

8-1-1999

A Stochastic Analog to the Richardson's Arms Race Model

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A Stochastic Analog to the
Richardson's Arms Race Model

A Thesis Presented to
the Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

by

John Fricks

August 1999

A Stochastic Analog to the
Richardson's Arms Race Model

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Acknowledgments

First, I greatly appreciate the guidance of committee members Dr. Daniel Biles and Dr. David Neal. Thanks also goes to Dr. Bruce Kessler for crucial aid in typesetting. Finally, I would like to thank Dr. Randall Swift for his leadership, support, and encouragement throughout this project.

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A Stochastic Analog to the
Richardson's Arms Race Model

John Fricks

August 1999

73 Pages

Directed by: Randall Swift, Daniel Biles, and David Neal

Department of Mathematics

Western Kentucky University

In this thesis, a stochastic version of the Richardson's arms race model is developed through the method of birth-death processes. The expected value of the model is explored and shown to be analogous to the original deterministic arms race model.

The numerical method of randomization is then expanded and applied to the stochastic model. A comparison is then made between outcomes of the deterministic and stochastic models.

Chapter One

Preliminaries and Background

■ 1.1 Introduction

The Richardson's Arms Race Model has been studied widely since its introduction in the 1930's by its namesake Lewis F. Richardson. The idea of using a mathematical model to describe a diplomatic situation was most novel at that time. Richardson was a British physicist who worked mostly in the area of meteorology. While working as an ambulance driver during World War I, Richardson began work on formulating a mathematical model of the causes of war. Richardson had a strong Quaker upbringing which inspired his pacifism and gave him impetus to find the causes of war in hopes that it could be prevented. Richardson believed that the central cause of war was the build up of nations' arms, thus his primary focus was on nations' armament stocks [7].

Richardson's basic model began with only two nations. Let us

imagine there are nations X and Y who are neighbors [9]. Since changes in X's and Y's war readiness may be an indicator of impending military action, we are interested in the contributing factors that will make these countries increase or decrease the amount of armaments that they own. We will use the variables x and y to denote the amounts of arms that X and Y respectively have. In light of these variable name choices, we will denote the rate of change in X's armaments over time as $\frac{dx}{dt}$ and the rate of change in Y's armaments as $\frac{dy}{dt}$.

Let us consider the perspective of country X. What should cause country X to increase or decrease its armament? The level of Y's armaments should have a positive impact on the change in X's armaments. Whether Y is a friend or foe, there is a natural fear of a neighbor having arms. So we introduce the term αy into the expression for $\frac{dx}{dt}$. If Y is a friend, then α will be a small nonnegative number. If Y is a foe, then α will be a larger nonnegative number. This term is called the "mutual fear" term.

If this term were the only one, then X would perpetually increase its armament. We know however that this situation does not mirror real

life. Nation X is constrained by the impact on its economy due to greater and greater military expenditures. We model this constraint by subtracting a "drag" term γx from the mutual fear term αy . The nonnegative "drag" coefficient γ may be interpreted as to how sensitive the policy makers of the nation are to economic drag due to increased military spending. A totalitarian government is probably less sensitive, so the drag coefficient would be small. On the other hand, many democracies or less wealthy nations may be more sensitive when military spending encroaches on the standard of living of the citizens, so the drag coefficient would be higher in these situations.

Under the scenario that has been set up so far, if neither nation had any armaments, then neither nation would arm itself. However if nation X had some grudge against nation Y, then it seems likely that X might arm itself. So we include a constant term ζ in the expression for $\frac{dx}{dt}$. If this term is positive, then it is interpreted as a natural animosity of country X toward country Y. If the term is negative, then it is interpreted as natural goodwill of country X toward country Y. An amalgam of possibilities may account for the goodwill or animosity,

including such things as border disputes, similarities or differences in religions, a common language, etc.

Arguments similar to the above can be made from country Y's perspective. The result is the basic Richardson's model consisting of a pair of differential equations describing the rates of change in arms of two countries over time.

$$\frac{dx}{dt} = \alpha y - \gamma x + \zeta$$

$$\frac{dy}{dt} = \beta x - \delta y + \eta , \quad (1)$$

for $\alpha, \beta, \gamma, \delta > 0$.

This model is a rather simple representation of the actual complexities involved in an arms race between two countries. Richardson explained that this formulation was a reduction of an earlier model that used many more variables as well as square and interactive terms; he eventually decided that the elements above were the most valuable in explaining the situation while keeping a focus on simplicity.

It should also be noted that Richardson allowed for the variables x and y to be negative. What does it mean to own a negative armament? To allow for a robust interpretation of his results, Richardson interpreted negative values for these variables as cooperation between the nations in terms of economic trade.

Richardson examined the results of the model with differing values for the various coefficients, especially to determine if the model stabilizes. Michael Olnick [7] summarizes these results succinctly in *An Introduction to Mathematical Models in the Social and Life Sciences*, and they are paraphrased here. There are two cases when ζ and η are positive; these parameters being positive may be interpreted as a natural animosity between the two nations. When $\gamma\delta-\alpha\beta$ is positive, there will be a stabilized arms race--the values for x and y will converge to set values as time goes on. On the other hand if $\gamma\delta-\alpha\beta$ is negative, then there is a runaway arms race, meaning as time goes on x and y become progressively larger. There are also two cases when the ζ and η are negative, which we interpreted above as meaning that the countries actually have goodwill toward one another. The first of these cases is

$\gamma\delta-\alpha\beta$ being positive; total disarmament of both nations will result. The second case, when $\gamma\delta-\alpha\beta$ is negative, yields one of two results. Either there will be total disarmament if the initial amount of arms is below a certain value, or there will be a runaway arms race if the initial amount of arms by both nations is above that certain value. An interesting point to note about the last case is that even though there is authentic goodwill between the two nations a runaway arms race can still occur.

■ 1.2 Variations

Richardson himself included a number of variations on the primary model discussed above [9]. One of the simplest variations was changing the first term of each of the equations so that mutual fear did not depend on the absolute amount of arms of another country but on the difference between one's own amount of arms and the rival's amount of arms. The differential equations would now appear as

$$\frac{dx}{dt} = \alpha(y - x) - \gamma x + \zeta$$

$$\frac{dy}{dt} = \beta(x - y) - \delta y + \eta,$$

for $\alpha, \beta, \gamma, \delta > 0$.

An interesting result of this variation, which Richardson calls the "rivalry" variation, is that it will always result in a stabilizing arms race.

Some of Richardson's variations included discussions of equations similar to the primary model for any number of nations, not just two. Also, the effects of improved communication and of the sizes of the nations were studied. In addition to variations on the model, Richardson offered evidence to support his theories by using expenditure figures of European powers and describing how these expenditures related to the outbreak of war. This attempt to gather real world support for his theories culminated in his work *The Statistics of Deadly Quarrels* [10].

Chapter Two

The Stochastic Model

■ 2.1 The Derivation

Clearly, the model under investigation is an oversimplification. So it is a natural extension to enrich the model by grouping the assorted effects not included explicitly in the model as a random influence on the model and then study the results.

Previous attempts have been made at stochastic extensions of the model. One of these extensions considers the deterioration of arms as a stochastic factor in the model [2]. Another approach was to include a uniform random variable in the model to account for all of the random perturbations [5].

We will approach the problem from the perspective of a birth-death process. We will develop a model and then show that the expected value of the stochastic model presented matches the original deterministic Richardson's model.

First, we must make some assumptions. Our model will assume that

expenditures of countries X and Y are discrete valued; thus we will denote the expenditures of X and Y by the variables m and n, respectively. We introduce two interdependent random variables, M(t) and N(t), whose outputs are the possible discrete levels of expenditure of X and Y. Also, we need to break apart the grievance term into two components, one nonnegative and one nonpositive. This separation will make our model capable of dealing with the analog of the original model when we have negative values for these grievance terms. Unlike the Richardson model however, we will consider only nonnegative values of m and n. Our model is expressed as the probability of being at given levels of expenditure, m and n, at time t and is given by

$$\begin{aligned}
 P_{m,n}(t + \Delta t) = & (\alpha n + \rho) (\Delta t) P_{m-1,n}(t) \\
 & + (\gamma(m+1) + \kappa) (\Delta t) P_{m+1,n}(t) \\
 & + (\beta m + \tau) (\Delta t) P_{m,n-1}(t) \\
 & + (\delta(n+1) + \lambda) (\Delta t) P_{m,n+1}(t) \\
 & + (1 - (\alpha n + \rho + \gamma m + \kappa \\
 & + \beta m + \tau + \delta n + \lambda)) (\Delta t) P_{m,n}(t),
 \end{aligned} \tag{3}$$

for $\alpha, \beta, \gamma, \delta, \rho, \tau, \kappa, \lambda > 0$, where

$$P_{m,n}(t) = P\{M(t) = m, N(t) = n\}.$$

As mentioned above, we have broken apart the grievance terms. So, the expression $\rho-\kappa$ is analogous to the parameter ζ in the original model, and the expression $\tau-\lambda$ is analogous to the parameter η in the original model.

We will assume that in any given time step (Δt) only one event will happen. The probability that the model is in a particular state in a given time $t+\Delta t$ is the sum of probabilities of all of the cases which may occur during Δt . Since we are only allowing for one event in a given time step, there are five possible cases:

1. Country X increases its armaments
2. Country Y increases its armaments
3. Country X decreases its armaments
4. Country Y decreases its armaments
5. Nothing happens

Each of these terms consists of two parts. The first factor is the probability that a given event happens. The second factor is the probability that the system was in a state where this step could happen. We also assume that these events are independent, so that the probability of both occurring is the product of these factors.

We will now see how each of the terms in our model correlates to the original model.

1. $(\alpha n + \rho) (\Delta t) P_{m-1, n} (t)$ -- This term corresponds to the case that the model is at one less of m and at the same level of n in the previous time step. The first factor corresponds to the mutual fear term αy plus the positive contribution ρ from the grievance term ζ in the first differential equation of the original Richardson model (1). The second factor states that the contribution from this term is proportional to the length of the current time step Δt . The last factor is the probability that the model has arrived at this state at this time step.

2. $(\gamma (m + 1) + \kappa) (\Delta t) P_{m+1, n} (t)$ -- This term corresponds to the case that the model is at one more of m and at the same level of n in the previous time step, t. The first factor corresponds to the sum of a term $\gamma(m+1)$ which corresponds to the drag term γx and the negative contribution κ which corresponds to the grievance term ζ in the first differential equation of the original Richardson model (1). Again, the second factor states that the contribution from this term is proportional to the length of the current time step Δt . The last factor is the probability that the model has arrived at this state at the previous time

step t.

3. $(\beta m + \tau) (\Delta t) P_{m,n-1}(t)$ -- This term corresponds to the case that the model is at one less of n and at the same level of m in the previous time step t. The first factor corresponds to the mutual fear term βx plus the positive contribution τ of the grievance term η of the second differential equation of the original Richardson model (1). The second factor consists of the length of the time step Δt . The last factor is the probability that the model has arrived at this state at this time step t.

4. $(\delta(n+1) + \lambda) (\Delta t) P_{m,n+1}(t)$ --This term corresponds to the case that the model is at one more of n and at the same level of m in the previous time step t. The first factor is analogous to the drag term δy plus the negative contribution λ from the grievance term of the second differential equation in the original Richardson's model (1).

5. $(1 - (\alpha n + \rho + \gamma m + \kappa + \beta m + \tau + \delta n + \lambda) (\Delta t)) P_{m,n}(t)$ -- This term considers the probability that nothing happens in this time step between t and $t+\Delta t$. Thus, we have one minus the sum of the probabilities of the other cases times the probability that we were originally in this state.

