HW 8 ungraded exam #3 practice problems
Math 527, spring 2018, University of New Hampshire

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Section: —

These are practice problems to help you prepare for exam #3. The actual exam will have fewer problems and different kinds of problems, but their level of difficulty will be about the same.

INSTRUCTIONS (worth some points)
1. Write your name legibly in pen on each page and name and section number on this page.
2. Show your work and put a box or circle around your answers.
3. Always write equations. Partial credit will be given only for work written clearly in equations.

Problem 1. Compute the Laplace transform or inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{e^{-as}\frac{1}{s^4}\right\} = \mathcal{U}(t-a)\left[\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}\right]_{t\to t-a}$

$$= \frac{1}{6}\mathcal{U}(t-a)\left[t^3\right]_{t\to t-a}$$

$$\boxed{\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s^4}\right\} = \frac{1}{6}\,\mathcal{U}(t-a)\,(t-a)^3}$$

(b) $\frac{1}{s^2-2s} = \frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$ cover-up $\Rightarrow$ $A = -1/2$, $B = 1/2$

$$= \frac{-1/2}{s} + \frac{1/2}{s-2}$$

so

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^2-2s}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$$= -\frac{1}{2}\cdot 1 + \frac{1}{2}e^{2t}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s}\right\} = \frac{1}{2}\left(e^{2t}-1\right)}$$

Problem 1, cont'd. Compute the Laplace transform or inverse Laplace transform.

(c) $\mathscr{L}\{\mathscr{U}(t-2)\, e^{-3t}\,(t+4)\} = e^{-2s}\mathcal{L}\{e^{-3(t+2)}(t+6)\}$

$$= e^{-6}e^{-2s}\mathcal{L}\{e^{-3t}(t+6)\}$$

$$= e^{-2(s+3)}\left[\mathcal{L}\{t+6\}\right]_{s\to s+3}$$

$$= e^{-2(s+3)}\left[\frac{1}{s^2}+\frac{6}{s}\right]_{s\to s+3}$$

$$\boxed{\mathcal{L}\{\mathcal{U}(t-2)\, e^{-3t}(t+4)\} = e^{-2(s+3)}\left(\frac{1}{(s+3)^2}+\frac{6}{s+3}\right)}$$

(d) $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{s^2+6s+34}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2(s+3)-1}{(s+3)^2+25}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+5^2}\Big|_{s\to s+3}\right\}$$

$$= e^{-3t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\}+\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\}\right)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = e^{-3t}\left(2\cos 5t+\frac{1}{5}\sin 5t\right)}$$

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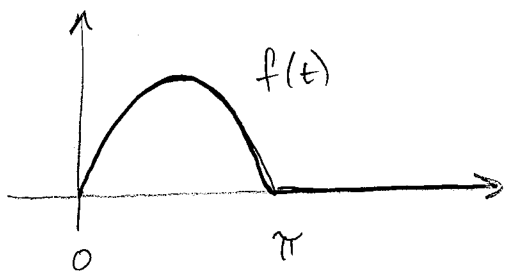
Problem 2. Express $f(t)$ in terms of Heaviside functions and then compute $\mathscr{L}\{f(t)\}$.

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t \end{cases}$$



$$f(t) = \left(1 - \mathcal{U}(t-\pi)\right)\sin t$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{\mathcal{U}(t-\pi)\sin t\}$$

$$= \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$$

$(\sin(t+\pi) = -\sin t)$

$$= \frac{1}{s^2+1} + e^{-\pi s}\mathcal{L}\{\sin t\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s^2+1}\left(1 + e^{-\pi s}\right)}$$

Problem 3. Find the solution of the initial value problem using Laplace transforms. Derivatives y', y'' are with respect to t.

$$y'' + 4y' + 8y = e^{-t}, \quad y(0) = 0,\ y'(0) = 1$$

$$\left(s^2 Y(s) - s\,\cancel{y(0)}^{0} - y'(0)^{1}\right) + 4\left(sY(s) - \cancel{y(0)}^{0}\right) + 8Y(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 8)Y(s) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{1}{(s+1)(s^2+4s+8)} + \frac{1}{s^2+4s+8}$$

$$= \frac{1 + (s+1)}{(s+1)(s^2+4s+8)} = \frac{s+2}{(s+1)(s^2+4s+8)}$$

$$\frac{s+2}{(s+1)(s^2+4s+8)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8}$$

cover-up: $A = \dfrac{+1}{1-4+8} = +\dfrac{1}{5}$

$$s+2 = \frac{1}{5}(s^2+4s+8) + (Bs+C)(s+1)$$

s^2 term: $B = -\dfrac{1}{5}$, 1 term: $C = \dfrac{2}{5}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = +\mathcal{L}^{-1}\left\{\frac{-1/5}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{-1/5\,s - 2/5}{s^2+4s+8}\right\}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{(s+2)-4}{(s+2)^2+4}\right\}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right]$$

$$\boxed{y(t) = \frac{1}{5}\left(e^{-t} + e^{-2t}(2\sin 2t - \cos 2t)\right)}$$

note: there will not be linear algebra problems on exam 3

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Problem 4.

(a) Compute the matrix-vector product

$$\underbrace{\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 0 \\ -2 & 5 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 9+2+5 \\ 12+1+0 \\ -6+5-5 \end{pmatrix}}_{b} = \begin{pmatrix} 16 \\ 13 \\ -6 \end{pmatrix}$$

$$\boxed{Ax = \begin{pmatrix} 16 \\ 13 \\ -6 \end{pmatrix}}$$

(b) Express the system of equations as an $\mathbf{Ax} = \mathbf{b}$ problem, where $\mathbf{A}$ is a matrix, $\mathbf{b}$ is a known vector, and $\mathbf{x}$ is an unknown vector.

$$\begin{aligned} x + y - 2z &= 14 \\ 2x - y + z &= 0 \\ 6x + 3y + 4z &= 1 \end{aligned}$$

$$\boxed{\begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 6 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \\ 1 \end{pmatrix}}$$

(c) Solve the $\mathbf{Ax} = \mathbf{b}$ equation from (b) using Gauusian elimination or Gauss-Jordan elimination. (Note: this problem is a natural follow-on to (b), but we haven't hit it in lecture yet, so don't expect it to appear on the exam).

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$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 14 \\ 2 & -1 & 1 & 0 \\ 6 & 3 & 4 & 1 \end{array}\right)$$

$$\begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 14 \\ 0 & -3 & 5 & -28 \\ 0 & -3 & 16 & -83 \end{array}\right)$$

$$\begin{array}{l} R_1 \\ -R_2 \\ R_3 - R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 14 \\ 0 & 3 & -5 & 28 \\ 0 & 0 & 11 & -55 \end{array}\right)$$

R_3: $11x_3 = -55$ $\qquad x_3 = -5$

R_2: $3x_2 - 5(-5) = 28$ $\qquad x_2 = 1$

R_1: $x_1 + 1 - 2(-5) = 14$ $\qquad x_1 = 3$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}} \quad \text{or} \quad \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}}$$