S722 HW7

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To save on typing, I will denote $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$.

4.20

a

$Y_1 = X_1^2 + X_2^2$
$Y_2 = \frac{X_1}{\sqrt{Y_1}}$

$\implies X_1 = Y_1^{1/2} Y_2$
and $X_2 = \sqrt{Y_1 - Y_1 Y_2^2}$

$\partial_{Y_1} X_1 = \frac{1}{2} Y_1^{-1/2} Y_2$ \$ $\partial_{Y_2} X_1 = Y_1^{1/2}$
$\partial_{Y_1} X_2 = \frac{1}{2} \sqrt{\frac{1-Y_2^2}{Y_1}}$
$\partial_{Y_2} X_2 = \frac{Y_2 Y_1^{1/2}}{\sqrt{1-Y_2^2}}$

Then $|J| = \frac{1}{2} \frac{Y_2^2}{\sqrt{1-Y_2^2}} - \frac{1}{2}\sqrt{1-Y^2} = \frac{1}{2\sqrt{1-Y_2^2}}$

Since the density functions for X_1 and X_2 are unimodal and symmetric, we just multiply this by 2.

Then we get $f(y_1, y_2) = (2\pi\sigma^2)^{-1} e^{-\frac{y_1}{2\sigma^2}} (1 - y_2^2)^{-1/2}$

b

Let $f_{Y_1}(y_1) \propto e^{-\frac{y_1}{2\sigma^2}}$ and $f_{Y_2}(y_2) \propto (1 - y_2^2)^{-1/2}$. Then we can see that $f(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$.

If we use polar coordinates, we can see that $Y_1 = R^2$ and $Y_2 = cos\theta$, and so Y_1 and Y_2 do not share any terms.

4.24

We can see that $Z_2 = X/Z_1$, so $X = Z_1 Z_2$. Then $Y = Z_1(1 - Z_2)$.

$\partial_{Z_1} X = Z_2$
$\partial_{Z_2} X = Z_1$
$\partial_{Z_1} Y = 1 - Z_2$
$\partial_{Z_2} Y = -Z_1$

$\implies |J| = |-Z_1 Z_2 - Z_1(1 - Z_2)| = Z_1$

Then $f(z_1, z_2) \propto (z_1 z_2)^{r-1} e^{-z_1 z_2} (z_1 - z_1 z_2)^{s-1} e^{-z_1 + z_1 z_2} z_1$
$= z_1^{r+s-1} e^{-z_1} \times z_2^{r-1} (1 - z_2)^{s-1}$

And we can identify the product of the kernels of the gamma and beta densities.

$Z_1 \sim Gamma(r + s, 1)$
$Z_2 \sim Beta(r, s)$

4.26

a

$P(Z \le z, W = 0) = P(\min(X,Y) \le z, Y \le X) = P(Y \le z, Y \le X)$
$= \int_0^z \int_y^\infty \lambda^{-1} e^{-x/\lambda} \mu^{-1} e^{-y/\mu} dxdy$
$= \frac{\lambda}{\mu+\lambda}(1 - e^{-(\mu^{-1}+\lambda^{-1})z})$

And similarly, $P(Z \le z, W = 1) = \frac{\mu}{\mu+\lambda}(1 - e^{-(\mu^{-1}+\lambda^{-1})z})$

b

So we can say $P(Z \le z, W = w) = \frac{\mu w + (1-w)\lambda}{\mu+\lambda} \times (1 - e^{-(\mu^{-1}+\lambda^{-1})z})$

and we can see that this is separable.

