S721 HW3

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Problem 1.45

We have:

$$P_X(X = x_i) = P(\{s_j \in S \mid X(s_j) = x_i\})$$

where each $x_i \in \mathcal{X}$ and $|\mathcal{X}| = m < \infty$.

i. For each x_i, $P_X(X = x_i) = P(\{s_j \in S \mid X(s_j) = x_i\})$, and $P(\{s_j \in S \mid X(s_j) = x_i\}) \geq 0$ since P is a probability measure. Similarly, let $A \subset \mathcal{X}$. Then $P_X(A) = \sum_{x_i \in A} P_X(X = x_i)$, and each $P_X(X = x_i)$ is nonnegative, so $P_X(A) \geq 0$.

ii. $P_X(\mathcal{X}) = P_X(X \in \cup_{i=1}^m x_i) = P(\cup_i^m \{s_j \in S | X(s_j) = x_i\})$. We know that every s_j maps to an x_i, so $X^{-1}(\mathcal{X}) = S$. Therefore, this is equal to $P(X^{-1}(\mathcal{X})) = P(S) = 1$.

iii. Since $\mathcal{X}$ is finite (and therefore disjoint), $\mathcal{B}$ is the set of all subsets of $\mathcal{X}$.
Let $A_1, A_2, ...$ be pairwise disjoint subsets of $\mathcal{X}$. They are also all $\in \mathcal{B}$. Since the A_is are disjoint and since X is a function, $B_i = X^{-1}(A_i)$ are all also pairwise disjoint (there cannot be one $s_j \in S$ that maps to two A_is since that would mean $X(s_j)$ can take two different values).
Then $P_X(\cup_i^\infty A_i) = P(\cup_i^\infty X^{-1}(A_i)) = \sum_i^\infty P(X^{-1}(A_i)) = \sum_i^\infty P_X(A_i)$.

Problem 1.47

Part d

$\lim_{x \to -\infty} 1 - \exp(-x) = 1 - 1 = 0$

$\lim_{x \to \infty} 1 - \exp(-x) = 1 - 0 = 1$

$\big(1 - \exp(-x)\big)' = \exp(-x) > 0$

Part e

We have for some $\epsilon \in (0, 1)$,

$$F_Y(y) = \begin{cases} \frac{1-\epsilon}{1+\exp(-y)} & y < 0 \\ \epsilon + \frac{1-\epsilon}{1+\exp(-y)} & y \geq 0 \end{cases}$$

For the left limit, $\lim_{y \to -\infty} \frac{1-\epsilon}{1+\exp(-y)} = 0$ since $\exp(-y) \to \infty$ as $y \to -\infty$.

For the right limit, $\lim_{y \to \infty} \epsilon + \frac{1-\epsilon}{1+\exp(-y)} = \epsilon + 1 - \epsilon = 1$ since $1 + \exp(-y) \to 1 + 0 = 1$ as $y \to \infty$.

$\left(\frac{1-\epsilon}{1+\exp(-y)}\right)' = (1-\epsilon)(-1)(-\exp(-y))(1+\exp(-y))^{-2} = \frac{(1-\epsilon)\exp(-y)}{(1+\exp(-y))^2}$. Since $\epsilon > 0$ and $\exp(.) > 0$, this expression is always positive.

$\epsilon + \frac{1-\epsilon}{1+\exp(-y)}$ is just the previous expression with a constant, so its derivative is the same.

Problem 1.49

We are given that $F_X(t) \leq F_Y(t)\ \forall t$.

$P(X > t) = 1 - P(X \leq t) = 1 - F_X(t) \geq 1 - F_Y(t) = 1 - P(Y \leq t) = P(Y > t)$

We are given that $F_X(t) < F_Y(t)$ for some t, i.e., $\exists t$ such that this is true. Suppose that this is true for $t = s$. Then like before,

$P(X > s) = 1 - P(X \leq s) = 1 - F_X(s) > 1 - F_Y(s) = 1 - P(Y \leq s) = P(Y > s)$

Problem 1.53

Part a

The support of Y is $y \geq 1$, so by definition, $\forall y < 1$, $F_Y(y) = 0$.

On the other hand, as $y \to \infty$, $y^{-2} \to 0$, so $1 - y^{-2} \to 1$.

$\left(1 - y^{-2}\right)' = 2y^{-3}$, and $y \geq 1$, so this is always positive.

Part b

We found the derivative of F_Y for $y \geq 1$ in part (a). For $y < 1$, F_Y is a constant (0), so the derivative is 0. Then

$$f_Y(y) = \begin{cases} 0 & y < 1 \\ 2y^{-3} & y \geq 1 \end{cases}$$

Part c

$F_Z(z) = P(Z \leq z) = P(10(Y-1) \leq z) = P(Y \leq z/10 + 1) = F_Y(z/10 + 1)$

Then $F_z(z) = F_Y(z/10 + 1) = 1 - \frac{1}{\left(\frac{z}{10}+1\right)^2}$

Problem 1.54

Part b

We require $\int ce^{-|x|}dx = 1$.

$1 = \int ce^{-|x|}dx$

$= c\left(\int_{-\infty}^{0} e^x dx + \int_0^\infty e^{-x} dx\right)$

$= c(1+1) = 2c$

Therefore, $c = 1/2$.

