S722 HW6

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To save on typing, I will denote $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$.

Part 1

1.33

$P(M|CB) = \frac{P(CB|M)P(M)}{P(CB|M)P(M)+P(CB|F)P(F)}$
$= \frac{(.05)(.5)}{(.05)(.5)+(.0025)(.5)}$
≈ 0.952

2.11

a

From example 2.1.7, $f_Y(y) = \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) = (2\pi y)^{-1/2} e^{-y/2}$.

Then $E[Y] = \int_0^\infty \sqrt{\frac{y}{2\pi}} e^{-y/2} dy$.
Let $u = \sqrt{y}$ and $dv = e^{-y/2} dy$. Then $du = \frac{1}{2} y^{-1/2}$ and $v = -2e^{-y/2}$.
Then the integral becomes $(2\pi)^{-1/2} \int_0^\infty y^{-1/2} e^{-y/2} dy$.
Let $u = \sqrt{y} \implies du = \frac{1}{2} y^{-1/2} dy$. Then the integral becomes
$(2\pi)^{-1/2} \int_0^\infty 2e^{-u^2/2} du = (2\pi)^{-1/2}(2)(\pi/2)^{1/2} = 1$.

b

$Y = |X| \implies X = \pm Y \implies |X'| = 1$
$\implies f_Y(y) = f_X(y) + f_X(-y) = \sqrt{2/\pi} e^{-y^2/2}$.

$E[Y] = \int_0^\infty y(2\pi)^{1/2} e^{-y^2/2} dy$
$u = y^2/2 \implies du = ydy$ so the above becomes
$= (2/\pi)^{1/2} \int_0^\infty e^{-u} du = (2/\pi)^{1/2}$.

$E[Y^2] = (2/\pi)^{1/2} \int_0^\infty y^2 e^{-y^2/2} dy$
$u = y$ and $dv = ye^{-y^2/2} \implies du = dy$ and $v = -e^{-y^2/2}$ so the above becomes
$= \sqrt{2/\pi} \int_0^\infty e^{-y^2/2} dy = 1$.

So $Var(Y) = E[Y^2] - (E[Y])^2 = 1 - \frac{2}{\pi}$.

2.33

a

$M_X(t) = E[e^{tX}] = \sum_x e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

$E[X] = M_X'(t=0) = e^{\lambda(e^t-1)} \lambda e^t|_{t=0} = e^{\lambda(1-1)} \lambda e^0 = \lambda$

$E[X^2] = M''_X(t=0) = \lambda e^t e^{\lambda(e^t-1)}\lambda e^t + \lambda e^t e^{\lambda(e^t-1)}|_{t=0} = \lambda^2 + \lambda$

$Var(X) = E[X^2] - (E[X])^2 = \lambda$

$\phi_X(t) = M_X(it) = e^{\lambda(e^{it}-1)}$

$E[X] = -i\phi'_X(0) = -ie^{\lambda(e^{it}-1)}\lambda i e^{it}|_{t=0} = (-1)i^2\lambda = \lambda$

$E[X^2] = -\phi''_X(0) = -i\lambda(e^{\lambda(e^{it}-1)}ie^{it}\lambda + e^{\lambda(e^{it}-1)}ie^{it})|_{t=0}$
$= -i\lambda(i\lambda + i) = \lambda^2 + \lambda$

$Var(X) = E[X^2] - (E[X])^2 = \lambda$

c

$M_X(t) = E[e^{tX}] = \int e^{tx}(2\pi\sigma^2)^{-1/2}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$
$= (2\pi\sigma^2)^{-1/2}\int e^{-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2-2\sigma^2 tx)}dx$
$= (2\pi\sigma^2)^{-1/2}e^{-\frac{\mu^2}{2\sigma^2}}\int e^{-\frac{1}{2\sigma^2}(x^2-2(\mu+\sigma^2 t)x+(\mu+\sigma^2 t)^2-(\mu+\sigma^2 t)^2)}dx$
$= (2\pi\sigma^2)^{-1/2}e^{-\frac{\mu^2}{2\sigma^2}}e^{\frac{(\mu+\sigma^2 t)^2}{2\sigma^2}}\int e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2}dx$
$= (2\pi\sigma^2)^{-1/2}(2\pi\sigma^2)^{1/2}e^{-\frac{\mu^2}{2\sigma^2}+\frac{\mu^2}{2\sigma^2}+\frac{\sigma^4 t^2}{2\sigma^2}+\frac{2\mu\sigma^2 t}{2\sigma^2}}$
$= e^{\frac{\sigma^2 t^2}{2}+\mu t}$

