Mass Conservation in the Current Configuration

We begin with the mass balance equation for the fluid phase in a porous medium, expressed in the current (spatial) configuration. The fluid occupies the pore space, and its mass conservation is governed by

$$\frac{\partial \rho^f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho^f \mathbf{v}^f) = 0,$$

where, ρ^f is the bulk fluid density, defined as the mass of the fluid divided by the total volume of the mixture, $\mathbf{v}^f$ is the velocity of the fluid phase, $\nabla_{\mathbf{x}}$ denotes the divergence operator with respect to the spatial coordinates $\mathbf{x}$,

Substituting $\rho^f = \phi^f \bar{\rho}^f$ into the equation, we obtain

$$\frac{\partial (\phi^f \bar{\rho}^f)}{\partial t} + \nabla_{\mathbf{x}} \cdot (\phi^f \bar{\rho}^f \mathbf{v}^f) = 0.$$

This form explicitly incorporates the porosity and intrinsic fluid density, setting the stage for further manipulation.

Relative Mass Flux

In a porous medium, the solid matrix also moves, with a velocity $\mathbf{v}^s$. To account for the relative motion between the fluid and solid phases, we define the relative mass flux $\mathbf{w}$

$$\mathbf{w} = \phi^f \bar{\rho}^f (\mathbf{v}^f - \mathbf{v}^s).$$

This represents the mass flow rate of the fluid relative to the solid per unit area in the current configuration. Solving for the fluid velocity

$$\mathbf{v}^f = \mathbf{v}^s + \frac{\mathbf{w}}{\phi^f \bar{\rho}^f}.$$

Substitute this expression into the mass balance equation

$$\frac{\partial (\phi^f \bar{\rho}^f)}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(\phi^f \bar{\rho}^f \left(\mathbf{v}^s + \frac{\mathbf{w}}{\phi^f \bar{\rho}^f} \right) \right) = 0.$$

Distribute the terms inside the divergence

$$\frac{\partial (\phi^f \bar{\rho}^f)}{\partial t} + \nabla_{\mathbf{x}} \cdot (\phi^f \bar{\rho}^f \mathbf{v}^s + \mathbf{w}) = 0.$$

Using the linearity of the divergence operator, we split it as

$$\frac{\partial (\phi^f \bar{\rho}^f)}{\partial t} + \nabla_{\mathbf{x}} \cdot (\phi^f \bar{\rho}^f \mathbf{v}^s) + \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0$$

This equation now separates the contributions of the solid motion and the relative flux, which is advantageous for analyzing the system on the motion of the solid.

Material Time Derivative with Respect to the Solid Phase

Since the solid matrix is deforming, it is natural to express the time derivative in a frame moving with the solid phase. The material time derivative following the solid velocity $\mathbf{v}^s$ is defined as

$$\frac{D^s}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}^s \cdot \nabla_{\mathbf{x}}.$$

Using the product rule to expand the divergence operator keeping the terms $\phi^f \bar{\rho}^f$ together we have

$$\frac{\partial(\phi^f \bar{\rho}^f)}{\partial t} + (\nabla_{\mathbf{x}} \phi^f \bar{\rho}^f) \cdot \mathbf{v}^s + \phi^f \bar{\rho}^f \nabla_{\mathbf{x}} \cdot \mathbf{v}^s = 0.$$

Recognize that

$$\frac{D^s}{Dt}(\phi^f \bar{\rho}^f) = \frac{\partial(\phi^f \bar{\rho}^f)}{\partial t} + (\mathbf{v}^s \cdot \nabla_{\mathbf{x}})(\phi^f \bar{\rho}^f),$$

so the mass balance becomes

$$\frac{D^s}{Dt}(\phi^f \bar{\rho}^f) + \phi^f \bar{\rho}^f \nabla_{\mathbf{x}} \cdot \mathbf{v}^s + \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0. \tag{1}$$

Now observe the following

$$\begin{aligned}\frac{D^s}{Dt}(J^s \phi^f \bar{\rho}^f) &= J^s \frac{D^s}{Dt}(\phi^f \bar{\rho}^f) + \frac{D^s J^s}{Dt}(\phi^f \bar{\rho}^f) \\ &= J^s \frac{D^s}{Dt}(\phi^f \bar{\rho}^f) + J^s (\phi^f \bar{\rho}^f) \nabla_{\mathbf{x}} \cdot \mathbf{v}^s \end{aligned} \tag{2}$$

using the identity $\dot{J}^s = J^s \nabla_{\mathbf{x}} \cdot \mathbf{v}^s$. Dividing (2) through by J^s gives

$$\frac{1}{J^s} \frac{D^s}{Dt}(J^s \phi^f \bar{\rho}^f) = \frac{D^s}{Dt}(\phi^f \bar{\rho}^f) + \phi^f \bar{\rho}^f \nabla_{\mathbf{x}} \cdot \mathbf{v}^s. \tag{3}$$

