Math 527 HW7

Solutions

1) $\mathcal{L}[(3t+1)\,\mathcal{U}(t-1)]$

$$
\begin{aligned}
3t+1 &= 3\left(t+\tfrac{1}{3}\right)\\
&= 3\left(t-1+1+\tfrac{1}{3}\right)\\
&= 3\left((t-1)+\tfrac{4}{3}\right)\\
&= 3(t-1)+4
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\left[(3(t-1)+4)\,\mathcal{U}(t-1)\right] &= e^{-s}\,\mathcal{L}[3t+4]\\
&= e^{-s}\left[\frac{3}{s^2}+\frac{4}{s}\right]
\end{aligned}
$$

2)
$$
\begin{aligned}
\mathcal{L}\left[e^{2t}(t-1)^2\right] &= \mathcal{L}\left[(t-1)^2\right]_{s\to s-2}\\
&= \mathcal{L}\left[t^2-2t+1\right]_{s\to s-2}\\
&= \left[\frac{2}{s^3}-\frac{2}{s^2}+\frac{1}{s}\right]_{s\to s-2}\\
&= \frac{2}{(s-2)^3}-\frac{2}{(s-2)^2}+\frac{1}{s-2}
\end{aligned}
$$

3) $\mathcal{L}^{-1}\left[\frac{2s+5}{s^2+6s+34}\right]$

Complete the square

$s^2+6s+9-9+34$

$(s+3)^2+25$

$$\mathcal{L}^{-1}\left[\frac{2s+5}{(s+3)^2+25}\right] = \mathcal{L}^{-1}\left[\frac{2\left(s+\frac{5}{2}\right)}{(s+3)^2+25}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2(s+3-3+5/2)}{(s+3)^2+25}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2(s+3)-1}{(s+3)^2+25}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+25}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+3)^2+25}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right]_{s\to s+3} - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right]_{s\to s+3}$$

$$= 2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t)$$

4) $\mathcal{L}^{-1}\left[\frac{s}{s^2+4}e^{-\pi/2\, s}\right] = u\left(t-\frac{\pi}{2}\right)\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]_{t\to t-\frac{\pi}{2}}$

$$= u\left(t-\frac{\pi}{2}\right)\cos(2t)\Big|_{t\to t-\frac{\pi}{2}}$$

$$= u(t-\pi/2)\cos\left(2\left(t-\frac{\pi}{2}\right)\right)$$

$$= u(t-\pi/2)\cos(2t-\pi)$$

$$= -u(t-\pi/2)\cos(2t)$$

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$$f(t) = \begin{cases} \sin(t) & 0 \le t < 2\pi \\ 0 & 2\pi \le t \end{cases}$$

$$f(t) = \sin(t) - \sin(t)\,\mathcal{U}(t-2\pi)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}[\sin(t)] - \mathcal{L}[\sin(t)\,\mathcal{U}(t-2\pi)]$$

$$= \frac{1}{s^2+1} - e^{-2\pi s}\,\mathcal{L}[\sin(t+2\pi)$$

$$= \frac{1}{s^2+1} - e^{-2\pi s}\,\mathcal{L}[\sin(t)]$$

$$= \frac{1}{s^2+1} - e^{-2\pi s}\,\frac{1}{s^2+1}$$

6

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t^2 & 1 \le t \end{cases}$$

$$f(t) = t^2\,\mathcal{U}(t-1)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2\,\mathcal{U}(t-1)] = e^{-s}\,\mathcal{L}\{(t+1)^2]$$

$$= e^{-s}\,\mathcal{L}[t^2+2t+1]$$

$$= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

#7

$$y'' - 5y' + 6y = u(t-1) \qquad y(0)=0 \quad y'(0)=1$$

$$\mathcal{L}[y'' - 5y' + 6y] = \mathcal{L}[u(t-1)]$$

$$s^2 Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{1} - 5[sY - \cancel{y(0)}^{0}] + 6Y = \frac{e^{-s}}{s}$$

$$s^2 Y - 1 - 5sY + 6Y = \frac{e^{-s}}{s}$$

$$Y(s^2 - 5s + 6) = 1 + \frac{e^{-s}}{s}$$

$$Y = \frac{1}{(s-3)(s-2)} + \frac{e^{-s}}{s(s-3)(s-2)}$$

$$\frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s-3)$$

$s=3$ $\quad$ $s=2$

$A=1$ $\quad$ $B=-1$

$$\frac{1}{s(s-3)(s-2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-2}$$

$$1 = A(s-3)(s-2) + B(s)(s-2) + C(s)(s-3)$$

$s=0$ $\quad$ $s=3$ $\quad$ $s=2$

$A = 1/6$ $\quad$ $B = \frac{1}{3}$ $\quad$ $C = -\frac{1}{2}$

$$Y = \frac{1}{s-3} - \frac{1}{s-2} + \left[\frac{\frac{1}{6}}{s} + \frac{\frac{1}{3}}{s-3} + \frac{\frac{1}{2}}{s-2}\right]e^{-s}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s}e^{-s}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-3}e^{-s}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-2}e^{-s}\right]$$

$$= e^{3t} - e^{2t} + \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s}\right]_{t\to t-1} u(t-1) + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]_{t\to t-1} u(t-1) - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]_{t\to t-1} u(t-1)$$

$$y(t) = e^{3t} - e^{2t} + \left[\frac{1}{6} + \frac{1}{3}e^{3(t-1)} - \frac{1}{2}e^{2(t-1)}\right]u(t-1)$$

