Problem of the Week 3

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Consider the set of 3×3 matrices with distinct entries. We form 6 numbers, 3 by reading off the digits in the row entries left to right, and 3 more by reading off the digits in the columns up to down. Note that we interpret a number like 037 as 37 (though we will see that something like this cannot happen).

We start by stating some lemmas that are common knowledge.

Lemma 1. If an integer n is divisible by 6, then n is even.

Proof. Assume $n \in \mathbb{Z}$ is divisible by 6. Then $n = 6m$ for some $m \in \mathbb{Z}$. But then $n = 2(3m)$. Thus, n is even.

□

Lemma 2. If an integer n is even, then the digit in the ones place of the number is $0, 2, 4, 6$ or 8.

Lemma 3. If an integer n is divisible by 5, then the digit in its ones place is either 5 or 0.

Proof. Let n be an integer divisible by 5. Then $n = 5m$ for some $m \in \mathbb{Z}$. But then we see that

$$n = \underbrace{5 + 5 + \cdots + 5}_{\text{m times}}$$

If m even, we see that this expression will end in 0. Otherwise, it will clearly end in a 5.

□

Using these three lemmas, we can now show that **there is only only number in the 3×3 matrix that is divisible by 5**.

Proof. Since we know that each number in the matrix is divisible by 6, by lemma 1 each number must be even. But by lemma 2, this means that their digits in the ones place must be even. Therefore, our matrix must look like this.

$$\begin{bmatrix} odd_1 & odd_2 & even_1 \\ odd_3 & odd_4 & even_2 \\ even_3 & even_4 & even_5 \end{bmatrix}$$

where $odd_i \in \{1,3,5,7,9\}$ and $even_i \in \{0,2,4,6,8\}$. But then, by lemma 3, we know in order to be divisible by 5, our number must end in 0 or 5. Since our number cannot end in 5, there is only one number divisible by 5. But since we must use all of our even numbers, we know 0 must occur in the ones place of exactly one number in our matrix. Therefore, there is one number in this matrix that is divisible by 5.

$\square$