S722 HW1

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To save on typing, I will denote $\frac{\partial^k}{\partial x^k} f(x) = \partial_x^k f(x)$.

Part 1

8.3

Let $y = \sum_i y_i$ be the number of successes. We showd that this is a sufficient statistic. Then we can write λ in terms of y:

$\lambda(y) = \begin{cases} 1 & y/m \leq \theta_0 \\ (\frac{\theta_0}{y/m})^y (\frac{1-\theta_0}{1-y/m})^{m-y} & y/m > \theta_0 \end{cases}$

And we reject H_0 if λ is too small, i.e., $\lambda < c$.

We can also see that we should reject when $y \leq m\theta_0$ since we would get $\lambda = 1$.

We can see that when $y > m\theta_0$, we have:

$\log \lambda = y \log \theta_0 - y \log y + y \log m + (m - y) \log(1 - \theta_0) - (m - y) \log(1 - y/m)$.

Derivating once, we get:

$(\log \lambda)' = \log \theta_0 - \log y - 1 + \log m - \log(1 - \theta_0) + \frac{1}{1-y/m} + \log(1 - y/m) - \frac{y/m}{1-y/m}$
$= \log \frac{\theta_0(1-y/m)}{my(1-\theta_0)}$.

Setting this negative is equivalent to setting the term inside the logarithm to less than 1, which gives us

$y > \frac{m\theta_0}{m^2(1-\theta_0)+\theta_0}$

or alternatively

$y/m = \bar{y} > \frac{\theta_0}{m^2(1-\theta_0)+\theta_0}$

Since the denominator is always positive, y/m is always less than θ_0.

8.5

Part a

$\log f(x_i \mid \theta, \nu) = \log \theta + \theta \log \nu - (\theta + 1) \log x_i$, so we get

$\ell(\theta, \nu) = n \log \theta + n\theta \log \nu - (\theta + 1) \sum_i x_i$

Then we get $\partial_\nu \ell = \frac{n\theta}{\nu} = 0 \implies \nu \to \infty$. However, $\nu \leq x_{(1)}$ so we can set $\hat{\nu} = x_{(1)}$.

$\partial_\theta \ell = n/\theta + n \log \hat{\nu} - \sum_i x_i = 0$
$\implies n + \theta(n \log x_{(1)} - \sum_i x_i) = 0$
$\implies \hat{\theta} = \frac{n}{\sum_i x_i - n \log x_{(1)}}$

Part b

In order to show this, we must show that λ has one global maximum.

First, we note that:

$T = \log \frac{\prod_i X_i}{X_{(1)}^n}$
$= \sum_i \log X_i - n \log X_{(1)}$
$\implies \hat{\theta} = n/T$

Under H_0, we get $L(\Theta_0) = \frac{x_{(1)}^n}{\prod_i x_i^2}$, so the likelihood ratio statistic is

$\lambda = \frac{X_{(1)}^n / \prod_i X_i^2}{(n/T)^n X_{(1)}^{n^2/T} / \prod_i X_i^{n/T+1}}$
$= \left(\frac{X_{(1)}^n}{\prod_i X_i}\right)^{1-n/T} \left(\frac{T}{n}\right)^n$ (after some algebra)
$= T^{n/T-1+n}/n^n$

Then $\log \lambda = (n/T - 1 + n)\log T - n \log n$ and
$(\log \lambda)' = -\frac{n}{T^2} \log T + \frac{n/T-1+n}{T}$

In order to show the desired result, we need to set this to zero and solve for T to obtain a single local maximum, but I don't know how to do this analytically. However, some algebra after setting to zero yields:

$\log T = 1 + (n^{-2} - n^{-1})T$

which is a logarithm on one side and a linear equation on the other. Since the left side is monotone increasing and the right side is monotone decreasing for $n > 1$ and they are both unbounded, they must meet at some point. And a logarithmic curve can meet a negative linear curve at at most one point, so there is only one maximum for λ (after checking the second derivative).

Part c

We saw in the previous part that T can be written as

$T = \sum_i \log X_i - n \log X_{(1)}$
$= \sum_i (\log X_i - \log X_{(1)})$

Let $Y = \log X_i$.
Then $X = e^Y$.
and $X' = e^Y$.
Then $f_Y(y) = \nu e^{-y} I(y \geq \nu)$.

