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Econ 899
20 September, 2022

Problem 1.

Solution: Consider a market in which goods are homogenous.

a) Let $P(\cdot)$ be an inverse demand function with constant elasticity. This notably implies that, $\forall Q$,

$$\varepsilon(P(Q)) = -\frac{P(Q)}{P'(Q)Q} = -\sigma$$

for some $\sigma > 0$. Rearranging this expression, we obtain:

$$P(Q) = \sigma P'(Q)Q \iff P(Q) - \sigma P'(Q)Q = 0$$

Applying the Implicit Function Theorem to the above expression, we have that

$$P'(Q) - \sigma(P'(Q) + P''(Q)Q) = 0$$

Hence,

$$P'(Q) + P''(Q)Q = \frac{P'(Q)}{\sigma} > 0$$

since both $P'(Q) > 0$ and $\sigma > 0$. It thus follows that, $\forall Q$, $P'(Q) + P''(Q)Q > 0$ for all inverse demand functions $P(\cdot)$ with constant elasticity. As such, assumption A1 of the Cournot model is violated.

b) Consider the Cournot model with N firms that each have identical cost functions $C(\cdot)$. Suppose that assumptions A1 and A2 hold. Namely, A1 requires that $0 \geq P''(Y)y_i + P'(Y)\ \forall y_i < Y$. A2 requires that $0 \geq P'(Y) - C''(y_i)\ \forall y_i \leq Y$. We wish to show that these two assumptions imply that the equilibrium price and firm quantities are decreasing in N.

We begin by deriving each firm's profit function. Firm i's profits when producing y_i

and facing aggregate output by opponents $Y_{-i} = \sum_{j \neq i} y_j$ are given by

$$\pi_i(y_i, Y_{-i}) = P(y_i + Y_{-i})y_i - C(y_i)$$

The first order condition for y_i is the following:

$$\frac{\partial \pi_i(y_i, Y_{-i})}{\partial y_i} = 0 \iff P'(y_i + Y_{-i})y_i + P(y_i + Y_{-i}) - C'(y_i) = 0$$
$$\iff y_i = \frac{C'(y_i) - P(y_i + Y_{-i})}{P'(y_i + Y_{-i})}$$

Based on the above first order condition, we define $R(Y_{-i}) = \frac{C'(y_i) - P(y_i + Y_{-i})}{P'(y_i + Y_{-i})}$ as the best reply function of firm i to opponent action profile Y_{-i}.

We must verify the optimality $R(Y_{-i})$ when facing opponent action profile Y_{-i} using the second derivative test. The second derivative of the profit function is given by

$$\frac{\partial^2 \pi_i(y_i, Y_{-i})}{\partial y_i^2} = P''(y_i + Y_{-i})y_i + P'(y_i + Y_{-i}) + P'(y_i + Y_{-i}) - C''(y_i)$$

Note by A1, $P''(y_i + Y_{-i})y_i + P'(y_i + Y_{-i}) \leq 0$. Similarly, by A2, $P'(y_i + Y_{-i} - C''(y_i)) \leq 0$. Jointly, this implies that

$$\frac{\partial^2 pi_i(y_i, Y_{-i})}{\partial y_i^2} = P''(y_i + Y_{-i})y_i + P'(y_i + Y_{-i}) + P'(y_i + Y_{-i}) - C''(y_i) \leq 0$$

Thus, firm i's objective function is locally concave in y_i when y_i satisfies the first order condition, implying that the (unique) solution to the FOC yields a maximum of the profit function. This verifies the optimality (and uniqueness) of the best reply function $R_i(Y_{-i}$.

We now must find the Nash equilibrium. Since each firm has identical cost functions, their best reply functions are identical. We therefore suppress the i subscript and instead write $R_i(Y_{-i}) = R(Y_{-i})$ for $i = 1, \ldots, N$. Given the symmetry of best response functions, we search for a symmetric equilibrium. That is, we suppose that $y_i = y_j = y^*$ $\forall i, j = 1, \ldots, N$. In order for $Y = (y^*, \ldots, y^*)$ to constitute a Nash equilibrium, we must have that

$$y^* = R((Y^*_{-i}) = R((N-1)y^*)$$

Now that we have obtained the equation which characterizes the Nash equilibrium, we can analyze the impact of the number of firms N on the equilibrium quantities y^*.