Simple equation manipulation gives

$$\begin{aligned}
 P_{m,n}(t + \Delta t) - P_{m,n}(t) &= (\alpha n + \rho)(\Delta t) P_{m-1,n}(t) \\
 &\quad + (\gamma(m+1) + \kappa)(\Delta t) P_{m+1,n}(t) \\
 &\quad + (\beta m + \tau)(\Delta t) P_{m,n-1}(t) \\
 &\quad + (\delta(n+1) + \lambda)(\Delta t) P_{m,n+1}(t) - \\
 &\quad (\alpha n + \rho + \gamma m + \kappa + \beta m + \tau + \delta n + \lambda)(\Delta t) P_{m,n}(t).
 \end{aligned} \tag{4}$$

Dividing both sides by Δt we obtain,

$$\begin{aligned}
 \frac{P_{m,n}(t + \Delta t) - P_{m,n}(t)}{\Delta t} &= (\alpha n + \rho) P_{m-1,n}(t) \\
 &\quad + (\gamma(m+1) + \kappa) P_{m+1,n}(t) \\
 &\quad + (\beta m + \tau) P_{m,n-1}(t) \\
 &\quad + (\delta(n+1) + \lambda) P_{m,n+1}(t) \\
 &\quad - (\alpha n + \rho + \gamma m + \kappa + \beta m + \tau + \delta n + \lambda) P_{m,n}(t).
 \end{aligned} \tag{5}$$

If we assume that the limit as Δt approaches 0 exists, then

$$\begin{aligned}
 \frac{dP_{m,n}(t)}{dt} &= \\
 \lim_{\Delta t \rightarrow 0} \frac{P_{m,n}(t + \Delta t) - P_{m,n}(t)}{\Delta t} &= \\
 (\alpha n + \rho) P_{m-1,n}(t)
 \end{aligned}$$

$$\begin{aligned}
& + (\gamma(m+1) + \kappa) P_{m+1,n}(t) \\
& + (\beta m + \tau) P_{m,n-1}(t) \\
& + (\delta(n+1) + \lambda) P_{m,n+1}(t) \\
& - (\alpha n + \rho + \gamma m + \kappa + \beta m + \tau + \delta n + \lambda) P_{m,n}(t),
\end{aligned}$$

for $m = 0, 1, 2, \dots$ $n = 0, 1, 2, \dots$

We therefore have an infinite system of differential equations (6).

Often these types of systems may be solved recursively given an initial condition. This method seems to be untenable in this case, since there is recursion in two variables. In the next section, we will introduce a tool that may be useful in analyzing this system.

■ 2.2 The Probability Generating Function

The probability generating function (pgf) is a valuable tool for analyzing discrete valued random variables. The pgf ϕ of our model is

$$\phi(t, z_1, z_2) =$$

$$E[z_1^m z_2^n] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n}(t). \quad (7)$$

Differentiation with respect to t yields

$$\partial_t \phi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n \frac{dP_{m,n}(t)}{dt} \quad (8)$$

Substitution from Equation (6) and separation of terms gives

$$\partial_t \phi = \alpha \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n z_1^m z_2^n P_{m-1,n}(t)$$

$$+ \rho \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m-1,n}(t)$$

$$+ \gamma \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1) z_1^m z_2^n P_{m+1,n}(t)$$

$$+ \kappa \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m+1,n}(t)$$

$$+ \beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m z_1^m z_2^n P_{m,n-1}(t)$$

$$+ \tau \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n-1}(t)$$

$$+ \delta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (n+1) z_1^m z_2^n P_{m,n+1}(t)$$

$$+ \lambda \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n+1}(t)$$

$$- \alpha \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n z_1^m z_2^n P_{m,n}(t)$$

$$- \rho \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n}(t)$$

$$- \gamma \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m z_1^m z_2^n P_{m,n}(t)$$

$$- \kappa \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n}(t)$$

$$- \beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m z_1^m z_2^n P_{m,n}(t)$$

$$- \tau \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n}(t)$$

$$- \delta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n z_1^m z_2^n P_{m,n}(t)$$

$$- \lambda \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n}(t) .$$

Rewriting some of the terms as partial derivatives of ϕ gives

$$\begin{aligned}
\partial_t \phi &= \alpha z_1 z_2 (\partial_{z_2} \phi) + \rho z_1 \phi + \gamma (\partial_{z_1} \phi) \\
&\quad + \frac{\kappa}{z_1} \phi + \beta z_1 z_2 (\partial_{z_1} \phi) + \tau z_2 \phi + \delta (\partial_{z_2} \phi) \\
&\quad + \frac{\lambda}{z_2} \phi - \alpha z_2 (\partial_{z_1} \phi) - \rho \phi - \gamma z_1 (\partial_{z_1} \phi) \\
&\quad - \kappa \phi - \beta z_1 (\partial_{z_2} \phi) - \tau \phi - \delta z_2 (\partial_{z_2} \phi) - \lambda \phi.
\end{aligned} \tag{10}$$

We now group by derivatives of ϕ :

$$\begin{aligned}
\partial_t \phi &= (\beta z_1 z_2 + \gamma - \gamma z_1 - \beta z_1) (\partial_{z_1} \phi) \\
&\quad + (\alpha z_1 z_2 + \delta - \alpha z_2 - \delta z_2) (\partial_{z_2} \phi) \\
&\quad + \left(\rho z_1 + \frac{\kappa}{z_1} + \tau z_2 + \frac{\lambda}{z_2} - \rho - \kappa - \tau - \lambda \right) \phi.
\end{aligned} \tag{11}$$

To obtain the pgf ϕ of the System (6), a solution of Equation (11) is required. However, this partial differential equation is at best difficult to solve. It remains an open problem to find an explicit solution to Equation (11). The expression in Equation (11) though can be used to gather information about the system.

■ 2.3 The Expected Value

The partial differential equation (11) yields useful information about our stochastic model. Using the fact that the expected value of a random variable can be obtained from the derivative of its pgf, we take the derivative in Equation (11) with respect to z_1 :

$$\begin{aligned}
 \partial_{z_1, t} \phi &= (\beta z_1 z_2 + \gamma - \gamma z_1 - \beta z_1) (\partial_{z_1, z_1} \phi) \\
 &\quad + (\beta z_2 - \gamma - \beta) (\partial_{z_1} \phi) \\
 &\quad + (\alpha z_1 z_2 + \delta - \alpha z_2 - \delta z_2) (\partial_{z_1, z_2} \phi) \\
 &\quad + (\alpha z_1) (\partial_{z_2} \phi) \\
 &\quad + \left(\rho z_1 + \frac{\kappa}{z_1} + \tau z_2 + \frac{\lambda}{z_2} - \rho - \kappa - \tau - \lambda \right) (\partial_{z_1} \phi) \\
 &\quad + (\rho - \kappa z_1^{-2}) \phi . \tag{12}
 \end{aligned}$$

Now, $\partial_{z_1, t} \phi = \partial_{t, z_1} \phi$ since ϕ is continuous where it exists. So letting $z_1=1$ and $z_2=1$ we obtain

$$\begin{aligned}
 \partial_{t, z_1} \phi (1, 1) &= \alpha (\partial_{z_2} \phi (1, 1)) \\
 &\quad - \gamma (\partial_{z_1} \phi (1, 1)) \\
 &\quad + (\rho - \kappa) \phi (1, 1) . \tag{13}
 \end{aligned}$$

From Equation (11) and the properties of a pgf we can write

$$\begin{aligned} \partial_t E[M(t)] &= \\ &\alpha E[N(t)] - \gamma E[M(t)] + (\rho - \kappa) . \end{aligned} \tag{14}$$

Equation (14) corresponds with the first differential equation in the original model (1) the change in country X's expected arms $\partial_t E[M(t)]$ is equal to α times the expected amount of arms of country Y minus γ times the expected amount of arms of country X plus the goodwill term $\rho-\kappa$ which we defined to be ζ .

Similarly, we can take the derivative in Equation (11) with respect to z_2 to obtain the equivalent of the second equation in the original model:

$$\begin{aligned} \partial_{z_2, t} \phi &= (\beta z_1 z_2 + \gamma - \gamma z_1 - \beta z_1) (\partial_{z_2, z_1} \phi) \\ &+ (\beta z_1) (\partial_{z_1} \phi) \\ &+ (\alpha z_1 z_2 + \delta - \alpha z_2 - \delta z_2) (\partial_{z_2, z_2} \phi) \\ &+ (\alpha z_1 - \alpha - \delta) (\partial_{z_2} \phi) \\ &+ \left(\rho z_1 + \frac{\kappa}{z_1} + \tau z_2 + \frac{\lambda}{z_2} - \rho - \kappa - \tau - \lambda \right) (\partial_{z_2} \phi) \\ &+ (\tau - \lambda z_2^{-2}) \phi . \end{aligned} \tag{15}$$

Again letting $z_1=1$ and $z_2=1$, we obtain

$$\begin{aligned}
\partial_{t, z_2} \phi (1, 1) &= \beta (\partial_{z_1} \phi (1, 1)) \\
&\quad - \delta (\partial_{z_2} \phi (1, 1)) \\
&\quad + (\tau - \lambda) (\phi (1, 1)) .
\end{aligned} \tag{16}$$

Using a property of a pgf and the continuity of ϕ , we may rewrite Equation (16) as

$$\partial_t E[N(t)] = \beta E[M(t)] - \delta E[N(t)] + (\tau - \lambda) . \tag{17}$$

As above, this equation agrees precisely with the second equation of the original model. The change in Y's expected armaments with respect to time $\partial_t E[N(t)]$ is equal to β times the expected amount of country X's armaments minus δ times the expected amount of country Y's armaments plus the goodwill term $\tau - \lambda$ which we defined to correspond to η .

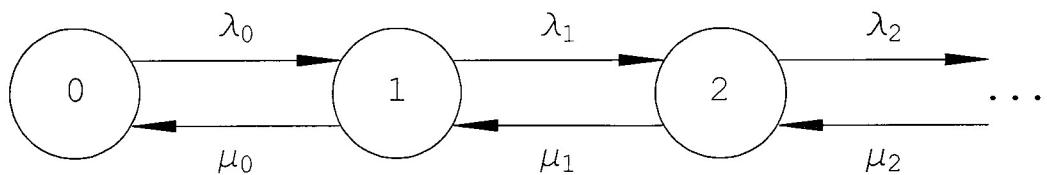
It should be clear that the model we have created in system (4) is indeed a stochastic version of the Richardson's arms race model. The expected value of our model corresponds to the deterministic orginal model. In the next chapter, we will present a numerical solution for the stochastic model presented in this chapter.

Chapter Three

Randomization--a Numerical Solution

■ 3.1 Introduction to Randomization

Randomization is a numerical method for finding an approximation of the transition probability function for a discrete valued, continuous time Markov chain. We shall use the method presented in Medhi [4] and Krinik [6] to approximate the transition probabilities for a simple birth-death process. A graphical representation of a birth-death process is shown in the following figure:



Similar to the derivation in the last chapter, the probability of being at a given state n at time $t+\Delta t$ would be

$$\begin{aligned}
P_n(t + \Delta t) &= \lambda_{n+1}(\Delta t) P_{n+1}(t) \\
&\quad + \mu_{n-1}(\Delta t) P_{n-1}(t) \\
&\quad + (1 - (\lambda_n + \mu_n)(\Delta t)) P_n(t).
\end{aligned} \tag{18}$$

So,

$$\begin{aligned}
\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} &= \lambda_{n+1} P_{n+1}(t) \\
&\quad + \mu_{n-1} P_{n-1}(t) \\
&\quad - (\lambda_n + \mu_n) P_n(t).
\end{aligned} \tag{19}$$

Taking the limit of both sides with $\Delta t \rightarrow 0$ yields

$$\begin{aligned}
P_n'(t) &= \lim_{\Delta t \rightarrow 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} \\
&= \lambda_{n+1} P_{n+1}(t) \\
&\quad + \mu_{n-1} P_{n-1}(t) \\
&\quad - (\lambda_n + \mu_n) P_n(t).
\end{aligned} \tag{20}$$

This gives us an infinite system of differential equations having one equation per state. We may write this system in the form of an infinite dimensional matrix. We will treat multiplication of an infinite dimensional matrix as we would an n-dimensional matrix.

$$P' (t) =$$

$$\begin{pmatrix} p_0'(t) & 0 & 0 & 0 & 0 & \dots \\ 0 & p_1'(t) & 0 & 0 & 0 & \dots \\ 0 & 0 & p_2'(t) & 0 & 0 & \dots \\ 0 & 0 & 0 & p_3'(t) & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}. \quad (21)$$

This matrix is equal to the product of two other matrices: the transition rate matrix

$$Q =$$

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -\lambda_1 - \mu_1 & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -\lambda_2 - \mu_2 & \lambda_2 & 0 & \dots \\ 0 & 0 & \mu_3 & -\lambda_3 - \mu_3 & \lambda_3 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (22)$$

and transition probability matrix

$$P(t) =$$

$$\begin{pmatrix} p_{0,0}(t) & p_{0,1}(t) & p_{0,2}(t) & p_{0,3}(t) & \dots \\ p_{1,0}(t) & p_{1,1}(t) & p_{1,2}(t) & p_{1,3}(t) & \dots \\ p_{2,0}(t) & p_{2,1}(t) & p_{2,2}(t) & p_{2,3}(t) & \dots \\ p_{3,0}(t) & p_{3,1}(t) & p_{3,2}(t) & p_{3,3}(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (23)$$

where $p_{i,j}(t)$ is the probability of moving from state i to state j at time t .

