MATH 527 HW #9 Solutions

1) $x' = x + 2y$
$y' = 4x + 3y$ $\rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

- To find eigenvalues, solve: $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0 \rightarrow \lambda_1 = \frac{4 + \sqrt{16+20}}{2} = 5 \qquad \lambda_2 = \frac{4 - \sqrt{16+20}}{2} = -1$$

- To find $\vec{v_1}$: solve $(A - \lambda_1 I)\vec{v_1} = \vec{0}$

Rewrite as $\begin{pmatrix} 1-5 & 2 \\ 4 & 3-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

a system w/ 1 free parameter

$-4x + 2y = 0$,
$2x = y$. Let $x = 1$, then $y = 2$. So, $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- To find $\vec{v_2}$, solve $(A - \lambda_2 I)\vec{v_2} = \vec{0}$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x + 2y = 0$
$x = -y$ Let $x = 1$, then $y = -1$. $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}}$$ where c_1, c_2 - arbitrary constants.

Alternatively, $\begin{cases} x(t) = c_1 e^{5t} + c_2 e^{-t} \\ y(t) = 2c_1 e^{5t} - c_2 e^{-t} \end{cases}$

2) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix} = A$

$\det(A - \lambda I) = 0$, solve for λ:

$$\det\begin{pmatrix} -4-\lambda & 2 \\ -5/2 & 2-\lambda \end{pmatrix} = 0$$

$$(-4-\lambda)(2-\lambda) + 5 = 0$$

$$-8 - 2\lambda + 4\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 + 2\lambda - 3 = 0$$

$(\lambda+3)(\lambda-1) = 0$, $\underline{\lambda_1 = -3}$, $\underline{\lambda_2 = 1}$

$(A - \lambda_1 I)\vec{v_1} = \vec{0}$

$\begin{pmatrix} -4+3 & 2 \\ -5/2 & 2+5 \end{pmatrix} \vec{v_1} = \vec{0}$

$\begin{pmatrix} -1 & 2 \\ -5/2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-x + 2y = 0$

Let $y = 1$, then $x = 2$

$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$(A - \lambda_2 I)\vec{v_2} = \vec{0}$

$\begin{pmatrix} -4-1 & 2 \\ -5/2 & 2-1 \end{pmatrix} \vec{v_2} = \vec{0}$

$\begin{pmatrix} -5 & 2 \\ -5/2 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-5x + 2y = 0$

Let $x = 2$, then $y = 5$

$\vec{v_2} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{t}}$$

c_1, c_2 - arbitrary constants

Alternatively, $x(t) = 2c_1 e^{-3t} + 2c_2 e^{t}$

$y(t) = c_1 e^{-3t} + 5c_2 e^{t}$

3) $x' = x + y$
$y' = -2x - y$ $\sim$ $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = A$

$\det(A - \lambda I) = 0$, solve for λ:

$$\det \begin{pmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) + 2 = 0$$

$$-1 + \lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 1 = 0, \quad \lambda = \pm i$$

In standard notation $\lambda = \mu \pm \omega i$
$\mu = 0$, $\omega = 1$

To find $\vec{v}_1$ for $\lambda_1 = i$:

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(1-i)x + y = 0$$

Let $x = 1$, then $y = -1 + i$. So, $\vec{v}_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$ for $\lambda_1 = i$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\vec{v}_2 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ for $\lambda_2 = -i$

$$\boxed{\vec{x}(t) = \tilde{c}_1 \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{it} + \tilde{c}_2 \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{-it}}$$

Alternatively,

$$x(t) = c_1(\vec{v}_r \cos \omega t - \vec{v}_i \sin \omega t) e^{\mu t} + c_2(\vec{v}_i \cos \omega t + \vec{v}_r \sin \omega t) e^{\mu t}$$

$$\boxed{\vec{x}(t) = c_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t\right) + c_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos \omega t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t\right)}$$

4) $x' = 5x + y$
$y' = -2x + 3y$ $\rightarrow$ $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = A$

$\det(A - \lambda I) = 0$

$\det \begin{pmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix} = 0$

$(5-\lambda)(3-\lambda) + 2 = 0$

$15 - 3\lambda - 5\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 8\lambda + 17 = 0$ $\lambda = \dfrac{8 \pm \sqrt{64 - 17 \cdot 4}}{2} = 4 \pm i$

$\lambda = \mu \pm \omega i$
$\mu = 4, \ \omega = 1$

To find $\vec{v}_1$ for $\lambda_1 = 4 + i$:

$[A - \lambda_1 I] \vec{v}_1 = \vec{0}$

$\begin{pmatrix} 5-4-i & 1 \\ -2 & 3-4-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda_1 = 4 + i$ $\quad \lambda_2 = 4 - i$

$(1-i)x + y = 0$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$

Let $x = 1$, then $y = -1 + i$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{\vec{x}(t) = \tilde{c}_1 \begin{bmatrix} 1 \\ -1+i \end{bmatrix} e^{(4+i)t} + \tilde{c}_2 \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{(4-i)t}}$$

Alternatively,

$$\boxed{\vec{x}(t) = c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) e^{4t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t \right) e^{4t}}$$

5) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad\qquad$ $\begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} = A$

$\det(A - \lambda I) = 0$

$\det \begin{pmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{pmatrix} = 0$

