$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}\,u - f = 0$$

$$\underbrace{-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)}_{\nabla\cdot(A\nabla u)}$$

$u = u(x,y)$ $\quad a_{ij}$ are the data

Models

1. Heat transfer
2. Irrotational flow of an ideal fluid
3. Groundwater flow through a permeable medium

Weak Form

$$0 = \int_\Omega \delta u \left[-\frac{\partial}{\partial x}(F_1) - \frac{\partial}{\partial y}(F_2) + a_{00}\,u - f \right] dx\,dy$$

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}, \qquad F_2 = a_{22}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

Integrate by parts

$$\frac{\partial}{\partial x}(\delta u\, F_1) = \frac{\partial u}{\partial x}F_1 + \delta u \frac{\partial F_1}{\partial x} \Rightarrow -\delta u\frac{\partial F_1}{\partial x} = \frac{\partial(\delta u)}{\partial x}F_1 - \frac{\partial}{\partial x}(\delta u\, F_1)$$

$$\frac{\partial}{\partial y}(\delta u\, F_2) = \frac{\partial u}{\partial y}F_2 + \delta u \frac{\partial F_2}{\partial y} \Rightarrow -\delta u\frac{\partial F_2}{\partial y} = \frac{\partial(\delta u)}{\partial y}F_2 - \frac{\partial}{\partial y}(\delta u\, F_2)$$

Diverge Theorem

$$\int_\Omega \frac{\partial}{\partial x}(\delta u\, F_1)\,dx\,dy = \oint_{\partial\Omega} \delta u\, F_1\, n_x\, dS$$

$$\int_\Omega \frac{\partial}{\partial y}(\delta u\, F_2)\,dx\,dy = \oint_{\partial\Omega} \delta u\, F_2\, n_y\, dS$$

n_x + n_y are components of the unit normals to $\partial\Omega$

$$\hat{n} = n_x\,\hat{e}_x + n_y\,\hat{e}_y \quad \text{on } \partial\Omega$$

$B(\delta u, u)$ =

$$0 = \iint_\Omega \left[\frac{\partial(\delta u)}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial(\delta u)}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}\,\delta u\, u - \delta u\, f \right] dx\,dy$$

$$- \oint_{\partial\Omega} \delta u \Bigg[\underbrace{n_x\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + n_y\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)}_{\equiv\, q_n} \Bigg] dS$$

$\equiv l(\delta u)$

$$B(\delta u, u) = \iint_\Omega \left[\frac{\partial(\delta u)}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial(\delta u)}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}\,\delta u\, u\right] dxdy$$

$$l(\delta u) = \int_\Omega \delta u\, f\, dxdy + \oint_{\partial\Omega} \delta u\, q_n\, dS$$

$$B(\delta u, u) = l(\delta u) \qquad B \text{ is symm. if } a_{12} = a_{21}$$

Let

$$C = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix}, \quad D(\,) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 1 \end{bmatrix} \qquad D(u) = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ u \end{bmatrix}$$

$$\left.\begin{aligned} B(\delta u, u) &= \int_\Omega \begin{bmatrix} \frac{\partial(\delta u)}{\partial x} \\ \frac{\partial(\delta u)}{\partial y} \\ \delta u \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ u \end{Bmatrix} dx\,dy = \int_\Omega \left(D(\delta u)^T\, C\, D(u)\right) dxdy \\ l(\delta u) &= \int_\Omega \{\delta u\}^T \vec{f}\, dxdy + \oint_{\partial\Omega} \{\delta u\}^T q_n\, dS \end{aligned}\right\} \text{Weak Form}$$

FE Model.

$$u \approx u^h(x,y) = N_J u_J \qquad \delta u_I = N_I$$

$$\Bigg\{ \underbrace{\int_\Omega \left[\frac{\partial N_I}{\partial x}\left(a_{11}\frac{\partial N_J}{\partial x} + a_{21}\frac{\partial N_J}{\partial y}\right) + \frac{\partial N_I}{\partial y}\left(a_{21}\frac{\partial N_J}{\partial x} + a_{22}\frac{\partial N_J}{\partial y}\right) + a_{00} N_I N_J \right] dxdy}_{K_{IJ}} \Bigg\} u_J$$

$$- \underbrace{\int_\Omega N_I f\, dx\, dy}_{f_J} - \underbrace{\oint_{\partial\Omega} N_I q_n\, dS}_{Q_I}$$

