MATH-M463

Homework 9

John Koo

Problem 1

[Exercise 4 in 5.2]

Problem 2

[Exercise 6 in 5.2]

Part a

Part b

Problem 3

[Exercise 6 in 5.3]

Part a

Part b

Problem 4

[Exercise 8 in 5.3]

Problem 5

Part a

$$
\begin{aligned}
\int f(x,y)dxdy &= \lambda^3 \int_0^\infty e^{-\lambda y} \int_0^y xdxdy \\
&= \frac{\lambda^3}{2} \int_0^\infty y^2 e^{-\lambda y} dy
\end{aligned}
$$

Let:
$u = y^2$, $du = 2ydy$
$v = -\frac{1}{\lambda} e^{-\lambda y}$, $dv = e^{-\lambda y} dy$

Then we have:

$$= \frac{\lambda^3}{2}\left(-\frac{y^2}{\lambda}e^{-\lambda y}\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty ye^{-\lambda y}dy\right)$$
$$= \lambda^2 \int_0^\infty ye^{-\lambda y}dy$$

Let:
$u = y$, $du = dy$
$v = -\frac{1}{\lambda}e^{-\lambda y}$, $dv = e^{-\lambda y}dy$

Then:

$$= \lambda^2\left(-\frac{y}{\lambda}e^{-\lambda y}\Big|_0^\infty + \int_0^\infty \frac{1}{\lambda}e^{-\lambda y}dy\right)$$
$$= \lambda \int_0^\infty e^{-\lambda y}dy$$
$$= -e^{-\lambda y}\Big|_0^\infty$$
$$-(0-1) = \boxed{1}$$

Part b

$$f_X(x) = \int_x^\infty f(x,y)dy$$
$$= \int_x^\infty \lambda^3 xe^{-\lambda y}dy$$
$$= \lambda^3 x\int_x^\infty e^{-\lambda y}dy$$
$$= \lambda^3 x\left(-\frac{1}{\lambda}e^{-\lambda y}\Big|_x^\infty\right)$$
$$= -\lambda^2 x(0 - e^{-\lambda x})$$
$$= \boxed{\lambda^2 xe^{-\lambda x}}$$

Part c

$$E[X] = \int_0^\infty xf_X(x)dx$$
$$= \lambda^2 \int_0^\infty x^2e^{-\lambda x}dx$$

Let:
$u = x^2$, $du = 2xdx$
$v = -\frac{1}{\lambda}e^{-\lambda x}$, $dv = e^{-\lambda x}dx$

Then:

$$= \lambda^2\left(-\frac{x^2}{\lambda}e^{-\lambda x}\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty xe^{-\lambda x}dx\right)$$

$$= 2\lambda \int_0^\infty x e^{-\lambda x} dx$$

Let:
$u = x,\ du = dx$
$v = -\frac{1}{\lambda} e^{-\lambda x},\ dv = e^{-\lambda x} dx$

Then:

$$\begin{aligned}
&= 2\lambda \left(-\frac{x}{\lambda} e^{-\lambda x} \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx \right) \\
&= 2 \int_0^\infty e^{-\lambda x} dx \\
&= -\frac{2}{\lambda}(0 - 1) \\
&= \boxed{\frac{2}{\lambda}}
\end{aligned}$$

Part d

$$\begin{aligned}
f_Y(y|X = x) &= \frac{f(x, y)}{f_X(x)} \\
&= \frac{\lambda^3 x e^{-\lambda y}}{\lambda^2 x e^{-\lambda x}} \\
&= \boxed{\lambda e^{-\lambda(y-x)}}
\end{aligned}$$

Part e

$$\begin{aligned}
E[Y|X = 1] &= \int_1^\infty \lambda y e^{-\lambda(y-1)} dy \\
&= \lambda e^\lambda \int_1^\infty y e^{-\lambda y} dy
\end{aligned}$$

Let:
$u = y,\ du = dy$
$v = -\frac{1}{\lambda} e^{-\lambda y},\ dv = e^{-\lambda y} dy$

Then

$$\begin{aligned}
&= \lambda e^\lambda \left(-\frac{y}{\lambda} e^{-\lambda y} \Big|_1^\infty + \frac{1}{\lambda} \int_1^\infty e^{-\lambda y} dy \right) \\
&= \lambda e^\lambda \left(\frac{e^{-\lambda}}{\lambda} - \frac{1}{\lambda^2} e^{-\lambda y} \Big|_1^\infty \right) \\
&= 1 - \frac{e^\lambda}{\lambda}(0 - e^{-\lambda}) \\
&= \boxed{1 + \frac{1}{\lambda}}
\end{aligned}$$

Problem 6

Part a

First note that since $P(A \cup B) = P(A) + P(B) - P(AB)$, $P(AB) = P(A) + P(B) - P(A \cup B) = .2 + .6 - .7 = .1$

Then $P(A|B) = \frac{P(AB)}{P(B)} = \frac{.1}{.6} = \boxed{\frac{1}{6}}$

Part b

$P(A)P(B) = .12 > .1 = P(AB)$, so they are $\boxed{\text{negatively dependent}}$

Part c

$cov(X, Y) = P(AB) - P(A)P(B) = .1 - .12 = .02$

Then note that $var(X) = E[X^2] - E[X]^2 = E[X] - E[X]^2 = P(A) - P(A)^2 = .2 - .04 = .16$

Similarly, $var(Y) = P(B) - P(B)^2 = .6 - .36 = .24$

Then $corr(X, Y) = \frac{.02}{\sqrt{.16 \times .24}} \approx \boxed{.102}$