S722 HW4

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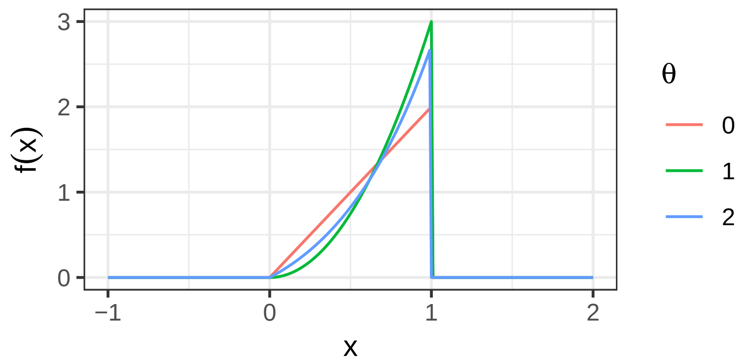
To save on typing, I will denote $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$.

Problem 1

a



$\frac{L(1)}{L(0)} = \frac{3}{2}x$ is increasing w.r.t. x, so we reject when this ratio is large. We can ignore the constant up front and just look at x by itself. So the UMP test would be to reject when $X > c$ for some $c = c(\alpha)$, and to obtain c:

$\alpha = .19 = P(X > c|H_0) = 1 - P(X \leq c|H_0)$
$= 1 - \int_0^c 2x dx = 1 - c^2$
$\implies c = \sqrt{1 - \alpha} = \sqrt{.81} = 0.9$

To obtain the power, we compute
$\beta(\theta = 1) = P(X > .9|\theta = 1)$
$= 1 - P(X \leq .9|\theta = 1) = 1 - \int_0^{\cdot} 93x^2 dx = 1 - .9^3 \approx 0.271$

b

From the plot, we can see that $f(x|1) > f(x|2)$ for the relevant region, so the UMP would use $\theta = 1$ since it is the MLE. So the test in part (a) is the UMP of size .19.

Problem 2

a

$L(\theta) = \prod \frac{2x_i}{\theta} \exp(-\frac{1}{\theta} x_i^2)$
$= (\frac{2}{\theta})^n (\prod x_i) \exp(-\frac{1}{\theta} \sum x_i^2)$
$\propto \theta^{-n} \exp(-\frac{1}{\theta} \sum x_i^2)$

Then the log-likelihood is
$\ell(\theta) = -n \log \theta - \frac{1}{\theta} \sum x_i^2 + C$

And to find the MLE for θ,
$0 = \ell'(\theta) = -n/\theta + \theta^{-2} \sum x_i^2$
$\implies \hat{\theta} = \frac{1}{n} \sum X_i^2$

The likelihood is maximized at
$L(\frac{1}{n} \sum x_i^2) \propto (\frac{1}{n} \sum x_i^2)^{-n} \exp(-n)$

So the LRT statistic is

$\lambda(X) = \frac{L(\hat{\theta})}{L(1)} = \frac{(\frac{1}{n} \sum X_i^2)^{-n} \exp(-n)}{\exp(-\sum X_i^2)}$
$\propto (\sum X_i^2)^{-n} \exp(\sum X_i^2)$

Letting $T = \sum X_i^2$, we get $\lambda(T) = T^{-n} \exp(T)$, and the LRT is $\phi(T) = 1 \iff T^{-n} \exp(T) > c$.

λ is not a monotone function of T, and it has a global minimum. So this test is equivalent to $\phi(T) = 1 \iff T < c_1$ or $T > c_2$.

To find c_1 and c_2, we solve the system of equations:

- $\alpha = P(T < c_1 | H_0) + P(T > c_2 | H_0)$
- $\lambda(c_1) = \lambda(c_2)$

b

A UMP size-α test does not exist in this case because it is a two-sided test. To be more specific, suppose this were a right-sided test. Since the MLE for θ is $\frac{1}{n} \sum X_i^2$, we would reject for large values of $\sum X_i^2$, i.e., when $T > c$ for some c. Similarly, if this were a left-sided test, we would reject when $T < c$ for some c. Then the UMP test would require knowing whether $\theta > 1$ or $\theta < 1$.

