Math 527 Spring 2018 Homework #6 Solutions

① $y' - y = 1$, $y(0) = 0$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$sY - \cancel{y(0)}^{0} - Y = \frac{1}{s}$$

$$(s-1)Y = \frac{1}{s}$$

$$Y = \frac{1}{s(s-1)}$$

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

$$1 = A(s-1) + Bs$$

$$0 \cdot s + 1 \cdot 1 = (A+B)s - A$$

$$A + B = 0 \Rightarrow B = 1$$

$$-A = 1 \Rightarrow A = -1$$

Hence,

$$Y = \frac{1}{s-1} - \frac{1}{s}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\boxed{y(t) = e^t - 1}$$

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② $y' + 6y = e^{4t}$, $y(0) = 2$

$$\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{e^{4t}\}$$

$$sY - y(0) + 6Y = \frac{1}{s-4}$$

$$sY - 2 + 6Y = \frac{1}{s-4}$$

$$(s+6)Y = \frac{1}{s-4} + 2$$

$$Y = \frac{1}{(s+6)(s-4)} + \frac{2}{s+6}$$

$$\frac{1}{(s+6)(s-4)} = \frac{A}{s+6} + \frac{B}{s-4}$$

$$1 = A(s-4) + B(s+6)$$

$$1 = (A+B)s + (6B - 4A)$$

$$A + B = 0 \Rightarrow B = -A$$

$$6B - 4A = 1 \Rightarrow -6A - 4A = 1$$

$$A = \frac{-1}{10}$$

$$B = \frac{1}{10}$$

$$Y = \left(\frac{-1}{10}\right)\frac{1}{s+6} + \left(\frac{1}{10}\right)\frac{1}{s-4} + \frac{2}{s+6}$$

$$Y = \left(\frac{19}{10}\right)\frac{1}{s+6} + \left(\frac{1}{10}\right)\frac{1}{s-4}$$

$$\mathcal{L}^{-1}\{Y\} = \left(\frac{19}{10}\right)\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} + \left(\frac{1}{10}\right)\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$\boxed{y(t) = \frac{19}{10}e^{-6t} + \frac{1}{10}e^{4t}}$$

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(3) $y'' + 5y' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2Y - s\cancel{y(0)}^{1} - \cancel{y'(0)}^{0} + 5sY - 5\cancel{y(0)}^{1} + 4Y = 0$$

$$(s^2 + 5s + 4)Y = s + 5$$

$$Y = \frac{s+5}{(s+4)(s+1)}$$

$$\frac{s+5}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$$

$$s + 5 = A(s+1) + B(s+4)$$

$$s + 5 = (A+B)s + (A+4B)$$

$$A + B = 1 \Rightarrow A + \frac{4}{3} = 1$$

$$\underline{A + 4B = 5} \qquad A = -\frac{1}{3}$$

$$-3B = -4$$

$$B = \frac{4}{3}$$

Hence,

$$Y = \left(\frac{-1}{3}\right)\frac{1}{s+4} + \left(\frac{4}{3}\right)\frac{1}{s+1} \Rightarrow \mathcal{L}^{-1}\{Y\} = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(t) = -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}}$$

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(4) $y'' + y = \sqrt{2}\sin\sqrt{2}t$, $y(0) = 10$, $y'(0) = 0$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sqrt{2}\,\mathcal{L}\{\sin\sqrt{2}t\}$$

$$s^2Y - sy(0) - y'(0) + Y = \sqrt{2}\left(\frac{\sqrt{2}}{s^2+2}\right)$$

$$s^2Y - 10s + Y = \frac{2}{s^2+2}$$

$$(s^2+1)Y = \frac{2}{s^2+2} + 10s$$

$$Y = \frac{2}{(s^2+2)(s^2+1)} + \frac{10s}{s^2+1}$$

$$\frac{2}{(s^2+2)(s^2+1)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+1}$$

