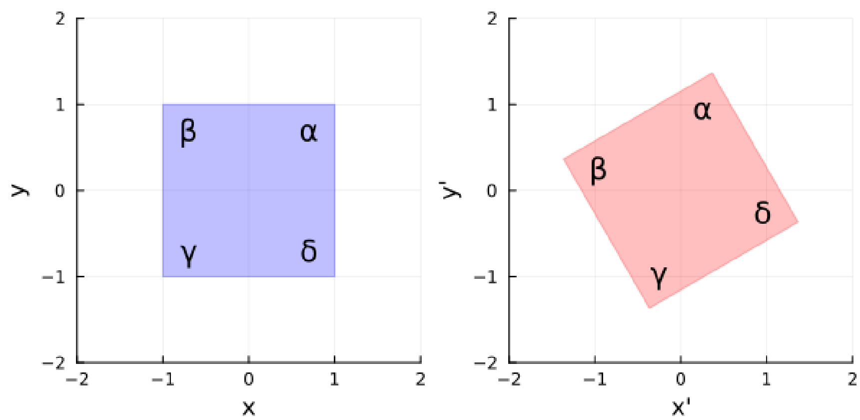
A figure in a plane can move in several ways without changing its shape.

- **rotation:** changing the tilt of the shape with respect to the coordinate axes
- **reflection:** picking up the shape, flipping it over, and placing it back down
- **translation:** shifting left/right or up/down with no change of orientation

With this worksheet you will develop mathematical representations of these transformations using geometry, algebra, vectors, and matrices.

1 Rotations



The figure above shows the rotation of a square by $\pi/6$ (or $30°$). We use (x, y) coordinates for the original square on the left and (x', y') coordinates for the rotated square on the right. Each point (x, y) in the original square moves to a point (x', y') in the rotated square according to equations of form

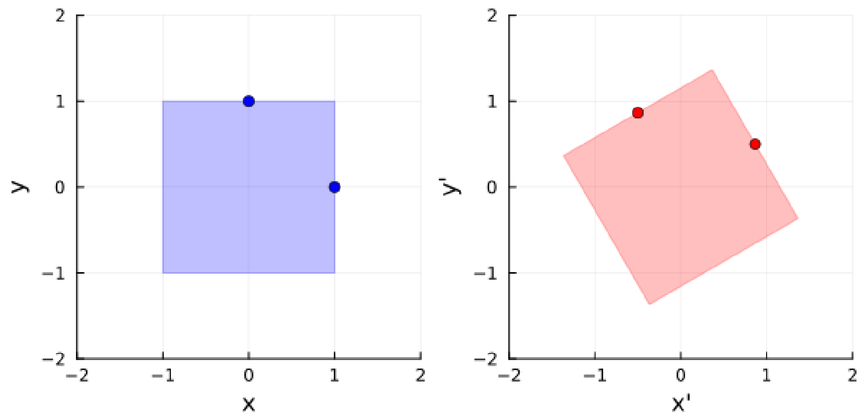
$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{1}$$

where $a, b, c,$ and d are constants determined by the rotation angle.

Problem 1: Figure out the values of $a, b, c,$ and d as functions of the rotation angle θ.

Strategy: There are four unknowns a, b, c, d, so we need four equations to solve for them. Choose a point on the original square, say vertex α at $(x, y) = (1, 1)$. Use trigonometry to find the position (x', y') of vertex α after rotation as a function of the rotation angle θ. Then plug those values of (x, y) and (x', y') into the above equations. That gives you two equations. Do the same for another point, say vertex β, to get two more equations. Solve the four equations to get a, b, c, d as trigonometric functions of θ.

Hint: If you use the points $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$ instead of the vertices α and β, the trigonometry and algebra are much simpler.



Problem 2: Rewrite your answer to problem 1 as a matrix-vector multiplication.

Guidance: The equations

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{2}$$

can be written as a matrix-vector multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3}$$

This is just a graphical way to represent the original equations.

$\begin{bmatrix} x' \\ y' \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ are *vectors* representing the coordinates (x', y') and (x, y) as points on a plane.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a *matrix* that specifies the functional relationship between the vectors (x', y') and (x, y).

