

S721 HW7

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From text (I)

7.11

Part a

From HW6, we obtained $\hat{\theta}_{MLE} = -\frac{n}{\sum_i \log X_i}$.

In class, we noted that if $X_i \sim \text{Beta}(\theta, 1)$, $-\log X_i \sim \text{Exp}(1/\theta)$. The sum of exponentially distributed random variables is gamma distributed, so $-\sum_i \log X_i = \text{Gamma}(n, 1/\theta)$. Therefore, $\hat{\theta}_{MLE} \sim -n \times \text{InvGamma}(n, 1/\theta)$.

If $Y \sim \text{InvGamma}(n, 1/\theta)$, then the density function is $f(y | n, 1/\theta) = \frac{\theta^n}{\Gamma(n)} (\frac{1}{y})^{n+1} \exp(-\frac{\theta}{y})$.

$$\begin{aligned} E[Y] &= \int_0^\infty \frac{\theta^n}{\Gamma(n)} y y^{-n-1} \exp(-\theta/y) dy \\ &= \frac{\theta^n}{\Gamma(n)} \int y^{-n} \exp(-\theta/y) dy \end{aligned}$$

Letting $u = y^{-1}$, we get $du = -y^{-2} dy$, so the expectation goes to:
 $-\frac{\theta^n}{\Gamma(n)} \int u^{n-2} \exp(-\theta u) du = -\frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} \int \frac{\theta^{n-1}}{\Gamma(n-1)} u^{(n-1)-1} \exp(-\theta u) du = -\frac{\theta}{n-1}$ since the term inside the integral is just the density function for a gamma-distributed U .

Then $E[\hat{\theta}_{MLE}] = E[-nY] = -nE[Y] = n\frac{\theta}{n-1}$.

$$E[Y^2] = \frac{\theta^n}{\Gamma(n)} \int y^{-n+1} \exp(-\theta/y) dy.$$

Then again letting $u = y^{-1}$, $du = -y^{-2}$, we get:

$$\begin{aligned} &-\frac{\theta^n}{\Gamma(n)} \int u^{n-3} \exp(-\theta u) du \\ &= -\frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} \int \frac{\theta^{n-2}}{\Gamma(n-2)} u^{(n-2)-1} \exp(-\theta u) du \\ &= -\frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} \\ &= -\frac{\theta^2}{(n-1)(n-2)}. \end{aligned}$$

Then $E[\hat{\theta}_{MLE}^2] = -n^2 E[Y^2] = n^2 \frac{\theta^2}{(n-1)(n-2)}$.

$$\begin{aligned} \text{Then } \text{Var}(\hat{\theta}_{MLE}) &= \frac{n^2}{(n-1)(n-2)} \theta^2 - \frac{n^2}{(n-1)^2} \theta^2 \\ &= \theta^2 \left(\frac{n^2(n-1) - n^2(n-2)}{(n-1)^2(n-2)} \right) \\ &= \theta^2 \left(\frac{n^3 - n^2 - n^3 + 2n^2}{(n-1)^2(n-2)} \right) \\ &= \frac{n^2}{(n-1)^2(n-2)} \theta^2. \end{aligned}$$

Then $\text{Var}(\hat{\theta}_{MLE}) \sim \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$.

Part b

We saw from previous homework that $E[X_i] = \frac{\theta}{\theta+1}$, so setting this equal to \bar{X} , we get $\hat{\theta}_{MOM} = \frac{\bar{X}}{1-\bar{X}}$.

7.12

Part a

$E[X] = \theta$, so $\hat{\theta}_{MOM} = \bar{X}$.

The likelihood function is $L(\theta | x) = \prod_i \theta^{x_i} (1 - \theta)^{1-x_i}$

$$\implies \ell(\theta | x) = \log \theta \sum_i x_i + \log(1 - \theta) \sum_i (1 - x_i)$$

$$\implies \ell'(\theta) = \frac{\sum_i x_i}{\theta} - \frac{\sum_i (1-x_i)}{1-\theta}, \text{ and setting this to zero, we get}$$

$$\hat{\theta}_{MLE} = \bar{X}.$$

But this is only true if $\bar{X} \leq 1/2$. When $\bar{X} > 1/2$, we note that $\ell'(\theta) > 0$, so L is an increasing function. We just set it at the highest possible value, $\hat{\theta}_{MLE} = 1/2$.