4.27

$U = X + Y$
$V = X - Y$

$\implies U + V = 2X \implies X = \frac{U+V}{2}$ and $Y = \frac{U-V}{2}$

$\partial_U X = 1/2$
$\partial_V X = 1/2$
$\partial_U Y = 1/2$
$\partial_V Y = -1/2$

$|J| = |-1/4 - 1/4| = 1/2$

$f(u,v) \propto e^{-\frac{1}{2\sigma^2}((\frac{u+v}{2}-\mu)^2 + (\frac{u-v}{2}-\gamma)^2)}$
$\propto e^{-\frac{1}{2\sigma^2}(u^2/4+uv+v^2/2-\mu u-\mu v+u^2/4-uv+v^2/4-\gamma u-\gamma v)}$
$= e^{-\frac{1}{2\sigma^2}(u^2/4+v^2/2-\mu u-\mu v+u^2/4+v^2/4-\gamma u-\gamma v)}$

and we can see that u and v are separable since there are no mixed terms.

4.31

a

$E[Y] = E[E[Y|X]] = E[nX] = nE[X] = n/2$

$Var(Y) = Var(E[Y|X]) + E[Var(Y|X)] = Var(nX) + E[nX(1-X)] = n^2/12 + n(E[X] - E[X^2]) = n^2/12 + n(1/2 - 1/12 - 1/4) = n^2/12 + n/6$

b

$f(x,y) = f(y|x)f(x) = \binom{n}{y} x^y (1-x)^{n-y} I(0 < x < 1)$

c

$f(y) = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} dx = \binom{n}{y} \int_0^1 x^{y+1-1}(1-x)^{n-y+1-1} dx = \binom{n}{y} B(y+1, n-y+1)$

4.44

$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = \mu_1 + \cdots + \mu_n$

$Var(X_1 + \cdots + X_n) = E[(X_1 + \cdots + X_n - \mu_1 - \cdots - \mu_n)^2]$
$= E[(\sum_i^n X_i - \mu_i)^2]$
$= E[\sum_i^n (X_i - \mu_i)^2 + 2\sum_{i \neq j}(X_i - \mu_i)(X_j - \mu_j)] = \sum_i E[(X_i - \mu_i)^2] + 2\sum_{i \neq j} E[(X_i - \mu_i)(X_j - \mu_j)]$
$= \sum_i Var(X_i) + 2\sum_{i \neq j} Cov(X_i, X_j)$

4.45

a

Let $u = \frac{x-\mu_X}{\sigma_X}$ and $v = \frac{y-\mu_Y}{\sigma_Y}$. Then $f_{XY}(u,v) \propto \exp^{-\frac{1}{2(1-\rho^2)}(u^2-2\rho uv+v^2)}$
$\implies f_X(x) \propto \int \exp^{-\frac{1}{2(1-\rho^2)}(u^2-2\rho uv+v^2)} dy$
$= \int \exp^{-\frac{1}{2(1-\rho^2)}(u^2-2\rho uv+v^2-\rho^2 u^2-\rho^2 u^2)} dy$
$= \int \exp^{-\frac{1}{2(1-\rho^2)}((v-\rho u)^2+(1-\rho^2)u^2)} dy$
$= \exp^{-u^2/2} \int \exp^{-\frac{1}{2(1-\rho^2)}(v-\rho u)^2} dy$
$\propto \exp^{-u^2/2} = \exp^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$
$\implies X \sim \mathcal{N}(\mu_X, \sigma_X^2)$

and by symmetry, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$

b

$f_{Y|X}(y) \propto \exp\Bigg(-\frac{1}{2(1-\rho^2)}\big((\frac{x-\mu_X}{\sigma_X})^2 - 2\rho\frac{x-\mu_X}{\sigma_X}\frac{y-\mu_Y}{\sigma_Y} + (\frac{y-\mu_Y}{\sigma_Y})^2\big) + \frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2\Bigg)$
$\propto \exp\Bigg(-\frac{1}{2\sigma_Y^2(1-\rho^2)}(y^2 - 2\mu_Y y - 2\rho\frac{\sigma_Y}{\sigma_X}(xy - \mu_X y))\Bigg)$
$= \exp\Bigg(-\frac{1}{2\sigma_Y^2(1-\rho^2)}(y^2 - 2(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x)))\Bigg)$
$\propto \exp\Bigg(-\frac{1}{2\sigma_Y^2(1-\rho^2)}\big(y - (\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X))\big)^2\Bigg)$
$\implies Y \mid x \sim \mathcal{N}\Bigg(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \sigma_Y^2(1-\rho^2)\Bigg)$

c

We know that $aX + bY$ is normally distributed since linear transformations of normals are normal.

