S721 HW1

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Problem 1.1

Part a

An ordered set of 4 elements, each of which can be "H" or "T". $|\Omega| = 2^4$.

Part b

A count is a natural number (including 0). So $\Omega \subset \mathbb{N}_0$.

Problem 1.2

Part a

Suppose $x \in A \setminus B$. Then $x \in A$ and $x \notin B \iff x \in B^c$. Then $x \in A$ and $x \in B^c$, i.e., $x \in A \cap B^c$.

Suppose $x \in A \setminus B$. Then $x \in A$ and $x \notin B \iff x \notin A \cap B$. Then $x \in A \setminus (A \cap B)$.

Part b

Suppose $x \in B$ and we don't know anything about its relation to A. Then x can either $\in A$ or $\notin A$. Then $x \in B \cap A$ or $x \in B \cap A^c$, i.e., $x \in (B \cap A) \cup (B \cap A^c)$.

Problem 1.3

Part a

$x \in A \cup B \iff x \in A$ or $x \in B \iff x \in B \cup A$

$x \in A \cap B \iff x \in A$ and $x \in B \iff x \in B \cap A$

Part b

$x \in A \cap (B \cap C) \iff x \in A$ and $x \in B \cap C \iff x \in A$ and $x \in B$ and $x \in C \iff x \in A \cap B$ and $x \in C \iff x \in (A \cap B) \cap C$

$x \in A \cup (B \cup C) \iff x \in A$ or $x \in B \cup C \iff x \in A$ or $x \in B$ or $x \in C \iff x \in A \cup B$ or $x \in C \iff x \in (A \cup B) \cup C$

Part c

$x \in (A \cup B)^c \iff x \notin A \cup B \iff x \notin A$ and $x \notin B \iff x \in A^c$ and $x \in B^c \iff x \in A^c \cap B^c$

$x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A$ or $x \notin B \iff x \in A^c$ or $x \in B^c \iff x \in A^c \cup B^c$

Problem 1.9

Part a

$x \in (\cup_\alpha A_\alpha)^c \iff x \notin \cup_\alpha A_\alpha \iff x \notin A_\alpha \ \forall\alpha \iff x \in A_\alpha^c \ \forall\alpha \iff x \in \cap_\alpha A_\alpha^c$

Part b

$x \in (\cap_\alpha A_\alpha)^c \iff x \notin \cap_\alpha A_\alpha \iff \exists\alpha$ such that $x \in A_\alpha^c \iff x \in \cup_\alpha A_\alpha^c$

Problem 1.11

Part a

i. $\emptyset \in \{\emptyset, S\} = \mathcal{B}$
ii. $\emptyset^c = S^c$ and vice versa. Both $\emptyset$ and $S \in \mathcal{B}$, so it is closed under complements.
iii. $\emptyset \cup S = S \in \mathcal{B}$

Part b

i. $\emptyset \subset S$, so $\emptyset \in \mathcal{B}$.
ii. If $A \subset S$, then $A^c = S \setminus A \subset S$. So $A^c \in \mathcal{B}$.
iii. If $A_i \in \mathcal{B}$, then $A_i \subset S \implies \cup_i A_i \subset S \in \mathcal{B}$.

Part c

Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are σ-algebras.

i. They both contain $\emptyset$ since they are σ-algebras, so the intersection contains $\emptyset$.
ii. Suppose $A \in \mathcal{B}_1 \cap \mathcal{B}_2$. Then it must be in both $\mathcal{B}_1$ and $\mathcal{B}_2$. Then A^c must also be in both $\mathcal{B}_1$ and $\mathcal{B}_2$. So $A^c \in \mathcal{B}_1 \cap \mathcal{B}_2$.
iii. Suppose $A_1, A_2, ... \in \mathcal{B}_1 \cap \mathcal{B}_2$. Then each A_i must be in both $\mathcal{B}_1$ and $\mathcal{B}_2$. Since they are both σ-algebras, $\cup_i A_i \in$ both $\mathcal{B}_1$ and $\mathcal{B}_2 \implies \cup_i A_i \in \mathcal{B}_1 \cap \mathcal{B}_2$.

Not from textbook

Problem 1

$[a, b]$

For $a < b$, let $A = \cap_{n=1}^{\infty} A_n$ where $A_n = (a - 1/n, b + 1/n)$.
Let $B = [a, b]$.

We can see that $B \subset A$ since $\forall n > 0$, $a - 1/n < a < b < b + 1/n$.

Suppose $x \in A$ but $x \notin B$. Then $x < a$ or $x > b$.

If $x \in A$, then $x \in A_n \ \forall n \in \mathbb{N}$.

If $x < a$, then $\exists N \in \mathbb{N}$ such that $a - x > 1/n \ \forall n > N$. This means that $x \notin A_n \ \forall n > N$. Therefore, $x \notin A$, which is a contradiction.

Similarly, if $x > b$, then $\exists N \in \mathbb{N}$ such that $x - b > 1/n \ \forall n > N$. This means that $x \notin A_n \ \forall n > N$. Therefore $x \notin A$, which is a contradiction.

Therefore, $[a, b] = \cap_{n=1}^{\infty}(a - 1/n, a + 1/n)$.

$[a, b)$

Let $A = \cap_{n=1}^{\infty}(a - 1/n, b) = \cap_{n=1}^{\infty} A_n$. Let $B = [a, b)$.

Since $a - 1/n < a < b$, $B \subset A$.

Suppose $\exists x \in A$ but $x \notin B$. Then $x < a \implies a - x > 1/n \ \forall n > N$ for some $N \in \mathbb{N}$. Like before, we can see that x is not in some A_n, so it cannot be in A, which is a contradiction. Therefore, $x \in B \implies A \subset B$.

Therefore, $[a, b) = \cap_{n=1}^{\infty}(a - 1/n, b)$.

$(a, b]$

By symmetry, we can use $A = \cup_{n=n_0}^{\infty}(a, b - 1/n)$ and proceed with the same reasoning as we did for $[a, b)$.

Problem 2

It is sufficient to show that any open interval (a, b) is generated by intervals of the form $(-\infty, a]$.

Using two such intervals, we can construct $(a + 1/n, b - 1/n] = (-\infty, b - 1/n] \cap (-\infty, a + 1/n]^c$, a half-open interval (and $n \geq \frac{2}{b-a}$).

Let $A = \cup_{n=n_0}^{\infty} A_n$ where $A_n = (a + 1/n, b - 1/n]$ and $n_0 \geq \frac{2}{b-a}$, and let $B = (a, b)$.

We can see that since $a < a + 1/n \leq b - 1/n < b$, so $A_n \subset B \ \forall n \geq n_0$, so $A \subset B$.

Conversely, suppose $x \in B$ but $x \notin A$. Then $\exists N > n_0$ such that $x \notin A_n \ \forall n > N$. But since $x \in B$, $a < x < b$, we can consider two situations, $x < a + 1/n$ and $x > b - 1/n$.

For the first case, $a < x \leq a + 1/n$, so $\exists \epsilon > 0$ such that $(a + 1/n) - a > \epsilon \implies 1/n > \epsilon \implies n < 1/epsilon$. But n is unbounded, so this cannot be true.

For the second case, $b - 1/n < x < b \implies \exists \epsilon$ such that $b - (b - 1/n) > \epsilon$. Like before, we can see that this implies $n < 1/\epsilon$, but ϵ is fixed while n is unbounded, so this cannot be true. Then $A \subset B$.