So,

$$P'(t) = Q P(t). \quad (24)$$

Thus the matrix solution of this differential equation is

$$P(t) = e^{tQ}. \quad (25)$$

The method of randomization replaces Q with a stochastic matrix P . To apply the method, we must also make a technical assumption. We assume that this transition rate matrix Q is uniformizable, *which means that the absolute values of the diagonal elements of Q have a finite maximum d .* In other words, Q has a dominant eigenvalue. Creating the matrix P is done by dividing each element of Q by the maximum d of the diagonal and adding to this new matrix an appropriately sized identity matrix I . So,

$$P = \left(\frac{1}{d} \right) Q + I. \quad (26)$$

Solving for Q , we obtain

$$Q = d(P - I). \quad (27)$$

Substituting into equation (25) above gives

$$P(t) = e^{d(P-I)t}. \quad (28)$$

With simple manipulation we have

$$P(t) = e^{-dt} e^{dtP}. \quad (29)$$

The exponential of a matrix may be defined as it often is for real numbers by its Taylor series. Thus, an approximation for the matrix $P(t)$ can be obtained from

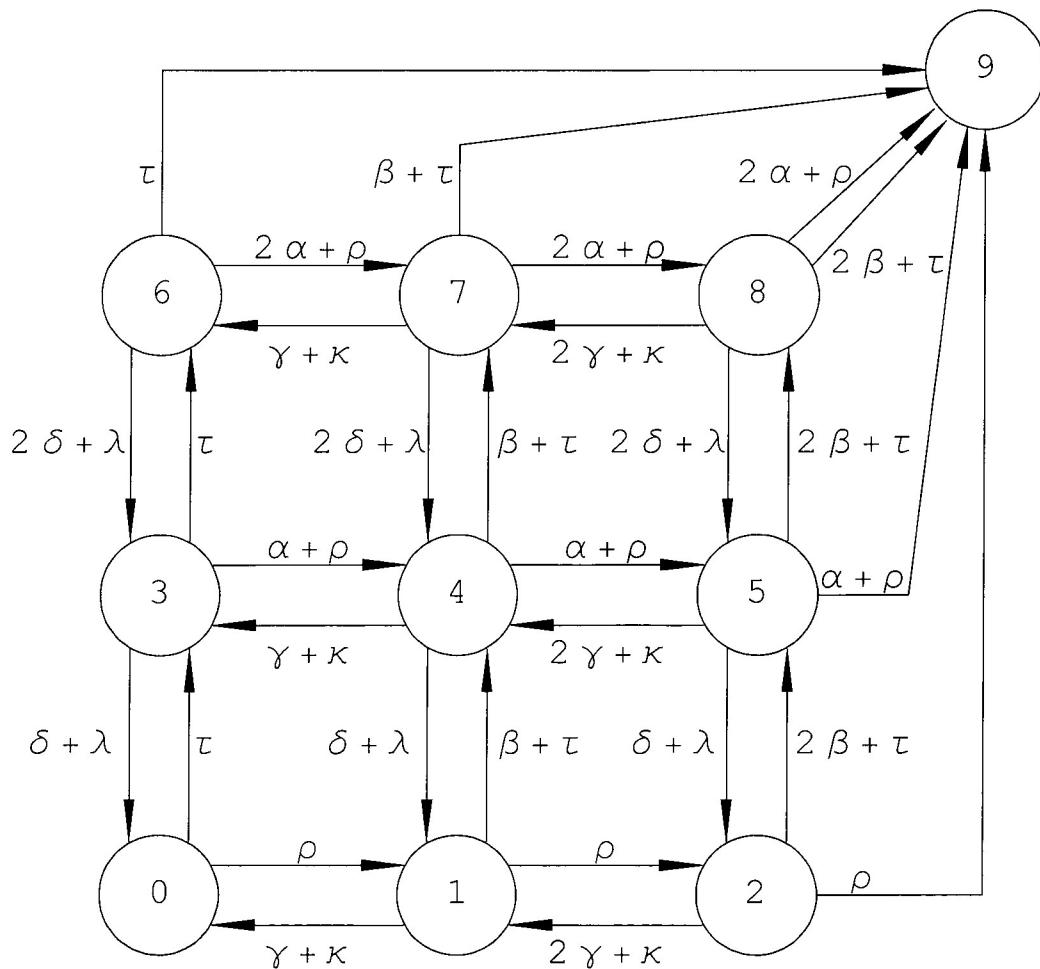
$$P(t) = e^{-dt} \sum_{n=0}^{\infty} \frac{(dt)^n}{n!} P^n. \quad (30)$$

We will of course have to limit the number of terms that we calculate in the Taylor series; we will therefore only have an approximation of $P(t)$. This power series expansion requires that d must be finite, which explains our earlier assumption of this fact.

We will also be unable to practically deal with an infinite matrix, so we may have only a finite number of states in the process. Using a large number of states may allow us to approximate an infinite system. Wherever we truncate the process, a single absorbing state is placed. The probability that we are in this state is a bound on the error for the other states due to truncating the process at any given time [3].

■ 3.2 Randomization for the Stochastic Richardson Model

How shall we apply this randomization method to the stochastic Richardson model? Our model differs from the situation presented in Section 3.1 in one very important respect. The process moves in not one but two directions. A graphical representation is given as



While the true situation of our model is that the number of states is infinite in both vertical and horizontal directions, the only hope we have to obtain a numerical solution will require artificially limiting the number of states. So, we will have one state that will accept all of the transitions to states that were truncated. Notice that the transitions are consistent with the possible movements of our model explained in Chapter 2.

Next, we need to "flatten" this two dimensional process into one dimension. This flattening is accomplished by numbering the states and then treating the states as if they were in one dimension.

If we take the convention of numbering the rows and columns starting with zero, then we find the row of a given state by performing integer division of the new numbering by the number of columns. Similarly, we find the column of a given state by finding the congruence modulo the number of columns . For example, state five is in row 1 since 5 divided by 3 is 1. State five is in column 2 since 5 modulo 3 is 2.

We are now in a position of being able to construct the Q matrix we

discussed in the last section. Once this construction is completed, we will use the randomization method outlined above.

■ 3.3 Numerical Comparison of the Original Richardson Model and the Stochastic Richardson Model

In this section we will apply the randomization method to obtain transient probabilities for the stochastic model. We will also compare these approximate probabilities with the deterministic model by proceeding through concrete examples using specific values for the parameters in the model. The examples are chosen specifically to correspond with the four general situations described by Olnick that were discussed in Chapter 1. They are a stabilized arms race, a runaway arms race, total disarmament, and the situation that yields disarmament or a runaway race depending on the starting point.

A few observations should be made about the implementation of the numerical method outlined in the last section. A lattice of size seven by seven is used to make estimations of the stochastic model. The result is a fifty by fifty matrix to be used in most calculations; a lattice of any larger size would take a great deal of computation time. Also, the Euler

method is implemented to obtain results for the deterministic model for comparison. Mathematica code that is used for both the randomization technique and the Euler method implemented are included in the appendices.

■ 3.3.1 The Stabilized Arms Race

We recall that a stabilized race results when $\gamma\delta-\alpha\beta$ is positive and ζ and η are positive. So, the parameters of the deterministic model are set as follows:

$$\alpha = 1 \quad \beta = 1$$

$$\gamma = 1.5 \quad \delta = 1.5$$

$$\zeta = 1 \quad \eta = 1.$$

These values for the deterministic model correspond to the following values in the stochastic model:

$$\alpha = 1 \quad \beta = 1$$

$$\gamma = 1.5 \quad \delta = 1.5$$

$$\lambda = 0 \quad \tau = 1$$

$$\kappa = 0 \quad \rho = 1.$$

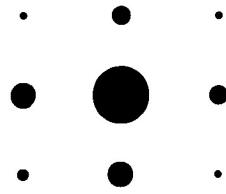
Also, we will assume a starting point at $t=0$ for countries X and Y to be at 3 weapons units each. (These are units of weapons not necessarily 3 weapons.) The results obtained are

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	2.95122	2.95122	2.95313	2.95313
0.2	2.90481	2.90481	2.9197	2.9197
0.5	2.77875	2.77875	2.89057	2.89057
0.8	2.67025	2.67025	2.90535	2.90535
1.0	2.60645	2.60645	2.92527	2.92527
1.5	2.47228	2.47228	2.99634	2.99634
2.0	2.36779	2.36779	3.08712	3.08712
2.5	2.28642	2.28642	3.18896	3.18896
3.0	2.22305	2.22305	3.29632	3.29632
3.5	2.1737	2.1737	3.40574	3.40574
4.0	2.13527	2.13527	3.51508	3.51508
4.5	2.10534	2.10534	3.62303	3.62303
5.0	2.08203	2.08203	3.72885	3.72885
5.5	2.06388	2.06388	3.83209	3.83209
6.0	2.04975	2.04975	3.93253	3.93253
6.5	2.03874	2.03874	4.03007	4.03007
7.0	2.03017	2.03017	4.12469	4.12469

7.5	2.0235	2.0235	4.21603	4.21603
8.0	2.0183	2.0183	4.29018	4.29018
8.5	2.01425	2.01425	4.19134	4.19134

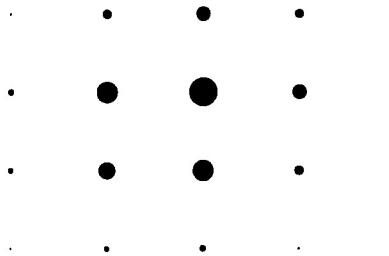
The expected values of the stochastic model do not seem to agree very well with the deterministic model. Up until $t=0.5$, the numbers seem to agree fairly well, but as the deterministic values seem to be getting closer to x and y being 2 units, the expected values of the stochastic models are inching up. The explanation is that the absorbing state holds in much of the probability that would normally go back into the system. A graphical representation of the probabilities makes this point clear. The absorbing state is to the right of the lattice, which starts at $(0,0)$ and goes up to $(6,6)$. The size of the dots reflects the magnitude of the probabilities.