Problem 3

8.37 from Casella & Berger

a

Under H_0, $\bar{X} \sim \mathcal{N}(\theta_0, \frac{\sigma^2}{n})$. So
$P(\bar{X} > \theta_0 + z_\alpha \sigma/\sqrt{n} | H_0)$
$= P(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > z_\alpha | H_0) = P(Z > z_\alpha) = \alpha$.

Using the LRT method, we have

$\lambda(X) = \frac{\exp(-\frac{1}{2\sigma^2} \sum (X_i - \theta_0)^2)}{\exp(-\frac{1}{2\sigma^2} \sum (X_i - \bar{X})^2)}$
$= \exp(-\frac{n}{2\sigma^2} (\bar{X} - \theta_0^2)^2)$

And we reject if this value is too small. We can also see that λ can be expressed purely in terms of $\bar{X}$, and λ is decreasing in $\bar{X}$, so this is equivalent to rejecting for large $\bar{X}$. Using the distribution of $\bar{X}$, we arrive at the test described.

b

From class, we arrived at the test described in (a) from the ratio $\frac{\sup_{\theta \in \Theta_0^C} L(\theta)}{\sup_{\theta \in \Theta_0} L(\theta)}$, so it is the UMP test.

c

Under H_0, $\frac{\bar{X}-\theta_0}{S/\sqrt{n}} \sim T_{n-1}$. So $P(\bar{X} > \theta_0 + t_{n-1,\alpha} S/\sqrt{n} | H_0) = P(\frac{\bar{X}-\theta_0}{S/\sqrt{n}} > t_{n-1,\alpha} | H_0) = P(T > t_{n-1,\alpha}) = \alpha$.

The MLE for σ^2 is $\frac{1}{n}\sum(X_i - \bar{X})^2$, and in the restricted case, $\hat{\sigma}_R^2 = \frac{1}{n}\sum(X_i - \theta_0)^2$. We can note that when we plug these into the likelihoods, the exponential term cancels out since we have $\sum(x_i - \bar{x})^2 / \sum(x_i - \bar{x})^2$ and likewise with θ_0 instead of $\bar{x}$. So the LRT test statistic is simply:

$\lambda(X) = (\hat{\sigma}^2/\hat{\sigma}_R^2)^{n/2}$

We can remove the power since we know it must be positive. So the term we must consider is:

$$
\begin{aligned}
&\hat{\sigma}^2/\hat{\sigma}_R^2 \\
&= \frac{\frac{1}{n}\sum(X_i-\bar{X})^2}{\frac{1}{n}\sum(X_i-\theta_0)^2} \\
&= \frac{\sum(X_i-\bar{X})^2}{\sum(X_i-\theta_0)^2} \\
&= \frac{(n-1)S^2}{\sum(X_i-\bar{X}+\bar{X}-\theta_0)^2} \\
&= \frac{(n-1)S^2}{\sum(X_i-\bar{X})^2+\sum(\bar{X}-\theta_0)^2+2\sum(X_i-\bar{X})(\bar{X}-\theta_0)} = \frac{(n-1)S^2}{(n-1)S^2+n(\bar{X}-\theta_0)^2} \\
&= \frac{n-1}{(n-1)+\frac{(\bar{X}-\theta_0)^2}{S^2/n}}
\end{aligned}
$$

This is decreasing in $\frac{(\bar{X}-\theta_0)^2}{S^2/n}$, so it is also decreasing in $\frac{\bar{X}-\theta_0}{S/\sqrt{n}}$. Then rejecting when the ratio is too small is equivalent to rejecting when this is too large.

Problem 4

8.41 from Casella & Berger

a

Under H_0, $\hat{\mu}_R = \frac{\sum X_i + \sum Y_i}{n+m}$. Then $\hat{\sigma}_R^2 = \frac{\sum(X_i-\hat{\mu}_R)^2+\sum(Y_i-\hat{\mu}_R)^2}{n+m}$

