S721 HW5

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From text (I)

Problem 1.27

Part a

We know that $\binom{n}{k} = \binom{n}{n-k}$, so for odd n, we can write this as:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{k} + \sum_{k=\frac{n-1}{2}+1}^{n}(-1)^k\binom{n}{k} = 0$$

For even n, we first note that for $0 < k < n$, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{n-k}$ since each $\binom{n}{k}$ is on Pascal's triangle and is the sum of the two values above it. Then we can write:

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k} \\
&= \binom{n}{0} + \binom{n}{n} + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k} \\
&= \binom{n}{0} + \binom{n}{n} + \sum_{k=1}^{n-1}(-1)^k\left(\binom{n-1}{k} + \binom{n-1}{n-k}\right)
\end{aligned}$$

Since n is even, $n-1$ is odd. So the first part in the summation would sum to 0 if it started from $k=0$, but since it starts from $k=1$, we are left with $\binom{n-1}{0}$, the $k=0$ term. On the other hand, the second part in the summation would sum to 0 if it ended at $k=n$, but since it ends at $k=n-1$, we are left with $\binom{n-1}{0}$. And since 1 and $n-1$ are both odd, $(-1)^k = -1$ when k is equal to either of those values. So we are left with:

$$\begin{aligned}
&\binom{n}{0} + \binom{n}{n} - \binom{n-1}{0} - \binom{n-1}{n-1} \\
&= 1 + 1 - 1 - 1 \\
&= 0
\end{aligned}$$

Part b

Note that:

$$\begin{aligned}
k\binom{n}{k} &= \frac{kn!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k)!} \\
&= \frac{n(n-1)!}{(k-1)!(n-k)!}
\end{aligned}$$

$$= n\binom{n-1}{k-1}$$

We also note that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

Then

$$\sum_{k=1}^{n} k\binom{n}{k}$$
$$= n\sum_{k=1}^{n}\binom{n-1}{k-1}$$
$$= n2^{n-1}$$

Part c

Using parts (a) and (b), we can say:

$$\sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k}$$
$$= \sum_{k=1}^{n}(-1)^{k+1}n\binom{n-1}{k-1}$$
$$= \sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j}n$$
$$= 0$$

Problem 1.28

First, note that $\int_0^n \log x dx = n\log n - n$ and $\int_1^{n+1}\log x dx = (n+1)\log(n+1) - (n+1) + 1$, so their average is

$$\frac{n\log n - n + (n+1)\log(n+1) - (n+1) + 1}{2}$$
$$= \frac{n\log n + (n+1)\log(n+1) - 2n}{2}$$
$$\approx (n + \frac{1}{2})\log n - n$$

Note that $\exp\Big((n+1/2)\log n - n\Big) = n^{n+1/2}e^{-n}$, which is the denominator of our expression. So

$$\log\left(\frac{n!}{n^{n+1/2}e^{-n}}\right)$$
$$= \log n! - \big((n+1/2)\log n - n\big)$$

Then we note that the difference between the n^{th} and $(n+1)^{\text{th}}$ terms is

$$\left(\log n! - \big((n+1/2)\log n - n\big)\right) - \left(\log(n+1)! - \big((n+1/2+1)\log(n+1) - n - 1\big)\right)$$
$$= (n+\frac{1}{2})\log\frac{n+1}{n} - 1$$

And by Taylor expansion, this is

$$\approx \left(n+\frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right) - 1$$
$$= 1 - \frac{1}{2n} + \frac{1}{3n^2} + \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{6n^3} - 1$$
$$= \frac{1}{12n^2} + \frac{1}{6n^3}$$

And this goes to 0 as $n \to \infty$. So each successive term adds less and less until it reaches 0. And since this term $\sim \frac{1}{n^2}$ for large n, we can say that the sum of these terms converges. Therefore, the sequence $\log\left(\frac{n!}{n^{n+1/2}e^{-n}}\right)$ converges to a constant.

Problem 2.9

We just need to calculate $F(x) = \int f(y)dy$. We can see that for the support, $\int_0^x \frac{y-1}{2}dy = \frac{x^2-2x+1}{4}$, so

$$F(x) = \begin{cases} 0 & x \le 1 \\ \left(\frac{x-1}{2}\right)^2 & x \in (1,3) \\ 1 & x > 3 \end{cases}$$

and $u(x) = F(x)$.