For $t=0.1$,

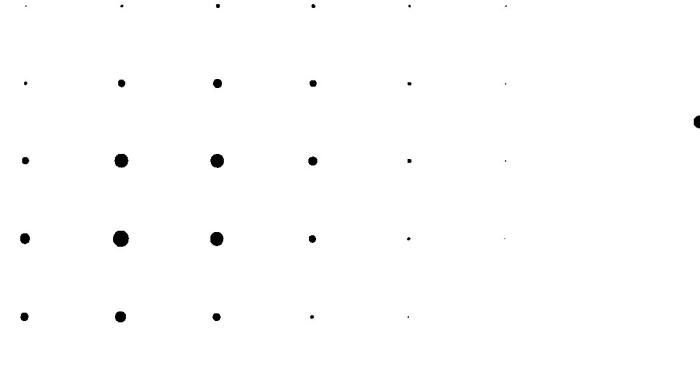


We note that the dots below and to the left of the starting point are larger than those that are above and to the right indicating the slight downward trend.

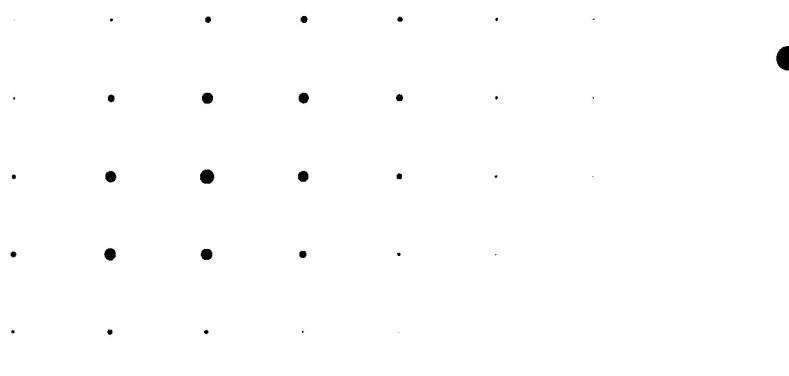
At $t=0.2$:



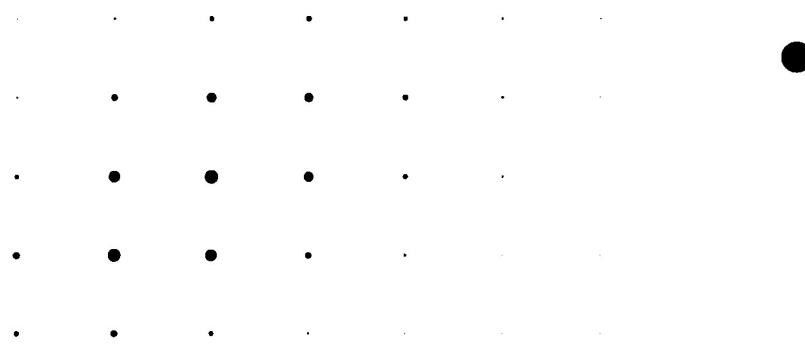
At $t=0.5$ the trend is going towards $(2,2)$, but the absorbing state is larger.



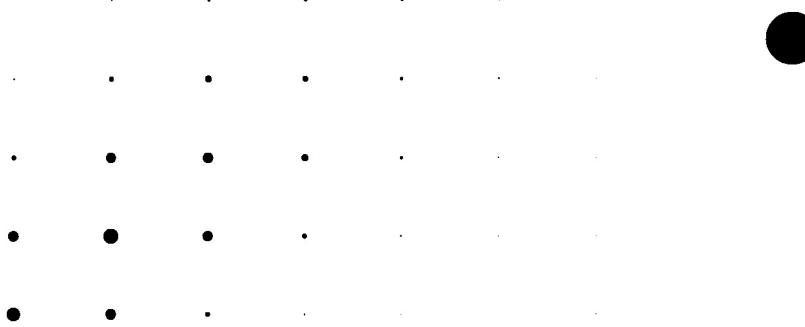
At $t=0.8$, the pattern is similar, but the absorbing state continues to grow.



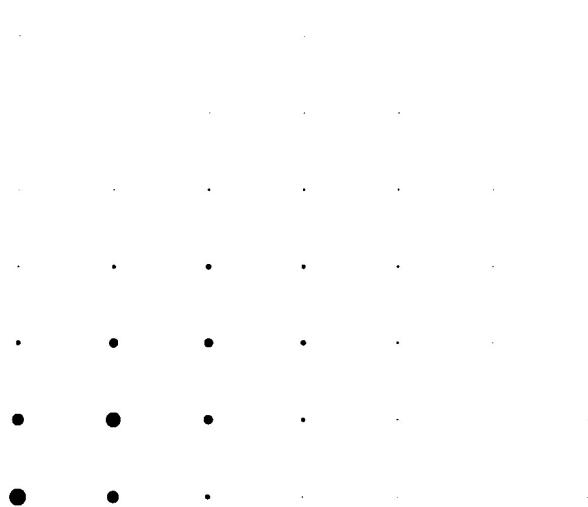
At $t=1$, most of the probability is very near $(2,2)$, but the absorbing state is much larger.



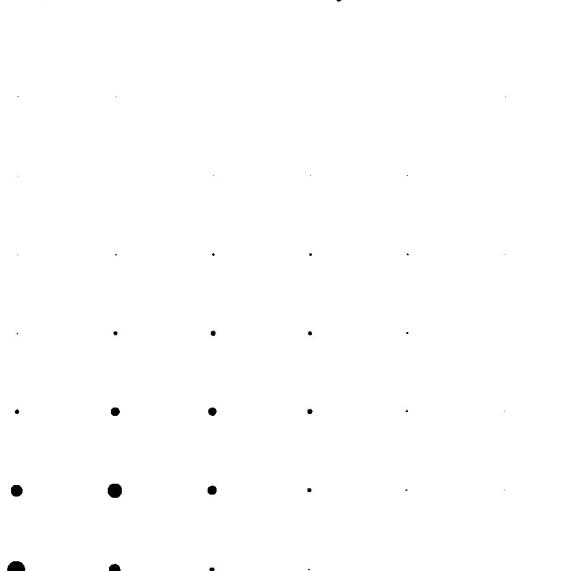
At $t=2$, things are still similar to above with a still larger absorbing state.



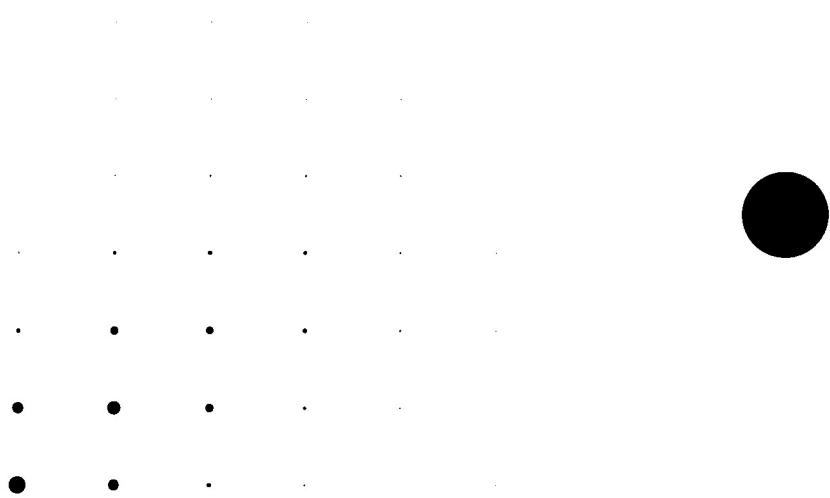
At $t=3$, things are still similar to above with a yet larger absorbing state.



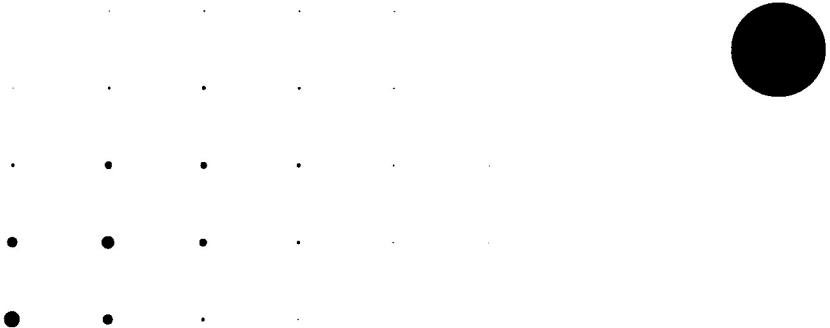
At t=4, the situation is very near to that at t=3.



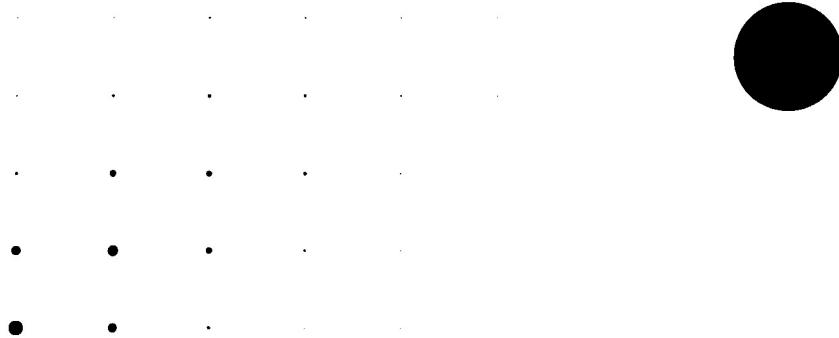
At t=5, the situation is nearly identical to that at t=4. This is important since it implies a more stable situation.



At $t=6$, the situation is still almost identical, but that absorbing state of course continues to grow. There is still a slight chance of getting away from the bottom left corner and eventually ending up in the absorbing state.



Finally looking at $t=8$, we see that the situation is very similar, but again we have a slightly larger absorbing state. In order to obtain good results, this absorbing must ideally be kept at bay by increasing the lattice size. As mentioned above however, doing so results in the problem of computing resources.



■ 3.3.2 The Runaway Arms Race

The next case that we will consider is the runaway arms race. This case occurs when in the deterministic model $\gamma\delta-\alpha\beta$ is negative and ζ and η are positive. The parameters of the deterministic model are set as

$$\alpha = 1 \quad \beta = 1$$

$$\gamma = 0.5 \quad \delta = 0.5$$

$$\zeta = 1 \quad \eta = 1 .$$

This corresponds to the following values in the stochastic model:

$$\alpha = 1 \quad \beta = 1$$

$$\gamma = 0.5 \quad \delta = 0.5$$

$$\lambda = 0 \quad \tau = 1$$

$$\kappa = 0 \quad \rho = 1 .$$

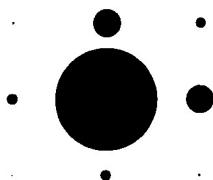
We will also take as a starting point 2 units of weapons each for countries X and Y. The following results were obtained:

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	2.20503	2.20503	2.20517	2.20517
0.2	2.42057	2.42057	2.4226	2.4226
0.5	3.13578	3.13578	3.18863	3.18863
0.8	3.9667	3.9667	4.06576	4.06576
1.0	4.59406	4.59406	4.6159	4.6159
1.5	6.46641	6.46641	5.66139	5.66139

Computations were discontinued at this point, since as would be expected, it is more likely that as time goes on the amounts of arms

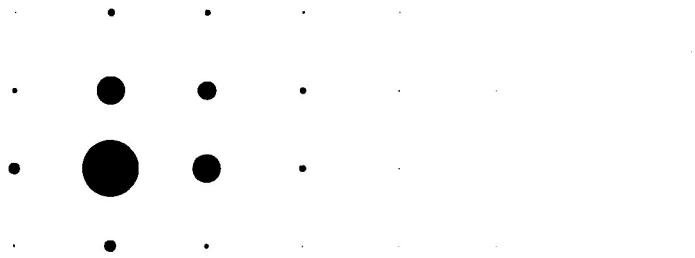
become larger for both countries, and our estimations are limited to seven. However, we note how close the numbers are for times of one and less. This process can be seen more clearly by diagrams of the lattice where the size of a location indicates the probability that the system is in this state at the given time. The general absorbing state is alone on the right.