Part b

The MSE can be broken up into bias and variance. The bias in this case is 0, so we just have the variance of $\bar{X} = \text{Var}(X)/n = \frac{\theta(1-\theta)}{n}$.

Part c

When $\bar{X} \leq 1/2$, the estimators are equivalent. When $\bar{X} > 1/2$, we get an invalid estimate from the method of moments. So the MLE estimator is preferred.

7.15

Part a

Writing down the likelihood function, we get

$L(\mu, \lambda | x) = \prod_i \left(\frac{\lambda}{2\pi x_i^3}\right)^{1/2} \exp(-\lambda(x_i - \mu)^2 / (2\mu^2 x_i))$, but it's easier to deal with the log-likelihood, which is

$$\ell(\mu, \lambda | x) = \frac{n}{2} \log \lambda - \frac{n}{2} \log 2\pi - \frac{3}{2} \sum_i \log x_i - \sum_i \frac{\lambda(x_i - \mu)^2}{2\mu^2 x_i}.$$

When considering μ , we can ignore everything that doesn't have μ in it, so we only need to consider

$$-\frac{\lambda}{2} \sum_i \frac{(x_i - \mu)^2}{\mu^2 x_i} = -\frac{\lambda}{2} \left(\sum_i x_i / \mu^2 - 2n/\mu + \sum_i x_i^{-1} \right),$$

and if we differentiate w.r.t. μ , we get

$$-\frac{\lambda}{2} (-\mu^{-3} \sum_i x_i + 2n\mu^{-2}).$$

Setting this to 0, we get

$$\mu = \sum_i x_i / n, \text{ so } \hat{\mu}_{MLE} = \bar{X}.$$

For λ , we can plug in our estimate for $\mu = \bar{x}$, so, considering only the terms that depend on λ , we get

$$\frac{n}{2} \log \lambda - \frac{\lambda}{2\bar{x}^2} \sum_i \frac{(x_i - \bar{x})^2}{x_i},$$

and if we find the derivative w.r.t. λ and set it to zero, we get

$$\begin{aligned} \frac{n}{2\lambda} - \frac{1}{2\bar{x}^2} \sum_i \frac{(x_i - \bar{x})^2}{x_i} &= 0 \\ \frac{n}{\lambda} &= \frac{1}{\bar{x}^2} \left(\sum_i x_i - 2n\bar{x} + \bar{x}^2 \sum_i x_i^{-1} \right) \\ \frac{n}{\lambda} &= \frac{1}{\bar{x}^2} (n\bar{x} - 2n\bar{x} + \bar{x}^2 \sum_i x_i^{-1}) \\ \frac{n}{\lambda} &= \frac{1}{\bar{x}} \left(\bar{x} \sum_i x_i^{-1} - n \right) \\ \frac{n}{\lambda} &= \sum_i x_i^{-1} - n\bar{x}^{-1} \\ \frac{n}{\lambda} &= \sum_i (x_i^{-1} - \bar{x}^{-1}) \end{aligned}$$

$$\Rightarrow \lambda = n \left(\sum_i (x_i^{-1} - \bar{x}^{-1}) \right)^{-1}$$

$$\text{So } \hat{\lambda}_{MLE} = n \left(\sum_i (X_i^{-1} - \bar{X}^{-1}) \right)^{-1}$$

2.34

Since the density functions for both X and Y are even, any odd-numbered moment for either must be 0.

If r is even, $E[Y] = \frac{1}{6}3^{r/2} \times 2 = 3^{r/2-1}$, so the first five moments of Y are 0, 1, 0, 3, 0.

From S620 notes, the moment generating function for the standard normal is $M_X(t) = \exp(t^2/2)$.

$$M'_X(t) = t \exp(t^2/2)$$

$$M''_X(t) = \exp(t^2/2) + t \exp(t^2/2) = (1 + t^2) \exp(t^2/2)$$

$$M'''_X(t) = 2t \exp(t^2/2) + (1 + t^2)t \exp(t^2/2) = (t^3 + 3t) \exp(t^2/2) \quad M_X^{(4)}(t) = (3t + 3) \exp(\dots) + (t^3 + 3t)t \exp(\dots) = (t^4 + 3t^2 + 6t + 3) \exp(\dots)$$

Then setting each of these to 0 (and noting that any odd-numbered moment is 0), we get 0, 1, 0, 3, 0.

