

STAT-S620

Assignment 7

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4.4.7

$$\psi(t) = \frac{1}{4}(3e^t + e^{-t})$$

$$\psi'(t) = \frac{1}{4}(3e^t - e^{-t})$$

$$\psi''(t) = \frac{1}{4}(3e^t + e^{-t})$$

Then $\mu = \mu_1 = \psi'(0) = 2/4 = \boxed{1/2}$.

Since $\psi''(t) = \psi(t)$, we know that $\mu_2 = \psi''(0) = \psi(0) = 1$. Therefore, $\text{var}(X) = 1 - 1/4 = \boxed{3/4}$.

4.4.11

For the pmf of a discrete X , the mgf is $\psi(t) = \sum_x P(X = x)e^{tx}$. Then $f(x)$ has support $\{1, 4, 8\}$, and $\boxed{f(1) = 1/5, f(4) = 2/5, f(8) = 2/5}$ (and 0 otherwise).

4.4.12

Using the same idea as exercise 4.4.11, we can say X is discrete and $\boxed{f(0) = 2/3, f(1) = 1/6, f(-1) = 1/6}$ (and 0 otherwise).

4.6.13

Part a

$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$. We know that $\text{cov}(X, Y) = (-1/6)(3)(2) = -1$. Then $\text{var}(X + Y) = 9 + 4 - 2 = \boxed{11}$.

Part b

$\text{var}(X - 3Y + 4) = \text{var}(X) + 9\text{var}(Y) + (2)(1)(-3)\text{cov}(X, Y) = \boxed{51}$.

4.7.7

$f_X(x) = \int_0^1 (x + y)dy = xy + \frac{y^2}{2} \Big|_{y=0}^1 = x + 1/2$. Then $g(y|x) = \frac{x+y}{x+1/2}$ (for $y \in [0, 1]$ and 0 otherwise).

$$E[Y|X] = \int_0^1 \frac{xy+y^2}{x+1/2} dy = \boxed{\frac{x/2 + 1/3}{x + 1/2}}.$$

$$E[Y^2|X] = \int_0^1 \frac{xy^2+y^3}{x+1/2} dy = \frac{x/3+1/4}{x+1/2}.$$

$$\text{Then } \text{var}(Y|X) = \frac{x/3+1/4}{x+1/2} - \left(\frac{x/2+1/3}{x+1/2}\right)^2 = \boxed{\frac{1}{12} - \frac{1}{36(2x+1)^2}} \text{ (via WolframAlpha).}$$

mgf of the normal distribution

Part 1

By definition:

$$\psi(t) = \int e^{tx} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Substituting $z = \frac{x-\mu}{\sigma}$, we get that $x = \mu + \sigma z$ and $dz = \frac{1}{\sqrt{\sigma^2}} dx$, so we are left with:

$$\begin{aligned} \psi(t) &= \int e^{t(\mu+\sigma z)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= \int \frac{1}{\sqrt{2\pi}} e^{-z^2/2 + \sigma t z + \mu t} dz \\ &= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2\sigma t z - \mu t)} dz \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \sigma t)^2} dz \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \end{aligned}$$

Part 2

$$\psi'(t) = (\mu + \sigma^2 t) \psi(t)$$

$$\text{Then } E[X] = \psi'(0) = \mu$$

Part 3

$$\psi''(t) = (\mu + \sigma^2 t)^2 \psi(t) + \sigma^2 \psi(t)$$

$$\text{Then } E[X^2] = \psi''(0) = \mu^2 + \sigma^2, \text{ and } \text{var}(X) = E[X^2] - E[X]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

Part 4

$$\text{We know that the mgf of } Z \text{ is } \psi_Z(t) = \psi_X(t)\psi_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}.$$

Not from text

Part 1

If, for when $n = 1$, $r = 1/2$, then the possible values for R when $n = 2$ are $1/2 * 1/2 = 1/4$ and $1/2 * 2 = 1$. On the other hand, if when $n = 1$, $r = 2$, then the possible values for R when $n = 2$ are $2 * 1/2 = 1$ and $2 * 2 = 4$.

For a given n and i , $r = 2^{2i-n}$.

Part 2

We can see this is a binomial distribution. $P(R = r) = \binom{n}{i} 2^{-n}$.

Part 3

By symmetry, $E[R] = 1$.

Part 4

Since $E[R] = 1$, the expected value of the fortune after n games is always the original value, c .