S721 HW2

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Problem 1.12

Part a

Let $A_n = \emptyset \; \forall n > 2$. Then $A_1, A_2, ..., A_n, ...$ are pairwise disjoint, and $P(\cup_i^\infty A_i) = \sum_i^\infty P(A_i)$.

Since $A_3, ..., A_n, ... = \emptyset$, $P(\cup_i^\infty A_i) = P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1 \cup A_2 \cup \emptyset \cup \emptyset \cup \cdots) = P(A_1 \cup A_2)$.

We also have $\forall n > 2$, $P(A_n) = P(\emptyset) = 0$. Then $\sum_i^\infty P(A_i) = P(A_1) + P(A_2) + 0 + 0 + \cdots = P(A_1) + P(A_2)$.

Therefore, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

Part b

Let $B_n = \cup_{i=n}^\infty A_i$. Then $B_n \subset B_{n-1} \subset \cdots \subset B_1$. We can also say $P(\cup_{i=1}^\infty A_i) = P((\cup_{i=1}^n A_i) \cup B_{n+1}) = \sum_{i=1}^n P(A_i) + P(B_{n+1})$ (note that we're looking at a *finite* n here).

As $n \to \infty$, $B_{n+1} \to \emptyset$ since $A_n \downarrow A$. Then we have $\lim_{n\to\infty} P(\cup_i^n A_i) = P(\cup_i^\infty A_i)$, and on the other hand, we have $\lim_{n\to\infty} \sum_i^n P(A_i) + P(B_{n+1}) = \sum_i^\infty P(A_i) + P(\emptyset) = \sum_i^\infty P(A_i)$. Putting it all together, we get:

$$P(\cup_i^\infty A_i) = \lim_{n\to\infty} P(\cup_i^n A_i) = \lim_{n\to\infty} \sum_i^n P(A_i) + P(B_{n+1}) = \sum_i^\infty P(A_i)$$

Problem 1.13

$P(B^c) = 1/4 \implies P(B) = 3/4$. Then if A and B are disjoint, $P(A \cup B) = P(A) + P(B) = 1/3 + 3/4 = 13/12 > 1$. Therefore, A and B cannot be disjoint.

Problem 1.14

Let $S = \{s_1, ..., s_n\}$ where s_i is the i^{th} element of S (although order doesn't matter). Then any subset of S can be constructed by going through each s_i and either including it or not. Since there are 2 choices per element and n elements, there are 2^n possible subsets.

Not from textbook

Problem 1

We can rewrite in terms of disjoint sets:

$A = (A \cap B) \cup (A \cap B^c)$
$B = (B \cap A) \cup (B \cap A^c)$

Then, again in terms of disjoint sets, $A \cup B = ((A \cap B) \cup (A \cap B^c)) \cup ((B \cap A) \cup (B \cap A^c)) = (A \cap B^c) \cup (B \cap A^c) \cup ((A \cap B) \cup (A \cap B)) = (A \cap B^c) \cup (B \cap A^c) \cup (A \cap B)$.

Then since the sets are disjoint, $P(A \cup B) = P(A \cap B^c) + P(B \cap A^c) + P(A \cap B)$.

We can also see that $P(A) = P(A \cap B) + P(A \cap B^c)$. Then $P(A \cap B^c) = P(A) - P(A \cap B)$, and vice versa.

Therefore, $P(A \cup B) = P(A \cap B^c) + P(B \cap A^c) + P(A \cap B) = (P(A) - P(A \cap B)) + (P(B) - P(B \cap A)) + P(A \cap B) = P(A) + P(B) - P(A \cap B)$.

Problem 2

Since $A_1 \supset A_2 \supset \cdots$, we can say $A_n = \cap_i^n A_i$. Then $P(A_n) = P(\cap_i^n A_i)$.

Taking $n \to \infty$ to both sides, we get:

$$\lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} P(\cap_i^n A_i) = P(\cap_i^\infty A_i) = P(A)$$

Problem 3

Since Ω is a discrete sample space, the collection of all possible subsets of Ω form a σ-algebra. Then for any $B \subset \mathbb{R}$ (I don't think we even need Borel subsets here), the pre-image $X^{-1}(B) \subset \Omega$. So all $X^{-1}(B) \in \mathcal{F}$.