3.16

Part a

$$\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} dx$$

Letting $u = x^\alpha$ and $dv = e^{-x}$, we get $du = \alpha x^{\alpha-1} dx$ and $v = -e^{-x}$. Then the above integral is equal to

$$-x^\alpha e^{-x} \Big|_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$$= \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$$= \alpha \Gamma(\alpha)$$

Part b

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} \exp(-x) dx$$

Let $u = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2} dx$. Then the above integral becomes

$$\int_0^\infty 2 \exp(-u^2) du$$

$$= 2\sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{\pi}} \exp(-u^2) du$$

$$= 2\sqrt{\pi} \frac{1}{2}$$

$$= \sqrt{\pi}$$

3.17

$$E[X^\nu] = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1+\nu} \exp(-x/\beta) dx$$

$$= \frac{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}}{\Gamma(\alpha)\beta^\alpha} \int \frac{1}{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}} x^{(\alpha+\nu)-1} \exp(-x/\beta) dx$$

$$= \frac{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}}{\Gamma(\alpha)\beta^\alpha}$$

$$= \frac{\beta^\nu \Gamma(\alpha+\nu)}{\Gamma(\alpha)}$$

3.20

Part a

$$E[X] = \frac{2}{\sqrt{2\pi}} \int_0^\infty x \exp(-x^2/2) dx.$$

Let $u = x^2/2 \implies du = x dx$. Then the above integral becomes

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \exp(-u) du$$

$$= \sqrt{\frac{2}{\pi}}.$$

$$E[X^2] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^2 \exp(-x^2/2) dx$$

Let $u = x \implies du = dx$ and $dv = x \exp(-x^2/2) dx \implies v = -\exp(-x^2/2)$. Then the above becomes

$$\sqrt{\frac{2}{\pi}} (-x \exp(-x^2/2)|_0^\infty + \int_0^\infty \exp(-x^2/2) dx)$$

$$= \sqrt{\frac{2}{\pi}} (0 + \sqrt{\frac{\pi}{2}})$$

$$= 1.$$

$$\text{Then } \text{Var}(X) = 1 - \frac{2}{\pi}.$$

Part b

The gamma distribution has the density function $f(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} \exp(-y/\beta)$.

Let $Y = g(X) = X^2$. Then $X = g^{-1}(Y) = Y^{1/2}$, so $\frac{d}{dy} g^{-1}(y) = \frac{1}{2} y^{-1/2}$. Then the density function for Y is

$$f_Y(y) = \sqrt{\frac{2}{\pi}} \exp(-y/2) \frac{1}{2} y^{-1/2}$$

$$= \frac{1}{\sqrt{\pi 2^{1/2}}} y^{1/2-1} \exp(-y/2).$$

Then since $\Gamma(1/2) = \sqrt{\pi}$, we can see that $\alpha = 1/2$ and $\beta = 2$.

3.24

Part a

$$f_X(x) = \frac{1}{\beta} \exp(-x/\beta)$$

If $Y = X^{1/\gamma}$, then $X = Y^\gamma$ and $X' = \gamma Y^{\gamma-1}$.

Plugging this into f_X , we get

$$\frac{\gamma}{\beta} y^{\gamma-1} \exp(-y^\gamma/\beta)$$

To evaluate $\int_0^\infty \frac{\gamma}{\beta} y^{\gamma-1} \exp(-y^\gamma/\beta) dy$,

we set $x = y^\gamma \implies dx = \gamma y^{\gamma-1} dy$ to obtain

$$\int_0^\infty \frac{1}{\beta} \exp(-x/\beta) dx = 1$$

since it is just the density function for an exponential random variable.

