MATH 527 HW4 Solutions

1) $y'' + 3y = x^3 - 1$

a) solve associated homog. eqn $y'' + 3y = 0$

$y = e^{\lambda x}$

$$\lambda^2 + 3 = 0 \quad , \quad \lambda = \pm\sqrt{3} \; ;$$

complementary solution:

$$y_c = C_1 \cos(\sqrt{3}\,x) + C_2 \sin(\sqrt{3}\,x)$$

b) ansatz for particular solution: $y_p = Ax^3 + Bx^2 + Cx + D$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

Subs. into eqn $y'' + 3y = x^3 - 1$:

$$6Ax + 2B + 3(Ax^3 + Bx^2 + Cx + D) = x^3 - 1$$

$$3A\,x^3 + 3Bx^2 + (6A + 3C)x + (2B + 3D) = x^3 - 1$$

This means: $3A\,x^3 = 1\,x^3 \rightarrow A = \frac{1}{3}$

$$3B\,x^2 = 0x^2 \rightarrow B = 0$$

$$(6A + 3C)x = 0x \rightarrow 6A + 3C = 0, \; C = -\tfrac{2}{3}$$

$$2B + 3D = -1 \rightarrow D = -\tfrac{1}{3}$$

Then $y_p = \frac{1}{3}x^3 - \frac{2}{3}x - \frac{1}{3}$

$$\boxed{y = C_1 \cos(\sqrt{3}\,x) + C_2 \sin(\sqrt{3}\,x) + \tfrac{1}{3}x^3 - \tfrac{2}{3}x - \tfrac{1}{3}}$$

2) $y'' + 4y' + 4y = te^{2t}$

a) solve associated homog. eqn $y'' + 4y' + 4y = 0$

$y = e^{\lambda t} \rightarrow \lambda^2 + 4\lambda + 4 = 0 \qquad \lambda_1 = \lambda_2 = -2$ (Repeated roots)

$y_c = C_1 e^{-2t} + C_2 t \cdot e^{-2t}$

b) Ansatz: $y_P = (At + B)e^{2t}$

$y_P' = 2(At+B)e^{2t} + Ae^{2t} = (2At + A + 2B)e^{2t}$

$y_P'' = 2(2At+A+2B)e^{2t} + 2Ae^{2t} = (4At+4A+4B)e^{2t}$

subst: $e^{2t}(4At+4A+4B) + 4e^{2t}(2At+A+2B) + 4e^{2t}(At+B) = t \cdot e^{2t}$

$e^{2t} \cdot (16At + 8A + 16B) = e^{2t} \cdot t$

which means: $16At = t \quad \rightarrow A = \frac{1}{16}$

$8A + 16B = 0 \quad B = -\frac{1}{32}$

$y_P = \left(\frac{1}{16}t - \frac{1}{32}\right)e^{2t}$

$$\boxed{y = C_1 e^{-2t} + C_2 t e^{-2t} + \left(16t - \frac{1}{32}\right)e^{2t}}$$

3) $y'' + 2y' + y = e^{-t}$

a) homog. eqn $y'' + 2y' + y = 0$;

$y = e^{\lambda t} \rightarrow \lambda^2 + 2\lambda + 1 = 0, \quad \lambda_1 = \lambda_2 = -1$

$y_c = C_1 e^{-t} + C_2 t e^{-t}$

b) Ansatz: $y_P = A \cdot e^{-t} \cdot t^2, \quad y_P' = 2Ate^{-t} - At^2e^{-t} = e^{-t}(-At^2 + 2At)$

$y_P'' = -e^{-t}(-At^2+2At) + e^{-t}(-2At+2A) = e^{-t}(At^2 - 4At + 2A)$

Subst: $e^{-t}(At^2 - 4At + 2A) + 2e^{-t}(-At^2 + 2At) + e^{-t}(At^2) = e^{-t}$

$e^{-t}(2A) = e^{-t} \rightarrow 2A = 1, \ A = \frac{1}{2}. \quad y_P = \frac{1}{2}t^2 e^{-t}$

$$\boxed{y = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2}t^2 e^{-t}}$$

4) $y'' + 4y = t\sin t$

a) solve $y'' + 4y = 0$

$y = e^{\lambda t} \to \lambda^2 + 4 = 0, \quad \lambda = \pm 2i$

$y_c = C_1 \cos 2t + C_2 \cdot \sin 2t$

b) Ansatz: $y_P = \left((At+B)\sin 2t + (Ct+D)\cos 2t\right)t$

$y_P = (At^2 + Bt)\sin 2t + (Ct^2 + Dt)\cos 2t$

$y_P' = (2At+B)\sin 2t + 2(At^2+Bt)\cos 2t + (2Ct+D)\cos 2t - 2(Ct^2+Dt)\sin 2t$

$= \sin 2t(2At + B - 2Ct^2 - 2Dt) + \cos 2t(2At^2 + 2Bt + 2Ct + D)$

$y_P'' = 2\cos 2t(At + B - 2Ct^2 - 2Dt) + \sin 2t(2A - 4Ct - 2D) +$ ... (next line)

