STAT-S675

Homework 5

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Problem 1

[Exercise 4.6.1 in the notes]

Want to show $\tau(\kappa(\Gamma)) = P\Gamma P$.

$P = (I - \frac{ee^T}{n})$
$\tau(X) = -\frac{1}{2}PXP$
$\kappa(X) = \text{diag}(X)e^T - 2X + e\text{diag}(X)^T$

Since Γ is a similarity matrix, $\text{diag}(\Gamma) = e$ (we can just normalize everything to this WLOG).

$$\begin{aligned}
\kappa(\Gamma) &= \text{diag}(\Gamma)e^T - 2\Gamma + e\text{diag}(\Gamma)^T \\
&= ee^T - 2\Gamma + ee^T \\
&= 2(ee^T - \Gamma)
\end{aligned}$$

So if we want to find $\tau(\kappa(\Gamma))$:

$$\begin{aligned}
\tau(\kappa(\Gamma)) &= \tau(2(ee^T - \Gamma)) \\
&= -\frac{1}{2}P(2(ee^T - \Gamma))P \\
&= P(\Gamma - ee^T)P \\
&= P\Gamma P - Pee^T P
\end{aligned}$$

So we just need to show that $Pee^T P = 0$. First, note that $e^T e = n$.

$$\begin{aligned}
Pee^T P &= (I - \frac{ee^T}{n})ee^T(I - \frac{ee^T}{n}) \\
&= (Iee^T - \frac{ee^T ee^T}{n})(I - \frac{ee^T}{n}) \\
&= (ee^T - \frac{nee^t}{n})(I - \frac{ee^T}{n}) \\
&= (ee^T - ee^T)(I - \frac{ee^T}{n}) \\
&= 0
\end{aligned}$$

Therefore $\tau(\kappa(\Gamma)) = P\Gamma P$.

Problem 2

[Exercise 4.6.3 in the notes]

```r
import::from(readr, read_table2)
import::from(magrittr, `%>%`, `%<>%`)
library(ggplot2)
import::from(GGally, ggpairs)
import::from(ggrepel, geom_label_repel)
import::from(gridExtra, grid.arrange)

# read the data
Delta <- read_table2('http://pages.iu.edu/~mtrosset/Courses/675/congress.dat',
                     col_names = FALSE) %>%
  as.matrix()

# code from the notes
Delta.cmds <- cmdscale(Delta, k = 3, eig = TRUE)
X <- Delta.cmds$points
```

Part a

```r
r <- sum(Delta.cmds$eig > 0)
sum.pos.eig.vals <- sum(sapply(Delta.cmds$eig, function(i) max(i, 0)))
sum.neg.eig.vals <- sum(sapply(Delta.cmds$eig, function(i) min(i, 0)))
print(c(sum.pos.eig.vals, sum.neg.eig.vals))
```

```
[1] 920.1218 -90.3218
```

9 out of 15 eigenvalues are positive. 0.911 of the variation is explained by the positive eigenvalues.

Part b

```r
sum(sort(Delta.cmds$eig, decreasing = TRUE)[1:2]) / sum.pos.eig.vals
```

```
[1] 0.6998389
```

```r
sum(sort(Delta.cmds$eig, decreasing = TRUE)[1:3]) / sum.pos.eig.vals
```

```
[1] 0.8116862
```

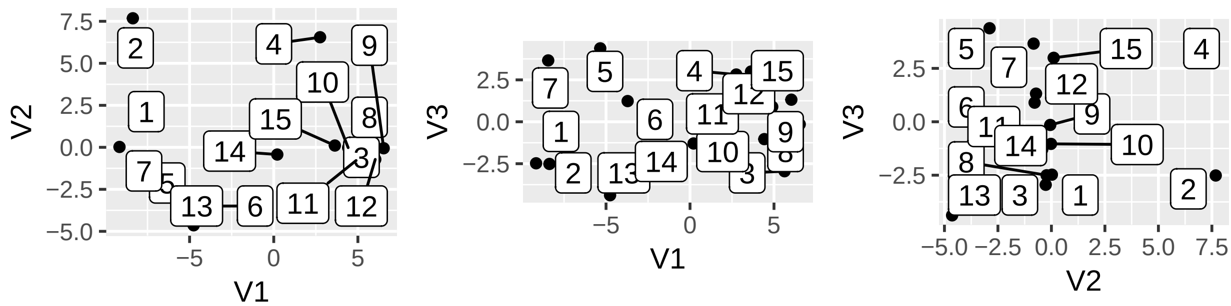
Two eigenvalues is ~70% of the sum of the positive eigenvalues while three is ~80%. If we want our approximation to capture at least 95% of the variation in the data, then neither $d = 2$ nor $d = 3$ are sufficient.

Part c

```r
X.df <- X %>%
  as.data.frame() %>%
  dplyr::mutate(id = rownames(.))

plot.12 <- ggplot(X.df, aes(x = V1, y = V2, label = id)) +
  geom_point() +
  geom_label_repel() +
  coord_fixed()
plot.13 <- ggplot(X.df, aes(x = V1, y = V3, label = id)) +
```

```r
  geom_point() +
  geom_label_repel() +
  coord_fixed()
plot.23 <- ggplot(X.df, aes(x = V2, y = V3, label = id)) +
  geom_point() +
  geom_label_repel() +
  coord_fixed()
grid.arrange(plot.12, plot.13, plot.23, ncol = 3)
```



When looking at just the first two components, we see that congressmen 2 and 4 are outliers whereas the others fall roughly in a line along V1. However, when we compare the first and third components, 1 and 2 are very close, as are 4 and 15. Furthermore, it appears that 1 is on the opposite side of the spectrum from 3 when looking at the first two components, but when looking and the second and third components, 1 and 3 are very similar.

The scatterplots using the three components seem to suggest that we lose information by truncating at the second component.