UNIVERSITY OF NAIROBI

SMA 306 - COMPLEX ANALYSIS NOTES

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Complex Numbers.

1. Definition.

complex number Z = ordered pair (x, y)

Any real number $x = (x, 0)$
The real numbers is a subset of the complex numbers

Imaginery unit $i = (0, 1)$

$$z = (x, y)$$

x = Real component of z or $(x, y) = Re.(z)$
y = Imaginary » » z ,, $(x, y) = Im.(z)$

Any pure imaginary number $y = (0, y)$

We have $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$
since $z_1 = (x_1, y_1)$ $z_2 = (x_2, y_2)$

$z = (x, y) = 0$ if and only if $x = 0$ and $y = 0$
and so $0 = (0, 0)$

α) $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

example : $(x, 0) + (0, y) = (x+0, 0+y) = (x, y)$

β) $z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

example.

$$(0, y) = (x_1 x_2 - y_1 y_2 , x_1 y_2 + x_2 y_1) \quad \text{So}$$

$$x_1 x_2 - y_1 y_2 = 0 \Rightarrow x_1 = \frac{y_1 y_2}{x_2} , \frac{y_1 y_2^2}{x_2} = x_1 y_2$$

$$x_1 y_2 + x_2 y_1 = y \Rightarrow \frac{y_1 y_2^2}{x_2} + x_2 y_1 = y$$

$$y_1 y_2^2 + y_1 x_2^2 = x_2 y \Rightarrow y_1 \cdot (x_2^2 + y_2^2) = x_2 \cdot y$$

Ordered pairs (x, y) of real numbers x, y that satisfy the conditions in preceding page are defined as Complex numbers.

$$z = x + iy = (x, y)$$

$$z^2 = z \cdot z \qquad z^3 = z^2 z = z \cdot z \cdot z$$

$$\left.\begin{aligned} &(0,1)^2 = (i)^2 = i^2 \\ &(0,1)^2 = (0 + i \cdot 1)^2 = i^2 \cdot 1^2 = i^2 \\ &(0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1, 0) \end{aligned}\right\} \Rightarrow i^2 = -1$$

2. Further properties

Subtraction:
$$z_1 = (x_1, y_1) = x_1 + iy_1$$
$$z_2 = (x_2, y_2) = x_2 + iy_2$$
$$z_3 = (x_3, y_3) = x_3 + iy_3$$

$$z_1 - z_2 = z_3 \Rightarrow z_2 + z_3 = z_1$$

$$(x_2, y_2) + (x_3, y_3) = (x_1, y_1)$$
$$(x_2 + x_3,\ y_2 + y_3) = (x_1, y_1) \Rightarrow x_2 + x_3 = x_1,\ y_2 + y_3 = y_1$$

$x_3 = x_1 - x_2 \qquad y_3 = y_1 - y_2 \qquad$ and so

$$z_3 = z_1 - z_2 = (x_1 - x_2,\ y_1 - y_2) = (x_1 - x_2) + i \cdot (y_1 - y_2)$$

Division :

$$\frac{z_1}{z_2} = z_3 \quad \Rightarrow \quad z_2 \cdot z_3 = z_1$$

$$(x_2 x_3 - y_2 y_3 \, , \, x_2 y_3 + x_3 y_2) = (x_1 , y_1)$$

$$x_2 x_3 - y_2 y_3 = x_1$$
$$x_3 \cdot y_2 + x_2 y_3 = y_1$$

$$\alpha_1 x + \beta_1 y = \gamma_1$$
$$\alpha_2 x + \beta_2 y = \gamma_2$$

$$y = \frac{\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}$$

$$x = \frac{\begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}$$

$$y_3 = \frac{\begin{vmatrix} x_2 & x_1 \\ y_2 & y_1 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{x_2 \cdot y_1 - x_1 \cdot y_2}{x_2^2 + y_2^2}$$

$$x_3 = \frac{\begin{vmatrix} -y_2 & x_1 \\ x_2 & y_1 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}$$

$$z_3 = \frac{z_1}{z_2} = x_3 + i y_3 = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 \cdot y_2}{x_2^2 + y_2^2} \cdot i$$

or $$z_3 = \frac{z_1}{z_2} = \frac{x_1 + i \cdot y_1}{x_2 + i \cdot y_2} = \frac{(x_1 + i y_1) \cdot (x_2 - i \cdot y_2)}{(x_2 + i y_2) \cdot (x_2 - i y_2)} =$$

$$= \frac{x_1 x_2 + i \cdot x_2 \cdot y_1 - x_1 y_2 \cdot i - i^2 \cdot y_1 \cdot y_2}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 + (x_2 y_1 - x_1 y_2) \cdot i}{x_2^2 + y_2^2}$$

which is the same as above.

a) Commutative law.

addition : $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + (y_1 + y_2)i =$
$= x_2 + x_1 + (y_2 + y_1)i = x_2 + y_2 \cdot i + x_1 + y_1 \cdot i =$
$= z_2 + z_1$

multiplication: $z_1 \cdot z_2 = (x_1 + i \cdot y_1) \cdot (x_2 + i \cdot y_2) =$
$= x_1 \cdot x_2 + i(y_1 \cdot x_2 + x_1 y_2) + i^2 \cdot y_1 y_2$
$= x_1 x_2 + i(y_1 x_2 + x_1 y_2) - y_1 y_2$
$= x_2 x_1 - y_2 \cdot y_1 + i \cdot (x_2 y_1 + x_1 y_2)$
$= z_2 \cdot z_1$

$i \cdot y = y \cdot i$ because

$(0 + 1 \cdot i) \cdot (y + 0 \cdot i) = x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) \cdot i = 1 \cdot i \cdot y = i \cdot y$

$(y + 0 \cdot i) \cdot (0 + 1 \cdot i) = x_2 x_1 - y_2 y_1 + (x_2 y_1 + x_1 y_2) i = y \cdot 1 \cdot i = y \cdot i$.

and so $z = x + i \cdot y$
$= x + y \cdot i$

b) Associative law. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

c) Distributive law of multiplication with respect to addition

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

$z_1 \cdot z_2 = 0$ implies $z_1 = 0$ or $z_2 = 0$

$$(x_1 + iy_1)\cdot(x_2 + i\cdot y_2) = x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1)\,i = 0$$

$x_1x_2 - y_1y_2 = 0$ and $x_1y_2 + x_2y_1 = 0$

$$\left.\begin{array}{l} x_1x_2 = y_1y_2 \\ x_1y_2 = -x_2y_1 \end{array}\right\} \Rightarrow \frac{x_2}{y_2} = \frac{y_2}{-x_2} \Rightarrow x_2^2 + y_2^2 = 0$$

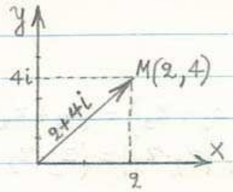
$$x_2 = y_2 = 0$$

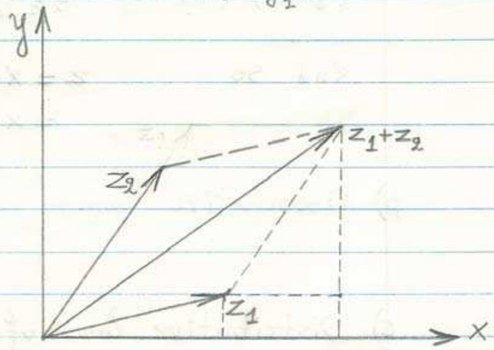
or

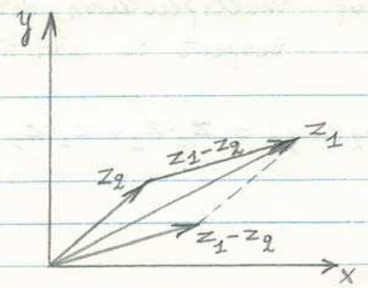
$$\left.\begin{array}{l} y_1y_2 = x_1x_2 \\ x_1y_2 = -x_2y_1 \end{array}\right\} \Rightarrow \frac{y_1}{x_1} = -\frac{x_1}{y_1} \Rightarrow x_1^2 + y_1^2 = 0$$

$$x_1 = y_1 = 0$$

3. Geometric Representation







$$z_2 + (z_1 - z_2) = z_1$$

4. Complex Conjugates.

complex number $z = (x,y) = x + i\cdot y$

$\bar{z}$ = complex conjugate of z or conjugate of z

$$\bar{z} = (x, -y) = x - i\cdot y$$

$$z_1 = x_1 + i\cdot y_1 \quad , \quad z_2 = x_2 + i\cdot y_2$$
$$\bar{z}_1 = x_1 - iy_1 \quad , \quad \bar{z}_2 = x_2 - i\cdot y_2$$

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 =$$
$$= x_1 + x_2 + i\cdot(y_1 + y_2)$$
$$\overline{z_1 + z_2} = x_1 + x_2 - i\cdot(y_1 + y_2) =$$
$$= x_1 - iy_1 + x_2 - iy_2 = \bar{z}_1 + \bar{z}_2$$

$$z = x + iy \quad , \quad \bar{z} = x - iy \quad , \quad \bar{\bar{z}} = x + iy = z$$

$$z + \bar{z} = x + i\cdot y + x - iy = 2\cdot x = 2\cdot Re(z)$$

$$z - \bar{z} = x + iy - x + iy = 2y\cdot i = 2\cdot Im(z)\cdot i$$

$$Im(z) = y$$

$$Re(z) = x$$

5. Absolute values

If x and y are real the nonnegative real number

$$|z| = \sqrt{x^2+y^2}$$

is called absolute value or modulus of z

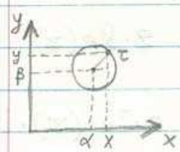
$$\left.\begin{aligned} z_1 &= x_1+iy_1 \\ z_2 &= x_2+iy_2 \end{aligned}\right\} \Rightarrow z_1-z_2 = x_1-x_2+i\cdot(y_1-y_2)$$

$$|z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

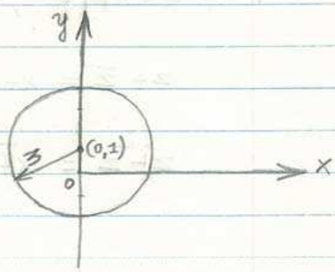
e.g. $|z-i| = 3 \Rightarrow |x+iy-i| = 3$

$|x+i\cdot(y-1)| = 3 \Rightarrow \sqrt{x^2+(y-1)^2} = 3$

$x^2+(y-1)^2 = 9$



$(x-\alpha)^2+(y-\beta)^2 = r^2$



$z = x + iy$ $\quad$ α) $|z| = \sqrt{x^2+y^2}$

β) $Re(z) = x$

γ) $Im(z) = y$

$$|z|^2 = x^2 + y^2 = [Re(z)]^2 + [Im(z)]^2$$

$|z|^2 \geqslant x^2 = [Re(z)]^2 \quad \Rightarrow \quad |z| \geq |Re(z)| \geq Re(z)$

$|z|^2 \geqslant y^2 = [Im(z)]^2 \quad \Rightarrow \quad |z| \geq |Im(z)| \geq Im(z)$

$\bar{z} = x - iy \quad \Rightarrow \quad z \cdot \bar{z} = (x+iy)\cdot(x-iy) =$
$= x^2 + y^2 = |z|^2 = |\bar{z}|^2$