Problem 3.11

Part a

We want to show that as $M/N \to p$ and $N \to \infty$ and $M \to \infty$, $\frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \to \binom{K}{x}p^x(1-p)^{K-x}$

$$\lim_{\substack{M/N\to p \\ M,N\to\infty}} \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$$
$$= \binom{K}{x} \lim_{\substack{M/N\to p \\ M,N\to\infty}} \frac{M!(N-M)!(N-K)!}{(M-x)!(N-M-K+x)!N!}$$

Then if we replace each factorial with its Stirling approximation, we get:

$$= \binom{K}{x} \lim_{\substack{M/N\to p \\ M,N\to\infty}} \frac{M^{M+1/2}(N-M)^{N-M+1/2}(N-K)^{N-K+1/2}}{(M-x)^{M-x+1/2}(N-M-K+x)^{N-M-K+x+1/2}N^{N+1/2}}$$

If we rearrange this, we get

$$
\begin{aligned}
\lim_{\substack{M/N\to p\\ M,N\to\infty}} & \\
& \binom{K}{x} \\
& \times\left(\frac{M}{M-x}\right)^{M}\left(\frac{N-M}{N-M-K+x}\right)^{N-M}\left(\frac{N-K}{N}\right)^{N} \\
& \times\left(\frac{M}{M-x}\right)^{1/2}\left(\frac{N-M}{N-M-K+x}\right)^{1/2}\left(\frac{N-K}{N}\right)^{1/2} \\
& \times\left(\frac{1}{(M-x)^{-x}}\frac{(N-K)^{-K}}{(N-M-K+x)^{-K+x}}\right)
\end{aligned}
$$

So we can break this down into 8 components:

- The first component doesn't depend on anything we are taking the limits to, so we can just leave it alone.
- The second component is equal to $(1-x/M)^{-M}\to e^{x}$
- The third component is equal to $(1-\frac{K-x}{N-M})^{-(N-M)}\to e^{K-x}$
- The fourth component is equal to $(1-K/N)^{N}\to e^{-K}$
- In the fifth component, M dominates x, so this goes to 1
- In the sixth component, $N-M$ dominates $-K+x$, so this goes to 1
- In the seventh component, N dominates K, so this goes to 1
- We will consider the eigth component shortly

So we are left with:

$$
\begin{aligned}
& \binom{K}{x}e^{x+K-x-K}(M-x)^{x}\frac{(N-K)^{-K}}{(N-M-K+x)^{-K+x}} \\
&= \binom{K}{x}(M-x)^{x}\frac{(N-M-K+x)^{K-x}}{(N-K)^{K}} \\
&= \binom{K}{x}\left(\frac{M-x}{N-K}\right)^{x}\left(\frac{N-M-K+x}{N-K}\right)^{K-x}
\end{aligned}
$$

Then as $N\to\infty$, $N-K\to N$, and as $M\to\infty$, $M-x\to M$, so we get

$$
=\binom{K}{x}\left(\frac{M}{N}\right)^{x}\left(\frac{N-M}{N}\right)^{K-x}
$$

And $M/N\to p$, so

$$
=\binom{K}{x}p^{x}(1-p)^{K-x}
$$

Part b

We have $K\to\infty$, $M/N=p\to\infty$, and $KM/N=Kp\to\lambda$, so the Poisson approximation is simply

$$
\begin{aligned}
& \frac{e^{-Kp}(Kp)^{x}}{x!} \\
&= \frac{e^{-\lambda}\lambda^{x}}{x!}
\end{aligned}
$$

Part c

Using Stirling's approximation, we can write the expression as

$$
\begin{aligned}
\approx &\frac{e^{-x}}{x!}\times \\
&\left(\frac{K}{K-x}\right)^{1/2}\left(\frac{M}{M-x}\right)^{1/2}\left(\frac{N-M}{N-M-K+x}\right)^{1/2}\left(\frac{N-K}{N}\right)^{1/2}\times \\
&\left(\frac{K}{K-x}\right)^{K}\left(\frac{M}{M-x}\right)^{M}\left(\frac{N-M}{N-M-K+x}\right)^{N-M}\left(\frac{N-K}{N}\right)^{N}\times \\
&(K-x)^x(M-x)^x(N-M-K+x)^{-K+x}(N-K)^{-K}
\end{aligned}
$$

- We will leave the first line alone.
- In the second line, one term dominates the other, so they all become $1^{1/2}=1$.
- In the third line, they become exponentials as in part (a).
- In the fourth line, one term dominates the other, so the smaller term drops out.

