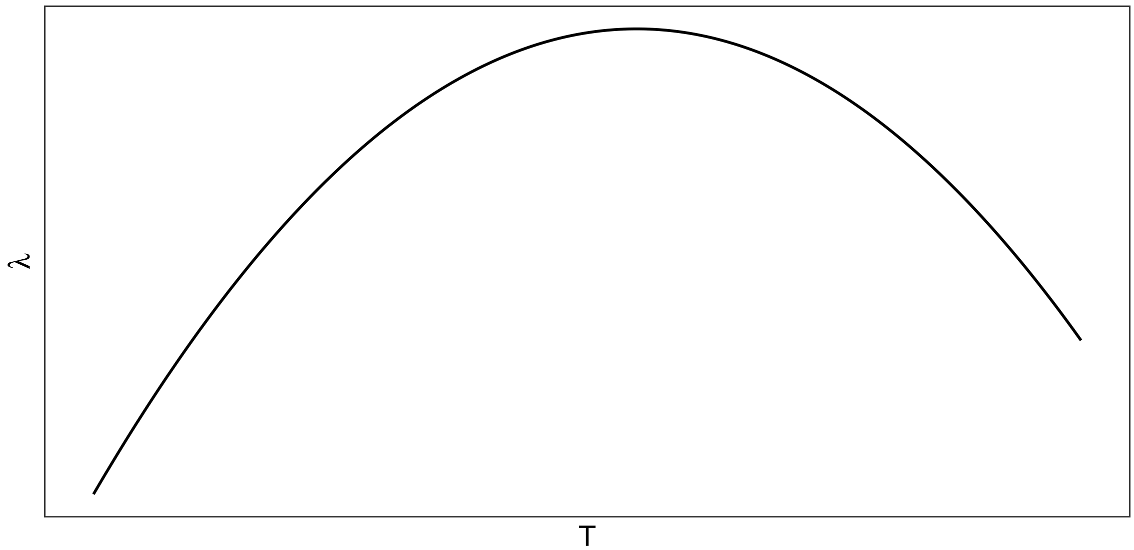
$= -(\frac{n+m}{m})^n (\frac{n+m}{m})^m (\frac{\sum \log x_i}{\sum \log x_i + \sum \log y_i})^n (\frac{\sum \log y_i}{\sum \log x_i + \sum \log y_i})^m e^{-n-m} e^n e^m$
$= -(\frac{n+m}{m})^n (\frac{n+m}{m})^m (\frac{\sum \log x_i}{\sum \log x_i + \sum \log y_i})^n (\frac{\sum \log y_i}{\sum \log x_i + \sum \log y_i})^m$

And as usual, we reject $H_0$ if this is $\leq c$ for some $c \in (0, 1)$.

#### b

Substituting $T = \frac{\sum \log x_i}{\sum \log x_i + \sum \log y_i}$, we get

$\lambda(T) = (\frac{n+m}{m})^n (\frac{n+m}{m})^m T^n (1 - T)^m$




So we reject if $T \geq c_1$ or $T \leq c_2$ for some $c_1$ and $c_2$ such that $\lambda(c_1) = \lambda(c_2)$.

### c

Under $H_0$:

$-\log X_i \sim Exp(1/\mu)$
$\implies -\sum \log X_i \sim Gamma(n, 1/\mu)$
$\implies \frac{\sum \log X_i}{\sum \log X_i + \sum \log Y_i} \sim Beta(n, m)$ (since $\mu = \theta$ under $H_0$)

So we set:

$\alpha = .1 = P(T \leq c_1) + P(T \geq c_2)$
$(1 - c_1)^m c_1^n = (1 - c_2)^m c_2^n$

