Final exam sample problems **Math 527 UNH, spring 2018**

These problems are provided as examples of the kind and of level of difficulty of problems that might appear on the final exam. Please note the following

- I've provided a lot of sample problems. The final exam will have fewer problems. Even so, the coverage of problem types and topics is not complete.
- Don't expect that the final exam will consist of variations of these problems.
- I do not expect to have time to write our solutions to the problems, so you will be on your own for figuring them out. You can check answers using Wolfram Alpha, `http://www.wolframalpha.com/`.
- For most problems you can check your answers by plugging the solution back into the differentil equation and verifying that the equation is satisfied.
- You can also review exams 1-3. Problems there are similar to what might appear on the final.

Problem 1: Find the general solution.

$$y' - 3y = 0$$

Problem 2: Find the general solution.

$$y'' - 3y = 0$$

Problem 3: Find the general solution.

$$y'' + 3y = 0$$

Problem 4: Find the general solution.

$$y' - 3y = x$$

Problem 5: Solve the initial-value problem.

$$y' - 3y = x, \quad y(0) = 1$$

Problem 6: Find the general solution.

$$y' + 2xy^2 = 0$$

Problem 7: Find the general solution.

$$x\frac{dy}{dx} = 2xe^x - y + 6x^2$$

Problem 8: Find the general solution.

$$y'' - 4y = 12e^{2x}$$

Problem 9: Find the general solution. Note that $\sinh x = (e^x - e^{-x})/2$.

$$y'' - y = \sinh 2x$$

Problem 10: Solve the initial-value problem .

$$y'' + 3y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Problem 11: Solve the initial-value problem

$$y'' + 4y = \begin{cases} 0, & 0 \le t < 2\pi \\ \sin t, & 2\pi \le t \end{cases}$$
$$y(0) = 1, \quad y'(0) = 0$$

For problems 12-15, express your answer in terms of real-valued functions.

Problem 12: Find the general solution.

$$\mathbf{x}' = \begin{pmatrix} 2 & -4 \\ 10 & 6 \end{pmatrix} \mathbf{x}$$

Problem 13: Find the general solution.

$$\frac{dx}{dt} = 3x - y$$
$$\frac{dy}{dt} = 9x - 3y$$

Problem 14: Solve the initial value problem.

$$\mathbf{x}' = \begin{pmatrix} 1/2 & 0 \\ 1 & -1/2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Problem 15: Find the general solution.

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Problem 16: The eigenvalue-eigenvector method of solving the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the n-dimensional vector $\mathbf{x}$ and $n \times n$ matrix $\mathbf{A}$ requires that we solve the equations $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ and $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$.
(a) Derive these two equations from $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and an appropriate ansatz.
(b) How many eigenvalue solutions λ will there be for the equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$?

Problem 17: This problem is a recapitulation of material presented in lecture. This problem is too long in its entirety for a final exam question, but any one or two parts of it would be appropriate.

The rotational dynamics of a spinning book are governed by

$$\begin{aligned} \Omega_1' &= k_1\, \Omega_2\, \Omega_3 \\ \Omega_2' &= -k_2\, \Omega_3\, \Omega_1 \\ \Omega_3' &= k_3\, \Omega_1 \Omega_2 \end{aligned}$$

where Ω_j is the angular velocity around the jth axis of the book and k_1, k_2, k_3 are positive constants determined by the height, thickness, and width of the book, and where the prime notation indicates differentiation in time.

(a) Rewrite these equations in vector form $\mathbf{x}' = f(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, x_3) = (\Omega_1, \Omega_2, \Omega_3)$.

(b) Show that $\mathbf{x} = (C, 0, 0)$, $\mathbf{x} = (0, C, 0)$, and $\mathbf{x} = (0, 0, C)$, are equilibria of $\mathbf{x}' = f(\mathbf{x})$. These three equilibrium solutions correspond to rotation about the three axes of the book, along its the shortest, middle, and longest dimensions, respectively.

(c) What is the matrix of partial derivatives Df as a function of $\mathbf{x}$?

(d) Evaluate the matrix of partial derivatives Df at $\mathbf{x} = (C, 0, 0)$. What are the eigenvalues of $Df(\mathbf{x})$? Is rotation about axis 1 stable or unstable?

(e) Evaluate the matrix of partial derivatives Df at $\mathbf{x} = (0, C, 0)$. What are the eigenvalues of $Df(\mathbf{x})$? Is rotation about axis 2 stable or unstable?

(f) Evaluate the matrix of partial derivatives Df at $\mathbf{x} = (0, 0, C)$. What are the eigenvalues of $Df(\mathbf{x})$? Is rotation about axis 3 stable or unstable?