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Math 527
HW 10 ungraded
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Prob 1. Find the general solution

$$\underline{x}' = \begin{pmatrix} 1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix} \underline{x}$$

$\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 1-\lambda & 4 & 2 \\ 4 & -1-\lambda & -2 \\ 0 & 0 & 6-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-1-\lambda)(6-\lambda) - (6-\lambda)\cdot 4 \cdot 4 = 0$$

$\boxed{\lambda_1 = 6}$ or

$(\lambda-1)(\lambda+1) - 16 = 0$

$\lambda^2 - 1 - 16 = 0 \qquad \boxed{\lambda_{2,3} = \pm\sqrt{17}}$

for $\lambda_1 = 6$, $\quad (A - \lambda_1 I)\underline{v} = 0$

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -7 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

adding 1st + 2nd rows $\Rightarrow \quad -x - 3y = 0 \quad \Rightarrow \quad x = -3y$

1st row $\quad -5x + 4y + 2z = 0$

$15y + 4y + 2z = 0$

$z = -\frac{19}{2} y \qquad$ choose $y = 2$, $x = -6$, $z = -19$

$$\boxed{\lambda_1 = 6, \quad \underline{v} = \begin{pmatrix} -6 \\ 2 \\ -19 \end{pmatrix}}$$

for $\lambda_2 = \sqrt{17}$

$$\begin{pmatrix} 1-\sqrt{17} & 4 & 2 \\ 4 & -1-\sqrt{17} & -2 \\ 0 & 0 & 6-\sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

last row $\Rightarrow$ $z = 0$

top row $\quad (1-\sqrt{17})x + 4y = 0 \quad$ choose $x = 4$, $y = \sqrt{17}-1$

$$\lambda_2 = \sqrt{17} \qquad \underline{v}_2 = \begin{pmatrix} 4 \\ \sqrt{17}-1 \\ 0 \end{pmatrix}$$

for $\lambda_3 = -\sqrt{17}$

$$\begin{pmatrix} 1+\sqrt{17} & 4 & 2 \\ 4 & -1+\sqrt{17} & -2 \\ 0 & 0 & 6+\sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

last row $\Rightarrow$ $z = 0$

top row $\quad (1+\sqrt{17})x + 4y = 0 \quad$ choose $x = -4$, $y = \sqrt{17}+1$

$$\lambda_3 = -\sqrt{17} \qquad \underline{v}_3 = \begin{pmatrix} -4 \\ \sqrt{17}+1 \\ 0 \end{pmatrix}$$

genl soln

$$\boxed{\underline{x}(t) = c_1 \begin{pmatrix} -6 \\ 2 \\ -19 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 4 \\ \sqrt{17}-1 \\ 0 \end{pmatrix} e^{\sqrt{17}t} + c_3 \begin{pmatrix} -4 \\ \sqrt{17}+1 \\ 0 \end{pmatrix} e^{-\sqrt{17}t}}$$

Problem 2: Find the general solution

$$\underline{x}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \underline{x}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{pmatrix} = 0$$

$$(\lambda+1)^2(2-\lambda) - 3(-1-\lambda) - (-1-\lambda) = 0$$

$$(\lambda+1)^2(\lambda-2) - 4(\lambda+1) = 0$$

$$\lambda = -1 \quad \text{or} \quad (\lambda+1)(\lambda-2) - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda-3)(\lambda+2) = 0$$

$$\boxed{\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = 3}$$

for $\lambda_1 = -1$, $(A - \lambda_1 I)\underline{v}_1 = \underline{0}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

row 1 $\Rightarrow$ $y = 0$, row 2 $\Rightarrow$ $x + z = 0$. Choose $x = 1$, $z = -1$

$$\boxed{\lambda_1 = -1, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

for $\lambda_2 = -2$, $(A - \lambda_2 I)\underline{v} = \underline{0}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

row 1 $\Rightarrow$ $x + y = 0$, row 3 $\Rightarrow$ $3y + z = 0$

choose $x = 1$, $y = -1$, $z = 3$

$$\boxed{\lambda_2 = -2 \qquad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}$$

for $\lambda_3 = 3$ $(A - \lambda_3 I)\underline{v}_3 = \underline{0}$

$$\begin{pmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

row 1 $\Rightarrow$ $-4x + y = 0$ row 3 $\Rightarrow$ $3y - 4z = 0$

choose $y = 4$, $x = 1$, $z = 3$

$$\boxed{\lambda_3 = 3, \qquad \underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}}$$

genl soln

$$\boxed{\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t}}$$

Problem 3. Find the general solution

$$\underline{x}' = \begin{pmatrix} 2 & 5 & 1 \\ -5 & -6 & -4 \\ 0 & 0 & 2 \end{pmatrix} \underline{x}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 2-\lambda & 5 & 1 \\ -5 & -6-\lambda & -4 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$$

$$-(\lambda-2)^2(\lambda+6) - 25(\lambda-2) = 0$$

$$\lambda_1 = 2 \quad \text{or} \quad (\lambda-2)(\lambda+6) + 25 = 0$$

$$\lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i$$

$$\boxed{\lambda_1 = 2, \quad \lambda_2 = -2+3i, \quad \lambda_3 = -2-3i}$$

for $\lambda_1 = 2$, $(A - \lambda_1 I)\underline{v} = \underline{0}$

$$\begin{pmatrix} 0 & 5 & 1 \\ -5 & -8 & -4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

row 1: $5y + z = 0$ choose $y = -1$, $z = 5$

row 2: $5x + 8y + 4z = 0$

$$5x - 8 + 20 = 0 \quad \Rightarrow \quad x = -12/5$$

$$\lambda_1 = 2 \qquad \underline{v}_1 = \begin{pmatrix} -12/5 \\ -1 \\ 5 \end{pmatrix}$$

for $\lambda_2 = -2+3i$, $(A-\lambda_2 I)\,\underline{v}_2 = \underline{0}$

$$\begin{pmatrix} 4-3i & 5 & 1 \\ -5 & -4-3i & -4 \\ 0 & 0 & 4-3i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