We can simplify $\hat{\sigma}_R^2$:

$$
\begin{aligned}
&\sum(X_i - \hat{\mu}_R)^2 = \sum(X_i - \tfrac{n\bar{X}+m\bar{Y}}{n+m})^2 \\
&= \sum(X_i - \bar{X} + \bar{X} - \tfrac{n\bar{X}+m\bar{Y}}{n+m})^2 \\
&= \sum(X_i - \bar{X})^2 + n(\bar{X} - \tfrac{n\bar{X}+m\bar{Y}}{n+m})^2 + 2(\bar{X} - \tfrac{n\bar{X}+m\bar{Y}}{n+m})\sum(X_i - \bar{X}) \\
&= \sum(X_i - \bar{X})^2 + n(\bar{X} - \tfrac{n\bar{X}+m\bar{Y}}{n+m})^2 \\
&= \sum(X_i - \bar{X})^2 + n\tfrac{(n\bar{X}+m\bar{X}-n\bar{X}-m\bar{Y})^2}{(n+m)^2} \\
&= \sum(X_i - \bar{X})^2 + \tfrac{nm^2}{(n+m)^2}(\bar{X} - \bar{Y})^2
\end{aligned}
$$

and similarly for the Y term. So we have:

$$
\begin{aligned}
&\hat{\sigma}_R^2 = \frac{\sum(X_i-\bar{X})^2+(Y_i-\bar{Y})^2}{n+m} + (\tfrac{nm^2}{(n+m)^3} + \tfrac{n^2m}{(n+m)^3})(\bar{X} - \bar{Y})^2 \\
&= \frac{\sum(X_i-\bar{X})^2+(Y_i-\bar{Y})^2}{n+m} + \tfrac{nm}{(n+m)^2}(\bar{X} - \bar{Y})^2
\end{aligned}
$$

Without the H_0 restriction, we get the usual $\hat{\mu}_X = \bar{X}$ and $\hat{\mu}_Y = \bar{Y}$ and $\hat{\sigma}^2 = \frac{\sum(X_i-\bar{X})^2+\sum(Y_i-\bar{Y})^2}{n+m}$.

Plugging these terms into the likelihood functions, similar to 8.37, we get terms canceling out in the exponentials. So we are left with simply

$\lambda(X,Y) = \left(\frac{\hat{\sigma}_R^2}{\hat{\sigma}^2}\right)^{-(n+m)/2}$

and we reject for small $\lambda(X,Y)$, which is equivalent to rejecting for large $\frac{\hat{\sigma}_R^2}{\hat{\sigma}^2}$.

$\frac{\hat{\sigma}_R^2}{\hat{\sigma}^2} = \frac{\sum(X_i-\bar{X})^2+\sum(Y_i-\bar{Y})^2+\frac{nm}{n+m}(\bar{X}-\bar{Y})^2}{\sum(X_i-\bar{X})^2+\sum(Y_i-\bar{Y})^2}$
$= 1 + \frac{nm}{n+m}\frac{(\bar{X}-\bar{Y})^2}{(n+m-2)S_p^2}$
$= 1 + \frac{(\bar{X}-\bar{Y})^2}{(n+m-2)(n^{-1}+m^{-1})S_p^2}$

We can ignore the leading constant term (1) as well as the factor of $(n+m-2)^{-2}$, and we are left with

$\frac{(\bar{X}-\bar{Y})^2}{(n^{-1}+m^{-1})S_p^2} > c$
$\implies \left|\frac{\bar{X}-\bar{Y}}{(n^{-1}+m^{-1})S_p}\right| > c'$

b

Under H_0, $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$ and $\bar{Y} \sim \mathcal{N}(\mu, \sigma^2/m)$, so $\bar{X} - \bar{Y} \sim \mathcal{N}(0, (n^{-1} + m^{-1})\sigma^2)$.

Since S_X^2 and S_Y^2 are χ^2 distributed with $n-1$ and $m-1$ degrees of freedom respectively, and since S_p^2 is just a linear combination of the two, $S_p^2 \sim \chi^2_{n+m-2}$. So $\frac{\bar{X}-\bar{Y}}{S_p\sqrt{n^{-1}+m^{-1}}} \sim T_{n+m-2}$.

