S722 HW2

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To save on typing, I will denote $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$.

Part 1

8.12

a

Suppose $\hat{\mu}_{MLE} \leq 0$. Then $\lambda(X) = 1$, so we do not have to consider this scenario. On the other hand, if $\hat{\mu}_{MLE} > 0$, then $\hat{\mu}_0 = 0$ and $\hat{\mu} = \bar{X}$. So we have:

$\lambda(X) = \frac{(2\pi\sigma^2)^{-n/2} \exp(-\frac{1}{2\sigma^2} \sum X_i^2)}{(2\pi\sigma^2)^{-n/2} \exp(-\frac{1}{2\sigma^2} \sum (X_i - \bar{X})^2)}$
$= \big(\exp(\frac{1}{\sigma^2} \bar{X} \sum X_i - \frac{1}{2\sigma^2} \sum (\bar{X})^2)\big)^{-1}$
$= \exp(-\frac{n}{2\sigma^2} (\bar{X})^2)$

So we reject when $\exp(-\frac{n}{2\sigma^2}(\bar{X})^2) < c$
$\implies -\frac{n}{2\sigma^2}(\bar{X})^2 < c$
$\implies \frac{n}{\sigma^2}(\bar{X})^2 > c$
$\implies \sqrt{n}\bar{X}/\sigma > c$

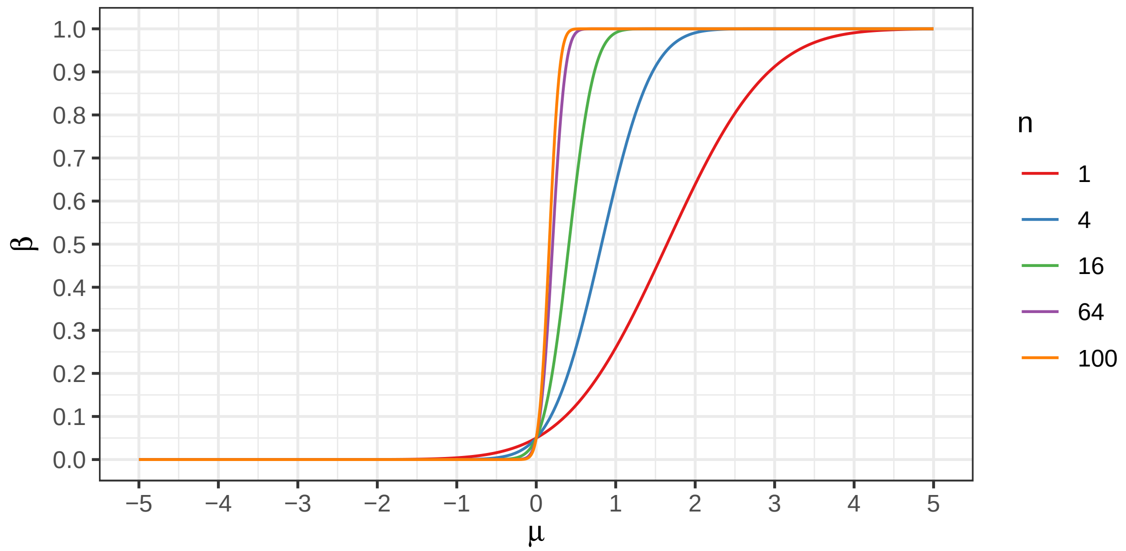
Since under H_0, $\frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim \mathcal{N}(0, 1)$ and $\mu_0 = 0$, $c = z_\alpha \approx 1.645$.

$\beta(\mu) = P(\text{reject } H_0 \mid \mu)$
$= P(Z > 1.645 - \frac{\mu}{\sigma/\sqrt{n}})$
$= 1 - P(Z \leq 1.645 - \frac{\mu}{\sigma/\sqrt{n}})$
$= 1 - \Phi(z_\alpha - \frac{\mu}{\sigma/\sqrt{n}})$