$\boxed{z \cdot \bar{z} = |z|^2}$, $\quad z \cdot \bar{z} = |z|^2 = |\bar{z}|^2 \Rightarrow |z| = |\bar{z}|$

e.g

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ $\quad$ neither z_1 nor z_2 is zero

$$|z_1 z_2|^2 = (z_1 z_2)\cdot(\overline{z_1 \cdot z_2}) = (z_1 \cdot z_2)\cdot(\bar{z}_1 \cdot \bar{z}_2) = (z_1 \cdot \bar{z}_1)\cdot(z_2 \cdot \bar{z}_2)$$

$$= |z_1|^2 \cdot |z_2|^2 \Rightarrow |z_1 \cdot z_2|^2 - (|z_1| \cdot |z_2|)^2 = 0$$

$$(|z_1 z_2| - |z_1| \cdot |z_2|)\cdot(|z_1 z_2| + |z_1| \cdot |z_2|) = 0 \Rightarrow$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

a) Triangle inequality. $\Rightarrow$ $\boxed{|z_1+z_2| \leq |z_1|+|z_2|}$

$$|z_1+z_2|^2 = (z_1+z_2)\cdot(\overline{z_1+z_2}) = (z_1+z_2)\cdot(\bar{z}_1+\bar{z}_2) =$$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + (z_1\bar{z}_2 + \bar{z}_1\cdot z_2)$$

$$z_1\cdot\bar{z}_2 = (x_1+iy_1)\cdot(x_2-iy_2) = x_1x_2+y_1y_2+i\cdot(x_2y_1-x_1y_2) = z_3$$

$$\bar{z}_1\cdot z_2 = (x_1-y_1i)\cdot(x_2+iy_2) = x_1x_2+y_1y_2-i\cdot(x_2y_1-x_1y_2) = \bar{z}_3$$

$$z_3+\bar{z}_3 = 2\,Re(z_3) \Rightarrow z_1\cdot\bar{z}_2+\bar{z}_1\cdot z_2 = 2\cdot Re(z_1\bar{z}_2)$$

and so : $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,Re(z_1\bar{z}_2)$

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|-2|z_1|\cdot|z_2|+2Re(z_1\bar{z}_2)$$

$|z|=|\bar{z}|$,

$$= (|z_1|+|z_2|)^2 - 2|z_1|\cdot|\bar{z}_2| + 2Re(z_1\bar{z}_2)$$

$$|z_1+z_2|^2-(|z_1|+|z_2|)^2 = -2\cdot\left[|z_1\cdot\bar{z}_2| - Re(z_1\cdot\bar{z}_2)\right]$$

We always have $|z_1\bar{z}_2| \geq Re(z_1\cdot\bar{z}_2)$ and so

$$|z_1+z_2|^2-(|z_1|+|z_2|)^2 \leq 0$$

$$\left[|z_1+z_2|-(|z_1|+|z_2|)\right]\cdot\left[\underbrace{|z_1+z_2|+|z_1|+|z_2|}_{>0 \text{ because } |z_1\cdot z_2|\neq 0}\right] \leq 0$$

$$\boxed{|z_1+z_2| \leq |z_1|+|z_2|}$$

β) $$|z_1 - z_2| \geq \big| |z_1| - |z_2| \big|$$

γ) $$|z_1 - z_2| \geq |z_1| - |z_2|$$

δ) $$|z_1 - z_2| \leq |z_1| + |z_2|$$

ε) $$|z_1 + z_2| \geq \big| |z_1| - |z_2| \big|$$

From the : $|z_1 + z_2| \leq |z_1| + |z_2|$

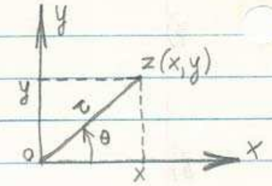
we have : $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$

and so :

$$\left| \sum_{K=1}^{n} z_K \right| \leq \sum_{K=1}^{n} |z_K|$$

with $n = 1, 2, 3, \cdots, K, \cdots$

6. The Polar Form.



$x = r\cdot\cos\theta$

$y = r\cdot\sin\theta$

$z = x + iy = r\cdot\cos\theta + i\cdot r\cdot\sin\theta = r\cdot(\cos\theta + i\cdot\sin\theta)$

$|z| = \sqrt{x^2+y^2} = r \geqslant 0$

r, θ = Polar coordinates

argument of $z = \theta \Rightarrow \tan\theta = \frac{y}{x}$

$$0 \leq \theta < 2\pi$$

α) $z=0 \Rightarrow |z|=0 \Rightarrow r=|z|=0$ and θ is arbitrary

β) $z = 2-2\cdot i \Rightarrow x=2,\ y=-2 \Rightarrow \tan\theta = \frac{-2}{2} = -1$

$|z| = r = \sqrt{x^2+y^2} = \sqrt{4+4} = \sqrt{8} = 2\cdot\sqrt{2}$

$\theta = -\frac{\pi}{4} \pm 2K\pi \qquad K = 0,1,2,\dots$

Τό πρόσημον ± ἀφορᾷ τήν φοράν περιστροφῆς.

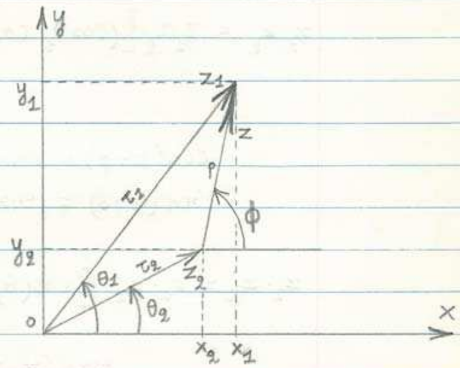
γ) $z=-i \Rightarrow x=0,\ y=-1 \Rightarrow \tan\theta \to -\infty \Rightarrow \theta = \frac{3\pi}{2}$

thus: $-i = \cos\frac{3\pi}{2} + i\cdot\sin\frac{3\pi}{2} = \cos\left(-\frac{\pi}{2}\right) + i\cdot\sin\left(-\frac{\pi}{2}\right)$

$$z = x + iy = r\cdot(\cos\theta + i\cdot\sin\theta)$$

$$\bar{z} = x - iy = r\cdot(\cos\theta - i\cdot\sin\theta) = r\cdot[\cos\theta + i\cdot\sin(-\theta)]$$
$$= r\cdot[\cos(-\theta) + i\cdot\sin(-\theta)]$$

$$\arg\bar{z} = -\theta = -\arg z$$

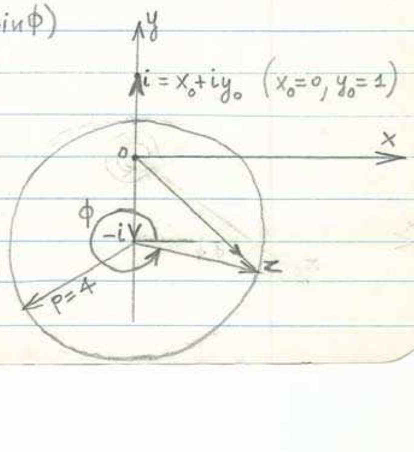


$$z_1 = r_1\cdot(\cos\theta_1 + i\cdot\sin\theta_1) = x_1 + i\cdot y_1$$
$$z_2 = r_2\cdot(\cos\theta_2 + i\sin\theta_2) = x_2 + iy_2$$
$$z = \rho\cdot(\cos\phi + i\cdot\sin\phi) = x_1 - x_2 + i\cdot(y_1 - y_2) = z_1 - z_2$$
$$= \rho\cdot\cos\phi + i\cdot\rho\cdot\sin\phi.$$

$$z + i = z - (-i) = 4\cdot(\cos\phi + i\sin\phi)$$
$$= x - x_1 + i(y - y_1)$$
$$= 4\cdot\cos\phi + i\,4\cdot\sin\phi$$

$$(x-0)^2 + (y-1)^2 = \rho^2$$
$$x^2 + (y-1)^2 = 4^2$$
$$x_0 = 0,\quad y_0 = 1 \qquad i = x_0 + i\cdot y_0$$

$$0 \le \phi < 2\pi$$



7. Products, Powers, Quotiens

a) Product.

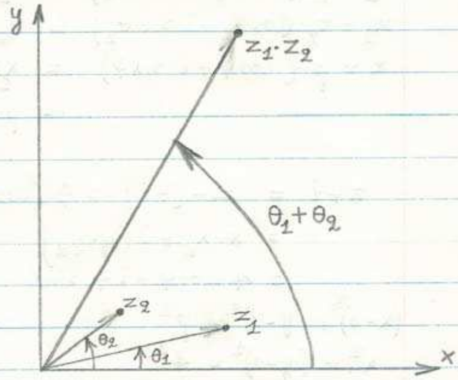
$$z_1 = x_1 + iy_1 = r_1 \cdot (\cos\theta_1 + i \cdot \sin\theta_1)$$
$$z_2 = x_2 + iy_2 = r_2 \cdot (\cos\theta_2 + i \cdot \sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \left[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + \right.$$
$$\left. + i \cdot (\cos\theta_1 \cdot \sin\theta_2 + \cos\theta_2 \cdot \sin\theta_1) \right]$$

$$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \left[\cos(\theta_1+\theta_2) + i \cdot \sin(\theta_1+\theta_2) \right]$$

$$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$$

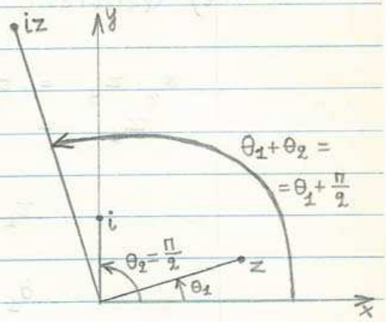


example : $i \cdot z = r_1 \cdot r_2 \cdot (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$i = 0 + 1 \cdot i = 1 \cdot \left(\cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2}\right)$

$z = x + y \cdot i = r \cdot (\cos\theta + i \cdot \sin\theta)$

$i \cdot z = r \cdot \left[\cos\left(\theta + \frac{\pi}{2}\right) + i \cdot \sin\left(\theta + \frac{\pi}{2}\right)\right]$



b) Power.

$z_1 \cdot z_2 \cdots z_n = r_1 \cdot r_2 \cdots r_n \cdot \left[\cos(\theta_1+\theta_2+\cdots+\theta_n) + i \cdot \sin(\theta_1+\theta_2+\cdots+\theta_n)\right]$

if $z_1 = z_2 = \cdots = z_n = z \quad (n = 1, 2, \cdots)$

$$\boxed{z^n = r^n \cdot (\cos n\theta + i \cdot \sin n\theta)}$$

when $r = 1 \Rightarrow \boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i \cdot \sin n\theta}$

De Moivre's theorem for $n > 0$

c) Quotient. $\frac{z_1}{z_2} = z_3 \Rightarrow z_2 \cdot z_3 = z_1$

$$z_2 \cdot z_3 = r_2 \cdot r_3 \cdot (\cos(\theta_2+\theta_3) + i \cdot \sin(\theta_2+\theta_3))$$
$$z_1 = r_1 \cdot (\cos\theta_1 + i \sin\theta_1)$$

$$r_1 = r_2 \cdot r_3 \Rightarrow r_3 = \frac{r_1}{r_2}$$

$$\theta_1 = \theta_2 + \theta_3 \Rightarrow \theta_3 = \theta_1 - \theta_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \left[\cos(\theta_1-\theta_2) + i \cdot \sin(\theta_1-\theta_2)\right] \quad , \quad (r_2 \neq 0)$$

when $z_1 = x_1 + iy_1 = 1 \Rightarrow r_1 = 1 \Rightarrow$

$$\frac{1}{z} = \frac{1}{r} \cdot \left[\cos(-\theta) + i \cdot \sin(-\theta)\right]$$

$$= \frac{1}{r} \cdot (\cos\theta - i \cdot \sin\theta)$$

$$\left(\frac{1}{z}\right)^n = \frac{1}{z^n} = \frac{1}{r^n} \cdot (\cos n\theta - i \cdot \sin n\theta)$$

$$\boxed{z^{-n} = r^{-n} (\cos n\theta - i \cdot \sin n\theta)}$$

when $r = 1 \Rightarrow$ $\boxed{(\cos\theta + i \cdot \sin\theta)^{-n} = \cos n\theta - i \cdot \sin n\theta}$, $n > 0$

De Moivre's theorem
for negative exponent $(m = -n)$

$$, \quad (\cos\theta + i \sin\theta)^{-n} = \cos(-n\theta) + i \cdot \sin(-n\theta)$$

8. Extraction of Roots

Complex number $z = x + iy = r\cdot(\cos\theta + i\sin\theta)$

n^{th} root of $z = \sqrt[n]{z} = z_0 \implies z_0^n = z$

$$z_0 = x_0 + iy_0 = r_0\cdot(\cos\theta_0 + i\cdot\sin\theta_0)$$

$$z_0^n = z$$

$$\left[r_0\cdot(\cos\theta_0 + i\sin\theta_0)\right]^n = r\cdot(\cos\theta + i\cdot\sin\theta)$$

