STAT-S675

Assignment 1

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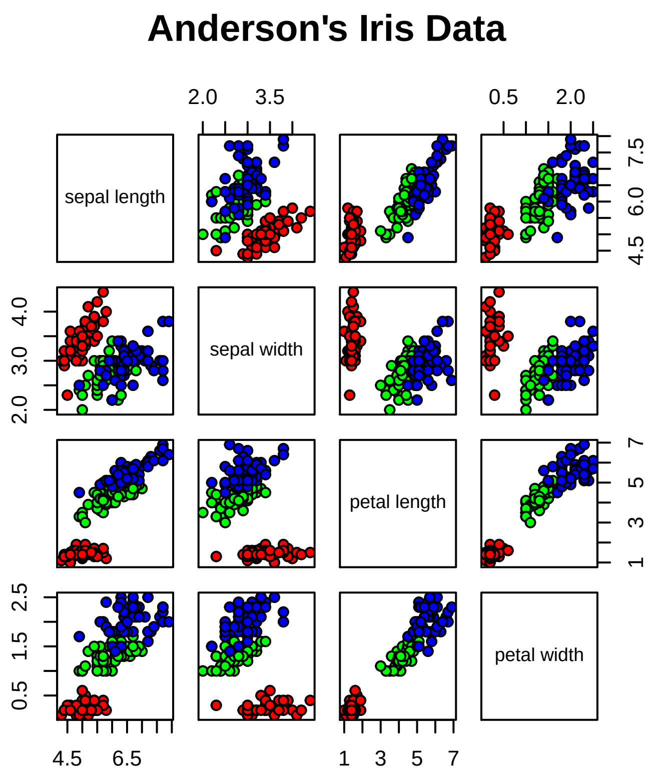
Section 1.5

Exercise 1



- It appears that identifying setosas from the rest is possible with just either petal length or petal width by setting a threshold value below which all points will be classified as "setosa".
- It appears that it's possible to come up with a reasonable method for classifying versicolors and virginicas with a straight line (e.g., support vector machine) using any two measurements other than sepal length and width as a pair. Using petal length and petal width seems like it would result in the best model.

Exercise 4

Part a

In $\mathbb{R}^2$

WLOG we can set $\overrightarrow{x_1} = (0, 0)$, $\overrightarrow{x_2} = (2, 0)$, and $\overrightarrow{x_3} = (1, \sqrt{3})$. Then we have to find $\overrightarrow{x_4} = (x, y)$ that satisfies Δ.

Our system of equations is then:

$$x^2 + y^2 = 1.21$$
$$(x-2)^2 + y^2 = 1.21$$
$$(x-1)^2 + (y-\sqrt{3})^2 = 1.21$$

From the first two equations:

$$x^2 + y^2 = (x-2)^2 + y^2$$
$$x^2 = (x-2)^2$$
$$x^2 = x^2 - 4x + 4$$
$$x = 1$$

Plugging this into the first equation from the system of equations yields:

$$1^2 + y^2 = 1.21$$
$$y = \pm\sqrt{.21}$$

So our possible solutions are $(1, \sqrt{.21})$ and $(1, -\sqrt{.21})$.

Finally, we can try both possiblities with our third equation:

$$(1-1)^2 + (\sqrt{.21} - \sqrt{3})^2 \approx 1.623 \neq 1.21$$
$$(1-1)^2 + (-\sqrt{.21} - \sqrt{3})^2 \approx 4.797 \neq 1.21$$

Therefore no combination of 4 vectors in $\mathbb{R}^2$ satisfies Δ.

In $\mathbb{R}^3$

Similar to the above, $\overrightarrow{x_1} = (0,0,0)$, $\overrightarrow{x_2} = (2,0,0)$, and $\overrightarrow{x_3} = (1, \sqrt{3}, 0)$. Then our system of equations is:

$$x^2 + y^2 + z^2 = 1.21$$
$$(x-2)^2 + y^2 + z^2 = 1.21$$
$$(x-1)^2 + (y-\sqrt{3})^2 + z^2 = 1.21$$

And we just need a nonempty solution space.