Here, I will use $\sigma = 1$.

```r
library(ggplot2)
import::from(magrittr, `%>%`)
theme_set(theme_bw())

alpha <- .05
sigma <- 1
n.vector <- c(1, 4, 16, 64, 100)

z_alpha <- qnorm(1 - alpha)

mu <- seq(-5, 5, .001)

out.df <- lapply(n.vector, function(n) {
  beta <- 1 - pnorm(z_alpha - mu / (sigma / sqrt(n)))
  dplyr::data_frame(n = factor(n, levels = n.vector), mu = mu, beta = beta)
}) %>%
  dplyr::bind_rows()
```

```r
ggplot(out.df) +
  geom_line(aes(x = mu, y = beta, colour = n)) +
  labs(x = expression(mu),
       y = expression(beta)) +
  scale_colour_brewer(palette = 'Set1') +
  scale_x_continuous(breaks = seq(-5, 5, 1)) +
  scale_y_continuous(breaks = seq(0, 1, .1))
```



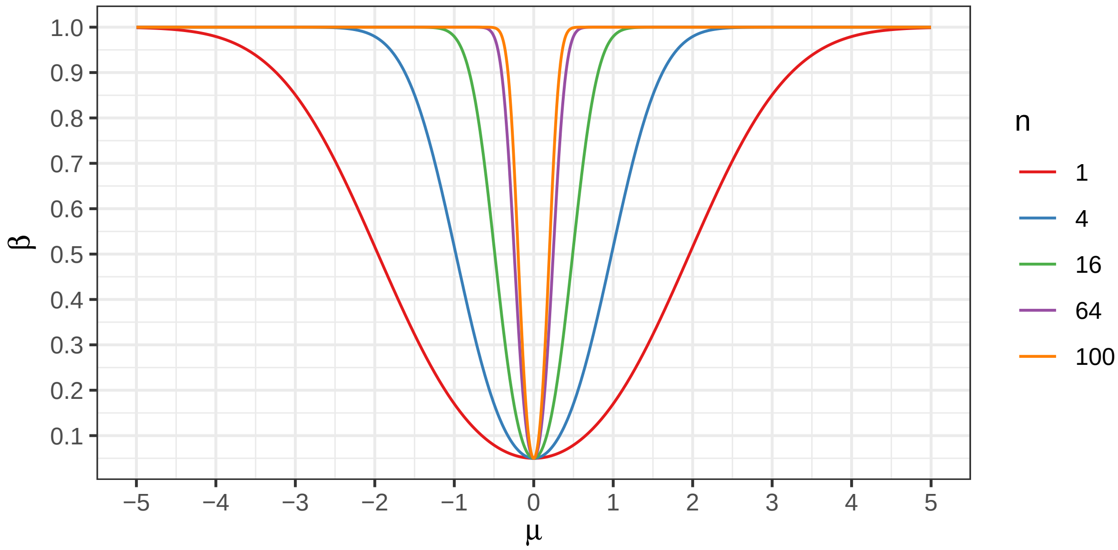
b

This is very similar to part (a) except $z_\alpha \to z_{\alpha/2}$ and it is two-sided. So now we have:

$\beta(\mu) = 1 - \left(\Phi(z_{\alpha/2} + \frac{\mu}{\sigma/\sqrt{n}}) - \Phi(-z_{\alpha/2} - \frac{\mu}{\sigma/\sqrt{n}})\right)$

Again, I will use $\sigma = 1$

```r
z_alpha_2 <- qnorm(1 - alpha / 2)

out.df <- lapply(n.vector, function(n) {
  beta <-
    1 -
    pnorm(z_alpha_2 +- mu / (sigma / sqrt(n))) +
    pnorm(-z_alpha_2 - mu / (sigma / sqrt(n)))
  dplyr::data_frame(n = factor(n, levels = n.vector), mu = mu, beta = beta)
}) %>%
  dplyr::bind_rows()

ggplot(out.df) +
  geom_line(aes(x = mu, y = beta, colour = n)) +
  labs(x = expression(mu),
       y = expression(beta)) +
  scale_colour_brewer(palette = 'Set1') +
  scale_x_continuous(breaks = seq(-5, 5, 1)) +
  scale_y_continuous(breaks = seq(0, 1, .1))
```



8.13

a

The level of ϕ_1 is $P_{\theta_0}(X_1 > .95) = 1 - P_{\theta_0}(X_1 \le .95) = 1 - .95 = .05$.