$E[aX + bY] = aE[X] + bE[Y] = a\mu_X + b\mu_Y$

$Var(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCov(X,Y) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$

4.46

a

$E[X] = a_X E[Z_1] + b_X E[Z_2] + c_X = c_X$
$Var(X) = a_X^2 Var(Z_1) + b_X^2 Var(Z_2) = a_X^2 + b_X^2$

Similarly, $E[Y] = c_Y$ and $Var(Y) = a_Y^2 + b_Y^2$

$Cov(X,Y) = \frac{1}{2}(Var(X+Y) - Var(X) - Var(Y))$
$= \frac{1}{2}(a_X^2 + 2a_X a_Y + a_Y^2 + b_X^2 + 2b_X b_Y + bY^2 - a_X^2 - a_Y^2 - b_X^2 - b_Y^2) = a_X a_Y + b_X b_Y$

b

$E[X] = c_X = \mu_X$
$Var(X) = \frac{1+\rho}{2}\sigma_X^2 + \frac{1-\rho}{2}\sigma_X^2 = \sigma_X^2$

Similarly, $E[Y] = \mu_Y$ and $Var(Y) = \sigma_Y^2$

$Cov(X,Y) = \frac{1+\rho}{2}\sigma_X\sigma_Y - \frac{1-\rho}{2}\sigma_X\sigma_Y = \rho\sigma_X\sigma_Y$
$\implies Cor(X,Y) = Cov(X,Y)/(\sigma_X\sigma_Y) = \rho$

c

After some algebra ...

$Z_1 = \frac{\frac{X-\mu_X}{\sigma_X} + \frac{Y-\mu_Y}{\sigma_Y}}{\sqrt{2(1+\rho)}}$

$Z_2 = \frac{\frac{X-\mu_X}{\sigma_X} + \frac{Y-\mu_Y}{\sigma_Y}}{\sqrt{2(1-\rho)}}$

... and additionally ...

$|J| = \frac{1}{\sigma_X\sigma_Y\sqrt{1-\rho^2}}$

Then $f_{XY}(x,y) \propto \exp\left(-\frac{1}{2(1-\rho^2)}\big((\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - 2\rho\frac{x-\mu_X}{\sigma_X}\frac{y-\mu_Y}{\sigma_Y}\big)\right)$

d

Since we have 5 equations for 6 variables, there are an infinite number of solutions.

4.50

We know that $E[X] = E[Y] = 0$ and $Var(X) = Var(Y) = 1$
$\implies E[X^2] = E[Y^2] = 1$

$Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY] = \int xyf(x,y)dxdy$
$= \int xyf(y|x)f(x)dxdy$
$= \int xf(x)dx \int yf(y|x)dy$
$= \int xf(x)\rho x dx$
$= \rho \int x^2 f(x) = \rho E[X^2] = \rho$

Then $Cor(X,Y) = Cov(X,Y)/\sqrt{Var(X)Var(Y)} = Cov(X,Y) = \rho$

$Cov(X^2, Y^2) = E[X^2Y^2] - E[X^2]E[Y^2] = E[X^2Y^2] - 1$

$E[X^2Y^2] = \int x^2y^2 f(y|x)f(x)dydx$
$= \int x^2 f(x)dx \int y^2 f(y|x)dy$
$= \int x^2 f(x)E[Y^2|x]dx$
$= \int x^2 f(x)(Var(Y|x) + (E[Y|x])^2)dx$ \$ $= \int x^2 f(x)(1 - \rho^2 + \rho^2x^2)dx$
$= \int x^2 f(x)dx - \rho^2 \int x^2 f(x)dx + \rho^2 \int x^4 f(x)dx$
$= E[X^2] - \rho^2 E[X^2] + \rho^2 E[X^4]$
$= 1 - \rho^2 + 3\rho^2 = 2\rho^2 + 1$
$\implies E[X^2Y^2] = 2\rho^2 + 1 - 1 = 2\rho^2$