$r_0^n\cdot(\cos n\theta_0 + i\cdot\sin n\theta_0) = r\cdot(\cos\theta + i\cdot\sin\theta)$ So we have

$$r_0^n = r \implies r_0 = \sqrt[n]{r}$$

$$n\theta_0 = \theta \pm 2K\pi \implies \theta_0 = \theta/n \pm 2K\pi/n$$

where : $n = 1, 2, 3, \ldots\ldots \rightarrow$ (one of them)

(always $n > K$) , $K = 0, 1, 2, \cdots (n-1) \longrightarrow$ (that is all the numbers bellow n

$+$ for counterclockwise ↶ or (left handed)

$-$ for clockwise ↷ or (right ״)

So we have : $\theta_0 = \dfrac{\theta}{n} + \dfrac{2K\pi}{n}$

$$z_0 = \sqrt[n]{r}\cdot\left(\cos\frac{\theta + 2K\pi}{n} + i\cdot\sin\frac{\theta + 2K\pi}{n}\right)$$

or : $$z_0 = \sqrt[n]{r}\cdot\left[\cos\left(\frac{\theta}{n} + \frac{2K\pi}{n}\right) + i\cdot\sin\left(\frac{\theta}{n} + \frac{2K\pi}{n}\right)\right]$$

We have : $z = 1 = \cos 0 + i \cdot \sin 0$

So $\sqrt[n]{1} = \cos\frac{2K\pi}{n} + i \cdot \sin\frac{2K\pi}{n}$, $K = 0, 1, 2, \cdots n-1$

$$|1| = |1^{1/2}| = |1^{1/3}| = |1^{1/4}| = \cdots = |1^{1/n}| = 1 = \text{μονάς}$$

$$|1| = |\sqrt[2]{1}| = |\sqrt[3]{1}| = |\sqrt[4]{1}| = \cdots = |\sqrt[n]{1}| = 1 = \text{μονάς}$$

$$|1| = |1^2| = |1^3| = |1^4| = \cdots = |1^n| = 1 = \text{μονάς}$$

we call : $\boxed{w = \cos\frac{2\pi}{n} + i \cdot \sin\frac{2\pi}{n}}$

$$w^K = \left[\cos\frac{2\pi}{n} + i \cdot \sin\frac{2\pi}{n}\right]^K$$

$$w^K = \cos\frac{2K\pi}{n} + i \cdot \sin\frac{2K\pi}{n}$$

$K=0 \Rightarrow w^0 = 1 = \cos 0 + i \cdot \sin 0 = \left(\sqrt[n]{1}\right)_1$

$K=1 \Rightarrow w^1 = w = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n} = \left(\sqrt[n]{1}\right)_2$

$K=2 \Rightarrow w^2 = w^2 = \cos\frac{2\cdot 2\cdot\pi}{n} + i \cdot \sin\frac{2\cdot 2\cdot \pi}{n} = \left(\sqrt[n]{1}\right)_3$

. . . .

. . . . . .

$K=n-1 \Rightarrow w^{n-1} = w^{n-1} = \cos\frac{2\cdot(n-1)\pi}{n} + i \cdot \sin\frac{2\cdot(n-1)\cdot\pi}{n} = \left(\sqrt[n]{1}\right)_n$

These are the n n^{th} roots of 1

$$1, w, w^2, w^3, \ldots, w^{n-1}$$

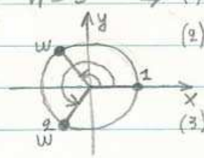
when $z=1$, $\quad w=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$

α) $n=1 \Rightarrow K=0 \Rightarrow w^0 = 1 = \cos 0 + i\sin 0$

β) $n=3 \Rightarrow$ (1) $K=0 \Rightarrow w^0 = \cos 0 + i\sin 0$

(2) $K=1 \Rightarrow w^1 = \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

(3) $K=2 \Rightarrow w^2 = \cos\frac{2\cdot 2\pi}{3}+i\sin\frac{2\cdot 2\pi}{3}$



γ) $n=6 \Rightarrow w=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}$

(1) $K=0 \Rightarrow w^0 = 1 = \cos 0 + i\sin 0 \Rightarrow \theta_1 = 0$

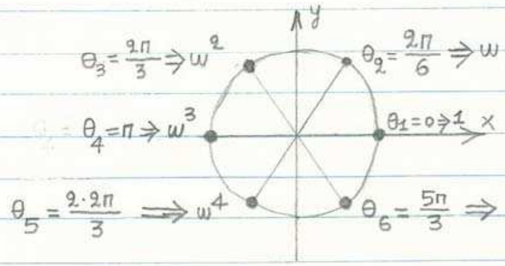
(2) $K=1 \Rightarrow w^1 = \cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6} \Rightarrow \theta_2 = \frac{2\pi}{6}$

(3) $K=2 \Rightarrow w^2 = \cos\frac{2\cdot 2\cdot\pi}{6}+i\cdot\sin\frac{2\cdot 2\cdot\pi}{6} \Rightarrow \theta_3 = \frac{2\pi}{3}$

(4) $K=3 \Rightarrow w^3 = \cos\frac{2\cdot 3\cdot\pi}{6}+i\cdot\sin\frac{2\cdot 3\cdot\pi}{6} \Rightarrow \theta_4 = \pi$

(5) $K=4 \Rightarrow w^4 = \cos\frac{2\cdot 4\cdot\pi}{6}+i\cdot\sin\frac{2\cdot 4\cdot\pi}{6} \Rightarrow \theta_5 = \frac{2\cdot 2\pi}{3}$

(6) $K=5 \Rightarrow w^5 = \cos\frac{2\cdot 5\cdot\pi}{6}+i\cdot\sin\frac{2\cdot 5\cdot\pi}{6} \Rightarrow \theta_6 = \frac{5\pi}{3}$



When $z \neq 1 \Rightarrow z = r\cdot(\cos\theta + i\sin\theta)$

For $K=0 \Rightarrow z_1 = \sqrt[n]{z} \Rightarrow$ the n n^{th} roots of z are

$$\underbrace{z_1}_{K=0},\ \underbrace{z_1 w}_{K=1},\ \underbrace{z_1 w^2}_{K=2},\ \underbrace{z_1 w^3}_{K=3},\ \dots,\ \underbrace{z_1 w^{n-1}}_{K=n-1}$$

$$z_1 = \sqrt[n]{r}\cdot\left[\cos\left(\frac{\theta}{n} + \frac{2K\pi}{n}\right) + i\cdot\sin\left(\frac{\theta}{n} + \frac{2K\pi}{n}\right)\right]$$

$$K=0 \Rightarrow \sqrt[n]{z} = z_1 = \sqrt[n]{r}\cdot\left(\cos\frac{\theta}{n} + i\cdot\sin\frac{\theta}{n}\right)$$

$$\boxed{\sqrt[n]{z} = z_1 = \sqrt[n]{r}\cdot\left(\cos\frac{\theta}{n} + i\cdot\sin\frac{\theta}{n}\right)}$$

$$w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$$z = r\cdot(\cos\theta + i\cdot\sin\theta)$$

$$z^m = r^m\cdot\left[\cos(m\theta+2k\pi) + i\cdot\sin(m\theta+2k\pi)\right]$$

$$(z^m)^{1/n} = \sqrt[n]{z^m} = \sqrt[n]{r^m}\cdot\left[\cos\left(\frac{m\theta+2k\pi^m}{n}+\frac{2k''\pi}{n}\right) + \right.$$

$$\left. + i\sin\left(\frac{m\theta+2k\pi^m}{n}+\frac{2k''\pi}{n}\right)\right]$$

$k'' = 0, 1, 2, \cdots, n-1$

$$(z^m)^{\frac{1}{n}} = \sqrt[n]{r^m}\left[\cos\left(\frac{m\theta}{n}+\frac{2(k+k'')\pi}{n}\right) + i\sin\left(\frac{m\theta}{n}+\frac{2(k+k'')\pi}{n}\right)\right]$$

we call $\Rightarrow$ $k+k'' = h = 0, 1, 2, \cdots, n-1$

$$(z^m)^{\frac{1}{n}} = \sqrt[n]{r^m}\cdot\left[\cos\left(\frac{m\theta}{n}+\frac{2h\pi}{n}\right) + i\cdot\sin\left(\frac{m\theta}{n}+\frac{2h\pi}{n}\right)\right]$$

$$= \sqrt[n]{r^m}\cdot\left[\cos\frac{m\theta}{n} + i\cdot\sin\frac{m\theta}{n}\right]\cdot\left[\cos\frac{2h\pi}{n} + i\sin\frac{2h\pi}{n}\right]$$

$$= \sqrt[n]{r^m}\cdot\left(\cos\frac{m\theta}{n} + i\cdot\sin\frac{m\theta}{n}\right)\cdot\underbrace{\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}_{w}{}^{h}$$

$$(z^m)^{\frac{1}{n}} = \sqrt[n]{z^m} = \sqrt[n]{r^m}\cdot\left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}\right)\cdot w^h$$

$h = 0, 1, 2, \cdots, (n-1)$

$$z^{\frac{1}{n}} = \sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i \cdot \sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

$$\left(z^{\frac{1}{n}}\right)^m = \sqrt[n]{r^m} \cdot \left[\cos\left(\frac{m(\theta+2k\pi)}{n} + 2k'\pi\right) + i\sin\left(\frac{m(\theta+2k\pi)}{n} + 2k'\pi\right)\right]$$

$$= \sqrt[n]{r^m} \cdot \left[\cos\left(\frac{m\theta}{n} + \frac{2(mk+k'n)\cdot\pi}{n}\right) + i \cdot \sin\left(\frac{m\theta}{n} + \frac{2(mk+k'n)\pi}{n}\right)\right]$$

$$= \sqrt[n]{r^m} \cdot \left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}\right) \cdot \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{\underbrace{(mk+k'n)}}$$

it is $\boxed{mk+k'n = h}$

So we have :

$$z^{\frac{m}{n}} = \sqrt[n]{r^m} \cdot \left[\cos\left(\frac{m\theta}{n} + \frac{m2k\pi}{n}\right) + i\sin\left(\frac{m\theta}{n} + \frac{m2k\pi}{n}\right)\right]$$

$$k = 0, 1, 2, \dots, n-1$$

α) $z^c = (x+iy)^{\alpha+ib}$ is defined in Sec. 28.

β) $z^{-m/n} = \dfrac{1}{z^{m/n}} = \left(z^{-m}\right)^{\frac{1}{n}} = \left(z^{\frac{1}{n}}\right)^{-m}$

Example

Find the $\sqrt[3]{z}$ where $z = 1+i$

$r = |z| = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2} \Rightarrow r = \sqrt{2}$

$r \geq 0$ always

$\left.\begin{array}{l} x=1 \\ y=1 \end{array}\right\}$ $z = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot i\right) = \sqrt{2}\cdot\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot i\right)$

$\cos\theta = \frac{\sqrt{2}}{2} \quad \sin\theta = \frac{\sqrt{2}}{2} \quad \theta = \frac{\pi}{4}$

$z = \sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right) = 1+i$

$z^{\frac{1}{n}} = \sqrt[n]{r}\cdot\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \quad k = 0,1,2,\dots n-1$

$n=3.$

$\sqrt[3]{z} = \sqrt[3]{r}\cdot\left[\cos\left(\frac{\theta}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{2k\pi}{3}\right)\right]$

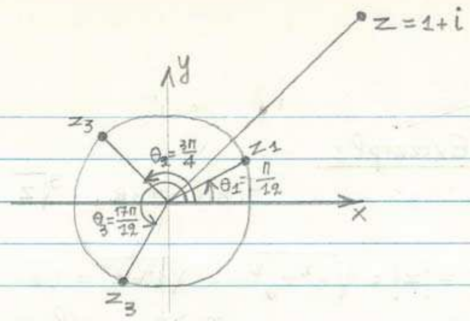
$\sqrt[3]{1+i} = \sqrt[3]{2^{1/2}}\cdot\left[\cos\left(\frac{\pi}{4\cdot 3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4\cdot 3} + \frac{2k\pi}{3}\right)\right]$

where $k = 0,1,2$

$k=0 \Rightarrow z_1 = \sqrt[6]{2}\cdot\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \Rightarrow \theta_1 = \frac{\pi}{12}$

$k=1 \Rightarrow z_2 = \sqrt[6]{2}\cdot\left[\cos\left(\frac{\pi}{12} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2\pi}{3}\right)\right] \Rightarrow \theta_2 = \frac{3\pi}{4}$

$k=2 \Rightarrow z_3 = \sqrt[6]{2}\cdot\left[\cos\left(\frac{\pi}{12} + \frac{4\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{12} + \frac{4\pi}{3}\right)\right] \Rightarrow \theta_3 = \frac{17\pi}{12}$



Home WorK : Page 12/1(d) , 12/2c , 12/3b , 13/10

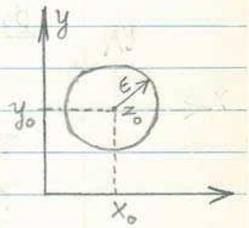
9. Regions in the complex plane

Neighborhood of the point z_0 is the set of all points z for which $|z-z_0| < \epsilon$
where ϵ = positive constant

$z = x+iy$, $z_0 = x_0+iy_0$ $\Rightarrow$ $z-z_0 = (x-x_0)+i(y-y_0)$

$$|z-z_0| = \left|(x-x_0)+i(y-y_0)\right| = \left[(x-x_0)^2+(y-y_0)^2\right]^{1/2} < \epsilon$$

$$(x-x_0)^2+(y-y_0)^2 < \epsilon^2$$

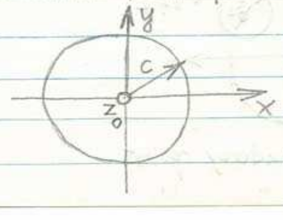


which includes all points in this disk and the center z_0 and excludes the points on the boundary circle.