The level of ϕ_2 is $P_{\theta_0}(X_1 + X_2 > C)$

The density of $Y = X_1 + X_2$ is $f(y) = 2 - y$, so $P(Y > C) = \int_C^2 (2-y)dy$ (since $X_1 + X_2$ is bounded from above by $2\theta_0$)
$= 4 - 2 - 2C + C^2/2$
$= \frac{(2-C)^2}{2}$

Then $\alpha_2 = \frac{(2-C)^2}{2} \implies C = 2 - \sqrt{.1}$

b

$\beta_1(\theta) = P_\theta(X_1 > .95)$
$= 1 - P_\theta(X_1 \le .95)$
$= 1 - \int_\theta^{.95} dx_1$
$= \theta + .05$

Since $\beta_1 \in [0, 1]$, we need to truncate this function, so we get:

$$\beta_1(\theta) = \begin{cases} 0 & \theta < -.05 \\ \theta + .05 & \theta \in [-.05, .95] \\ 1 & \theta > .95 \end{cases}$$

We have the distribution of Y under H_0 in part (a). For arbitrary θ, we can just draw a triangle:

```
ggplot() +
  geom_segment(aes(x = 0, y = 0, xend = 1, yend = 2)) +
  geom_segment(aes(x = 0, y = 0, xend = 2, yend = 0)) +
  geom_segment(aes(x = 1, y = 2, xend = 2, yend = 0)) +
  labs(x = 'y', y = 'f(y)') +
```

```r
  scale_x_continuous(labels = c(expression(2 * theta),
                                expression(2 * theta + 2)),
                     breaks = c(0, 2)) +
  scale_y_continuous(labels = NULL)
```

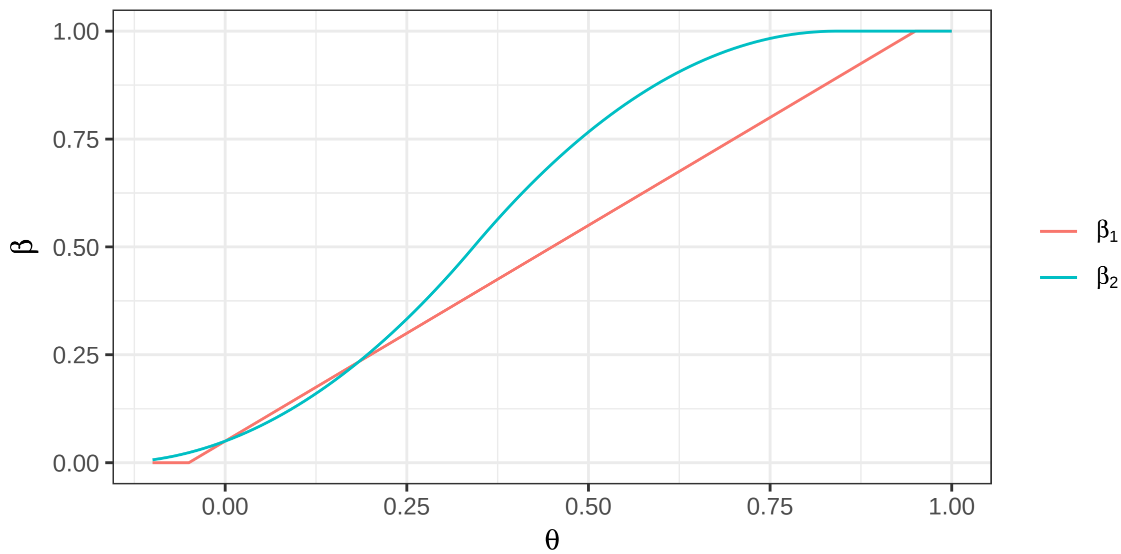
This is because the minimum value for Y is 2θ while the maximum value is $2\theta + 2$. Integrating to get an area of 1, we get:

$$f(y) = \begin{cases} 0 & y < 2\theta \\ y - 2\theta & y \in [2\theta, 2\theta + 1) \\ 2\theta + 2 - y & y \in [2\theta + 1, 2\theta + 2) \\ 0 & y \geq 2\theta + 2 \end{cases}$$

