Math 527 Spring 2018 Homework # 1 solutions

① $\frac{dy}{dt} = (1+t)(1+y)$, Use seperation of variables.

$$\frac{1}{1+y}\frac{dy}{dt} = (1+t) \quad , \; y \neq -1$$

$$\int \frac{1}{1+y}\frac{dy}{dt}\,dt = \int (1+t)\,dt$$

$$\int \frac{1}{1+y}\,dy = \int (1+t)\,dt$$

$$\ln|1+y| = t + \frac{t^2}{2} + C$$

$$|1+y| = e^{t+\frac{t^2}{2}+C}$$

$$|1+y| = e^{C} e^{t+\frac{t^2}{2}}$$

$$1+y = \pm e^{C} e^{t+\frac{t^2}{2}}$$

Singular Solution

$$y \equiv -1$$

$$\frac{dy}{dt} = 0 = (1+t)(0) = (1+t)(1+y)$$

Thus every solution is of the form,

$$\boxed{y = Ke^{t+\frac{t^2}{2}} - 1 \quad , \; K \in \mathbb{R}}$$

② $\frac{dy}{dt} = 1 - t + y^2 - ty^2$ (Seperable)

$$\frac{dy}{dt} = (1-t) + y^2(1-t)$$

$$\frac{dy}{dt} = (1-t)(1+y^2)$$

$$\frac{1}{1+y^2}\frac{dy}{dt} = (1-t)$$

$$\int \frac{1}{1+y^2}\frac{dy}{dt}\,dt = \int (1-t)\,dt$$

$$\int \frac{1}{1+y^2}\,dy = \int (1-t)\,dt$$

Not Required

$$\arctan(y) = t - \frac{t^2}{2} + C \quad , \quad -\frac{\pi}{2} < t - \frac{t^2}{2} + c < \frac{\pi}{2}$$

$$\boxed{y = \tan\left(t - \frac{t^2}{2} + c\right)}$$

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3) $\frac{dy}{dx} = e^{x+y+3} = e^y \cdot e^{x+3}$ (seperable)

$$e^{-y}\frac{dy}{dx} = e^{x+3}$$

$$\int e^{-y}\frac{dy}{dx}\,dx = \int e^{x+3}\,dx$$

$$\int e^{-y}\,dy = \int e^{x+3}\,dx$$

$$-e^{-y} = e^{x+3} + C$$

$$e^{-y} = -e^{x+3} + D \quad , \quad D = -C$$

$$-y = \ln\left(D - e^{x+3}\right) \quad , \quad D - e^{x+3} > 0$$ Not Required

$$\boxed{y = \ln\left(\frac{1}{D - e^{x+3}}\right)}$$

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4) $\frac{dy}{dt} = \frac{2t}{y + yt^2}$, $y(2) = 3$ (seperable)

$$\frac{dy}{dt} = \frac{2t}{y(1+t^2)} \iff y\frac{dy}{dt} = \frac{2t}{1+t^2} \quad , \quad y \neq 0$$

$$\int y \frac{dy}{dt}\, dt = \int \frac{2t}{1+t^2}\, dt \quad , \text{ let } u = 1+t^2$$
$$du = 2t\, dt$$

$$\int y\, dy = \int \frac{du}{u}$$

$$\frac{y^2}{2} = \ln|u| + C$$

$$\frac{y^2}{2} = \ln|1+t^2| + C$$

$$y^2 = 2\ln|1+t^2| + 2C$$

$$y^2 = \ln(1+t^2)^2 + D \quad , \quad D = 2C$$

$$y = \pm\sqrt{\ln(1+t^2)^2 + D}$$

Now $y(2) = 3 \Rightarrow y = \sqrt{\ln(1+t^2)^2 + D}$ and further,

$$y(2) = \sqrt{\ln(1+2^2)^2 + D}$$
$$= \sqrt{\ln(25) + D} = 3$$

$$\Rightarrow \ln(25) + D = 9$$
$$\Rightarrow D = 9 - \ln(25)$$

$$y = \sqrt{\ln(1+t^2)^2 - \ln(25) + 9}$$

$$\boxed{y = \sqrt{\ln\left(\frac{1+t^2}{5}\right)^2 + 9}}$$

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(5) $\frac{dy}{dt} + y\cos t = 0$, $p(t) = \cos t \Rightarrow e^{\int p(t)dt} = e^{\int \cos t\, dt}$
$= e^{\sin t}$

mult. by $e^{\sin t}$.

