# S626

HW3

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```r
library(ggplot2)
import::from(magrittr, `%>%`)

theme_set(theme_bw())
set.seed(626)
doMC::registerDoMC(8)
```

## 4.2

### a

From problem 3.3, we know:

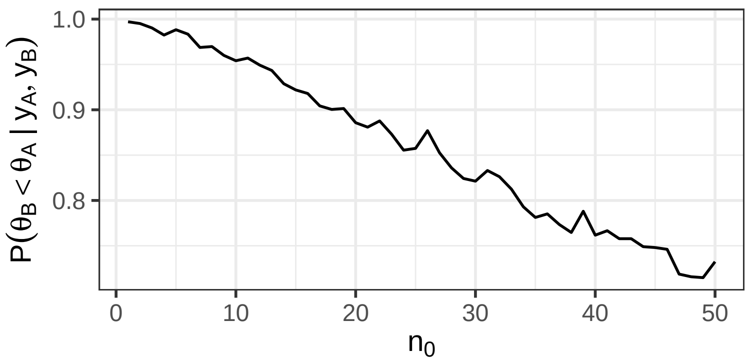
- $\theta_A \mid y_A \sim Gamma(120 + \sum y_{A,i}, 10 + n_A)$
- $\theta_B \mid y_B \sim Gamma(12 + \sum y_{B,i}, 1 + n_B)$

```r
# data
y.a <- c(12, 9, 12, 14, 13, 13, 15, 8, 15, 6)
y.b <- c(11, 11, 10, 9, 9, 8, 7, 10, 6, 8, 8, 9, 7)

# statistics
n.a <- length(y.a)
n.b <- length(y.b)
sum.y.a <- sum(y.a)
sum.y.b <- sum(y.b)

# prior parameters
a.a <- 120
a.b <- 10
b.a <- 12
b.b <- 1

# sample size
n <- 2 ** 10

# generate sample and compute probability
theta.a <- rgamma(n, a.a + sum(y.a), a.b + n.a)
theta.b <- rgamma(n, b.a + sum(y.b), b.b + n.b)
mean(theta.b < theta.a)
```

```
[1] 0.9941406
```

### b

```r
# values of n_0 to try
n.0.vector <- seq(50)

sensitivity.df <- plyr::ldply(n.0.vector, function(n.0) {
  theta.a <- rgamma(n, a.a + sum(y.a), a.b + n.a)
  theta.b <- rgamma(n, 12 * n.0 + sum(y.b), n.0 + n.b)
  p <- mean(theta.b < theta.a)
  dplyr::data_frame(n.0 = n.0, p = p)
}, .parallel = TRUE)

ggplot(sensitivity.df) +
  geom_line(aes(x = n.0, y = p)) +
  labs(x = expression(n[0]),
       y = expression(P(theta[B] < theta[A]~'|'~y[A], y[B])))
```



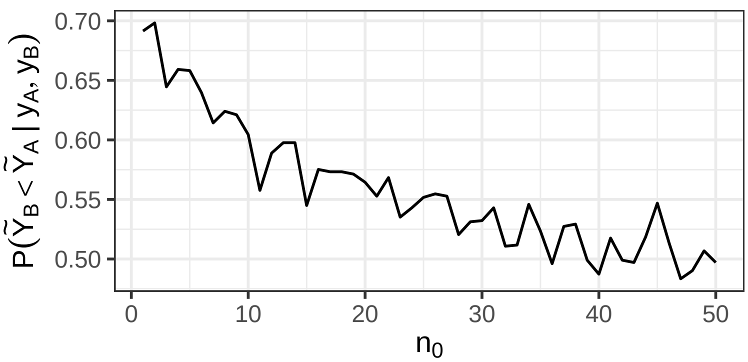

As $n_0$ increases, we are more sure that $\theta_B = E[Y_B] = 12$, and since $\bar{Y}_A < 12$, even though $\bar{Y}_B < \bar{Y}_A$, our posterior estimate for $\theta_B$ approaches 12, so the probability decreases.

### c

From class, we know:

- $\tilde{Y}_A \mid y_A \sim NB(120 + \sum y_{A,i}, 10 + n_A)$
- $\tilde{Y}_B \mid y_B \sim NB(12n_0 + \sum y_{B,i}, n_0 + n_B)$

