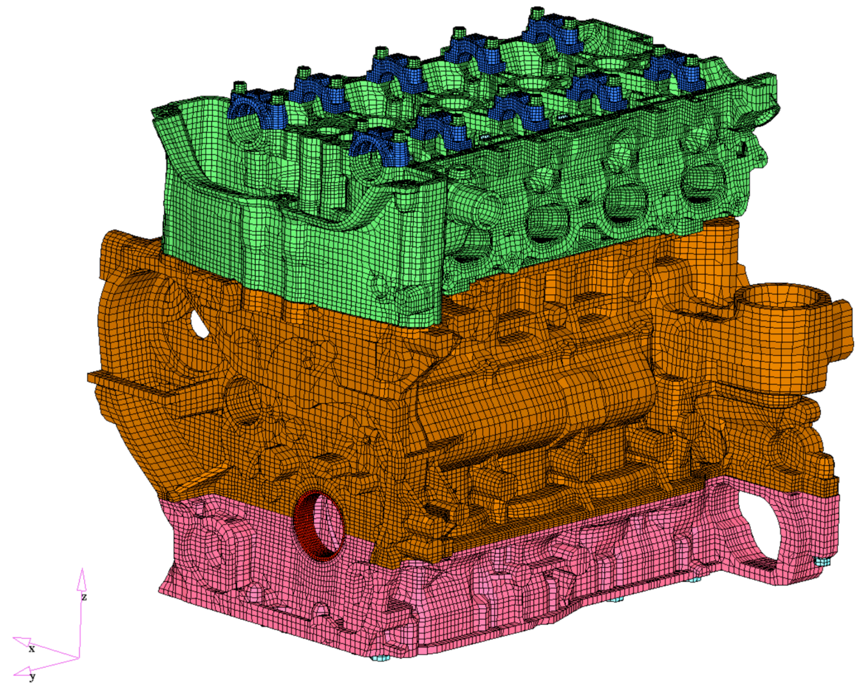
$$\sigma''_{ij} = \sigma_{ij} + \alpha p \, \delta_{ij}$$

$$\sigma_{ij} = \underline{\sigma''} - \alpha p \, \delta_{ij}$$

$$\sigma''_{ij} = \underline{\underline{C_{ijkl}}} \, \varepsilon_{kl}$$

Introduction to Finite Element Analysis



“The purpose of computing is insight, not numbers.”

--Book Dedication: RW Hamming (1971). Introduction to Applied Numerical Analysis. McGraw Hill.

“The purpose of analysis is insight, not numbers.”

What is analysis?

- From the Greek word *analyein*, meaning *“to break up”*
- An informal definition in the context of science and engineering would be *“probing into, or simulating nature”*

Why do analysis?

- Analysis is the key to effective design

Why do analysis?

- Analysis is the key to effective design
  - What is an effective design?

Why do analysis?

- Analysis is the key to effective design
  - What is an effective design?
    - One that works!

http://www.youtube.com/watch?v=_ve4M4UsJQo

Why do analysis?

- Analysis is the key to effective design
  - What is an effective design?
    - ~~One that works!~~

Why do analysis?

- Analysis is the key to effective design
  - What is an effective design?
    - One that performs the task efficiently

Why do analysis?

- Analysis is the key to effective design
  - What is an effective design?
    - One that performs the task efficiently
    - Economical