Limit point z_0 for a set of points if every neighborhood of z_0 contains points other than z_0 of the set.

each point of the set $|z| = c \Rightarrow |x+iy| = c \Rightarrow x^2+y^2 = c^2$
is a limit point for the set $|z| < c \Rightarrow$ $x^2+y^2 < c^2$
(κάθε σημείον της περιφερείας είναι limit point των εντός της περιφερείας σημείων)

Zero is the limit point for the set $z = \frac{1}{n}$, $n = 1,2,3\ldots$



<u>Interior point</u> of a set S is a point of S such that some neighborhood of that point contains only points in S.

An Interior point is always a limit point

<u>Boundary point</u> is a point z_0 of S when this z_0 point is a limil point and it is not an interior point.

Then each neighborhood of z_0 contains a point not in S as well as points in S.

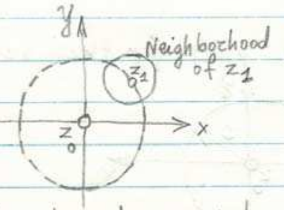
Every limit point that does not belong to the set is a boundary point.

e.g. The origin $z_0 = 0$ and each point of the Set $|z| = 1 \Rightarrow x^2 + y^2 = 1$, is a boundary point for either of the sets:

a) $0 < |z| < 1 \Rightarrow 0 < x^2 + y^2 < 1$

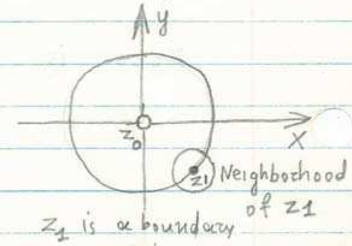
b) $0 < |z| \leq 1 \qquad 0 < x^2 + y^2 \leq 1$

a)



z_1 is a boundary point

z_0 " " "



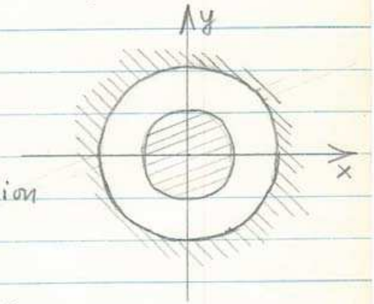
z_1 is a boundary point

Connected. A region is connected if each pair of its points can be joined by some continuous chain of a finite number of line segments all points of which lie in the region.

The open region consisting of all points interior to the circle $|z| = 1 \Rightarrow |x+iy| = 1 \Rightarrow x^2+y^2 = 1$ that is all the points $|z| < 1 \Rightarrow x^2+y^2 < 1$

and all the points exterior to the $|z| = 2$ that is all the points $x^2+y^2 > 2$

are not connected.

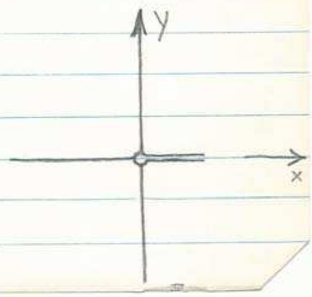


Domain. A connected open region is called a domain.

example: the domain:

$|z| > 0$ and $0 < \arg z < 2\pi$

or $x^2+y^2 > 0 \Rightarrow r > 0$, $0 < \theta < 2\pi$



is a domain which contains all points of the place except the origin and points of the positive X axis

EXERCISES

5/ **1.** Verify:

(a) $(\sqrt{2} - i) - i(1 - i\sqrt{2}) = -2i$; (b) $(2,-3)(-2,1) = (-1,8)$;

(c) $(3,1)(3,-1)(\frac{1}{5}, \frac{1}{10}) = (2,1)$; (d) $\dfrac{1 + 2i}{3 - 4i} + \dfrac{2 - i}{5i} = -\dfrac{2}{5}$;

(e) $\dfrac{5}{(1 - i)(2 - i)(3 - i)} = \dfrac{1}{2}i$; (f) $(1 - i)^4 = -4$.

6/ **2.** Exhibit the numbers z_1, z_2, $z_1 + z_2$, and $z_1 - z_2$ graphically, when
(a) $z_1 = 2i$, $z_2 = \frac{2}{3} - i$; (b) $z_1 = (-\sqrt{3}, 1)$, $z_2 = (\sqrt{3}, 0)$;
(c) $z_1 = (-3,1)$, $z_2 = (1,4)$; (d) $z_1 = x_1 + y_1 i$, $z_2 = x_1 - y_1 i$.

3. If $z \neq 0$ in parts (a) and (b) below, prove that

(a) $\dfrac{z}{z} = 1$; (b) $\dfrac{1}{1/z} = z$; (c) $\mathscr{I}(iz) = \mathscr{R}(z)$.

4. Show that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

5. Establish formulas (3) of Sec. 2.

6. Prove the commutative law $z_1 z_2 = z_2 z_1$.

7. Prove the associative laws (5) and (6), Sec. 2.

8. Prove the distributive law (7), Sec. 2.

9. If k is a real number and $z = (x,y)$, show that $kz = (kx,ky)$ and hence that $-z = -x - yi$, where $-z$ denotes $(-1)z$.

10. Establish the first of formulas (8), Sec. 2.

11. Prove that $z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3$.

12. Show that the product of three numbers z_1, z_2, and z_3 does not depend on which two factors are multiplied together first, so that the product may be written $z_1 z_2 z_3$.

13. If $z_1 z_2 z_3 = 0$, prove that at least one of the three factors is zero.

14. Prove that $(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4)$.

15. Establish the second of formulas (8), Sec. 2, and show that

$$\frac{zz_1}{zz_2} = \frac{z_1}{z_2} \qquad (z \neq 0,\ z_2 \neq 0).$$

16. Show that the point represented by $\frac{1}{2}(z_1 + z_2)$ is the mid-point of the line segment between points z_1 and z_2.

17. Prove that $(1 + z)^2 = 1 + 2z + z^2$.

18. Use induction to prove the binomial formula

$$(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}z^k + \cdots + z^n,$$

where n and k are positive integers.

Exercises

5/1.

α). $(\sqrt{2}-i)-(1-i\cdot\sqrt{2})\cdot i = \sqrt{2}-i-i+i^2\cdot\sqrt{2}$
$= \sqrt{2}-2\cdot i-\sqrt{2} = -2\cdot i$

c) $(3+i)\cdot(3-i)\left(\frac{1}{5}+\frac{i}{10}\right) = \left(9-i^2\right)\left(\frac{1}{5}+\frac{i}{10}\right) =$

$= (9+1)\cdot\left(\frac{1}{5}+\frac{i}{10}\right) = 10\cdot\left(\frac{1}{5}+\frac{i}{10}\right) = \frac{10}{2}+i = 2+i = (2,1)$

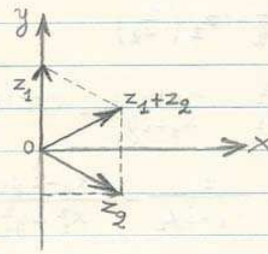
b) $(2,-3)\cdot(-2,1) = (2-3i)\cdot(-2+i) =$
$= -4+2\cdot i+6\cdot i-3\cdot i^2 = -1+8\cdot i = (-1,8)$

f) $(1-i)^4 = 1-4\cdot 1^3\cdot i+6\cdot 1\cdot i^2-4\cdot 1\cdot i^3+i^4 =$
$= 1-4i-6+4\cdot i+1 = -4$

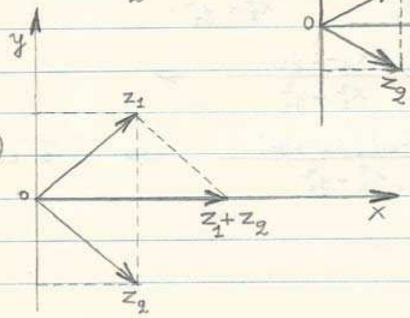
$(\alpha+\beta)^4 = \alpha^4+4\alpha^3\beta+6\cdot\alpha^2\beta^2+4\alpha\beta^3+\beta^4$

6/2.

α) $z_1 = 2\cdot i = 0+2i$
$z_2 = \frac{3}{2}-i$



d)



6/3.

α) $z = x + y\cdot i \Rightarrow \frac{z}{z} = \frac{x+y\cdot i}{x+yi} = \frac{(x+yi)(x-iy)}{(x+yi)(x-iy)} = \frac{x^2+y^2}{x^2+y^2} = 1$

b) $\frac{1}{z} = \frac{x-iy}{(x+iy)\cdot(x-iy)} = \frac{x-iy}{x^2+y^2}$

$\frac{1}{\frac{1}{z}} = \frac{x^2+y^2}{x-yi} = \frac{(x^2+y^2)\cdot(x+iy)}{(x-iy)\cdot(x^2+iy)} = \frac{x^2+y^2}{x^2+y^2}(x+iy) = z$

c) $Re(z) = Im(iz)$

$z = x+iy \Rightarrow Re(z) = x$

$iz = ix + i^2 y = -y + ix \Rightarrow Im(iz) = x.$

6/4. $z_1 = 1+i \qquad z_2 = 1-i$

$z_1^2 - 2z_1 + 2 = 0 \Rightarrow (1+i)^2 - 2\cdot(1+i) + 2 = 1+2i-1-2$
$-2i+2 = 0$

$z_2^2 - 2z_2 + 2 = 0 \Rightarrow (1-i)^2 - 2(1-i) + 2 =$
$= 1-2i-1-2+2i+2 = 0$

6/5. $\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)\cdot(x_2-iy_2)}{x_2^2+y_2^2} = \frac{x_1x_2 + y_1y_2 + i\cdot(x_2y_1 - x_1y_2)}{x_2^2+y_2^2}$

$\frac{1}{z_2} = \frac{1}{x_2+iy_2} = \frac{x_2-iy_2}{x_2^2+y_2^2} \Rightarrow$

$z_1\cdot\left(\frac{1}{z_2}\right) = \frac{(x_1+iy_1)\cdot(x_2-iy_2)}{x_2^2+y_2^2} \Rightarrow$ ὅ. ἔ. δ.