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \theta \geq C/2 - 1 \\ (2\theta + 2 - C)^2 & \theta \in (C/2 - 1, (C-1)/2] \\ 1 - (C - 2\theta)^2/2 & \theta \in ((C-1)/2, C/2] \\ 1 & \theta > C/2 \end{cases}$$

```r
C. <- 2 - sqrt(.1)

beta.1 <- function(theta) {
  if (theta <= -.05) {
    return(0)
  } else if (theta <= .95) {
    return(theta + .05)
  } else {
    return(1)
  }
}

beta.2 <- function(theta) {
  if (theta <= C. / 2 - 1) {
    return(0)
  } else if (theta < (C. - 1) / 2) {
    return((2 * theta + 2 - C.) ^ 2 / 2)
```

```r
  } else if (theta <= C. / 2) {
    return(1 - (C. - 2 * theta) ^ 2 / 2)
  } else{
    return(1)
  }
}

theta <- seq(-.1, 1, .01)
beta.1.vec <- sapply(theta, beta.1)
beta.2.vec <- sapply(theta, beta.2)

ggplot() +
  geom_line(aes(x = theta, y = beta.1.vec, colour = 'beta1')) +
  geom_line(aes(x = theta, y = beta.2.vec, colour = 'beta2')) +
  labs(x = expression(theta), y = expression(beta), colour = NULL) +
  scale_colour_discrete(labels = c(expression(beta[1]),
                                   expression(beta[2])))
```



c

The plot shows that ϕ_2 is more powerful in most of the range plotted, but ϕ_1 is more powerful from around $\theta = 0$ to $\theta = .15$.

d

We should reject H_1 if either X_1 or X_2 is greater than 1 since $P_{\theta=0}(X > 1) = 0$. So a more powerful test might be

$$\phi_3(X_1, X_2) = \begin{cases} 1 & X_1 + X_2 > C \text{ or } X_1 > 1 \text{ or } X_2 > 1 \\ 0 & \text{otherwise} \end{cases}$$

8.14

$\sum X_i \sim \mathcal{N}(np, np(1-p))$
$\implies \frac{\sum X_i - np}{\sqrt{np(1-p)}} \sim \mathcal{N}(0,1)$

Similar to problem 8.13, our test is simply $\sum X_i > c$.

We want $\beta(p = .49) = .01$ and $\beta(p = .51) = .99$.

Since we have standard normal probabilities, we have $\beta(p|n, c) = 1 - \Phi(\frac{c-np}{\sqrt{np(1-p)}})$

```r
f <- function(x) {
  c((x[1] - x[2] * .49) / sqrt(x[2] * .49 * .51),
    (x[1] - x[2] * .51) / sqrt(x[2] * .49 * .51)) -
    c(qnorm(.99), qnorm(.01))
}

pracma::fsolve(f, c(1000, 1000), maxiter = 1000)
```

```
$x
[1]  6762.162 13524.324

$fval
[1] 2.042961e-10 2.050871e-10
```

So we have $c \approx 6762.162$ and $n = 13525$.

8.15

The UMP test is of the form:
$\frac{L(\sigma_1)}{L(\sigma_0)} > k$
$\frac{(2\pi\sigma_1^2)^{-n/2} e^{-\sum x_i/(2\sigma_1^2)}}{(2\pi\sigma_0^2)^{-n/2} e^{-\sum x_i/(2\sigma_0)^2}}$
$= (\sigma_0/\sigma_1)^n \exp(\frac{1}{2}(\sigma_0^{-2} - \sigma_1^{-1}) \sum x_i^2) > k$
$\implies \sum x_i^2 > c$ since the exponential function is increasing.

Since $X_i \overset{iid}{\sim} \mathcal{N}(0, \sigma_0^2)$ under the null hypothesis, $\frac{1}{\sigma_0^2} \sum X_i \sim \chi_n^2 \implies c = \sigma_0^2 \chi_{n,\alpha}^2$ where α is the area under the curve to the right.

8.16

a

$1 = P(\text{reject } H_0) = P(\text{reject } H_0 \mid H_0) = P(\text{reject } H_0 \mid H_1)$, so size = power = 1.

b

$0 = P(\text{reject } H_0) = P(\text{reject } H_0 \mid H_0) = P(\text{reject } H_0 \mid H_1)$, so size = power = 0.

