STAT-S620

Assignment 5

John Koo

3.4.3

Part a

```r
import::from(magrittr, `%>%`)

expand.grid(x = seq(-2, 2), y = seq(-2, 2)) %>%
  dplyr::mutate(f = abs(x + y)) %>%
  .$f %>%
  sum()
```

```
[1] 40
```

Therefore, $\boxed{c = 1/40}$.

Part b

$f(0, -2) = 2/40 = \boxed{1/20}$.

Part c

$P(X = 1) = \sum_y f(1, y)$

```r
expand.grid(x = seq(-2, 2), y = seq(-2, 2)) %>%
  dplyr::mutate(f = abs(x + y) / 40) %>%
  dplyr::filter(x == 1) %>%
  .$f %>%
  sum()
```

```
[1] 0.175
```

Therefore, $\boxed{P(X = 1) = 0.175 = 7/40}$.

Part d

We can create a new variable $Z = |X - Y|$. Then we want $P(Z \leq 1)$.

```r
expand.grid(x = seq(-2, 2), y = seq(-2, 2)) %>%
  dplyr::mutate(z = abs(x - y)) %>%
  dplyr::mutate(f = abs(x + y) / 40) %>%
  dplyr::filter(z <= 1) %>%
  .$f %>%
  sum()
```

```
[1] 0.7
```

Therefore, $P(|X-Y| \leq 1) = \boxed{0.7 = 7/10}$.

3.4.4

Part a

$\int_{x=0}^{2}\int_{y=0}^{1} cy^2 dxdy = 2c\int_0^1 y^2 dy = \frac{2c}{3}$. Therefore, $\boxed{c = 3/2}$.

Part b

$P(X+Y > 2) = \int_{x=1}^{2} dx \int_{y=2-x}^{1} 3/2y^2 dy = \int_{x=1}^{2} y^3/2|_{y=2-x}^{1} dx = 1/2\int_1^2 1-(2-x)^3 dx = x/2+(2-x)^4/8\big|_1^2 = 1 - 1/2 - 1/8 = \boxed{3/8}$.

Part c

$P(Y < 1/2) = \int_0^2 dx \int_0^{1/2} 3y^2/2dy = \int_0^2 1/2 * (1/2)^3 dx = 2/16 = \boxed{1/8}$.

Part d

$P(X \leq 1) = \int_0^1 dx \int_0^1 3y^2/2dy = \int_0^1 1/2dx = \boxed{1/2}$.

Part e

$P(X = 3Y) = \boxed{0}$ since this is a continuous distribution.

3.5.4

Part a

$$f_X(x) = \int_0^{1-x^2} 15x^2/4dy = \boxed{15x^2(1-x^2)/4}$$

For f_Y, we have to consider the minimum and maximum values of x. Since $y \leq 1-x^2$, we have $y \in [-\sqrt{1-y}, \sqrt{1-y}]$.

$$\begin{aligned} f_Y(y) &= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} 15x^2/4dx \\ &= \frac{5}{4}2(\sqrt{1-y})^3 \\ &= \boxed{\frac{5}{2}(1-y)^{3/2}} \end{aligned}$$

Part b

$f_X(x)f_Y(y) = \frac{75}{8}(1-x^2)(1-y)^{3/2} \neq f(x,y)$. Therefore, X and Y are not independent.

3.5.5

Part a

Since X and Y are independent, $P(X=x, Y=y) = P(X=x)P(Y=y) = \boxed{p_x p_y}$.

Part b

Since X and Y are independent, $P(X=Y) = \sum_{i=0}^{3} P(X=i)^2$. Then this is $.1^2 + .2^2 + .4^2 + .3^2 = \boxed{.3}$.

Part c

$P(X>Y) = \sum_{x>y} P(X=x)P(Y=y) = \frac{1}{2}\sum_{x\neq y} P(X=x)P(Y=y) = \frac{1}{2}\big(1 - \sum_{x=y} P(X=x, Y=y)\big)$
$= \frac{1}{2}\big(1 - P(X=Y)\big)$

We already found $P(X=Y)$ in part (b). Therefore, $P(X>Y) = \frac{1}{2}(1-.3) = \boxed{.35}$.

3.6.8

Part a

$\int_0^1 dy \int_{.8}^1 \frac{2}{5}(2x+3y)dx = \int_0^1 \frac{2}{5}(1-\frac{16}{25}) + \frac{6}{5}\frac{1}{5}ydy = \int_0^1 \frac{18}{125} + \frac{6}{25}ydy = \frac{18}{125} + \frac{3}{25} = \boxed{33/125}$

Part b

We first need the marginal of Y. $f_Y(y) = \int_0^1 \frac{2}{5}(2x+3y)dx = \frac{2}{5} + \frac{3}{5}y$ for $y \in [0,1]$.