```r
pred.posterior.df <- plyr::ldply(n.0.vector, function(n.0) {
  y.a.tilde <- rnbinom(n, a.a + sum(y.a), mu = (a.a + sum(y.a)) / (a.b + n.a))
  y.b.tilde <- rnbinom(n, 12 * n.0 + sum(y.b),
                       mu = (12 * n.0 + sum(y.b)) / (n.0 + n.b))
  dplyr::data_frame(n.0 = n.0, p = mean(y.b.tilde < y.a.tilde))
}, .parallel = TRUE)

ggplot(pred.posterior.df) +
  geom_line(aes(x = n.0,y = p)) +
  labs(x = expression(n[0]),
       y = expression(P(tilde(Y)[B] < tilde(Y)[A]~'|'~y[A], y[B])))
```




Similar to above, the probability decreases as we become more and more sure that $\theta_B > \theta_A$.

## 4.5

### a

Given

- $Y_i \mid \theta, x_i \sim Poisson(\theta x_i)$
- $\theta \sim Gamma(a, b)$

Then

- $p(\theta|x, y) = p(\theta) \prod p(y_i|\theta, x_i)$
  $\propto \theta^{a-1} e^{-b\theta} \theta^{\sum y_i} e^{-\theta \sum x_i}$
  $= \theta^{\sum y_i + a - 1} e^{-(b + \sum x_i)\theta}$
  $\implies \theta \mid x, y \sim Gamma(\sum y_i + a, \sum x_i + b)$

### b

```
react.df <- readr::read_table(
  'http://www2.stat.duke.edu/~pdh10/FCBS/Exercises/cancer_react.dat'
)
noreact.df <- readr::read_table(
  'http://www2.stat.duke.edu/~pdh10/FCBS/Exercises/cancer_noreact.dat'
)

apply(react.df, 2, sum)
```

```
  x   y
 95 256
```

```
apply(noreact.df, 2, sum)
```

```
   x    y
1037 2285
```

So $\theta_1 \mid x, y \sim Gamma(256 + a_1, 85 + b_1)$
and $\theta_2 \mid x, y \sim Gamma(2285 + a_2, 1037 + b_2)$

### c

For the calculations of $P(\theta_2 > \theta_1 | x, y)$, I will use Monte-Carlo estimation instead of computing the integrals. The plots will be done last for better comparison.

Before we compute any posterior probabilities or estimates, it's worth computing the MLEs as a comparison:

- $\hat{\theta}_1 = 2285/1037 \approx 2.203$
- $\hat{\theta}_2 = 256/95 \approx 2.695$

#### i

```
# statistics
sum.x2 <- sum(react.df$x)
sum.y2 <- sum(react.df$y)
sum.x1 <- sum(noreact.df$x)
sum.y1 <- sum(noreact.df$y)

# parameters
a1 <- a2 <- 2.2 * 100
b1 <- b2 <- 100

# expected values
(sum.y1 + a1) / (sum.x1 + b1)
```

```
[1] 2.203166
```

```
(sum.y2 + a2) / (sum.x2 + b2)
```

```
[1] 2.441026
```

```
# intervals
qgamma(c(.025, .975), sum.y1 + a1, sum.x1 + b1)
```

```
[1] 2.117726 2.290273
```

```
qgamma(c(.025, .975), sum.y2 + a2, sum.x2 + b2)
```

```
[1] 2.226633 2.665131
```

```
# p(theta_1 > theta_2 | x, y)
theta.1 <- rgamma(n, sum.y1 + a1, sum.x1 + b1)
theta.2 <- rgamma(n, sum.y2 + a2, sum.x2 + b2)
mean(theta.2 > theta.1)
```

```
[1] 0.9765625
```

#### ii

```
# parameters
a1 <- 2.2 * 100
b1 <- 100
a2 <- 2.2
b2 <- 1
```

```r
# expected values
(sum.y1 + a1) / (sum.x1 + b1)
```

```
[1] 2.203166
```

```r
(sum.y2 + a2) / (sum.x2 + b2)
```

```
[1] 2.689583
```

```r
# intervals
qgamma(c(.025, .975), sum.y1 + a1, sum.x1 + b1)
```

```
[1] 2.117726 2.290273
```

```r
qgamma(c(.025, .975), sum.y2 + a2, sum.x2 + b2)
```

```
[1] 2.371497 3.027397
```

```r
# p(theta_1 > theta_2 | x, y)
theta.1 <- rgamma(n, sum.y1 + a1, sum.x1 + b1)
theta.2 <- rgamma(n, sum.y2 + a2, sum.x2 + b2)
mean(theta.2 > theta.1)
```