Similar to the $\mathbb{R}^2$ case, we can combine the first two equations to obtain $x = 1$. Plugging that into the first and third equations yields:

$$1^2 + y^2 + z^2 = 0^2 + (y-\sqrt{3})^2 + z^2$$
$$\implies 1 + y^2 = (y-\sqrt{3})^2$$

And solving for y gives us $y = \frac{1}{\sqrt{3}}$.

Finally, we can plug x and y into either the first or second equation to obtain a value for z:

$$1 + \frac{1}{3} + z^2 = 1.21$$
$$z^2 = .21 - \frac{1}{3}$$

But this implies that $z^2 < 0$. Therefore no combination of 4 vectors in $\mathbb{R}^3$ satisfies Δ.

Part b

In $\mathbb{R}^2$

Similar to part a, we can start with some configuration of three vectors that satisfies part of Δ and then solve for the last vector. In this case, we can set $\overrightarrow{x_1} = (0,0)$, $\overrightarrow{x_2} = (1,0)$, $\overrightarrow{x_4} = (1,1)$, and then solve for $\overrightarrow{x_3}$:

$$x^2 + y^2 = 4$$
$$(x-1)^2 + y^2 = 3$$
$$(x-1)^2 + (y-1)^2 = 5$$

Combining the first two equations as we've done in the past problems yields $x = 1$. Plugging that into the second equation yields $y = \pm\sqrt{3}$. Finally, we can try both $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ with the third equation:

$$(1-1)^2 + (\sqrt{3}-1)^2 \approx .536 \neq 5$$
$$(1-1)^2 + (-\sqrt{3}-1)^2 \approx 7.464 \neq 5$$

Therefore there is no set of 4 vectors in $\mathbb{R}^2$ that works with Δ.

In $\mathbb{R}^3$

To solve for $\overrightarrow{x_3} = (x, y, z)$:

$$x^2 + y^2 + z^2 = 4$$
$$(x-1)^2 + y^2 + z^2 = 3$$
$$(x-1)^2 + (y-1)^2 + z^2 = 5$$

Combining the first two yields $x^2 - (x-1)^2 = 1 \implies x = 1$.

Plugging $x = 1$ into the second and third equations and combining them yields $y^2-(y-1)^2 = -2 \implies y = -\frac{1}{2}$.

Plugging $x = 1$ and $y = -\frac{1}{2}$ into the second equation yields $z = \pm\frac{\sqrt{11}}{2}$. So we need to see if $(1, \frac{1}{2}, \pm\frac{\sqrt{11}}{2})$ is consistent with the first and third equations:

$$1 + \frac{1}{4} + \frac{11}{4} = 4$$
$$0 + \frac{1}{4} + \frac{11}{4} = 3$$

Therefore there are two sets of 4 vectors that works with Δ.

Exercise 5

We can generalize the methods used in exercise 4 for $\mathbb{R}^2$. Let $\Delta \in \mathbb{R}^{4\times4}$ be a dissimilarity matrix where $\delta ij = |\overrightarrow{x_i} - \overrightarrow{x_j}|_2$. Then we can start with $\overrightarrow{x_1} = (0,0)$ and $\overrightarrow{x_2} = (\delta_{1,2}, 0)$. Then to solve for $\overrightarrow{x_3} = (x, y)$ we have the system of equations:

$$x^2 + y^2 = \delta_{1,3}^2$$
$$(x - \delta_{1,2})^2 + y^2 = \delta_{2,3}^2$$

Assuming there is a solution, we end up with:

$$x = \frac{\delta_{1,2}^2 + \delta_{1,3}^2 - \delta_{2,3}^2}{2\delta_{1,2}}$$

$$y = \pm\sqrt{\delta_{1,3}^2 - \frac{(\delta_{1,2}^2 + \delta_{1,3}^2 - \delta_{2,3}^2)^2}{4\delta_{1,2}^2}}$$

Due to symmetry, we only need to consider one of these two solutions. Note that for there to be a solution for $y \in \mathbb{R}$, the term under the suqare root needs to be positive.

