STAT-S620

Assignment 7

John Koo

4.4.7

$\psi(t) = \frac{1}{4}(3e^t + e^{-t})$

$\psi'(t) = \frac{1}{4}(3e^t - e^{-t})$

$\psi''(t) = \frac{1}{4}(3et + e^{-t})$

Then $\mu = \mu_1 = \psi'(0) = 2/4 = \boxed{1/2}$.

Since $\psi''(t) = \psi(t)$, we know that $\mu_2 = \psi''(0) = \psi(0) = 1$. Therefore, $var(X) = 1 - 1/4 = \boxed{3/4}$.

4.4.11

For the pmf of a discrete X, the mgf is $\psi(t) = \sum_x P(X = x)e^{tx}$. Then $f(x)$ has support $\{1, 4, 8\}$, and $\boxed{f(1) = 1/5, f(4) = 2/5, f(8) = 2/5}$ (and 0 otherwise).

4.4.12

Using the same idea as exercise 4.4.11, we can say X is discrete and $\boxed{f(0) = 2/3, f(1) = 1/6, f(-1) = 1/6}$ (and 0 otherwise).

4.6.13

Part a

$var(X + Y) = var(X) + var(Y) + 2cov(X, Y)$. We know that $cov(X, Y) = (-1/6)(3)(2) = -1$. Then $var(X + Y) = 9 + 4 - 2 = \boxed{11}$.

Part b

$var(X - 3Y + 4) = var(X) + 9var(Y) + (2)(1)(-3)cov(X, Y) = \boxed{51}$.

4.7.7

$f_X(x) = \int_0^1 (x + y)dy = xy + \frac{y^2}{2}|_{y=0}^1 = x + 1/2$. Then $g(y|x) = \frac{x+y}{x+1/2}$ (for $y \in [0, 1]$ and 0 otherwise).

$E[Y|X] = \int_0^1 \frac{xy+y^2}{x+1/2}dy = \boxed{\dfrac{x/2 + 1/3}{x + 1/2}}$.

$E[Y^2|X] = \int_0^1 \frac{xy^2+y^3}{x+1/2} dy = \frac{x/3+1/4}{x+1/2}$.

Then $var(Y|X) = \frac{x/3+1/4}{x+1/2} - \left(\frac{x/2+1/3}{x+1/2}\right)^2 = \boxed{\frac{1}{12} - \frac{1}{36(2x+1)^2}}$ (via WolframAlpha).

mgf of the normal distribution

Part 1

By definition:

$$\psi(t) = \int e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Substituting $z = \frac{x-\mu}{\sigma}$, we get that $x = \mu + \sigma z$ and $dz = \frac{1}{\sqrt{\sigma^2}} dx$, so we are left with:

$$\begin{aligned}
\psi(t) &= \int e^{t(\mu+\sigma z)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \int \frac{1}{\sqrt{2\pi}} e^{-z^2/2+\sigma tz+\mu t} dz \\
&= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2-2\sigma tz-2\mu t)} dz \\
&= e^{\mu t+\frac{\sigma^2 t^2}{2}} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} dz \\
&= e^{\mu t+\frac{\sigma^2 t^2}{2}}
\end{aligned}$$

Part 2

$\psi'(t) = (\mu + \sigma^2 t)\psi(t)$

Then $E[X] = \psi'(0) = \mu$

Part 3

$\psi''(t) = (\mu + \sigma^2 t)^2 \psi(t) + \sigma^2 \psi(t)$

Then $E[X^2] = \psi''(0) = \mu^2 + \sigma^2$, and $var(X) = E[X^2] - E[X]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$.

Part 4

We know that the mgf of Z is $\psi_Z(t) = \psi_X(t)\psi_Y(t) = e^{\mu_1 t+\frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t+\frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1+\mu_2)t+\frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$.

Not from text

Part 1

If, for when $n = 1$, $r = 1/2$, then the possible values for R when $n = 2$ are $1/2 * 1/2 = 1/4$ and $1/2 * 2 = 1$. On the other hand, if when $n = 1$, $r = 2$, then the possible values for R when $n = 2$ are $2 * 1/2 = 1$ and $2 * 2 = 4$.

For a given n and i, $r = 2^{2i-n}$.

Part 2

We can see this is a binomial distribution. $P(R = r) = \binom{n}{i}2^{-n}$.

Part 3

$\psi(t) = E[e^{tr}] = \sum_r e^{tr}\binom{n}{i}2^{-n}$ where $r = 2^{2i-n}$.

Part 4

$E[R] = \sum_{i=1}^n r\binom{n}{i}2^{-n}$
$= \sum_i^n \binom{n}{i}2^{2i-2n}$
$= \sum_i^n \binom{n}{i}4^{i-n}$
$= \frac{1}{4^n}\sum_i^n \binom{n}{i}4^i$
$= \frac{1}{4^n}(4+1)^n = (\frac{5}{4})^n$

Part 5

Since $E[R] = (\frac{5}{4})^n$ and $C = cR$, $E[C] = c(\frac{5}{4})^n$.