$$\frac{1}{z_2} = \frac{1}{x_2 + iy_2} = \frac{x_2 - iy_2}{x_2^2 + y_2^2} \qquad \frac{1}{z_3} = \frac{x_3 - iy_3}{x_3^2 + y_3^2}$$

$$\left(\frac{1}{z_2}\right)\cdot\left(\frac{1}{z_3}\right) = \frac{(x_2 - iy_2)\cdot(x_3 - iy_3)}{(x_2^2+y_2^2)(x_3^2+y_3^2)} = \frac{x_2x_3 - y_2y_3 - i\cdot(x_3y_2 + x_2y_3)}{x_2^2x_3^2 + x_2^2y_3^2 + y_2^2x_3^2 + y_2^2y_3^2}$$

$$z_2 z_3 = (x_2 + i\cdot y_2)\cdot(x_3 + iy_3) = x_2x_3 - y_2y_3 + i\cdot(x_3y_2 + x_2y_3)$$

$$\frac{1}{z_2\cdot z_3} = \frac{1}{x_2x_3 - y_2y_3 + i\cdot(x_3y_2 + x_2y_3)} = \frac{x_2x_3 - y_2y_3 - i(x_3y_2 + x_2y_3)}{(x_2x_3 - y_2y_3)^2 + (x_3y_2 + y_3x_2)^2}$$

$$= \frac{x_2x_3 - y_2y_3 - i\cdot(x_3y_2 + x_2\cdot y_3)}{x_2^2x_3^2 + y_2^2y_3^2 + x_3^2y_2^2 + y_3^2x_2^2}$$

and so we have

$$\frac{1}{z_2\cdot z_3} = \left(\frac{1}{z_2}\right)\cdot\left(\frac{1}{z_3}\right)$$

6/6. $z_1z_2 = z_2z_1$

$$(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i\cdot(x_1y_2 + x_2y_1)$$
$$= (x_2x_1 - y_2y_1) + i\cdot(x_2y_1 + x_1y_2)$$
$$= (x_2 + iy_2)\cdot(x_1 + iy_1).$$

6/7. α) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$$x_1 + iy_1 + (x_2 + iy_2 + x_3 + iy_3) = x_1 + iy_1 + x_2 + iy_2 + x_3 + iy_3 =$$
$$= x_1 + x_2 + x_3 + i\cdot(y_1 + y_2 + y_3) = x_1 + x_2 + i\cdot(y_1 + y_2) + x_3 + iy_3$$
$$= (x_1 + iy_1 + x_2 + iy_2) + (x_3 + iy_3) = (z_1 + z_2) + z_3$$

β) $z_1 \cdot (z_2 z_3) = (z_1 z_2) \cdot z_3$

$(z_2 \cdot z_3) = [(x_2 + iy_2) \cdot (x_3 + iy_3)] = [x_2 x_3 - y_2 y_3 + i \cdot (x_2 y_3 + x_3 y_2)]$
$(z_1 \cdot z_2) = (x_1 + iy_1) \cdot (x_2 + iy_2) = [x_1 x_2 - y_1 y_2 + i \cdot (x_1 y_2 + x_2 y_1)]$

$z_1 \cdot (z_2 \cdot z_3) = (x_1 + y_1 i) \cdot [x_2 x_3 - y_2 y_3 + i \cdot (x_2 y_3 + x_3 y_2)] =$

$= x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - x_3 y_1 y_2 +$
$+ i \cdot (y_1 x_2 x_3 - y_1 y_2 y_3 + x_1 x_2 y_3 + x_1 x_3 y_2)$

$(z_1 \cdot z_2) \cdot z_3 = [x_1 x_2 - y_1 y_2 + i \cdot (x_1 y_2 + x_2 y_1)] \cdot (x_3 + y_3 i) =$

$= x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - x_2 y_1 y_3 +$
$+ i \cdot (x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 x_3 y_2 + x_2 x_3 y_1)$

it is true.

6/8 $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$
$z_1 \cdot z_3 = (x_1 + y_1 i)(x_3 + y_3 i) = x_1 x_3 - y_1 y_3 + i \cdot (x_1 y_3 + y_1 x_3)$
$z_2 + z_3 = x_2 + y_2 i + x_3 + y_3 i = x_2 + x_3 + i \cdot (y_2 + y_3)$
$z_1 \cdot (z_2 + z_3) = (x_1 + y_1 i) \cdot [(x_2 + x_3) + i \cdot (y_2 + y_3)] =$
$= x_1 x_2 + x_1 x_3 - y_1 y_2 - y_1 y_3 +$
$+ i \cdot (y_1 x_2 + y_1 x_3 + x_1 y_2 + x_1 y_3)$

$z_1 \cdot z_2 + z_1 z_3 = x_1 x_2 - y_1 y_2 + i \cdot (x_1 y_2 + x_2 y_1) +$
$+ x_1 x_3 - y_1 y_3 + i \cdot (x_1 y_3 + y_1 x_3) =$
$= x_1 x_2 + x_1 x_3 - y_1 y_2 - y_1 y_3 +$
$+ i \cdot (y_1 x_2 + y_1 x_3 + x_1 y_2 + x_1 y_3)$

It is true

6/9. K = real number and $z = (x, y)$ then

$Kz = (Kx, Ky)$

$Kz = K\cdot(x+iy) = Kx + i\cdot Ky = (Kx, Ky)$

If $K=-1 \Rightarrow -z = (-1)\cdot z = -1\cdot x - 1\cdot iy = -x-iy$

6/10.
$$\frac{z_1+z_2}{z_3} = \frac{x_1+iy_1+x_2+iy_2}{x_3+iy_3} = \frac{x_1+x_2+i\cdot(y_1+y_2)}{x_3+iy_3} =$$
$$= \frac{(x_1+x_2)x_3+(y_1+y_2)\cdot y_3+i\cdot(x_3y_1+x_3y_2-x_1y_3-x_2y_3)}{x_3^2+y_3^2}$$
$$= \frac{x_1x_3+x_2x_3+y_1y_3+y_2y_3+i\cdot(x_3y_1+x_3y_2-x_1y_3-x_2y_3)}{x_3^2+y_3^2}$$
$$= \frac{x_1x_3+y_1y_3+i\cdot(x_3y_1-x_1y_3)}{x_3^2+y_3^2} + \frac{x_2x_3+y_2y_3+i\cdot(x_3y_2-x_2y_3)}{x_3^2+y_3^2}$$
$$= \frac{(x_1+iy_1)\cdot(x_3-iy_3)}{(x_3+iy_3)\cdot(x_3-iy_3)} + \frac{(x_2+iy_2)\cdot(x_3-iy_3)}{(x_3+iy_3)\cdot(x_3-iy_3)} =$$
$$= \frac{x_1+iy_1}{x_3+iy_3} + \frac{x_2+iy_2}{x_3+iy_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$$

6/11.
$$z\cdot(z_1+z_2+z_3) = (x+iy)\cdot(x_1+y_1\cdot i+x_2+y_2\cdot i+x_3+iy_3) =$$
$$= x\cdot(x_1+x_2+x_3)-y\cdot(y_1+y_2+y_3) +$$
$$+ i\cdot(xy_1+xy_2+xy_3+yx_1+yx_2+yx_3) =$$
$$= xx_1-yy_1+i\cdot(xy_1+yx_1) +$$
$$+xx_2-yy_2+i\cdot(xy_2+yx_2) +$$
$$+xx_3-yy_3+i\cdot(xy_3+yx_3) =$$
$$= (x+iy)\cdot(x_1+iy_1)+(x+iy)\cdot(x_2+iy_2)+(x+iy)\cdot(x_3+iy_3)$$
$$= z\cdot z_1 + z\cdot z_2 + z\cdot z_3$$

6/12. $(z_1 \cdot z_2) \cdot z_3 = (z_1 \cdot z_3) \cdot z_2 = (z_3 \cdot z_1) \cdot z_2 = \cdots = z_1 \cdot z_2 \cdot z_3$

$$(z_1 \cdot z_2) \cdot (z_3) = \left[(x_1 + iy_1) \cdot (x_2 + iy_2)\right](x_3 + iy_3) =$$

$$= \left[(x_1 x_2 - y_1 y_2) + i \cdot (x_1 y_2 + x_2 y_1)\right] \cdot (x_3 + iy_3) =$$

$$= (x_1 x_2 - y_1 y_2) x_3 - y_3 \cdot (x_1 y_2 + x_2 y_1) +$$
$$+ i \left[(x_1 x_2 - y_1 y_2) y_3 + x_3 \cdot (x_1 y_2 + x_2 y_1)\right] =$$

$$= x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 +$$
$$+ i \cdot (x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3)$$

$$z_1 \cdot z_2 \cdot z_3 = (x_1 + iy_1) \cdot (x_2 + iy_2)(x_3 + iy_3) =$$
$$= x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 +$$
$$+ i \cdot (x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3)$$

it is true.

6/13. $z_1\cdot z_2\cdot z_3=0$

$$(x_1+iy_1)\cdot(x_2+i\cdot y_2)\cdot(x_3+iy_3)=\left[x_1x_2-y_1y_2+i\cdot(x_1y_2+x_2y_1)\right]\cdot$$

$$(x_3+iy_3)=x_1x_2x_3-y_1\cdot y_2\cdot x_3-x_1y_2y_3-x_2y_1y_3+$$

$$+\,i\cdot\left(x_1x_2y_3-y_1y_2y_3+x_1y_2x_3+x_2x_3y_1\right)=0$$

$$\left.\begin{array}{l} x_1x_2x_3-x_1y_2y_3-x_2y_1y_3-y_1\cdot y_2\cdot x_3=0 \\ x_1x_2y_3+x_1y_2x_3-y_1y_2y_3+y_1x_2x_3=0 \end{array}\right\}$$

$$\begin{array}{l|l} x_1\cdot(x_2x_3-y_2y_3)-y_1\cdot(x_2y_3+y_2x_3)=0 & x_1 \\ y_1\cdot(x_2x_3-y_2y_3)+x_1\cdot(x_2y_3+y_2\cdot x_3)=0 & y_1 \end{array}$$

$$\left.\begin{array}{l} x_1^2\cdot(x_2x_3-y_2y_3)-x_1y_1\cdot(x_2y_3+y_2\cdot x_3)=0 \\ y_1^2\cdot(x_2x_3-y_2y_3)+x_1y_1\cdot(x_2y_3+y_2x_3)=0 \end{array}\right\}\Rightarrow$$

$$(x_1^2+y_1^2)\cdot(x_2x_3-y_2\cdot y_3)=0$$

α) $x_1^2+y_1^2=0 \Rightarrow x_1=y_1=0$ or

β) $x_2x_3-y_2y_3=0 \Rightarrow \left.\begin{array}{l} x_2y_3+y_2\cdot x_3=0 \\ x_2x_3-y_2\cdot y_3=0 \end{array}\right\}\begin{array}{l} y_3 \\ x_3 \end{array}$

$$\left.\begin{array}{l} x_2y_3^2+\cancel{y_2x_3y_3}=0 \\ x_2x_3^2-\cancel{y_2x_3y_3}=0 \end{array}\right\}\Rightarrow x_2\cdot(x_3^2+y_3^2)=0, \Rightarrow$$

$\beta_1)$ $x_3^2 + y_3^2 = 0 \Rightarrow x_3 = y_3 = 0$

$\beta_2)$ $x_2 = 0 \Rightarrow$ $-x_1 y_2 y_3 - y_1 y_2 x_3 = 0$

$-y_1 y_2 y_3 + x_1 y_2 x_3 = 0$

$x_1 \cdot y_2 \cdot y_3 + y_1 y_2 x_3 = 0 \Rightarrow y_2 \cdot (x_1 y_3 + y_1 x_3) = 0$

$y_1 y_2 y_3 - x_1 \cdot y_2 \cdot x_3 = 0 \Rightarrow y_2 \cdot (y_1 y_3 - x_1 y_3) = 0$

$\beta_2)_\alpha$ $y_2 = 0.$

$\beta_2)_\beta$ $\left.\begin{matrix} x_1 \cdot y_3 + y_1 x_3 = 0 \\ y_1 x_3 - x_1 y_3 = 0 \end{matrix}\right\} \Rightarrow y_1 \cdot x_3 = 0$ κ' $x_1 y_3 = 0$

ή $y_1 = 0$ $x_1 = 0$ ή $x_3 = 0$ $y_3 = 0$

$y_1 = 0$ $y_3 = 0$ $x_3 = 0$ $x_1 = 0$

τά όποῖα ἀποκλείονται ἄρα ἰσχύει τό $y_2 = 0$.