$Var(X^2) = E[X^4] - (E[X^2])^2 = 3 - 1 = 2$

$\implies Cor(X^2, Y^2) = \frac{2\rho^2}{\sqrt{2\times 2}} = \rho^2$

5.10

a

$\theta_1 = E[X_i] = \mu$
$\theta_2 = E[(X_i - \mu)^2] - \sigma^2$
$\theta_2 = E[(X_i - \mu)^3] = 0$ since odd central moments of normally distributed variables are 0
$\theta_4 = E[(X_i - \mu)^4] = 3\sigma^4$ (from S620 notes)

b

$Var(S^2) = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2)$
$= \frac{1}{n}(3\sigma^4 - \frac{n-3}{n-1}\sigma^4) = \frac{2\sigma^4}{n-1}$

c

Let $\Sigma \sim \chi^2_{n-1}$. Then $Var(\Sigma) = 2(n-1)$.

$S^2 = \sigma^2\Sigma/(n-1)$
$\implies Var(S^2) = \frac{\sigma^4}{(n-1)^2}Var(\Sigma)$
$= \frac{\sigma^4}{(n-1)}2(n-1) = \frac{2\sigma^4}{n-1}$

5.14

a

Suppose $Cov(\sum a_{ij}X_j, \sum b_{rj}X_j) = 0$.

This is also equal to $E[(\sum a_{ij}X_j)(\sum b_{rj}X_j)] - E[\sum a_{ij}X_j]E[\sum b_{rj}X_j]$

Then $E[(\sum a_{ij}X_j)(\sum b_{rj}X_j)] = E[\sum a_{ij}X_j]E[\sum b_{rj}X_j] \implies \sum a_{ij}X_j$ and $\sum b_{rj}X_j$ are independent.

b

$Cov(\sum a_{ij}X_j, \sum b_{rj}X_j) = E[(\sum a_{ij}(X_j - \mu_j))(\sum b_{rj}(X_j - \mu_j))]$
$= E[\sum a_{ij}b_{rj}\sigma_j^2 Z_j]$
$= \sum a_{ij}b_{rj}\sigma_j^2$

5.18

a

$E[X] = E[Z]\frac{1}{p}E[\Sigma^{-1/2}]$ where $Z \sim \mathcal{N}(0,1)$ and $\Sigma \sim \chi_p^2$ and they are independent. $E[Z] = 0$, so the entire expression is 0.

b

$Z^2 \sim \chi_1^2$, so $X^2 = Z^2/(\Sigma/p) = (Z^2/1)/(\Sigma/p) \sim F_{1,p}$

c

$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(p/2)\sqrt{p\pi}}(1 + x^2/p)^{-(p+1)/2}$

By Stirling's approximation ...

- $\Gamma(\frac{p+1}{2}) \to (2\pi\frac{p-1}{2})^{1/2}(\frac{p-1}{2})^{\frac{p-1}{2}}e^{-\frac{p-1}{2}}$
- $\Gamma(p/2) \to (2\pi\frac{p-2}{2})^{1/2}(\frac{p-2}{2})^{\frac{p-2}{2}}e^{-\frac{p-2}{2}}$

Dividing the two and letting $p - 1 \approx p - 2$, we are left with $2^{-1/2}$.

And $(1 + x^2/p)^{-\frac{p+1}{2}} \to e^{-x^2/2}$

So the expression becomes ...

$= (2\pi)^{-1/2}e^{-x^2/2}$

d

$X \to \mathcal{N}(0,1)$, so $X^2 \to \chi_1^2$

e

$F_{q,p}$ would be the sum of q t_p distributed random variables, so as $p \to \infty$, $F_{q,p} \to \chi_q^2$