$(-1-\lambda)(5-\lambda) + 9 = 0$

$-5 - 5\lambda + \lambda + \lambda^2 + 9 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)^2 = 0 \rightarrow \underline{\lambda_1 = \lambda_2 = 2}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{pmatrix} -1-2 & 3 \\ -3 & 5-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-3x + 3y = 0$

$x = y$ Let $x = 1$, then $y = 1$

$\underline{\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ only 1 linearly independent eigen vec for this system

To find a generalized eigenvector ($\vec{u}$):

$(A - \lambda I)\vec{u} = \vec{v}$ (solve for $\vec{u}$)

$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$-3x + 3y = 1$ Let $x = 0$, then $y = \frac{1}{3}$; $\vec{u} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$

$\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \right] e^{2t}$

6) $x' = 12x - 9y$ $\rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. $\begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} = A$

$y' = 4x$

- $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{pmatrix} = 0$$

$$(12-\lambda)(-\lambda) + 36 = 0$$

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)^2 = 0 \qquad \underline{\lambda_1 = \lambda_2 = 6}$$

- $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 12-6 & -9 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4x - 6y = 0$

$2x = 3y$ Let $x = 3$, then $y = 2$. $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

- to find a generalized eigenvector solve for $\vec{u}$:

$(A - \lambda I)\vec{u} = \vec{v}$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$6x - 9y = 3$. Let $y = 0$, then $x = \frac{1}{2}$, $\vec{u} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$$\vec{x}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right] e^{6t}$$

7) $x' = -3x - y$, $x(0)=3$
$y' = 9x - 3y$, $y(0)=5$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} -3 & -1 \\ 9 & -3 \end{pmatrix} = A$$

• $\det(A - \lambda I) = 0$, solve for λ

$$\det\begin{pmatrix} -3-\lambda & -1 \\ 9 & -3-\lambda \end{pmatrix} = 0$$

$$(-3-\lambda)(-3-\lambda) + 9 = 0$$

$$\lambda^2 + 6\lambda + 18 = 0 \qquad \lambda = \frac{-6 \pm \sqrt{36 - 18\cdot 4}}{2} = -3 \pm 3i$$

• $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$, solve for $\vec{v}_1$

$$\begin{pmatrix} -3-(-3+3i) & -1 \\ 9 & -3-(-3+3i) \end{pmatrix}\vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} -3i & -1 \\ 9 & -3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3ix - y = 0$ Let $x = -1$, then $y = 3i$

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 3i \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Real-valued form : $\vec{x}(t) = c_1\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ 3 \end{bmatrix}\sin 3t\right)e^{-3t} + c_2\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\cos 3t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin 3t\right)e^{-3t}$

or $\quad x(t) = e^{-3t}(-c_1\cos 3t - c_2 \sin 3t)$
$\quad\quad y(t)\ e^{-3t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$

$x(0) = 3$, so $\quad 3 = 1(-c_1 \cdot 1 - 0) \quad \rightarrow \quad c_1 = -3$

$y(0) = 5$, so $\quad 5 = 1(0 + 3c_2) \quad \rightarrow \quad c_2 = 5/3$

$$\boxed{\begin{aligned} x(t) &= e^{-3t}\left(3\cos 3t - \tfrac{5}{3}\sin 3t\right) \\ y(t) &= e^{-3t}(9\sin 3t + 5\cos 3t) \end{aligned}}$$

/ complex form on next page /

7) cont

Since we have $\lambda_1 = -3+3i$ $\quad$ $\lambda_2 = -3-3i$

$\vec{V_1} = \begin{pmatrix} -1 \\ 3i \end{pmatrix}$ $\quad$ $\vec{V_2} = \begin{pmatrix} -1 \\ -3i \end{pmatrix}$

Then $\vec{X}(t) = \tilde{c}_1 \begin{bmatrix} -1 \\ 3i \end{bmatrix} e^{(-3+3i)t} + \tilde{c}_2 \begin{bmatrix} -1 \\ -3i \end{bmatrix} e^{(-3-3i)t}$

$\vec{X}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$\tilde{c}_1 \begin{bmatrix} -1 \\ 3i \end{bmatrix} \cdot 1 + \tilde{c}_2 \begin{bmatrix} -1 \\ -3i \end{bmatrix} \cdot 1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Solve for $\tilde{c}_1, \tilde{c}_2$:

$$\begin{bmatrix} -1 & -1 \\ 3i & -3i \end{bmatrix} \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & -1 & 3 \\ 3i & -3i & 5 \end{array}\right] \xrightarrow[R_1 \cdot 3i + R_2]{R_1 \cdot (-1)} \left[\begin{array}{cc|c} 1 & 1 & -3 \\ 0 & -6i & 5+9i \end{array}\right]$$

$-6i\,\tilde{c}_2 = 5+9i \quad | \cdot i$

$6c_2 = 5i - 9, \quad c_2 = \dfrac{-9+5i}{6}$

$\tilde{c}_1 + \tilde{c}_2 = -3$, then $\tilde{c}_1 = -3 - c_2 = \dfrac{-18}{6} - \dfrac{-9+5i}{6} = \dfrac{-18+9-5i}{6}$

$$\tilde{c}_1 = \frac{-9-5i}{6}$$

$$\boxed{\vec{X}(t) = \frac{-9-5i}{6} \begin{bmatrix} -1 \\ 3i \end{bmatrix} e^{(-3+3i)t} + \frac{-9+5i}{6} \begin{bmatrix} -1 \\ -3i \end{bmatrix} e^{(-3-3i)t}}$$