Not from textbook

Problem 1

i. Consider $x < y$ and $F(x) = P(X \leq x)$ and $F(y) = P(X \leq y)$.
Note that for any $z \in \mathbb{R}$, $P(X \leq z) = P((-\infty, z])$.
For $x < y$, $(-\infty, x] \subset (-\infty, y]$. Therefore, $P((-\infty, x]) \leq P((-\infty, y])$.

ii. We showed (in class and previous homework) that if $A_1 \supset A_2 \supset \cdots$ then $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$.
Consider the sequence of intervals $(-\infty, x + 1/n]$. We can see that each interval is a subset of the previous intervals and that as $n \to \infty$, the interval goes to $(-\infty, x]$.
$P((-\infty, x + 1/n]) = F(x + 1/n)$ by definition, and $\lim_{n\to\infty} F(x + 1/n) = \lim_{n\to\infty} P((-\infty, x + 1/n]) = P((-\infty, x]) = F(x)$. We also know that F is monotone increasing (not necessarily strictly). So if we let $\delta = 1/n$, then we can see that $F(x) = \lim_{n\to\infty} F(x + 1/n) = \lim_{\delta\to 0} F(x + \delta)$.

iii. Consider the sequence of intervals $(-\infty, x - n]$ for some constant x. We can see that each interval is a subset of the interval before it, and as $n \to \infty$, this interval becomes empty (Since each interval is a subset of the previous intervals, $(-\infty, x - N] = \cap_n^N (-\infty, x - n]$. Assume that y is in the interval where $n \to \infty$. Then $\exists N \in \mathbb{N}$ such that $-N < y$, so y cannot be in $\cap_n^\infty (-\infty, x - n]$).
Let $A_n = X^{-1}((-\infty, x - n]))$ (we can do this since such intervals generate the Borel σ-algebra). Since each interval is a subset of the previous intervals, $A_1 \supset A_2 \supset \cdots$ as well. Since the interval approaches the empty set, $A_n \to \emptyset$.
$\lim_{n\to\infty} P((-\infty, x - n]) = \lim_{n\to\infty} F(x - n) = \lim_{y\to-\infty} F(y)$. On the other hand, $\lim_{n\to\infty} P((-\infty, x - n]) = \lim_{n\to\infty} P(A_n) = P(\emptyset) = 0$. Therefore, $\lim_{y\to-\infty} F(y) = 0$.
Similarly, for $\lim_{x\to\infty} F(x)$, consider the intervals $(-\infty, x + n]$. Then as $n \to \infty$, the union of the intervals (or equivalently, the last interval, since each interval is a subset of the next interval), approaches $\mathbb{R}$.
Let $B_n = X^{-1}((-\infty, x + n])$. Then since each interval is a subset of the next, $B_1 \subset B_2 \subset \cdots$ and $B_n = \cup_i^n B_i$.
By De Morgan's laws, $B_n^c = \cap_i^n B_i^c$. $B^c = \lim_{n\to\infty} B_n^c$ is empty since if we suppose that there is some $s \in B^c$, then $X(s) \in \lim_{n\to\infty} (-\infty, x + n]^c = \emptyset$. So $P(B^c) = P(\emptyset) = 0$, and $P(B^c) = P(\lim_{n\to\infty} B_n^c)$ $= P(\lim_{n\to\infty} (-\infty, x + n]^c) = \lim_{y\to\infty} P(X \leq y) = \lim_{y\to\infty} F(y)$ (letting $y = x + n$).

iv. Consider the intervals $(-\infty, x - 1/n)$. As $n \to \infty$, the interval approaches $(-\infty, x)$. We can also see that $P((-\infty, x)) = P(X < x)$.
Since the i^{th} interval is a subset of the $(i + 1)^{\text{th}}$ interval, each interval is also the union of itself with all of its preceeding intervals. Then $(-\infty, x) = \cup_n^\infty (-\infty, x - 1/n)$ and $P(X < x) = P(\cup_n^\infty (-\infty, x - 1/n))$.
Let $\delta = 1/n$. Then as $\delta \to \infty$, $P((-\infty, x - 1/n)) = P((-\infty, x - \delta)) \to P((-\infty, x)) = F(x^-)$.

v. $P(X = x) = P(X \leq x) - P(X < x) = F(x) - F(x^-)$

Problem 2

It is sufficient to show that $X^{-1}((-\infty, x]) \in \mathcal{F}$.
Note that since F is strictly increasing, F^{-1} exists and is also strictly increasing.

$X^{-1}((-\infty, x]) = \{\omega \in \Omega \mid X(\omega) \leq x\}$
$= \{\omega \in \Omega \mid F^{-1}(\omega) \leq x\}$
$= \{\omega \in \Omega \mid \omega \leq F(x)\}$

This set is just $[0, F(x)]$ (since $\Omega = [0, 1]$), which is in $\mathcal{B}$.

$P(X(\omega) \leq x)$
$= P(\{\omega \in \Omega | X(\omega) \leq x\})$

$= P([0, F(x)])$
$= F(x)$ since P is the Lebesgue measure.