MATH-M463

Homework 8

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Problem 1

Part a

$\lambda(t) = t^2$
$G(t) = \exp(-\int_0^t \lambda(u)du) = \exp(-\int_0^t u^2 du) = \exp(-t^3/3)$.

$P(T > 10) = G(10) = \boxed{\exp(-1000/3)}$

Part b

$f(t) = -\frac{dG}{dt} = -\frac{d}{dt}\exp(-t^3/3)$
$= -(-t^2)\exp(-t^3/3) = \boxed{t^2 \exp(-t^3/3)}$

Part c

$E[T^3] = \int_0^\infty t^5 \exp(-t^3/3)dt$

Let:
$u = t^3$, $du = 3t^2 dt$
$v = -\exp(-t^3/3)$, $dv = t^2 \exp(-t^3/3)dt$

Then we get $-t^3 \exp(-t^3/3)|_0^\infty + 3\int_0^\infty t^2 \exp(-t^3/3)dt$

The first term goes to 0 and so we're left with $3\int_0^\infty t^2 \exp(-t^3/3)dt$. Then let:

$u = -t^3/3$
$du = -t^2 dt$

Then this becomes $-3\int_0^\infty \exp(u)du$
$= -3\exp(u)|_{u=0}^\infty$
$= -3(0-1) = \boxed{3}$

Problem 2

Part a

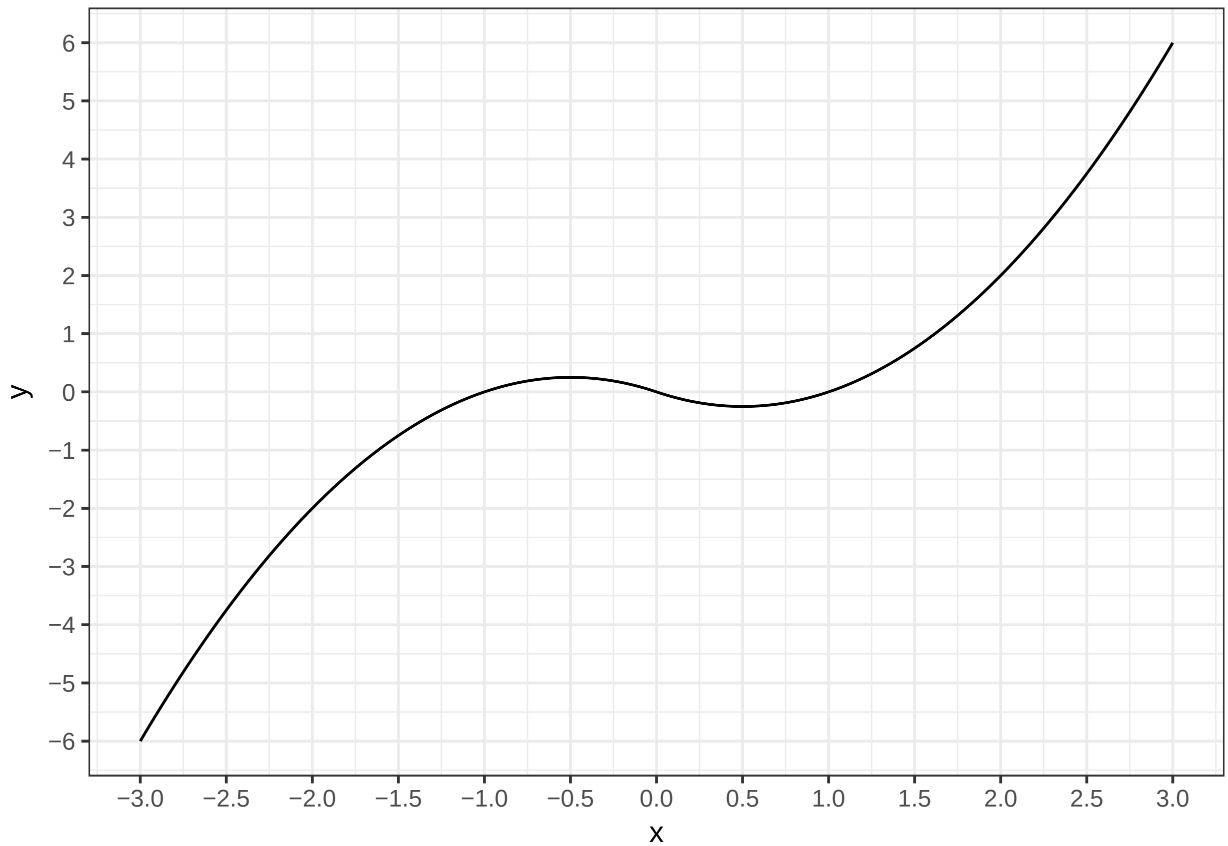
$P(T_2 > 16) = P(N_{16} < 1)$
$= e^{-8/3}(1 + 8/3) = \boxed{\frac{11}{3}e^{-8/3} \approx 0.255}$

Part b

This is equivalent to an example from the book/lectures, so we have $m = x/\lambda = 6x$ and $x \approx 1.675$, so $m \approx 10.05$ or $\boxed{\text{10 weeks}}$.

Problem 3

Part a



Part b

$$\frac{dy}{dx} = \begin{cases} 2x - 1 & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

Part c

Solving for x wrt y for each of the cases, we get:

$$x = \frac{1 \pm \sqrt{1 \pm 4y}}{2}$$

With $(+,+)$ when $x \geq 1/2$, $(-,+)$ when $0 \leq x < 1/2$, $(+,-)$ when $-1/2 \leq x < 0$, and $(-,-)$ when $x < 1/2$.

