

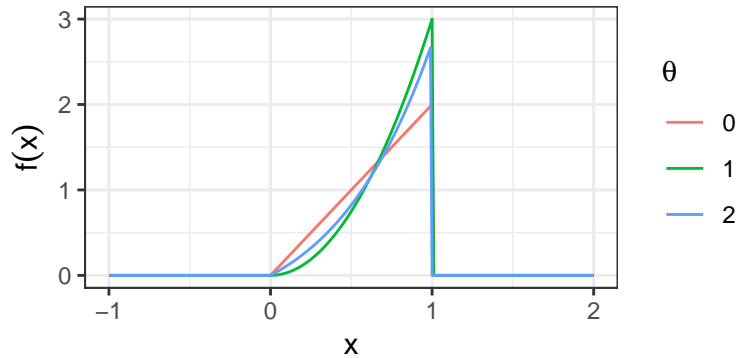
# S722 HW4

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To save on typing, I will denote  $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$ .

## Problem 1

a



$\frac{L(1)}{L(0)} = \frac{3}{2}x$  is increasing w.r.t.  $x$ , so we reject when this ratio is large. We can ignore the constant up front and just look at  $x$  by itself. So the UMP test would be to reject when  $X > c$  for some  $c = c(\alpha)$ , and to obtain  $c$ :

$$\begin{aligned}\alpha &= .19 = P(X > c|H_0) = 1 - P(X \leq c|H_0) \\ &= 1 - \int_0^c 2x dx = 1 - c^2 \\ \implies c &= \sqrt{1 - \alpha} = \sqrt{.81} = 0.9\end{aligned}$$

To obtain the power, we compute

$$\begin{aligned}\beta(\theta = 1) &= P(X > .9|\theta = 1) \\ &= 1 - P(X \leq .9|\theta = 1) = 1 - \int_0^{.9} 93x^2 dx = 1 - .9^3 \approx 0.271\end{aligned}$$

b

From the plot, we can see that  $f(x|1) > f(x|2)$  for the relevant region, so the UMP would use  $\theta = 1$  since it is the MLE. So the test in part (a) is the UMP of size .19.

## Problem 2

a

$$\begin{aligned}L(\theta) &= \prod \frac{2x_i}{\theta} \exp(-\frac{1}{\theta}x_i^2) \\ &= (\frac{2}{\theta})^n (\prod x_i) \exp(-\frac{1}{\theta} \sum x_i^2) \\ &\propto \theta^{-n} \exp(-\frac{1}{\theta} \sum x_i^2)\end{aligned}$$

Then the log-likelihood is

$$\ell(\theta) = -n \log \theta - \frac{1}{\theta} \sum x_i^2 + C$$

And to find the MLE for  $\theta$ ,  
 $0 = \ell'(\theta) = -n/\theta + \theta^{-2} \sum x_i^2$   
 $\implies \hat{\theta} = \frac{1}{n} \sum X_i^2$

The likelihood is maximized at  
 $L(\frac{1}{n} \sum x_i^2) \propto (\frac{1}{n} \sum x_i^2)^{-n} \exp(-n)$

So the LRT statistic is

$$\lambda(X) = \frac{L(\hat{\theta})}{L(1)} = \frac{(\frac{1}{n} \sum X_i^2)^{-n} \exp(-n)}{\exp(-\sum X_i^2)} \\ \propto (\sum X_i^2)^{-n} \exp(\sum X_i^2)$$

Letting  $T = \sum X_i^2$ , we get  $\lambda(T) = T^{-n} \exp(T)$ , and the LRT is  $\phi(T) = 1 \iff T^{-n} \exp(T) > c$ .

$\lambda$  is not a monotone function of  $T$ , and it has a global minimum. So this test is equivalent to  $\phi(T) = 1 \iff T < c_1$  or  $T > c_2$ .

To find  $c_1$  and  $c_2$ , we solve the system of equations:

- $\alpha = P(T < c_1 | H_0) + P(T > c_2 | H_0)$
- $\lambda(c_1) = \lambda(c_2)$

**b**

A UMP size- $\alpha$  test does not exist in this case because it is a two-sided test. To be more specific, suppose this were a right-sided test. Since the MLE for  $\theta$  is  $\frac{1}{n} \sum X_i^2$ , we would reject for large values of  $\sum X_i^2$ , i.e., when  $T > c$  for some  $c$ . Similarly, if this were a left-sided test, we would reject when  $T < c$  for some  $c$ . Then the UMP test would require knowing whether  $\theta > 1$  or  $\theta < 1$ .