At t=0.1:

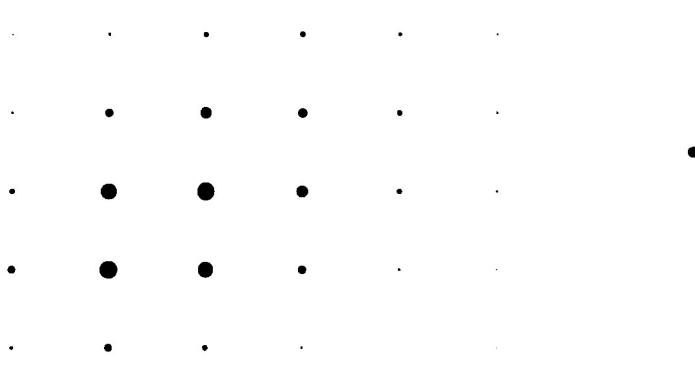


In such a short time, it is very unlikely that the system has moved from its original location.

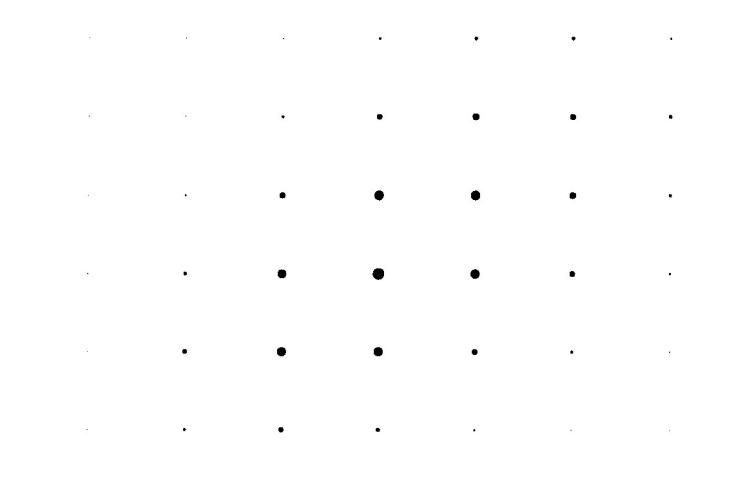
At t=0.2, things are a bit more dispersed mostly going up.



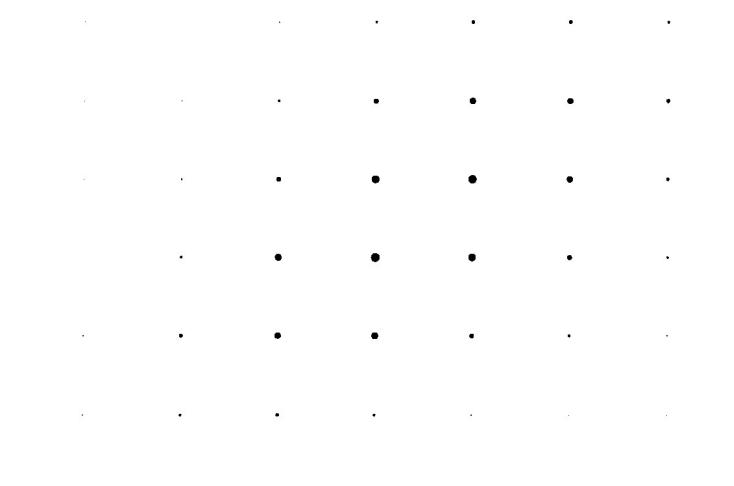
At $t=0.5$:



At $t=1$, we note how large the absorbing state has become, meaning that the probability of being larger than seven is quite high at this point.



At $t=1.5$, it is almost guaranteed that in both the x and y directions that both are greater than seven.



We could enlarge the lattice to obtain better results at this time interval, but as was mentioned before, computation times increase rapidly. Also, we know that the expected value of the stochastic model

agrees with the deterministic model. Since these values for the parameters will yield a runaway arms race in the deterministic model, the situation above will continue to happen even if the size of the lattice were increased, though it would take more time. The values for x and y would eventually become larger than the arbitrary size of the lattice.

■ 3.3.3 Total Disarmament

The third case which we will be observing is the one that yields total disarmament. This situation occurs when in the deterministic model $\gamma\delta-\alpha\beta$ is positive and ζ and η are negative. The parameters of the deterministic model are set as

$$\alpha = 0.5 \quad \beta = 0.5$$

$$\gamma = 1 \quad \delta = 1$$

$$\zeta = -1 \quad \eta = -1 .$$

This corresponds to the following parameter settings for the stochastic model:

$$\alpha = 0.5 \quad \beta = 0.5$$

$$\gamma = 1 \quad \delta = 1$$

$$\lambda = 1 \quad \tau = 0$$

$$\kappa = 1 \quad \rho = 0$$

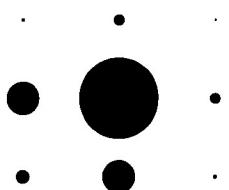
Starting at x=3, y=3 yields these results.

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	2.75609	2.75609	2.75627	2.75627
0.2	2.52407	2.52407	2.52555	2.52555
0.5	1.89376	1.89376	1.91664	1.91664
0.8	1.35126	1.35126	1.43044	1.43044
1.0	1.03227	1.03227	1.16837	1.16837
1.5	0.36139	0.36139	0.693114	0.693114
2.0	-0.161063	-0.161063	0.409251	0.409251
2.5	-0.567924	-0.567924	0.247015	0.247015
3.0	-0.884768	-0.884768	0.157059	0.157059
3.5	-1.13151	-1.13151	0.108198	0.108198
4.0	-1.32366	-1.32366	0.0820276	.0820276
4.5	-1.4733	-1.4733	0.0681449	.0681449
5.0	-1.58983	-1.58983	0.0608295	.0608295
5.5	-1.68058	-1.68058	0.0569925	.0569925
6.0	-1.75125	-1.75125	0.0549865	.0549865
6.5	-1.80629	-1.80629	0.0539401	.0539401
7.0	-1.84915	-1.84915	0.0533952	.0533952
7.5	-1.88252	-1.88252	0.0531118	.0531118

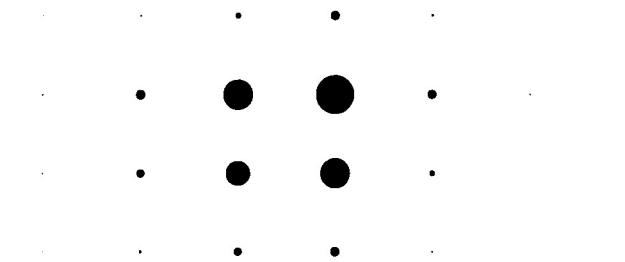
8.0	-1.90851	-1.90851	0.0529645	.0529645
8.5	-1.92875	-1.92875	0.0528879	.0528879

Perusing these results should quickly show an inconsistency; the numbers for the deterministic model become negative. The probability of ever being negative for the stochastic model is zero, since this is how we modeled the problem. However the numbers remain close until the deterministic numbers get close to zero. Also, the behavior is practically identical--that is, while the deterministic model shows disarmament, the expected values of the stochastic model show that this scenario is still the most likely one. The diagrams make this point even clearer.

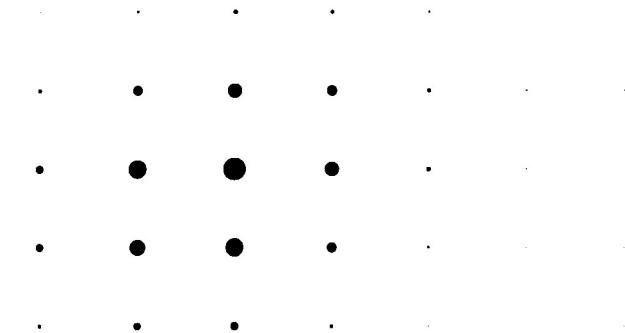
At t=0.1:



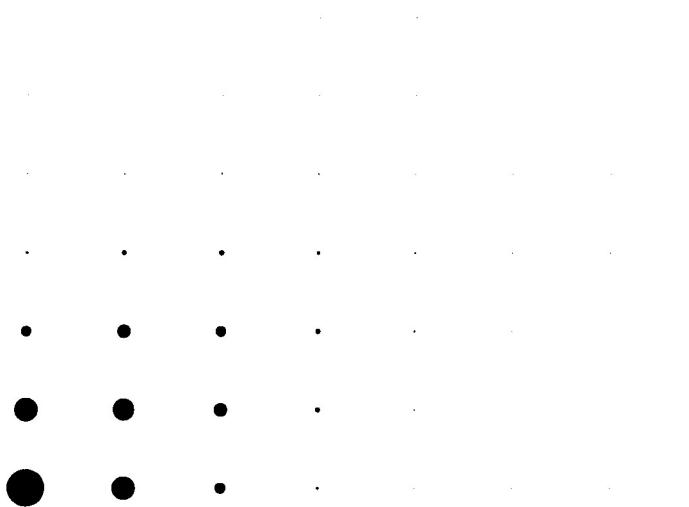
At $t=0.2$, we notice the almost complete downward trend.



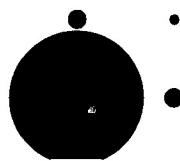
At $t=0.5$:



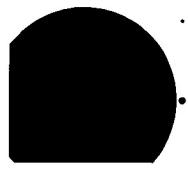
At $t=1$:



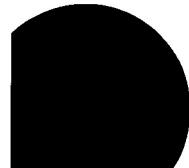
At $t=2$, we see that it is highly likely the model is at $(0,0)$.



At $t=3$, notice we are even more likely to be at $(0,0)$.



At t=8, we see that this behavior continues to be the trend.



■ 3.3.4 Total Disarmament or Runaway Arms Race

This situation is quite different than the others. The end result of the deterministic model depends on the initial conditions, the amount of arms with which countries X and Y start. This case occurs when $\gamma\delta-\alpha\beta$ is negative and ζ and η are negative. We will set the parameters of the deterministic model to

$$\alpha = 1 \quad \beta = 1$$

$$\gamma = 0.5 \quad \delta = 0.5$$

$$\zeta = -1.5 \quad \eta = -1.5 .$$

The corresponding stochastic parameters are

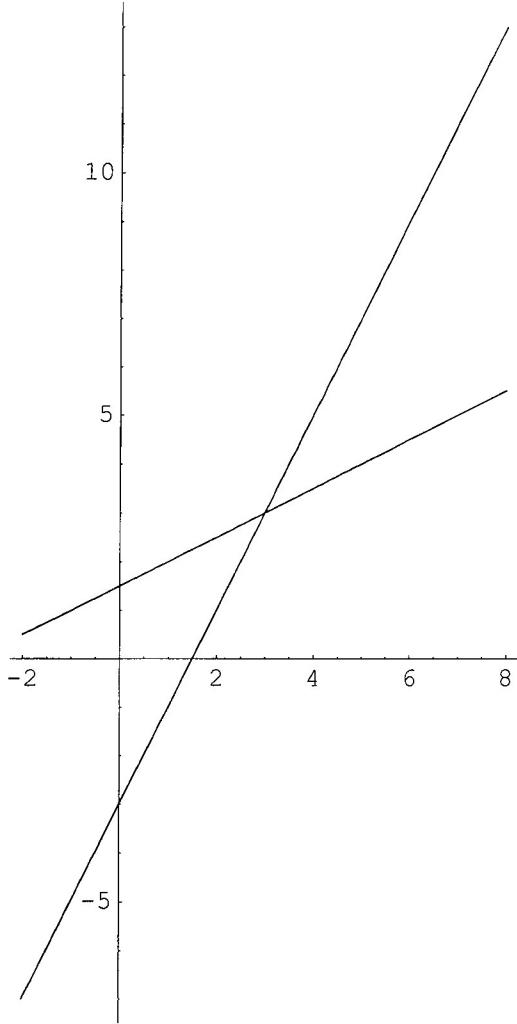
$$\alpha=1 \quad \beta=1$$

$$\gamma=0.5 \quad \delta=0.5$$

$$\lambda=1.5 \quad \tau=0$$

$$\kappa=1.5 \quad \rho=0 .$$

Let us look more closely at the deterministic case before comparing it to the stochastic version. Four regions are created by the two lines formed when each of the differential equations in the system is set equal to zero, i.e., the *isoclines* of the system of differential equations.