$$E[Y] = \int \frac{\gamma}{\beta} y^\gamma \exp(-y^\gamma/\beta) dy$$

Letting $x = y^\gamma \implies dx = \gamma y^{\gamma-1} dy$ as before, we get

$$\int \frac{1}{\beta} y \exp(-x/\beta) dx$$

$$= \frac{1}{\beta} \int x^{1/\gamma} \exp(-x/\beta) dx = \frac{1}{\beta} \int x^{(1/\gamma+1)-1} \exp(-x/\beta) dx$$

$$= \beta^{-1} \Gamma(1 + 1/\gamma) \beta^{1+1/\gamma}$$

$$= \beta^{1/\gamma} \Gamma(1 + 1/\gamma)$$

$$E[Y^2] = \int \frac{\gamma}{\beta} y^{\gamma+1} \exp(-y^\gamma/\beta) dy$$

Letting $x = y^\gamma \implies dx = \gamma y^{\gamma-1} dy$ as before, we get

$$\begin{aligned}
& \beta^{-1} \int y^2 \exp(-x/\beta) dx \\
&= \beta^{-1} \int x^{2/\gamma} \exp(-x/\beta) dx \\
&= \beta^{-1} \int x^{(1+2/\gamma)-1} \exp(-x/\beta) dx \\
&= \beta^{-1} \Gamma(1 + 2/\gamma) \beta^{1+2/\gamma} \\
&= \beta^{2/\gamma} \Gamma(1 + 2/\gamma) \\
\text{Then } Var(Y) &= \beta^{2/\gamma} \Gamma(1 + 2/\gamma) - \beta^{2/\gamma} \Gamma(1 + 1/\gamma)^2 \\
&= \beta^{2/\gamma} (\Gamma(1 + 2/\gamma) - \Gamma(1 + 1/\gamma)^2)
\end{aligned}$$

Part c

$$Y = 1/X \implies X = 1/Y \implies X' = -Y^{-2}$$

$$\begin{aligned}
\text{Then } f(y) &= f(x(y)) |x'(y)| = \frac{1}{\Gamma(a)b^a} y^{1-a} \exp(-\frac{1}{by}) y^{-2} \\
&= \frac{1}{\Gamma(a)b^a} \frac{1}{y^{a+1}} \exp(-\frac{1}{by})
\end{aligned}$$

$$\begin{aligned}
& \int_0^\infty \frac{1}{\Gamma(a)b^a} \frac{1}{y^{a+1}} \exp(-\frac{1}{by}) dy \\
\text{Let } x &= y^{-1} \implies dx = -y^{-2} dy. \text{ Then the above becomes} \\
& \int_\infty^0 \frac{1}{\Gamma(a)b^a} y^{-a-1} (-y^2) \exp(-x/b) dx \\
&= \int_0^\infty \frac{1}{\Gamma(a)b^a} y^{-a+1} \exp(-x/b) dx \\
&= \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-1} \exp(-x/b) dx \\
&= 1 \text{ since this is just the integral of } Gamma(a, b).
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \int_0^\infty \frac{1}{\Gamma(a)b^a} y^{-a} \exp(-\frac{1}{by}) dy \\
\text{Again, let } x &= y^{-1} \implies dx = -y^{-2} dy. \text{ Then the above becomes} \\
& \int_\infty^0 \frac{1}{\Gamma(a)b^a} x^a (-x^{-2}) \exp(-x/b) dx \\
&= \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-2} \exp(-x/b) dx \\
&= \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} \int_0^\infty \frac{1}{\Gamma(a-1)b^{a-1}} x^{(a-1)-1} \exp(-x/b) dx \\
&= \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} \\
&= \frac{1}{(a-1)b}
\end{aligned}$$

For $E[Y^2]$, we can extrapolate the results for $E[Y]$ to obtain:

$$\begin{aligned}
E[Y^2] &= \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-3} \exp(-x/b) dx \\
&= \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} \int_0^\infty \frac{1}{\Gamma(a-2)b^{a-2}} x^{(a-2)-1} \exp(-x/b) dx \\
&= \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} \\
&= \frac{1}{(a-1)(a-2)b^2}.
\end{aligned}$$

$$\begin{aligned}
\text{Then } Var(Y) &= \frac{1}{(a-1)(a-2)b^2} - \frac{1}{(a-1)^2 b^2} \\
&= \frac{(a-1)-(a-2)}{(a-1)^2 (a-2)b^2} \\
&= \frac{1}{(a-1)^2 (a-2)b^2}
\end{aligned}$$

Not from text (II)