Comparing the right hand side of (3) with the first two terms in (1) then

$$\frac{1}{J^s} \frac{D^s}{Dt}(J^s \phi^f \bar{\rho}^f) + \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0 \tag{4}$$

Transformation to the Reference Configuration

To simplify the analysis, especially when dealing with constitutive relationships defined in the undeformed state, we transform the equation to the reference configuration of the solid, denoted by $\mathbf{X}^s$. The Jacobian J^s relates the current volume element $\mathrm{d}v$ to the reference volume element $\mathrm{d}V^s$ via $\mathrm{d}v = J^s \mathrm{d}V^s$.

First, transform the relative mass flux $\mathbf{w}$ using the Piola transform. Define the reference mass flux $\mathbf{W}$

$$\mathbf{W} = J^s \mathbf{w} (\mathbf{F}^s)^{-T}.$$

The divergence transforms as

$$\nabla_{\mathbf{X}^s} \cdot \mathbf{W} = J^s \nabla_{\mathbf{x}} \cdot \mathbf{w},$$

or equivalently

$$\nabla_{\mathbf{x}} \cdot \mathbf{w} = \frac{1}{J^s} \nabla_{\mathbf{X}^s} \cdot \mathbf{W}.$$

Now, define the reference (Lagrangian) porosity $\Phi = J^s \phi^f$, which represents the fluid volume per unit reference volume of the solid. In the reference frame, the material derivative simplifies to the partial derivative with respect to time at fixed $\mathbf{X}^s$

$$\frac{D^s}{Dt} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}^s}.$$

Substitute into the mass balance

$$\frac{1}{J^s}\frac{\partial}{\partial t}(\Phi \bar{\rho}^f) + \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0,$$

using the transformed divergence,

$$\frac{1}{J^s}\frac{\partial}{\partial t}(\Phi \bar{\rho}^f) + \frac{1}{J^s}\nabla_{\mathbf{X}^s} \cdot \mathbf{W} = 0.$$

Now multiply through by J^s

$$\frac{\partial}{\partial t}(\Phi \bar{\rho}^f) + \nabla_{\mathbf{X}^s} \cdot \mathbf{W} = 0.$$

This is the mass balance equation in the reference configuration, where $\Phi \bar{\rho}^f$ is the fluid mass per unit reference volume, and $\mathbf{W}$ is the reference mass flux.

Incorporating the Barotropic Assumption

Assume the fluid is barotropic, meaning its intrinsic density depends only on pressure: $\bar{\rho}^f = \bar{\rho}^f(p^f)$. The compressibility is characterized by the bulk modulus K^f

$$\frac{d\bar{\rho}^f}{dp^f} = \frac{\bar{\rho}^f}{K^f}.$$

Thus,

$$\frac{\partial \bar{\rho}^f}{\partial t} = \frac{d\bar{\rho}^f}{dp^f}\frac{\partial p^f}{\partial t} = \frac{\bar{\rho}^f}{K^f}\frac{\partial p^f}{\partial t}.$$

Expand the time derivative in the mass balance

$$\frac{\partial}{\partial t}(\Phi \bar{\rho}^f) = \frac{\partial \Phi}{\partial t}\bar{\rho}^f + \Phi \frac{\partial \bar{\rho}^f}{\partial t} = \frac{\partial \Phi}{\partial t}\bar{\rho}^f + \Phi \frac{\bar{\rho}^f}{K^f}\frac{\partial p^f}{\partial t},$$

and substitute into the mass balance

$$\frac{\partial \Phi}{\partial t}\bar{\rho}^f + \Phi \frac{\bar{\rho}^f}{K^f}\frac{\partial p^f}{\partial t} + \nabla_{\mathbf{X}^s} \cdot \mathbf{W} = 0.$$

Factor out $\bar{\rho}^f$

$$\bar{\rho}^f \left(\frac{\partial \Phi}{\partial t} + \frac{\Phi}{K^f}\frac{\partial p^f}{\partial t} \right) + \nabla_{\mathbf{X}^s} \cdot \mathbf{W} = 0. \tag{5}$$

Porosity Constitutive Equation

Using a constitutive equation for the Lagrangian porosity (c.f. [1] eq. (4.19b) assuming isothermal)

$$\Phi - \Phi_0 = B \ln(J^s) + \frac{B - \Phi}{K^s}(p - p_0) \tag{6}$$

Taking the time derivative of both sides of (6) gives

$$\frac{\partial \Phi}{\partial t} = \frac{\partial B}{\partial t} \ln(J^s) + \frac{B}{J^s}\frac{\partial J^s}{\partial t} + \frac{1}{K^s}\left[\left(\frac{\partial B}{\partial t} - \frac{\partial \Phi}{\partial t}\right)(p - p_0) + (B - \Phi)\frac{\partial p}{\partial t}\right].$$