$-2\sin 2t(2At^2 + 2Bt + 2Ct + D) + \cos 2t(4At + 2B + 2C)$

$y_P'' = \cos 2t(4At + 2B - 4Ct^2 - 4Dt + 4At + 2B + 2C) + \sin 2t(-4At^2 - 4Bt - 4Ct - 2D + 2A - 4Ct - 2D)$

Subs.: $\cos 2t(8At + 4B - 4Ct^2 - 4Dt + 2C) + \sin 2t(-4At^2 - 4Bt - 8Ct - 4D + 2A) +$ ... (next line)

$+ 4\sin 2t(At^2 + Bt) + 4\cos 2t(Ct^2 + Dt) = t \cdot \sin t$

$\cos 2t(8At + 4B + 2C) + \sin 2t(-8Ct + 2A - 4D) = t \cdot \sin t$

It follows, then:

$8At \cdot \cos 2t = 0 \cdot t \cdot \cos 2t \to A = 0$

$(4B + 2C)\cdot \cos 2t = 0 \cdot \cos 2t \to 4B + 2C = 0$

$-8Ct \cdot \sin 2t = 1 \cdot t \cdot \sin 2t \to C = -\frac{1}{8}$

$(2A - 4D)\cdot \sin 2t = 0 \cdot \sin 2t \to 2A - 4D = 0$

$\to A = 0, \quad B = \frac{1}{16}, \quad C = -\frac{1}{8}, \quad D = 0$

$y_P = \frac{1}{16} t \cdot \sin 2t - \frac{1}{8} t^2 \cos 2t$

c) $\boxed{y = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{16} t \cdot \sin 2t - \frac{1}{8} t^2 \cos 2t}$

5) $y'' - 2y' + 5y = \underbrace{2\cos^2 x}$, can use double angle formula

$y'' - 2y' + 5y = \cos(2x) + 1$

a) Solving homog. eqn $y'' - 2y' + 5y = 0$

$y = e^{\lambda x} \rightarrow \lambda^2 - 2\lambda + 5 = 0 \qquad \lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$y_c = C_1 e^x \cdot \cos 2x + C_2 e^x \cdot \sin 2x$

б) $y_p = A\cos 2x + B\sin 2x + C$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

Subst.: $-4A\cos 2x - 4B\sin 2x - 2(-2A\sin 2x + 2B\cos 2x) + 5(A\cos 2x + B\sin 2x + C) = \cos 2x + 1$

$(-4A - 4B + 5A)\cos 2x = 1\cdot\cos 2x \rightarrow A = 1 + 4B$

$(-4B + 4A + 5B)\sin 2x = 0\cdot\sin 2x \rightarrow 4A = -B$

$5C = 1 \rightarrow C = \frac{1}{5}$

$A = \frac{1}{17}$

$B = -\frac{4}{17}$

$C = \frac{1}{5}$

$y_p = \frac{1}{17}\cos 2x - \frac{4}{17}\sin 2x + \frac{1}{5}$

c) $\boxed{y = C_1 e^x \cos 2x + C_2 e^x \sin 2x + \frac{1}{17}\cos 2x - \frac{4}{17}\sin 2x + \frac{1}{5}}$

6) $y'' + y' - 6y = \sin t + t \cdot e^{2t}$

a) Solving $\quad y'' + y' - 6y = 0$

$y = e^{\lambda t} \rightarrow \lambda^2 + \lambda - 6 = 0 \qquad \lambda_1 = -3, \ \lambda_2 = 2$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

b) Since the RHS is a sum of 2 indep. functions,

we can guess $\quad y_{P_1} = A \sin t + B \cos t$

$y_{P_2} = (Ct + D) \cdot e^{2t} \cdot t$

and $\quad y_P = y_{P_1} + y_{P_2}$

$y_{P_1} = A \sin t + B \cos t$

$y_{P_1}' = A \cos t - B \sin t$

$y_{P_1}'' = -A \sin t - B \cos t$

Subst.: $\quad -A \sin t - B \cos t + A \cos t - B \sin t - 6(A \sin t + B \cos t) = \sin t$