Now let $Z_i = Y_i - Y_{(1)}$.
Then the indicator function goes away, so we have $f_Z(z) = e^{-z}$, i.e., $Z_i \sim \mathit{Exponential}(1)$. Therefore, $T \sim \mathit{Gamma}(n-1, 1)$, and $2T \sim \mathit{Gamma}(n-1, 2) = \chi^2_{2(n-1)}$.

8.6

Part a

The MLE for the parameter of an exponential distribution is simply the sample mean, which is also a sufficient statistic. Therefore, we get

$\hat{\theta}_R = \frac{\sum X_i + \sum Y_i}{n+m}$

$\hat{\theta} = \frac{1}{n}\sum X_i$
$\hat{\mu} = \frac{1}{m}\sum Y_i$

Then we get:

$$\lambda = \frac{(\frac{n+m}{\sum X_i + \sum Y_i})^{n+m}\exp(-(\frac{n+m}{\sum X_i + Y_i})(\sum X_i + \sum Y_i))}{(\frac{n}{\sum X_i})^n (\frac{m}{\sum Y_i})^m \exp(-\frac{n}{\sum X_i}\sum X_i - \frac{m}{\sum Y_i}\sum Y_i)}$$
$$= (\frac{n+m}{\sum X_i + \sum Y_i})^{n+m}(\frac{\sum X_i}{n})^n(\frac{\sum Y_i}{m})^m$$

Part b

Continuing with the algebra from part (a), we get:

$$\lambda = \frac{(n+m)^{nm}}{n^n m^m}(\frac{\sum X_i}{\sum X_i + Y_i})^n(\frac{\sum Y_i}{\sum X_i + \sum Y_i})^m$$
$$= \frac{(n+m)^{nm}}{n^n m^m} T^n (1-T)^m$$

Differentiating once and setting to zero, we get the maximum at:

$$\lambda'(T) = C\big(nT^{n-1}(1-T)^m - mT^n(1-T)^{m-1}\big) = 0$$
$$\implies n(1-T) = mT$$
$$\implies T = \frac{n}{n+m}$$

Part c

Since X_i and Y_i are exponentially distributed,

$\sum X_i \sim Gamma(n, \theta)$
$\sum Y_i \sim Gamma(m, \theta)$

Therefore, $\sum X_i + \sum Y_i \sim Gamma(n+m, \theta)$
$\implies T \sim Beta(n, m)$ (using diagram from back of book)

8.7

Part a

Without any restriction, we have:

$$\log f(x \mid \theta, \lambda) = -\log\lambda - x/\lambda + \theta/\lambda$$
$$\implies \ell(\theta, \lambda) = -n\log\lambda - \frac{1}{\lambda}\sum x_i + \frac{n\theta}{\lambda}$$

This is increasing in θ, so $\hat{\theta} = X_{(1)}$.

We also get:

$$0 = \partial_\lambda \ell = -n/\lambda + \sum x_i/\lambda - n\theta/\lambda$$
$$\implies \hat{\lambda} = \frac{\sum X_i - nX_{(1)}}{n}$$

With the null hypothesis restriction, we have the same result as long as $x_{(1)} \le 0$. Otherwise, we set $\hat{\theta} = 0$ and so $\hat{\lambda} = \sum X_i/n$. Therefore, we can write the LRT statistic as

$$\lambda(\overrightarrow{x}) = \begin{cases} 1 & x_{(1)} \le 0 \\ \frac{L(\bar{x}, 0)}{L(\frac{\sum x_i - nx_{(1)}}{n}, x_{(1)})} & x_{(1)} > 0 \end{cases}$$

For the latter case, we have:

$\lambda = \frac{\bar{x}^{-n} e^{-\sum x_i/\bar{x}}}{(\frac{n}{\sum x_i - n x_{(1)}})^n \exp(-\frac{\sum x_i - x_{(n)}}{\frac{\sum x_i - x_{(n)}}{n}})}$
$= (\frac{\bar{x} - x_{(1)}}{\bar{x}})^n$
$= (1 - \frac{x_{(1)}}{\bar{x}})^n$

Now let our statistic be $T = \frac{X_{(1)}}{\bar{X}}$. We can see that λ is a decreasing function of T, so we reject when $T > c$.

Part b

We already know that for an exponential random variable ($\gamma = 0$), the MLE is $\hat{\beta} = \bar{x}$.