If the initial point is in the section lower and to the left of the intersection of the two lines, then total disarmament will occur. If the initial point is in the section above and to the right of the intersection, then there will be a runaway arms race. If the initial point is in either of the other sections, the model will eventually end up in one of the first two sections described depending on where in these sections things begin, so that there is disarmament or a runaway race. So, we are left

with three subcases to be examined.

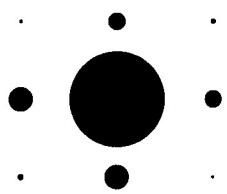
First, we start in the lower, left section. So, country X and country Y each have two units of weapons. The results are

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	1.94874	1.94874	1.94973	1.94973
0.2	1.89486	1.89486	1.90151	1.90151
0.5	1.71605	1.71605	1.78591	1.78591
0.8	1.50832	1.50832	1.71498	1.71498
1.0	1.35148	1.35148	1.68454	1.68454
1.5	0.883397	0.883397	1.63867	1.63867
2.0	0.282397	0.282397	1.61339	1.61339
2.5	-0.489253	-0.489253	1.59756	1.59756
3.0	-1.48001	-1.48001	1.58699	1.58699
3.5	-2.75209	-2.75209	1.57973	1.57973
4.0	-4.38536	-4.38536	1.57466	1.57466
4.5	-6.4824	-6.4824	1.57107	1.57107
5.0	-9.17488	-9.17488	1.56853	1.56853
5.5	-12.6319	-12.6319	1.56672	1.56672
6.0	-17.0705	-17.0705	1.56543	1.56543
6.5	-22.7694	-22.7694	1.56451	1.56451
7.0	-30.0865	-30.0865	1.56385	1.56385
7.5	-39.4813	-39.4813	1.56338	1.56338

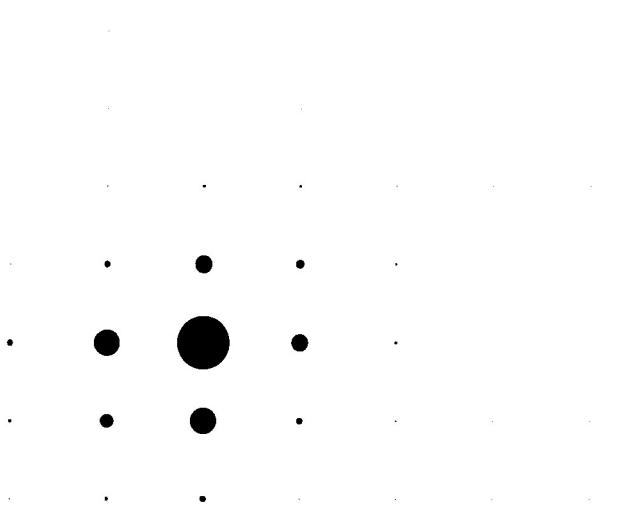
8.0	-51.5436	-51.5436	1.56304	1.56304
8.5	-67.031	-67.031	1.5628	1.5628

Notice that the behavior of the numbers is very similar to the straight total disarmament case in Section 3.3.3. However, the expected values of the stochastic model do not get as close to zero as before. More of the probability has gotten "stuck" in the absorbing state. However, this case seems that disarmament is the most likely outcome. This outcome becomes clearer by considering the graphical representations.

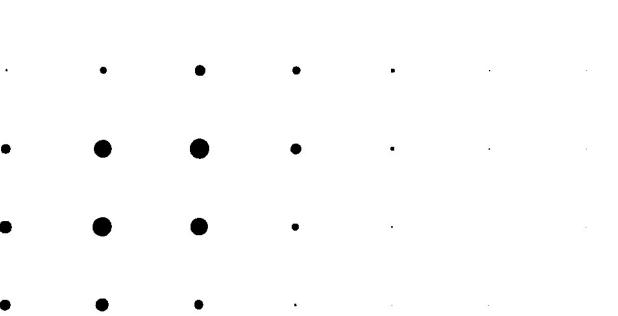
At t=0.1:



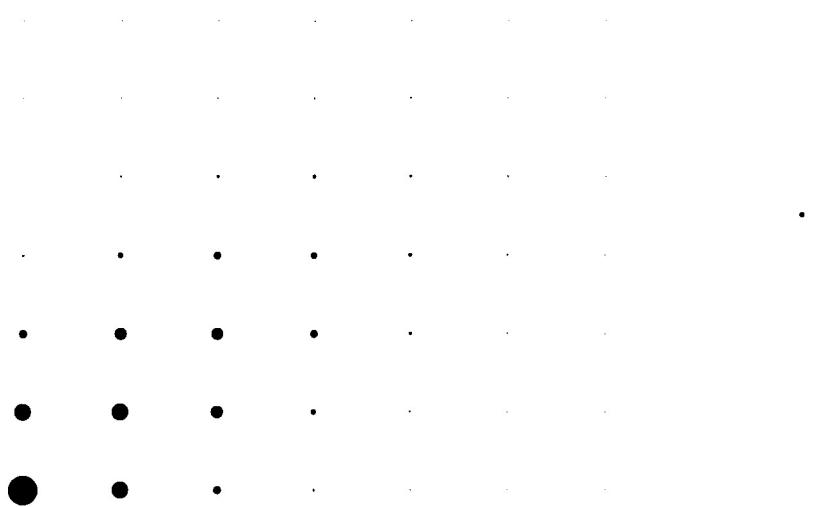
At t=0.2:



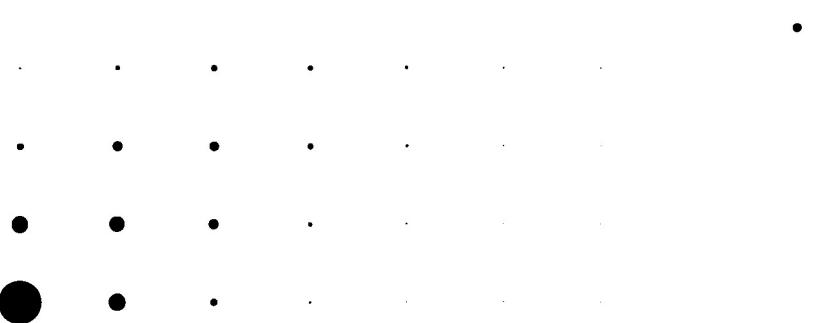
At $t=0.5$:



At $t=0.8$, the more likely outcome is becoming clearer.



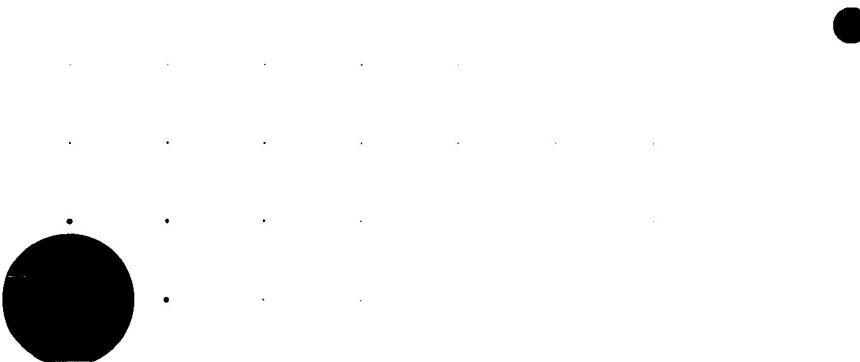
At $t=1$:



At $t=2$:



At t=3:



Finally at $t=8$, there is very little change from the situation at $t=3.0$, but the absorbing state is fairly substantial, which may indicate that there is a possibility, though small, of having a runaway arms race.

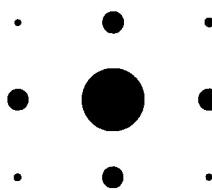
The second subcase occurs when the initial condition is above and to the right of the intersection. So, we will take as a starting point (4,4). With this the following results are obtained:

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	4.05126	4.05126	4.06796	4.06796
0.2	4.10514	4.10514	4.17805	4.17805
0.5	4.28395	4.28395	4.52421	4.52421
0.8	4.49168	4.49168	4.74684	4.74684
1.0	4.64852	4.64852	4.84002	4.84002
1.5	5.1166	5.1166	4.96612	4.96612
2.0	5.7176	5.7176	5.01622	5.01622
2.5	6.48925	6.48925	5.03497	5.03497
3.0	7.48001	7.48001	5.04078	5.04078

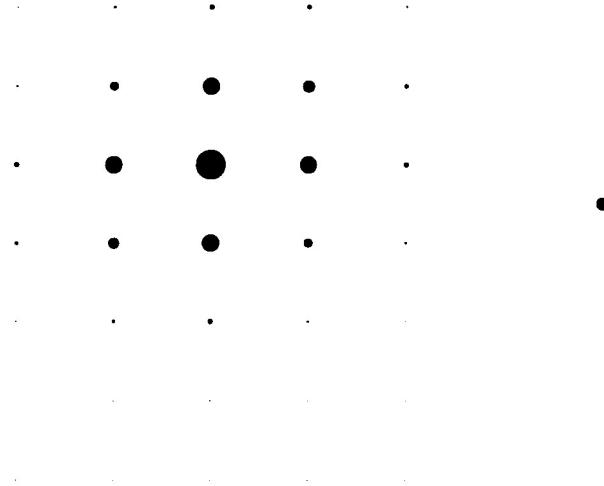
3.5	8.75209	8.75209	5.04148	5.04148
4.0	10.3854	10.3854	5.04038	5.04038
4.5	12.4824	12.4824	5.03885	5.03885
5.0	15.1749	15.1749	5.03741	5.03741
5.5	18.6319	18.6319	5.03622	5.03622

The numbers remain reasonably close until the deterministic model goes well out of range from the stochastic model between $t=1.5$ and $t=2$. We may be able tell more from the graphs.

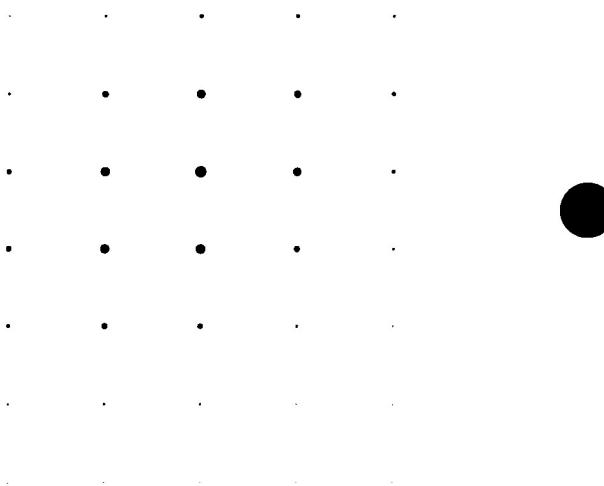
At $t=0.1$,



At $t=0.2$,



At $t=0.5$, we see that the absorbing state has already become large.



At $t=1$, the probability of being in the absorbing state has grown drastically.

At t=2, the probability of disarmament has grown.

At t=4, the probability of disarmament has grown a bit more.



At $t=6$, the probability of disarmament has grown only slightly.



So, the most likely result from this subcase is a runaway arms race, just like in the deterministic case. However, there is a substantial chance that disarmament will occur.

We are now to the final subcase in this section. Our starting point

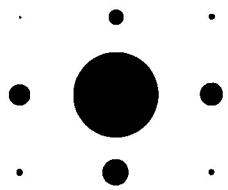
will need to be either above and to the left or below and to the right. Since our parameters are symmetric, two examples from one of these sections should be sufficient to show both situations. For the first example, we will take (2,3) as the initial values; this point lies in the area above and to the left.