So we are left with:

$$
\begin{aligned}
&\approx \frac{1}{x!}e^{-x}e^{x}e^{x}e^{K-x}e^{-K}K^xM^x(\frac{1}{N-M})^{K-x}\frac{1}{N^K} \\
&= \frac{1}{x!}(KM)^x(\frac{1}{N-M})^{K-x}\frac{1}{N^K}
\end{aligned}
$$

Then multiplying by N^x/N^x, we get

$$
= \frac{1}{x!}\left(\frac{KM}{N}\right)^x\left(\frac{N-M}{N}\right)^{K-x}
$$

$\left(\frac{KM}{N}\right)=\lambda$, so we can say

$$
= \frac{1}{x!}\lambda^x\left(\frac{N-M}{N}\right)^{K-x}
$$

Since $K\gg x$, $(K-x)\to K$.

$$
\approx \frac{1}{x!}\lambda^x\left(\frac{N-M}{N}\right)^{K}
$$

$$
\begin{aligned}
&= \frac{1}{x!}\lambda^x(1-M/N)^K \\
&= \frac{1}{x!}\lambda^x\left(1-\frac{(M/N)K}{K}\right)^K \\
&= \frac{1}{x!}\lambda^x\left(1-\frac{KM/N}{K}\right)^K \\
&\approx \frac{1}{x!}\lambda^x e^{-KM/N} \\
&= \frac{1}{x!}\lambda^x e^{-\lambda}
\end{aligned}
$$

Not from text (II)

$f_X(x \mid \theta) = \theta x^{\theta-1}$

$Y = -\log X \to X = \exp(-Y) \to X' = -\exp(-Y)$

$X \in [0,1] \to Y \in [0,\infty)$

Then $f_Y(y \mid \theta) = \theta(e^{-y})^{\theta-1}e^{-y}$
$= \theta e^{-\theta y}$

And $F_Y(y \mid \theta) = \int_0^y f_Y(u)du = 1 - e^{-\theta y}$, so $F_Y^{-1}(q) = -\frac{1}{\theta}\log(1-q)$

From HW4, we have $F_X(x) = x^\theta$ and $F_X^{-1}(q) = q^{1/\theta}$.

```
library(ggplot2)

theme_set(theme_bw())

# as specified by the problem
theta <- 2.5
n <- 1e5

draw.x <- function(n, theta) {
  runif(n) ^ (1 / theta)
}

# draw X and then transform to Y
X <- draw.x(n, theta)
Y <- -log(X)

# pdf of Y as derived above
y <- seq(0, 5, by = 1e-3)
p <- theta * exp(-theta * y)

ggplot() +
  geom_histogram(aes(x = Y, y = ..density..),
                 colour = 'black', fill = 'white') +
  geom_density(aes(x = Y)) +
  geom_line(aes(x = y, y = p), linetype = 2)
```

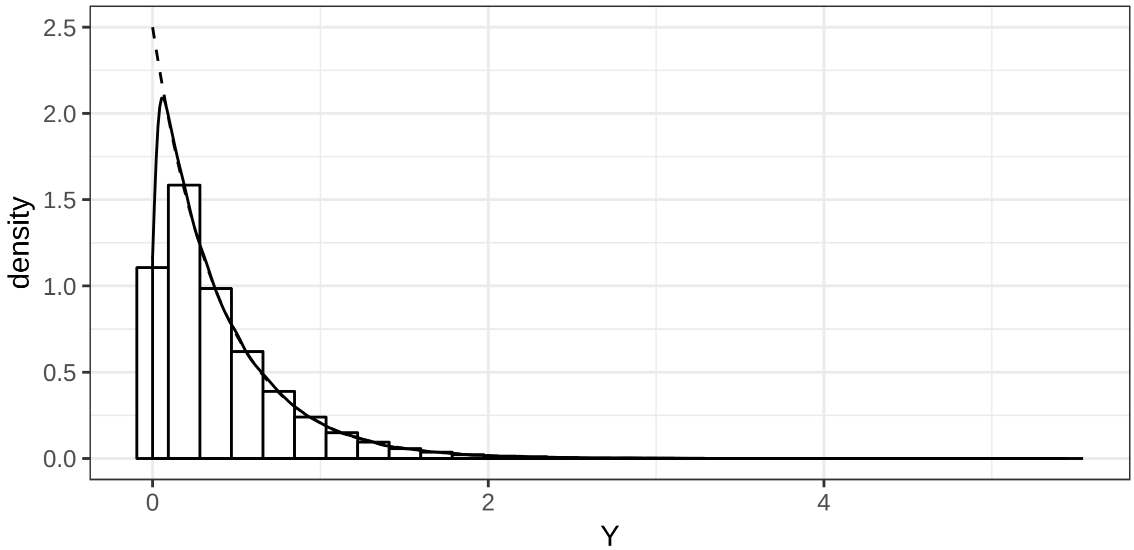

Figure 1: Solid line is density estimation, dashed line is ground truth

$P(X > .3) = 1 - F_X(x) = 1 - .3^{2.5} = 0.951$

Or empirically, we can use `mean(X > .3)` $= 0.951$.