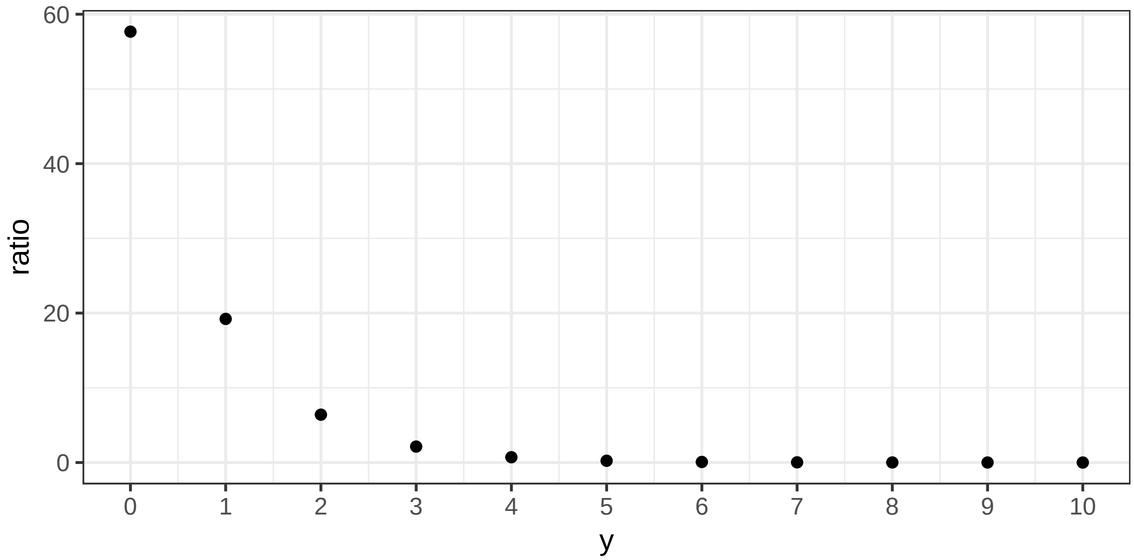
Since the beta CDF doesn't have an easy to use closed form, this needs to be solved numerically.

## 8.22

### a

We have $Y = \sum X_i \sim Binomial(10, p)$

$\frac{L(1/4)}{L(1/2)} = (1/2)^y (3/2)^{10-y}$




Since this is decreasing in $y$, we reject $H_0$ for small values of $y$. We need $c$ such that $P(Y \leq c|H_0) = .0547$.

```
qbinom(.0547, 10, .5)
```

```
[1] 3
```

```
pbinom(3, 10, .5)
```

```
[1] 0.171875
```

```
pbinom(2, 10, .5)
```

```
[1] 0.0546875
```

So we use $c = 2$ (we could be more precise but 0.0546875 is close enough to .0547).

The power is $P(Y \leq 2|H_1) = \sum_0^2 \binom{10}{y} .25^y .75^{10-y} \approx 0.526$.

### b

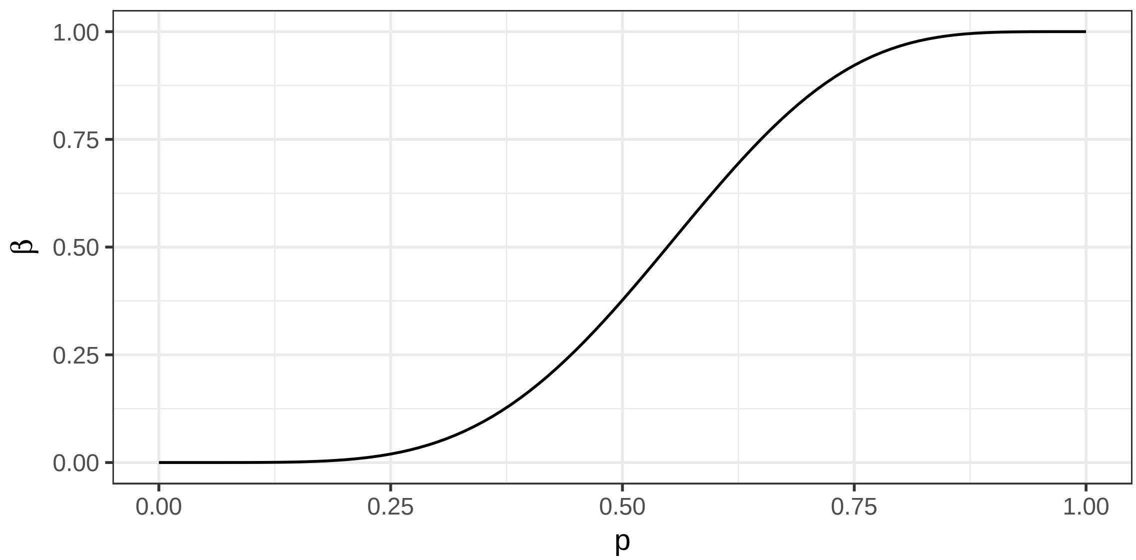
When $\sum X_i = 6$, $\hat{p} = 3/5 \in \Theta_0^C$. So $\hat{p}_0 = 1/2$.

