# S721 HW7

*John Koo*

## From text (I)

### 7.11

### Part a

From HW6, we obtained $\hat{\theta}_{MLE} = -\frac{n}{\sum_i^n \log X_i}$.

In class, we noted that if $X_i \sim Beta(\theta, 1)$, $-\log X_i \sim Exp(1/\theta)$. The sum of exponentially distributed random variables is gamma distributed, so $-\sum_i^n \log X_i = Gamma(n, 1/\theta)$. Therefore, $\hat{\theta}_{MLE} \sim -n \times InvGamma(n, 1/\theta)$.

If $Y \sim InvGamma(n, 1/\theta)$, then the density function is $f(y \mid n, 1/\theta) = \frac{\theta^n}{\Gamma(n)}(\frac{1}{y})^{n+1}\exp(-\frac{\theta}{y})$.

$E[Y] = \int_0^\infty \frac{\theta^n}{\Gamma(n)} y y^{-n-1}\exp(-\theta/y)dy$
$= \frac{\theta^n}{\Gamma(n)} \int y^{-n}\exp(-\theta/y)dy$

Letting $u = y^{-1}$, we get $du = -y^{-2}dy$, so the expectation goes to:
$-\frac{\theta^n}{\Gamma(n)}\int u^{n-2}\exp(-\theta u)du = -\frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-1)}{\theta^{n-1}}\int \frac{\theta^{n-1}}{\Gamma(n-1)}u^{(n-1)-1}\exp(-\theta u)du = -\frac{\theta}{n-1}$ since the term inside the integral is just the density function for a gamma-distributed $U$.

Then $E[\hat{\theta}_{MLE}] = E[-nY] = -nE[Y] = n\frac{\theta}{n-1}$.

$E[Y^2] = \frac{\theta^n}{\Gamma(n)}\int y^{-n+1}\exp(-\theta/y)dy$.
Then again letting $u = y^{-1}$, $du = -y^{-2}$, we get:
$-\frac{\theta^n}{\Gamma(n)}\int u^{n-3}\exp(-\theta u)du$
$= -\frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-2)}{\theta^{n-2}}\int \frac{\theta^{n-2}}{\Gamma(n-2)}u^{(n-2)-1}\exp(-\theta u)du$
$= -\frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-2)}{\theta^{n-2}}$
$= -\frac{\theta^2}{(n-1)(n-2)}$.

Then $E[\hat{\theta}^2_{MLE}] = -n^2 E[Y^2] = n^2\frac{\theta^2}{(n-1)(n-2)}$.

Then $Var(\hat{\theta}_{MLE}) = \frac{n^2}{(n-1)(n-2)}\theta^2 - \frac{n^2}{(n-1)^2}\theta^2$
$= \theta^2(\frac{n^2(n-1)-n^2(n-2)}{(n-1)^2(n-2)})$
$= \theta^2(\frac{n^3-n^2-n^3+2n^2}{(n-1)^2(n-2)})$
$= \frac{n^2}{(n-1)^2(n-2)}\theta^2$.

Then $Var(\hat{\theta}_{MLE}) \sim \frac{1}{n} \to 0$ as $n \to \infty$.

### Part b

We saw from previous homework that $E[X_i] = \frac{\theta}{\theta+1}$, so setting this equal to $\bar{X}$, we get $\hat{\theta}_{MOM} = \frac{\bar{X}}{1-\bar{X}}$.

## 7.12

### Part a

$E[X] = \theta$, so $\hat{\theta}_{MOM} = \bar{X}$.

The likelihood function is $L(\theta \mid x) = \prod_i \theta^{x_i}(1-\theta)^{1-x_i}$
$\implies \ell(\theta \mid x) = \log\theta \sum_i x_i + \log(1-\theta)\sum_i(1-x_i)$
$\implies \ell'(\theta) = \frac{\sum_i x_i}{\theta} - \frac{\sum_i 1-x_i}{1-\theta}$, and setting this to zero, we get
$\hat{\theta}_{MLE} = \bar{X}$.