Problem 1

$$\begin{aligned}
R(\theta, W) &= E[(\theta - W)^2] \\
&= E[\theta^2 - 2\theta W + W^2] \\
&= \theta^2 - 2\theta E[W] + E[W^2] \text{ (since } \theta \text{ is a constant)} \\
&= \theta^2 - 2\theta E[W] + Var(W) + E[W]^2 \text{ (since } Var(W) = E[W^2] - E[W]^2) \\
&= Var(W) + (\theta^2 - 2\theta E[W] + E[W]^2)
\end{aligned}$$

$$\begin{aligned}
&= \text{Var}(W) + (\theta - E[W])^2 \\
&= \text{Var}(W) + (\text{Bias}(W))^2
\end{aligned}$$

Problem 2

Part a

$$\begin{aligned}
&\frac{1}{2n(n-1)} \sum_i \sum_j (X_i - X_j)^2 \\
&= \frac{1}{2n(n-1)} \sum_i \sum_j ((X_i - \bar{X}) - (X_j - \bar{X}))^2 \\
&= \frac{1}{2n(n-1)} \left(n \sum_i (X_i - \bar{X})^2 - 2 \sum_i (X_i - \bar{X}) \sum_j (X_j - \bar{X}) + n \sum_j (X_j - \bar{X})^2 \right)
\end{aligned}$$

Note that the middle term is 0 since $\sum_i (X_i - \bar{X}) = 0$. We also note that $\sum_i (X_i - \bar{X}) = (n-1)S^2$. Then we get:

$$\begin{aligned}
&\frac{1}{2n(n-1)} \left(n(n-1)S^2 + n(n-1)S^2 \right) \\
&= \frac{1}{2n(n-1)} 2n(n-1)S^2 \\
&= S^2
\end{aligned}$$

Part b

We can find this by computing $E[(S^2)^2] - (E[S^2])^2$, which means we have to first compute $E[S^2]$ and $E[(S^2)^2]$.

First, we can rewrite:

$$\begin{aligned}
S^2 &= \frac{1}{n-1} \sum_i (X_i - \bar{X})^2 \\
&= \frac{1}{n-1} \sum_i \left(X_i - \frac{1}{n} \sum_j X_j \right)^2 \\
&= \frac{1}{n-1} \sum_i \left(X_i^2 - \frac{2}{n} X_i \sum_j X_j + \frac{1}{n^2} (\sum_j X_j)^2 \right) \\
&= \frac{1}{n-1} \left(\sum_i X_i^2 - \frac{2}{n} \sum_i X_i \sum_j X_j + n \frac{1}{n^2} (\sum_j X_j)^2 \right) \\
&= \frac{1}{n-1} \left(\sum_i X_i^2 - \frac{1}{n} (\sum_i X_i)^2 \right) \\
&= \frac{1}{n(n-1)} (n \sum_i X_i^2 - (\sum_i X_i)^2)
\end{aligned}$$

We will, w.l.o.g., take $\theta_1 = E[X_i] = 0$.

$$\begin{aligned}
\text{Then } E[S^2] &= \frac{1}{n(n-1)} (n \sum_i E[X_i^2] - E[(\sum_i X_i)^2]) \\
&= \frac{1}{n(n-1)} (n^2 \theta_2 - n \theta_2) \\
&= \frac{1}{n-1} (n \theta_2 - \theta_2) \\
&= \theta_2
\end{aligned}$$

$$\begin{aligned}
\text{Then } E[(S^2)^2] &= \frac{1}{n^2(n-1)^2} E[(n \sum_i X_i^2 - (\sum_i X_i)^2)^2] \\
&= \frac{1}{n^2(n-1)^2} E[n^2 (\sum_i X_i^2)^2 - 2n (\sum_i X_i^2) (\sum_i X_i)^2 + (\sum_i X_i)^4]
\end{aligned}$$

which we can break down into three components inside the expectation.

$$\begin{aligned}
\text{First, } E[(\sum_i X_i^2)^2] &= E[(X_1^2 + \dots + X_n^2)^2] \\
&= E[\sum_i X_i^4 + \sum_{i \neq j} X_i^2 X_j^2] \\
&= \sum_i E[X_i^4] + \sum_{i \neq j} E[X_i^2] E[X_j^2] \text{ (since this is an iid sample)} \\
&= n \theta_4 + n(n-1) \theta_2^2 \text{ (since there are } n \text{ possible } i\text{'s and then } n-1 \text{ possible } j\text{'s since } i \neq j\text{).}
\end{aligned}$$

$$\begin{aligned}
\text{Second, } E[(\sum_i X_i^2) (\sum_i X_i)^2] &= E[(X_1^2 + \dots + X_n^2) (X_1^2 + \dots + X_n^2 + \sum_{i \neq j} X_i X_j)] \\
&= E[(\sum_i X_i^2)^2] + E[(\sum_i X_i^2) (\sum_{i \neq j} X_i X_j)]
\end{aligned}$$