### Problem 3

8.37 from Casella & Berger

**a**

Under  $H_0$ ,  $\bar{X} \sim \mathcal{N}(\theta_0, \frac{\sigma^2}{n})$ . So  
 $P(\bar{X} > \theta_0 + z_\alpha \sigma / \sqrt{n} | H_0)$   
 $= P(\frac{\bar{X} - \theta_0}{\sigma / \sqrt{n}} > z_\alpha | H_0) = P(Z > z_\alpha) = \alpha$ .

Using the LRT method, we have

$$\lambda(X) = \frac{\exp(-\frac{1}{2\sigma^2} \sum (X_i - \theta_0)^2)}{\exp(-\frac{1}{2\sigma^2} \sum (X_i - \bar{X})^2)} \\ = \exp(-\frac{n}{2\sigma^2} (\bar{X} - \theta_0^2)^2)$$

And we reject if this value is too small. We can also see that  $\lambda$  can be expressed purely in terms of  $\bar{X}$ , and  $\lambda$  is decreasing in  $\bar{X}$ , so this is equivalent to rejecting for large  $\bar{X}$ . Using the distribution of  $\bar{X}$ , we arrive at the test described.

**b**

From class, we arrived at the test described in (a) from the ratio  $\frac{\sup_{\theta \in \Theta_0^C} L(\theta)}{\sup_{\theta \in \Theta_0} L(\theta)}$ , so it is the UMP test.

**c**

Under  $H_0$ ,  $\frac{\bar{X}-\theta_0}{S/\sqrt{n}} \sim T_{n-1}$ . So  $P(\bar{X} > \theta_0 + t_{n-1,\alpha}S/\sqrt{n} | H_0) = P(\frac{\bar{X}-\theta_0}{S/\sqrt{n}} > t_{n-1,\alpha} | H_0) = P(T > t_{n-1,\alpha}) = \alpha$ .

The MLE for  $\sigma^2$  is  $\frac{1}{n} \sum (X_i - \bar{X})^2$ , and in the restricted case,  $\hat{\sigma}_R^2 = \frac{1}{n} \sum (X_i - \theta_0)^2$ . We can note that when we plug these into the likelihoods, the exponential term cancels out since we have  $\sum (x_i - \bar{x})^2 / \sum (x_i - \bar{x})^2$  and likewise with  $\theta_0$  instead of  $\bar{x}$ . So the LRT test statistic is simply:

$$\lambda(X) = (\hat{\sigma}^2 / \hat{\sigma}_R^2)^{n/2}$$

We can remove the power since we know it must be positive. So the term we must consider is:

$$\begin{aligned} & \hat{\sigma}^2 / \hat{\sigma}_R^2 \\ &= \frac{\frac{1}{n} \sum (X_i - \bar{X})^2}{\frac{1}{n} \sum (X_i - \theta_0)^2} \\ &= \frac{\sum (X_i - \bar{X})^2}{\sum (X_i - \theta_0)^2} \\ &= \frac{(n-1)S^2}{\sum (X_i - \bar{X} + \bar{X} - \theta_0)^2} \\ &= \frac{(n-1)S^2}{\sum (X_i - \bar{X})^2 + \sum (\bar{X} - \theta_0)^2 + 2 \sum (X_i - \bar{X})(\bar{X} - \theta_0)} = \frac{(n-1)S^2}{(n-1)S^2 + n(\bar{X} - \theta_0)^2} \\ &= \frac{n-1}{(n-1) + \frac{(\bar{X} - \theta_0)^2}{S^2/n}} \end{aligned}$$

This is decreasing in  $\frac{(\bar{X} - \theta_0)^2}{S^2/n}$ , so it is also decreasing in  $\frac{\bar{X} - \theta_0}{S/\sqrt{n}}$ . Then rejecting when the ratio is too small is equivalent to rejecting when this is too large.

## Problem 4

8.41 from Casella & Berger

**a**

Under  $H_0$ ,  $\hat{\mu}_R = \frac{\sum X_i + \sum Y_i}{n+m}$ . Then  $\hat{\sigma}_R^2 = \frac{\sum (X_i - \hat{\mu}_R)^2 + \sum (Y_i - \hat{\mu}_R)^2}{n+m}$