But this is only true if $\bar{X} \leq 1/2$. When $\bar{X} > 1/2$, we note that $\ell'(\theta) > 0$, so $L$ is an increasing function. We just set it at the highest possible value, $\hat{\theta}_{MLE} = 1/2$.

### Part b

The MSE can be broken up into bias and variance. The bias in this case is 0, so we just have the variance of $\bar{X} = Var(X)/n = \frac{\theta(1-\theta)}{n}$.

### Part c

When $\bar{X} \leq 1/2$, the estimators are equivalent. When $\bar{X} > 1/2$, we get an invalid estimate from the method of moments. So the MLE estimator is preferred.

## 7.15

### Part a

Writing down the likelihood function, we get
$L(\mu, \lambda \mid x) = \prod_i (\frac{\lambda}{2\pi x_i^3})^{1/2} \exp(-\lambda(x_i - \mu)^2/(2\mu^2 x_i))$, but it's easier to deal with the log-likelihood, which is
$\ell(\mu, \lambda \mid x) = \frac{n}{2}\log\lambda - \frac{n}{2}\log 2\pi - \frac{3}{2}\sum_i \log x_i - \sum_i \frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i}$.

When considering $\mu$, we can ignore everything that doesn't have $\mu$ in it, so we only need to consider
$-\frac{\lambda}{2}\sum_i \frac{(x_i-\mu)^2}{\mu^2 x_i}$
$= -\frac{\lambda}{2}(\sum_i x_i/\mu^2 - 2n/\mu + \sum_i x_i^{-1})$,
and if we differentiate w.r.t. $\mu$, we get
$-\frac{\lambda}{2}(-\mu^{-3}\sum_i x_i + 2n\mu^{-2})$.
Setting this to 0, we get
$\mu = \sum_i x_i/n$, so $\hat{\mu}_{MLE} = \bar{X}$.

For $\lambda$, we can plug in our estimate for $\mu = \bar{x}$, so, considering only the terms that depend on $\lambda$, we get
$\frac{n}{2}\log\lambda - \frac{\lambda}{2\bar{x}^2}\sum_i \frac{(x_i-\bar{x})^2}{x_i}$,
and if we find the derivative w.r.t. $\lambda$ and set it to zero, we get
$\frac{n}{2\lambda} - \frac{1}{2\bar{x}^2}\sum_i \frac{(x_i-\bar{x})^2}{x_i} = 0$
$\frac{n}{\lambda} = \frac{1}{\bar{x}^2}(\sum_i x_i - 2n\bar{x} + \bar{x}^2\sum_i x_i^{-1})$
$\frac{n}{\lambda} = \frac{1}{\bar{x}^2}(n\bar{x} - 2n\bar{x} + \bar{x}^2\sum_i x_i^{-1})$
$\frac{n}{\lambda} = \frac{1}{\bar{x}}(\bar{x}\sum_i x_i^{-1} - n)$
$\frac{n}{\lambda} = \sum_i x_i^{-1} - n\bar{x}^{-1}$
$\frac{n}{\lambda} = \sum_i (x_i^{-1} - \bar{x}^{-1})$

$\implies \lambda = n\left( \sum_i (x_i^{-1} - \bar{x}^{-1})\right)^{-1}$

So $\hat{\lambda}_{MLE} = n\left( \sum_i (X_i^{-1} - \bar{X}^{-1})\right)^{-1}$

## 2.34

Since the density functions for both $X$ and $Y$ are even, any odd-numbered moment for either must be 0.

If $r$ is even, $E[Y] = \frac{1}{6}3^{r/2} \times 2 = 3^{r/2-1}$, so the first five moments of $Y$ are 0, 1, 0, 3, 0.

From S620 notes, the moment generating function for the standard normal is $M_X(t) = \exp(t^2/2)$.
$M_X'(t) = t\exp(t^2/2)$
$M_X''(t) = \exp(t^2/2) + t\exp(t^2/2) = (1+t^2)\exp(t^2/2)$
$M_X'''(t) = 2t\exp(t^2/2) + (1+t^2)t\exp(t^2/2) = (t^3+3t)\exp(t^2/2)$
$M_X^{(4)}(t) = (3t+3)\exp(...) + (t^3+3t)t\exp(...) = (t^4+3t^2+3t+3)\exp(...)$

Then setting each of these to 0 (and noting that any odd-numbered moment is 0), we get 0, 1, 0, 3, 0.

