

**Problem 1.**

*Solution:*

- a) Given that  $P(Q) = a_0 - a_1Q + \nu$ , we compute that the elasticity of demand is given by

$$\varepsilon(P) = -\frac{P(Q)}{P'(Q)Q} = -\frac{a_0 - a_1Q + \nu}{-a_1Q} = \frac{a_0 - a_1Q + \nu}{a_1Q} = \frac{a_0 + \nu}{a_1Q} - 1$$

We observe that this is increasing in  $a_0$  and  $\nu$  and decreasing in  $a_1$  and  $Q$ .

- b) Using our result from the previous homework, we have that the first order condition for each firm is the following:

$$q_i = \frac{a_0 - a_1(q_i + q_{-i}) + \nu - (b_0 - \eta)}{a_1}$$

By symmetry, we have that  $q_i = q_j = q^* \forall i \neq j$  in equilibrium. Hence, we can impose this to solve for  $q^*$ :

$$\begin{aligned} q^* &= \frac{(a_0 - a_1Nq^* + \nu) - (b_0 - \eta)}{a_1} \\ \iff (1 + N)q^* &= \frac{a_0 + \nu - (b_0 - \eta)}{a_1} \\ \iff q^* &= \frac{a_0 + \nu - (b_0 - \eta)}{(N + 1)a_1} \end{aligned}$$

It follows that total industry output is given by

$$Q^* = Nq^* = \frac{N}{N + 1} \frac{a_0 + \nu - (b_0 - \eta)}{a_1}$$

Hence, the price is

$$\begin{aligned} P^* &= a_0 - a_1 \left( \frac{N}{N+1} \frac{a_0 + \nu - (b_0 - \eta)}{a_1} \right) + \nu \\ &= a_0 - \frac{N}{N+1} (a_0 + \nu - (b_0 - \eta)) + \nu \\ &= \frac{a_0 + \nu + N(b_0 - \eta)}{N+1} \end{aligned}$$

We now use these to compute firm profits in equilibrium for fixed  $N$  and  $F$ :

$$\begin{aligned} \Pi^* &= (P^* - (b_0 - \eta))q^* - F \\ &= \left( \frac{a_0 + \nu + N(b_0 - \eta)}{N+1} - (b_0 - \eta) \right) \frac{a_0 + \nu - (b_0 - \eta)}{(N+1)a_1} - F \\ &= \left( \frac{a_0 + \nu - (b_0 - \eta)}{N+1} \right) \frac{a_0 + \nu - (b_0 - \eta)}{(N+1)a_1} - F \\ &= \frac{1}{a_1} \left( \frac{(a_0 + \nu - (b_0 - \eta))}{N+1} \right)^2 - F \end{aligned}$$

c) The Lerner index  $L_i$  is given by

$$L_i = \frac{P^* - C'(q^*)}{P^*} = \frac{\frac{a_0 + \nu + N(b_0 - \eta)}{N+1} - (b_0 - \eta)}{\frac{a_0 + \nu + N(b_0 - \eta)}{N+1}} = \frac{\frac{a_0 + \nu - (b_0 - \eta)}{N+1}}{\frac{a_0 + \nu + N(b_0 - \eta)}{N+1}} = \frac{a_0 + \nu - (b_0 - \eta)}{a_0 + \nu + N(b_0 - \eta)}$$

The industry Lerner index is thus

$$L_I = \sum_{i=1}^N s_i L_i = \frac{1}{N} \sum_{i=1}^N \frac{1}{N} \frac{a_0 + \nu - (b_0 - \eta)}{a_0 + \nu + N(b_0 - \eta)} = \frac{1}{N} \frac{a_0 + \nu - (b_0 - \eta)}{a_0 + \nu + N(b_0 - \eta)}$$

