MATH-M463

Homework 9

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Problem 1

[Exercise 4 in 5.2]

Part a

$$
\begin{aligned}
P(X \le x, Y \le y) &= \int_0^x \int_0^y f(x,y) dx dy \\
&= \int_0^x 2e^{-2x} dx \int_0^y 3e^{-3y} dy \\
&= \boxed{(1 - e^{-2x})(1 - e^{-3y})}
\end{aligned}
$$

Part b

We can see that this is just a product of two exponential distributions. Therefore, $\boxed{f_X(x) = 2e^{-2x}}$

Part c

Likewise, $\boxed{f_Y(y) = 3e^{-3y}}$

Part d

$X \perp Y$ since $f(x,y) = f_X(x) f_Y(y)$

Problem 2

[Exercise 6 in 5.2]

Part a

$$
\begin{aligned}
P(Y > 2X) &= \int_{x=0}^{1/2} \int_{y=2x}^{1} 90(y-x)^8 dy dx \\
&= \int_0^{1/2} 10(y-x)^9 \Big|_{y=2x}^{1} dx \\
&= \int_0^{1/2} 10\big((1-x)^9 - x^9\big) dx
\end{aligned}
$$

$$= -(1-x)^{10} - x^{10}\Big|_0^{1/2}$$

$$= -\frac{1}{2^{10}} + 1 - \frac{1}{2^{10}}$$

$$= \boxed{1 - \frac{1}{2^9}}$$

Part b

$$f_X(x) = \int_{y=x}^{1} 90(y-x)^8 dy$$

$$= 10(y-x)^9\Big|_{y=x}^{1}$$

$$= \boxed{10(1-x)^9}$$

Problem 3

[Exercise 6 in 5.3]

Part a

Since X and Y are iid standard normal, $3X + 2Y \sim \mathcal{N}(0, 3^2 + 2^2)$. Then

$$P(3X + 2Y > 5) = P(\frac{3X + 2Y}{\sqrt{13}} > \frac{5}{\sqrt{13}})$$

$$= \boxed{1 - \Phi(\frac{5}{\sqrt{13}}) \approx .083}$$

Part b

$$P(\min(X, Y) < 1)$$

This is just the probability of the complement of the event that both X and Y are greater than 1. We can also use the fact that $X \perp Y$.

$$= 1 - P(X > 1, Y > 1)$$

$$= 1 - P(X > 1)P(Y > 1)$$

$$= \boxed{1 - (1 - \Phi(1))^2 \approx .975}$$

Problem 4

[Exercise 8 in 5.3]

Part a

Let $X \sim (0, 5)$ be the time when Peter arrives and $Y \sim \mathcal{N}(2, 3)$ be the time when Paul arrives.

Then $P(X < Y) = P(X - Y < 0)$.

$X - Y \sim \mathcal{N}(-2, 5^2 + 3^2)$. Then:

$$P(X - Y < 0) = P(\frac{X - Y + 2}{\sqrt{34}} < \frac{0 + 2}{\sqrt{34}})$$
$$= \boxed{\Phi(\frac{2}{\sqrt{34}}) \approx .634}$$

Part b

Since X and Y are independent,

$$P(|X| < 3, |Y| < 3) = P(|X| < 3)P(|Y| < 3)$$
$$= P(\frac{-3}{5} < X < \frac{3}{5})P(\frac{-1}{3} < Y < \frac{5}{3})$$
$$= \Big(\Phi(3/5) - \Phi(-3/5)\Big)\Big(\Phi(5/3) - \Phi(-1/3)\Big)$$
$$\approx \boxed{.263}$$

Part c

We already saw that $X - Y \sim \mathcal{N}(-2, 34)$.

$$P(-3 < X - Y < 3) = P\Bigg(\frac{-3 + 2}{\sqrt{34}} < \frac{X - Y + 2}{\sqrt{34}} < \frac{3 + 2}{\sqrt{34}}\Bigg)$$
$$= \boxed{\Phi(5/\sqrt{34}) - \Phi(-1/\sqrt{34}) \approx .372}$$

Problem 5

Part a

$$\int f(x, y)dxdy = \lambda^3 \int_0^\infty e^{-\lambda y} \int_0^y xdxdy$$
$$= \frac{\lambda^3}{2} \int_0^\infty y^2 e^{-\lambda y} dy$$

Let:
$u = y^2$, $du = 2ydy$
$v = -\frac{1}{\lambda}e^{-\lambda y}$, $dv = e^{-\lambda y}dy$

Then we have:

$$
\begin{aligned}
&= \frac{\lambda^3}{2}\bigg(-\frac{y^2}{\lambda}e^{-\lambda y}\bigg|_0^\infty + \frac{2}{\lambda}\int_0^\infty ye^{-\lambda y}dy\bigg) \\
&= \lambda^2 \int_0^\infty ye^{-\lambda y}dy
\end{aligned}
$$