$$2 = (As+B)(s^2+1) + (Cs+D)(s^2+2)$$

$$2 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2 + 2Cs + 2D$$

$$2 = (A+C)s^3 + (B+D)s^2 + (A+2C)s + (B+2D)$$

$A + C = 0 \Rightarrow C = -A$

$B + D = 0 \Rightarrow D = -B$

$A + 2C = 0 \rightarrow A - 2A = 0 \Rightarrow A = 0$

$B + 2D = 2 \Rightarrow B - 2B = 2 \Rightarrow B = -2$

$C = 0$

$D = 2$

$$Y = \frac{-2}{s^2+2} + \frac{2}{s^2+1} + \frac{10s}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y\} = -2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+10}\right\}$$

$$\mathcal{L}^{-1}\{Y\} = \frac{-2}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$\boxed{y(t) = -\sqrt{2}\sin\sqrt{2}t + 2\sin t + 10\cos t}$$

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(5) $y'' - 6y' + 9y = t$, $y(0)=0$, $y'(0)=1$

$$\mathcal{L}\{y''\} - 6\,\mathcal{L}\{y'\} + 9\,\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2Y - s\cancel{y(0)}^{0} - y'(0) - 6sY + 6y(0) + 9Y = \frac{1}{s^2}$$

$$s^2Y - 1 - 6sY + 9Y = \frac{1}{s^2}$$

$$(s^2 - 6s + 9)Y = \frac{1}{s^2} + 1$$

$$Y = \frac{1}{s^2(s^2-6s+9)} + \frac{1}{s^2-6s+9}$$

$$Y = \frac{1}{s^2(s-3)^2} + \frac{1}{(s-3)^2}$$

$$\frac{1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$1 = As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2$$

$$1 = A(s^3 - 6s^2 + 9s) + B(s^2 - 6s + 9) + C(s^3 - 3s^2) + Ds^2$$

$$1 = (A+C)s^3 + (-6A + B - 3C + D)s^2 + (9A - 6B)s + 9B$$

$A + C = 0 \Rightarrow C = -A \Rightarrow C = -\frac{2}{27}$

$-6A + B - 3C + D = 0$

$9A - 6B = 0 \Rightarrow 9A - \frac{6}{9} = 0 \Rightarrow 81A = 6$

$9B = 1 \Rightarrow B = \frac{1}{9}$

$A = \frac{6}{81} = \frac{2}{27}$

$D = 6A - B + 3C$

$D = \frac{12}{27} - \frac{3}{27} - \frac{6}{27}$

$D = \frac{3}{27} = \frac{1}{9}$

$$Y = \left(\frac{2}{27}\right)\frac{1}{s} + \left(\frac{1}{9}\right)\frac{1}{s^2} - \left(\frac{2}{27}\right)\frac{1}{s-3} + \left(\frac{1}{9}\right)\frac{1}{(s-3)^2} + \frac{1}{(s-3)^2}$$

$\mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}}$$

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(6) $y'' - 4y' + 4y = t^3e^{2t}$, $y(0) = 0$, $y'(0) = 0$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3e^{2t}\}$$

$$s^2Y - sy(0) - y'(0) - 4sY + 4y(0) + 4Y = \frac{3!}{(s-2)^4}$$

$$(s^2 - 4s + 4)Y = \frac{6}{(s-2)^4} \implies (s-2)^2 Y = \frac{6}{(s-2)^4}$$

$$Y = \frac{6}{(s-2)^6} \implies y(t) = 6\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^6}\right\}$$

$$y(t) = \frac{6}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{(s-2)^6}\right\}$$

$$y(t) = \frac{6}{120}t^5e^{2t}$$

$$\boxed{y(t) = \frac{1}{20}t^5e^{2t}}$$

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(6)(Modified) $y'' - 4y' + 4y = te^{2t}$, $y(0) = 0$, $y'(0) = 0$

$$(s^2 - 4s + 4)Y = \frac{1}{(s-2)^2}$$

$$(s-2)^2 Y = \frac{1}{(s-2)^2}$$

$$Y = \frac{1}{(s-2)^4}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\}$$

$$y(t) = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\}$$

$$\boxed{y(t) = \frac{1}{6}t^3e^{2t}}$$