We note that the first order condition implies that

$$\frac{\partial \pi_i(R(Y_{-i}), Y_{-i})}{\partial y_i} = 0$$

Using the Implicit Function Theorem, we obtain that

$$R'(Y_{-i}) = -\frac{\partial^2 \pi_i(R(Y_{-i}))/\partial Y_{-i} \partial y_i}{\partial^2 \pi_i(R(Y_{-i}), Y_{-i})/\partial y_i^2}$$

Thus,

$$R'(Y_{-i}) = -\frac{P'(y_i + Y_{-i}) + y_i P''(y_i + Y_{-i})}{y_i P''(y_i + Y_{-i}) + 2P'(y_i + Y_{-i}) - C''(y_i)}$$

Note that A1 directly implies the numerator is negative. Furthermore, we have previously demonstrated how A1 and A2 jointly imply that the denominator is negative. Together, it follows that $R'(Y_{-i}) \leq 0$. It now remains to show that $R'(Y_{-i}) \geq -1$. Note that

$$\begin{aligned}
R'(Y_{-i}) &= -\frac{P'(y_i + Y_{-i}) + y_i P''(y_i + Y_{-i})}{y_i P''(y_i + Y_{-i}) + 2P'(y_i + Y_{-i}) - C''(y_i)} \\
&\geq -\frac{P'(y_i + Y_{-i}) + y_i P''(y_i + Y_{-i}) + P'(y_i + Y_{-i}) - C''(y_i)}{y_i P''(y_i + Y_{-i}) + 2P'(y_i + Y_{-i}) - C''(y_i)} \\
&= -1
\end{aligned}$$

where we the inequality comes from the fact that we added $P'(y_i + Y_{-i}) - C''(y_i)$ on the top, which is a negative quantity according to A2. This makes the overall expression (which is negative) increase. Finally, the top and bottom divide and we are left with 1.

Hence, we have shown that $R'(Y_{-i}) \in (-1, 0)$. This implies that the best reply function is a contraction mapping and thus that the Nash equilibrium exists, is unique, and is globally stable! We now consider the implications of this for the equilibrium prices and quantities. The Nash equilibrium is characterized by the equation $y^* = R((N-1)y^*)$, where $y_i = y^* \; \forall i = 1, \ldots, N$ (such an solution is guaranteed to exist by the above argument). Applying the fact that $R'(Y_{-i}) < 0$, it follows that

$$R'((N-1)y^*) \leq 0$$

This implies that the equilibrium per-firm quantity y^* is decreasing in the number of firms N. It finally remains to show that the equilibrium price is decreasing. Let y^*_N indicate the per-firm equilibrium quantity with N firms. We claim that Ny^*_N is increasing in N.

To do so, we define the mapping

$$\varphi : Y_{-i} \mapsto (R(Y_{-i}) + Y_{-i}) \left(\frac{N-1}{N} \right)$$

It is clear that $\varphi(Y_{-i}, N)$ is strictly increasing in N, as $\frac{N-1}{N} = 1 - \frac{1}{N}$ is strictly increasing in N. It is also strictly increasing in Y_{-i}, as

$$\varphi'(Y_{-i}) = (R'(Y_{-i}) + 1) \left(\frac{N-1}{N} \right) \geq (-1+1) \frac{N-1}{N} = 0$$

Combining these arguments, we note that φ has increasing differences in Y_{-i} and N and is therefore supermodular.

Furthermore, we observe that $(N-1)y_N^*$ is a fixed point of φ. Indeed,

$$\varphi((N-1)y_N^*)) = (y_N^* + (N-1)y_N^*)\left(\frac{N-1}{N}\right) = Ny_N^*\left(\frac{N-1}{N}\right) = y_N^*(N-1)$$

Now for the punchline! As φ is supermodular in N and Y_{-i} and is increasing in N, its fixed points will also be increasing in N. Hence, $(N-1)y_N^*$ is increasing in N. It follows that total industry output is increasing in N, even though per-firm outputs decrease. This works because even though each firm produces less as the number of firms increases, the reduction in quantity is outweighed by the fact that an additional firm is producing. Hence, total industry output increases.