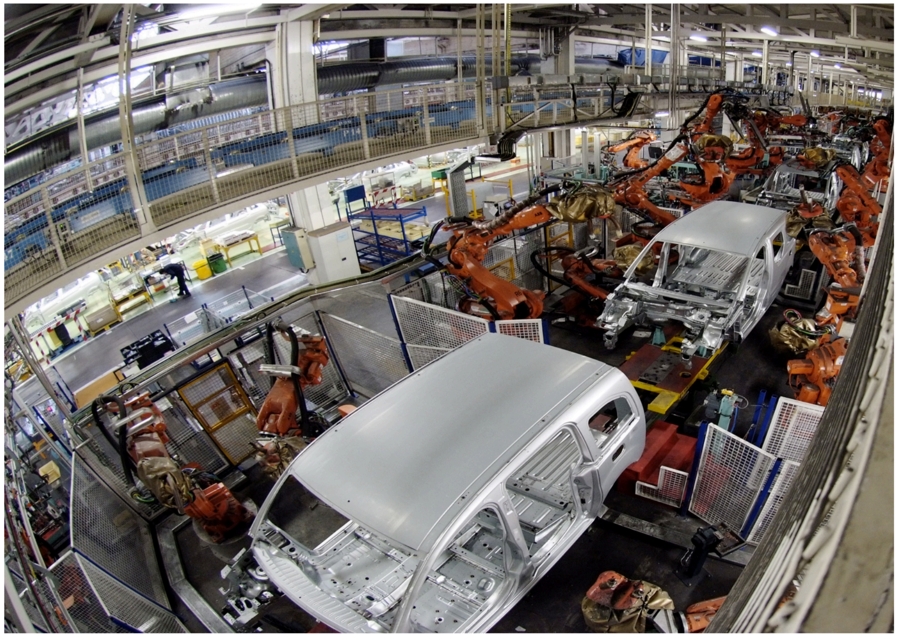
VS

Why do analysis?

- Analysis is the key to effective design
  - What is an effective design?
    - One that performs the task efficiently
    - Economical
    - Safe

Why do analysis?

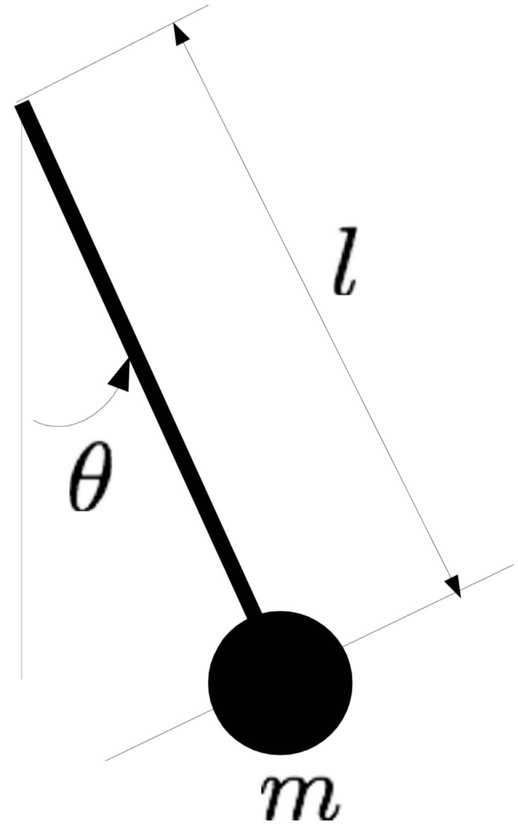
- Analysis is the key to effective design
  - What is an effective design?
    - One that performs the task efficiently
    - Economical
    - Safe
    - Manufacturable



Why do analysis?

