STAT-S620

Assignment 8

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5.8.5

$$
\begin{aligned}
E[X^r(1-X)^S] &= \int_0^1 x^r(1-x)^s \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+r)-1}(1-x)^{(\beta+s)-1}dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+r)\Gamma(\beta+s)}{\Gamma(\alpha+r+\beta+s)}
\end{aligned}
$$

Not from text

Problem 1

We are given $f(x) = (\theta+1)x^\theta$ for $x \in [0,1]$. Then

$$
\begin{aligned}
E[X] &= \int_0^1 (\theta+1)x^{\theta+1}dx \\
&= (\theta+1)\int_0^\infty x^{\theta+1}dx \\
&= \frac{\theta+1}{\theta}
\end{aligned}
$$

Using the method of moments, we set $E[X] = \bar{x}$. Then

$$
\begin{aligned}
\bar{x} &= \frac{\hat{\theta}+1}{\hat{\theta}} \\
\hat{\theta} &= \frac{1}{\bar{x}-1}
\end{aligned}
$$

Problem 2

If $X \sim Gamma(\alpha, \beta)$, then

$$
\begin{aligned}
f(x) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \\
&= \frac{x^{\alpha-1}}{\Gamma(\alpha)} \exp(-\alpha\log\beta - x/\beta)
\end{aligned}
$$

Then we can set:

- $\eta(\beta) = 1/\beta$
- $T(x) = -x$
- $B(\beta) = \alpha\log\beta$

Part a

We can see from $\eta(\beta)$ that $\beta \neq 0$, and from $B(\beta)$ that $\beta > 0$.

Part b

In order to get the canonical form, we need to replace $B(\beta)$ with $A(\eta)$. Since $\eta(\beta) = 1/\beta$, $\beta = 1/\eta$. Then $A(\eta) = \alpha \log \frac{1}{\eta} = -\alpha \log \eta$.

$E[T(X)] = E[-X]$, but we also know

$$\begin{aligned} E[T(X)] = E[-X] &= \frac{\partial}{\partial \eta} A(\eta) \\ &= \frac{\partial}{\partial \eta}(-\alpha \log \eta) \\ &= -\alpha/\eta \\ \\ &= -\alpha\beta \end{aligned}$$

Problem 3

We know that $f(\theta|x) \propto f(x|\theta)f(\theta)$. We are also given

- $f(x|\theta) = (2\pi\theta)^{-1/2} \exp(-\frac{(x-\mu)^2}{2\theta})$
- $f(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}$

Then ignoring constant factors,

$$\begin{aligned} f(\theta|x) &\propto \theta^{-1/2} \exp\left(-\frac{(x-\mu)^2}{2\theta}\right) \theta^{-\alpha-1} \exp(-\beta/\theta) \\ &\propto \theta^{-(\alpha+\frac{1}{2}+1)} \exp\left(-\frac{\frac{(x-\mu)^2}{2} + \beta}{\theta}\right) \end{aligned}$$

Then we get

$$\theta|x \sim InvGamma\left(\alpha + \frac{1}{2}, \frac{(x-\mu)^2}{2} + \beta\right)$$