STAT-S632

Assignment 0

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Problem 1

Bias is defined as:

$$bias(\hat{y}|x) = \left(x_{1i}^T(X_1^T X_1)^{-1}X_1^T X_2^T - x_{2i}^T\right)\beta_2$$

Then:

$$\sum_{i=1}^n \left(bias(\hat{y}|x)\right)^2 = \sum \beta_2^T \left(x_{1i}^T(X_1^T X_1)^{-1}X_1^T X_2 - x_{2i}^T\right)^T \left(x_{1i}^T(X_1^T X_1)^{-1}X_1^T X_2 - x_{2i}^T\right)\beta_2$$

$$= \sum \beta_2^T \left(X_2^T X_1 (X_1^T X_1)^{-1} x_{1i} - x_{2i}\right)\left(x_{1i}^T(X_1^T X_1)^{-1}X_1^T X_2 - x_{2i}^T\right)\beta_2$$

If we FOIL expand the expression:

$$= \sum \beta_2^T X_2^T \left(X_1(X_1^T X_1)^{-1} x_{1i} x_{1i}^T (X_1^T X_1)^{-1} X_1^T\right) X_2 \beta_2 - 2\sum \beta_2^T X_2^T X_1 (X_1^T X_1)^{-1} x_{1i} x_{2i}^T \beta_2 + \sum \beta_2^T x_{2i} x_{2i}^T \beta_2$$

Then using the definition of matrix multiplication:

$$= \beta_2^T X_2^T X_1 (X_1^T X_1)^{-1} X_1^T X_1 (X_1^T X_1)^{-1} X_1^T X_2 \beta_2 - 2\beta_2 X_2^T X_1 (X_1^T X_1)^{-1} X_1^T X_2 \beta_2 + \beta_2^T X_2^T X_2 \beta_2$$

The first term contains a $(X_1^T X_1)^{-1}(X_1^T X_1)$ which reduces to I:

$$= \beta_2^T X_2^T \left(X_1 I (X_1^T X_1)^{-1} X_1^T\right) X_2 \beta_2 - 2\beta_2 X_2^T \left(X_1 (X_1^T X_1)^{-1} X_1^T\right) X_2 \beta_2 + \beta_2^T X_2^T X_2 \beta_2$$

And we can recognize that $H_1 = X_1(X_1^T X_1)^{-1} X_1^T$:

$$= (X_2\beta_2)^T H_1 (X_2\beta_2) - 2(X_2\beta_2)^T H_1 (X_2\beta_2) + (X_2\beta_2)^T I (X_2\beta_2)$$

$$= (X_2\beta_2)^T I (X_2\beta_2) - (X_2\beta_2)^T H_1 (X_2\beta_2)$$

$$= (X_2\beta_2)^T (I - H_1)(X_2\beta_2)$$

Problem 2

Part a

Recall from S631 that $(H - H_1)$ is symmetric and idempotent.

Lemma: If matrix A is symmetric and idempotent, then it is postiive semidefinite

Proof: Let z be a nonzero vector. Then since A is symmetric and idempotent,
$z^T Az = z^T AAz = z^T A^T Az = (Az)^T (Az)$.
Define $y = Az$. Then the above is $y^T y \geq 0$. Therefore, A is positive semidefinite.

Since $(H - H_1)$ is symmetric and idempotent, it is positive semidefinite.

Part b

If $z = e_i$ where e_i is a column vector such that the i^{th} element is 1 and the rest are 0, then for any matrix A, $z^T Az = [A]_{ii}$. Then $z^T (H - H_1)z = [H - H_1]_{ii} = h_{ii}^F - h_{ii}^R$. Since $H - H_1$ is positive semidefinite, $z^T (H - H_1)z \geq 0$. Therefore, $h_{ii}^F - h_{ii}^R \geq 0 \implies h_{ii}^F \geq h_{ii}^R$.