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	2.04407	2.90468	2.04517	2.90486
0.2	2.0771	2.81775	2.08534	2.81995
0.5	2.12198	2.59408	2.20308	2.63792
0.8	2.1037	2.40462	2.30336	2.54663
1.0	2.0643	2.28718	2.35183	2.51423
1.5	1.88909	1.99431	2.41819	2.47534
2.0	1.61636	1.66604	2.43905	2.45853
2.5	1.24365	1.2671	2.44168	2.44817
3.0	0.75446	0.765531	2.43833	2.44046
3.5	0.121343	0.12657	2.43375	2.43375
4.0	-0.693916	-0.691448	2.42956	2.42978
4.5	-1.74178	-1.74062	2.42616	2.42623
5.0	-3.08772	-3.08717	2.42355	2.42357
5.5	-4.81607	-4.81581	2.42161	2.42161
6.0	-7.0353	-7.03518	2.42018	2.42018
6.5	-9.88473	-9.88467	2.41914	2.41915

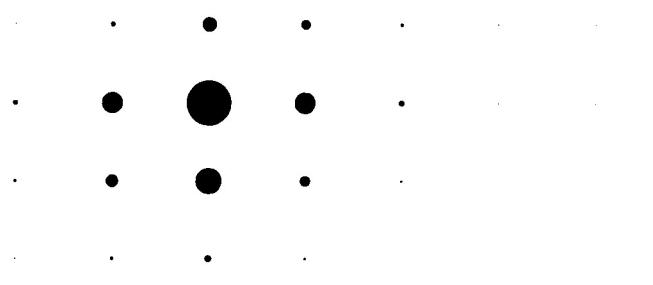
7.0	-13.5433	-13.5432	2.4184	2.4184
7.5	-18.2406	-18.2406	2.41785	2.41785
8.0	-24.2718	-24.2718	2.41747	2.41747
8.5	-32.0155	-32.0155	2.41719	2.41719

We see in the deterministic case that from this initial point disarmament occurs. As we have seen in previous cases, the numbers agree rather closely until the deterministic model approaches zero. Also, notice that the expected value in the stochastic case remains fairly high. As we will see, the reason for this situation is a fair amount of accumulated probability that gets stuck in the absorbing state.

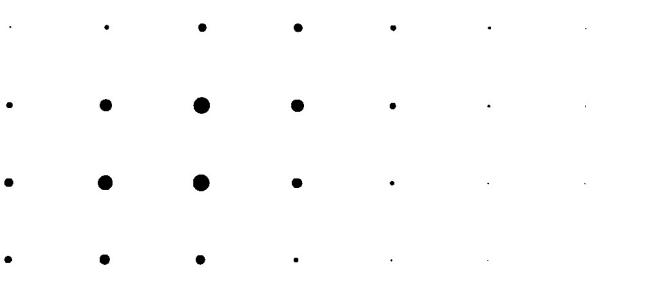
At t=0.1, the situation is



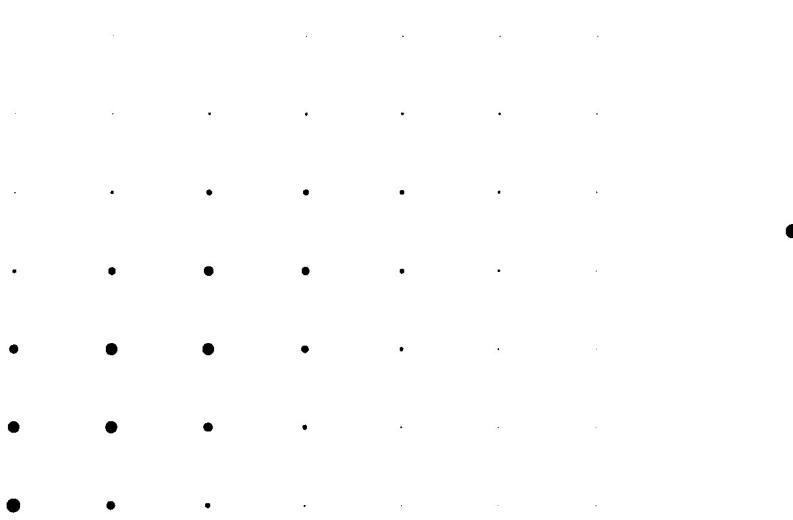
At t=0.2:



At $t=0.5$,



At $t=0.8$, the absorbing state begins to increase in size.



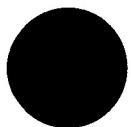
At t=1, the (0,0) disarmament state increases in size.



At t=2, the trend is to disarmament with a substantial chance of being in the absorbing state.



At t=4, the trend remains the same but to a greater extent.



At t=8, we see little has changed.

So, the unusually high expected values incommensurate with the deterministic model may be attributed to the substantial amount of probability in the absorbing state.

The second example in this subcase has a starting point of (3,4).

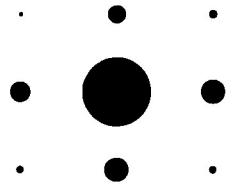
The results obtained are as follows:

t	Deterministic		Stochastic	
	X	Y	E[X]	E[Y]
0.1	3.09532	3.95593	3.10603	3.95775
0.2	3.18225	3.9229	3.23441	3.23441
0.5	3.40592	3.87802	3.62242	3.98921
0.8	3.59538	3.8963	3.88642	4.07822
1.0	3.71282	3.9357	4.00015	4.12604
1.5	4.00569	4.11091	4.15495	4.19942
2.0	4.33396	4.38364	4.21502	4.23057

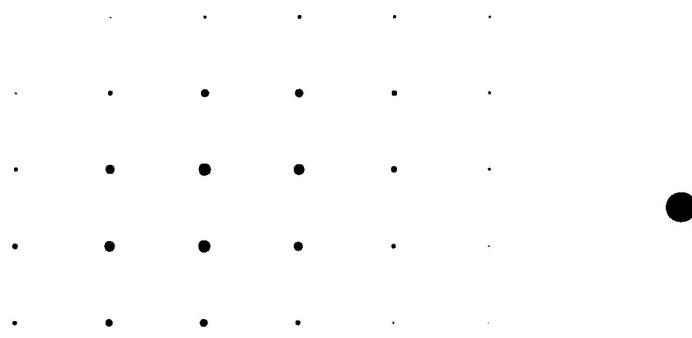
2.5	4.38364	4.75635	4.23626	4.24159
3.0	5.23447	5.24554	4.24202	4.24381
3.5	5.87343	5.87866	4.242	4.24259
4.0	6.69145	6.69392	4.24014	4.24033
4.5	7.74062	7.74178	4.23799	4.23805
5.0	9.08717	9.08772	4.23607	4.23609
5.5	10.8158	10.8161	4.23453	4.23453
6.0	13.0352	13.0353	4.23334	4.23334
6.5	15.8847	15.8847	4.23246	4.23246
7.0	19.5432	19.5433	4.23181	4.23181
7.5	24.2406	24.2406	4.23133	4.23133
8.0	30.2718	30.2718	4.23099	4.23099
8.5	38.0155	38.0155	4.23074	4.23074

The deterministic model escapes the restrictions of the lattice fairly quickly, and the result is a runaway arms race. Similar to previous cases, the expected values of the stochastic model agree until the deterministic model begins to escape the restrictions of 6 weapons units for either country. We will see a reverse case from the last example in that most of the probability will end up in the absorbing state, while there is still a chance of complete disarmament.

At t=0.1,



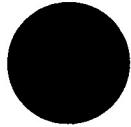
By $t=0.5$, a substantial amount of probability is already in the absorbing state.



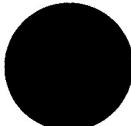
By $t=1$, an even larger portion has gone to the absorbing state.

By $t=2$, we see that the $(0,0)$ disarmament has substantially increased.

This situation remains fairly stable through $t=4$.



At $t=8$, we see more clearly that this seems to indeed be a stable situation.



So, in this second example we most likely obtain a runaway arms race, but there is a substantial possibility of disarmament.

■ 3.4 Conclusion

We have seen through the above examples that randomization yields promising results while the corresponding deterministic model is within the bounds of the lattice. When the corresponding deterministic model is inside the bounds of the model, the behavior of the stochastic model seems to be similar to the deterministic. When the deterministic model yields a runaway arms race or disarmament, the stochastic model dictates that the indicated outcome is more likely.

There are many open questions that remain. One problem is to obtain a bound on the error in approximating the expected value of the stochastic model. Another would be further experimentation with larger lattices, which might be possible with the method applied on more powerful computing systems. Also, while some efforts were made to optimize the computer code, more could be done on this front.

Though we have applied the randomization technique to the stochastic Richardson's model, the technique could be readily applied to

a variety of birth-death processes. The computer code is very flexible and could be easily adapted for other processes.

Appendix A

This appendix contains an explicit expression for the Q matrix on a 3X3 grid.

Q = q_{i,j}, such that

$$q_{1,1} = -\rho - \tau$$

$$q_{1,2} = \rho$$

$$q_{1,3} = 0$$

$$q_{1,4} = \tau$$

$$q_{1,5} = 0$$

$$q_{1,6} = 0$$

$$q_{1,7} = 0$$

$$q_{1,8} = 0$$

$$q_{1,9} = 0$$

$$q_{1,10} = 0$$

$$q_{2,1} = \gamma + \kappa$$

$$q_{2,2} = -\beta - \gamma - \kappa - \rho - \tau$$

$$q_{2,3} = \rho$$

$$\mathbf{q}_{2,4} = 0$$

$$\mathbf{q}_{2,5} = \beta + \tau$$

$$\mathbf{q}_{2,6} = 0$$

$$\mathbf{q}_{2,7} = 0$$

$$\mathbf{q}_{2,8} = 0$$

$$\mathbf{q}_{2,9} = 0$$

$$\mathbf{q}_{2,10} = 0$$

$$\mathbf{q}_{3,1} = 0$$

$$\mathbf{q}_{3,2} = 2\gamma + \kappa$$

$$\mathbf{q}_{3,3} = -2\beta - 2\gamma - \kappa - \rho - \tau$$

$$\mathbf{q}_{3,4} = 0$$

$$\mathbf{q}_{3,5} = 0$$

$$\mathbf{q}_{3,6} = 2\beta + \tau$$

$$\mathbf{q}_{3,7} = 0$$

$$\mathbf{q}_{3,8} = 0$$

$$\mathbf{q}_{3,9} = 0$$

$$\mathbf{q}_{3,10} = \rho$$

$$\mathbf{q}_{4,1} = \delta + \lambda$$

$$\mathbf{q_{4,2}} = 0$$

$$\mathbf{q_{4,3}} = 0$$

$$\mathbf{q_{4,4}} = -\alpha - \delta - \lambda - \rho - \tau$$

$$\mathbf{q_{4,5}} = \alpha + \rho$$

$$\mathbf{q_{4,6}} = 0$$

$$\mathbf{q_{4,7}} = \tau$$

$$\mathbf{q_{4,8}} = 0$$

$$\mathbf{q_{4,9}} = 0$$

$$\mathbf{q_{4,10}} = 0$$

$$\mathbf{q_{5,1}} = 0$$

$$\mathbf{q_{5,2}} = \delta + \lambda$$

$$\mathbf{q_{5,3}} = 0$$

$$\mathbf{q_{5,4}} = \gamma + \kappa$$

$$\mathbf{q_{5,5}} = -\alpha - \beta - \gamma - \delta - \kappa - \lambda - \rho - \tau$$