The size is $P(Y \geq 6|H_0) = P(Y \geq 6|1/2) \approx \sum_6^1 0\binom{10}{y}.5^1 0 = 0.377$.

The power function is $\beta(p) = P(Y \geq 6|p)$:

```
p <- seq(0, 1, .01)
beta <- 1 - pbinom(5, 10, p)

ggplot() +
  geom_line(aes(x = p, y = beta)) +
  labs(y = expression(beta))
```




**c**

```r
MASS::fractions(pbinom(seq(0, 10), 10, .5))
```

```
 [1]    1/1024   11/1024     7/128     11/64   193/512   319/512     53/64
 [8]   121/128 1013/1024 1023/1024         1
```

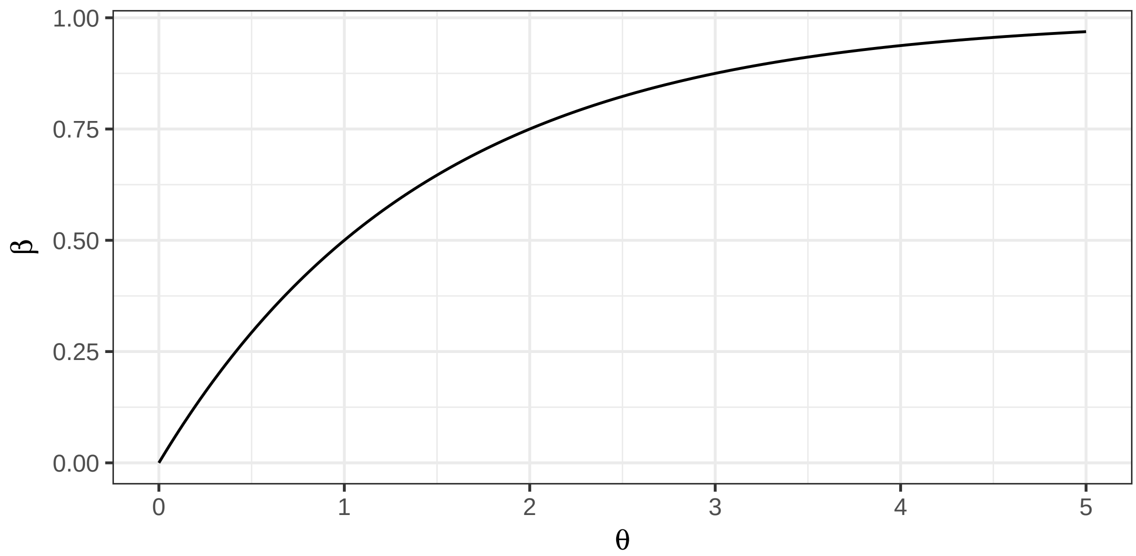
## 8.23

**a**

$\beta(\theta) = P(X > 1/2|\theta)$

```r
theta <- seq(0, 5, .01)
beta <- 1 - pbeta(.5, theta, 1)

ggplot() +
  geom_line(aes(x = theta, y = beta)) +
  labs(x = expression(theta), y = expression(beta))
```




**b**

$\frac{L(2)}{L(1)} = \frac{B(2,1)^{-1}x}{B(1,1)^{-1}} = 2x$

So we reject $H_0$ if $2x > k \implies x > k/2$.

To determine $k/2$:

$\alpha = P(X > k/2|1) = \int_{k/2}^{1} dx = 1 - k/2$
$\implies k/2 = 1 - \alpha$

So we reject $H_0$ when $X > 1 - \alpha$.

**c**

By Karlin-Rubin, the UMP test for this is the same as in part (b).

## 8.25

**a**

$\frac{L(\theta_1)}{L(\theta_0)} = \exp(-\frac{1}{2\sigma^2}((x-\theta_1)^2 - (x-\theta_0)^2)) \propto \exp(\frac{\theta_1-\theta_0}{\sigma^2}x)$

This is monotone w.r.t. $x$.

**b**

$\frac{L(\theta_1)}{L(\theta_0)} = \frac{e^{-\theta_1}\theta_1^x}{e^{-\theta_0}\theta_0^x}$
$\propto (\theta_1/\theta_0)^x$

This is monotone w.r.t. $x$.