8.18

a

This is very similar to problem 8.12, and just copying that while changing θ_0 and c, we get:

$\beta(\theta) = 1 - \left(\Phi(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}) - \Phi(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}})\right)$

b

$\alpha = .05$, so $c = z_{\alpha/2}$.

$\beta(\theta_0 + \sigma) = 1 - \Phi(c - \sqrt{n}) + \Phi(-c - \sqrt{n}) = .75$

```
c. <- qnorm(.975)
f <- function(n) {
  1 - pnorm(c. - sqrt(n)) + pnorm(-c. - sqrt(n)) - .75
}

uniroot(f, c(1, 100))
```

```
$root
[1] 6.940316

$f.root
[1] 3.04201e-07

$iter
[1] 10

$init.it
[1] NA

$estim.prec
[1] 6.103516e-05
```

So $n = 7$.

8.19

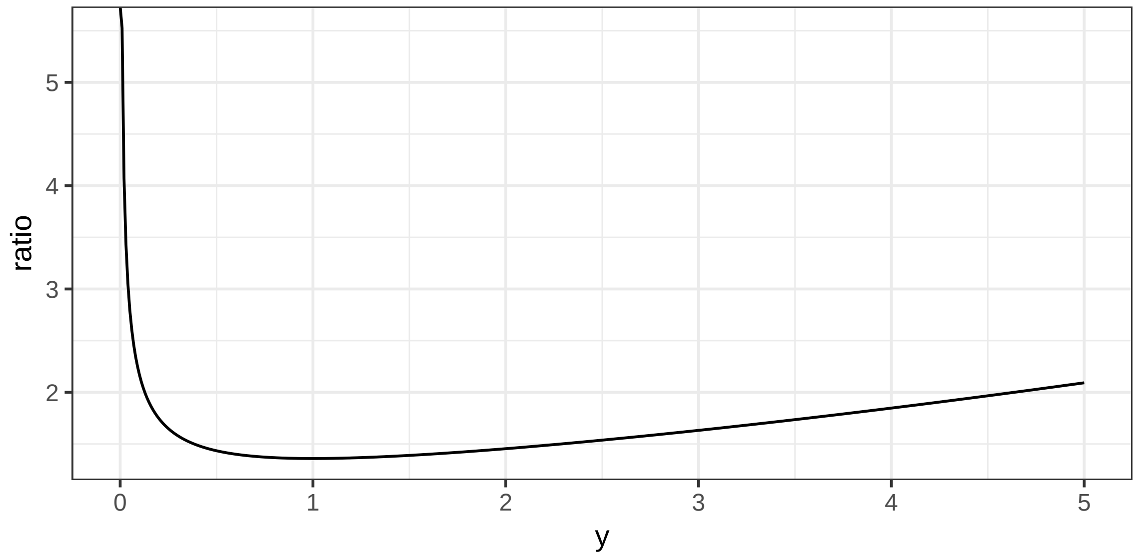
$Y = X^\theta$
$\implies X = Y^{1/\theta}$
$\implies X' = \frac{Y^{1/\theta - 1}}{\theta}$
$\implies f(y) = \frac{1}{\theta} y^{1/\theta - 1} e^{-y^{1/\theta}}$ (inverse gamma?)

The UMP test is to reject when $\frac{L(2)}{L(1)} > k$.

$\frac{L(2)}{L(1)} = \frac{\frac{1}{2} y^{1/2-1} e^{-y^{1/2}}}{y^0 e^{-y}}$
$= \frac{1}{2} y^{-1/2} e^{y^{1/2}}$

```
y <- seq(0, 5, .01)
ratio <- .5 * y ^ -.5 * exp(y ^ .5)
```

```r
ggplot() +
  geom_line(aes(x = y, y = ratio))
```



So if the likelihood ratio is greater than some k, then that means $y < c_1$ or $y > c_2$ where the value of the ratio is equivalent at $y = c_1$ and $y = c_2$.

