MATH 527
SPRING 2015
HW #7 SOLNS.
Variation of Parameters and LI.

1) $2y'' - 3y' + y = (t^2+1)e^{2t}$

constant coefficients.
2nd order problem
non-homog.

<u>ansatz</u>: $y_c = e^{\lambda t}$

$$2\lambda^2 - 3\lambda + 1 = 0$$

$$(2\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = \tfrac{1}{2}, 1 \qquad f(t) = \frac{(t^2+1)e^{2t}}{2}$$

$$\boxed{\begin{aligned} y_c(t) &= c_1 e^{\frac{1}{2}t} + c_2 e^t \\ y_p(t) &= u_1(t) e^{\frac{1}{2}t} + u_2(t) e^t \end{aligned}}$$

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{\frac{1}{2}t} & e^t \\ \frac{1}{2}e^{\frac{1}{2}t} & e^t \end{vmatrix}$$

$$W = e^{\frac{1}{2}t+t} - \tfrac{1}{2}e^{\frac{1}{2}t+t}$$

$$\boxed{W = \tfrac{1}{2}e^{\frac{1}{2}t+t} = \tfrac{1}{2}e^{\frac{3}{2}t}}$$

$$u_1'(t) = \frac{-y_2(t)f(t)}{W}$$

$$u_1'(t) = \frac{-e^t(t^2+1)e^{2t}\left(\frac{1}{2}\right)}{\frac{1}{2}e^{\frac{3}{2}t}}$$

$$u_1'(t) = -\frac{2(t^2+1)e^{\frac{3}{2}t}}{2}$$

$$u_1(t) = -\int (t^2+1)e^{\frac{3}{2}t}\,dt$$

IBP $u = t^2+1$ $\quad v = \frac{2}{3}e^{3/2t}$

$du = 2t\,dt \quad dv = e^{3/2t}\,dt$

$$u_1(t) = -\left((t^2+1)\tfrac{2}{3}e^{\frac{3}{2}t} - \int \tfrac{2}{3}e^{\frac{3}{2}t}2t\,dt\right)$$

$$-\tfrac{4}{3}\int te^{\frac{3}{2}t}\,dt$$

IBP: $u = t$ $\quad v = \frac{2}{3}e^{\frac{3}{2}t}$

$du = dt \quad dv = e^{3/2t}\,dt$

$$-\left[\tfrac{2}{3}(t^2+1)e^{\frac{3}{2}t} - \tfrac{4}{3}\left(t\tfrac{2}{3}e^{\frac{3}{2}t} - \int \tfrac{2}{3}e^{\frac{3}{2}t}\,dt\right)\right]$$

$$-\tfrac{4}{9}e^{\frac{3}{2}t}$$

$$\boxed{u_1(t) = -\tfrac{2}{3}(t^2+1)e^{\frac{3}{2}t} + \tfrac{8}{9}te^{\frac{3}{2}t} - \tfrac{16}{27}e^{\frac{3}{2}t}}$$

$$u_2'(t) = \frac{y_1(t)f(t)}{W} = \frac{e^{\frac{1}{2}t}(t^2+1)e^{2t}\left(\frac{1}{2}\right)}{\frac{1}{2}e^{\frac{3}{2}t}}$$

$$u_2'(t) = (t^2+1)e^t$$

$$u_2(t) = \int (t^2+1)e^t\,dt$$

IBP $u = t^2+1$ $\quad v = e^t$

$du = 2t\,dt \quad dv = e^t\,dt$

$$u_2(t) = \left((t^2+1)e^t - \int e^t 2t\,dt\right)$$

$$u_2(t) = \left[(t^2+1)e^t - 2\int te^t\,dt\right]$$

IBP $u = t$ $v = e^t$
$du = dt$ $dv = e^t dt$

$$-\left(te^t - \int e^t dt\right)$$
$$-e^t$$

$$u_2(t) = \left[(t^2+1)e^t - 2te^t + 2e^t\right]$$

$$y_p(t) = \left(-\frac{2}{3}(t^2+1)e^{\frac{3}{2}t} + \frac{8}{9}te^{\frac{3}{2}t} - \frac{16}{27}e^{\frac{3}{2}t}\right)e^{\frac{1}{2}t}$$
$$+\left((t^2+1)e^t - 2te^t + 2e^t\right)e^t$$

