STAT-S631

Assignment 9

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Problem 1

[From ALR 5.14]

Problem 2

We are given:

$X = [X_1|X_2]$
$H = X(X^TX)^{-1}X^T$
$H_R = X_1(X_1^TX_1)^{-1}X_1^T$

Part a

Show $H_RX_1 = X_1$

$H_RX_1 = X_1(X_1^TX_1)^{-1}X_1^TX_1 = X_1(X_1^TX_1)^{-1}(X_1^TX_1) = X_1$

Show $HX_1 = X_1$

Consider HX. We know that $HX = X$, so we can say:

$$\begin{aligned} HX &= H[x_0, x_1, x_2, ..., x_p] \\ &= [Hx_0, Hx_1, ..., Hx_p] \end{aligned}$$

Where x_i is the i^{th} column vector of X.

But then $HX = X = [x_0, ..., x_p]$. Therefore:

$$\begin{aligned} [Hx_0, ..., Hx_p] &= [x_0, ..., x_p] \\ \implies Hx_i &= x_i \end{aligned}$$

Then if we consider HX_1:

$$\begin{aligned} HX_1 &= H[x_0, ..., x_q] \\ &= [Hx_0, ..., Hx_q] \\ &= [x_0, ..., x_q] \\ &= H_1 \end{aligned}$$

Show $HH_R = H_R$

$HH_R = H\big(X_1(X_1^TX_1)^{-1}X_1^T\big) = (HX_1)(X_1^TX_1)^{-1}X_1^T = X_1(X_1^TX_1)^{-1}X_1^T = H_R$

Part b

Show $H - H_R$ is symmetric

$H - H_R$ is symmetric iff $H - H_R = (H - H_R)^T$.

We also know that H and H_R are symmetric.

Therefore, $(H - H_R)^T = H^T - H_R^T = H - H_R$.

Show $H - H_R$ is idempotent

$H - H_R$ is idempotent iff $(H - H_R)^2 = H - H_R$

We know that H and H_R are idempotent.

Therefore:

$$\begin{aligned}(H - H_R)(H - H_R) &= HH - HH_R - H_RHH_RH_R \\ &= H - H_R - H_RH + H_R \\ &= H - H_RH\end{aligned}$$

Consider that H and H_R are symmetric and $HH_R = H_R$. Therefore, $H_R = H_R^T = (HH_R)^T = H_R^TH^T = H_RH$ $\implies H_R = H_RH$.

Therefore:

$$\begin{aligned}H - H_RH &= H - H_R \\ \implies (H - H_R)^2 &= H - H_R\end{aligned}$$

Part c

$$\begin{aligned}\frac{SSreg}{\sigma^2} &= \frac{RSS_R - RSS_F}{\sigma^2} \\ &= \frac{Y^T(I - H_R)Y - Y^T(I - H)Y}{\sigma^2} \\ &= \frac{Y^T(H - H_R)Y}{\sigma^2} \\ &= \frac{(Y - X_1\hat{\beta}_1)^T(H - H_R)(Y - X_1\hat{\beta}_1)}{\sigma^2}\end{aligned}$$

We know that $Y - X_1\hat{\beta}_1 \sim \mathcal{N}(0, \sigma^2(I - H_R))$. Furthermore, we know that $rank(H - H_R) = rank(H) - rank(H_R) = p + 1 - (p + 1 - q) = q$ (assuming H and H_R are full rank).

Then $\frac{SSreg}{\sigma^2} \sim \chi_q^2$ if $(\frac{H - H_R}{\sigma^2})(\sigma^2(I - H_R)) = (H - H_R)(I - H_R)$ is idempotent. But $(H - H_R)(I - H_R) = H - HH_R - H_R + H_RH_R = H - H_R - H_R + H_R = H - H_R$ which we already know to be idempotent. Therefore,

$$\frac{SSreg}{\sigma^2} \sim \chi_q^2$$

Part d

$$\hat{\sigma}^2 = \frac{RSS}{n-p-1}$$
$$= Y^T \frac{I-H}{n-p-1} Y$$

So we have to show:

$$\left(\frac{H-H_R}{\sigma^2}\right)\left(\sigma^2(I-H)\right)\left(\frac{I-H}{n-p-1}\right) = 0$$

We know that the product of the first two components is $H - H_R$. Therefore,

$$\left(\frac{H-H_R}{\sigma^2}\right)\left(\sigma^2(I-H)\right)\left(\frac{I-H}{n-p-1}\right) = (H-H_R)(I-H)\frac{1}{n-p-1}$$
$$= (H - H - H_R + H_R)\frac{1}{n-p-1}$$
$$= 0$$

Part e

We know that $\frac{SSreg}{\sigma^2} \sim \chi^2_q$ and $\frac{RSS}{\sigma^2} \sim \chi^2_{n-p-1}$. Then we know that $\frac{\frac{SSreg}{\sigma^2}/q}{\frac{RSS}{\sigma^2}/(n-p-1)} = \frac{SSreg/q}{RSS/(n-p-1)} \sim F_{q,n-p-1}$

Problem 3

[From ALR 6.4]