For the case where $\gamma \neq 0$, we have

$f(x \mid \gamma, \beta) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}$
$\implies \log f = \log\gamma - \log\beta + (\gamma - 1)\log x - x^\gamma/\beta$
$\implies \ell(\gamma, \beta) = n\log\gamma - n\log\beta + (\gamma - 1)\sum \log x_i - \sum x_i^\gamma/\beta$

To find the MLEs, we differentiate and set to 0:

$\partial_\beta \ell = -n/\beta + \frac{1}{\beta^2}\sum x_i^\gamma = 0$
$\implies \hat{\beta} = \frac{1}{n}\sum x_i^\gamma$

There is no closed form solution for $\hat{\gamma}$.

The LRT statistic is then:

$\lambda = \frac{\bar{x}^{-n}}{\sup_\gamma (\gamma/\frac{1}{n}\sum x_i^\gamma)^n \prod x_i^{\gamma-1}}$

Again, since we cannot evaluate $\hat{\gamma}$ analytically, this cannot be evaluated without some sort of numerical method.

8.8

Part a

In the unrestricted case, our MLEs are solved as follows:

$f(x \mid \theta, a) = (2\pi a\theta)^{-1/2}\exp(-\frac{(x-\theta)^2}{2a\theta})$
$\implies \log f = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log a - \frac{1}{2}\log\theta - \frac{(x-\theta)^2}{2a\theta}$
$\implies \ell(\theta, a) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log a - \frac{n}{2}\log\theta - \frac{1}{2a\theta}\sum(x_i - \theta)^2$
$= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log a - \frac{n}{2}\log\theta - \frac{\sum x_i^2}{2a\theta} + \frac{\sum x_i}{a} - \frac{n\theta}{2a}$

Differentiation yields

$\partial_a \ell = -\frac{n}{2a} + \frac{\sum(x_i - \theta)^2}{2a^2\theta}$
$\partial_\theta \ell = -\frac{n}{2\theta} + \frac{\sum x_i^2}{2a\theta^2} - \frac{n}{2a}$

Setting the first to 0 and solving for a yields:

$\hat{a} = \frac{1}{n\theta}\sum(x_i - \theta)^2$

Some algebra on the second yields:

$n\theta^2 + na\theta = \sum x_i^2$

Substituting the first part into the second yields:

$n\theta^2 + \frac{n\theta}{n\theta}\sum(x_i - \theta)^2 = \sum x_i$
$\implies n\theta^2 + \sum x_i^2 - 2\theta\sum x_i + n\theta^2 = \sum x_i^2$
$\implies \hat{\theta} = \frac{1}{n}\sum x_i = \bar{x}$

Substituting this into our expression for $\hat{a}$ yields:

$\hat{a} = \frac{\sum(x_i-\theta)^2}{\sum x_i}$

Under the null hypothesis $\alpha = 1$, we have

$\ell_R = -\frac{n}{2}\log\theta - \frac{\sum x_i^2}{2\theta} + \sum x_i - \frac{n\theta}{2}$
$\implies \ell_R' = -\frac{n}{2\theta} + \frac{\sum x_i^2}{2\theta^2} - \frac{n}{2}$

Setting this to 0, we get

$\hat{\theta}_R = \frac{-1+\sqrt{1+\frac{4\sum x_i^2}{n}}}{2}$ (we only consider one solution since $\theta > 0$).

For the LRT statistic

$\lambda = \frac{\hat{\theta}_R^{-n/2}\exp(-\frac{(x-\hat{\theta}_R)^2}{2\hat{\theta}_R})}{\hat{a}^{-n/2}\hat{\theta}^{-n/2}\exp(-\frac{(x-\hat{\theta})^2}{2\hat{a}\hat{\theta}})}$

We reject H_0 if this value is above some value c.

Part b

The MLE under the null hypothesis is unchanged.

For the unrestricted case, now we have

$\ell(\theta, a) = -\frac{n}{2}\log 2\pi - n\log a - \frac{n}{2}\log\theta - \frac{1}{2a\theta^2}\sum(x_i - \theta)^2$

So the derivatives are:

$\partial_\theta \ell = -\frac{n}{2\theta} + \frac{\sum x_i^2}{a\theta^3} - \frac{\sum x_i}{2a\theta^2}$
$\partial_a \ell = -\frac{n}{a} + \frac{\sum(x_i-\theta)^2}{2a^2\theta^2}$

Setting to 0, the second yields

$\hat{a} = \frac{1}{n\theta^2}\sum(x_i - \theta)^2$

The second can be turned into

$na\theta^2 = 2\sum x_i^2 - \theta\sum x_i$

Substituting our expression for $\hat{a}$ and solving for θ yields:

$\hat{\theta} = \sqrt{\frac{1}{n}\sum x_i^2}$

And again, for the LRT statistic, we have

$\lambda = \frac{L(\hat{\theta}_R)}{L(\hat{a},\hat{\theta})}$

which results in a messy expression. We reject H_0 if $lambda > c$ for some value c.