$$y_p(t) = \left(-\frac{2}{3}(t^2+1) + \frac{8}{9}t - \frac{16}{27}\right)e^{2t}$$
$$+\left((t^2+1) - 2t + 2\right)e^{2t}$$

$$y_p(t) = \left[\left(\frac{-2}{3} + \frac{3}{3}\right)(t^2+1) + \left(\frac{8}{9} - \frac{18}{9}\right)t\right.$$
$$\left. + \left(\frac{-2}{3} - \frac{16}{27} + 1 + 2\right)\right]e^{2t}$$

$$y_p(t) = \left[\frac{1}{3}(t^2+1) - \frac{10}{9}t\right.$$
$$\left. + \left(-\frac{18}{27} - \frac{16}{27} + \frac{27}{27} + \frac{54}{27}\right)\right]e^{2t}$$

$$y_p(t) = \left[\frac{1}{3}(t^2+1) - \frac{10}{9}t + \frac{47}{27}\right]e^{2t}$$

$$\boxed{y_p(t) = \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}}$$

$$\boxed{y_{gen'l}(t) = c_1e^{\frac{1}{2}t} + c_2e^t + \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}}$$

2) $y'' + y = \sec(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$y_c(t) = e^{\lambda t}$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_c(t) = c_1 e^{it} + c_2 e^{-it}$ OR

$y_c(t) = \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t)$

$y_p(t) = u_1(t)\cos(t) + u_2(t)\sin(t)$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$

$W = \cos^2 t + \sin^2 t = 1$

$u_1'(t) = \frac{-y_2(t) f(t)}{W}$

$= \frac{-\sin(t)\sec(t)}{1}$

$u_1(t) = -\int \frac{\sin(t)}{\cos(t)}\, dt$

$=$ $u = \cos(t)$

$du = -\sin(t)\, dt$

$u_1(t) = \int \frac{1}{u} dt = \ln|\cos(t)| + \cancel{C}^{0}$

$u_1(t) = \ln|\cos(t)|$, $\cos(t) > 0$ for

$\boxed{u_1(t) = \ln(\cos(t))}$ $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$u_2'(t) = \frac{y_1(t) f(t)}{W} = \frac{\cos(t)\sec(t)}{1}$

$u_2'(t) = 1$

$u_2(t) = \int 1\, dt$

$u_2(t) = t + \cancel{C}^{0}$

$\boxed{u_2(t) = t}$

$\boxed{y_p(t) = \ln(\cos(t))\cos(t) + t\sin(t)}$

$$\boxed{\begin{aligned} y_{gen'l}(t) &= \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t) \\ &+ \ln(\cos(t))\cos(t) + t\sin(t) \end{aligned}}$$

3) $y'' - 3y' + 2y = te^{3t} + 1$

$y_c(t) = e^{\lambda t}$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1, 2$

$$\boxed{\begin{array}{l} y_c(t) = c_1 e^t + c_2 e^{2t} \\ y_p(t) = u_1 e^t + u_2 e^{2t} \end{array}}$$

$$W = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = 2e^{3t} - e^{3t}$$

$\boxed{W = e^{3t}}$

$$u_1'(t) = \frac{-y_2(t) f(t)}{W}$$

$$= \frac{-e^{2t}(te^{3t} + 1)}{e^{3t}}$$

$$= -(te^{3t} + 1)e^{-t}$$

$u_1'(t) = -te^{2t} - e^{-t}$

$u_1(t) = -\int te^{2t}\,dt - \int e^{-t}\,dt$

IBP $u = t$ $v = \frac{1}{2}e^{2t}$

$du = dt$ $dv = e^{2t}\,dt$

$-\left(\frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t}\,dt\right) - \int e^{-t}\,dt$

$u_1 = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t} + \cancel{C}^{0}$

$\boxed{u_1(t) = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}}$

$$u_2'(t) = \frac{y_1(t) f(t)}{W} = e^t(te^{3t} + 1)e^{-3t}$$

$$= (te^{4t} + e^t)e^{-3t}$$

$$= te^t + e^{-2t}$$

$u_2(t) = \int te^t\,dt + \int e^{-2t}\,dt$

$u_2(t) = te^t - e^t + -\frac{1}{2}e^{-2t} + \cancel{C}^{0}$

$\boxed{u_2(t) = te^t - e^t - \frac{1}{2}e^{-2t}}$

$$y_p(t) = \left(-\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}\right)e^t$$
$$+ \left(te^t - e^t - \frac{1}{2}e^{-2t}\right)e^{2t}$$

$$y_p(t) = -\frac{1}{2}te^{3t} + \frac{1}{4}e^{3t} + e^0$$
$$+ te^{3t} - e^{3t} - \frac{1}{2}e^0$$