Let:
$u = y$, $du = dy$
$v = -\frac{1}{\lambda}e^{-\lambda y}$, $dv = e^{-\lambda y}dy$

Then:

$$
\begin{aligned}
&= \lambda^2\bigg(-\frac{y}{\lambda}e^{-\lambda y}\bigg|_0^\infty + \int_0^\infty \frac{1}{\lambda}e^{-\lambda y}dy\bigg) \\
&= \lambda \int_0^\infty e^{-\lambda y}dy \\
&= -e^{-\lambda y}\bigg|_0^\infty \\
&-(0-1) = \boxed{1}
\end{aligned}
$$

Part b

$$
\begin{aligned}
f_X(x) &= \int_x^\infty f(x,y)dy \\
&= \int_x^\infty \lambda^3 x e^{-\lambda y}dy \\
&= \lambda^3 x \int_x^\infty e^{-\lambda y}dy \\
&= \lambda^3 x\bigg(-\frac{1}{\lambda}e^{-\lambda y}\bigg|_x^\infty\bigg) \\
&= -\lambda^2 x(0 - e^{-\lambda x}) \\
&= \boxed{\lambda^2 x e^{-\lambda x}}
\end{aligned}
$$

Part c

$$
\begin{aligned}
E[X] &= \int_0^\infty x f_X(x)dx \\
&= \lambda^2 \int_0^\infty x^2 e^{-\lambda x}dx
\end{aligned}
$$

Let:
$u = x^2$, $du = 2xdx$
$v = -\frac{1}{\lambda}e^{-\lambda x}$, $dv = e^{-\lambda x}dx$

Then:

$$
= \lambda^2\bigg(-\frac{x^2}{\lambda}e^{-\lambda x}\bigg|_0^\infty + \frac{2}{\lambda}\int_0^\infty xe^{-\lambda x}dx\bigg)
$$

$$= 2\lambda \int_0^\infty xe^{-\lambda x} dx$$

Let:
$u = x$, $du = dx$
$v = -\frac{1}{\lambda}e^{-\lambda x}$, $dv = e^{-\lambda x}dx$

Then:

$$= 2\lambda \left(-\frac{x}{\lambda}e^{-\lambda x}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}dx \right)$$
$$= 2\int_0^\infty e^{-\lambda x}dx$$
$$= -\frac{2}{\lambda}(0-1)$$
$$= \boxed{\frac{2}{\lambda}}$$

Part d

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)}$$
$$= \frac{\lambda^3 xe^{-\lambda y}}{\lambda^2 xe^{-\lambda x}}$$
$$= \boxed{\lambda e^{-\lambda(y-x)}}$$

Part e

$$E[Y|X=1] = \int_1^\infty \lambda y e^{-\lambda(y-1)}dy$$
$$= \lambda e^\lambda \int_1^\infty ye^{-\lambda y}dy$$

Let:
$u = y$, $du = dy$
$v = -\frac{1}{\lambda}e^{-\lambda y}$, $dv = e^{-\lambda y}dy$

Then

$$= \lambda e^\lambda \left(-\frac{y}{\lambda}e^{-\lambda y}\Big|_1^\infty + \frac{1}{\lambda}\int_1^\infty e^{-\lambda y}dy \right)$$
$$= \lambda e^\lambda \left(\frac{e^{-\lambda}}{\lambda} - \frac{1}{\lambda^2}e^{-\lambda y}\Big|_1^\infty \right)$$
$$= 1 - \frac{e^\lambda}{\lambda}(0 - e^{-\lambda})$$
$$= \boxed{1 + \frac{1}{\lambda}}$$

Problem 6

Part a

First note that since $P(A \cup B) = P(A) + P(B) - P(AB)$, $P(AB) = P(A) + P(B) - P(A \cup B) = .2 + .6 - .7 = .1$

Then $P(A|B) = \frac{P(AB)}{P(B)} = \frac{.1}{.6} = \boxed{\frac{1}{6}}$

Part b

$P(A)P(B) = .12 > .1 = P(AB)$, so they are $\boxed{\text{negatively dependent}}$

Part c

$cov(X, Y) = P(AB) - P(A)P(B) = .1 - .12 = -.02$

Then note that $var(X) = E[X^2] - E[X]^2 = E[X] - E[X]^2 = P(A) - P(A)^2 = .2 - .04 = .16$

Similarly, $var(Y) = P(B) - P(B)^2 = .6 - .36 = .24$

Then $corr(X, Y) = \frac{-.02}{\sqrt{.16 \times .24}} \approx \boxed{-.102}$