## 3.16

### Part a

$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x} dx$
Letting $u = x^\alpha$ and $dv = e^{-x}$, we get $du = \alpha x^{\alpha-1}dx$ and $v = -e^{-x}$. Then the above integral is equal to
$-x^\alpha e^{-x}|_0^\infty + \alpha \int x^{\alpha-1}e^{-x}dx$
$= \alpha \int_0^\infty x^{\alpha-1}e^{-x}dx$
$= \alpha\Gamma(\alpha-1)$

### Part b

$\Gamma(1/2) = \int_0^\infty x^{-1/2}\exp(-x)dx$
Let $u = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}dx$. Then the above integral becomes
$\int_0^\infty 2\exp(-u^2)du$
$= 2\sqrt{\pi}\int_0^\infty \frac{1}{\sqrt{\pi}}\exp(-u^2)du$
$= 2\sqrt{\pi}\frac{1}{2}$
$= \sqrt{\pi}$

## 3.17

$E[X^\nu] = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1+\nu}\exp(-x/\beta)dx$
$= \frac{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}}{\Gamma(\alpha)\beta^\alpha}\int \frac{1}{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}} x^{(\alpha+\nu)-1}\exp(-x/\beta)dx$
$= \frac{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}}{\Gamma(\alpha)\beta^\alpha}$
$= \frac{\beta^\nu\Gamma(\alpha+\nu)}{\Gamma(\alpha)}$

## 3.20

### Part a

$E[X] = \frac{2}{\sqrt{2\pi}} \int_0^\infty x \exp(-x^2/2)dx.$
Let $u = x^2/2 \implies du = xdx$. Then the above integral becomes
$= \sqrt{\frac{2}{\pi}} \int_0^\infty \exp(-u)du$
$= \sqrt{\frac{2}{\pi}}.$

$E[X^2] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^2 \exp(-x^2/2)dx$
Let $u = x \implies du = dx$ and $dv = x\exp(-x^2/2)dx \implies v = -\exp(-x^2/2)$. Then the above becomes
$\sqrt{\frac{2}{\pi}}(-x\exp(-x^2/2)|_0^\infty + \int_0^\infty \exp(-x^2/2)dx)$
$= \sqrt{\frac{2}{\pi}}(0 + \sqrt{\frac{\pi}{2}})$
$= 1.$
Then $Var(X) = 1 - \frac{2}{\pi}$.

### Part b

The gamma distribution has the density function $f(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}\exp(-y/\beta)$.

Let $Y = g(X) = X^2$. Then $X = g^{-1}(Y) = Y^{1/2}$, so $\frac{d}{dy}g^{-1}(y) = \frac{1}{2}y^{-1/2}$. Then the density function for $Y$ is
$f_Y(y) = \sqrt{\frac{2}{\pi}}\exp(-y/2)\frac{1}{2}y^{-1/2}$
$= \frac{1}{\sqrt{\pi}2^{1/2}} y^{1/2-1}\exp(-y/2).$
Then since $\Gamma(1/2) = \sqrt{\pi}$, we can see that $\alpha = 1/2$ and $\beta = 2$.

## 3.24

### Part a

$f_X(x) = \frac{1}{\beta}\exp(-x/\beta)$

If $Y = X^{1/\gamma}$, then $X = Y^\gamma$ and $X' = \gamma Y^{\gamma-1}$.

Plugging this into $f_X$, we get
$\frac{\gamma}{\beta}y^{\gamma-1}\exp(-y^\gamma/\beta)$

To evaluate $\int_0^\infty \frac{\gamma}{\beta}y^{\gamma-1}\exp(-y^\gamma/\beta)dy$,
we set $x = y^\gamma \implies dx = \gamma y^{\gamma-1}dy$ to obtain
$\int_0^\infty \frac{1}{\beta}\exp(-x/\beta)dx = 1$
since it is just the density function for an exponential random variable.