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$\boxed{y_p(t) = \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \cancel{e^{2t}} + \frac{1}{2}}$

$$\boxed{y_{gen'l}(t) = c_1 e^t + c_2 e^{2t} + \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \frac{1}{2}}$$

4) $3y'' + 4y' + y = e^{-t}\sin t$, $y(0) = 1$, $y'(0) = 0$

$y_c(t) = e^{\lambda t}$

$\boxed{f(t) = \frac{e^{-t}\sin t}{3}}$

$3\lambda^2 + 4\lambda + 1 = 0$

$(3\lambda + 1)(\lambda + 1) = 0$

$\lambda = -\frac{1}{3}, -1$

$$\boxed{\begin{aligned} y_c(t) &= c_1 e^{-\frac{1}{3}t} + c_2 e^{-t} \\ y_p(t) &= u_1(t) e^{-\frac{1}{3}t} + u_2(t) e^{-t} \end{aligned}}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{3}t} & e^{-t} \\ -\frac{1}{3}e^{-\frac{1}{3}t} & -e^{-t} \end{vmatrix}$$

$W = -e^{-\frac{1}{3}t - t} + +\frac{1}{3}e^{-\frac{1}{3}t-t}$

$\boxed{W = -\frac{2}{3}e^{-\frac{4}{3}t}}$

$$u_1'(t) = \frac{-y_2(t)f(t)}{W} = \frac{-e^{-t}\frac{e^{-t}\sin(t)}{3}}{-\frac{2}{3}e^{-\frac{4}{3}t}}$$

$u_1'(t) = \frac{1}{3}\frac{3}{2}\sin(t)e^{-2t+\frac{4}{3}t}$

$u_1'(t) = \frac{1}{2}\sin(t)e^{-\frac{2}{3}t}$

$u_1(t) = \frac{1}{2}\int \sin(t)e^{-\frac{2}{3}t}\,dt$

$u = \sin(t)$ $\quad v = -\frac{3}{2}e^{-\frac{2}{3}t}$

$du = \cos(t)\,dt$ $\quad dv = e^{-\frac{2}{3}t}\,dt$

$= \frac{1}{2}\left(-\frac{3}{2}\sin(t)e^{-\frac{2}{3}t} - \int -\frac{3}{2}e^{-\frac{2}{3}t}\cos(t)\,dt\right)$

$u = \cos(t)$ $\quad v = -\frac{3}{2}e^{-\frac{2}{3}t}$

$du = -\sin(t)\,dt$ $\quad dv = e^{-\frac{2}{3}t}\,dt$

$u_1(t) = \frac{1}{2}\left(-\frac{3}{2}\sin(t)e^{-\frac{2}{3}t} + \frac{3}{2}\left(\cos(t)\left(-\frac{3}{2}e^{-\frac{2}{3}t}\right) - \int +\frac{3}{2}e^{-\frac{2}{3}t}\sin t\,dt\right)\right)$

Let $I = \int \sin(t)e^{-\frac{2}{3}t}\,dt$

$u_1(t) = \frac{1}{2}I$

$\frac{1}{2}I = \frac{1}{2}\left(-\frac{3}{2}\sin(t)e^{-\frac{2}{3}t} - \frac{9}{4}\cos(t)e^{-\frac{2}{3}t} - \frac{9}{4}I\right)$

$\left(\frac{4}{4} + \frac{9}{4}\right)I = \frac{3}{2}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}$

$I = \frac{4}{13}\cdot\frac{3}{2}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}$

$I = \frac{6}{13}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t} + \cancel{C}^{0}$

$\boxed{u_1(t) = \frac{1}{2}I = \frac{3}{13}\left(-\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}}$

$$u_2'(t) = \frac{y_1(t)f(t)}{W} = \frac{e^{-\frac{1}{3}t}\left(e^{-t}\sin(t)\frac{1}{3}\right)}{-\frac{2}{3}e^{-\frac{4}{3}t}}$$