8.9

Part a

The Y_i's are exponentially distributed.
Under H_0, the MLE is $\hat{\lambda}_R = \frac{1}{\bar{Y}}$.
The unrestricted MLEs are $\hat{\lambda}_i = \frac{1}{Y_i}$.

The LRT statistic is

$\lambda = \frac{(1/\bar{y})^n}{\prod y_i^{-1}}$

$\lambda \leq 1$, so we get:

$(1/\bar{y})^n \leq (\prod y_i)^{-1}$
$\implies \bar{y}^n \geq \prod y_i$
$\implies \bar{y} \geq (\prod y_i)^{1/n}$

Part b

$X_i = 1/Y_i$
$\implies Y_i = 1/X_i$
$\implies Y_i' = -X_i^{-2}$

Then $f_i(x) = \lambda_i e^{-\lambda_i/x_i}/x_i^2$
$\implies \log f_i = \log \lambda_i - 2\log x_i - \lambda_i/x_i$,
so under the alternative hypothesis, we can just differentiate and set to zero to obtain $\hat{\lambda}_i = x_i$.

Under the null hypothesis, we have
$\ell = n \log \lambda - 2\sum \log x_i - \lambda \sum x_i^{-1}$
$\implies \ell' = n/\lambda - \sum x_i^2$
$\implies \hat{\lambda}_R = \frac{n}{\sum 1/x_i}$

For the LRT statistic, some simplification gets us:

$\lambda = \frac{\hat{\lambda}_R^n / \prod x_i^2}{\prod x_i / \prod x_i^2}$
$= \hat{\lambda}_R^n / \prod x_i$
$= \frac{(n/\sum x_i^{-1})^n}{\prod x_i}$

Again, bounding $\lambda \leq 1$, we get

$\frac{(n/\sum x_i^{-1})^n}{\prod x_i} \leq 1$
$\implies (\frac{n}{\sum 1/x_i})^n \leq \prod x_i$
$\implies \frac{n}{\sum 1/x_i} \leq (\prod x_i)^{1/n}$

Part 2

For simplicity of notation, let $\kappa(\overrightarrow{x}) = \frac{\sup_{\Theta_0} L(\theta|x)}{\sup_{\Theta_0^c} L(\theta|x)}$

We can see that when the unrestricted MLE $\hat{\theta} \notin \Theta_0^c$, the two expressions are equivalent. So we will consider the case where the unrestricted MLE is in Θ_0.

Then immediately, we can see that $\lambda = 1$ (and we always fail to reject (unless for some reason we set $c = 1$). And in this case, the numerator of κ is the unrestricted likelihood and so is greater than the denominator, so $\kappa \geq 1$.

Part 3

We can write $f(\overrightarrow{x} \mid \theta) = g(T(\overrightarrow{x}) \mid \theta)h(\overrightarrow{x}) = L(\theta \mid \overrightarrow{x})$, so the LRT statistic becomes:

$\lambda = \frac{\sup_{\Theta_0} L}{\sup_{\Theta} L}$
$= \frac{\sup_{\Theta_0} f(\overrightarrow{x}|\theta)}{\sup_{\Theta} f(\overrightarrow{x}|\theta)}$
$= \frac{\sup_{\Theta_0} g(T(\overrightarrow{x})|\theta)h(\overrightarrow{x})}{\sup_{\Theta} g(T(\overrightarrow{x})|\theta)h(\overrightarrow{x})}$
$= \frac{\sup_{\Theta_0} g(T(\overrightarrow{x})|\theta)}{\sup_{\Theta} g(T(\overrightarrow{x})|\theta)}$

Now we note that $g(T \mid \theta)$ is the density function of T given θ, which is then a likelihood function of θ given T. So this is just the ratio of two likelihoods as a function of T:

$= \lambda^*(T(\overrightarrow{x}))$