```
[1] 1
```

**iii**

```r
# parameters
a1 <- a2 <- 2.2
b1 <- b2 <- 1

# expected values
(sum.y1 + a1) / (sum.x1 + b1)
```

```
[1] 2.203468
```

```r
(sum.y2 + a2) / (sum.x2 + b2)
```

```
[1] 2.689583
```

```r
# intervals
qgamma(c(.025, .975), sum.y1 + a1, sum.x1 + b1)
```

```
[1] 2.114081 2.294680
```

```r
qgamma(c(.025, .975), sum.y2 + a2, sum.x2 + b2)
```

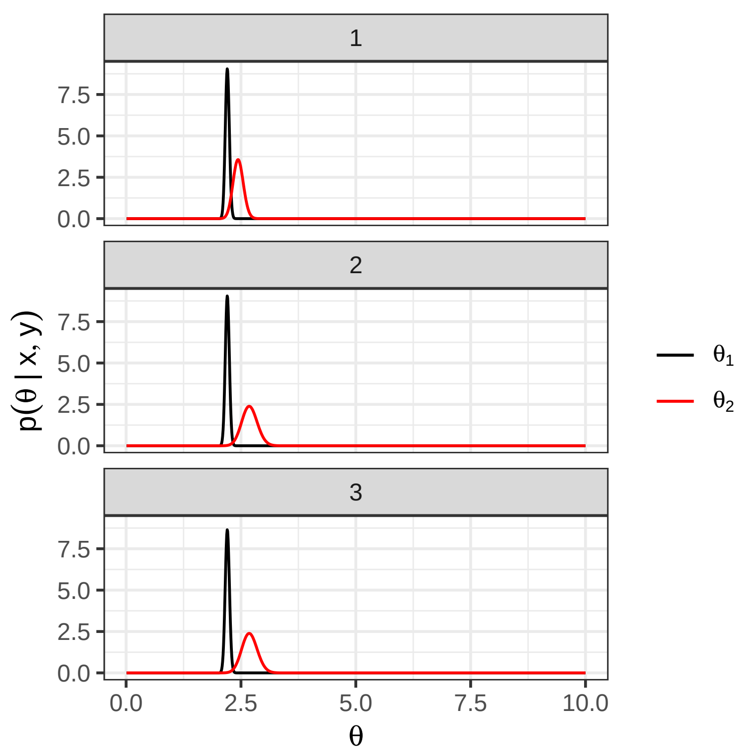
```
[1] 2.371497 3.027397
```

```r
# p(theta_1 > theta_2 | x, y)
theta.1 <- rgamma(n, sum.y1 + a1, sum.x1 + b1)
theta.2 <- rgamma(n, sum.y2 + a2, sum.x2 + b2)
mean(theta.2 > theta.1)
```

```
[1] 0.9990234
```

**posterior plots and comparison**

```r
# support of theta
theta <- seq(.01, 10, .01)

# parameters
a1 <- c(2.2 * 100, 2.2 * 100, 2.2)
a2 <- c(2.2 * 100, 2.2, 2.2)
b1 <- c(100, 100, 1)
b2 <- c(100, 1, 1)

out.df <- plyr::ldply(seq(3), function(i) {
  p.1 <- dgamma(theta, a1[i] + sum.y1, b1[i] + sum.x1)
  p.2 <- dgamma(theta, a2[i] + sum.y2, b2[i] + sum.x2)
  dplyr::data_frame(theta = theta, p.1 = p.1, p.2 = p.2, scenario = i)
}, .parallel = TRUE)

ggplot(out.df) +
  facet_wrap(~ scenario, ncol = 1) +
  geom_line(aes(x = theta, y = p.1, colour = '1')) +
  geom_line(aes(x = theta, y = p.2, colour = '2')) +
  labs(x = expression(theta), y = expression(p(theta~'|'~x, y))) +
  scale_colour_manual(labels = c(expression(theta[1]),
                                 expression(theta[2])),
                      values = seq(2),
                      name = NULL)
```




The prior for $\theta_1$ is very close to its MLE, and we have a lot of data for $\theta_1$, so there isn't much change in our estimate as we shift the prior certainty. On the other hand, the prior for $\theta_2$ is lower than its MLE, and we don't have as much data for $\theta_2$, so as we make the prior weaker, there is a shift toward the MLE.

## 5.1

