STAT-S631

Assignment 5

John Koo

Part a

Start by expanding $(Y - X\hat{\beta})^T(Y - X\hat{\beta})$

$$(Y - X\hat{\beta})^T(Y - X\hat{\beta})$$
$$= Y^TY - Y^T(X\hat{\beta}) - (X\hat{\beta})^TY + (X\hat{\beta})^T(X\hat{\beta})$$

Note that $x^Ty = y^Tx$ when $x, y \in \mathbb{R}^n$. So the above is equal to

$$Y^TY - 2Y^T(X\hat{\beta}) + (X\hat{\beta})^T(X\hat{\beta})$$

Expanding this out to series:

$$= \sum_i^n y_i^2 - 2\sum_i^n y_i(x_{i,\cdot}\hat{\beta}) + \sum_i^n (x_{i,\cdot}\hat{\beta})^2$$

$$= \sum_i^n y_i^2 - 2y_i(x_{i,\cdot}\hat{\beta}) + (x_{i,\cdot}\hat{\beta})^2$$

Which is just an expansion of:

$$\sum_i^n (y_i - x_{i,\cdot}\hat{\beta})^2$$

Part b

If H is symmetric, then $H = H^T$.

$$H^T = [X(X^TX)^{-1}X^T]^T$$
$$= [X^T]^T[(X^TX)^{-1}]^T[X]^T$$
$$= [X][(X^TX)^T]^{-1}[X^T]$$
$$= X[(X^T)(X^T)^T]^{-1}X^T$$
$$= X(X^TX)^{-1}X^T$$
$$= H$$

Therefore $H = H^T$.

Part c

Show symmetric

$(I - H)^T = I^T - H^T = I - H$

Show idempotent

We know that $HH = H$.

$(I - H)^2 = II - IH - HI + H^2 = I - H - H + H = I - H$

Part d

$HX = [X(X^TX)^{-1}X^T]X = X(X^TX)^{-1}(X^TX) = XI = X$

Part e

$$\begin{aligned}(I - H)(Y - X\hat{\beta}) &= Y - X\hat{\beta} - HY + HX\hat{\beta} \\ &= Y - X\hat{\beta} - HY + X\hat{\beta} \\ &= Y - HY = (I - H)Y\end{aligned}$$

Part f

$$\begin{gathered}(Y - X\hat{\beta})^T(I - H)(Y - X\hat{\beta}) \\ = [(I - H)^T(Y - X\hat{\beta})]^T(Y - X\hat{\beta}) \\ = [(I - H)(Y - X\hat{\beta})]^T(Y - X\hat{\beta}) \\ = [(I - H)Y]^TY \\ = Y^T(I - H)^TY \\ = Y^T(I - H)Y\end{gathered}$$

Part g

$$\begin{gathered}RSS(\hat{\beta}) = (Y - X\hat{\beta})^T(Y - X\hat{\beta}) \\ = (Y - HY)^T(Y - HY) \\ = Y^TY - Y^THY - (HY)^TY + (HY)^THY \\ = Y^TY - Y^THY - Y^TH^TY + Y^TH^THY \\ = Y^TY - Y^THY - Y^THY + Y^THY \\ = Y^TY - Y^THY \\ = (Y^T - Y^TH)Y \\ = Y^T(I - H)Y\end{gathered}$$