We can simplify  $\hat{\sigma}_R^2$ :

$$\begin{aligned} \sum (X_i - \hat{\mu}_R)^2 &= \sum (X_i - \frac{n\bar{X} + m\bar{Y}}{n+m})^2 \\ &= \sum (X_i - \bar{X} + \bar{X} - \frac{n\bar{X} + m\bar{Y}}{n+m})^2 \\ &= \sum (X_i - \bar{X})^2 + n(\bar{X} - \frac{n\bar{X} + m\bar{Y}}{n+m})^2 + 2(\bar{X} - \frac{n\bar{X} + m\bar{Y}}{n+m}) \sum (X_i - \bar{X}) \\ &= \sum (X_i - \bar{X})^2 + n(\bar{X} - \frac{n\bar{X} + m\bar{Y}}{n+m})^2 \\ &= \sum (X_i - \bar{X})^2 + n \frac{(n\bar{X} + m\bar{X} - n\bar{X} - m\bar{Y})^2}{(n+m)^2} \\ &= \sum (X_i - \bar{X})^2 + \frac{nm^2}{(n+m)^2} (\bar{X} - \bar{Y})^2 \end{aligned}$$

and similarly for the  $Y$  term. So we have:

$$\begin{aligned} \hat{\sigma}_R^2 &= \frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2}{n+m} + (\frac{nm^2}{(n+m)^3} + \frac{n^2m}{(n+m)^3})(\bar{X} - \bar{Y})^2 \\ &= \frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2}{n+m} + \frac{nm}{(n+m)^2} (\bar{X} - \bar{Y})^2 \end{aligned}$$

Without the  $H_0$  restriction, we get the usual  $\hat{\mu}_X = \bar{X}$  and  $\hat{\mu}_Y = \bar{Y}$  and  $\hat{\sigma}^2 = \frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2}{n+m}$ .

Plugging these terms into the likelihood functions, similar to 8.37, we get terms canceling out in the exponentials. So we are left with simply

$$\lambda(X, Y) = \left( \frac{\hat{\sigma}_R^2}{\hat{\sigma}^2} \right)^{-(n+m)/2}$$

and we reject for small  $\lambda(X, Y)$ , which is equivalent to rejecting for large  $\frac{\hat{\sigma}_R^2}{\hat{\sigma}^2}$ .

$$\begin{aligned} \frac{\hat{\sigma}_R^2}{\hat{\sigma}^2} &= \frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2 + \frac{nm}{n+m} (\bar{X} - \bar{Y})^2}{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2} \\ &= 1 + \frac{nm}{n+m} \frac{(\bar{X} - \bar{Y})^2}{(n+m-2)S_p^2} \\ &= 1 + \frac{(\bar{X} - \bar{Y})^2}{(n+m-2)(n^{-1}+m^{-1})S_p^2} \end{aligned}$$

We can ignore the leading constant term (1) as well as the factor of  $(n+m-2)^{-2}$ , and we are left with

$$\begin{aligned} \frac{(\bar{X} - \bar{Y})^2}{(n^{-1}+m^{-1})S_p^2} &> c \\ \implies \left| \frac{\bar{X} - \bar{Y}}{(n^{-1}+m^{-1})S_p} \right| &> c' \end{aligned}$$

**b**

Under  $H_0$ ,  $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$  and  $\bar{Y} \sim \mathcal{N}(\mu, \sigma^2/m)$ , so  $\bar{X} - \bar{Y} \sim \mathcal{N}(0, (n^{-1} + m^{-1})\sigma^2)$ .

Since  $S_X^2$  and  $S_Y^2$  are  $\chi^2$  distributed with  $n-1$  and  $m-1$  degrees of freedom respectively, and since  $S_p^2$  is just a linear combination of the two,  $S_p^2 \sim \chi_{n+m-2}^2$ . So  $\frac{\bar{X} - \bar{Y}}{S_p \sqrt{n^{-1} + m^{-1}}} \sim T_{n+m-2}$ .

**c**

```
# data
core <- c(1294, 1279, 1274, 1264, 1263, 1254, 1251,
          1251, 1248, 1240, 1232, 1220, 1218, 1210)
periphery <- c(1284, 1272, 1256, 1254, 1242,
               1274, 1264, 1256, 1250)

n <- length(core)
m <- length(periphery)

# sample means
core.mean <- mean(core)
periphery.mean <- mean(periphery)

# pooled variance
s.p <- sqrt(
  (sum((core - core.mean) ** 2) + sum((periphery - periphery.mean) ** 2)) /
  (n + m - 2))

# t-statistic
t.stat <- (core.mean - periphery.mean) / (s.p * sqrt(1 / n + 1 / m))
t.stat
```

```
[1] -1.290656
```

So we would reject  $H_0$  if our chosen  $c$  is less than 1.291.

