MATH-M463

Homework 6

John Koo

Problem 1

$p = \frac{1}{6}$, $q = \frac{5}{6}$

Part a

Let's say that the third player wins on the first try. The probability of this is q^2p. Similarly, the probably of the third player winning on the second try is q^5p, third try is q^8p, i$^{\text{th}}$ try is $q^{3i-1}p$. So $P(\text{third player wins}) = \sum_{i=1} q^{3i-1}p = p\sum_{i=0} q^{3i+2} = pq^2\sum_{i=0} q^{3i} = \frac{pq^2}{1-q^3} = \boxed{\frac{25}{91}}$

Part b

Similar to part (a), the probably of the second player winning on the i$^{\text{th}}$ try is just pq^{3i-2}, so $P(\text{second player wins}) = pq\sum_{i=0} q^{3i} = \frac{pq}{1-q^3} = \boxed{\frac{30}{91}}$

Part c

Similar to parts (a) and (b), the probability of the first player winning on the i$^{\text{th}}$ try is pq^{3i-3}, so $P(\text{first player wins}) = p\sum_{i=0} q^{3i} = \frac{p}{1-q^3} = \boxed{\frac{36}{91}}$

Problem 2

Let $W = \frac{XY}{Y} = X$. Since W is a function of X and Y but not Z and X and Y are independent of Z, W is independent of Z. But W is just X, so X and Z are pairwise independent. The same goes for the other two pairs by symmetry.

Problem 3

Let $W = X + Y + Z$. Then $E[W] = E[X] + E[Y] + E[Z]$.

For brevity, denote the respective expected values by μ_X, μ_Y, and μ_Z.

Consider $(W - E[W])^2$:

$$(W - E[W])^2 = ((X - \mu_X) + (Y - \mu_Y) + (Z - \mu_Z))^2$$

$$= (X - \mu_X)^2 + (Y - \mu_Y)^2 + (Z - \mu_Z)^2 + 2((X - \mu_X)(Y - \mu_Y) + (X - \mu_X)(Z - \mu_Z) + (Y - \mu_Y)(Z - \mu_Z))$$

So taking the expected value of this, we get (by the addition rule):

$$E[(W - E[W])^2]$$
$$= var(X) + var(Y) + var(Z) + 2E[(X - \mu_X)(Y - \mu_Y) + (X - \mu_X)(Z - \mu_Z) + (Y - \mu_Y)(Z - \mu_Z)]$$

Since X, Y, and Z are independent, we can separate out each of the cross products, e.g., $E[(X-\mu_X)(Y-\mu_Y)] = E[X - \mu_X]E[Y - \mu_Y]$. But for each of the random variables, the expected value of the variable minus its expected value is just 0, e.g., $E[X - \mu_X] = E[X] - \mu_X = \mu_X - \mu_X = 0$. So all of the cross terms cancel out to 0, and we're left with:

$$var(X + Y + Z) = E[(W - E[W])^2]$$
$$= var(X) + var(Y) + var(Z)$$

Problem 3 (again?)

10" is $|\frac{10-21.88}{7.58}| \approx 1.567$ standard deviations from the mean.

Part a

By Markov's inequality, we know that $P(|X - E[X]| \geq k\sigma_X) \leq \frac{1}{k^2}$. So if we want the probability that X is more than 1.567 standard deviations from the mean, we can say that this probability is no more than $\frac{1}{1.567^2}$ ≈ 0.407. In the extreme case where we have a heavily skewed distribution where the density more than 1.567 standard deviations above the mean is 0, all of 0.407 must be 1.567 standard deviations below the mean, so $\boxed{P(X \leq 10) \leq 0.407}$.

Part b

On the other hand, if the distribution is symmetric about the mean, then half of this density is 1.567 standard deviations above the mean while half of it is 1.567 standard deviations below the mean. Then $\boxed{P(X \leq 10) \leq 0.204}$.

Problem 4

Part a

Let $N \sim poisson(\mu)$. We know that $E[N] = \mu$ and $var(N) = \mu$, so $E[N^2] = var(N) + E[N]^2 = \mu + \mu^2$.

Consider $E[N(N-1)(N-2)]$. This is equal to:

$$\sum_{i=0} i(i-1)(i-2)e^{-\mu}\mu^i/i!$$

When $i \in \{0, 1, 2\}$, the corresponding terms are 0. So this is equal to:

$$\sum_{i=3} i(i-1)(i-2)e^{-\mu}\mu^i/i!$$

Then

$$= e^{-\mu}\mu^3 \sum_{i=3} \mu^{i-3}/(i-3)!$$
$$= e^{-\mu}\mu^3 \sum_{i=0} \mu^{i}/i!$$
$$= e^{-\mu}\mu^3 e^{\mu} = \mu^3$$

.

On the other hand, this is equal to $E[N^3] - 3E[N^2] + 2E[N]$ after some expansion of the original term. So solving for $E[N^3]$:

$$E[N^3] = \mu^3 + 3\mu^2 + \mu$$

If we expand $E[(X-Y)^3]$ and separate the cross terms using the fact that $X \perp Y$, we get:

$$E[X^3] - 3E[X^2]E[Y] + 3E[X]E[Y^2] - E[Y^3]$$

$E[X] = 2,\ E[X^2] = 2 + 2^2 = 6,\ E[X^3] = 2^3 + 3(2^2) + 2 = 22$

$E[Y] = 3,\ E[Y^2] = 3 + 3^2 = 12,\ E[Y^3] = 3^3 + 3(3^2) + 3 = 57$

So

$$E[X^3] - 3E[X^2]E[Y] + 3E[X]E[Y^2] - E[Y^3]$$
$$= 22 - 3(6)(3) + 3(2)(12) - 57$$
$$= \boxed{-17}$$

Part b

Since $X \perp Y$, $5X \perp Y$, so $var(5X+Y) = 25var(X) + var(Y) = 25(2) + 3 = \boxed{53}$.

Problem 5

This is equivalent to $P(\text{first two are face cards}) = \frac{12}{52}\frac{11}{51} = \boxed{\frac{1}{17}}$.

Problem 6

Part a

$P(RRB) = \frac{5}{7}\frac{4}{6}\frac{2}{5} = \boxed{\frac{4}{21}}$

Part b

$P(BRB) = \frac{2}{7}\frac{5}{6}\frac{1}{5} = \boxed{\frac{1}{21}}$

Part c

$P(RBB) = \frac{5}{7}\frac{2}{6}\frac{1}{5} = \boxed{\dfrac{1}{21}}$

Part d

We can just sum up our answers from parts (a), (b), and (c): $P(\text{third is } B) = \frac{4+1+1}{21} = \boxed{\dfrac{2}{7}}$

We also know that $P(\text{third is } B) = P(B) = \boxed{\dfrac{2}{7}}$.

Problem 7

This is just the collector's problem. $E[X] = 10\sum_i^{10} \frac{1}{i} \approx \boxed{29.29}$.

Problem 8

For this scenario, we need one out in the first seven bats, followed by two outs. So one possibility is out, not out six times, out, out. This has probability pq^6pp where $p = 0.6$ and $q = 0.4$. We can then see that all possibilities have the same probability p^3q^6 and the first seven can be ordered however. Then there are $\binom{7}{1}$ possibilities. So $P(\text{2nd and 3rd outs are made by 8th and 9th players}) = \binom{7}{1}0.6^30.4^6 \approx \boxed{6.193 \times 10^{-3}}$ We can also just think of this as $P(T_2 = 8)p$.