$E[Y] = \int \frac{\gamma}{\beta}y^\gamma \exp(-y^\gamma/\beta)dy$
Letting $x = y^\gamma \implies dx = \gamma y^{\gamma-1}dy$ as before, we get
$\int \frac{1}{\beta}y\exp(-x/\beta)dx$
$= \frac{1}{\beta}\int x^{1/\gamma}\exp(-x/\beta)dx = \frac{1}{\beta}\int x^{(1/\gamma+1)-1}\exp(-x/\beta)dx$
$= \beta^{-1}\Gamma(1+1/\gamma)\beta^{1+1/\gamma}$
$= \beta^{1/\gamma}\Gamma(1+1/\gamma)$

$E[Y^2] = \int \frac{\gamma}{\beta}y^{\gamma+1}\exp(-y^\gamma/\beta)dy$
Letting $x = y^\gamma \implies dx = \gamma y^{\gamma-1}dy$ as before, we get

$\beta^{-1} \int y^2 \exp(-x/\beta)dx$
$= \beta^{-1} \int x^{2/\gamma} \exp(-x/\beta)dx$
$= \beta^{-1} \int x^{(1+2/\gamma)-1} \exp(-x/\beta)dx$
$= \beta^{-1}\Gamma(1+2/\gamma)\beta^{1+2/\gamma}$
$= \beta^{2/\gamma}\Gamma(1+2/\gamma)$
Then $Var(Y) = \beta^{2/\gamma}\Gamma(1+2/\gamma) - \beta^{2/\gamma}\Gamma(1+1/\gamma)^2$
$= \beta^{2/\gamma}(\Gamma(1+2/\gamma) - \Gamma(1+1/\gamma)^2)$

**Part c**

$Y = 1/X \implies X = 1/Y \implies X' = -Y^{-2}$

Then $f(y) = f(x(y))|x'(y)| = \frac{1}{\Gamma(a)b^a} y^{1-a} \exp(-\frac{1}{by})y^{-2}$
$= \frac{1}{\Gamma(a)b^a} \frac{1}{y^{a+1}} \exp(-\frac{1}{by})$

$\int_0^\infty \frac{1}{\Gamma(a)b^a} \frac{1}{y^{a+1}} \exp(-\frac{1}{by})dy$
Let $x = y^{-1} \implies dx = -y^{-2}dy$. Then the above becomes
$\int_\infty^0 \frac{1}{\Gamma(a)b^a} y^{-a-1}(-y^2) \exp(-x/b)dx$
$= \int_0^\infty \frac{1}{\Gamma(a)b^a} y^{-a+1} \exp(-x/b)dx$
$= \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-1} \exp(-x/b)dx$
$= 1$ since this is just the integral of $Gamma(a, b)$.

$E[Y] = \int_0^\infty \frac{1}{\Gamma(a)b^a} y^{-a} \exp(-\frac{1}{by})dy$
Again, let $x = y^{-1} \implies dx = -y^{-2}dy$. Then the above becomes
$\int_\infty^0 \frac{1}{\Gamma(a)b^a} x^a(-x^{-2}) \exp(-x/b)dx$
$= \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-2} \exp(-x/b)dx$
$= \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} \int_0^\infty \frac{1}{\Gamma(a-1)b^{a-1}} x^{(a-1)-1} \exp(-x/b)dx$
$= \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a}$
$= \frac{1}{(a-1)b}$

For $E[Y^2]$, we can extrapolate the results for $E[Y]$ to obtain:
$E[Y^2] = \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-3} \exp(-x/b)dx$
$= \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} \int_0^\infty \frac{1}{\Gamma(a-2)b^{a-2}} x^{(a-2)-1} \exp(-x/b)dx$
$= \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a}$
$= \frac{1}{(a-1)(a-2)b^2}$.
Then $Var(Y) = \frac{1}{(a-1)(a-2)b^2} - \frac{1}{(a-1)^2b^2}$
$= \frac{(a-1)-(a-2)}{(a-1)^2(a-2)b^2}$
$= \frac{1}{(a-1)^2(a-2)b^2}$