Similarly, we compute that

$$\varepsilon(P^*) = \frac{a_0 - a_1 Q^* + \nu}{a_1 Q^*} = \frac{\frac{a_0 + \nu + N(b_0 - \eta)}{N+1}}{\frac{N}{N+1} (a_0 + \nu - (b_0 - \eta))} = \frac{a_0 + \nu + N(b_0 - \eta)}{N(a_0 + \nu - (b_0 - \eta))}$$

Finally, since firms have the same output, the HHI is simply  $\frac{1}{N}$ .

d) To see how many firms enter before it is no longer profitable to do so, we must find

the  $N^*$  such that firm profits are zero. That is, we need to solve for  $N^*$  such that

$$\begin{aligned} \frac{1}{a_1} \left( \frac{(a_0 + \nu - (b_0 - \eta))}{N^* + 1} \right)^2 &= F \\ \iff (N^* + 1)^2 \frac{1}{a_1 F} ((a_0 + \nu - (b_0 - \eta)))^2 & \\ \iff N^* + 1 &= \frac{1}{\sqrt{a_1 F}} (a_0 + \nu - (b_0 - \eta)) \\ \iff N^* &= \frac{1}{\sqrt{a_1 F}} (a_0 + \nu - (b_0 - \eta)) - 1 \end{aligned}$$

- e) Under perfect collusion, firms agree to jointly produce the monopoly quantity and split the profits evenly. To determine the monopoly quantity, we solve

$$\max_Q P(Q)Q - C(Q)$$

This has first order condition

$$\begin{aligned} P'(Q)Q + P(Q) - (b_0 - \eta) &= 0 \\ \iff -a_1 Q + a_0 - a_1 Q + \nu + (b_0 - \eta) &= 0 \\ \iff 2a_1 Q &= a_0 + \nu + (b_0 - \eta) \\ \iff Q &= \frac{a_0 + \nu + (b_0 - \eta)}{2a_1} \end{aligned}$$

Hence,  $Q^c = \frac{a_0 + \nu + (b_0 - \eta)}{2a_1}$  and thus each firm produces  $q^c = \frac{a_0 + \nu + (b_0 - \eta)}{2Na_1}$ . The profit to each firm under collusion is given by

$$\begin{aligned} \Pi^c &= \frac{1}{N} (P(Q^c) - (b_0 - \eta)) Q^c - F \\ &= \frac{1}{N} \left( \frac{a_0 + \nu - (b_0 - \eta)}{2} \right) \frac{a_0 + \nu + (b_0 - \eta)}{2a_1} - F \\ &= \frac{1}{N} \left( \frac{(a_0 + \nu - (b_0 - \eta))^2}{4a_1} \right) - F \end{aligned}$$

This implies that firms will enter until the number of firms is equal to  $N^c$ , where  $N^c$  is defined so that profits under collusion are zero:

$$\frac{1}{N^c} \left( \frac{(a_0 + \nu - (b_0 - \eta))^2}{4a_1} \right) - F = 0 \iff N^c = \frac{1}{F} \left( \frac{(a_0 + \nu - (b_0 - \eta))^2}{4a_1} \right)$$

We then compute the elasticity of demand at the collusive price as follows:

$$\varepsilon(P^c) = \frac{a_0 + \nu}{\frac{a_0 + \nu - (b_0 - \eta)}{2}} - 1 = \frac{a_0 + \nu + (b_0 - \eta)}{a_0 + \nu - (b_0 - \eta)}$$

The Lerner index for each firm is thus

$$L_i = \frac{s_i}{\varepsilon(P^c)} = \frac{1}{N^c} \frac{a_0 + \nu + (b_0 - \eta)}{a_0 + \nu - (b_0 - \eta)}$$

Hence, the industry Lerner index is

$$L_I = \sum_{i=1}^{N^c} \frac{1}{N^c} \frac{a_0 + \nu + (b_0 - \eta)}{a_0 + \nu - (b_0 - \eta)} = \frac{1}{N^c} \frac{a_0 + \nu + (b_0 - \eta)}{a_0 + \nu - (b_0 - \eta)}$$

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## Problem 2.

*Solution:*