6/14. $$(z_1 \cdot z_2)\cdot(z_3 \cdot z_4) = (z_1 z_3)\cdot(z_2 \cdot z_4)$$

$$(z_1 z_2) = [(x_1+iy_1)\cdot(x_2+iy_2)] = [(x_1x_2 - y_1y_2) + i\cdot(x_1y_2 + x_2y_1)]$$

$$(z_3 z_4) = [(x_3+iy_3)\cdot(x_4+iy_4)] = [(x_3x_4 - y_3y_4) + i\cdot(x_3y_4 + x_4y_3)]$$

$$\begin{aligned}(z_1\cdot z_2)\cdot(z_3\cdot z_4) = {} & (x_1x_2 - y_1y_2)\cdot(x_3x_4 - y_3y_4) - \\ & - (x_1y_2 + x_2y_1)\cdot(x_3y_4 + x_4y_3) + \\ & + i\cdot[(x_1x_2 - y_1y_2)\cdot(x_3y_4 + x_4y_3) + \\ & + (x_3x_4 - y_3y_4)\cdot(x_1y_2 + x_2y_1)] =\end{aligned}$$

$$\begin{aligned}= {} & x_1\cdot x_2\cdot x_3\cdot x_4 - y_1y_2x_3x_4 - x_1x_2y_3y_4 + y_1y_2y_3y_4 - \\ & - x_1y_2x_3y_4 - y_1x_2x_3y_4 - x_1y_2y_3x_4 - y_1x_2y_3x_4 + \\ & + i\cdot(x_1x_2x_3y_4 - y_1y_2x_3y_4 + x_1x_2y_3x_4 - y_1y_2y_3x_4 + \\ & + x_1y_2x_3x_4 - x_1y_2y_3y_4 + y_1x_2x_3x_4 - y_1x_2y_3\cdot y_4)\end{aligned}$$

$$\begin{aligned}(z_1z_3)\cdot(z_2z_4) = {} & [(x_1x_3 - y_1y_3) + i\cdot(x_1y_3 + x_3y_1)]\cdot \\ & [(x_2x_4 - y_2y_4) + i\cdot(x_2y_4 + x_4y_2)] =\end{aligned}$$

$$= (x_1x_3 - y_1y_3)\cdot(x_2x_4 - y_2y_4) - (x_1y_3 + x_3y_1)(x_2y_4 + x_4y_2)$$

$$+ i\cdot[(x_1x_3 - y_1y_3)\cdot(x_2y_4 + x_4y_2) + (x_2x_4 - y_2y_4)\cdot(x_1y_3 + x_3y_1)]$$

$$\begin{aligned}= {} & x_1x_2x_3x_4 + y_1y_2y_3y_4 - x_1y_2x_3y_4 - y_1x_2y_3x_4 - \\ & - x_1x_2y_3y_4 - x_1y_2x_3y_4 - y_1x_2x_3y_4 - y_1y_2x_3x_4 + \\ & + i\cdot(x_1x_2x_3y_4 + x_1y_2x_3x_4 - y_1x_2y_3y_4 - y_1y_2y_3x_4 + \\ & + x_1x_2y_3x_4 - x_1y_2y_3y_4 + y_1x_2x_3x_4 - y_1y_2x_3y_4)\end{aligned}$$

it is true

6/15. $$\frac{z_1 z_2}{z_3 \cdot z_4} = \left(\frac{z_1}{z_3}\right)\cdot\left(\frac{z_2}{z_4}\right)$$

$$\frac{z_1 z_2}{z_3 z_4} = \frac{\left[(x_1x_2 - y_1y_2) + i\cdot(x_2y_1 + x_1y_2)\right]\cdot\left[(x_3x_4 - y_3y_4) - i(x_3y_4 + x_4y_3)\right]}{\left[(x_3x_4 - y_3y_4) + i\cdot(x_3y_4 + x_4y_3)\right]\cdot\left[(x_3x_4 - y_3y_4) - i(x_3y_4 + x_4y_3)\right]}$$

$$= \frac{\left[(x_1x_2 - y_1y_2)(x_3x_4 - y_3y_4) - (x_2y_1 + x_1y_2)(x_3y_4 + x_4y_3)\right] + i\cdot\left[\,A\,\right]}{\left[(x_3x_4 - y_3y_4)^2 + (x_3y_4 + x_4y_3)^2\right] = \Pi_1}$$

$$= \frac{x_1x_2x_3x_4 - x_1x_2y_3y_4 - y_1y_2x_3x_4 + y_1y_2y_3y_4 - y_1x_2x_3y_4 - y_1x_2y_3x_4 + x_1y_2x_3y_4 - x_1y_2y_3x_4}{\Pi}$$

$$+\, i\cdot\frac{y_1x_2x_3x_4 + x_1y_2x_3x_4 - y_1x_2y_3y_4 - x_1y_2y_3y_4 - x_1x_2x_3y_4 + y_1y_2x_3y_4 - x_1x_2y_3x_4 + y_1y_2y_3x_4}{\Pi}$$

$$\frac{z_1}{z_3} = \frac{(x_1 + iy_1)\cdot(x_3 - iy_3)}{(x_3 + iy_3)(x_3 - iy_3)} = \frac{(x_1x_3 + y_1y_3) + i\cdot(x_3y_1 - x_1y_3)}{x_3^2 + y_3^2} = \frac{\alpha_1}{\omega_1}$$

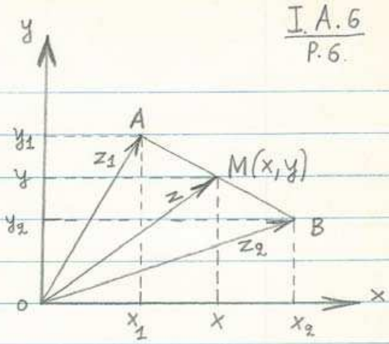
$$\frac{z_2}{z_4} = \frac{(x_2 + iy_2)(x_4 - iy_4)}{(x_4 + iy_4)\cdot(x_4 - iy_4)} = \frac{(x_2x_4 + y_2y_4) + i\cdot(x_4y_2 - x_2y_4)}{x_4^2 + y_4^2} = \frac{\alpha_2}{\omega_2}$$

$$\left(\frac{z_1}{z_3}\right)\cdot\left(\frac{z_2}{z_4}\right) = \frac{\alpha_1}{\omega_1}\cdot\frac{\alpha_2}{\omega_2} = \frac{\alpha_1\alpha_2}{\omega_1\omega_2} = \frac{\alpha_1\alpha_2}{\Pi}$$

it is true

$$\frac{z\cdot z_1}{z\cdot z_2} = \frac{z_1}{z_2}.$$ From above we have:

$$\frac{z\cdot z_1}{z\cdot z_2} = \left(\frac{z}{z}\right)\cdot\left(\frac{z_1}{z_2}\right) = 1\cdot\frac{z_1}{z_2} = \frac{z_1}{z_2}$$



6/16. It is :

$$\frac{x-x_1}{x_2-x_1} = \frac{MA}{AB} = \frac{y_1-y}{y_1-y_2} \,.$$

$$x = \frac{1}{2}\cdot(x_1+x_2)$$
$$y = \frac{1}{2}\cdot(y_1+y_2)$$

$$z = \frac{1}{2}(x_1+x_2) + i\cdot\frac{1}{2}\cdot(y_1+y_2) =$$
$$= \frac{1}{2}\cdot\left[x_1+iy_i + x_2+iy_2\right] = \frac{1}{2}\cdot(z_1+z_2)$$

6/17. $z = x+iy \qquad 1+z = 1+x+iy$

$$(1+z)^2 = (1+x+iy)^2 = (1+x)^2 + 2(1+x)\cdot iy + i^2y^2 =$$
$$= 1+x^2+2x+2\cdot y\cdot i + 2xyi - y^2$$
$$= 1+(x^2+2xyi-y^2)+2\cdot(x+iy) =$$
$$= 1+2\cdot(x+iy)+(x+iy)^2 = 1+2\cdot z+z^2$$

6/18.

$$(1+\alpha)^1 = 1+\alpha$$
$$(1+\alpha)^2 = 1+2\alpha+\alpha^2$$
$$(1+\alpha)^3 = 1+3\alpha+3\alpha^2+\alpha^3$$
$$(1+\alpha)^4 = 1+4\alpha+6\alpha^2+4\alpha^3+\alpha^4$$
$$(1+\alpha)^5 = 1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4+\alpha^5$$

$$(1+\alpha)^n = 1^n + n\cdot 1^{n-1}\cdot\alpha^1 + \frac{n\cdot(n-1)}{2}\cdot 1^{n-2}\cdot\alpha^2 + \frac{n\cdot(n-1)\cdot(n-2)}{3}\cdot 1^{n-3}\cdot\alpha^3 +$$
$$+ \frac{n\cdot(n-1)(n-2)\cdot(n-3)}{4}\cdot 1^{n-4}\cdot\alpha^4 + \cdots\cdots +$$
$$+ \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots(n-K+1)}{K}\cdot 1^{n-K}\cdot\alpha^K + \cdots\cdots +$$
$$+ \frac{n(n-1)\cdot(n-2)\cdot(n-3)\cdots(n-K+1)\cdots(n-n+K+1)}{n-K}\cdot 1^{n-n+K}\cdot\alpha^{n-K} +$$
$$+\cdots + \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-K+1)\cdots(K+1)\cdots(n-n+2)}{n-n+2}\cdot 1^{n-n+1}\cdot\alpha^{n-1} + \cdots$$
$$+\cdots + \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-K+1)\cdots(K+1)\cdots 2\cdot(n-n+1)}{n-n+1}\, 1^{n-n}\cdot\alpha^{n} + \cdots +$$

$$(1+\alpha)^n = 1 + n\cdot\alpha + \frac{n\cdot(n-1)}{2}\cdot\alpha^2 + \frac{n\cdot(n-1)\cdot(n-2)}{3}\cdot\alpha^3 + \cdots\cdots +$$
$$+ \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-K+1)}{K}\cdot\alpha^K + \cdots +$$
$$+ \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-K+1)\cdots 2\cdot 1}{n}\cdot\alpha^n$$

$$P_{n-1} = 1\cdot 2\cdot 3\cdot 4\cdot\ \cdots\ (n-k)\cdots\ (n-2)\cdot(n-1) \ = (n-1)!$$

$$P_k = 1\cdot 2\cdot 3\cdot 4\cdots\ k \ = k!$$

$$(n-k+1)\cdots(n-2)\cdot(n-1)\cdot \ = \frac{1\cdot 2\cdot 3\cdots(n-k+1)\cdots(n-2)\cdot(n-1)}{1\cdot 2\cdot 3\cdots(n-k)}$$

$$= \frac{(n-1)!}{(n-k)!}$$

από το τέλος: $\overset{o\varsigma}{1}, \overset{o\varsigma}{2}, \overset{o\varsigma}{3} \cdots k-1, \overset{o\varsigma}{k}, k+1, \ldots, n-k-1, \overbrace{n-k}^{o\varsigma}, n-k+1, \cdots n-2, n-1, n$

$n, \ n-1, \ n-2, \cdots n-k+2, n-k+1, n-k, \cdots, k+2, \ k+1, \ k, \cdots, 3, 2, 1$

$$(1+z)^n = (1+x+iy)^n = [(1+x)+iy]^n$$

$$= (1+x)^n + n\cdot(1+x)^{n-1}\cdot(iy) + \frac{n\cdot(n-1)}{2}\cdot(1+x)^{n-2}\cdot(iy)^2 + \cdots\cdots +$$

$$+ \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k}\cdot(1+x)^{n-k}\cdot(iy)^k + \cdots\cdots +$$

$$+ \frac{n\cdot(n-1)\cdots(n-k+1)\cdots 2}{n-n+2}\cdot(1+x)^1\cdot(iy)^{n-1} +$$

$$+ \frac{n\cdot(n-1)\cdots(n-k+1)\cdots 2\cdot 1}{n-n+1}\cdot(1+x)^0\cdot(iy)^n$$

$$(1+z)^n = (1+x)^n + n\cdot(1+x)^{n-1}\cdot(iy) + \frac{n\cdot(n-1)}{2}\cdot(1+x)^{n-2}\cdot(iy)^2 + \cdots$$

$$+ \cdots + \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k}\cdot(1+x)^{n-k}\cdot(iy)^k + \cdots +$$

$$+ \cdots + \frac{n\cdot(n-1)\cdots(n-k+1)\cdots 2}{2}\cdot(1+x)\cdot(iy)^{n-1} +$$

$$+ \frac{n\cdot(n-1)\cdots(n-k+1)\cdots 2\cdot 1}{1}\cdot(iy)^n$$

$$(1+x)^n = 1 + n\cdot x + \frac{n\cdot(n-1)}{2}\cdot x^2 + \cdots \frac{n\cdot(n-1)\cdots(n-k+1)}{k}\cdot x^k + \cdots + x^n$$

$$(1+x)^{n-1} = 1 + (n-1)x + \frac{(n-1)\cdot(n-2)}{2}\cdot x^2 + \cdots \frac{(n-1)\cdot(n-2)\cdots(n-k)}{k}\cdot x^k + \cdots + x^{n-1}$$

$$(1+x)^{n-2} =$$

etc.

