STAT-S620

Assignment 9

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MLE for binomial distribution

Part a

$X \sim Binom(n,p)$, so $L(p|x) = \binom{n}{x} p^x (1-p)^{n-x}$. Then $\ell(p|x) = \log \binom{n}{x} + x \log p + (n-x)\log(1-p)$ and $\ell'(p|x) = \frac{x}{p} - \frac{n-x}{1-p} = 0 \implies x(1-p) = p(n-x) \implies \boxed{\hat{p} = \dfrac{x}{n}}$

Part b

$X_1, ..., X_m \overset{iid}{\sim} Binom(n,p)$. Then $L(p) = \prod_{i=1}^m \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$ and $\ell(p) = \sum_i^m \Bigg(\log \binom{n}{x_i} + x_i \log p + (n - x_i)\log(1-p)\Bigg)$. Then $\ell'(p) = \frac{1}{p}\sum_i x_i - \frac{1}{1-p}\sum_i (n - x_i) = 0 \implies (1-p)\sum_i x_i = p\sum_i n - p \sum_i x$ $\implies \sum_i x_i = pnm \implies \boxed{\hat{p} = \dfrac{\bar{x}}{n}}$

Second question

Part a

$\mu_1 = E[X] = \int_0^1 (\theta+1)x^{\theta+1}dx = \frac{\theta+1}{\theta+2}$

Then if we set $\bar{X} = \frac{\theta+1}{\theta+2} \implies (\bar{X}-1)\theta = 1 - 2\bar{X} \implies \boxed{\hat{\theta} = \dfrac{1-2\bar{X}}{\bar{X}-1}}$.

Part b

Let $X_1, ...X_n \overset{iid}{\sim} f(x) = (\theta+1)x^\theta$. Then $L(\theta) = \prod_i^n (\theta+1)x_i^\theta \implies \ell(\theta) = \sum_i^n \log(\theta+1) + \theta \log x_i$ $= n\log(\theta+1) + \theta \sum_i^n \log x_i$. Then $\ell'(\theta) = \frac{n}{\theta+1} + \sum_i x_i = 0 \implies -\bar{x}(\theta+1) = 1 \implies \hat{\theta} = -1 - \frac{1}{\bar{x}} = \frac{-\bar{x}-1}{\bar{x}}$. But $\theta > -1$, so this does not work. Furthermore, we can see that $\ell(\theta)$ is actually an increasing function, so the maximum is at $\theta \to \infty$.

5.7.4

$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}\exp(-x/\beta)$, then $f'(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} x^{\alpha-2}(\alpha - 1 - \frac{x}{\beta}) = 0$. Then solving for x, we get $x = 0$ or $x = \beta(\alpha-1)$. The latter solution only works when $\alpha \geq 1$ since $x \geq 0$. Furthermore, from our $f'(x)$, we can see that the coefficient, exponential, and $x^{\alpha-2}$ terms are all positive, so if $\alpha < 1$, $f'(x) < 0$. In that case, the only solution is $x = 0$.

7.4.4

From a previous example, our estimator for $\theta|X$ is $\frac{\alpha+\sum_i X_i}{\alpha+\beta+n}$.

$$\begin{aligned}
&\frac{\alpha+\sum_i X_i}{\alpha+\beta+n} \\
&= \frac{\alpha}{\alpha+\beta+n} + \frac{\sum_i X_i}{\alpha+\beta+n} \\
&= \frac{(\alpha+\beta)\mu_0}{\alpha+\beta+n} + \frac{n\bar{X}_n}{\alpha+\beta+n} \\
&= \frac{n}{\alpha+\beta+n}\bar{X}_n + \left(1 - \frac{n}{\alpha+\beta+n}\right)\mu_0
\end{aligned}$$

Then if we let $\gamma_n = \frac{n}{\alpha+\beta+n}$, we get:

$$\gamma_n \bar{X}_n + (1-\gamma_n)\mu_0$$

Furthermore, $\lim_{n\to\infty} \frac{n}{\alpha+\beta+n} = \frac{1}{\alpha/n+\beta/n+1} = 1$.