# Not from text (II)

## Problem 1

$R(\theta, W) = E[(\theta - W)^2]$
$= E[\theta^2 - 2\theta W + W^2]$
$= \theta^2 - 2\theta E[W] + E[W^2]$ (since $\theta$ is a constant)
$= \theta^2 - 2\theta E[W] + Var(W) + E[W]^2$ (since $Var(W) = E[W^2] - E[W]^2$)
$= Var(W) + (\theta^2 - 2\theta E[W] + E[W]^2)$

$= Var(W) + (\theta - E[W])^2$
$= Var(W) + (Bias(W))^2$

## Problem 2

### Part a

$\frac{1}{2n(n-1)} \sum_i \sum_j (X_i - X_j)^2$
$= \frac{1}{2n(n-1)} \sum_i \sum_j \big((X_i - \bar{X}) - (X_j - \bar{X})\big)^2$
$= \frac{1}{2n(n-1)} \Bigg( n \sum_i (X_i - \bar{X})^2 - 2 \sum_i (X_i - \bar{X}) \sum_j (X_j - \bar{X}) + n \sum_j (X_j - \bar{X})^2 \Bigg)$
Note that the middle term is 0 since $\sum_i (X_i - \bar{X}) = 0$. We also note that $\sum_i (X_i - \bar{X})^2 = (n-1)S^2$. Then we get:
$\frac{1}{2n(n-1)} \Bigg( n(n-1)S^2 + n(n-1)S^2 \Bigg)$
$= \frac{1}{2n(n-1)} 2n(n-1)S^2$
$= S^2$

### Part b

We can find this by computing $E[(S^2)^2] - (E[S^2])^2$, which means we hae to first compute $E[S^2]$ and $E[(S^2)^2]$.

First, we can rewrite:
$S^2 = \frac{1}{n-1} \sum_i (X_i - \bar{X})^2$
$= \frac{1}{n-1} \sum_i (X_i - \frac{1}{n} \sum_j X_j)^2$
$= \frac{1}{n-1} \sum_i (X_i^2 - \frac{2}{n} X_i \sum_j X_j + \frac{1}{n^2} (\sum_j X_j)^2)$
$= \frac{1}{n-1} (\sum_i X_i^2 - \frac{2}{n} \sum_i X_i \sum_j X_j + n \frac{1}{n^2} (\sum_j X_j)^2)$
$= \frac{1}{n-1} (\sum_i X_i^2 - \frac{1}{n} (\sum_i X_i)^2)$
$= \frac{1}{n(n-1)} (n \sum_i X_i - (\sum_i X_i)^2)$

We will, w.l.o.g., take $\theta_1 = E[X_i] = 0$.

Then $E[S^2] = \frac{1}{n(n-1)} (n \sum_i E[X_i^2] - E[(\sum_i X_i)^2])$
$= \frac{1}{n(n-1)} (n^2\theta_2 - n\theta_2)$
$= \frac{1}{n-1} (n\theta_2 - \theta_2)$
$= \theta_2$

Then $E[(S^2)^2] = \frac{1}{n^2(n-1)^2} E[(n \sum_i X_i^2 - (\sum_i X_i)^2)^2]$
$= \frac{1}{n^2(n-1)^2} E[n^2 (\sum_i X_i^2)^2 - 2n (\sum_i X_i^2)(\sum_i X_i)^2 + (\sum_i X_i)^4]$
which we can break down into three components inside the expectation.

First, $E[(\sum_i X_i^2)^2] = E[(X_1^2 + \cdots + X_n^2)^2]$
$= E[\sum_i X_i^4 + \sum_{i \neq j} X_i^2 X_j^2]$
$= \sum_i E[X_i^4] + \sum_{i \neq j} E[X_i^2] E[X_j^2]$ (since this is an iid sample)
$= n\theta_4 + n(n-1)\theta_2^2$ (since there are $n$ possible $i$'s and then $n-1$ possible $j$'s since $i \neq j$).