The second term is zero since $E[X_i X_j] = E[X_i] E[X_j]$, and we set $\theta_1 = 0$. Then we are left with $E[(\sum_i X_i^2)^2] = n \theta_4 + n(n-1) \theta_2^2$ as in the previous part.

$$\begin{aligned}
\text{Third, } E[(\sum_i X_i)^4] &= E[(X_1 + \dots + X_n)^4] \\
&= E[\sum_i X_i^4 + 3 \sum_{i \neq j} X_i^2 X_j^2] \text{ (noting that each pair of } X_i \text{ and } X_j \text{ are independent so we can separate out the}
\end{aligned}$$

expectation, and $E[X_i] = 0$)
 $= n\theta_4 + 3n(n-1)\theta_2^2$

Putting it all together, we get:

$$\begin{aligned} & \frac{n^2(n\theta_4+n(n-1)\theta_2^2)-2n(n\theta_4+n(n-1)\theta_2^2)+n\theta_4+3n(n-1)\theta_2^2}{n^2(n-1)^2} \\ &= \frac{n^3\theta_4+n^4\theta_2^2-n^3\theta_2^2-2n^2\theta_4-2n^3\theta_2^2+2n\theta_2^2+n\theta_4+3n^2\theta_2^2-3n\theta_2^2}{n^2(n-1)^2} \\ &= \frac{(n^3-2n^2+n)\theta_4+(n^4-3n^3+5n^2-n)\theta_2^2}{n^2(n-1)^2} \\ &= \frac{n(n-1)^2\theta_4+n(n-1)(n^2-2n+3)\theta_2^2}{n^2(n-1)^2} \\ &= \frac{(n-1)\theta_4+(n^2-2n+3)\theta_2^2}{n(n-1)} \end{aligned}$$

Then $E[(S^2)^2] - (E[S^2])^2$
 $= \frac{(n-1)\theta_4+(n^2-2n+3)\theta_2^2}{n(n-1)} - \theta_2^2$
 $= \frac{(n-1)\theta_4+(n^2-2n+3)\theta_2^2-(n^2-n)\theta_2^2}{n(n-1)}$
 $= \frac{(n-1)\theta_4-(n-3)\theta_2^2}{n(n-1)}$
 $= \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2)$

Part c

$$\begin{aligned} Cov(\bar{X}, S^2) &= \frac{1}{2n^2(n-1)} E[\sum_i X_i \sum_j \sum_k (X_j - X_k)^2] \\ &= \frac{1}{2n^2(n-1)} E[\sum_i X_i \sum_{j \neq k} (X_j - X_k)^2] \\ &= \frac{1}{2n^2(n-1)} E[\sum_{j \neq k} (X_i X_j^2 - 2X_i X_j X_k + X_i X_k^2)] \end{aligned}$$

Note that the middle term, $2E[X_i X_j X_k]$, is zero since $E[X_j] = 0$ and we force $X_j \neq X_k$. Similarly, the first and third terms are nonzero only when $X_i = X_j$ and $X_i = X_k$ respectively. So we are left with:

$$\begin{aligned} & \frac{1}{2n^2(n-1)} \sum_{j \neq k} (E[X_j^3] + E[X_k^3]) = \frac{2n(n-1)}{2n^2(n-1)} E[X_i^3] \text{ (since there are } n(n-1) \text{ nonzero terms in each sum, and} \\ & \text{there are two sums)} \\ &= \frac{1}{n} \theta_3 \end{aligned}$$

This is nonzero when the third moment is nonzero.