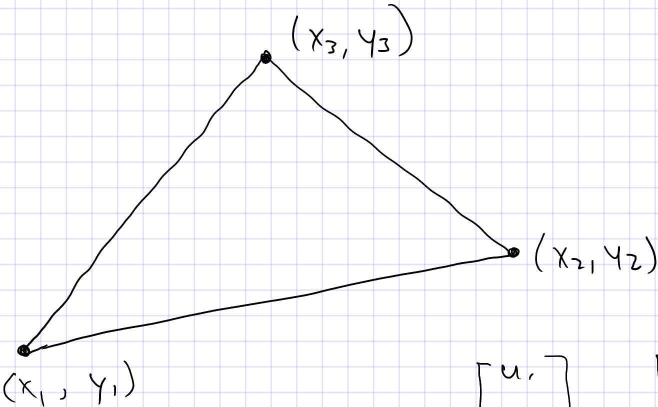
$$K_{IJ} u_J = f_J + Q_I$$

$$K\vec{u} = \vec{f} + \vec{Q}$$

$$K = \int B^T C\, B\, dxdy, \qquad \vec{f} = \int N^T \vec{f}\, dxdy, \qquad \vec{Q} = \oint_{\partial\Omega} N^T \vec{q}_n\, dS$$

$$B = D(N^T) = \begin{bmatrix} N_{1,x} & N_{2,x}, & \dots, & N_{n,x} \\ N_{1,y} & N_{1,y}, & \dots, & N_{n,y} \\ N_1, & N_2, & \dots, & N_n \end{bmatrix}$$

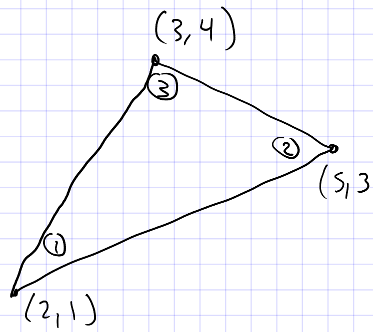
Constant Strain Triangle (CST) 3-nodes



$$u^h = C_1 + C_2 x + C_3 y$$

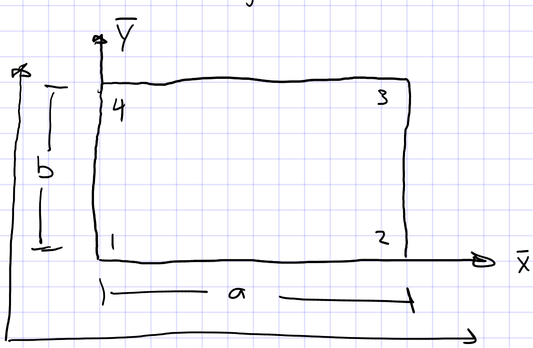
$$\vec{X} = [1, x, y]$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}^{A} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$



$$N = X^T \cdot A^{-1}$$

Linear Rectangle Element (QUAD4)

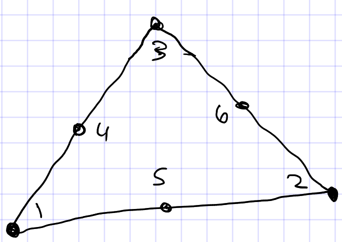


$$u_n(x,y) = C_1 + C_2 x + C_3 y + C_4 xy$$

$$N_1 = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right) \qquad N_2 = \frac{\bar{x}}{a}\left(1 - \frac{\bar{y}}{b}\right)$$

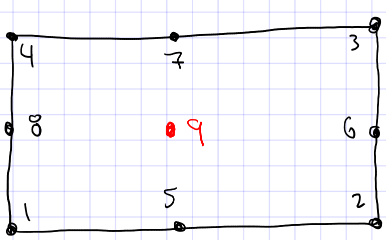
$$N_3 = \frac{\bar{x}}{a}\,\frac{\bar{y}}{b} \qquad N_4 = \left(1 - \frac{\bar{x}}{b}\right)\frac{\bar{y}}{b}$$

Quadraditic Triangle



$$u^h(x,y) = C_1 + C_2 x + C_3 y + C_4 xy + C_5 x^2 + C_6 y^2$$

(QUAD 8) (Serendipity)



$$u_n(x,y) = C_1 + C_2 x + C_3 y + C_4 xy + C_5 x^2 + C_6 y^2 + C_7 xy^2 + C_8 x^2 y + C_9 x^2 y^2$$