Second, $E[(\sum_i X_i^2)(\sum_i X_i)^2] = E[(X_1^2 + \cdots + X_n^2)(X_1^2 + \cdots + X_n^2 + \sum_{i \neq j} X_i X_j)]$
$= E[(\sum_i X_i^2)^2] + E[(\sum_i X_i^2)(\sum_{i \neq j} X_i X_j)]$
The second term is zero since $E[X_i X_j] = E[X_i]E[X_j]$, and we set $\theta_1 = 0$. Then we are left with $E[(\sum_i X_i^2)^2] = n\theta_4 + n(n-1)\theta_2^2$ as in the previous part.

Third, $E[(\sum_i X_i)^4] = E[(X_1 + \cdots + X_n)^4]$
$= E[\sum_i X_i^4 + 3 \sum_{i \neq j} X_i^2 X_j^2]$ (noting that each pair of $X_i$ and $X_j$ are independent so we can separate out the

expectation, and $E[X_i] = 0$)
$= n\theta_4 + 3n(n-1)\theta_2^2$

Putting it all together, we get:
$\frac{n^2(n\theta_4+n(n-1)\theta_2^2)-2n(n\theta_4+n(n-1)\theta_2^2)+n\theta_4+3n(n-1)\theta_2^2}{n^2(n-1)^2}$
$= \frac{n^3\theta_4+n^4\theta_2^2-n^3\theta_2^2-2n^2\theta_4-2n^3\theta_2^2+2n\theta_2^2+n\theta_4+3n^2\theta_2^2-3n\theta_2^2}{n^2(n-1)^2}$
$= \frac{(n^3-2n^2+n)\theta_4+(n^4-3n^3+5n^2-n)\theta_2^2}{n^2(n-1)^2}$
$= \frac{n(n-1)^2\theta_4+n(n-1)(n^2-2n+3)\theta_2^2}{n^2(n-1)^2}$
$= \frac{(n-1)\theta_4+(n^2-2n+3)\theta_2^2}{n(n-1)}$

Then $E[(S^2)^2] - (E[S^2])^2$
$= \frac{(n-1)\theta_4+(n^2-2n+3)\theta_2^2}{n(n-1)} - \theta_2^2$
$= \frac{(n-1)\theta_4+(n^2-2n+3)\theta_2^2-(n^2-n)\theta_2^2}{n(n-1)}$
$= \frac{(n-1)\theta_4-(n-3)\theta_2^2}{n(n-1)}$
$= \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2)$

### Part c

$Cov(\bar{X}, S^2) = \frac{1}{2n^2(n-1)}E[\sum_i X_i \sum_j \sum_k (X_j - X_k)^2]$
$= \frac{1}{2n^2(n-1)}E[\sum_i X_i \sum_{j\neq k}(X_j - X_k)^2]$
$= \frac{1}{2n^2(n-1)}E[\sum_{\substack{i\\ j\neq k}}(X_iX_j^2 - 2X_iX_jX_k + X_iX_k^2)]$
Note that the middle term, $2E[X_iX_jX_k]$, is zero since $E[X_j] = 0$ and we force $X_j \neq X_k$. Similarly, the first and third terms are nonzero only when $X_i = X_j$ and $X_i = X_k$ respectively. So we are left with:
$\frac{1}{2n^2(n-1)}\sum_{j\neq k}(E[X_j^3] + E[X_k^3]) = \frac{2n(n-1)}{2n^2(n-1)}E[X_i^3]$ (since there are $n(n-1)$ nonzero terms in each sum, and there are two sums)
$= \frac{1}{n}\theta_3$

This is nonzero when the third moment is nonzero.

