STAT-S675

Homework 7

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Link to assignment

```r
library(ggplot2)
dp <- loadNamespace('dplyr')
import::from(magrittr, `%>%`, `%<>%`)
import::from(ggrepel, geom_text_repel)
import::from(viridis, scale_colour_viridis)
import::from(readr, read_table)

theme_set(ggthemes::theme_base())

source('http://pages.iu.edu/~mtrosset/Courses/675/stress.r')
source('http://pages.iu.edu/~mtrosset/Courses/675/manifold.r')
```

Problem 1

Figure ??(a)

```r
# parameters/constants
a <- sqrt(2 - 2 * cos(pi / 3))
b <- sqrt(2 + 2 * cos(pi / 3))

# construct the data
input.df <- dplyr::data_frame(i = seq(6)) %>%
  dp$mutate(theta = (i - 1) * pi / 3) %>%
  dp$mutate(x = cos(theta), y = sin(theta)) %>%
  # attach i+1, i+2, and i+3 for the edge weights
  dp$mutate(x.next = lead(x), y.next = lead(y),
            x.next.2 = lead(x, 2), y.next.2 = lead(y, 2),
            x.next.3 = lead(x, 3), y.next.3 = lead(y, 3))

# plot
figure.a <- ggplot(input.df) +
  coord_fixed() +
  # scale_colour_viridis() +
  scale_colour_distiller(palette = 'Spectral') +
  labs(x = NULL, y = NULL, colour = 'dissimilarity') +
  # edge weights
  geom_segment(aes(x = x, y = y,
                   xend = x.next, yend = y.next,
                   colour = a)) +
  geom_segment(aes(x = x, y = y,
                   xend = x.next.2, yend = y.next.2,
                   colour = b)) +
  geom_segment(aes(x = x, y = y,
```

```
                   xend = x.next.3, yend = y.next.3,
                   colour = 2)) +
  # vertices
  geom_point(aes(x = x, y = y)) +
  geom_text_repel(aes(x = x, y = y, label = i))

figure.a
```