$(-B + A - 6B) \cdot \cos t = 0 \longrightarrow \quad A = 7B$

$(-A - B - 6A) \cdot \sin t = 1 \cdot \sin t \longrightarrow \quad -7A = 1 + B$

$\rightarrow \quad A = -\frac{7}{50}, \quad B = -\frac{1}{50}$

$y_{P_1} = -\frac{7}{50} \sin t - \frac{1}{50} \cos t$

$y_{P_2} = (Ct^2 + Dt) \cdot e^{2t}$

$y_{P_2}' = e^{2t}(2Ct + D) + 2(Ct^2 + Dt)e^{2t} = e^{2t}(2Ct^2 + 2Dt + 2Ct + D)$

$y_{P_2}'' = 2e^{2t}(2Ct^2 + 2Dt + 2Ct + D) + e^{2t}(4Ct + 2D + 2C)$

Subs.: $\quad e^{2t}(4Ct^2 + 4Dt + 4Ct + 2D + 4Ct + 2D + 2C + 2Ct^2 + 2Dt + 2Ct + D - 6Ct^2 - 6Dt) = t \cdot e^{2t}$

$e^{2t}(10Ct + 5D + 2C) = t \cdot e^{2t}$

$10Ct = t \qquad C = \frac{1}{10}$

$5D + 2C = 0 \qquad D = -\frac{1}{25}$

$y_{P_2} = \left(\frac{1}{10}t^2 - \frac{1}{25}t\right)e^{2t}$

$y_P = -\frac{7}{50} \sin t - \frac{1}{50} \cos t + \left(\frac{1}{10}t^2 - \frac{1}{25}t\right)e^{2t}$

c) $\boxed{y = C_1 e^{-3t} + C_2 e^{2t} - \frac{7}{50} \sin t - \frac{1}{50} \cos t + \left(\frac{1}{10}t^2 - \frac{1}{25}t\right)e^{2t}}$

7) $y'' - y = \cosh(x)$, $y(0) = 2$, $y'(0) = 12$

a) associated homog. eq.n $y'' - y = 0$

$y = e^{\lambda x}$, $\lambda^2 - 1 = 0$, $\lambda_1 = 1$, $\lambda_2 = -1$

$y_c = C_1 e^{x} + C_2 e^{-x}$

b) Since $\cosh(x) = \dfrac{e^{-x} + e^{x}}{2}$

Ansatz: $y_p = (A \cdot e^{-x} + B e^{x}) \cdot x = Ax \cdot e^{-x} + Bx \cdot e^{x}$

$y_p' = Ae^{-x} - Axe^{-x} + Be^{x} + Bxe^{x}$

$y_p'' = -2Ae^{-x} + Axe^{-x} + 2Be^{x} + Bxe^{x}$

Subst: $-2Ae^{-x} + Axe^{-x} + 2Be^{x} + Bxe^{x} - Axe^{-x} - Bxe^{x} = \frac{e^{-x}}{2} + \frac{e^{x}}{2}$

$-2Ae^{-x} + 2Be^{x} = \frac{e^{-x}}{2} + \frac{e^{x}}{2}$

$A = -\frac{1}{4}$ $\quad B = \frac{1}{4}$

$y_p = -\frac{1}{4}xe^{-x} + \frac{1}{4}xe^{x}$

c) $y = C_1 e^{x} + C_2 e^{-x} - \frac{1}{4}xe^{-x} + \frac{1}{4}xe^{x}$

$y' = C_1 e^{x} - C_2 e^{-x} + \frac{1}{4}xe^{-x} - \frac{1}{4}e^{-x} + \frac{1}{4}xe^{x} + \frac{1}{4}e^{x}$

$y(0) = 2$, so $\begin{cases} 2 = C_1 + C_2 \\ 12 = C_1 - C_2 - \frac{1}{4} + \frac{1}{4} \end{cases}$ $\rightarrow \begin{cases} 14 = 2C_1 \\ -10 = 2C_2 \end{cases}$ $\quad \begin{cases} C_1 = 7 \\ C_2 = -5 \end{cases}$

$y'(0) = 12$, so

$$\boxed{y = 7e^{x} - 5e^{-x} - \frac{1}{4}xe^{-x} + \frac{1}{4}xe^{x}}$$

Or

$$\boxed{y = 7e^{x} - 5e^{-x} + \frac{1}{2}x \cdot \sinh(x)}$$