## Problem 3

### Part 1

$W_1 = \bar{X}$

Then we have:
$E[\frac{(\theta-\bar{X})^2}{1+\theta^2}]$
$= \frac{1}{1+\theta^2}E[(\theta - \bar{X})^2]$
$= \frac{1}{1+\theta^2}(\theta^2 - 2\theta E[\bar{X}] + E[\bar{X}^2]) = \frac{1}{1+\theta^2}(\theta^2 - 2\theta^2 + Var(\bar{X}) + \theta^2)$
$= \frac{Var(\bar{X})}{1+\theta^2}$
$= \frac{Var(X)}{n(1+\theta^2)}$
$= \frac{\theta}{n(1+\theta^2)}$

### Part 2

$W_2 = \frac{\sum_i X_i + \sqrt{n/2}}{n+\sqrt{n/2}}$

Then we have:
$(1+\theta^2)^{-1}(n+\sqrt{n/2})^{-2}E[(n+\sqrt{n/2})\theta - n\bar{X} + \sqrt{n/2})^2]$
The part inside the expectation is, if we expand:
$n^2\theta^2 + \theta(n/2) + n^2\bar{X}^2 + n/2 + 2n\theta^2\sqrt{n/2} - 2n^2\theta\bar{X} + 2n\theta\sqrt{n/2} - 2\theta\sqrt{n/2}n\bar{X} + 2\theta(n/2) - 2n\bar{X}\sqrt{n/2}$
And taking the expectation of this (noting that $E[\bar{X}^2] = Var(\bar{X}) + \theta^2 = \theta/2 + \theta^2$), we get:
$n^2\theta^2 + \theta^2(n/2) + n^2(\theta/n) + n^2\theta^2 + n/2 + 2n\theta^2\sqrt{n/2} - 2n^2\theta^2 + 2n\theta\sqrt{n/2} - 2\theta^2\sqrt{n/2}n + 2\theta(n/2) - 2n\theta\sqrt{n/2}$
$= (n^2 + n/2 + n^2 + 2n\sqrt{n/2} - 2n^2 - 2\sqrt{n/2}n)\theta^2 + (n + 2n\sqrt{n/2} + n - 2n\sqrt{n/2})\theta + n/2$
$= (n/2)\theta^2 + n/2$

Then, adding in the coefficients, we get:
$(1+\theta^2)^{-1}(n+\sqrt{n/2})^{-2}(n/2)(\theta^2+1)$
$= \frac{n/2}{(n+\sqrt{n/2})^2}$
$= \frac{1}{(\sqrt{2/n}n + \sqrt{2/n}\sqrt{n/2})^2}$
$= \frac{1}{(1+\sqrt{2n})^2}$

**Part 3**

$W_3 = 1$

This is straightforward. The risk is:
$\frac{(\theta-1)^2}{\theta^2+1}$

**Part 4**

$W_4 = S_n^2$

First, we note that $E[S^2] = \theta$, $E[(\theta - S^2)^2] = Var(S^2)$.

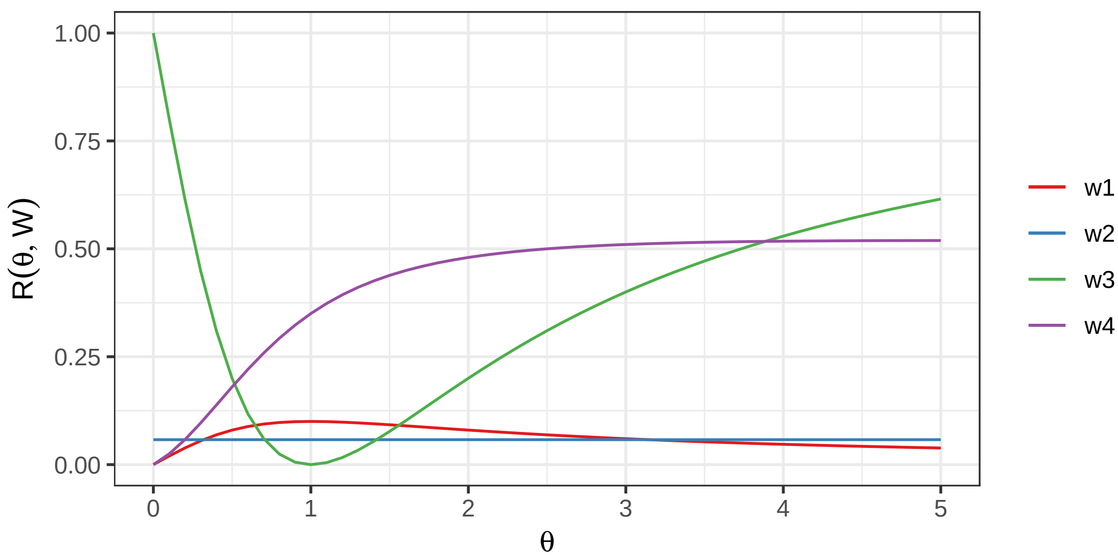
From a previous problem, we saw that $Var(S^2) = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2) = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta^2)$ (since $\theta_2 = \theta$).