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$$y' + 2y = f(t) \qquad y(0) = 0 \qquad f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t \end{cases}$$

$$f(t) = t - t\,u(t-1)$$

$$y' + 2y = t - t\,u(t-1)$$

$$\mathcal{L}[y' + 2y] = \mathcal{L}[t - t\,u(t-1)]$$

$$sY - \cancel{y(0)}^{0} + 2Y = \frac{1}{s^2} - e^{-s}\mathcal{L}[t+1]$$

$$Y(s+2) = \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$Y = \frac{1}{(s+2)s^2} - e^{-s}\left[\frac{s+1}{s^2(s+2)}\right]$$

$$\frac{1}{(s+2)s^2} = \frac{A}{s+2} + \frac{B}{s} + \frac{C}{s^2} \qquad \frac{s+1}{s^2(s+2)} = \frac{A}{s+2} + \frac{B}{s} + \frac{C}{s^2}$$

$$1 = As^2 + B(s)(s+2) + C(s+2) \qquad s+1 = As^2 + B(s)(s+2) + C(s+2)$$

$s = -2$ $\quad$ $s = 0$ $\quad$ $2B + C = 0$ $\qquad$ $s = -2$

$A = \frac{1}{4}$ $\quad$ $C = \frac{1}{2}$ $\quad$ $B = -\frac{1}{4}$ $\qquad$ $A = -\frac{1}{4}$ $\quad$ $s = 0$ $\quad$ $C = \frac{1}{2}$ $\quad$ $B = -\frac{1}{4}$

$$Y = \frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{4}}{s} - e^{-s}\left[-\frac{1}{4}\Big/_{s+2} + \frac{1}{2}\Big/_{s^2} - \frac{1}{4}\Big/_{s}\right]$$

$$y(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}\mathcal{L}\left[\frac{1}{s^2}\right] - \frac{1}{4}\mathcal{L}\left[\frac{1}{s}\right] + \frac{1}{4}\mathcal{L}\left[\frac{1}{s+2}\right]_{t \to t-1} u(t-1)$$

$$-\frac{1}{2}\mathcal{L}\left[\frac{1}{s^2}\right]_{t \to t-1} u(t-1)$$

$$y(t) = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4} + \left[\frac{1}{4}e^{-2(t-1)} - \frac{1}{2}t + \frac{1}{2}\right]u(t-1) + \frac{1}{4}\mathcal{L}\left[\frac{1}{s}\right]_{t \to t-1} u(t-1)$$

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$$y'' + 2y' + y = f(t) \quad y(0) = 0 \quad y'(0) = 1$$

$$f(t) = \begin{cases} 0 & 0 \le t < 3 \\ 2 & 3 \le t \end{cases}$$

$$f(t) = 2\,u(t-3)$$

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{2u(t-3)\}$$

$$s^2 Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{1} + 2[sY - \cancel{y(0)}^{0}] + Y = \frac{2e^{-3s}}{s}$$

$$s^2 Y - 1 + 2sY + Y = \frac{2e^{-3s}}{s}$$

$$Y(s^2 + 2s + 1) = 1 + \frac{2e^{-3s}}{s}$$

$$Y = \frac{1}{(s+1)^2} + \frac{2e^{-3s}}{s(s+1)^2}$$

$$\frac{2}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$2 = A(s+1)^2 + B(s)(s+1) + Cs$$

$s = 0$ $s = -1$ $A + B = 0$

$A = 2$ $C = -2$ $B = -2$

$$Y = \frac{1}{(s+1)^2} + \frac{2e^{-3s}}{s} - \frac{2e^{-3s}}{s+1} - \frac{2e^{-3s}}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+1}\right] - 2\mathcal{L}^{-1}\frac{e^{-3s}}{(s+1)^2}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]_{s \to s+1} + 2\mathcal{L}^{-1}\left[\frac{1}{s}\right]_{t \to t-3} u(t-3) - 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]_{t \to t-3} u(t-3)$$

$$-2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]_{\substack{s \to s+1 \\ t \to t-3}} u(t-3)$$

$$y(t) = e^{-t}t + \left[2 - e^{-(t-3)} - 2e^{-(t-3)}(t-3)\right]u(t-3)$$

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$$y'' + 4y' + 5y = \delta(t-2\pi) \qquad y(0) = y'(0) = 0$$

$$\mathcal{L}[y'' + 4y' + 5y] = \mathcal{L}[\delta(t-2\pi)]$$

$$s^2Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 4(sY - \cancel{y(0)}^{0}) + 5Y = e^{-2\pi s}$$

$$Y(s^2 + 4s + 5) = e^{-2\pi s}$$

$$Y = \frac{1}{s^2+4s+5} e^{-2\pi s}$$

complet the square

$$\left(\frac{4}{2}\right)^2 = 4$$

$$(s^2 + 4s + 4) - 4 + 5 = (s+2)^2 + 9$$

$$Y = \frac{1}{(s+2)^2+1} e^{-2\pi s}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1} e^{-2\pi s}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right]_{t \to t-2\pi} u(t-2\pi)$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]_{\substack{t \to t-2\pi \\ s \to s+2}} u(t-2\pi)$$

$$= e^{-2(t-2\pi)} \sin(t-2\pi)\, u(t-2\pi)$$

$$= e^{-2(t-2\pi} \sin(t)\, u(t-2\pi)$$