Finally, we can let $\overrightarrow{x_4} = (u, v)$ and then solve:

$$u^2 + v^2 = \delta_{1,4}^2$$
$$(u - \delta_{1,2})^2 + v^2 = \delta_{2,4}^2$$
$$(u - x)^2 + (v - y)^2 = \delta_{3,4}^2$$

```
G <- matrix(c(1, .1054, .0019, .0183,
              .1054, 1, .0183, .0019,
              .0019, .0183, 1, .1054,
              .0183, .0019, .1054, 1),
            nrow = 4, ncol = 4)
```

Part a

```
Delta <- 1 - G

print(Delta)
```

```
       [,1]   [,2]   [,3]   [,4]
[1,] 0.0000 0.8946 0.9981 0.9817
[2,] 0.8946 0.0000 0.9817 0.9981
[3,] 0.9981 0.9817 0.0000 0.8946
[4,] 0.9817 0.9981 0.8946 0.0000
```

```
x1 <- c(0, 0)
x2 <- c(Delta[1, 2], 0)
x3 <- c(
  (Delta[1, 2]^2 + Delta[1, 3]^2 - Delta[2, 3]^2) / 2 / Delta[1, 2],
  sqrt(
    Delta[1, 3]^2 -
      (Delta[1, 2]^2 +
         Delta[1, 3]^2 -
         Delta[2, 3]^2)^2 /
      4 / Delta[1, 2]^2
    )
)
```

Let $\overrightarrow{x_1}$, $\overrightarrow{x_2}$, and $\overrightarrow{x_3}$ be defined as in the previous R chunk. We can verify that this works:

```
dist(rbind(x1, x2, x3))
```

```
       x1     x2
x2 0.8946
x3 0.9981 0.9817
```

Then we can use this to solve for $\overrightarrow{x_4}$ (boring algebra done elsewhere):

```r
x4 <- 1:2
x4[1] <- (Delta[1, 4]^2 - Delta[2, 4]^2 + Delta[1, 2]^2) / 2 / Delta[1, 2]
x4[2] <- sqrt(Delta[1, 4]^2 - x4[1]^2)

# x4 has two solutions--calling the second one "x5"
x5 <- x4
x5[2] <- -x5[2]
```

And then we can check to see if either solution works:

```r
dist(rbind(x1, x2, x3, x4))
```

```
           x1         x2         x3
x2 0.89460000
x3 0.99810000 0.98170000
x4 0.98170000 0.99810000 0.03629412
```

```r
dist(rbind(x1, x2, x3, x5))
```

```
        x1      x2      x3
x2 0.89460
x3 0.99810 0.98170
x5 0.98170 0.99810 1.76623
```

In either case, $\delta_{1,4}$ and $\delta_{2,4}$ work out (as it should since I solved for $\overrightarrow{x_4}$ using the first two) but $\delta 3, 4$ is incorrect, implying that there is no configuration that works.

Part b

We can do the same thing we did in part a:

```r
Delta <- -log(G)

print(Delta)
```

```
         [,1]     [,2]     [,3]     [,4]
[1,] 0.000000 2.249993 6.265901 4.000854
[2,] 2.249993 0.000000 4.000854 6.265901
[3,] 6.265901 4.000854 0.000000 2.249993
[4,] 4.000854 6.265901 2.249993 0.000000
```

```r
x1 <- c(0, 0)
x2 <- c(Delta[1, 2], 0)
x3 <- c(
  (Delta[1, 2]^2 + Delta[1, 3]^2 - Delta[2, 3]^2) / 2 / Delta[1, 2],
  sqrt(
    Delta[1, 3]^2 -
      (Delta[1, 2]^2 +
         Delta[1, 3]^2 -
         Delta[2, 3]^2)^2 /
      4 / Delta[1, 2]^2
```

```
    )
)
```

```
Warning in sqrt(Delta[1, 3]^2 - (Delta[1, 2]^2 + Delta[1, 3]^2 - Delta[2, :
NaNs produced
```

But in thise case, $\overrightarrow{x_3} \notin \mathbb{R}^2$, so we can stop here.