STAT-S620

Assignment 4

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3.1.2

We know that $\sum_{x=1}^5 cx = 1$. We also know that $\sum_{x=1}^5 cx = c\frac{6 \times 5}{2} = 15c$. Then $\boxed{c = \frac{1}{15}}$.

3.1.3

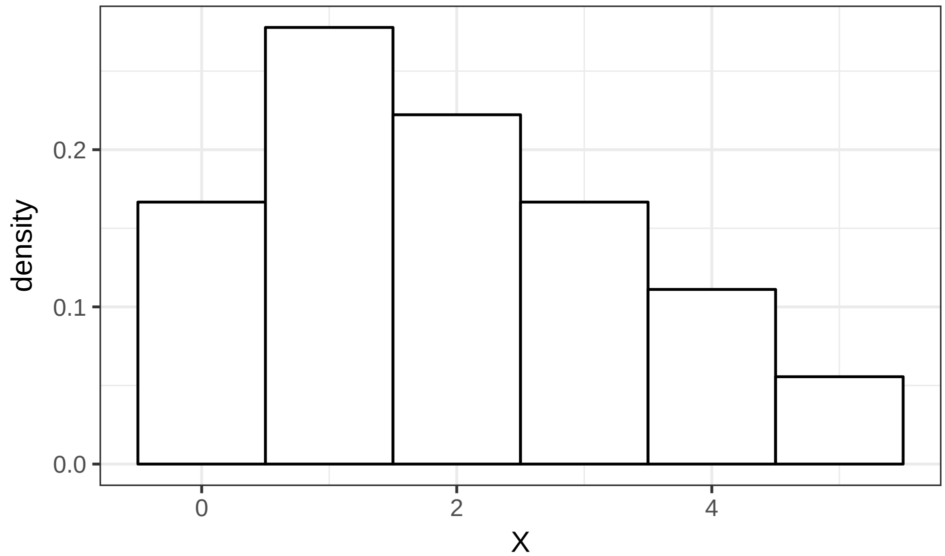
```
# packages, etc.
import::from(magrittr, `%>%`)
dp <- loadNamespace('dplyr')
import::from(MASS, fractions)
import::from(xtable, xtable)
library(ggplot2)
theme_set(theme_bw())

# create sample space
sample.space.df <- expand.grid(x1 = seq(6), x2 = seq(6)) %>%
  dp$mutate(abs.diff = abs(x1 - x2))

# rel freq table
sample.space.df$abs.diff %>%
  {table(.) / length(.)} %>%
  fractions()
```

```
.
   0    1    2    3    4    5
 1/6 5/18  2/9  1/6  1/9 1/18
```

```
# plot
ggplot(sample.space.df) +
  geom_histogram(aes(x = abs.diff, y = ..density..),
                 binwidth = 1, colour = 'black', fill = 'white') +
  labs(x = 'X')
```



3.1.10

Part a

$1 = \sum_{x=0}^{7} c(x+1)(8-x) = c(7\sum x - \sum x^2 + \sum 8) = c(7\frac{(8)(7)}{2} - \frac{(7)(8)(15)}{6} + (8)(8)) = 120c$. Then $c = \frac{1}{120} \implies$
$\boxed{f(x) = \begin{cases} \frac{(x+1)(8-x)}{120} & x = 0, 1, 2, ..., 7 \\ 0 & \text{otherwise} \end{cases}}$

Part b

$P(X \geq 5) = \sum_{x=5}^{7} f(x)$

```
pf <- function(x) (x + 1) * (8 - x) / 120

sum(pf(seq(5, 7))) %>%
  fractions()
```

```
[1] 1/3
```

