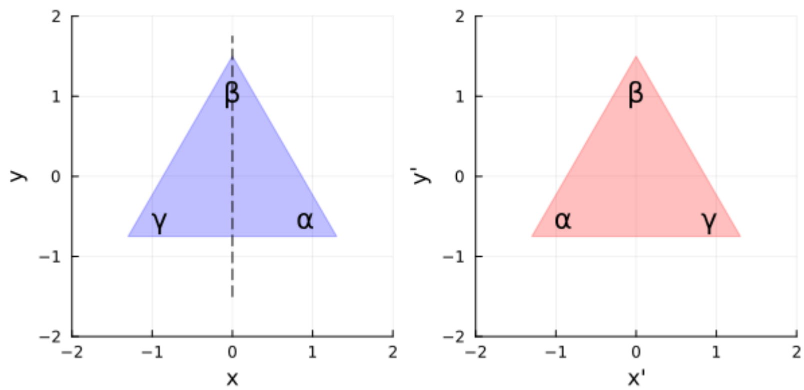
1 Symmetries: the mathematics of patterns

We all recognize the left-right symmetry of a face, or the black/white symmetry of a chessboard. But as we've seen with the Escher patterns, there are more complicated kinds of symmetries. To understand these more complicated symmetries, we need to describe them mathematically.

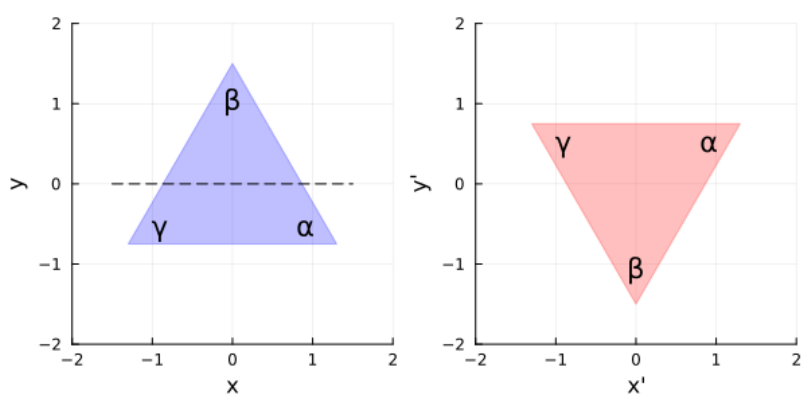
Definition: A shape is *symmetric* if it does not change under a given a coordinate transformation. Such a coordinate transformation is called a *symmetry* of the shape. The shape is *invariant* under the symmetry.

Example: a reflection symmetry. The equilateral triangle below is invariant under reflection about the vertical axis. Some points swap positions, such as α and γ. But the shaded regions in the two plots contain the same points.

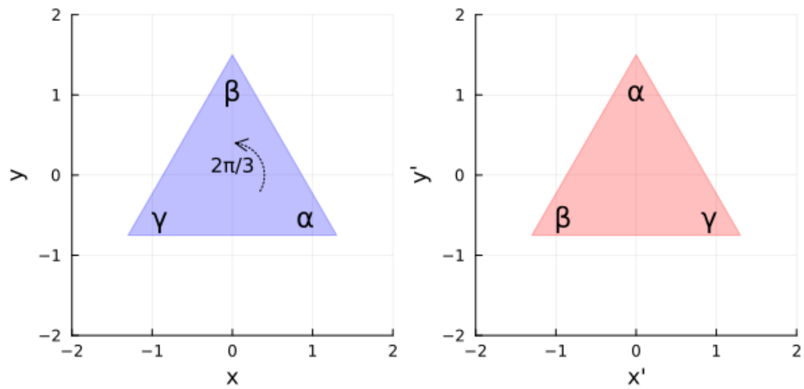


As we've seen in previous sessions, the coordinate transformation that performs this reflection is $\mathbf{x}' = S_y\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $S_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. So S_y is a symmetry of the triangle. Equivalently, the triangle is S_y-symmetric.