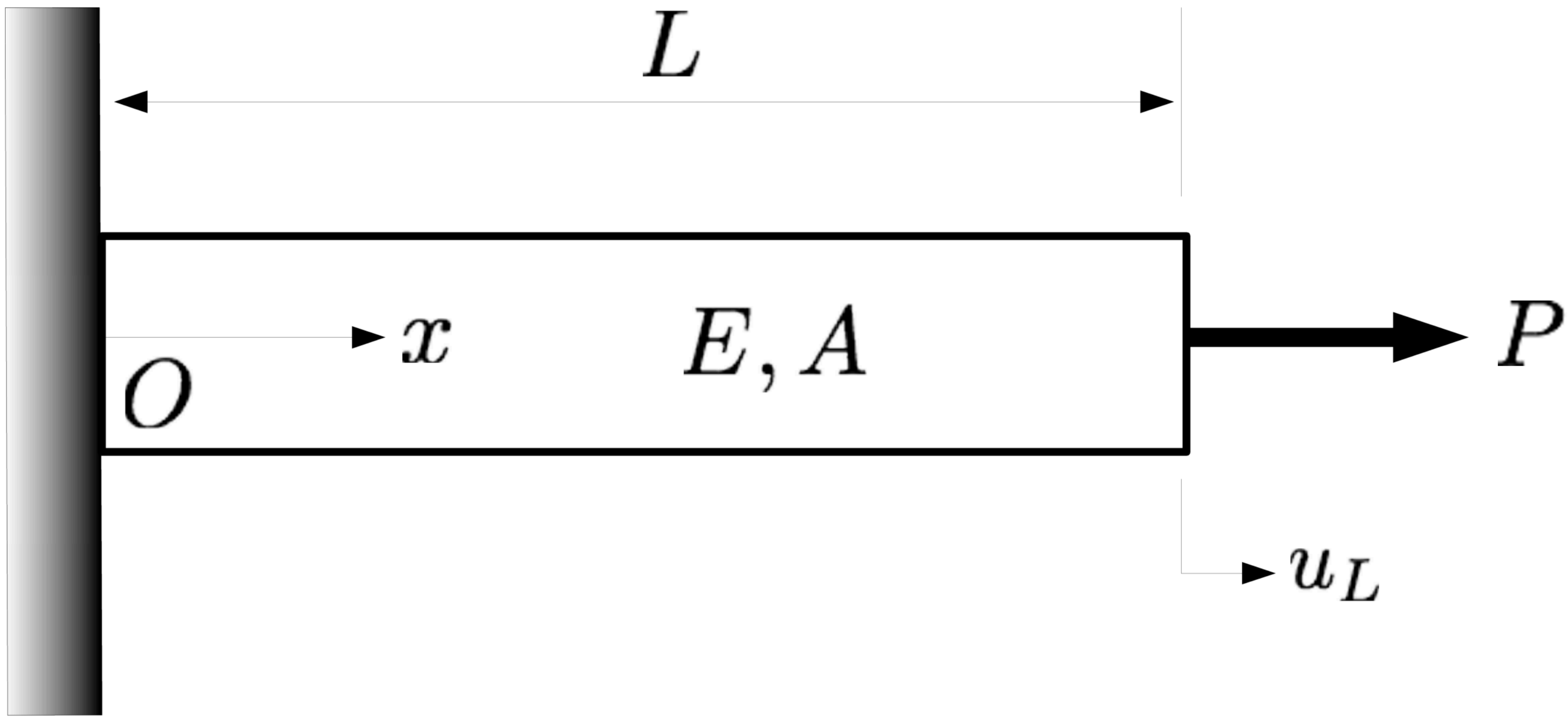
- Analysis is the key to effective design
  - What is an effective design?
    - One that performs the task efficiently
    - Economical
    - Safe
    - Manufacturable
    - Appealing

Analysis is performed by utilizing mathematical models

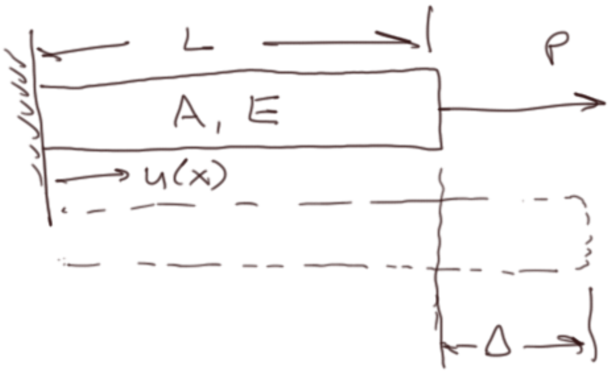


$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

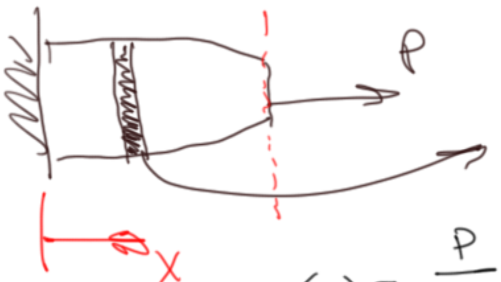
An example from solid mechanics

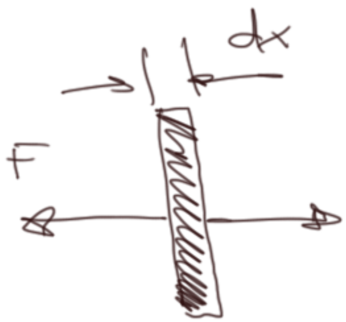


What is the equation of motion in terms of displacement u ?