c

```
# data
core <- c(1294, 1279, 1274, 1264, 1263, 1254, 1251,
          1251, 1248, 1240, 1232, 1220, 1218, 1210)
periphery <- c(1284, 1272, 1256, 1254, 1242,
               1274, 1264, 1256, 1250)
n <- length(core)
m <- length(periphery)

# sample means
core.mean <- mean(core)
periphery.mean <- mean(periphery)

# pooled variance
s.p <- sqrt(
    (sum((core - core.mean) ** 2) + sum((periphery - periphery.mean) ** 2)) /
      (n + m - 2))

# t-statistic
t.stat <- (core.mean - periphery.mean) / (s.p * sqrt(1 / n + 1 / m))
t.stat
```

```
[1] -1.290656
```

So we would reject H_0 if our chosen c is less than 1.291.

d

We can compute the two-sided p-value using the t-distribution:

```r
2 * pt(t.stat, df = n + m - 2)
```

```
[1] 0.2108527
```

Failing to reject H_0 would be equivalent to performing a UMP level-.21 test.

e

```r
# number of simulations
n.sim <- 1e3

# assumed parameters
grand.mean <- mean(c(core, periphery))
pooled.sd <- s.p

p.value.sim <- sapply(seq(n.sim), function(i) {
  # generate data
  core <- rnorm(n, grand.mean, pooled.sd)
  periphery <- rnorm(m, grand.mean, pooled.sd)

  # compute statistics
  core.mean <- mean(core)
  periphery.mean <- mean(periphery)
  s.p <- sqrt(
    (sum((core - core.mean) ** 2) + sum((periphery - periphery.mean) ** 2)) /
      (n + m - 2))

  # t-statistic
  t.stat.i <- (core.mean - periphery.mean) / (s.p * sqrt(1 / n + 1 / m))

  # p-value
  return(abs(t.stat.i) > abs(t.stat))
}) %>%
  mean()

p.value.sim
```

```
[1] 0.213
```

Problem 5

11.13 from Casella & Berger

Similar to the example from class, we have

$L(\theta, \sigma^2|y) = \prod_i^k \prod_j^{n_i} (2\pi\sigma^2)^{-1/2} \exp(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2)$
$= (2\pi\sigma^2)^{-N/2} \exp(-\frac{1}{2\sigma^2} \sum_i \sum_j (y_{ij} - \theta_i)^2)$
$= (2\pi\sigma^2)^{-N/2} \exp(-\frac{1}{2\sigma^2} (\sum_i \sum_j (y_{ij} - \bar{y}_i)^2 + \sum_i \sum_j (\bar{y}_i - \theta_i)^2))$

Under H_1, the second term is 0 since $\hat{\theta}_i = \bar{y}_i$. Under H_0, $\hat{\theta}_R = \bar{y}$ where $\bar{y}$ is the overall mean.

Simiarly, under H_0, we get $\hat{\sigma}_R^2 = \frac{1}{N}(\sum_i \sum_j (y_{ij} - \bar{y}_i)^2 + \sum_i \sum_j (\bar{y}_i - \bar{y})^2)$
and under H_1, we get $\hat{\sigma}^2 = \frac{1}{N} \sum_i \sum_j (y_{ij} - \bar{y}_i)^2$ since the second term is 0.

Similar to before, the terms in the exponentials disappear when we plug in the MLEs for θ_i and σ^2. So the likelihood ratio is:

$\lambda(Y) = (\hat{\sigma}_R^2/\hat{\sigma}^2)^{-N/2}$

And we reject when this value is small. This is equivalent to rejecting when $\hat{\sigma}_R^2/\hat{\sigma}^2$ is large.

$\hat{\sigma}_R^2/\hat{\sigma}^2 = \frac{\sum_i \sum_j (y_{ij}-\bar{y}_i)^2 + \sum_i \sum_j (\bar{y}_i - \bar{y})^2}{\sum_i \sum_j (y_{ij}-\bar{y}_i)^2}$
$= 1 + \frac{\sum_i \sum_j (\bar{y}_i - \bar{y})^2}{\sum_i \sum_j (y_{ij}-\bar{y}_i)^2}$

We can ignore the leading constant of 1. Then we note that this is a ratio of two χ^2 distributed random variables (times a constant), with the numerator having $k-1$ degrees of freedom and the denominator having $N-k$ degrees of freedom. Thus this ratio is $F_{k-1,N-k}$-distributed, and we reject when it is greater than $F_{k-1,N-k,\alpha}$ for a level-α test.