In addition, for a 0.1-level test, we need $P_{\theta_0}(Y < c_1) + P_{\theta_0}(Y > c_2) = .1$
$\implies .1 = 1 - e^{-c_1} + e^{-c_2}$

```r
f <- function(x) {
  c(.5 * x[1] ^ -.5 * exp(x[1] ^ .5) - .5 * x[2] ^ -.5 * exp(x[2] ^ .5),
    1 - exp(-x[1]) + exp(-x[2]) - .1)
}

sol <- pracma::fsolve(f, c(.5, 1.5), maxiter = 1000, tol = .Machine$double.eps)
print(sol$x)
```

```
[1] 0.09982121 5.29847588
```

The type II error is given by:

$\int_{c_1}^{c_2} \frac{1}{2} y^{-1/2} e^{-y^{1/2}} dy$
Let $u = y^{1/2} \implies du = \frac{1}{2} y^{-1/2} dy$
So the integral becomes $\int_{\sqrt{c_1}}^{\sqrt{c_2}} e^{-u} du$
$= e^{-c_1} - e^{-c_2}$
≈ 0.9

8.20

We can first compute the likelihood ratios:

x	$L(H_1)/L(H_0)$
1	6
2	5
3	4

x	$L(H_1)/L(H_0)$
4	3
5	2
6	1
7	0.84

The likelihood ratio is decreasing, so the test is $L(H_1)/L(H_0) < c$.

If we want a size-.04 test, then $P(X \leq c \mid H_0) = .04 \implies c = 4$.

The type II error probability is $P(X > 4 \mid H_1) = .2 + .1 + .79 = .82$.

Part 2

Given:

- Simple hypothesis test
  - $H_0 : \theta = \theta_0$
  - $H_1 : \theta = \theta_1$
- Joint pdf $f(\overrightarrow{x} \mid \theta_i)$
- Rejection region R such that for some $k \geq 0$,
  - $\overrightarrow{x} \in R$ if $f(\overrightarrow{x}|\theta_1) > kf(\overrightarrow{x}|\theta_0)$
  - $\overrightarrow{x} \in R$ if $f(\overrightarrow{x}|\theta_1) < kf(\overrightarrow{x}|\theta_0)$
- $\alpha = P_{\theta_0}(\overrightarrow{X} \in R)$

Then:

- This is a UMP level-α test
- If such a test exists for $k > 0$, then every UMP level-α test is a size-α test and every UMP level-α test has this type of rejection region (except perhaps on a set A such that $P_{\theta_0}(\overrightarrow{X} \in A) = P_{\theta_1}(\overrightarrow{X} \in A) = 0$).

Proof:

- Let $\phi(x)$ be a test function that satisfies the requirements with power function β.
- Let $\phi^*(x)$ be a test function for a level-α test with power function β^*.
- Note that
  - $f(x|\theta_1) > kf(x|\theta_0) \implies \phi(x) = 1$
  - $f(x|\theta_1) < kf(x|\theta_0) \implies \phi(x) = 0$
- Therefore, $\phi(x) - \phi^*(x)$ and $f(x|\theta_1) - kf(x|\theta_0)$ must have the same sign
- So $(\phi(x) - \phi^*(x))(f(x|\theta_1) - kf(x|\theta_0)) \geq 0$.
  $\implies 0 \leq \int(\phi(x) - \phi^*(x))(f(x|\theta_1) - kf(x|\theta_0))dx$
  $= \beta(\theta_1) - \beta^*(\theta_1) - k(\beta(\theta_0) - \beta^*(\theta_1))$ since $E[\phi(X)|\theta_i] = \beta(\theta_i)$.
- $\beta(\theta_0) = \alpha$
- $\beta^*(\theta_0) \leq \alpha$
- Therefore, $\beta(\theta_0) - \beta^*(\theta_0) \geq 0$
  $\implies 0 \leq \beta(\theta_1) - \beta^*(\theta_1) - k(\beta(\theta_0) - \beta^*(\theta_1)) \leq \beta(\theta_1) - \beta^*(\theta_1)$
- So ϕ has greater power than ϕ^* for any α and $\theta_1 \implies \phi(x)$ is UMP level-α
- Suppose that $\phi^*(x)$ is a UMP level-α test.
  Then $\beta(\theta_1) = \beta^*(\theta_1)$.

- Then that would imply $\beta(\theta_0) - \beta^*(\theta_0) \leq 0$, which can only be true when $\beta(\theta_0) - \beta^*(\theta_0) \leq 0$, which can only be true if $\phi^*(x)$ is a size-α test, which implies that ϕ and ϕ^* are the same test, so ϕ is a UMP level-α test.