12

EXERCISES

1. Show that

(a) $\overline{\bar{z} + 3i} = z - 3i$; (b) $\overline{iz} = -i\bar{z}$;

(c) $\dfrac{(2 + i)^2}{3 - 4i} = 1$; (d) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}\,|2z + 5|$.

2. Find one value of arg z when

(a) $z = \dfrac{z_1}{z_2}$ $(z_2 \neq 0)$; (b) $z = z_1^n$ $(n = 1, 2, \ldots)$;

(c) $z = \dfrac{-2}{1 + i\sqrt{3}}$; (d) $z = \dfrac{i}{-2 - 2i}$; (e) $z = (\sqrt{3} - i)^6$.

Ans. (a) $\arg z_1 - \arg z_2$; (b) $n \arg z_1$; (c) $2\pi/3$; (e) π.

3. Use the polar form to show that

(a) $i(1 - i\sqrt{3})(\sqrt{3} + i) = 2 + 2i\sqrt{3}$; (b) $\dfrac{5i}{2 + i} = 1 + 2i$;

(c) $(-1 + i)^7 = -8(1 + i)$;

(d) $(1 + i\sqrt{3})^{-10} = 2^{-11}(-1 + i\sqrt{3})$.

4. Let z_0 be a fixed complex number and R a positive constant. Show why point z lies on a circle of radius R with center at $-z_0$ when z satisfies

Exercises

12/1. α) $z = x+iy \Rightarrow \bar{z} = x-iy \Rightarrow \bar{z}+3i = x+i\cdot(3-y)$

$\overline{\bar{z}+3i} = x-i(3-y) = x+iy-3i = z-3i$

β) $iz = ix-y \Rightarrow \overline{iz} = -y-ix = i^2 y - ix$

$= -i\cdot(x-iy) = -i\cdot\bar{z}$

c) $\overline{2+i} = 2-i \Rightarrow (\overline{2+i})^2 = (2-i)^2 = 4-1-4i = 3-4i$

$\frac{(\overline{2+i})^2}{3-4i} = \frac{3-4i}{3-4i} = \frac{z}{z} = 1$

d) $z = x+iy \Rightarrow \bar{z} = x-iy$

$2\bar{z}+5 = 2x-2y\cdot i+5 = 2x+5-2y\cdot i$

$P = (2\bar{z}+5)\cdot(\sqrt{2}-i) = [(2x+5)-2yi]\cdot(\sqrt{2}-i) =$

$= \sqrt{2}\cdot(2x+5) - 2y - i[(2x+5)+2\sqrt{2}\cdot y]$

$= 2\sqrt{2}\cdot x + 5\sqrt{2} - 2y - i\cdot(2x+5+2\sqrt{2}\cdot y)$

$|P| = [8x^2+50+4y^2+40\cdot x - \cancel{8\sqrt{2}\cdot xy} - \cancel{20\sqrt{2}\cdot y} +$

$4x^2+25+8y^2+20x+\cancel{8\sqrt{2}\cdot xy}+\cancel{20\sqrt{2}\cdot y}]^{1/2}$

$= \sqrt{3}\cdot(4x^2+25+4y^2+20x)^{1/2} =$

$= \sqrt{3}\cdot[(2x)^2+5^2+2\cdot 5\cdot 2x+(2y)^2]^{1/2} =$

$= \sqrt{3}\cdot[(2x+5)^2+(2y)^2]^{1/2} =$

$= \sqrt{3}\cdot|2x+5+i\cdot 2y| = \sqrt{3}\cdot|2\cdot(x+iy)+5| =$

$= \sqrt{3}\cdot|2\cdot z+5|$

19/2. α) $$z = \frac{z_1}{z_2} = \frac{x_1 + i y_1}{x_2 + i y_2} = \frac{x_1 x_2 - y_1 y_2 + i\cdot(x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2}$$

$$z = \frac{x_1 x_2 - y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2} = r\cdot(\cos\theta + i \sin\theta)$$

$$\tan\theta = \frac{x_1 y_2 + x_2 y_1}{x_1 x_2 - y_1 y_2}.$$

 or :

$$z = \frac{z_1}{z_2} = \frac{r_1\cdot(\cos\theta_1 + i \sin\theta_1)}{r_2\cdot(\cos\theta_2 + i \sin\theta_2)} = \frac{r_1}{r_2}\cdot\left[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right]$$

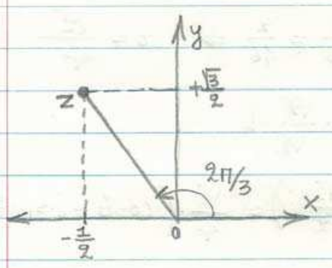
$$\arg z = \arg z_1 - \arg z_2$$
$$\theta = \theta_1 - \theta_2$$

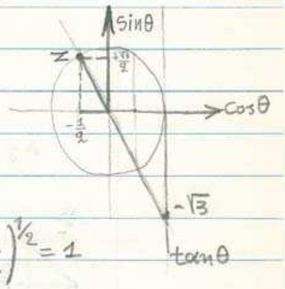
b) $$z_1 = r_1\cdot(\cos\theta_1 + i \sin\theta_1)$$
$$z_1\cdot z_1 = r_1\cdot r_2\cdot(\cos(\theta_1+\theta_1) + i \sin(\theta_1+\theta_1))$$

$$z = z_1^n = r_1^n\cdot\left[\cos(n\theta_1) + i\cdot\sin(n\theta_1)\right] \qquad n = 1, 2, \dots$$

$$\arg z = n\cdot\arg z_1 \Rightarrow \theta = n\cdot\theta_1$$

(c). $z = \dfrac{-2}{1+i\cdot\sqrt{3}} = \dfrac{-2(1-i\sqrt{3})}{1+(\sqrt{3})^2} = \dfrac{-2+2\sqrt{3}\cdot i}{4} = -\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\cdot i$





$r = \left(\frac{1}{4}+\frac{3}{4}\right)^{1/2} = 1$

$z = -\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i = 1\cdot(\cos\theta + i\cdot\sin\theta) = \cos\theta + i\cdot\sin\theta$

$\Rightarrow \cos\theta = -\frac{1}{2} \qquad \sin\theta = +\frac{\sqrt{3}}{2}$

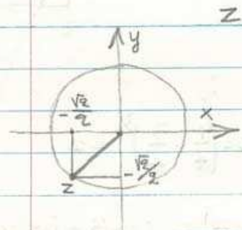
$\theta = \frac{2\pi}{3}$

d) $$z = \frac{i}{-2-2i} = \frac{i\cdot(-2+2\cdot i)}{(-2)^2+(-2)^2} = \frac{-2i+2i^2}{4+4}$$

$$= \frac{-2-2\cdot i}{8} = -\frac{1}{4} - \frac{1}{4}\cdot i$$

$$r^2 = \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 = \frac{2}{16} = \frac{2}{16} \Rightarrow r = \frac{\sqrt{2}}{4}$$

$$z = -\frac{1}{4}\cdot\frac{\sqrt{2}}{\sqrt{2}} - \frac{1}{4}\cdot\frac{\sqrt{2}}{\sqrt{2}}\cdot i$$



$$= \frac{\sqrt{2}}{4}\cdot\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot i\right) = \frac{\sqrt{2}}{4}\cdot\left(\cos\theta + i\cdot\sin\theta\right)$$

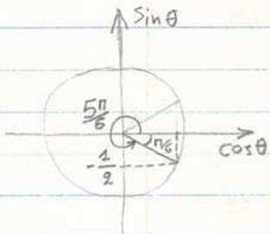
$$\cos\theta = -\frac{\sqrt{2}}{2} = \sin\theta \Rightarrow \theta = \frac{5\pi}{4}$$

e) $$z = \left(\sqrt{3}-i\right)^6$$

$$\left|\sqrt{3}-i\right| = \left(3+1\right)^{1/2} = 2$$

$$z_1 = \sqrt{3}-i = 2\cdot\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot i\right) = 2\cdot\left(\cos\theta_1 + i\sin\theta_1\right)$$

$$\cos\theta_1 = \frac{\sqrt{3}}{2}, \quad \sin\theta_1 = -\frac{1}{2}$$



$$z = z_1^n = 2^n\cdot\left(\cos n\theta_1 + i\sin n\theta_1\right)$$

$$n = 6$$

$$z = 2^6\cdot\left(\cos 6\theta_1 + i\sin 6\theta_1\right)$$

$$\theta_1 = 5\pi/6$$

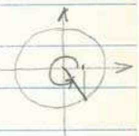
$$\theta = 6\cdot\theta_1 = 5\pi \qquad \theta = \pi.$$

12/3. α) $i\cdot(1-i\sqrt{3})\cdot(\sqrt{3}+i) = (i+\sqrt{3})(\sqrt{3}+i) =$
$-1+3+2\sqrt{3}\cdot i = 2+2\sqrt{3}\cdot i$

WITH POLAR FORM.

$z_1 = i = x_1+iy_1 = 0+1\cdot i \Rightarrow \tau_1 = 1 \,,\ tag\,\theta_1 = +\infty \Rightarrow \theta_1 = \frac{\pi}{2}$

$z_2 = 1-i\cdot\sqrt{3} \Rightarrow \tau_2 = \sqrt{1+3} = 2,\quad z_2 = 2\cdot\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i\right)$
$\cos\theta_2 = \frac{1}{2} \qquad \sin\theta_2 = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta_2 = +\frac{5\pi}{3}$



$z_3 = \sqrt{3}+i \Rightarrow \tau_3 = \sqrt{3+1} = 2,\quad z_3 = 2\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot i\right)$
$\cos\theta_3 = \frac{\sqrt{3}}{2},\quad \sin\theta_3 = \frac{1}{2} \quad \Rightarrow \quad \theta_3 = \frac{\pi}{6}$

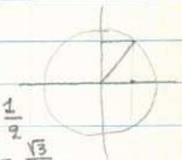
$$z_1\cdot z_2\cdot z_3 = \tau_1\cdot\tau_2\cdot\tau_3\cdot\left[\cos(\theta_1+\theta_2+\theta_3)+i\cdot\sin(\theta_1+\theta_2+\theta_3)\right]$$

$$= 1\cdot 2\cdot 2\cdot\left[\cos\left(\frac{\pi}{2}+\frac{5\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{2}+\frac{5\pi}{3}+\frac{\pi}{6}\right)\right]$$

$$= 4\cdot\left(\cos\left(\frac{\pi}{2}+\frac{11\pi}{6}\right)+i\cdot\sin\left(\frac{\pi}{2}+\frac{11\pi}{6}\right)\right)$$

$$= 4\cdot\left(\cos\frac{3\pi+11\pi}{6}+i\sin\frac{3\pi+11\pi}{6}\right)$$

$$= 4\cdot\left[\cos\left(\frac{\pi}{3}+2\pi\right)+i\sin\left(\frac{\pi}{3}+2\pi\right)\right]$$



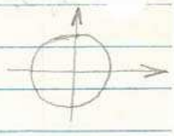
$$= 4\cdot\left[\cos\frac{\pi}{3}+i\cdot\sin\frac{\pi}{3}\right]$$