$\boxed{P(X \geq 5) = \frac{1}{3}}$

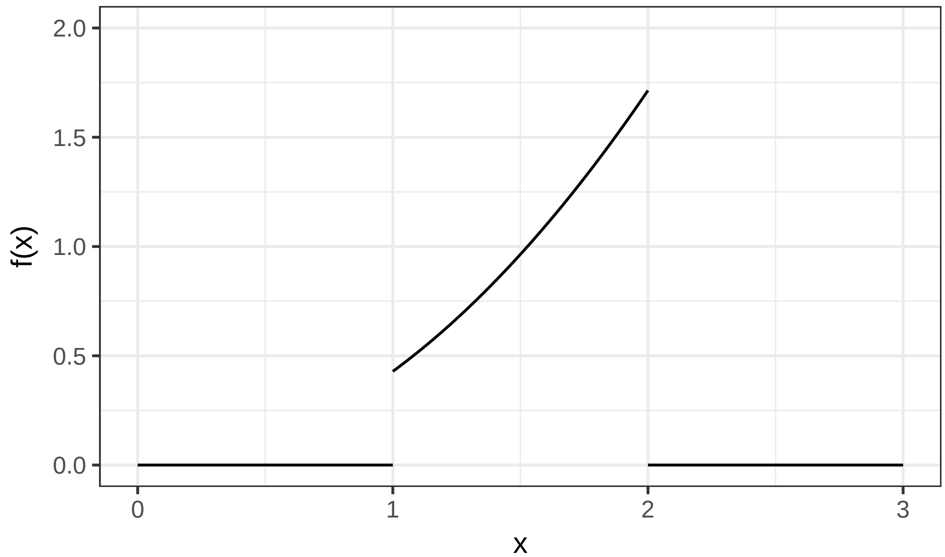
3.2.4

Part a

$1 = \int_1^2 cx^2 dx = \frac{c}{3}x^3\Big|_1^2 = \frac{7}{3}c$. Then $c = \frac{3}{7}$ and $\boxed{f(x) = \begin{cases} \frac{3}{7}x^2 & x \in [1, 2] \\ 0 & \text{otherwise} \end{cases}}$.

```r
X <- seq(1, 2, .01)
Y <- 3 / 7 * X ** 2

ggplot() +
  geom_line(aes(x = X, y = Y)) +
  ylim(0, 2) +
  labs(x = 'x', y = 'f(x)') +
  geom_segment(aes(x = 1, xend = 0, y = 0, yend = 0)) +
  geom_segment(aes(x = 2, xend = 3, y = 0, yend = 0))
```



Part b

$P(X \geq 3/2) = \int_{3/2}^{2} \frac{3}{7}x^2 dx = \frac{1}{7}x^3\big|_{3/2}^{2} = \frac{1}{7}(8 - \frac{27}{8}) = \boxed{\frac{37}{56}}$.

3.2.6

$P(Y = 0) = \int_0^{.5} \frac{x}{8} dx = x^2/16|_0^{.5} = 1/64$

For $i = 1, 2, 3$, $P(Y = i) = \int_{i-.5}^{i+.5} \frac{x}{8} dx = \frac{i}{8}$

$P(Y = 4) = \int_{3.5}^{4} \frac{x}{8} dx = \frac{1}{16}(16 - 49/4) = 15/64$.

Then $\boxed{f(y) = \begin{cases} 1/64 & y = 0 \\ y/8 & y = 1, 2, 3 \\ 15/64 & y = 4 \\ 0 & \text{otherwise} \end{cases}}$.