For simplicity, assume $p << K_s$ so that

$$\frac{\partial \Phi}{\partial t} \approx \frac{\partial B}{\partial t} \ln(J^s) + \frac{B}{J^s}\frac{\partial J^s}{\partial t} + \frac{B - \Phi}{K^s}\frac{\partial p}{\partial t}. \tag{7}$$

Substitute (7) into (5)

$$\frac{\partial B}{\partial t}(\ln J^s) + \frac{B}{J^s}\frac{\partial J^s}{\partial t} + \left(\frac{B - \Phi}{K^s} + \frac{\Phi}{K^f}\right)\frac{\partial p}{\partial t} + \frac{1}{\bar{\rho}^f}\nabla_{\mathbf{X}^s} \cdot \mathbf{W} = 0.$$

Or equivalently

$$\frac{\partial B}{\partial t}(\ln J^s) + \frac{B}{J^s}\frac{\partial J^s}{\partial t} + \frac{1}{M}\frac{\partial p}{\partial t} + \frac{1}{\bar{\rho}^f}\nabla_{\mathbf{X}^s} \cdot \mathbf{W} = 0 \tag{8}$$

where

$$M = \frac{K^s K^f}{K^f(B - \Phi) + K^s \Phi}.$$

Fluid Momentum Balance and Darcy's Law

Recall the momentum balance equation for the fluid is

$$\rho^f \mathbf{a}^f = \rho^f \mathbf{b} + \mathbf{h}^f - \phi^f \nabla_{\mathbf{x}} p,$$

if we assume $\mathbf{h} = -\frac{\mu}{\bar{\rho}^f}\mathbf{k}^{-1}\mathbf{w}$ (i.e. Darcy's Law) and rearrange, we have

$$\begin{aligned}(\mathbf{v} - \mathbf{v}^s) =& \frac{\mathbf{k}}{\mu}\left[\bar{\rho}^f\left(\mathbf{b} - \mathbf{a}^f\right) - \nabla_{\mathbf{x}} p\right], \\ \mathbf{w} =& \bar{\rho}^f\frac{\mathbf{k}}{\mu}\left[\bar{\rho}^f\left(\mathbf{b} - \mathbf{a}^f\right) - \nabla_{\mathbf{x}} p\right],\end{aligned}$$

assuming intertial forces are negligible ($\mathbf{a}^f = \mathbf{0}$) and pulling back to the reference configuration we have

$$\mathbf{W} = \bar{\rho}^f\frac{\mathbf{K}}{\mu}\left[\bar{\rho}^f\left(\mathbf{F}^s\right)^{-\mathsf{T}}\mathbf{b} - \nabla_{\mathbf{X}^s} p\right],$$

where

$$\mathbf{K} = J^s\left(\mathbf{F}^s\right)^{-1}\mathbf{k}\left(\mathbf{F}^s\right)^{-\mathsf{T}}.$$

Linearization

To linearize (9) we assume B is constant therfore $\frac{\partial B}{\partial t} = 0$. Additionally, substitute in the identity $\dot{J}^s = J^s \nabla_{\mathbf{x}} \cdot \mathbf{v}^s$ and push-forward the last term to the current configuration with $J^s \approx 1$.

$$B \nabla_{\mathbf{x}} \cdot \mathbf{v}^s + \frac{1}{M} \frac{\partial p}{\partial t} + \frac{1}{\bar{\rho}^f} \nabla_{\mathbf{x}} \cdot \mathbf{w} = 0 \tag{9}$$

Note that $\nabla_{\mathbf{x}} \cdot \mathbf{v}^s = \mathrm{tr}(\mathbf{D})$, where $\mathbf{D}$ is the rate-of-deformation tensor; which, when linearized, is approximately equivalent to the small strain rate tensor, i.e. $D_{ii} \approx \dot{\varepsilon}_{ii}$. Finally, if $\bar{\rho}^f$ varies slowly in space and we ignore all body forces, then the final linearized mass balance equation is

$$B \mathrm{tr}(\dot{\varepsilon}) + \frac{1}{M} \frac{\partial p}{\partial t} - \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{k}}{\mu} \nabla_{\mathbf{x}} p \right) = 0,$$

which can be combined with the lineared equalibrium equation

$$\nabla_{\mathbf{x}} \cdot \left(\boldsymbol{\sigma}''^s - Bp\mathbf{I} \right) = 0.$$

to solve for displacements $\mathbf{u}$ and pressure p.

References

[1] Olivier Coussy. *Poromechanics.* John Wiley & Sons, 2004.