$E[X] = M'_X(0) = e^{\sigma^2 t^2/2+\mu t}(\sigma^2 t+\mu)|_{t=0} = \mu$

$E[X^2] = M''_X(0) = e^{\sigma^2 t^2/2+\mu t}(\sigma^2 t+\mu)^2 + e^{\sigma^2 t^2/2+\mu t}\sigma^2|_{t=0} = \mu^2+\sigma^2$

$Var(X) = E[X^2] - (E[X])^2 = \sigma^2$

$\phi_X(t) = M_X(it) = e^{-\sigma^2 t^2/2+i\mu t}$

$E[X] = -i\phi'_X(0) = -ie^{-\sigma^2 t^2/2+i\mu t}(-\sigma^2 t+i\mu)|_{t=0} = -i(i\mu) = \mu$

$E[X^2] = -\phi''_X(0) = -e^{-\sigma^2 t^2/2+i\mu t}((-\sigma^2 t+i\mu)^2-\sigma^2)|_{t=0} = \mu^2+\sigma^2$

$Var(X) = \mu^2+\sigma^2-\mu^2 = \sigma^2$

2.38

a

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\binom{r+x-1}{x}p^r(1-p)^x$
$= p^r\sum_x\binom{r+x-1}{r}\left(e^t(1-p)\right)^r$
$= \frac{p^r}{\left(1-e^t(1-p)\right)^r}$

b

$M_Y(t) = M_X(2pt) = \left(\frac{p}{1-e^{2pt}(1-p)}\right)^r$

$\lim\limits_{p\to 0}\frac{p}{1-e^{2pt}(1-p)} = \lim\limits_{p\to 0}\frac{1}{e^{2pt}-2te^{2pt}(1-p)} = \frac{1}{1-2t}$
$\implies \lim\limits_{p\to 0}\left(\frac{p}{1-e^{2pt}(1-p)}\right)^r = \left(\frac{1}{1-2t}\right)^r$

3.24

a

$Y = X^{1/\gamma} \implies X = Y^\gamma \implies X' = \gamma Y^{\gamma-1}$
$\implies f_Y(y) = f_X(y^\gamma)\gamma y^{\gamma-1} = \frac{\gamma}{\beta}e^{-y^\gamma/\beta}y^{\gamma-1}$

b

$Y = (2X/\beta)^{1/2} \implies X = \beta Y^2/2 \implies X' = \beta Y$
$\implies f_Y(y) = ye^{-y^2/2}$

c

$Y = 1/X \implies X = 1/Y \implies |X'| = Y^{-2}$
$\implies f_Y(y) = \frac{1}{\Gamma(a)b^a}y^{-(a+1)}e^{-\frac{1}{by}}$

d

$Y = (X/\beta)^{1/2} \implies X = \beta Y^2 \implies X' = \beta Y/2$
$\implies f_Y(y) = \frac{2}{\Gamma(3/2)}y^2e^{y^2}$

e

$Y = \alpha - \gamma \log X \implies X = \exp(\frac{\alpha-Y}{\gamma}) \implies |X'| = \frac{1}{\gamma}\exp(\frac{\alpha-Y}{\gamma})$
$\implies f_Y(y) = \frac{1}{\gamma}\exp(-e^{\frac{\alpha-y}{\gamma}} + \frac{\alpha-y}{\gamma})$