3rd row $\Rightarrow$ $z=0$. 1st row $\Rightarrow$ $(4-3i)x + 5y = 0$

choose $x=5$, $y=-4+3i$

$\lambda_2 = -2+3i$ $\quad \underline{v}_2 = \begin{pmatrix} 5 \\ -4+3i \\ 0 \end{pmatrix}$ $\quad \underline{v}_r = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix}$ $\quad \underline{v}_i = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ $\quad \mu = -2$, $\omega = 3$

$\lambda_3 = -2-3i$ $\quad \underline{v}_3 = \begin{pmatrix} 5 \\ -4-3i \\ 0 \end{pmatrix}$

gen'l soln, complex form

$$\boxed{\underline{x}(t) = c_1 \begin{pmatrix} -12/5 \\ -1 \\ 5 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 5 \\ -4+3i \\ 0 \end{pmatrix} e^{(-2+3i)t} + c_3 \begin{pmatrix} 5 \\ -4-3i \\ 0 \end{pmatrix} e^{(-2-3i)t}}$$

gen'l soln, real form

$$\boxed{\begin{aligned} \underline{x}(t) = {} & c_1 \begin{pmatrix} -12/5 \\ -1 \\ 5 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \sin 3t \right] e^{-2t} \\ & + c_3 \left[\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \cos 3t + \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \sin 3t \right] e^{-2t} \end{aligned}}$$

Problem 4: Van der Pol oscillator

$$x'' - \mu(1-x^2)x' + x = 0 \qquad \mu \geq 0$$

(a) convert to a nonlinear system $\underline{x}' = f(\underline{x})$

let $y = x'$ and $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

then $\underline{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} x' \\ \mu(1-x^2)x' - x \end{pmatrix} = \begin{pmatrix} y \\ \mu(1-x^2)y - x \end{pmatrix}$

so $\underline{x}' = f(\underline{x})$ where $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $f(x) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} y \\ \mu(1-x^2)y - x \end{pmatrix}$

b) show that the origin is the only eqb

$\underline{x}' = f(\underline{x}) = 0$ requires $f_x = y = 0 \Rightarrow y = 0$

$f_y = \mu(1-x^2)\overset{0}{\cancel{y}} - x = 0 \Rightarrow x = 0$

so $\underline{x} = \underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only eqb.

c) linearize the dynamics about the origin

Write $\underline{x}(t)$ as $\underline{x} = \underline{0} + \underline{x}$ for small $\underline{x}$

Then plug into $\underline{x}' = f(\underline{x})$

$$(\underline{0} + \underline{x})' = \overset{0}{\cancel{f(\underline{0})}} + Df|_{\underline{0}}\, \underline{x} + \cancel{O(\|\underline{x}\|^2)} \text{ ignore}$$

$$\underline{x}' = Df|_{\underline{0}}\, \underline{x}$$

where

$$Df = \begin{pmatrix} \partial f_x/\partial x & \partial f_x/\partial y \\ \partial f_y/\partial x & \partial f_y/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\mu x y - 1 & \mu(1-x^2) \end{pmatrix}$$

$$Df|_{\underline{0}} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \qquad \text{(plugging in } x=0,\ y=0 \text{ to prev eqn)}$$

so for small perturbations about origin, dynamics are

$$\underline{x}' = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \underline{x}$$

d) find the gen'l soln to this system for $\mu = 1$

$$\underline{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \underline{x}$$

$\det(Df|_{\underline{0}} - \lambda I) = 0$: $\quad \det \begin{pmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = 0$

$$\lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$(A - \lambda I)\underline{v} = \underline{0}$: $\quad$ for $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$$\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2} i & 1 \\ -1 & \frac{3}{2} - \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{choose } \begin{matrix} x = 2 \\ y = 1 - \sqrt{3} i \end{matrix}$$

gen'l soln (complex form)

$$\boxed{\underline{x}(t) = c_1 \begin{pmatrix} 2 \\ 1+\sqrt{3}i \end{pmatrix} e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} + c_2 \begin{pmatrix} 2 \\ 1-\sqrt{3}i \end{pmatrix} e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)t}}$$

The eqb at the origin is <u>unstable</u> because the real part of the eigenvalues $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ is <u>positive</u>.

e) Where do the λ's change form, what kind of soln does the system have for what region of μ?

$$Df|_{\underline{0}} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

$$\det\left(Df|_{\underline{0}} - \lambda I\right) = 0 \Rightarrow \det\begin{pmatrix} -\lambda & 1 \\ -1 & \mu-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \mu\lambda + 1 = 0$$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

For $\mu = 0$, $\lambda = \pm i$ purely imaginary! $\Rightarrow$ steady oscillations

For $0 < \mu < 2$, λ is complex with positive real part

$\Rightarrow$ growing oscillations (unstable)

For $2 < \mu$, have distinct real λ's, one positive, one negative

$\Rightarrow$ unstable hyperbolic growth