$= -\frac{1}{3}\frac{3}{2}\sin(t)e^{-\frac{1}{3}t - t + \frac{4}{3}t}$

$u_2'(t) = -\frac{1}{2}\sin(t)e^{0t}$

$u_2(t) = -\frac{1}{2}\int \sin(t)\,dt$

$u_2(t) = -\frac{1}{2}\left(-\cos(t) + \cancel{C}^{0}\right)$

$\boxed{u_2(t) = \frac{1}{2}\cos(t)}$

$$y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+\left(\frac{3}{13}\left(\frac{3}{2}\cos(t) - \sin(t)\right)e^{-\frac{2}{3}t}\right)e^{-\frac{1}{3}t}$$

$$+\left(\frac{1}{2}\cos(t)\right)e^{-t}$$

$$y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+\frac{3}{13}\left(\frac{3}{2}\cos(t) - \sin(t)\right)e^{-t}$$

$$+\left(\frac{1}{2}\cos(t)\right)e^{-t}$$

$$\boxed{y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t} + \left[\left(\frac{9}{26} + \frac{13}{26}\right)\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t}}$$

$$\boxed{y_{gen'l}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t} + \left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t}}$$

1c) $y(0) = 1$, $y'(0) = 0$

$y(0) = 1$

$$1 = c_1 + c_2 + \frac{2}{13}$$

$$\boxed{c_1 = \frac{11}{13} - c_2}$$

$y'(0) = 0$

$$y'(t) = c_1\left(-\frac{1}{3}\right)e^{-\frac{1}{3}t} - c_2 e^{-t}$$

$$-\left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t} + e^{-t}\left(-\frac{2}{13}\sin(t) - \frac{3}{13}\cos(t)\right)$$

$y'(0) = 0$

$$0 = -\frac{1}{3}c_1 - c_2 - \frac{2}{13} - \frac{3}{13}$$

$$0 = -\frac{1}{3}\left(\frac{11}{13} - c_2\right) - c_2 - \frac{5}{13}$$

$$\frac{5}{13} + \frac{11}{39} = \left(+\frac{1}{3} - 1\right)c_2$$

$$\frac{15}{39} + \frac{11}{39}$$

$$\frac{26}{39}$$

$$\frac{2}{3} = -\frac{2}{3}c_2$$

$$\boxed{c_2 = -1} \rightarrow c_1 = \frac{11}{13} - \left(-\frac{13}{13}\right)$$

$$\boxed{c_1 = \frac{24}{13}}$$

$$y(t) = \frac{24}{13}e^{-\frac{1}{3}t} - e^{-t} + \left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t}$$

$$\boxed{y(t) = \frac{1}{13}e^{-t}\left(24e^{\frac{2}{3}t} - 13 + 2\cos(t) - 3\sin(t)\right)}$$

Homework 5 problem 5 solution: **Math 527, UNH spring 2018**

(a) What property must an operator L satisfy to be *linear*?

$L(\alpha f + \beta g) = \alpha L f + \beta L g$

where α, β are constants and f, g are functions.

(b) Why is linearity important for the solution of linear differential equations?

Linearity is important because it allows you to express the general solution of the linear ODE as a sum of the linearly independent solutions, e.g. $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

(c) How many linearly independent solutions does an nth order linear homogeneous equation have?

An nth-order linear homogeneous equation has n linearly independent solutions.

(d) When you integrate u_1' and u_2' in variation of parameters, why can you always set the integration constant to zero?

You can always set these integration constants to zero because u_1 and u_2 are coefficients of the homogeneous solutions y_1 and y_2 in the ansatz

$y_p = u_1 y_1 + u_2 y_2.$

Thus any constant included in the value of u_1 or u_2 could just be absorbed into the constants in front of y_1 and y_2 in the general solution. E.g.

$$\begin{aligned}
c_1 y_1 + c_2 y_2 + y_p &= c_1 y_1 + c_2 y_2 + y_p \\
&= c_1 y_1 + c_2 y_2 + (u_1 + a) y_1 + (u_2 + b) y_2 \\
&= (c_1 + a) y_1 + (c_2 + a) y_2 + u_1 y_1 + u_2 y_2
\end{aligned}$$

(e) What is Euler's formula?

$e^{ix} = \cos x + i \sin x$

(f) How would you prove Euler's formula? Don't do the proof, just describe the proof in a sentence or two.

Substitute ix in place of x in the power series expansion of e^x, then simplify and regroup so that the even terms become the power series for $\cos x$ and the odd terms become i times the power series for $\sin x$.