```
# data from https://www2.stat.duke.edu/~pdh10/FCBS/Exercises/
# for some reason i couldn't extract the data directly
school1 <- readr::read_lines('~/dev/stats-hw/stat-s626/school1.dat') %>%
  as.numeric()
school2 <- readr::read_lines('~/dev/stats-hw/stat-s626/school2.dat') %>%
  as.numeric()
school3 <- readr::read_lines('~/dev/stats-hw/stat-s626/school3.dat') %>%
  as.numeric()

schools <- list(school1, school2, school3)
```

### a

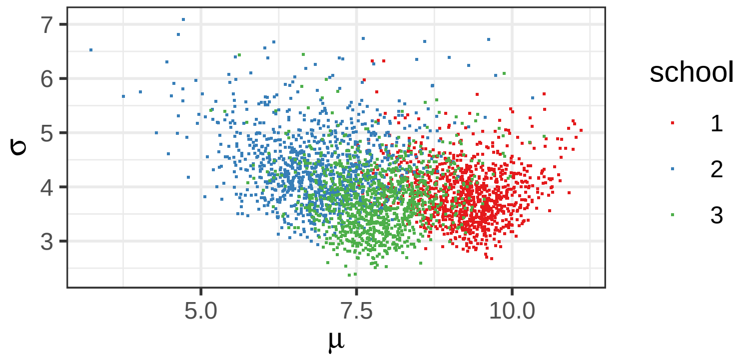
From class, we had:

- $\mu \mid \phi, y \sim \mathcal{N}(\mu_n, \frac{1}{\kappa_n \phi})$
- $\phi \mid y \sim Gamma(\nu_n/2, SS_n/2)$

We will use Monte Carlo estimation instead of evaluating integrals.

```
# MC params
sample.size <- 2 ** 10

# parameters
mu.0 <- 5
sigma2.0 <- 4
kappa.0 <- 1
nu.0 <- 2

# transformed parameters
ss.0 <- sigma2.0 * nu.0

# summary statistics
mc.df <- plyr::ldply(seq_along(schools), function(i) {
  # posterior params
  n <- length(schools[[i]])
  kappa.n <- kappa.0 + n
  mu.n <- (kappa.0 * mu.0 + sum(schools[[i]])) / kappa.n
  nu.n <- nu.0 + n
  y.bar <- mean(schools[[i]])
  ss.n <-
    ss.0 + sum((schools[[i]] - y.bar) ** 2) +
    n * kappa.0 / kappa.n * (y.bar - mu.0) ** 2

  # sample phi
  phi <- rgamma(sample.size, nu.n / 2, ss.n / 2)
  # sample mu
  mu <- rnorm(sample.size, mu.n, 1 / kappa.n / phi)
```

```r
  dplyr::data_frame(school = i, mu = mu, phi = phi, i = seq(sample.size))
}, .parallel = TRUE)

ggplot(mc.df) +
  geom_point(aes(x = mu, y = phi ** -.5, colour = factor(school)),
             alpha = 1, shape = '.') +
  labs(x = expression(mu), y = expression(sigma), colour = 'school') +
  scale_colour_brewer(palette = 'Set1')
```




```r
mc.df %>%
  dplyr::group_by(school) %>%
  dplyr::summarise(`$E(\\theta | y)$` = mean(mu),
                   `$E(\\sigma | y)$` = mean(phi ** -.5),
                   `$lower(\\theta | y)$` = quantile(mu, .025),
                   `$upper(\\theta | y)$` = quantile(mu, .975),
                   `$lower(\\sigma | y)$` = quantile(phi ** -.5, .025),
                   `$upper(\\sigma | y)$` = quantile(phi ** -.5, .975)) %>%
  dplyr::ungroup() %>%
  knitr::kable()
```

| school | $E(\theta \mid y)$ | $E(\sigma \mid y)$ | $lower(\theta \mid y)$ | $upper(\theta \mid y)$ | $lower(\sigma \mid y)$ | $upper(\sigma \mid y)$ |
|---:|---:|---:|---:|---:|---:|---:|
| 1 | 9.267307 | 3.883951 | 7.937426 | 10.523413 | 3.017915 | 5.087884 |
| 2 | 6.875541 | 4.395978 | 5.223745 | 8.564604 | 3.325591 | 5.973142 |
| 3 | 7.839043 | 3.743190 | 6.475512 | 9.245500 | 2.792640 | 5.070188 |

## b

