Math 527 HW #3 Solutions

① $6y'' - 7y' + y = 0$

guess $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

(plug in) $6\lambda^2 e^{\lambda x} - 7\lambda e^{\lambda x} + e^{\lambda x} = 0$

$6\lambda^2 - 7\lambda + 1 = 0$

$(6\lambda - 1)(\lambda - 1) = 0$

$6\lambda_1 - 1 = 0 \qquad \lambda_2 - 1 = 0$

$\lambda_1 = \frac{1}{6} \qquad \lambda_2 = 1$

$y(x) = c_1 e^{\frac{x}{6}} + c_2 e^{x}$

② $y'' + 2y' + 3y = 0$

guess $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

(plug in) $\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 3e^{\lambda x} = 0$

$\lambda^2 + 2\lambda + 3 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} = \dfrac{-2 \pm \sqrt{4 - 12}}{2}$

$\lambda = \dfrac{-2 \pm 2\sqrt{-2}}{2} = -1 \pm 2i$

$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$

③ $y'' - 6y' + 9y = 0$

guess $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

(plug in) $\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 9e^{\lambda x} = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0 \qquad \lambda_{1,2} = 3$

$y = c_1 e^{3x} + c_1 x e^{3x}$

④ $2y'' + y' - 10y = 0 \quad y(1) = 5 \quad y'(1) = 2$

$y = e^{\lambda x} \Rightarrow 2\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 10e^{\lambda x} = 0$

$2\lambda^2 + \lambda - 10 = 0$

$(2\lambda + 5)(\lambda - 2) \Rightarrow \lambda_1 = -5/2 \quad \lambda_2 = 2$

$y = c_1 e^{-5/2 x} + c_2 e^{2x}$

$-2(5 = c_1 e^{-5/2} + c_2 e^2)$

$2 = -\frac{5c_1}{2} e^{-5/2} + 2c_2 e^2$

$-10 = -2c_1 e^{-5/2} - 2c_2 e^2$

$-8 = -\frac{9}{2} c_1 e^{-5/2}$

$c_1 = \frac{16}{9} e^{5/2}$

$5 = \frac{16}{9} e^{5/2} e^{-5/2} + c_2 e^2$

$5 = \frac{16}{9} + c_2 e^2$

$c_2 = \frac{29}{9} e^{-2}$

$y = \frac{1}{9}(16 e^{-5/2 x + 5/2} + 29 e^{2x+2})$

⑤ $4y'' - 4y' + y = 0 \quad y(0) = 0 \quad y'(0) = 3$

$y = e^{\lambda x} \quad 4\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + e^{\lambda x} = 0$

$4\lambda^2 - 4\lambda + 1 = 0$

$\lambda = \frac{4 \pm \sqrt{4^2 - 4(4)(1)}}{8} = \frac{1}{2} \quad \lambda_{1,2} = \frac{1}{2}$

$y = c_1 e^{\frac{x}{2}} + c_2 x e^{x/2}$

$0 = c_1 e^0 + c_2(0)e^0 \Rightarrow c_1 = 0$

$y = c_2 x e^{x/2}$

$y' = c_2 e^{x/2} + c_2 \frac{x}{2} e^{x/2}$

$3 = c_2 e^0 + c_2(0) e^{0/2} \Rightarrow c_2 = 3$

$y = 3x e^{x/2}$

(6) $y'' + y' + 2y = 0 \qquad y(0) = 1 \quad y'(0) = 2$

$y = e^{\lambda x} \Rightarrow \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + 2e^{\lambda x} = 0$

$\lambda^2 + \lambda + 2 = 0$

$\lambda = \dfrac{-1 \pm \sqrt{1-4(2)(1)}}{2} \qquad \lambda = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

$y = e^{-\frac{x}{2}}\left(c_1 \cos(\sqrt{7}/2\, x) + c_2 \sin(\sqrt{7}/2\, x)\right)$

$y' = -\frac{1}{2} e^{-x/2}\left(c_1 \cos(\sqrt{7}/2\, x) + c_2 \sin(\sqrt{7}/2\, x)\right)$
$\quad + e^{-x/2} \sqrt{7}/2 \left(c_2 \cos(\sqrt{7}/2\, x) + c_1 \sin(\sqrt{7}/2\, x)\right)$