**c**

$\frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^x(1-\theta_1)^{n-x}}{\theta_0^x(1-\theta_0)^{n-x}}$
$\propto \left(\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right)^x$

This is monotone w.r.t. $x$.

## 8.26

**a**

If $f$ has an MLR, then $\frac{f(x|\theta_1)}{f(x|\theta_0)}$ is monotone.

We need to show that $F(x|\theta_0) \geq F(x|\theta_1)$ for $\theta_0 > \theta_1$.

Consider $\big(F(x|\theta_0) \geq F(x|\theta_1)\big)'$
$= f(x|\theta_0) - f(x|\theta_1)$
$= f(x|\theta_1)\big(\frac{f(x|\theta_0)}{f(x|\theta_1)} - 1\big)$.

Since the ratio is monotone, $\big(\frac{f(x|\theta_0)}{f(x|\theta_1)} - 1\big)$ must stay positive or negative. And since $f(x|\theta_1) \geq 0$, the whole expression cannot change sign. And the only time the expression can go to 0 is when $x \to \pm\infty$. Therefore, $F(x|\theta_0) \geq F(x|\theta_1)$ is always either increasing or decreasing.

## 8.27

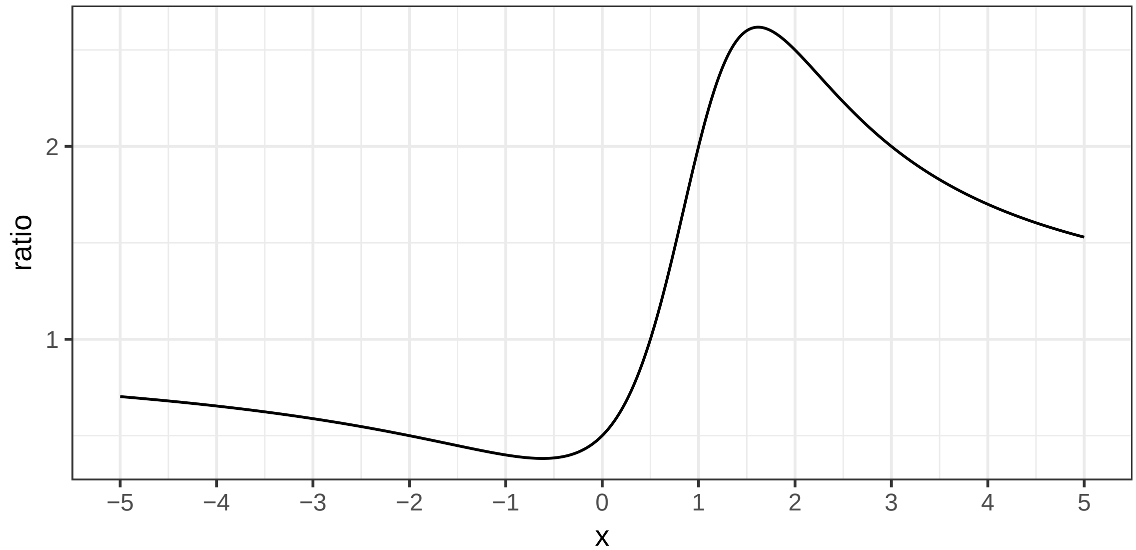
$\frac{g(t|\theta_1)}{g(t|\theta_0)} \propto e^{(w(\theta_1)-w(\theta_0))t}$, which is monotone w.r.t. $t$.

Examples include the bernoulli, binomial, and poisson distributions.

## 8.29

**a**

$\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{1+(x-\theta_0)^2}{1+(x-\theta_1)^2}$




So there exists at least some $\theta_0$, $\theta_1$ such that this ratio is not monotone.

#### b

$\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{1+(x-\theta_0)^2}{1+(x-\theta_1)^2} = \frac{1+x^2}{x^2-2x+2}$

Taking the derivative and setting equal to 0 yields:

$0 = (2x-2)(1+x^2) - (x^2-2x+2)(2x) = 2x^2 - 2x - 2$
$\implies x^2 - x - 1 = 0$
$\implies x = \frac{1 \pm \sqrt{5}}{2}$

From the plot in part (a), we can see that the ratio is decreasing when $x \in (-\infty, \frac{1-\sqrt{5}}{2}] \cup (\frac{1+\sqrt{5}}{2}, \infty)$ and increasing otherwise.