7.4.5

Let $Y = \sum_i X_i$ Then:

$Y|\theta \sim Poisson(n\theta)$
$\theta \sim Gamma(\alpha, \beta)$

Then:

$$\begin{aligned}
&f(\theta|y) \propto f(y|\theta)f(\theta) \\
&\propto \frac{e^{-n\theta}(n\theta)^y}{y!}\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta} \\
&\propto \theta^{y+\alpha-1}e^{-(n+1/\beta)\theta}
\end{aligned}$$

Then $\theta|Y \sim Gamma\big(y+\alpha, \frac{\beta}{n\beta+1}\big) \implies E[\theta|Y] = (y+\alpha)\frac{n\beta+1}{\beta} = (13+3)\frac{1}{\cdot 1+1} = \boxed{8/3}$

7.5.5

Part a

$L(\theta) = \prod_i^n \frac{e^{-\theta}\theta^{x_i}}{x_i!}$. Then $\ell(\theta) = \sum_i^n \big(-\theta + x_i\log\theta - \log x_i!\big) = -n\theta + n\bar{x}\log\theta - \sum_i^n \log x_i!$. Then $\ell'(\theta) = -n + \frac{n\bar{x}}{\theta} = 0 \implies \boxed{\hat{\theta} = \bar{x}}$

Part b

If every x_i is 0, then $\bar{x} = 0$ and we get $\ell'(\theta) = -n = 0$, but $n > 0$, which is a contradiction.

7.5.9

$L(\theta) = \prod_i^n \theta x_i^{\theta-1}$, and $\ell(\theta) = n\log\theta + \sum_i^n(\theta-1)\log x_i$. Then $\ell'(\theta) = \frac{n}{\theta} + \sum_i \log x_i = 0 \implies \boxed{\hat{\theta} = -\frac{n}{\sum_i^n \log x_i}}$

7.6.3

$X_1, ..., X_n \overset{iid}{\sim} Exp(\beta)$. Then $L(\beta) = \prod_i^n \beta e^{-\beta x_i}$ and $\ell(\beta) = n\log\beta - \beta\sum_i^n x_i \implies \ell'(\beta) = \frac{n}{\beta} - n\bar{x} = 0 \implies \hat{\beta} = \frac{1}{\bar{x}}$.

Next, we plug this into our pdf and find the median, which is the value m such that $1/2 = \int_0^m f(x)dx$:

$$\begin{aligned} 1/2 &= \int_0^m \beta e^{-\beta x} dx \\ &= 1 - e^{-\beta m} \\ \implies -\beta m &= -\log 2 \\ \implies m &= \frac{\log 2}{\beta} \end{aligned}$$

And if we plug in $\hat{\beta}$, we get $\boxed{\hat{m} = \bar{x}\log 2}$.

7.6.8

We are given $X_1, ..., X_n \overset{iid}{\sim} Gamma(\alpha, 1)$. Then $L(\alpha) = \prod_i^n \frac{1}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-x_i}$ and $\ell(\alpha) = -n\log\Gamma(\alpha) + (\alpha - 1)\sum_i \log x_i - \sum_i x_i$. Then $\ell'(\alpha) = -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_i \log x_i = 0 \implies \boxed{\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \frac{1}{n}\sum_i^n \log x_i}$.

7.6.9

We have $L(\alpha, \beta) = \prod_i^n \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta}$. Then $\ell(\alpha, \beta) = -n\log\Gamma(\alpha) - n\alpha\log\beta + (\alpha - 1)\sum_i \log x_i - \beta^{-1}\sum_i x_i$.

$\partial_\beta \ell = -\frac{n\alpha}{\beta} + \beta^{-2}\sum_i x_i = 0 \implies n\alpha\beta = \sum_i x_i \implies \boxed{\alpha\beta = \bar{x}}$

7.6.10

We have $L(\alpha, \beta) = \prod_i^n \frac{1}{B(\alpha,\beta)} x_i^{\alpha-1}(1-x_i)^{\beta-1}$ and $\ell(\alpha, \beta) = -n\log B(\alpha, \beta) + (\alpha - 1)\sum_i x_i + (\beta - 1)\sum_i \log(1 - x_i)$. We can then rewrite this as $\ell(\alpha, \beta) = n\log\Gamma(\alpha + \beta) - n\log\Gamma(\alpha) - n\log\Gamma(\beta) + (\alpha - 1)\sum_i \log x_i + (\beta - 1)\sum_i \log(1 - x_i)$. Then

$\partial_\alpha \ell = \frac{n\Gamma'(\alpha+\beta)}{\Gamma(\alpha+\beta)} - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_i \log x_i = 0$ and $\partial_\beta \ell = \frac{n\Gamma'(\alpha+\beta)}{\Gamma(\alpha+\beta)} - \frac{n\Gamma'(\beta)}{\Gamma(\beta)} + \sum_i \log(1 - x_i) = 0$. Subtracting the top from the bottom, we get: $\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{n\Gamma'(\beta)}{\Gamma(\beta)} + \sum_i \log\frac{1-x_i}{x_i} = 0 \implies \boxed{\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\beta)}{\Gamma(\beta)} = \frac{1}{n}\sum_i \log\frac{x_i}{1-x_i}}$