$$\mathbf{q_{5,6}} = \alpha + \rho$$

$$\mathbf{q_{5,7}} = 0$$

$$\mathbf{q_{5,8}} = \beta + \tau$$

$$\mathbf{q_{5,9}} = 0$$

$$\mathbf{q}_{5,10} = 0$$

$$\mathbf{q}_{6,1} = 0$$

$$\mathbf{q}_{6,2} = 0$$

$$\mathbf{q}_{6,3} = \delta + \lambda$$

$$\mathbf{q}_{6,4} = 0$$

$$\mathbf{q}_{6,5} = 2\gamma + \kappa$$

$$\mathbf{q}_{6,6} = -\alpha - 2\beta - 2\gamma - \delta - \kappa - \lambda - \rho - \tau$$

$$\mathbf{q}_{6,7} = 0$$

$$\mathbf{q}_{6,8} = 0$$

$$\mathbf{q}_{6,9} = 2\beta + \tau$$

$$\mathbf{q}_{6,10} = \alpha + \rho$$

$$\mathbf{q}_{7,1} = 0$$

$$\mathbf{q}_{7,2} = 0$$

$$\mathbf{q}_{7,3} = 0$$

$$\mathbf{q}_{7,4} = 2\delta + \lambda$$

$$\mathbf{q}_{7,5} = 0$$

$$\mathbf{q}_{7,6} = 0$$

$$\mathbf{q}_{7,7} = -2\alpha - 2\delta - \lambda - \rho - \tau$$

$$\mathbf{q}_{7,8}=2\,\alpha+\rho$$

$$\mathbf{q}_{7,9}=0$$

$$\mathbf{q}_{7,10}=\tau$$

$$\mathbf{q}_{8,1}=0$$

$$\mathbf{q}_{8,2}=0$$

$$\mathbf{q}_{8,3}=0$$

$$\mathbf{q}_{8,4}=0$$

$$\mathbf{q}_{8,5}=2\,\delta+\lambda$$

$$\mathbf{q}_{8,6}=0$$

$$\mathbf{q}_{8,7}=\gamma+\kappa$$

$$\mathbf{q}_{8,8}=-2\,\alpha-\beta-\gamma-2\,\delta-\kappa-\lambda-\rho-\tau$$

$$\mathbf{q}_{8,9}=2\,\alpha+\rho$$

$$\mathbf{q}_{8,10}=\beta+\tau$$

$$\mathbf{q}_{9,1}=0$$

$$\mathbf{q}_{9,2}=0$$

$$\mathbf{q}_{9,3}=0$$

$$\mathbf{q}_{9,4}=0$$

$$\mathbf{q}_{9,5}=0$$

$$\mathbf{q}_{9,6} = 2\delta + \lambda$$

$$\mathbf{q}_{9,7} = 0$$

$$\mathbf{q}_{9,8} = 2\gamma + \kappa$$

$$\mathbf{q}_{9,9} = -2\alpha - 2\beta - 2\gamma - 2\delta - \kappa - \lambda - \rho - \tau$$

$$\mathbf{q}_{9,10} = 2\alpha + 2\beta + \rho + \tau$$

$$\mathbf{q}_{10,1} = 0$$

$$\mathbf{q}_{10,2} = 0$$

$$\mathbf{q}_{9,3} = 0$$

$$\mathbf{q}_{10,4} = 0$$

$$\mathbf{q}_{10,5} = 0$$

$$\mathbf{q}_{10,6} = 0$$

$$\mathbf{q}_{10,7} = 0$$

$$\mathbf{q}_{10,8} = 0$$

$$\mathbf{q}_{10,9} = 0$$

$$\mathbf{q}_{10,10} = 0$$

Appendix B

This is *Mathematica* code that accomplishes a number of things. First, the Q-matrix described in chapter 3 is built. The randomization method is then applied to this matrix. The expected values for the horizontal and vertical directions can be obtained given a starting point and a time. Last, the code can generate the graphs from Chapter 3 to visualize the probability of being in a state at a given time.

```
q = 7;

Clear[\alpha, \beta, \gamma, \delta, \lambda, \tau, \kappa, \rho];

(* Initialize parameters *)
\alpha = 1;
\beta = 1;
\gamma = 1;
\delta = 1;
\lambda = 0;
\tau = 1;
\kappa = 0;
\rho = 1;

(* Set the number of terms for the power series *)
taylorterms = 300;

h[m_, n_] := \delta n + \lambda; (* loss of n-axis *)
k[m_, n_] := \beta m + \tau; (* gain of n-axis *)
g[m_, n_] := \gamma m + \kappa; (* loss of m-axis *)
p[m_, n_] := \alpha n + \rho; (* gain of m-axis *)
```

```

f[i_, j_] := Which[

(* very last row of matrix-- nothing
   ever comes out-- the master absorbing state*)
(i == q*q),
0,

(* top left corner of matrix
-- (0,0) state of the lattice*)

(i == 0),

Which[j - i == q, k[Mod[i, q], IntegerPart[i/q]],
j - i == 1, p[Mod[i, q], IntegerPart[i/q]],
j - i == 0, -(p[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]]),
True, 0],

(*bottom right corner of matrix--
 last state in lattice-- top row, last state *)
(i == q*q - 1),

Which[j - i == -q, h[Mod[i, q], IntegerPart[i/q]],
j - i == -1, g[Mod[i, q], IntegerPart[i/q]],
j - i == 0, -(h[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]] +
g[Mod[i, q], IntegerPart[i/q]] +
p[Mod[i, q], IntegerPart[i/q]]),
j == q*q, (p[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]]),
True, 0],

(* bottom row of top left n square of matrix--
 last state in the bottom row of lattice*)
(i == q - 1),

```

```

Which[j - i == -1, g[Mod[i, q], IntegerPart[i/q]],

j - i == 0, -(p[Mod[i, q], IntegerPart[i/q]] + k[Mod[i, q], IntegerPart[i/q]] + g[Mod[i, q], IntegerPart[i/q]]), j - i == q, k[Mod[i, q], IntegerPart[i/q]], j == q*q, p[Mod[i, q], IntegerPart[i/q]], True, 0],

(*bottom q block of matrix--
first state of top row in lattice*)
i == q*(q-1),


Which[j - i == -q, h[Mod[i, q], IntegerPart[i/q]],

j - i == 0, -(h[Mod[i, q], IntegerPart[i/q]] + p[Mod[i, q], IntegerPart[i/q]] + k[Mod[i, q], IntegerPart[i/q]]), j - i == 1, p[Mod[i, q], IntegerPart[i/q]], j == q*q, k[Mod[i, q], IntegerPart[i/q]], True, 0],


(*top left q square of matrix-- bottom row in lattice*)
(i < q),


Which[j - i == -1, g[Mod[i, q], IntegerPart[i/q]],

j - i == 0, -(p[Mod[i, q], IntegerPart[i/q]] + g[Mod[i, q], IntegerPart[i/q]] + k[Mod[i, q], IntegerPart[i/q]]), j - i == 1, p[Mod[i, q], IntegerPart[i/q]], j - i == q, k[Mod[i, q], IntegerPart[i/q]], True, 0],


(*bottom right q block of matrix-- top row in lattice*)
i > q*(q-1),


Which[j - i == -q, h[Mod[i, q], IntegerPart[i/q]],

j - i == -1, g[Mod[i, q], IntegerPart[i/q]], j - i == 0, -(h[Mod[i, q], IntegerPart[i/q]] +

```

```

g[Mod[i, q], IntegerPart[i/q]] +
p[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]]),
j - i == 1, p[Mod[i, q], IntegerPart[i/q]],
j == q*q, k[Mod[i, q], IntegerPart[i/q]],
True, 0],

(*last row of each q block of matrix--
last column in lattice *)
Mod[i, q] == q - 1,
Which[j - i == -q, h[Mod[i, q], IntegerPart[i/q]],
j - i == -1, g[Mod[i, q], IntegerPart[i/q]],
j - i == 0, -(h[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]] +
g[Mod[i, q], IntegerPart[i/q]] +
p[Mod[i, q], IntegerPart[i/q]]),
j == q*q, p[Mod[i, q], IntegerPart[i/q]],
j - i == q, k[Mod[i, q], IntegerPart[i/q]],
True, 0],

(*first row of each q block of matrix--
first column in lattice*)
Mod[i, q] == 0,
Which[j - i == -q, h[Mod[i, q], IntegerPart[i/q]],
j - i == 0, -(h[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]] +
p[Mod[i, q], IntegerPart[i/q]]),
j - i == 1, p[Mod[i, q], IntegerPart[i/q]],
j - i == q, k[Mod[i, q], IntegerPart[i/q]],
True, 0],

(* everything else *)

j - i == -q, h[Mod[i, q], IntegerPart[i/q]],
j - i == -1, g[Mod[i, q], IntegerPart[i/q]],
j - i == 0, -(h[Mod[i, q], IntegerPart[i/q]] +
k[Mod[i, q], IntegerPart[i/q]] +
g[Mod[i, q], IntegerPart[i/q]] +

```

```

p[Mod[i, q], IntegerPart[i/q]]),
j - i == 1, p[Mod[i, q], IntegerPart[i/q]],
j - i == q, k[Mod[i, q], IntegerPart[i/q]],
True, 0];

(* the Q matrix for randomization *)
Q = Table[f[i, j], {i, 0, q*q}, {j, 0, q*q}] ;

(* maximum
   of the absolute value of the diagonal of the Q-matrix*)
m = Max[Table[Abs[f[i, i]], {i, 0, q*q}]];

(* the P matrix for randomization *)
P = Q / m + IdentityMatrix[q*q + 1];

(* create a list of increasing
   powers of P for the powers series expansion *)
B = {IdentityMatrix[q*q + 1], P};
NewP = P;
For[i = 1, i < taylorterms, i = i + 1,
  NewP = NewP . P;
  B = Append[B, NewP];];

(* the result of randomization-- a time-
   dependent approximation of a function that
   yields transition probabilities for time t *)
F[t_] := e^-m t *  $\sum_{s=0}^{taylorterms} \frac{m^s t^s}{s!} * B[[s+1]]$ ;

(* use variable A to hold specific values for F[t] *)
A = F[0.5] ;

```

```

(* use "correct row" to hold the row number in
   A representing the correct strating state *)
convertinitial[x_, y_] := y*q + x + 1;
correctrow = convertinitial[1, 2];

(* horizontal expected value m-direction *)
N[ $\sum_{j=1}^{q^2} \text{Mod}[j - 1, q] * \mathbf{A}[[\text{correctrow}]][[j]] +$ 
 q *  $\mathbf{A}[[\text{correctrow}]][[q * q + 1]]]$ 

(* vertical expected value n-direction *)
N[ $\sum_{j=1}^{q^2+1} \text{IntegerPart}[(j - 1) / q] * \mathbf{A}[[\text{correctrow}]][[j]]$ ]

(* sum the row to check and see if this is 1-- if this is
   not one more terms are needed in the power series *)
N[ $\sum_{j=1}^{q^2+1} \mathbf{A}[[\text{correctrow}]][[j]]$ ]

(* produce the diagrams seen in chapter 3 *)
printout = {};
For[i = 0, i < q*q, i = i + 1,
 printout = Append[printout,
   AbsolutePointSize[80 * (A[[correctrow]][[i + 1]])]];
 printout = Append[printout,
   Point[{Mod[i, q], IntegerPart[i/q]}]];
 printout = Append[printout,
   AbsolutePointSize[80 * (A[[correctrow]][[q*q + 1]])]];
 printout = Append[printout, Point[{q + 1, q/2}]];
 Show[Graphics[printout],
 PlotRange -> {{-0.75, q + 2}, {-0.75, q - 0.25}}]

```

Appendix C

This is *Mathematica* code that implements the Euler method for the deterministic Richardson model.

```
Clear[x, y, α, β, γ, δ, ξ, η]
x = 2;
y = 2;
α = 1;
β = 1;
γ = 0.5;
δ = 0.5;
ξ = -1.5;
η = -1.5;

Δt = 0.001;
steps = 2000;
For[i = 0, i < steps, i = i + 1,
  newx = x + (α y - γ x + ξ) Δt; newy = y + (β x - δ y + η) Δt;
  x = newx; y = newy;]

x
y
```

References

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