In contrast the triangle is not invariant under reflection about the horizontal axis. The shaded regions in the two plots differ. The coordinate transformation for reflection about the horizontal axis $S_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So S_x is not a symmetry of the triangle, and the triangle is not S_x-symmetric.

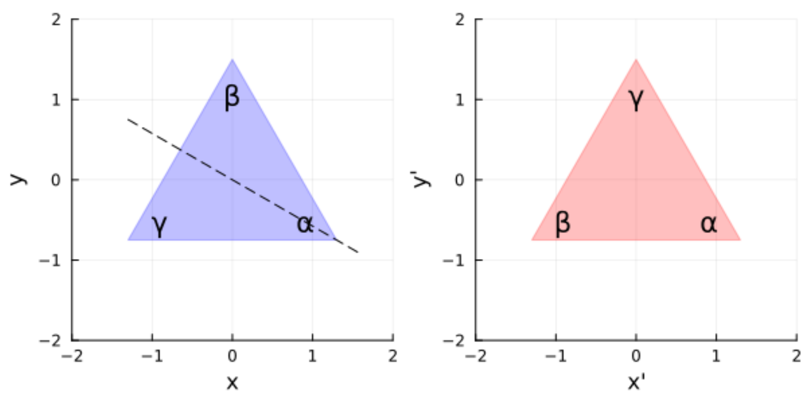


Example: a rotation symmetry The triangle is invariant under the rotation $R_{2\pi/3}$, where $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ and thus $R_{2\pi/3} = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$.



The triangle is thus $R_{2\pi/3}$-symmetric.

What other symmetries does the triangle have? We could also reflect it about the axis going through any of the triangle's vertices (shown below), or rotate it any multiple of $2\pi/3$.



Problem 1: What are all the symmetries of the triangle?

Use a cut-out, labeled triangle to figure out all possible rotations and reflections that leave the triangle as a whole unchanged. For each, draw a picture of two triangles with labeled vertices, like the ones above.

You don't have to figure out the matrices that represent the symmetries. Just give the symmetries names, like S_α for reflection about the axis going through vertex α, and $R_{2\pi/3}$ for rotation by $2\pi/3$.

Hint: the triangle has six symmetries. Don't forget the identity symmetry, $I = R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Answers:

2 Finding symmetries by substitution

How can we know how many symmetries an object has, and how can we find all of them? There is a systematic way to answer these questions. Let T be the set of all points in the triangle,

$$T = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\} = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \ldots \right\}.$$

Next let ST represent the coordinate transformation S applied to all the points in T,

$$S\,T = S\,\{\mathbf{x}_1, \mathbf{x}_2, \ldots\} = \{S\mathbf{x}_1, S\mathbf{x}_2, \ldots\}.$$

If the triangle is unchanged by a coordinate transformation S, we have $T = S\,T$. For example, the triangle's symmetry about the y axis and its symmetry under rotation by $2\pi/3$ can be expressed as

$$T = S_y\,T \qquad \text{and} \qquad T = R_{2\pi/3}\,T.$$

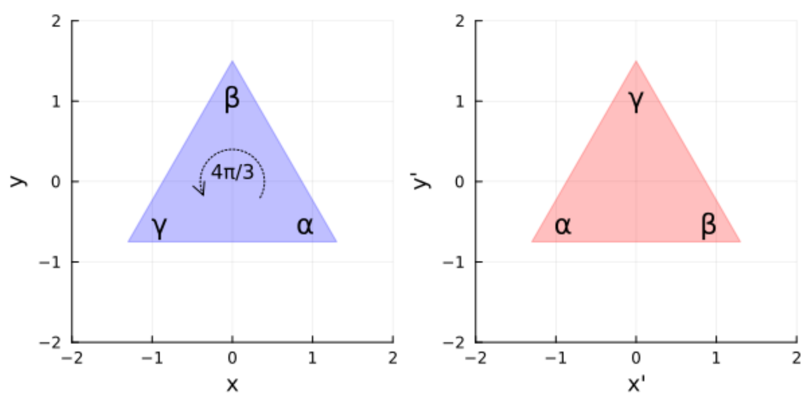
In words, the triangle is invariant under both S_y and $R_{2\pi/3}$, and it has these both as symmetries.