We can also see that the ratio is equal to 2 when $x = 1$ or $x = 3$. So $\phi(x) = 1 \iff 1 < x < 3$ is equivalent to $\phi(x) = 1 \iff \text{ ratio } > 2$.

By Neyman-Pearson, this is the UMP for a certain size.

Type I error:

```r
pcauchy(3, 0) - pcauchy(1, 0)
```

```
[1] 0.1475836
```

Type II error:

```r
1 - (pcauchy(3, 1) - pcauchy(1, 1))
```

```
[1] 0.6475836
```

#### c

The ratio is not monotone, so it cannot be UMP for a composite hypothesis test.

## 8.32

### a

From the class notes, we simply have $\phi = 1 \iff \bar{X} < \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$

### b

When $\theta < \theta_0$, the test from part (a) is UMP, but it is not UMP when $\theta > \theta_0$. Similarly, when $\theta > \theta_0$, $\phi = 1 \iff \bar{X} < \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$ is UMP but not when $\theta < \theta_0$.

If we try $\phi = 1 \iff \bar{X} \in (-\infty, \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n}}) \cup (\bar{X} < \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}, \infty)$, we get a size-$2\alpha$ test.

Trying $\phi = 1 \iff \bar{X} \in (-\infty, \theta_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}) \cup (\bar{X} < \theta_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \infty)$ results in a size-$\alpha$ test, but it is less powerful than the one-sided tests.

## 8.33

### a

Under $P(Y_n \geq 0 | H_0) = 0$, so we only need to consider $Y_1 \geq k$ (assuming $k \in (0,1)$).

$\alpha = P(Y_1 \geq k | H_0) = \int_k^1 n(1-y)^{n-1} dy = (1-k)^n \implies k = 1 - a^{1/n}$.

### b

When $\theta \leq k - 1$, we cannot reject $H_0$, so $\beta(\theta) = 0$.
When $\theta > k$, $H_0$ cannot be true, so $\beta(\theta) = 1$.

When $\theta \in (k-1, 0]$, $Y_n < 1$, so we only need to consider $P(Y \geq k)$. So $\beta(\theta) = \int_k^{\theta+1} n(1 - y + \theta)^{n-1} dy$ $= (1 - k + \theta)^n$.

When $\theta \in (0, k]$, $\beta(\theta) = P(Y_1 \geq k \cup Y_n \geq 1)$
$= P(Y_1 \geq k) + P(Y_1 < k \cap Y_n \geq 1)$.
We already computed the first term, and the second term is $= \int_k^{\theta+1} n(1 - y_1 + \theta)^{n-1} dy_1 + \int_\theta^k dy_1 \int_1^{\theta+1} dy_n n(n-1)(y_n - y_1)^{n-2}$
$= \int_\theta^k n(\theta + 1 - y_1)^{n-1} dy_1 - \int_\theta^k (1 - y_1)^{n-1} dy_1$
$= 1 - (1-\theta)^n - (\theta + 1 - k)^n + (1-k)^n$.
The last term is $\alpha$, and the third term cancels out with the first part, so we are left with $\beta(\theta) = 1 - (1-\theta)^n + \alpha$.

$$\beta(\theta) = \begin{cases} 0 & \theta \leq k - 1 \\ (1 - k + \theta)^n & \theta \in (k-1, 0] \\ 1 - (1-\theta)^n + \alpha & \theta \in (0, k] \\ 1 & \theta > k \end{cases}$$

### c

The sufficient statistic for iid uniform random variables is the min and max, so $(Y_1, Y_n)$ is sufficient.

$f(y_1, y_n | \theta) = n(n-1)(y_n - y_1)^{n-2} I(\theta < y_1 < y_n < \theta + 1)$,
so $f(y_1, y_n | \theta) = n(n-1)(y_n - y_1)^{n-2} I(0 < y_1 < y_n < 1)$.

Since only the indicator part is different between $H_0$ and $H_1$, we only have to consider that part. So we have to make sure that when $Y_n \geq 1$ or $Y_1 \geq k$, $I(\theta < y_1 < y_n < \theta + 1) > k' I(0 < y_1 < y_n < 1)$, and when both $Y_n < 1$ and $Y_1 < k$, $I(\theta < y_1 < y_n < \theta + 1) < k' I(0 < y_1 < y_n < 1)$.