From S620 notes, $\theta_4 = \theta(1+3\theta)$, so plugging this in, we get:
$\frac{1}{n}(\theta(1+3\theta) - \frac{n-3}{n-1}\theta^2)$
$= \frac{1}{n}(\theta + \frac{\theta^2}{n-1}(3n\theta^2 - 3\theta^2 - n\theta^2 + 3\theta^2))$
$= \frac{1}{n}(\theta + \frac{2n\theta^2}{n-1})$
$= \frac{2\theta^2}{n-1} + \frac{\theta}{n}$
Then the risk becomes:
$\frac{1}{\theta^2+1}(\frac{2\theta^2}{n-1} + \frac{\theta}{n})$

**Plot**

```
library(ggplot2)
theme_set(theme_bw())

# sample size (arbitrary?)
n <- 5

# values of theta
theta <- seq(0, 5, .1)

# risk values
r1 <- theta / (n * (1 + theta ** 2))
```

```r
r2 <- (1 + sqrt(2 * n)) ** -2
r3 <- (theta - 1) ** 2 / (theta ** 2 + 1)
r4 <- (theta ** 2 + 1) ** -1 * (2 * theta ** 2 / (n - 1) + theta / n)

ggplot() +
  geom_line(aes(x = theta, y = r1, colour = 'w1')) +
  geom_line(aes(x = theta, y = r2, colour = 'w2')) +
  geom_line(aes(x = theta, y = r3, colour = 'w3')) +
  geom_line(aes(x = theta, y = r4, colour = 'w4')) +
  scale_colour_discrete(labels = c(expression(W[1]),
                                   expression(W[2]),
                                   expression(W[3]),
                                   expression(W[4]))) +
  labs(x = expression(theta), y = expression(R(theta, W)), colour = NULL) +
  scale_colour_brewer(palette = 'Set1')
```




The first estimator has minima at 0 and infinity.
The second estimator is flat, so it doesn't have a preference for any estimate.
The third estimator is minimized at $\theta = 1$. Note that this one does not depend on $n$ (or any type of sample).
The fourth estimator is minimized at 0.

# Not from text (III)

## Problem 1

From class, we know that for a simple random variable $X$,

$$E[X] = \sum_{i}^{m} x_i P(A_i)$$

If $E[X] \leq E[Y]$, then $E[X] - E[Y] \leq 0$. And again from class, we know:

$$E[X] - E[Y] = E[X - Y]$$

and

$$E[X - Y] = \sum_i^m \sum_j^n (x_i - y_j)P(A_i \cap B_j)$$

Each $P(A_i \cap B_j \geq 0$ since $P$ is a probability measure.

Since $X \leq Y$, each $x_i - y_j \leq 0$.

Therefore, this is a sum of negative numbers, which must be negative.

$$\sum_i^m \sum_j^n (x_i - y_j)P(A_i \cap B_j) \leq 0$$

So $E[X] - E[Y] \leq 0 \implies E[X] \leq E[Y]$.

## Problem 2

We can write:

$$E[XY] = \sum_i^m \sum_j^n x_i y_j P(A_i \cap B_j)$$

Since $X$ and $Y$ are independent, each $A_i$ and $B_j$ are independent. Therefore, $P(A_i \cap B_j) = P(A_i)P(B_j)$, so the above becomes:

$$\begin{aligned} E[XY] &= \sum_i^m \sum_j^n x_i y_j P(A_i)P(B_j) \\ &= \sum_i^m x_i P(A_i) \sum_j^n y_j P(B_j) \\ &= E[X]E[Y] \end{aligned}$$