Problem 4

Part a

When $y > 1/4$, $y(x) = x^2 - x$ and $x(y) = \frac{1+\sqrt{1+4y}}{2}$. Then $f_Y(y) = f_X(x)/|\frac{dy}{dx}| = f_X(x)/|2x-1|$
$= \boxed{f_X(\frac{1+\sqrt{1+4y}}{2}) \Big/ \sqrt{1+4y}}$

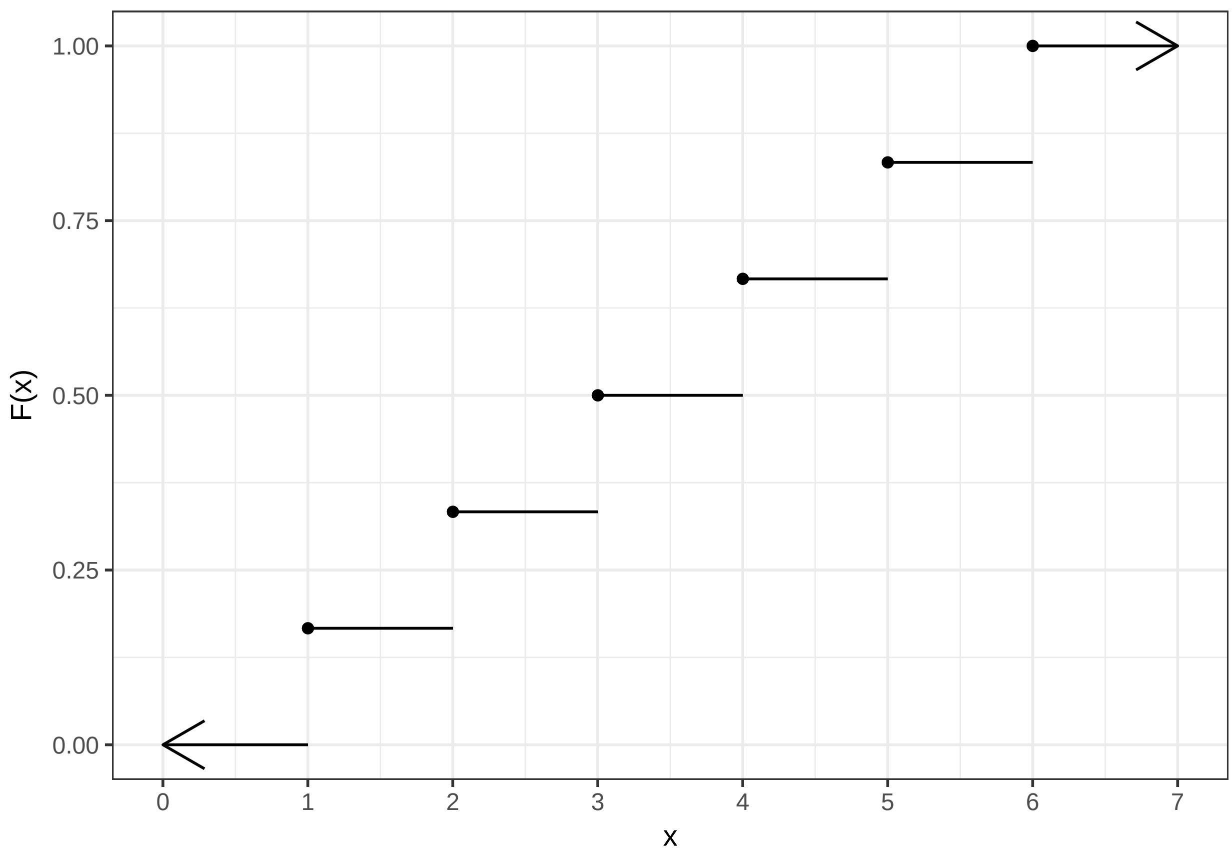
Part b

When $0 < y < 1/4$, $x \in [-1, 0] \cup [1, \infty)$. In terms of y, this defines 3 of the regions specified in the previous problem. Then:

$$\boxed{f_Y(y) = \frac{f_X(1/2 + \sqrt{1+4y}/2)}{\sqrt{1+4y}} + \frac{f_X(1/2 + \sqrt{1-4y}/2) + f_X(1/2 - \sqrt{1-4y}/2)}{\sqrt{1-4y}}}$$

Problem 5

Part a



Part b

$F_{max}(x) = P(X_1 \leq x)^4$, so

$$F_{max}(x) = \begin{cases} 0 & \text{if } x < 1 \\ \lfloor \frac{x}{6} \rfloor^4 & \text{if } 1 \leq x < 6 \\ 1 & \text{otherwise} \end{cases}$$

Part c

$F_{min}(x) = (1 - P(X_1 \leq x))^4$, so

$$F_{min}(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - (1 - \lfloor \frac{x}{6} \rfloor)^4 & \text{if } 1 \leq x < 6 \\ 1 & \text{otherwise} \end{cases}$$

Problem 6

Part a

$f(x) = \frac{d}{dx} \frac{4}{\pi} \arctan(x)$
$= \boxed{\dfrac{4}{\pi} \dfrac{1}{x^2 + 1}}$

Part b

$$\boxed{f_{(3)}(x) = 5 \frac{4}{\pi} \frac{1}{x^2 + 1} \binom{4}{2} (\frac{4}{\pi} \arctan(x))^2 (1 - \frac{4}{\pi} \arctan(x))^2}$$