Now here's the key idea: **We can find *all* the symmetries of the triangle by substitution!**

Example: Since $T = R_{2\pi/3}\,T$ we can substitute $R_{2\pi/3}\,T$ for T on the right-hand side of the equation to get

$$\begin{aligned} T &= R_{2\pi/3}\,(R_{2\pi/3}\,T), \\ &= (R_{2\pi/3} R_{2\pi/3})\,T, \\ &= R^2_{2\pi/3}\,T. \end{aligned}$$

Thus the triangle is invariant under two rotations by $2\pi/3$, or equivalently, one rotation by $4\pi/3$, and is $R^2_{2\pi/3} = R_{4\pi/3}$ symmetric.



$R_{4\pi/3}$ should be one of your answers for problem 1.

From here on let's drop the subscripts, let $R = R_{2\pi/3}$ and $R^2 = R_{4\pi/3}$, so that our equations of the triangle's invariance under reflection and rotation are

$$T = S_y T \qquad \text{and} \qquad T = RT,$$

and the manipulations of the example above amount to $T = RT = R(RT) = R^2T$.

Problem 2: Substituting $T = S_y T$ into $T = RT$ gives $T = R(S_y T)$. Since matrix multiplication is associative, $T = (RS_y)T$. The triangle thus has RS_y symmetry: it is invariant under reflection about the y axis followed by rotation by $2\pi/3$.

Verify that your cut-out triangle is RS_y-symmetric by performing the reflection about the y axis followed by the rotation, and seeing that the whole shape is unchanged. Draw a picture that shows the transformation and symmetry. Of the symmetries you listed for problem 1, which symmetry is the same as RS_y?

Answer:

Problem 3: Substituting $T = RT$ into $T = S_y T$ gives $T = S_y(RT) = (S_y R)T$. Thus the triangle is $S_y R$ symmetric.

Verify that your cut-out triangle is $S_y R$-symmetric by performing the rotation followed by reflection about the y axis. Draw a picture that shows the transformation and the symmetry. Of the symmetries you listed for problem 1, which is the same as $S_y R$?

Answer:

It is not hard to derive the equations

$$T = R^m T, \qquad T = S_y^n T, \qquad T = R^m S_y^n T, \qquad T = S_y^n R^m T$$

for any integers $m, n \geq 0$ from $T = T$ and repeated substitutions of $T = RT$ and $T = S_y T$.

Note that since a repeated reflection undoes itself, $S_y^2 = I$. Thus there is no point of going beyond $n = 1$ in S_y^n. Similarly, there is no point going beyond $m = 2$ in R^m, since since three rotations by $2\pi/3$ is the same as rotation by zero. We will show in problem 5 that symmetries of the form $S_y^n R^m$ are redundant with those of form $R^m S_y^n$.

The complete set of symmetries of the triangle is thus I, R, R^2, S_y, RS_y, and $R^2 S_y$.

Problem 4: For each of I, R, R^2, S_y, RS_y, and $R^2 S_y$, perform the transformation on the cut-out triangle, draw pictures, and indicate which of your answers from problem 1 is the same.

3 Symmetry groups

Just as rotation R and reflection S_y can be performed in sequence to form the product RS_y, each of the symmetries of problem 4 can be performed in sequence to form a product.

Problem 5: Using your cut-out triangle, fill out the multiplication table. Start with the triangle in its original position, perform the transformation in the column, and then the transformation in the row. Then look in your answers to problem 4, and see which of I, R, R^2, S_y, RS_y, or R^2S_y you got, and write that in the corresponding spot in the table.

For example, for transformation S_y (column) followed by transformation R (row), you should get RS_y as shown.

This is the *group multiplication* table of the symmetries of the triangle. Note that

1. The product of any two symmetries is a symmetry. The set is closed under multiplication.
2. One of the symmetries is the identity I.
3. The identity I appears once in every row and every column: every symmetry has an inverse.

These, plus associativity of multiplication, are the key properties of a group.