Problem 3

Part 1

$$W_1 = \bar{X}$$

Then we have:

$$\begin{aligned} & E\left[\frac{(\theta - \bar{X})^2}{1 + \theta^2}\right] \\ &= \frac{1}{1 + \theta^2} E[(\theta - \bar{X})^2] \\ &= \frac{1}{1 + \theta^2} (\theta^2 - 2\theta E[\bar{X}] + E[\bar{X}^2]) = \frac{1}{1 + \theta^2} (\theta^2 - 2\theta^2 + Var(\bar{X}) + \theta^2) \\ &= \frac{Var(\bar{X})}{1 + \theta^2} \\ &= \frac{Var(X)}{n(1 + \theta^2)} \\ &= \frac{\theta}{n(1 + \theta^2)} \end{aligned}$$

Part 2

$$W_2 = \frac{\sum_i X_i + \sqrt{n/2}}{n + \sqrt{n/2}}$$

Then we have:

$$(1 + \theta^2)^{-1} (n + \sqrt{n/2})^{-2} E[(n + \sqrt{n/2})\theta - n\bar{X} + \sqrt{n/2})^2]$$

The part inside the expectation is, if we expand:

$$n^2\theta^2 + \theta(n/2) + n^2\bar{X}^2 + n/2 + 2n\theta^2\sqrt{n/2} - 2n^2\theta\bar{X} + 2n\theta\sqrt{n/2} - 2\theta\sqrt{n/2}n\bar{X} + 2\theta(n/2) - 2n\bar{X}\sqrt{n/2}$$

And taking the expectation of this (noting that $E[\bar{X}^2] = Var(\bar{X}) + \theta^2 = \theta/2 + \theta^2$), we get:

$$\begin{aligned} & n^2\theta^2 + \theta^2(n/2) + n^2(\theta/n) + n^2\theta^2 + n/2 + 2n\theta^2\sqrt{n/2} - 2n^2\theta^2 + 2n\theta\sqrt{n/2} - 2\theta^2\sqrt{n/2}n + 2\theta(n/2) - 2n\theta\sqrt{n/2} \\ &= (n^2 + n/2 + n^2 + 2n\sqrt{n/2} - 2n^2 - 2\sqrt{n/2}n)\theta^2 + (n + 2n\sqrt{n/2} + n - 2n\sqrt{n/2})\theta + n/2 \\ &= (n/2)\theta^2 + n/2 \end{aligned}$$

Then, adding in the coefficients, we get:

$$(1 + \theta^2)^{-1} (n + \sqrt{n/2})^{-2} (n/2)(\theta^2 + 1)$$

$$\begin{aligned} &= \frac{n/2}{(n + \sqrt{n/2})^2} \\ &= \frac{1}{(\sqrt{2/n} + \sqrt{2/n}\sqrt{n/2})^2} \\ &= \frac{1}{(1 + \sqrt{2n})^2} \end{aligned}$$

Part 3

$$W_3 = 1$$

This is straightforward. The risk is:

$$\frac{(\theta-1)^2}{\theta^2+1}$$

Part 4

$$W_4 = S_n^2$$

First, we note that $E[S^2] = \theta$, $E[(\theta - S^2)^2] = Var(S^2)$.

From a previous problem, we saw that $Var(S^2) = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2) = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta^2)$ (since $\theta_2 = \theta$).

From S620 notes, $\theta_4 = \theta(1 + 3\theta)$, so plugging this in, we get:

$$\begin{aligned} & \frac{1}{n}(\theta(1 + 3\theta) - \frac{n-3}{n-1}\theta^2) \\ &= \frac{1}{n}(\theta + \frac{\theta^2}{n-1}(3n\theta^2 - 3\theta^2 - n\theta^2 + 3\theta^2)) \\ &= \frac{1}{n}(\theta + \frac{2n\theta^2}{n-1}) \\ &= \frac{2\theta^2}{n-1} + \frac{\theta}{n} \end{aligned}$$

Then the risk becomes:

$$\frac{1}{\theta^2+1}(\frac{2\theta^2}{n-1} + \frac{\theta}{n})$$

Plot

```
library(ggplot2)
theme_set(theme_bw())

# sample size (arbitrary?)
n <- 5

# values of theta
theta <- seq(0, 5, .1)

# risk values
r1 <- theta / (n * (1 + theta ** 2))
```