$$\Delta = \frac{PL}{AE}$$





$$F + \frac{\partial F}{\partial x} dx$$

$x=0$

$$\sigma(L) = \frac{P}{A(L)}$$

$$-\cancel{F} + \cancel{F} + \frac{\partial F}{\partial x}\cancel{dx} = \rho A \cancel{dx} \cancel{\frac{\partial^2 u}{\partial t^2}} \to 0$$

$$F = \sigma A(x)$$

$$= E \varepsilon A(x)$$

$$= E \frac{\partial u}{\partial x} A(x)$$

$$\frac{\partial}{\partial x}(F) = 0$$

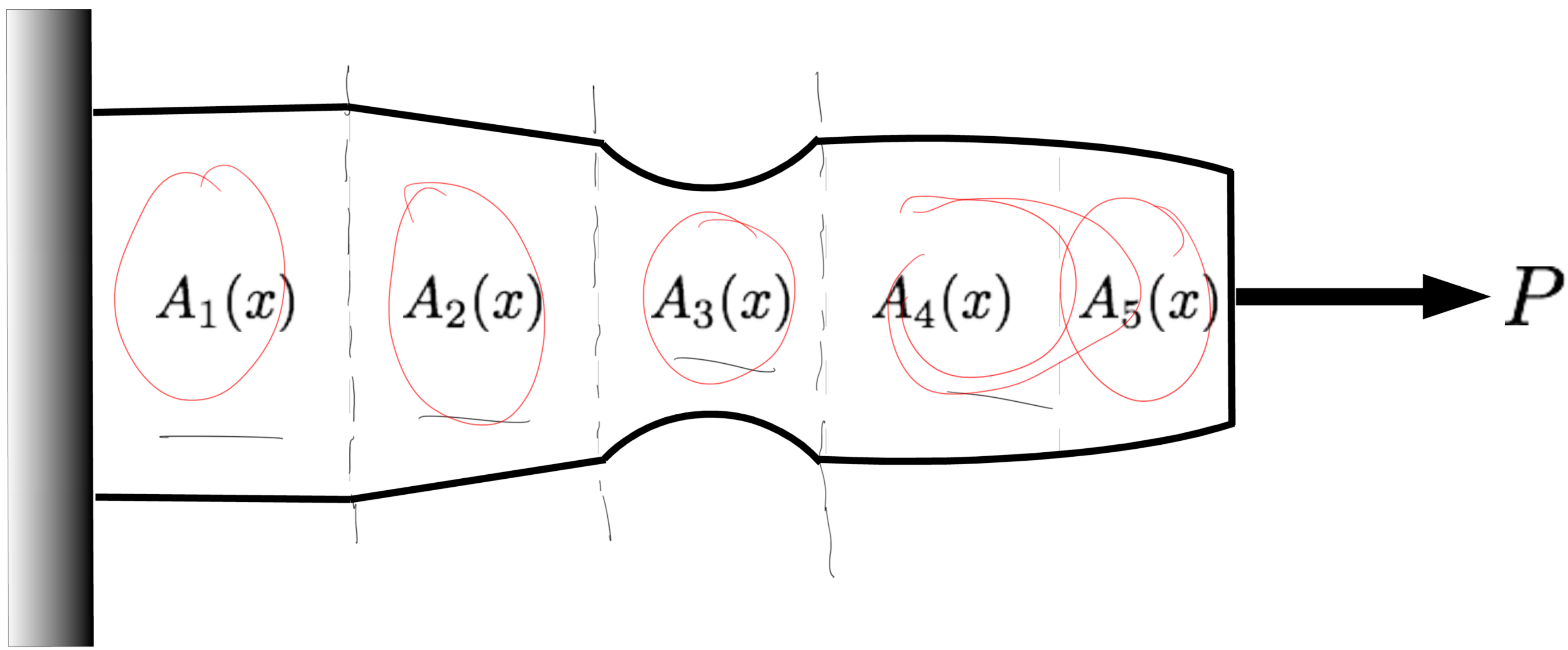
$$\boxed{\frac{\partial}{\partial x}\left[E A(x) \frac{\partial u}{\partial x} \right] = 0}$$