3.48

$f(x+1) = \binom{n}{x+1}p^{x+1}(1-p)^{n-x-1}$
$= \binom{n}{x}\frac{n-x}{x+1}\frac{p}{1-p}p^x(1-p)^{n-x}$
$= \frac{n-x}{x+1}\frac{p}{1-p}f(x)$

3.49

a

$E[g(X)(X-\alpha\beta)] = \int_0^\infty g(x)(x-\alpha\beta)\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}dx$
$u = g(x)$ and $dv = (x-\alpha\beta)x^{\alpha-1}e^{-x/\beta}e^{-x/\beta}dx \implies du = g'(x)dx$ and $v = -\beta x^\alpha e^{-x/\beta}$ so the above becomes
$\frac{1}{\Gamma(\alpha)\beta^\alpha}\beta\int g'(x)x^\alpha e^{-x/\beta}dx$
$= \beta\int xg'(x)f(x)dx = \beta E[Xg'(X)]$

4.4

a

$1 = \int_0^1 \int_0^2 C(x+2y)dxdy = 4C \implies C = 1/4$

b

$f_X(x) = \int_0^1 \frac{1}{4}(x+2y)dy = \frac{x+1}{4}$

c

$F(x,y) = \int_0^x \int_0^y \frac{s+2t}{4} dsdt = \frac{x^2y+2y^2x}{8}$

d

$Z = 9/(X+1)^2 \implies X = 3Z^{-1/2} - 1 \implies |X'| = \frac{3}{2}Z^{-3/2}$
$\implies f_Z(z) = \frac{1}{4}(3z^{-1/2} - 1 + 1)\frac{3}{2}z^{-3/2} = \frac{9}{8}z^{-2}$

4.40

a

In the case where a, b, and c are natural numbers ...

$\int\int x^{a-1}y^{b-1}(1-x-y)^{c-1}dxdy = \Gamma(a)\int y^{b-1}(1-y)^{c+a-1}dy = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$
$\implies C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$

b

By symmetry, we only have to show this for one, say, X.

Reusing some of the math from part (a), we get $f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)dy = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\Gamma(b)x^{a-1}(1-x)^{c+b-1}$
$\implies X \sim Beta(a, b+c)$

c

$f_{Y|X}(y) \propto f_{X,Y}(x,y) \propto y^{b-1}(1-x-y)^{c-1}$
$= y^{b-1}\sum_{i=0}^{c-1}(-y)^i(1-x)^{c-1-i}$
$\propto y^{b-1}\sum_i(-y)^i = y^{b-1}(1-y)^{c-1}$
$\implies Y \mid X \sim Beta(b,c)$

d

$E[X,Y] = \int\int \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^a y^b (1-x-y)^{c-1}dx$
$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c)}{\Gamma(a+b+c+2)}$
$= \frac{ab}{(a+b+c+1)(a+b+c)}$

Part 2

1

$E[e^{itX}] \leq E[(e^{itX})^2] = E[\sin^2 tX + \cos^2 tX] = E[1] = 1$

2

$\phi_{aX+b}(t) = E[e^{it(aX+b)}] = E[e^{itb}e^{itaX}] = e^{itb}E[e^{i(at)X}] = e^{itb}\phi_X(at)$

3

$\phi_{\sum_j X_j}(t) = E[e^{it\sum_j X_j}] = E[\prod_j e^{itX_j}] = \prod_j E[e^{itX_j}] = \prod_j \phi_{X_j}(t)$

4

Let $\delta > 0$.

$|\phi(t+\delta) - \phi(t)|^2 = |E[e^{i(t+\delta)X} - e^{itX}]|^2 = |E[e^{itX}(e^{i\delta X} - 1)]|^2 \leq E[|e^{itX}(e^{i\delta X} - 1)|^2] = E[(e^{i\delta X} - 1)^2]$
$\leq E[\delta^2 X^2] = \delta^2 E[X^2]$

So letting $\epsilon = \delta\sqrt{E[X^2]}$, we get $|\phi(t+\delta) - \phi(t)| \leq \epsilon$, assuming that the second moment exists.