- Suppose $Y_n \geq 1$. Then we are in the rejection region, so we look at $I(\theta < y_1 < y_n < \theta + 1) > k' I(0 < y_1 < y_n < 1)$. And here we can immediately see that the right hand side is 0, so this is true.
- Suppose $Y_1 \geq k$. We are again in the rejection region , so we look at $I(\theta < y_1 < y_n < \theta + 1) > k' I(0 < y_1 < y_n < 1)$. If $k > 1$, again, the right hand side is 0. If $k \in (0, 1)$, we can consider just the case where $Y_n > Y_1 > k$ otherwise again the right hand side is 0. In that case, the left and right hand sides (without the $k'$ are equal, so we can just set $k' < 1$.
- Suppose $Y_n < 1$ and $Y_1 < k$. Then as long as $Y_1 < Y_n$, the right hand side is 1. On the other hand, the left hand side could be 0 or 1, so the inequality still holds when choosing $k' < 1$.

#### d

If $\theta > k = 1 - .1^{1/n}$, then $\beta(\theta) = 1 > .8$.

## Part 2

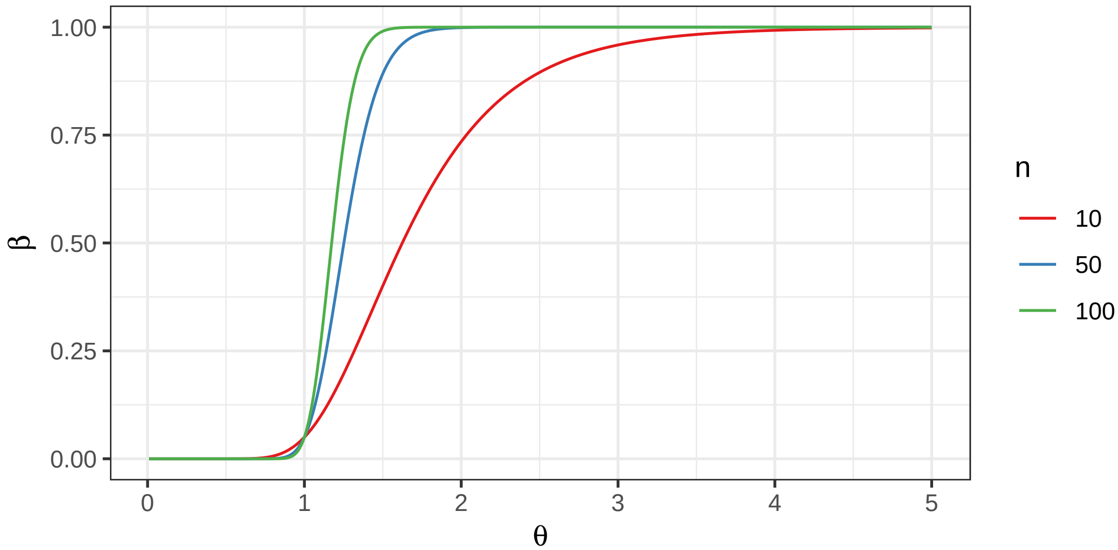
$\frac{L(\theta)}{L(1)} = \frac{1}{\theta} e^{(1-\frac{1}{\theta}) \sum x_i}$, which is monotone w.r.t. $x$. Under $H_1$, this is increasing w.r.t. $x$, so we reject when $\sum X_i > c$ for some $c$.

Next, we note that $Y = \sum X_i \sim Gamma(n, \theta)$. Under $H_0$, $Y = \sum X_i \sim Gamma(n, 1)$.
So $\alpha = .05 = P(Y > c | H_0)$, and we can solve for $c = c(n)$ numerically.

The power function is $\beta(\theta) = P(Y > c | \theta)$ where we use the same $c$ from before.

```
n.vector <- c(10, 50, 100)
theta <- seq(.01, 5, .01)
alpha <- .05
theta.0 <- 1

lapply(n.vector, function(n) {
  c. <- qgamma(1 - alpha, shape = n, scale = theta.0)
  beta <- 1 - pgamma(c., shape = n, scale = theta)
  dplyr::data_frame(n = n, theta = theta, beta = beta)
}) %>%
  dplyr::bind_rows() %>%
  ggplot() +
  geom_line(aes(x = theta, y = beta, colour = factor(n))) +
  scale_colour_brewer(palette = 'Set1') +
  labs(x = expression(theta), y = expression(beta), colour = 'n')
```


1.00
0.75
0.50
0.25
0.00
β
0
1
2
3
4
5
θ
n
10
50
100