MATH-M463

Homework 9

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Problem 1

[Exercise 4 in 5.2]

Part a

$$\begin{aligned} P(X \leq x, Y \leq y) &= \int_0^x \int_0^y f(x,y) dx dy \\ &= \int_0^x 2e^{-2x} dx \int_0^y 3e^{-3y} dy \\ &= \boxed{(1 - e^{-2x})(1 - e^{-3y})} \end{aligned}$$

Part b

We can see that this is just a product of two exponential distributions. Therefore, $\boxed{f_X(x) = 2e^{-2x}}$

Part c

Likewise, $\boxed{f_Y(y) = 3e^{-3y}}$

Part d

$X \perp Y$ since $f(x,y) = f_X(x) f_Y(y)$

Problem 2

[Exercise 6 in 5.2]

Part a

$$\begin{aligned} P(Y > 2X) &= \int_{x=0}^{1/2} \int_{y=2x}^{1} 90(y-x)^8 dy dx \\ &= \int_0^{1/2} 10(y-x)^9 \Big|_{y=2x}^{1} dx \\ &= \int_0^{1/2} 10\big((1-x)^9 - x^9\big) dx \end{aligned}$$

$$= -(1-x)^{10} - x^{10}\Big|_0^{1/2}$$

$$= -\frac{1}{2^{10}} + 1 - \frac{1}{2^{10}}$$

$$= \boxed{1 - \frac{1}{2^9}}$$

Part b

$$f_X(x) = \int_{y=x}^{1} 90(y-x)^8 dy$$

$$= 10(y-x)^9\Big|_{y=x}^{1}$$

$$= \boxed{10(1-x)^9}$$

Problem 3

[Exercise 6 in 5.3]

Part a

Since X and Y are iid standard normal, $3X + 2Y \sim \mathcal{N}(0, 3^2 + 2^2)$. Then

$$P(3X + 2Y > 5) = P(\frac{3X + 2Y}{\sqrt{13}} > \frac{5}{\sqrt{13}})$$

$$= \boxed{1 - \Phi(\frac{5}{\sqrt{13}}) \approx .083}$$

Part b

$$P(\min(X, Y) < 1)$$

This is just the probability of the complement of the event that both X and Y are greater than 1. We can also use the fact that $X \perp Y$.

$$= 1 - P(X > 1, Y > 1)$$

$$= 1 - P(X > 1)P(Y > 1)$$

$$= \boxed{1 - (1 - \Phi(1))^2 \approx .975}$$

Problem 4

[Exercise 8 in 5.3]

Problem 5

Part a

$$\int f(x,y)dxdy = \lambda^3 \int_0^\infty e^{-\lambda y} \int_0^y x dx dy$$
$$= \frac{\lambda^3}{2} \int_0^\infty y^2 e^{-\lambda y} dy$$

Let:
$u = y^2$, $du = 2ydy$
$v = -\frac{1}{\lambda} e^{-\lambda y}$, $dv = e^{-\lambda y} dy$

Then we have:

$$= \frac{\lambda^3}{2} \left(-\frac{y^2}{\lambda} e^{-\lambda y} \bigg|_0^\infty + \frac{2}{\lambda} \int_0^\infty y e^{-\lambda y} dy \right)$$
$$= \lambda^2 \int_0^\infty y e^{-\lambda y} dy$$

Let:
$u = y$, $du = dy$
$v = -\frac{1}{\lambda} e^{-\lambda y}$, $dv = e^{-\lambda y} dy$

Then:

$$= \lambda^2 \left(-\frac{y}{\lambda} e^{-\lambda y} \bigg|_0^\infty + \int_0^\infty \frac{1}{\lambda} e^{-\lambda y} dy \right)$$
$$= \lambda \int_0^\infty e^{-\lambda y} dy$$
$$= -e^{-\lambda y} \bigg|_0^\infty$$
$$-(0-1) = \boxed{1}$$