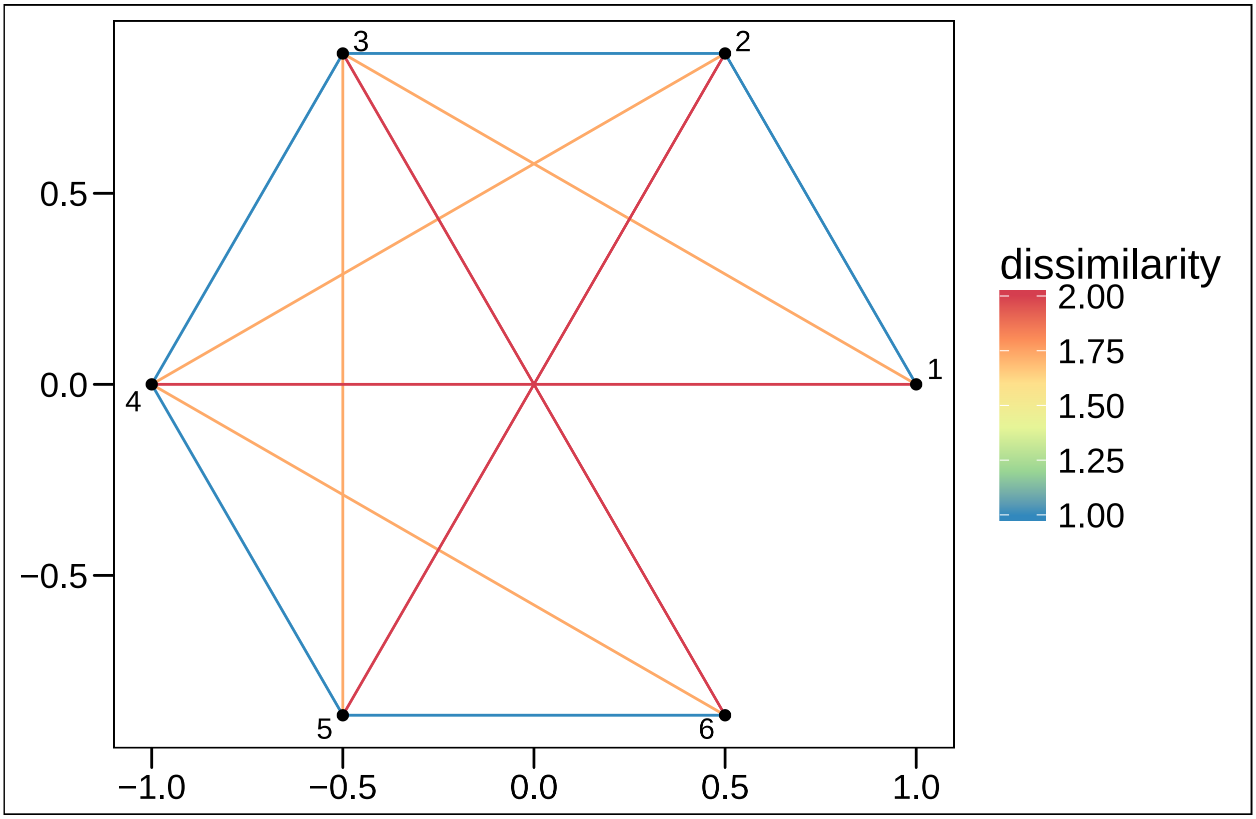
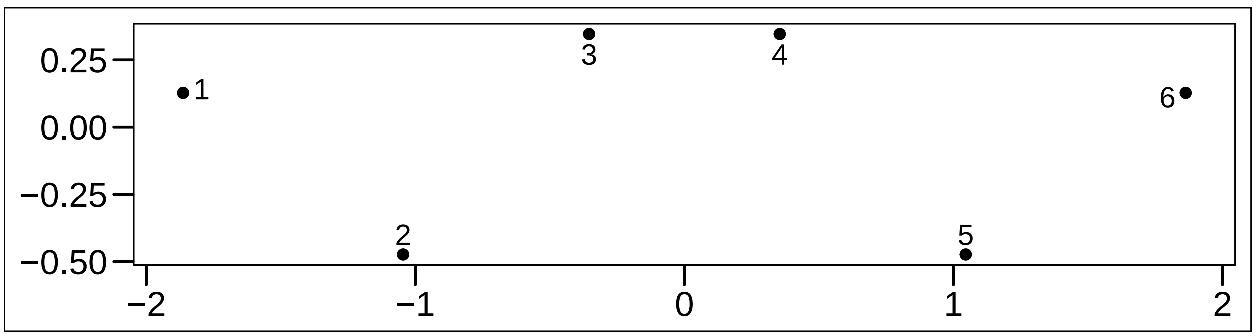


Figure ??(b)

```
# paramters/constants
N.iter <- 20

# specified dissimilarity matrix
Delta <- rbind(c(0, a, b, 2, a + 2, b + 2),
               c(a, 0, a, b, 2, a + 2),
               c(b, a, 0, a, b, 2),
               c(2, b, a, 0, a, b),
               c(a + 2, 2, b, a, 0, a),
               c(b + 2, a + 2, 2, b, a, 0))

# initialize the configuration using CMDS
X <- cmdscale(as.dist(Delta))
stress <- mds.stress.raw.eq(X, Delta)
gma.df <- as.data.frame(X) %>%
  dp$transmute(id = seq(6), x = V1, y = V2, iter = 0, stress)
```

```r
# GMA optimization
for (i in seq(N.iter)) {
  # iterate
  X <- mds.guttman.eq(X, Delta)
  # compute stress
  stress <- mds.stress.raw.eq(X, Delta)
  # compile into data frame
  temp.df <- as.data.frame(X) %>%
  dp$transmute(id = seq(6), x = V1, y = V2, iter = i, stress)
  # attach data frame to original
  gma.df %<>% dp$bind_rows(temp.df)
}

# plot final configuration
gma.df %>%
  dp$filter(iter == N.iter) %>%
  ggplot() +
  labs(x = NULL, y = NULL) +
  coord_fixed() +
  geom_point(aes(x = x, y = y)) +
  geom_text_repel(aes(x = x, y = y, label = id))
```