Then $f_{X|Y}(x|y) = \frac{2x+3y}{1+3y}$ for x and $y \in [0,1]$. Then $\boxed{f_{X|Y}(x|y=.3) = \dfrac{2x+.9}{1.9}}$ for $x \in [0,1]$.

Part c

We need the marginal of X. $f_X(x) = \int_0^1 \frac{2}{5}(2x+3y)dy = \frac{2}{5}(2x+\frac{3}{2})$

Then $f_{Y|X}(y|x) = \frac{2x+3y}{2x+\frac{3}{2}}$, and $f_{Y|X}(y|x=.3) = \frac{3y+.6}{2.1}$.

Then $P(Y>.8|X=.3) = \int_{.8}^1 \frac{3y+.6}{2.1}dy = 3/2.1 \times (1-.8^2)/2 + .6/2.1 \times .2 \approx \boxed{.314}$

3.8.1

If $Y = 1 - X^2$ for positive values of X, then we can say $X = \sqrt{1-Y}$.

The derivative of this is $-\frac{1}{2\sqrt{1-y}}$. Then we have:

$$g(y) = 3(\sqrt{1-y})^2 \times \frac{1}{2\sqrt{1-y}}$$
$$= \boxed{\frac{3}{2}\sqrt{1-y}}$$

(for $y \in (0,1)$ and $g(y) = 0$ otherwise)

3.8.8

We have $x = y^2$ and $dx/dy = 2y$. Then $\boxed{g(y) = 2ye^{-y^2}}$ for $y > 0$ and 0 otherwise.

Not from text

Sum of binomials

Let $X_1 \sim Binom(n_1, p)$ and $X_2 \sim Binom(n_2, p)$. Let $Y = X_1 + X_2$. Then $P(Y = y) = \sum_x^y P(X_1 = x)P(X_2 = y - x)$.

$$\sum_{x=0}^{y} P(X_1 = x)P(X_2 = y-x) = \sum_{x}^{y} \binom{n_1}{x} p^x q^{n_1 - x} \binom{n_2}{y-x} p^{y-x} q^{n_2 - y + x}$$
$$= \sum_{x}^{y} \binom{n_1 + n_2}{y} p^y q^{n_1 + n_2 - y}$$

Which is just $P(Y = y)$ for $Y \sim Binom(n_1 + n_2, p)$.

Sum of exponentials

Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} Exp(\lambda)$, and let $Y = X_1 + X_2$. Then $f(y) = \int_0^y \lambda e^{-\lambda x} \lambda e^{-\lambda(y-x)} dx$

$$\int_0^y \lambda e^{-\lambda x} \lambda e^{-\lambda(y-x)} dx = \int_0^y \lambda^2 e^{-\lambda y} dy$$
$$= \lambda^2 y e^{-\lambda y}$$

We know that $\Gamma(2) = 1$. If we substitute $\beta = 1/\lambda$, then we get:

$$f(y) = \frac{1}{\Gamma(2)\beta^2} y e^{-y/\beta}$$

Which is the pdf of $Y \sim Gamma(2, \beta)$.

Prove $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$

Proof by induction

Case $n = 1$

$\Gamma(1) = \int_0^\infty e^{-x} dx = -(0-1) = 1$

$0! = 1$

Therefore, $\Gamma(1) = 0!$

Case $n = k+1$, assuming it holds for $n = k$

Assume $\Gamma(k) = (k-1)!$

Consider $\Gamma(k+1)$:

$$\Gamma(k+1) = \int_0^\infty x^k e^{-x} dx$$

Let $u = x^k$, $du = kx^{k-1}$, $dv = e^{-x}dx$, and $v = -e^{-x}$ Then

$$\Gamma(k+1) = -x^k e^{-x}\Big|_0^\infty + k\int_0^\infty x^{k-1}e^{-x}dx$$

The first term goes to 0 since $0^k = 0$ and $e^{-\infty} = 0$. So we are left with:

$$\begin{aligned}\Gamma(k+1) &= k\int_0^\infty x^{k-1}e^{-x}dx \\ &= k\Gamma(k) \\ &= k(k-1)! \\ &= k!\end{aligned}$$

Therefore, $\Gamma(k+1) = k!$, and we can say that the property holds.

Show $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}dx$$

Let $u = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}dx$. Then we get:

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty 2e^{-u^2}du \\ &= 2\frac{\sqrt{\pi}}{2} \\ &= \sqrt{\pi}\end{aligned}$$