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Problem 1.

Solution:

a) Given that $P(Q) = a_0 - a_1 Q + \nu$, we compute that the elasticity of demand is given by

$$\varepsilon(P) = -\frac{P(Q)}{P'(Q)Q} = -\frac{a_0 - a_1 Q + \nu}{-a_1 Q} = \frac{a_0 - a_1 Q + \nu}{a_1 Q} = \frac{a_0 + \nu}{a_1 Q} - 1$$

We observe that this is increasing in a_0 and ν and decreasing in a_1 and Q.

b) Using our result from the previous homework, we have that the first order condition for each firm is the following:

$$q_i = \frac{a_0 - a_1(q_i + q_{-i}) + \nu) - (b_0 - \eta)}{a_1}$$

By symmetry, we have that $q_i = q_j = q^* \ \forall i \neq j$ in equilibrium. Hence, we can impose this to solve for q^*:

$$\begin{aligned} q^* &= \frac{(a_0 - a_1 N q^* + \nu) - (b_0 - \eta)}{a_1} \\ \iff (1+N)q^* &= \frac{a_0 + \nu - (b_0 - \eta)}{a_1} \\ \iff q^* &= \frac{a_0 + \nu - (b_0 - \eta)}{(N+1)a_1} \end{aligned}$$

It follows that total industry output is given by

$$Q^* = Nq^* = \frac{N}{N+1}\frac{a_0 + \nu - (b_0 - \eta)}{a_1}$$

Hence, the price is

$$
\begin{aligned}
P^* &= a_0 - a_1\left(\frac{N}{N+1}\frac{a_0+\nu-(b_0-\eta)}{a_1}\right)+\nu \\
&= a_0 - \frac{N}{N+1}(a_0+\nu-(b_0-\eta))+\nu \\
&= \frac{a_0+\nu+N(b_0-\eta)}{N+1}
\end{aligned}
$$

We now use these to compute firm profits in equilibrium for fixed N and F:

$$
\begin{aligned}
\Pi^* &= (P^*-(b_0-\eta))q^*-F \\
&= \left(\frac{a_0+\nu+N(b_0-\eta)}{N+1}-(b_0-\eta)\right)\frac{a_0+\nu-(b_0-\eta)}{(N+1)a_1}-F \\
&= \left(\frac{a_0+\nu-(b_0-\eta)}{N+1}\right)\frac{a_0+\nu-(b_0-\eta)}{(N+1)a_1}-F \\
&= \frac{1}{a_1}\left(\frac{(a_0+\nu-(b_0-\eta))}{N+1}\right)^2-F
\end{aligned}
$$

c) The Lerner index L_i is given by

$$
L_i = \frac{P^*-C'(q^*)}{P^*} = \frac{\frac{a_0+\nu+N(b_0-\eta)}{N+1}-(b_0-\eta)}{\frac{a_0+\nu+N(b_0-\eta)}{N+1}} = \frac{\frac{a_0+\nu-(b_0-\eta)}{N+1}}{\frac{a_0+\nu+N(b_0-\eta)}{N+1}} = \frac{a_0+\nu-(b_0-\eta)}{a_0+\nu+N(b_0-\eta)}
$$

The industry Lerner index is thus

$$
L_I = \sum_{i=1}^{N} s_i L_i = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\frac{a_0+\nu-(b_0-\eta)}{a_0+\nu+N(b_0-\eta)} = \frac{1}{N}\frac{a_0+\nu-(b_0-\eta)}{a_0+\nu+N(b_0-\eta)}
$$

Similarly, we compute that

$$
\varepsilon(P^*) = \frac{a_0-a_1Q^*+\nu}{a_1Q^*} = \frac{\frac{a_0+\nu+N(b_0-\eta)}{N+1}}{\frac{N}{N+1}(a_0+\nu-(b_0-\eta)} = \frac{a_0+\nu+N(b_0-\eta)}{N(a_0+\nu-(b_0-\eta))}
$$

Finally, since firms have the same output, the HHI is simply $\frac{1}{N}$.

d) To see how many firms enter before it is no longer profitable to do so, we must find the N^* such that firm profits are zero. That is, we need to solve for N^* such that

$$
\begin{aligned}
&\frac{1}{a_1}\left(\frac{(a_0+\nu-(b_0-\eta))}{N^*+1}\right)^2 = F \\
\iff &(N^*+1)^2\frac{1}{a_1F}\left((a_0+\nu-(b_0-\eta))\right)^2 \\
\iff &N^*+1 = \frac{1}{\sqrt{a_1F}}(a_0+\nu-(b_0-\eta)) \\
\iff &N^* = \frac{1}{\sqrt{a_1F}}(a_0+\nu-(b_0-\eta)) - 1
\end{aligned}
$$

e) Under perfect collusion, firms agree to jointly produce the monopoly quantity and split the profits evenly. To determine the monopoly quantity, we solve

$$\max_Q P(Q)Q - C(Q)$$

This has first order condition

$$
\begin{aligned}
&P'(Q)Q + P(Q) - (b_0-\eta) = 0 \\
\iff &-a_1Q + a_0 - a_1Q + \nu + (b_0-\eta) = 0 \\
\iff &2a_1Q = a_0+\nu+(b_0-\eta) \\
\iff &Q = \frac{a_0+\nu+(b_0-\eta)}{2a_1}
\end{aligned}
$$

Hence, $Q^c = \frac{a_0+\nu+(b_0-\eta)}{2a_1}$ and thus each firm produces $q^c = \frac{a_0+\nu+(b_0-\eta)}{2Na_1}$. The profit to each firm under collusion is given by

$$
\begin{aligned}
\Pi^c &= \frac{1}{N}\left(P(Q^c)-(b_0-\eta)\right)Q^c - F \\
&= \frac{1}{N}\left(\frac{a_0+\nu-(b_0-\eta)}{2}\right)\frac{a_0+\nu+(b_0-\eta)}{2a_1} - F \\
&= \frac{1}{N}\left(\frac{(a_0+\nu-(b_0-\eta))^2}{4a_1}\right) - F
\end{aligned}
$$

This implies that firms will enter until the number of firms is equal to N^c, where N^c is defined so that profits under collusion are zero:

$$\frac{1}{N^c}\left(\frac{(a_0+\nu-(b_0-\eta))^2}{4a_1}\right)-F=0 \iff N^c=\frac{1}{F}\left(\frac{(a_0+\nu-(b_0-\eta))^2}{4a_1}\right)$$

We then compute the elasticity of demand at the collusive price as follows:

$$\varepsilon(P^c)=\frac{a_0+\nu}{\frac{a_0+\nu-(b_0-\eta)}{2}}-1=\frac{a_0+\nu+(b_0-\eta)}{a_0+\nu-(b_0-\eta)}$$

The Lerner index for each firm is thus

$$L_i=\frac{s_i}{\varepsilon(P^c)}=\frac{1}{N^c}\frac{a_0+\nu-(b_0-\eta)}{a_0+\nu+(b_0-\eta)}$$

Hence, the industry Lerner index is

$$L_I=\sum_{i=1}^{N^c}\frac{1}{N^{c2}}\frac{a_0+\nu-(b_0-\eta)}{a_0+\nu+(b_0-\eta)}=\frac{1}{N^c}\frac{a_0+\nu-(b_0-\eta)}{a_0+\nu+(b_0-\eta)}$$

$\square$