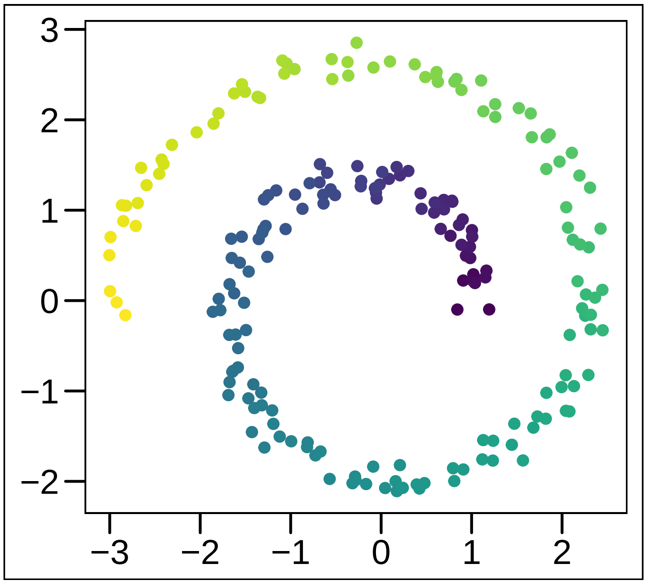


Problem 2

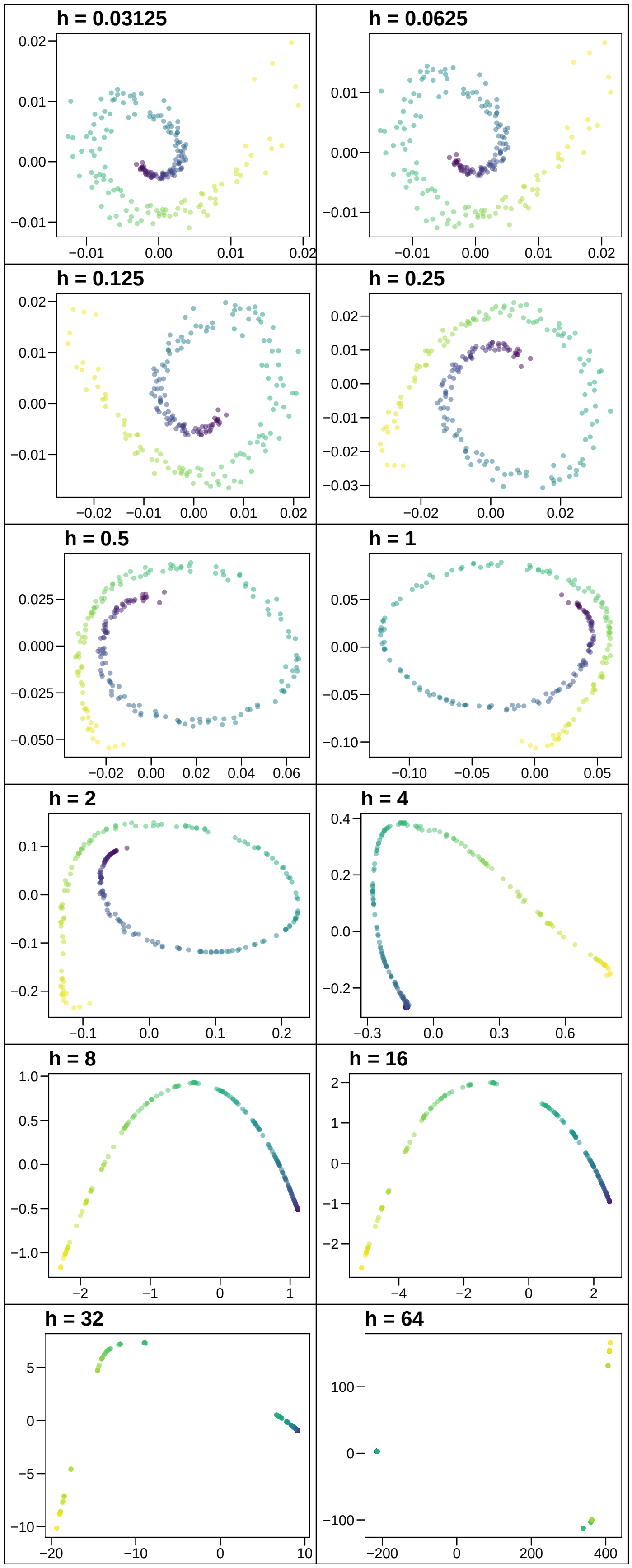
[Exercise 6.8.1 from the text]

```r
# load data
spiral.df <- read_table('http://pages.iu.edu/~mtrosset/Courses/675/X.spiral',
                        col_names = FALSE)

# save the number of rows of the data
# for when we want the 2 smallest eigenvalues
n <- nrow(spiral.df)

# plot the data
spiral.df %>%
  dp$mutate(id = as.numeric(rownames(.))) %>%
  ggplot() +
  viridis::scale_colour_viridis() +
  geom_point(aes(x = X1, y = X2, colour = id)) +
  coord_fixed() +
```

```r
  theme(legend.position = 'none') +
  labs(x = NULL, y = NULL)
```




```r
# values of h to try
h.vector <- 2 ** seq(-5, 6)

# construct plots
eigenmaps <- lapply(h.vector, function(h) {
  # i <- i + 1
  W <- exp(-h * as.matrix(dist(spiral.df)) ** 2)

  L <- graph.laplacian(W)

  L.eigen <- eigen(L)

  eigenmap <- cbind(L.eigen$vectors[, n - 1] / sqrt(L.eigen$values[n - 1]),
                    L.eigen$vectors[, n - 2] / sqrt(L.eigen$values[n - 2])) %>%
    as.data.frame() %>%
    dp$mutate(id = as.numeric(rownames(.))) %>%
    ggplot() +
    geom_point(aes(x = V1, y = V2, colour = id),
               alpha = .5) +
    # coord_fixed() +
    viridis::scale_colour_viridis() +
    labs(x = NULL, y = NULL, title = paste('h =', h)) +