Matrix-vector multiplication is defined by the "across and down" rule. For example, the product

$$\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

is found by multiplying each entry *across* the first row of the matrix by the entries going *down* the vector and adding,

$$\begin{bmatrix} 2 & 1 \\ & \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 + 1 \cdot 6 \\ \end{bmatrix} = \begin{bmatrix} 16 \\ \end{bmatrix},$$

and then doing the same going across the bottom row of the matrix to get

$$\begin{bmatrix} & \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \\ 3 \cdot 5 - 4 \cdot 6 \end{bmatrix} = \begin{bmatrix} \\ -9 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \end{bmatrix}.$$

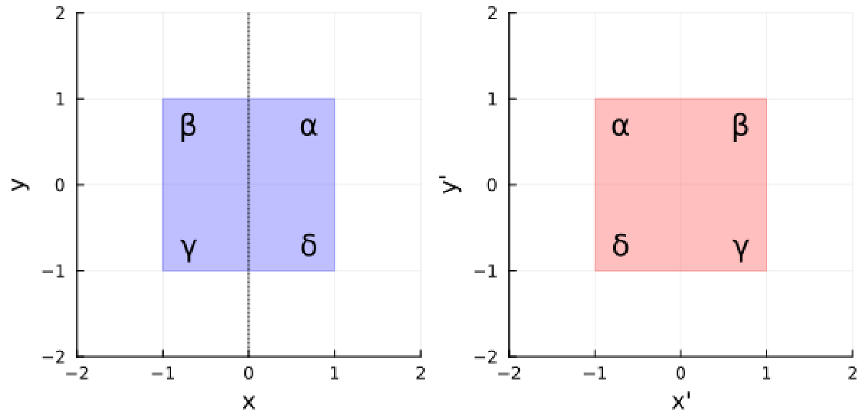
Similarly,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}. \tag{4}$$

This is just equation 2 written as an equality of two vectors. So to get your answer for problem 2, just substitute the trigonometric functions for a, b, c, d that you derived in problem 1 into equation 3.

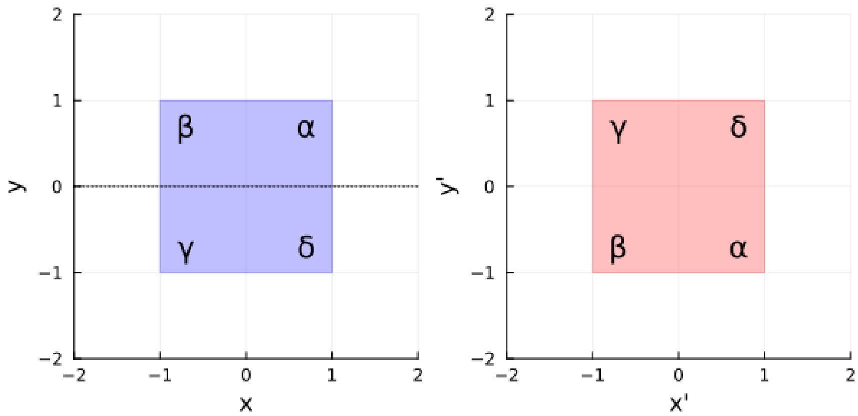
2 Reflections

Problem 3: Using similar methods as in problems 1 and 2, derive a matrix-vector representation of reflection about the vertical axis.

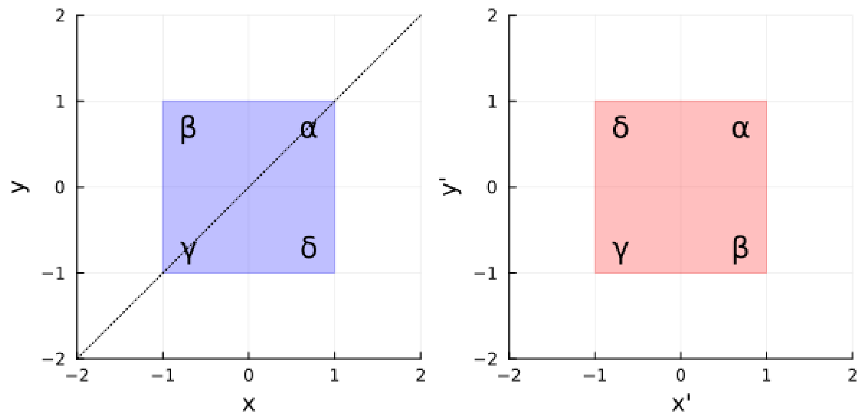


Your answer will be of the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with specific numeric values for a, b, c, d.