$$e^{\sin t}\frac{dy}{dt} + e^{\sin t}\cos t\, y = 0$$

$$\frac{d}{dt}\left(e^{\sin t}\,y\right) = 0$$

$$\int \frac{d}{dt}\left(e^{\sin t}\,y\right) dt = \int 0\, dt$$

$$e^{\sin t}\,y = C$$

$$\boxed{y = C e^{-\sin t}}$$

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(6) $\dfrac{dy}{dx} + \dfrac{2xy}{1+x^2} = \dfrac{1}{1+x^2}$, $p(x) = \dfrac{2x}{1+x^2}$

Then ,

$$e^{\int p(x)dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\ln|1+x^2|} \quad (\text{see } 4)$$

$$= |1+x^2| = 1+x^2$$

mult. by $1+x^2$,

$$(1+x^2)\frac{dy}{dx} + 2xy = 1$$

$$\frac{d}{dx}\left((1+x^2)y\right) = 1$$

$$\int \frac{d}{dx}\left((1+x^2)y\right) dx = \int 1\, dx$$

$$(1+x^2)y = x + C$$

$$\boxed{y = \frac{x+C}{1+x^2}}$$

(7) $(1+t^2)\frac{dy}{dt} + ty = (1+t^2)^{5/2}$

$\boxed{\div \text{ by } (1+t^2).}$

$$\frac{dy}{dt} + \frac{t}{1+t^2}\,y = (1+t^2)^{3/2}$$

$$p(t) = \frac{t}{1+t^2} \Rightarrow \int p(t)\,dt = \int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{2t}{1+t^2}\,dt$$
$$= \frac{1}{2}\ln|1+t^2| \quad (\text{see } 4)$$
$$= \ln\sqrt{1+t^2}$$

So the int. factor is $e^{\ln\sqrt{1+t^2}} = \sqrt{1+t^2} = (1+t^2)^{1/2}$,

$$(1+t^2)^{1/2}\frac{dy}{dt} + \frac{t}{(1+t^2)^{1/2}}\,y = (1+t^2)^2$$

$$\frac{d}{dt}\left(\sqrt{1+t^2}\,y\right) = (1+t^2)^2$$

$$\int \frac{d}{dt}\left(\sqrt{1+t^2}\,y\right)dt = \int (1+t^2)^2\,dt$$

$$\sqrt{1+t^2}\,y = \int (1+2t^2+t^4)\,dt$$

$$\sqrt{1+t^2}\,y = t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C$$

$$\boxed{y = \frac{t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C}{\sqrt{1+t^2}}}$$

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(8) $\frac{dy}{dx} - 2xy = x$, $y(0) = 1$

$$p(x) = -2x \Rightarrow e^{\int p(x)\,dx} = e^{\int -2x\,dx}$$
$$= e^{-x^2}$$

mult. by int. factor,

$$e^{-x^2}\frac{dy}{dx} - 2x\,e^{-x^2}y = x\,e^{-x^2}$$

$$\frac{d}{dx}\left(e^{-x^2}y\right) = x\,e^{-x^2}$$

$$\int \frac{d}{dx}\left(e^{-x^2}y\right)dx = \int x\,e^{-x^2}dx \quad , \quad \text{let } u = -x^2$$

$$du = -2x\,dx$$

$$-\tfrac{1}{2}du = x\,dx$$

$$e^{-x^2}y = -\tfrac{1}{2}\int e^u\,du$$

$$e^{-x^2}y = -\tfrac{1}{2}e^u + C$$

$$e^{-x^2}y = -\tfrac{1}{2}e^{-x^2} + C$$

$$y = -\tfrac{1}{2} + Ce^{x^2}$$

Using $y(0)=1 \Rightarrow y(0) = -\tfrac{1}{2} + Ce^0 = -\tfrac{1}{2} + C = 1$

$$\Rightarrow C = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$

$$\Rightarrow \boxed{y = -\tfrac{1}{2} + \tfrac{3}{2}e^{x^2}}$$

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9) $\dfrac{dy}{dx} = x\sqrt{1-y^2}$ (separable)

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = x,\ y \neq \pm 1 \Rightarrow \int \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}\,dx = \int x\,dx$$

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int x\,dx$$

Not Required

$$\arcsin(y) = \frac{x^2}{2} + C \quad , \quad -\frac{\pi}{2} < \frac{x^2}{2} + C < \frac{\pi}{2}$$

$$\boxed{y = \sin\left(\frac{x^2}{2} + C\right) \quad \text{(general solution)}}$$

We have singular solutions where $\boxed{y = \pm 1}$, it can be verified,

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \sqrt{1-y^2} = 0$$

$$\Rightarrow \frac{dy}{dx} = x\sqrt{1-y^2}$$