$$\boxed{u(0) = 0}$$

$$\boxed{E A(L) \frac{\partial u(L)}{\partial x} = P}$$

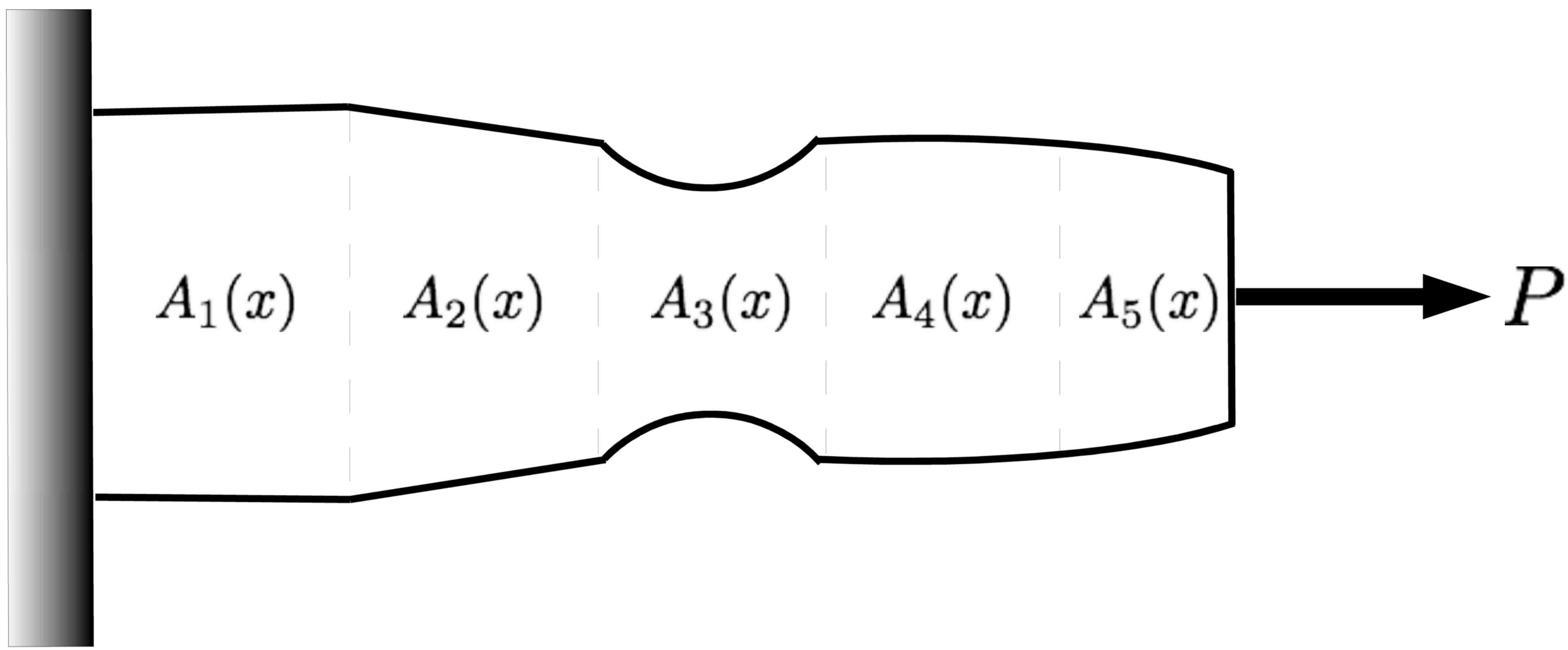
$$E\varepsilon(L) = \frac{P}{A(L)} \Rightarrow E \frac{\partial u(L)}{\partial x} = \frac{P}{A(L)}$$

What if A is nonuniform?



We *discretize* the domain.

What if A is nonuniform?



Does the shape of each subdomain look familiar?

The Finite Element Method (FEM) in a nutshell

- The domain of the problem is represented by a collection of simple subdomains, called *finite elements.*
  - The collection of finite elements is called the *finite element mesh.*
- Over each finite element, the physical process is approximated by functions (polynomials or otherwise) and algebraic equations relating physical quantities at selective points, called *nodes,* are developed.
- The element equations are *assembled* using continuity and/or “balance” of physical quantities and solved.