Problem 4: Do the same as problem 3 for reflection about the horizontal axis.

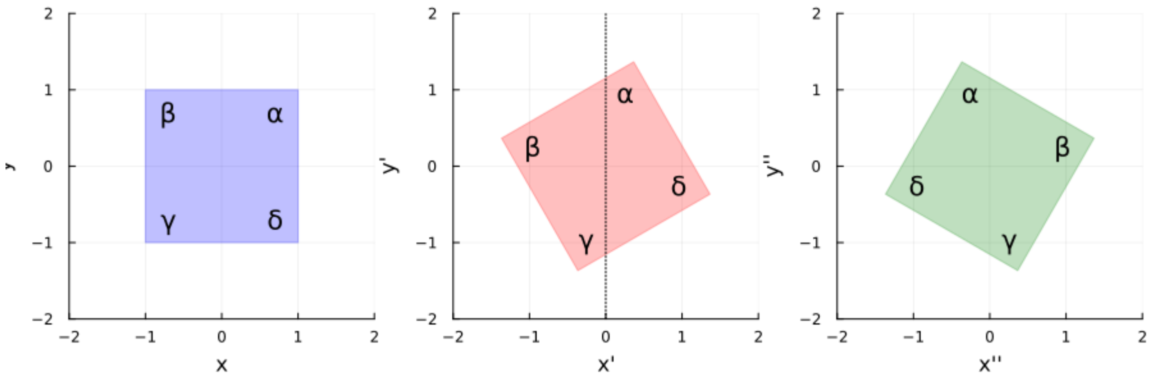


Problem 5 (challenge): Derive a matrix-vector representation of a reflection about a diagonal line at angle $\pi/4$ (45°).



3 Chained transformations, matrix-matrix multiplication.

Here we see a rotation of a square by $\theta = \pi/6$ followed by a reflection about the vertical axis.



Problem 6 (challenge): Express the chained transformation from $\mathbf{x}$ to $\mathbf{x}''$ as a matrix-vector multiplication.

Strategy: You should have the reflection as a matrix-vector product from problem 3, and you can get the rotation matrix by plugging in $\theta = \pi/6$ to your answer for problem 1.

Let the rotation and reflection matrices and the coordinate vector (x, y) be labeled as follows

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \qquad S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Your R and S matrices will have specific numerical values instead of symbols, but I'll use symbols as above to avoid giving away the answers for previous problems!

Then the rotation transformation is $\mathbf{x}' = R\mathbf{x}$, and the reflection is $\mathbf{x}'' = S\mathbf{x}'$, and the chained transformation is $\mathbf{x}'' = S\mathbf{x}' = S(Rx)$. Writing that out in long-hand form,

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \left(\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \tag{5}$$

Again, you will have specific numebrs for the matrix coefficients instead of symbols.

Now do across-and-down on the Rx multiplication to get Rx expressed as a vector. Then do across-and-down on the $S(Rx)$ multiplication. You should end up with a vector whose components are sums of x and y times some numeric coefficients. Then rewrite that vector as a single matrix-vector multiplication, where the coefficients go in the matrix, and the vector is $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. The answer should look like

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{6}$$

with specific numbers in place of the question marks.

Problem 7 (challenge): Do the same as in problem 6, but using symbols rather than specific numbers for the matrices. Start directly from equation 5, and end up with something like equation 6. But instead of numbers for the matrix elements, you will have expressions involving r_{11}, r_{12}, $\ldots$ and s_{11}, s_{12}, $\ldots$.

A question to ponder: Why do mirrors reverse images left to right, but not upside down?