Part 3

a

$\frac{L(2)}{L(1)} = \frac{2^{-n}\exp(-\frac{1}{2}\sum x_i)}{\exp(-\sum x_i)} > c$
$\implies \exp(\frac{1}{2}\sum x_i) > c'$
$\implies \sum x_i > c''$

Under the null hypothesis ($\theta = 1$), $\sum X_i \sim Gamma(n, 1)$.

A gamma-distributed random variable is χ^2-distributed with $2n$ degrees of freedom when the second parameter is 2. Using a transformation, we get that then $2\sum X_i \sim \chi^2_{2n}$, so $\alpha = P(\chi^2_{2n} > 2c) \implies c = \frac{1}{2}\chi^2_{2n,\alpha}$ (where α is the area to the right).

b

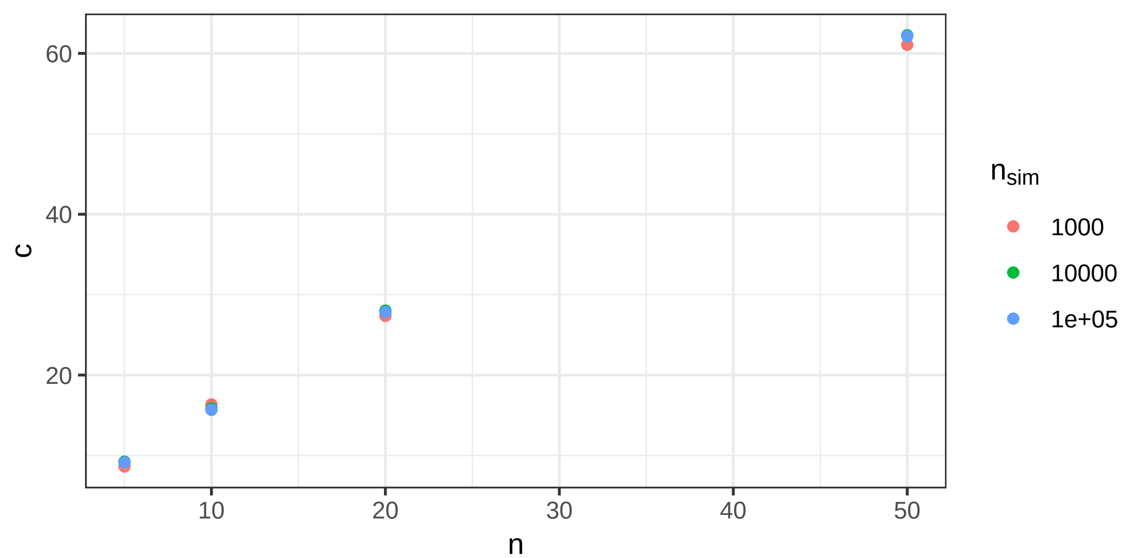
Using $\alpha = .05$

```
import::from(foreach, foreach, `%dopar%`)
import::from(xtable, xtable)

doMC::registerDoMC(4)

alpha <- .05

n.vector <- c(5, 10, 20, 50)
nsim.vector <- c(1e3, 1e4, 1e5)

theta.0 <- 1
theta.1 <- 2
```

```
out.df <- foreach(n = n.vector, .combine = dplyr::bind_rows) %dopar% {
  lapply(nsim.vector, function(nsim) {
    c.empirical <- sapply(seq(nsim), function(i) {
      rexp(n, rate = 1 / theta.0) %>%
        sum()
    }) %>%
      quantile(1 - alpha)
    dplyr::data_frame(n = n, nsim = nsim, c.empirical = c.empirical)
  })
}

out.df %>%
  xtable() %>%
  print(include.rownames = FALSE)
```

n	nsim	c.empirical
5.00	1000.00	8.62
5.00	10000.00	9.23
5.00	100000.00	9.13
10.00	1000.00	16.35
10.00	10000.00	15.85
10.00	100000.00	15.68
20.00	1000.00	27.35
20.00	10000.00	28.01
20.00	100000.00	27.83
50.00	1000.00	61.07
50.00	10000.00	62.25
50.00	100000.00	62.15

```
ggplot(out.df) +
  geom_point(aes(x = n, y = c.empirical, colour = factor(nsim))) +
  labs(y = 'c', colour = expression(n[sim]))
```



c

Exact

$\beta(\theta_1) = P_{\theta_1}(\sum X_i > \frac{1}{2}\chi^2_{2n,\alpha})$

Under H_1, $\frac{2\sum X_i}{\theta_1} = \sum X_i \sim \chi^2_{2n}$.