```r
# generate all six permutations
permutations <- gtools::permutations(3, 3, seq(3)) %>%
  {lapply(seq_len(nrow(.)), function(i) .[i, ])}

# compute probability for each permutation
plyr::ldply(permutations, function(permutation) {
  wide.df <- mc.df %>%
    dplyr::select(i, school, mu) %>%
    tidyr::spread(school, mu) %>%
    dplyr::select(-i)
```

```r
  p <- mean((wide.df[permutation[1]] < wide.df[permutation[2]]) &
              (wide.df[permutation[2]] < wide.df[permutation[3]]))

  dplyr::data_frame(permutation = paste(permutation, collapse = ', '),
                    `$P(\\theta_i < \\theta_j < \\theta_k | y)$` = p)
}, .parallel = TRUE) %>%
  knitr::kable()
```

| permutation | $P(\theta_i < \theta_j < \theta_k \mid y)$ |
|---|---|
| 1, 2, 3 | 0.0009766 |
| 1, 3, 2 | 0.0039062 |
| 2, 1, 3 | 0.0654297 |
| 2, 3, 1 | 0.7441406 |
| 3, 1, 2 | 0.0087891 |
| 3, 2, 1 | 0.1767578 |

### c

```r
plyr::ldply(permutations, function(permutation) {
  wide.df <- mc.df %>%
    dplyr::group_by(school, i) %>%
    dplyr::mutate(y.tilde = rnorm(1, mu, phi ** -.5)) %>%
    dplyr::ungroup() %>%
    dplyr::select(i, school, y.tilde) %>%
    tidyr::spread(school, y.tilde) %>%
    dplyr::select(-i)

  p <- mean((wide.df[permutation[1]] < wide.df[permutation[2]]) &
              (wide.df[permutation[2]] < wide.df[permutation[3]]))

  dplyr::data_frame(permutation = paste(permutation, collapse = ', '),
                    `$P(\\tilde{Y}_i < \\tilde{Y}_j < \\tilde{Y}_k | y)$` = p)
}, .parallel = TRUE) %>%
  knitr::kable()
```

| permutation | $P(\tilde{Y}_i < \tilde{Y}_j < \tilde{Y}_k \mid y)$ |
|---|---|
| 1, 2, 3 | 0.1123047 |
| 1, 3, 2 | 0.1025391 |
| 2, 1, 3 | 0.1972656 |
| 2, 3, 1 | 0.2734375 |
| 3, 1, 2 | 0.1435547 |
| 3, 2, 1 | 0.2060547 |

### d

```r
posterior.df <- mc.df %>%
    dplyr::select(i, school, mu) %>%
    tidyr::spread(school, mu) %>%
```

```r
    dplyr::select(-i)

pred.posterior.df <- mc.df %>%
    dplyr::group_by(school, i) %>%
    dplyr::mutate(y.tilde = rnorm(1, mu, phi ** -.5)) %>%
    dplyr::ungroup() %>%
    dplyr::select(i, school, y.tilde) %>%
    tidyr::spread(school, y.tilde) %>%
    dplyr::select(-i)

p.posterior <- mean((posterior.df[1] > posterior.df[2]) &
                      (posterior.df[1] > posterior.df[3]))
p.pred.posterior <- mean((pred.posterior.df[1] > pred.posterior.df[2]) &
                           (pred.posterior.df[1] > pred.posterior.df[3]))

dplyr::data_frame(
  `$P(\\theta_1 > \\theta_2, \\theta_1 > \\theta_3 | y)$` = p.posterior,
  `$P(\\tilde{Y}_1 > \\tilde{Y}_2, \\tilde{Y}_1 > \\tilde{Y}_3 | y)$` =
    p.pred.posterior
) %>%
  knitr::kable()
```

| $P(\theta_1 > \theta_2, \theta_1 > \theta_3 \vert y)$ | $P(\tilde{Y}_1 > \tilde{Y}_2, \tilde{Y}_1 > \tilde{Y}_3 \vert y)$ |
|---:|---:|
| 0.9208984 | 0.4599609 |