**d**

We can compute the two-sided  $p$ -value using the  $t$ -distribution:

```
2 * pt(t.stat, df = n + m - 2)
```

```
[1] 0.2108527
```

Failing to reject  $H_0$  would be equivalent to performing a UMP level-.21 test.

e

```
# number of simulations
n.sim <- 1e3

p.value.vector <- sapply(seq(n.sim), function(i) {
  # bootstrap
  core.bs <- sample(core, n, replace = TRUE)
  periphery.bs <- sample(periphery, m, replace = TRUE)

  # compute statistics
  core.mean <- mean(core.bs)
  periphery.mean <- mean(periphery.bs)
  s.p <- sqrt(
    (sum((core.bs - core.mean) ** 2) +
     sum((periphery.bs - periphery.mean) ** 2)) /
    (n + m - 2))

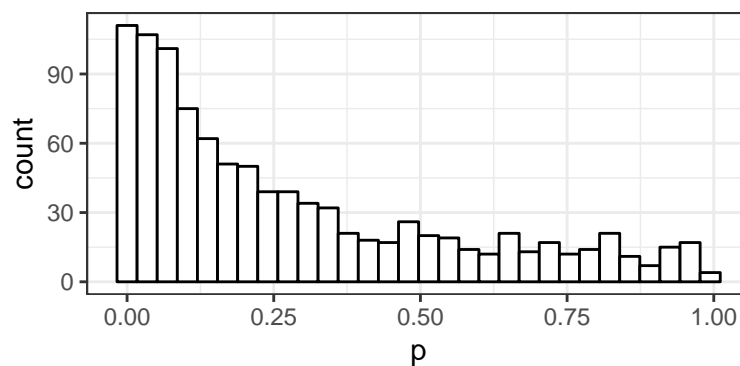
  # t-statistic
  t.stat <- (core.mean - periphery.mean) / (s.p * sqrt(1 / n + 1 / m))

  # p-value
  2 * pt(-abs(t.stat), df = n + m - 2)
})

summary(p.value.vector)

      Min.   1st Qu.   Median     Mean   3rd Qu.     Max.
0.0000057 0.0627951 0.1833563 0.2813335 0.4501091 0.9938876

ggplot() +
  geom_histogram(aes(x = p.value.vector), fill = 'white', colour = 'black') +
  labs(x = 'p')
```



## Problem 5

11.13 from Casella & Berger

Similar to the example from class, we have

$$\begin{aligned} L(\theta, \sigma^2 | y) &= \prod_i^k \prod_j^{n_i} (2\pi\sigma^2)^{-1/2} \exp(-\frac{1}{2\sigma^2} (y_{ij} - \theta_i)^2) \\ &= (2\pi\sigma^2)^{-N/2} \exp(-\frac{1}{2\sigma^2} \sum_i \sum_j (y_{ij} - \theta_i)^2) \\ &= (2\pi\sigma^2)^{-N/2} \exp(-\frac{1}{2\sigma^2} (\sum_i \sum_j (y_{ij} - \bar{y}_i)^2 + \sum_i \sum_j (\bar{y}_i - \theta_i)^2)) \end{aligned}$$

Under  $H_1$ , the second term is 0 since  $\hat{\theta}_i = \bar{y}_i$ . Under  $H_0$ ,  $\hat{\theta}_R = \bar{y}$  where  $\bar{y}$  is the overall mean.

Simiarly, under  $H_0$ , we get  $\hat{\sigma}_R^2 = \frac{1}{N} (\sum_i \sum_j (y_{ij} - \bar{y}_i)^2 + \sum_i \sum_j (\bar{y}_i - \bar{y})^2)$   
and under  $H_1$ , we get  $\hat{\sigma}^2 = \frac{1}{N} \sum_i \sum_j (y_{ij} - \bar{y}_i)^2$  since the second term is 0.

Similar to before, the terms in the exponentials disappear when we plug in the MLEs for  $\theta_i$  and  $\sigma^2$ . So the likelihood ratio is:

$$\lambda(Y) = (\hat{\sigma}_R^2 / \hat{\sigma}^2)^{-N/2}$$

And we reject when this value is small. This is equivalent to rejecting when  $\hat{\sigma}_R^2 / \hat{\sigma}^2$  is large.

$$\begin{aligned} \hat{\sigma}_R^2 / \hat{\sigma}^2 &= \frac{\sum_i \sum_j (y_{ij} - \bar{y}_i)^2 + \sum_i \sum_j (\bar{y}_i - \bar{y})^2}{\sum_i \sum_j (y_{ij} - \bar{y}_i)^2} \\ &= 1 + \frac{\sum_i \sum_j (\bar{y}_i - \bar{y})^2}{\sum_i \sum_j (y_{ij} - \bar{y}_i)^2} \end{aligned}$$

We can ignore the leading constant of 1. Then we note that this is a ratio of two  $\chi^2$  distributed random variables (times a constant), with the numerator having  $k - 1$  degrees of freedom and the denominator having  $N - k$  degrees of freedom. Thus this ratio is  $F_{k-1, N-k}$ -distributed, and we reject when it is greater than  $F_{k-1, N-k, \alpha}$  for a level- $\alpha$  test.