    theme(legend.position = 'none')
  return(eigenmap)
})

.gridarrange <- function(...) gridExtra::grid.arrange(..., ncol = 2)
do.call(.gridarrange, eigenmaps)
```


h = 0.03125
h = 0.0625
h = 0.125
h = 0.25
h = 0.5
h = 1
h = 2
h = 4
h = 8
h = 16
h = 32
h = 64

We can see that very small values of h result in a Laplacian eigenmap that preserves the original structure, and increasing values of h unravels the spiral up to a point. Once the spiral is unraveled, the points along the curve start to separate into three clusters, which seem to correspond to the “weak points” of the original spiral.

If we want an approximate representation of the original data in one dimension, $h \approx 8$ seems like the best choice.

An attempt at explaining what's going on

When h is very small, the similarity between any two points is close to 1 since $lim_{h \to 0} \exp(-hd_{ij}^2) = \exp(0) = 1$. On the other hand, when $h \to \infty$, the similarities go to 0. In addition, $\exp(-x)$ is approximately linear around $x = 0$. So when h is very small, we just have $\gamma_{ij} \sim -hd_{ij}^2 \; \forall i, j$. On the other hand, when h is very large, we have this relationship for only when d_{ij}^2 is very small, with $\gamma_{ij} \approx 0$ otherwise.

Problem 3