Not from text (III)

```
# params
n.vector <- 10^seq(2, 7)
a <- 0
b <- 1
p <- .5

# binomial probs
probs <- sapply(n.vector, function(n) {
  ex <- n * p
  vx <- n * p * (1 - p)
  pbinom(b * sqrt(vx) + ex, n, p) - pbinom(a * sqrt(vx) + ex - 1, n, p)
})

corrected.probs <- sapply(n.vector, function(n) {
  ex <- n * p
  vx <- n * p * (1 - p)
  pbinom(b * sqrt(vx) + ex - 1/2, n, p) - pbinom(a * sqrt(vx) + ex - 1 + 1/2, n, p)
})

# normal probs
exact.prob <- pnorm(b) - pnorm(a)

ggplot() +
  geom_point(aes(x = n.vector, y = probs)) +
  geom_point(aes(x = n.vector, y = corrected.probs), shape = 2) +
```

```r
scale_x_log10() +
geom_hline(yintercept = exact.prob)
```

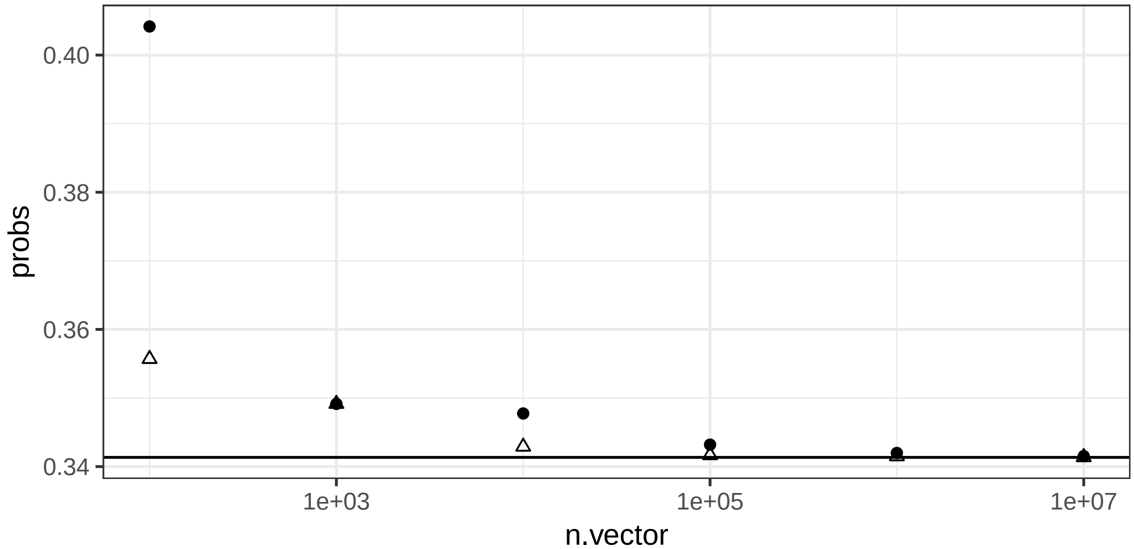

Figure 2: Solid line is normal probability.

The triangles are adjusted probabilities (continuity correction).

Not from text (IV)

Part 1

First, we note the Taylor expansion of $e^x = \sum_k^\infty \frac{x^k}{k!}$.

Then

$$
\begin{aligned}
&\sum_{k=0}^{\infty} P(X = k) \\
&= \sum_k \frac{e^{-\theta}\theta^k}{k!} \\
&= e^{-\theta} \sum_k \frac{\theta^k}{k!} \\
&= e^{-\theta} e^{\theta} \\
&= 1
\end{aligned}
$$

Part 2

By definition, $E[X] = \sum_k kP(X = k)$.

$$\sum_{k=0}^{\infty} k\frac{e^{-\theta}\theta^k}{k!}$$
$$= e^{-\theta}\theta\sum_{k=1}^{\infty}\frac{\theta^{k-1}}{(k-1)!}$$
$$= e^{-\theta}\theta\sum_{j=0}^{\infty}\frac{\theta^j}{j!}$$
$$= e^{-\theta}\theta e^{\theta}$$
$$= \theta$$

Part 3

Note that $E[X(X-1)] = E[X^2] - E[X]$, and $E[X^2] = Var(X) + E[X]^2$, so $Var(X) = E[X(X-1)] - E[X]^2 + E[X]$.

$$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1)\frac{e^{-\theta}\theta^k}{k!}$$
$$= e^{-\theta}\sum_{k=2}^{\infty} k(k-1)\frac{\theta^k}{k!} \text{ (since the first two terms are 0)}$$
$$= e^{-\theta}\theta^2\sum_{j=0}^{\infty}\frac{\theta^j}{j!}$$
$$= e^{-\theta}\theta^2 e^{\theta}$$
$$= \theta^2$$

So . . .

$$Var(X) = E[X(X-1)] - E[X]^2 + E[X]$$
$$= \theta^2 - \theta^2 + \theta$$
$$= \theta$$