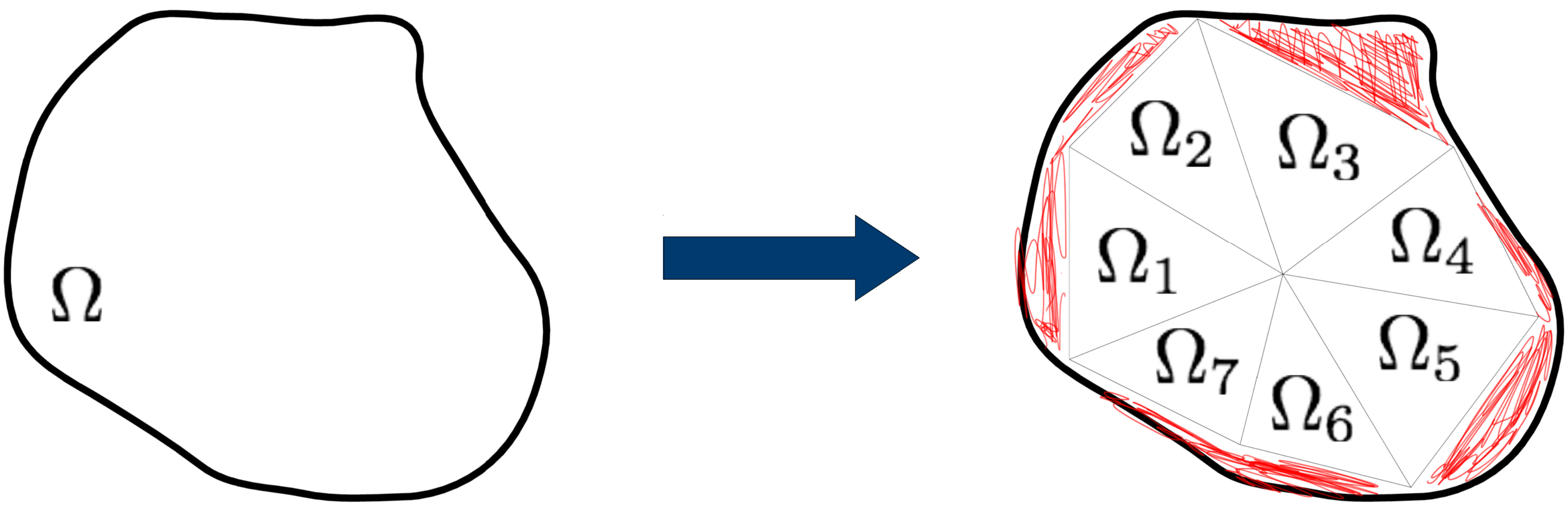
Notice I said the physical processes are *approximated over an element.*

- In the axial deformation problem posed earlier we solved the differential equations exactly.
- This is typically neither feasible nor efficient.
- In FEM we seek an approximation over the element of the form:

$$u \approx u_h = \sum_{j=1}^{n} u_j \psi_j + \sum_{j=1}^{m} c_j \phi_j$$

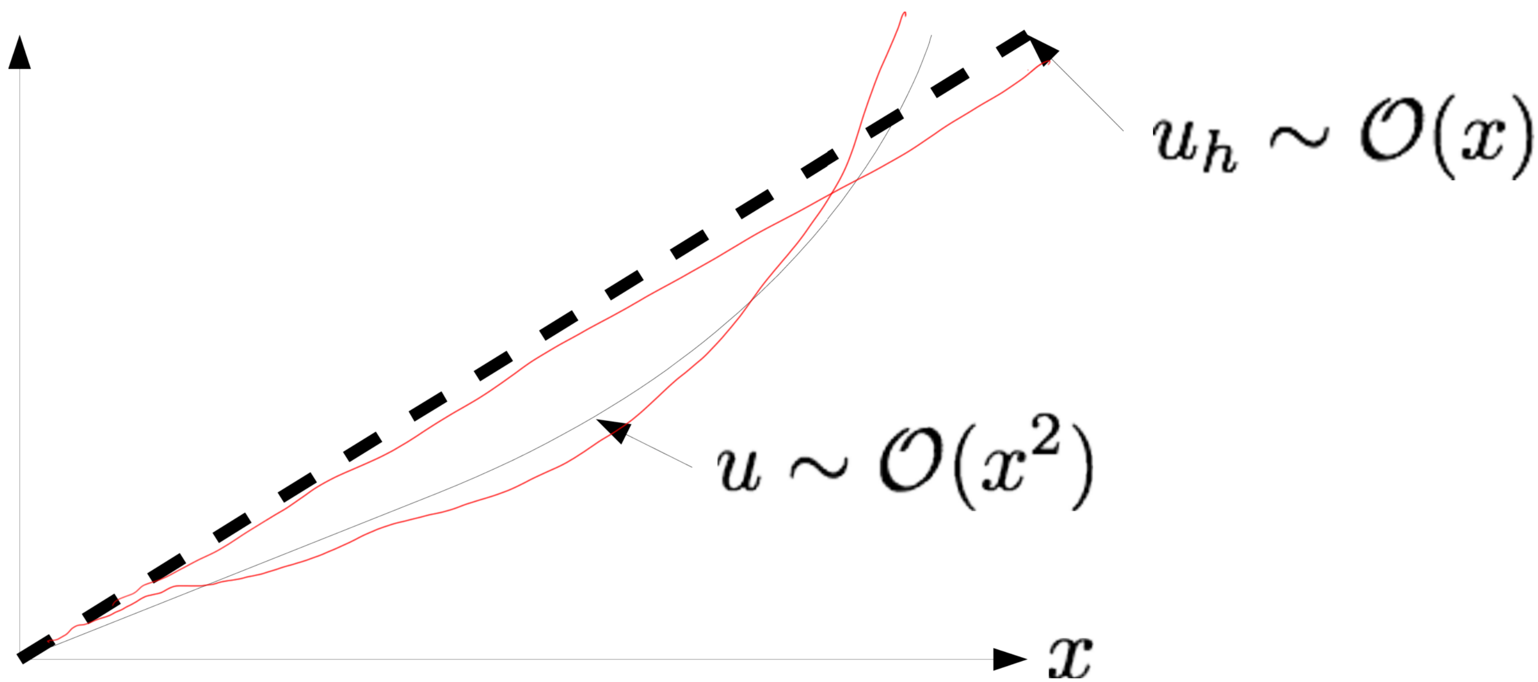
Sources of error

- Error due to the approximation of the domain – *discretization error*



Sources of error

- Error due to the approximation of the domain – *discretization error*
- Error due to approximation of the solution – truncation error

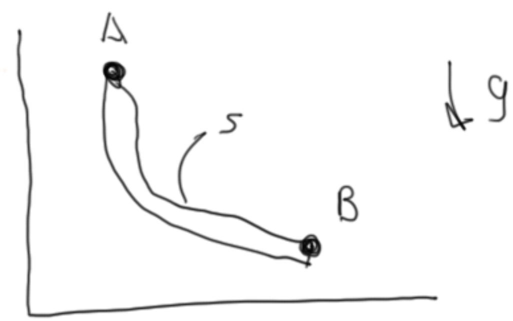


Sources of error

- Error due to the approximation of the domain – *discretization error*
- Error due to approximation of the solution – truncation error
- Computer related errors – roundoff error

Other remarks on FEM

- After assembly the resulting equations are usually singular, we have to impose boundary conditions in order to solve.
- For time-dependent problems there are two stages:
  - Use FEM to reduce PDE's to ODE's in time.
  - The ODE's in time are solved exactly or further approximated, typically with finite difference methods, to obtain algebraic equations which are then solved for the nodal values.



$$t = \frac{d}{v} = \int_{y_1}^{y_2} \frac{ds}{v} = \int_{y_1}^{y_2} \frac{\sqrt{1+y'(x)^2}}{v}\,dx$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2}\,dx$$

Conv of Energy

$$KE = PE$$

$$\frac{1}{2}mv^2 = mgy$$

$$v = \sqrt{2gy}$$

$$\boxed{t = \int_{y_1}^{y_2} \frac{\sqrt{1+y'(x)^2}}{\sqrt{2gy(x)}}\,dx}$$