$1 = e^0 (c_1 \cos(0) + c_2 \sin(0)) \Rightarrow c_1 = 1$

$2 = -\frac{1}{2} e^0 (\cos(0) + c_2 \sin(0))$
$\quad + e^0 \sqrt{7}/2 (c_2 \cos(0) + \sin(0))$

$2 = -\frac{1}{2} + \frac{\sqrt{7}}{2} c_2 \Rightarrow c_2 = \frac{5}{\sqrt{7}}$

$y = e^{-x/2}\left(\cos(\sqrt{7}/2\, x) + \frac{5}{\sqrt{7}} \sin(\sqrt{7}/2\, x)\right)$

(7) Newtons Law of cooling $\dfrac{dT}{dt} = k(T - T_m) \qquad T(0) = 70$
$T(30) = 50$

$\dfrac{dT}{dt} = k(T-10)$

$\displaystyle\int \frac{1}{T-10}\, dT = \int k\, dt$

$\ln|T-10| = kt + c$

$T - 10 = c e^{kt}$

$T = c e^{kt} + 10$

$70 = c e^{0k} + 10 \qquad c = 60$

$T = 60e^{kt} + 10$

$50 = 60e^{30k} + 10 \qquad e^{30k} = \frac{2}{3}$

$k = \dfrac{\ln(\frac{2}{3})}{30}$

$$T = 60e^{\ln(\frac{2}{3})\frac{t}{30}} + 10$$

$$T(60) = 60e^{2\ln(\frac{2}{3})} + 10$$
$$= 60e^{\ln(\frac{4}{9})} + 10$$
$$= 60\left(\frac{4}{9}\right) + 10$$
$$= 36.\overline{6}$$

$$15 = 60 + e^{\ln(2/3)\frac{t}{30}} + 10$$
$$\frac{1}{12} = e^{\ln(2/3)\frac{t}{30}}$$
$$\frac{1}{12} = \left(\frac{2}{3}\right)^{t/30}$$
$$\ln(1/12) = T/30 \ln(2/3)$$
$$t = \frac{30\ln(1/12)}{\ln(2/3)} = 183.86 \text{ seconds}$$

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$$\frac{dA}{dt} = 4 - \frac{A}{50}$$
$$\int \frac{1}{4-\frac{A}{50}}\, dA = \int dt$$

$w = 4 - \frac{A}{50}$

$dw = -\frac{1}{50}\, dA$

$-50\, dw = dA$

$$-50\ln\left|4 - \frac{A}{50}\right| = t + C$$
$$4 - \frac{A}{50} = Ce^{-\frac{t}{50}}$$
$$A = 200 - Ce^{-t/50}$$

$$A(t) = 200 - Ce^{-t/5}$$

$$\lim_{t \to \infty} A(t) = 200 \text{ grams of salt}$$

(9) $\frac{dc}{dt} = k\left(100 - \frac{2}{3}c\right)\left(50 - \frac{1}{3}c\right)$ $\qquad c(0) = 0$

$= k\,\frac{2}{3}(150 - c)\,\frac{1}{3}(150 - c)$ $\qquad c(10) = 10$

$$\frac{dc}{dt} = k(150-c)^2$$

$$\int \frac{1}{(150-c)^2}\,dc = \int k\,dt$$

$u = 150 - c$
$du = -dc$

$$-\int \frac{1}{u^2}\,du = kt + C_1$$

$$u^{-1} = kt + C_1$$

$$\frac{1}{(150-c)} = kt + C_1$$

$$150 - c = \frac{1}{kt + C_1}$$

$$c = 150 - \frac{1}{kt + C_1}$$

$$0 = 150 - \frac{1}{k(0) + C_1}$$

$$C_1 = \frac{1}{150} \implies c(t) = 150 - \frac{150}{150kt + 1}$$

$$10 = 150 - \frac{150}{150k(10) + 1}$$

$$-140 = -\frac{150}{150k(10) + 1}$$

$$1500k + 1 = \frac{150}{140}$$

$$1500k = \frac{150 - 140}{140}$$

$$1500k = \frac{1}{14}$$

$$k = \frac{1}{21{,}000}$$

$$c(t) = 150 - \frac{150}{\frac{t}{140} + 1}$$

$$c(t) = 150 - \frac{21000}{t + 140}$$

$$\lim_{t \to \infty} c(t) = 150\text{s}$$

$75 = 150 - \frac{21000}{t + 140}$ $\qquad -75 = \frac{-21000}{t + 140}$

$$t + 140 = 280$$
$$-140 \quad -140$$

$t = 140$ minutes