Since total industry output is increasing in N, it follows that $P(Ny^*)$ is decreasing in N (as $P'(\cdot) < 0$).

□

Problem 2.

Solution:

a) The game has players $I = \{1, 2\}$. Players each have valuation $V > 0$, and this valuation is common knowledge. Players submit bids $b_i \in B_i = [0, -\infty)$. The joint strategy space is $B = B_1 \times B_2$. Player i's utility function is a mapping $u : B \to \mathbb{R}$ given by

$$u_i(b_i, b_{-i}) = \begin{cases} V - b_i & \text{if } b_i > b_{-i} \\ (V - b_i)/2 & \text{if } b_i = b_{-i} \\ 0 & \text{if } b_i < b_j \end{cases}$$

where b_{-i} indicates the bid of the other player. I assume that if players tie, they each get a 50 percent chance of winning.

We want to find the Nash equilibrium of this game. We begin by noting that, regardless of what the other player plays, playing $b_i > V$ is strictly dominated by playing $b_i = 0$. That is, $\forall i$, $\forall b_{-i}$, $\forall b_i > V$, $u(b_i, b_{-i}) < 0 = u(0, b_{-i})$. As such, no equilibrium can contain $b_i > V$ for either player.

We now consider equilibria of the form (b_i, b_{-i}) where $b_i, b_{-i} \in (0, V)$. Suppose first that $b_i = b_{-i}$. We argue that this is not an equilibrium, since player i could instead bid $b_i + \varepsilon$ for some small $\varepsilon > 0$ and be certain to win. Their payoff from bid b_i is $(V - b_i)/2$, whereas their payoff from bid $b_i + \varepsilon$ is $V - b_i - \varepsilon$, which exceeds the tie payoff for sufficiently small ε. This means that player i could strictly improve by deviating, implying that this cannot be an equilibrium.

As a second case, suppose without loss of generality that $0 < b_i < b_{-i} < V$. We argue that this cannot be an equilibrium since player i would achieve a strictly higher payoff by bidding $b_i' = b_{-i} + \varepsilon$ for $\varepsilon < V - b_i$. As such, this cannot be an equilibrium.

Now consider equilibria of the form $(b_i, 0)$ for some player i. This is also not an equilibrium, since i could strictly benefit by playing $b_i = \varepsilon$ for some $\varepsilon \in (0, b_i)$ and achieve a strictly higher payoff.

This leaves us with only $b_i = b_{-i} = V$ remaining. This gives expected payoff of 0 to both players. To see why this is a Nash equilibrium, we note that either player cannot strictly improve by playing $b_i < V$, as this would guarantee they lose and give them 0 payoff. Furthermore, bidding over V is strictly dominated. Thus, neither player has a unilateral incentive to deviate, implying that this is a Nash equilibrium.

b) We now assume the seller uses an all-pay auction. The payoffs to player 1 for the pair of bids (b_1, b_2) are given by

$$u_1(b_1, b_2) = \begin{cases} V - b_1 & \text{if } b_1 > b_2 \\ V/2 - b_1 & \text{if } b_1 = b_2 \\ -b_1 & \text{if } b_1 < b_2 \end{cases}$$

As before, we assume that agents have an equal chance of winning if they submit tied bids.

c) We argue that no pure strategy Nash equilibrium of the all pay auction exists. To do so, we proceed by considering a number of cases.

As in the first price auction, playing $b_i > V$ is strictly dominated for all players and will never be played in equilibrium.

First consider the case where $b_i, b_j \in (0, V]$ with $b_i < b_j$. Player i's payoff is $-b_i$ because they do not win the auction. They could strictly improve by playing $b_i = 0$, which gives them payoff 0.