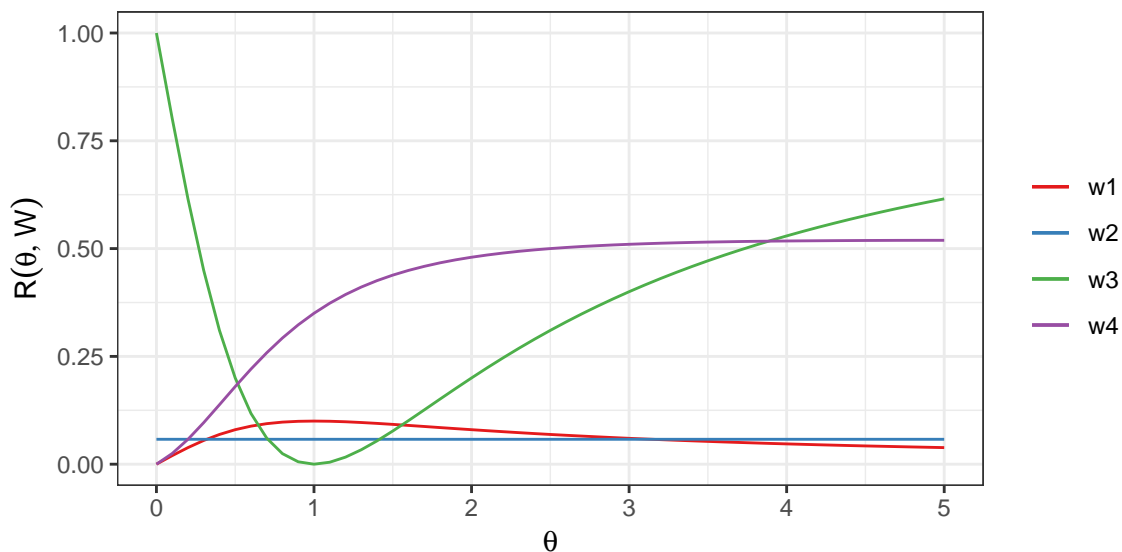


```

r2 <- (1 + sqrt(2 * n)) ** -2
r3 <- (theta - 1) ** 2 / (theta ** 2 + 1)
r4 <- (theta ** 2 + 1) ** -1 * (2 * theta ** 2 / (n - 1) + theta / n)

ggplot() +
  geom_line(aes(x = theta, y = r1, colour = 'w1')) +
  geom_line(aes(x = theta, y = r2, colour = 'w2')) +
  geom_line(aes(x = theta, y = r3, colour = 'w3')) +
  geom_line(aes(x = theta, y = r4, colour = 'w4')) +
  scale_colour_discrete(labels = c(expression(W[1]),
                                     expression(W[2]),
                                     expression(W[3]),
                                     expression(W[4]))) +
  labs(x = expression(theta), y = expression(R(theta, W)), colour = NULL) +
  scale_colour_brewer(palette = 'Set1')

```



The first estimator has minima at 0 and infinity.

The second estimator is flat, so it doesn't have a preference for any estimate.

The third estimator is minimized at $\theta = 1$. Note that this one does not depend on n (or any type of sample).

The fourth estimator is minimized at 0.

Not from text (III)

Problem 1

From class, we know that for a simple random variable X ,

$$E[X] = \sum_i^m x_i P(A_i)$$

If $E[X] \leq E[Y]$, then $E[X] - E[Y] \leq 0$. And again from class, we know:

$$E[X] - E[Y] = E[X - Y]$$

and

$$E[X - Y] = \sum_i^m \sum_j^n (x_i - y_j) P(A_i \cap B_j)$$

Each $P(A_i \cap B_j) \geq 0$ since P is a probability measure.

Since $X \leq Y$, each $x_i - y_j \leq 0$.

Therefore, this is a sum of negative numbers, which must be negative.

$$\sum_i^m \sum_j^n (x_i - y_j) P(A_i \cap B_j) \leq 0$$

So $E[X] - E[Y] \leq 0 \implies E[X] \leq E[Y]$.

Problem 2

We can write:

$$E[XY] = \sum_i^m \sum_j^n x_i y_j P(A_i \cap B_j)$$

Since X and Y are independent, each A_i and B_j are independent. Therefore, $P(A_i \cap B_j) = P(A_i)P(B_j)$, so the above becomes:

$$\begin{aligned} E[XY] &= \sum_i^m \sum_j^n x_i y_j P(A_i) P(B_j) \\ &= \sum_i^m x_i P(A_i) \sum_j^n P(B_j) \\ &= E[X] E[Y] \end{aligned}$$