So $\beta(\theta_1) = P(\chi^2_{2n} > \frac{1}{2}\chi^2_{2n,\alpha})$

```
power.df <- dplyr::data_frame(n = n.vector) %>%
  dplyr::mutate(power = 1 - pchisq(q = .5 * qchisq(1 - alpha, 2 * n),
                                   2 * n)) %>%
  xtable() %>%
  print(include.rownames = FALSE)
```

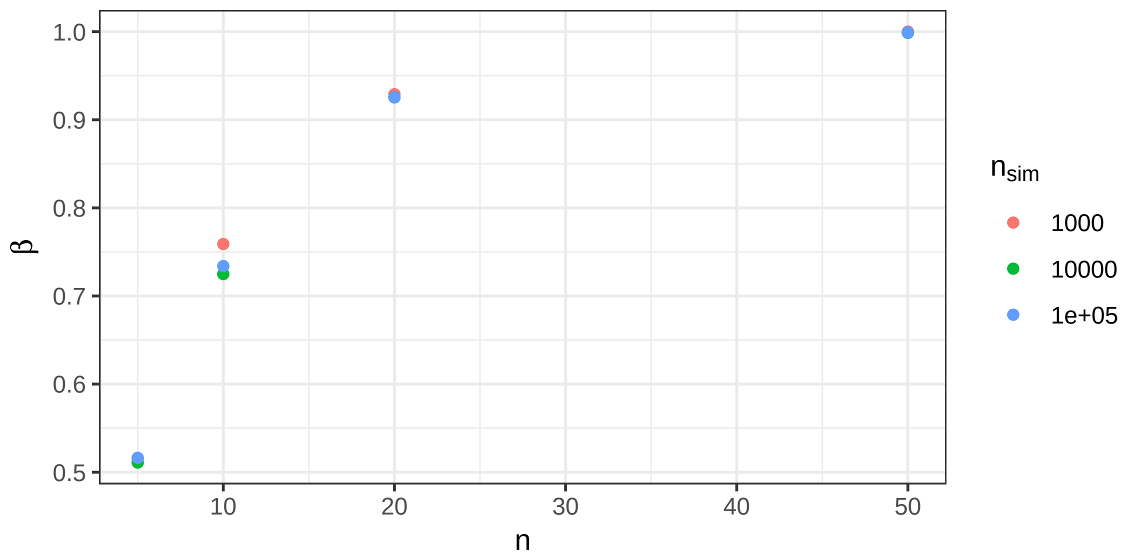
n	power
5.00	0.52
10.00	0.73
20.00	0.93
50.00	1.00

Monte Carlo

```
out.df <- foreach(n = n.vector, .combine = dplyr::bind_rows) %dopar% {
  c. <- .5 * qchisq(1 - alpha, 2 * n)
  lapply(nsim.vector, function(nsim) {
    power <- sapply(seq(nsim), function(i) {
      rexp(n, rate = 1 / theta.1) %>%
        sum()
    }) %>%
      magrittr::is_less_than(c.) %>%
      mean() %>%
      {1 - .}
    dplyr::data_frame(n = n, nsim = nsim, power = power)
  })
}

out.df %>%
  xtable() %>%
  print(include.rownames = FALSE)
```

n	nsim	power
5.00	1000.00	0.52
5.00	10000.00	0.51
5.00	100000.00	0.52
10.00	1000.00	0.76
10.00	10000.00	0.72
10.00	100000.00	0.73
20.00	1000.00	0.93
20.00	10000.00	0.93
20.00	100000.00	0.93
50.00	1000.00	1.00
50.00	10000.00	1.00
50.00	100000.00	1.00

```
ggplot(out.df) +
  geom_point(aes(x = n, y = power, colour = factor(nsim))) +
  labs(y = expression(beta), colour = expression(n[sim]))
```



1.0
0.9
0.8
0.7
0.6
0.5
β
10
20
30
40
50
n
n_{sim}
1000
10000
1e+05