We now consider $b_i = b_j \in [0, V]$. Each player has expected payoff of $V/2 - b_i$. If $b_i > V/2$, i's payoff will be negative, implying that $b_i > V/2$ is strictly dominated by playing $b_i' = 0$ (which gives payoff zero). Thus, an equilibrium could only consist of $b_i = b_j \leq V/2$. We argue that this is not an equilibrium, since b_i could strictly benefit by playing $b_i' = b_j + \varepsilon$ for $\varepsilon \leq \min\{V/2, V - b_j\}$, which allows them to win the auction and have strictly higher payoff than $V/2 - b_i$. In summary, we cannot have $b_i = b_j \in [0, V]$.

Now consider equilibria of the form $(b_i, 0)$, where $b_i \in (0, V]$. Player i will win the auction and get payoff $V - b_i$. However, they could strictly improve by bidding $b_i' = \varepsilon > 0$, which allows them to still win while paying less (so long as $\varepsilon < b_i$). This implies that $(b_i, 0)$ with $b_i > 0$ cannot be an equilibrium.

The above cases exhaust the possible pure strategy equilibrium candidates, and thus we conclude that there is no pure strategy Nash equilibrium.

d) We wish to find the mixed strategy Nash equilibrium. To do so, we assume that each player plays with a bidding strategy $G_i(b)$, where $G_i(\cdot) = P(b_i \leq b)$ is the CDF corresponding to the distribution of player i's bids. Since each player has the same valuation V and this valuation is common knowledge, we impose the restriction that $G_i(b) = G_j(b) = G(b)$. That is, players play with a common bid function.

Furthermore, given the linear form of the payoff function, we conjecture that G has the form $G(b) = \alpha b + \beta$ for some constants α, β to be determined. It follows that $g(b) = \alpha$ is the pdf of a player's bids. Note that this implies that $b_i = b_j$ with probability zero and we can thus neglect this possibility in players' expected utility.

Player 1's expected payoff from bidding strategy $G(\cdot)$ is given by

$$
\begin{aligned}
E[u(b_1, b_j)] &= \int_0^V \int_{b_1}^V (V - b_1)\alpha \, db_2 \alpha \, db_1 + \int_0^V \int_0^{b_1} (-b_1)\alpha \, db_2 \alpha \, db_1 \\
&= \alpha^2 \int_0^V (V - b_1)^2 \, db_1 - \alpha^2 \int_0^V b_1^2 \, db_1 \\
&= \alpha^2 \left[V^3 - V^2 + \frac{V^3}{3} \right] - \alpha^2 \frac{V^3}{3} \\
&= \frac{\alpha V^3}{3} - \alpha^2 \frac{V^3}{3} \\
&= 0
\end{aligned}
$$

(player 2's expected payoff is of course identical). It makes sense that the expected value is zero: by the Revenue Equivalence Theorem, we know that any bidder's expected value will be the same regardless of auction type (as long as they are normal auctions). The first price auction considered in part a) has the same expected payoff (0), so it is encouraging that these align.

It now remains to solve for α and β. Since playing $b_i > V$ is a dominated strategy, we must have that $G(V) = 1$. Hence,

$$\alpha V + \beta = 1 \iff \beta = 1 - \alpha V$$

We now must find α. Given that no player can submit a negative bid, G must also obey the condition that $G(0) = 0$. As such,

$$G(0) = 0 \iff \alpha(0) + \beta = 0 \iff 1 - \alpha V = 0 \iff \alpha = \frac{1}{V}$$

Consequently, $\beta = 1 - \alpha V = 1 - \alpha\frac{1}{\alpha} = 0$. It follows that $G(b) = \frac{b}{V}$ is the equilibrium bidding strategy.

e) We will directly compute the expected revenue to the seller (although we already know by the Revenue Equivalence Theorem that it will be equal to the expected revenue of the auction in part a), which is V). To do so, we simply take the expectation of the bids of each player given that they are playing with the bid function $G(\cdot)$ found in the previous section.

As $G(b) = \frac{b}{V}$ is the cdf of a uniform random variable on the interval $[0, V]$, the expectation of each player's bid is $\frac{V}{2}$. Adding the expected revenue from each player, we see that the expected revenue to the seller is $\frac{V}{2} + \frac{V}{2} = V$. This aligns with the auction in part a, which has the same expected revenue (due to the RET).

$\square$

Problem 3.

Solution:

$\square$