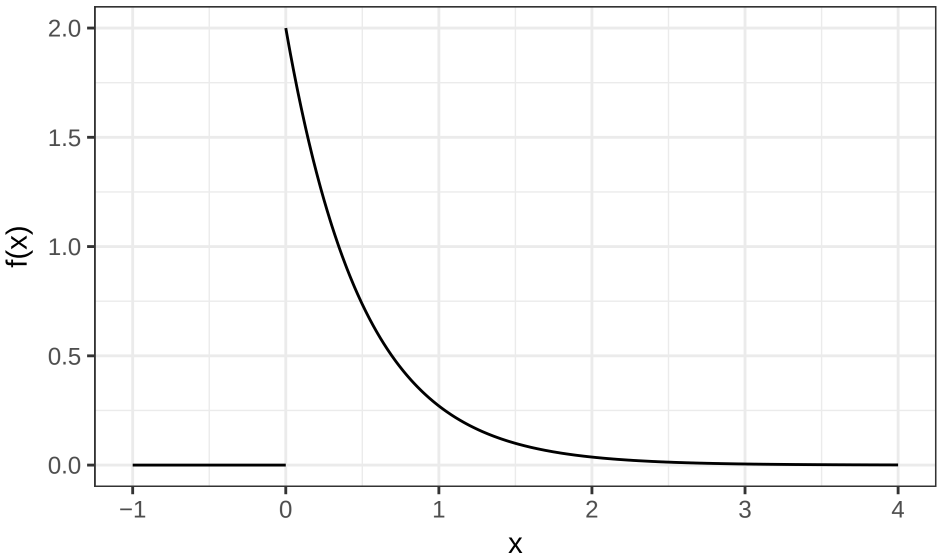
3.2.8

Part a

This is just the exponential distribution, so $\boxed{c = 2}$.

```
X <- seq(0, 4, .01)
Y <- 2 * exp(-2 * X)

ggplot() +
  geom_line(aes(x = X, y = Y)) +
  labs(x = 'x', y = 'f(x)') +
  geom_segment(aes(x = 0, xend = -1, y = 0, yend = 0))
```



Part b

$P(1 < X < 2) = \int_1^2 2e^{-2x}dx = -e^{-2x}|_1^2 = \boxed{e^{-2} - e^{-4} \approx 0.117}$

3.3.4

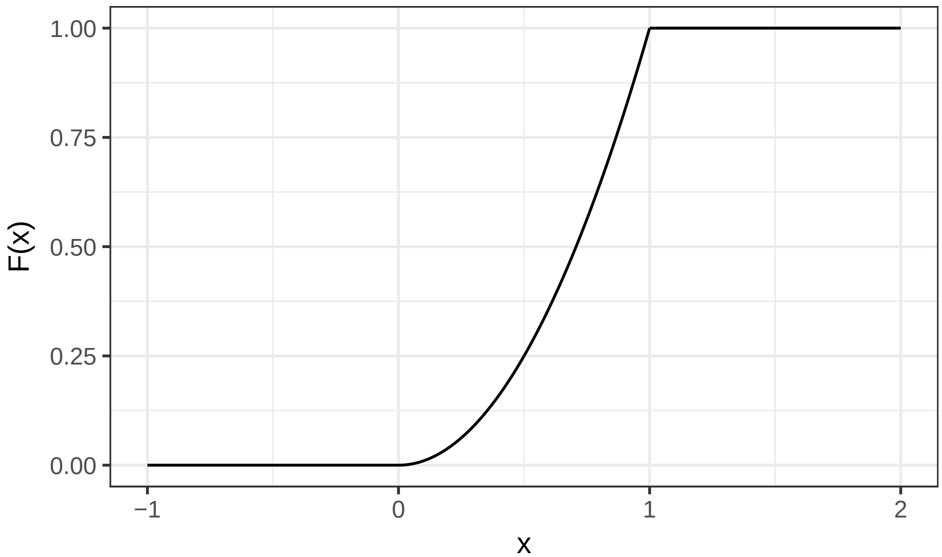
a. .1
b. .1
c. .2
d. 0
e. .8 - .2 = .6
f. .6 - .2 = .4
g. .8 - .1 = .7
h. 0
i. 0
j. 0
k. 0
l. 0

3.3.15

$F(x) = \int_0^x 2u du = x^2$ for $x \in (0, 1)$, $F(x) = 0$ for $x \leq 0$, and $F(x) = 1$ for $x \geq 1$.

```
X <- seq(0, 1, .01)
Y <- X ** 2

ggplot() +
  geom_line(aes(x = X, y = Y)) +
  geom_segment(aes(x = 0, xend = -1, y = 0, yend = 0)) +
  geom_segment(aes(x = 1, xend = 2, y = 1, yend = 1)) +
  labs(x = 'x', y = 'F(x)')
```



Not from text

Part a

$P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = \boxed{e^{-1}}$

Part b

$\frac{d}{dx}(1 - e^{-x^2}) = 2xe^{-x^2}$. Therefore, $\boxed{f(x) = \begin{cases} 0 & x < 0 \\ 2xe^{-x^2} & x \geq 0 \end{cases}}$

Part c

$P(X > 1) = \int_1^\infty 2xe^{-x^2} dx$

Let $u = x^2$. Then $du = 2xdx$. Then the above integral becomes $\int_1^\infty e^{-u} du = \boxed{e^{-1}}$

Part d

Let $p = 1 - e^{-(F^{-1}(p))^2}$. Then $\boxed{F^{-1}(p) = \sqrt{-\log(1-p)}}$.

Part e

$F^{-1}(.85) = \boxed{\sqrt{-\log(.15)} \approx 1.377}$