Part b

$$f_X(x) = \int_x^\infty f(x,y) dy$$
$$= \int_x^\infty \lambda^3 x e^{-\lambda y} dy$$
$$= \lambda^3 x \int_x^\infty e^{-\lambda y} dy$$
$$= \lambda^3 x \left(-\frac{1}{\lambda} e^{-\lambda y} \bigg|_x^\infty \right)$$
$$= -\lambda^2 x (0 - e^{-\lambda x})$$
$$= \boxed{\lambda^2 x e^{-\lambda x}}$$

Part c

$$E[X] = \int_0^\infty x f_X(x)dx$$
$$= \lambda^2 \int_0^\infty x^2 e^{-\lambda x} dx$$

Let:
$u = x^2$, $du = 2xdx$
$v = -\frac{1}{\lambda}e^{-\lambda x}$, $dv = e^{-\lambda x}dx$

Then:

$$= \lambda^2 \bigg(-\frac{x^2}{\lambda}e^{-\lambda x}\bigg|_0^\infty + \frac{2}{\lambda}\int_0^\infty xe^{-\lambda x}dx \bigg)$$
$$= 2\lambda \int_0^\infty xe^{-\lambda x}dx$$

Let:
$u = x$, $du = dx$
$v = -\frac{1}{\lambda}e^{-\lambda x}$, $dv = e^{-\lambda x}dx$

Then:

$$= 2\lambda \bigg(-\frac{x}{\lambda}e^{-\lambda x}\bigg|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}dx \bigg)$$
$$= 2\int_0^\infty e^{-\lambda x}dx$$
$$= -\frac{2}{\lambda}(0-1)$$
$$= \boxed{\frac{2}{\lambda}}$$

Part d

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)}$$
$$= \frac{\lambda^3 xe^{-\lambda y}}{\lambda^2 xe^{-\lambda x}}$$
$$= \boxed{\lambda e^{-\lambda(y-x)}}$$

Part e

$$E[Y|X=1] = \int_1^\infty \lambda y e^{-\lambda(y-1)}dy$$
$$= \lambda e^{\lambda}\int_1^\infty ye^{-\lambda y}dy$$

Let:
$u = y$, $du = dy$
$v = -\frac{1}{\lambda}e^{-\lambda y}$, $dv = e^{-\lambda y}dy$

Then

$$
\begin{aligned}
&= \lambda e^{\lambda}\left(-\frac{y}{\lambda}e^{-\lambda y}\bigg|_1^{\infty} + \frac{1}{\lambda}\int_1^{\infty} e^{-\lambda y}dy\right) \\
&= \lambda e^{\lambda}\left(\frac{e^{-\lambda}}{\lambda} - \frac{1}{\lambda^2}e^{-\lambda y}\bigg|_1^{\infty}\right) \\
&= 1 - \frac{e^{\lambda}}{\lambda}(0 - e^{-\lambda}) \\
&= \boxed{1 + \frac{1}{\lambda}}
\end{aligned}
$$

Problem 6

Part a

First note that since $P(A \cup B) = P(A) + P(B) - P(AB)$, $P(AB) = P(A) + P(B) - P(A \cup B) = .2 + .6 - .7 = .1$

Then $P(A|B) = \frac{P(AB)}{P(B)} = \frac{.1}{.6} = \boxed{\frac{1}{6}}$

Part b

$P(A)P(B) = .12 > .1 = P(AB)$, so they are $\boxed{\text{negatively dependent}}$

Part c

$cov(X,Y) = P(AB) - P(A)P(B) = .1 - .12 = -.02$

Then note that $var(X) = E[X^2] - E[X]^2 = E[X] - E[X]^2 = P(A) - P(A)^2 = .2 - .04 = .16$

Similarly, $var(Y) = P(B) - P(B)^2 = .6 - .36 = .24$

Then $corr(X,Y) = \frac{-.02}{\sqrt{.16 \times .24}} \approx \boxed{-.102}$