$\cos\frac{\pi}{3} = \frac{1}{2}$
$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$= 4\cdot\left(\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right) =$$

$$= 2+2\cdot i\cdot\sqrt{3}$$

b) $z_1 = 5i \Rightarrow r_1 = 5 \Rightarrow z_1 = 5(0+i)$



$\cos\theta_1 = 0 \qquad \sin\theta_1 = 1 \qquad \theta_1 = \frac{\pi}{2}$

$z_2 = 2+i \Rightarrow r_2 = \sqrt{4+1} = \sqrt{5} \Rightarrow z_2 = \sqrt{5}\cdot\left(\frac{2\sqrt{5}}{5} + \frac{\sqrt{5}}{5}\cdot i\right)$

$\cos\theta_2 = \frac{2\sqrt{5}}{5} \qquad \sin\theta_2 = \frac{\sqrt{5}}{5} \qquad \theta_2 > 0$ (1^{st} quadrant)

$\theta_2 = \arcsin\frac{\sqrt{5}}{5} = \arccos\frac{2\sqrt{5}}{5}$

$$z = \frac{z_1}{z_2} = \frac{5}{\sqrt{5}}\cdot\left[\cos\left(\frac{\pi}{2} - \arcsin\frac{\sqrt{5}}{5}\right) + i\cdot\sin\left(\frac{\pi}{2} - \arccos\frac{2\sqrt{5}}{5}\right)\right]$$

$$= \sqrt{5}\cdot\left(\sin\arcsin\frac{\sqrt{5}}{5} + i\cdot\cos\arccos\frac{2\sqrt{5}}{5}\right)$$

$$= \sqrt{5}\cdot\left(\frac{\sqrt{5}}{5} + i\cdot\frac{2\sqrt{5}}{5}\right) = 1 + 2\cdot i$$

c) $z_1 = -1+i \Rightarrow r_1 = \sqrt{2} \Rightarrow z_1 = \sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$\cos\theta_1 = -\frac{\sqrt{2}}{2} \qquad \sin\theta_1 = \frac{\sqrt{2}}{2} \Rightarrow \theta_1 = \frac{3\pi}{4}$

$$z = z_1^7 = (\sqrt{2})^7\cdot\left(\cos\frac{3\pi\cdot 7}{4} + i\sin\frac{3\pi\cdot 7}{4}\right)$$

$$= 2^3\sqrt{2}\cdot\left(\cos\frac{21\pi}{4} + i\sin\frac{21\pi}{4}\right)$$

$$= 8\sqrt{2}\cdot\left[\cos\left(4\pi + \frac{5\pi}{4}\right) + i\cdot\sin\left(4\pi + \frac{5\pi}{4}\right)\right]$$

$$= 8\sqrt{2}\cdot\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right]$$

$$= 8\sqrt{2}\cdot\left(-\cos\frac{\pi}{4} - i\cdot\sin\frac{\pi}{4}\right)$$

$$= 8\cdot\sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 8\cdot(-1 - i\cdot 1)$$

$$= -8\cdot(1+i)$$

d) $z_1 = 1 + i\sqrt{3} \Rightarrow r_1 = \sqrt{4} = 2 \Rightarrow z_1 = 2 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i\right)$

$\cos\theta_1 = \frac{1}{2} \qquad \sin\theta_1 = \frac{\sqrt{3}}{2} \qquad \theta_1 = \frac{\pi}{3}$

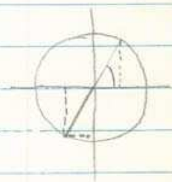
$$z^{-10} = r_1^{-10} \cdot \left[\cos(-10\theta_1) + i \cdot \sin(-10\theta_1)\right]$$

$$= \frac{1}{2^{10}} \cdot \left[\cos\left(\frac{10\pi}{3}\right) + i\,\sin\left(-\frac{10\pi}{3}\right)\right]$$

$$= \frac{1}{2^{10}} \cdot \left[\cos\frac{10\pi}{3} - i \cdot \sin\left(\frac{10\pi}{3}\right)\right]$$

$$= \frac{1}{2^{10}} \cdot \left[\cos\left(2\pi + \frac{4\pi}{3}\right) - i\,\sin\left(2\pi + \frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{2^{10}} \cdot \left[\cos\frac{4\pi}{3} - i \cdot \sin\frac{4\pi}{3}\right]$$

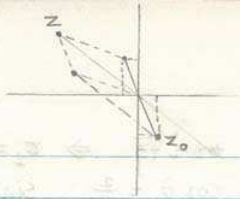


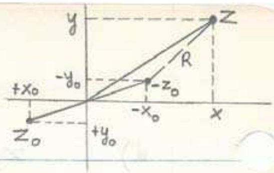
$$= \frac{1}{2^{10}} \cdot \left(-\cos\frac{\pi}{3} + i \cdot \sin\frac{\pi}{3}\right)$$

$$= 2^{-10} \cdot \left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right) =$$

$$= 2^{-11} \cdot \left(-1 + i \cdot \sqrt{3}\right)$$

$R = |z - (-z_0)|$
$= |z + z_0|$





12/4. a) $|z + z_0| = R$

$|x + iy + x_0 + iy_0| = \left[(x+x_0)^2 + (y+y_0)^2\right]^{1/2} = R$

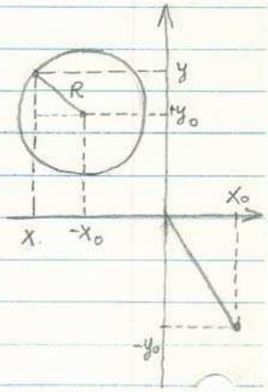
$(x+x_0)^2 + (y+y_0)^2 = R^2$

if $x_0 \gtreqless 0$, $y_0 \gtreqless 0$

circle with center in

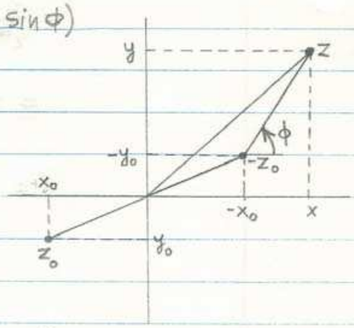
$(-x_0, -y_0) = -z_0$

and radius R.



b) $z + z_0 = R \cdot (\cos\phi + i \cdot \sin\phi)$

$z - (-z_0) = R \cdot (\cos\phi + \sin\phi)$



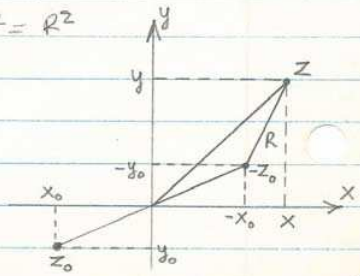
c) $z\bar{z} + \bar{z}_0 \cdot z + z_0\bar{z} + z_0\bar{z}_0 = R^2$

$|z|^2 + 2\mathrm{Re}(z_0\bar{z}) + |z_0|^2 = R^2$

$x^2 + y^2 + 2(x_0 x + y_0 y) + x_0^2 + y_0^2 = R^2$

$x^2 + 2xx_0 + x_0^2 + y^2 + 2yy_0 + y_0^2 = R^2$

$(x+x_0)^2 + (y+y_0)^2 = R^2$



13/

any one of the equations

(a) $|z + z_0| = R$; (b) $z + z_0 = R(\cos\phi + i\sin\phi)$,

where ϕ is real; (c) $z\bar{z} + \bar{z}_0 z + z_0\bar{z} + z_0\bar{z}_0 = R^2$.

5. Prove that (a) z is real if $\bar{z} = z$; (b) z is either real or pure imaginary if $z^2 = (\bar{z})^2$.

6. In Sec. 4, establish (a) formula (3); (b) formula (4).

7. Prove that (a) $\overline{z_1 z_2 z_3} = \bar{z}_1\bar{z}_2\bar{z}_3$; (b) $\overline{(z^4)} = (\bar{z})^4$.

8. Prove property (7), Sec. 5, on the absolute value of a quotient.

9. If $z_2 z_3 \neq 0$, show that

$$(a)\ \overline{\left(\frac{z_1}{z_2 z_3}\right)} = \frac{\bar{z}_1}{\bar{z}_2\bar{z}_3};\qquad (b)\ \left|\frac{z_1}{z_2 z_3}\right| = \frac{|z_1|}{|z_2||z_3|}.$$

10. Give an algebraic proof of triangle inequality (9), Sec. 5.

11. If $|z_2| \neq |z_3|$, prove that

$$\left|\frac{z_1}{z_2 + z_3}\right| \leq \frac{|z_1|}{\big||z_2| - |z_3|\big|}.$$

12. Prove that $|z|\sqrt{2} \geq |\mathcal{R}(z)| + |\mathcal{I}(z)|$.

13. Given that $z_1 z_2 \neq 0$, use the polar form with arguments measured in radians to prove that

$$\mathcal{R}(z_1\bar{z}_2) = |z_1|\,|z_2|$$

if and only if $\arg z_2 = \arg z_1 \pm 2n\pi$ $\quad(n = 0, 1, 2, \ldots)$.

14. Given that $z_1 z_2 \neq 0$, use the result in Exercise 13 to prove that

$$|z_1 + z_2| = |z_1| + |z_2|$$

if and only if $\arg z_2 = \arg z_1 \pm 2n\pi$. Also, note the geometric verification of this statement.

15. Given that $z_1 z_2 \neq 0$, use the result in Exercise 13 to prove that

$$|z_1 - z_2| = \big||z_1| - |z_2|\big|$$

if and only if $\arg z_2 = \arg z_1 \pm 2n\pi$. Also, note the geometric verification of this statement.

16. Establish the formula

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1),$$

for the sum of a finite geometric series; then derive the formulas

$$(a)\ 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2\sin(\theta/2)},$$

$$(b)\ \sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos[(n + \frac{1}{2})\theta]}{2\sin(\theta/2)},$$

where $0 < \theta < 2\pi$.

13/5. a) $\bar{z} = z \Rightarrow x - iy = x + iy \Rightarrow 0 = -x + iy + x + iy$

$0 = 2iy \Rightarrow y = 0 \Rightarrow z = x$ real.

b) $z^2 = (\bar{z})^2 \Rightarrow z^2 - (\bar{z})^2 = 0 \Rightarrow$

$(z - \bar{z}) \cdot (z + \bar{z}) = 0$

i. $z - \bar{z} = 0 \Rightarrow z = x$ real

or ii. $z + \bar{z} = 0 \Rightarrow x + iy + x - iy = 2x = 0 \Rightarrow x = 0$

$z = iy \Rightarrow$ pure imaginary

13/6. a) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i \cdot (x_1 y_2 + x_2 y_1)$

$\overline{z_1 z_2} = x_1 x_2 - y_1 y_2 - i \cdot (x_1 y_2 + x_2 y_1)$

$\bar{z}_1 \cdot \bar{z}_2 = (x_1 - iy_1) \cdot (x_2 - iy_2) = x_1 x_2 - y_1 y_2 - i \cdot (x_1 y_2 + x_2 y_1)$

and so $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

b) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \left[\cos(\theta_1 - \theta_2) + i \cdot \sin(\theta_1 - \theta_2)\right]$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{r_1}{r_2} \cdot \left[\cos(\theta_1 - \theta_2) - i \cdot \sin(\theta_1 - \theta_2)\right]$

$\frac{\bar{z}_1}{\bar{z}_2} = \frac{r_1 \cdot (\cos\theta_1 - i\sin\theta_1)}{r_2 \cdot (\cos\theta_2 - i\sin\theta_2)} = \frac{r_1 \cdot (\cos(-\theta_1) + i \cdot \sin(-\theta_1))}{r_2 \cdot (\cos(-\theta_2) + i \sin(-\theta_2))}$

$= \frac{r_1}{r_2}\left[\cos(-\theta_1 + \theta_2) + i \sin(-\theta_1 + \theta_2)\right]$

$= \frac{r_1}{r_2} \cdot \left[\cos(\theta_1 - \theta_2) - i \cdot \sin(\theta_1 - \theta_2)\right]$ it is true

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

13/8.

i) $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot\left[\cos(\theta_1-\theta_2)+i\cdot\sin(\theta_1-\theta_2)\right] \Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$$

$$\frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \qquad \text{it is true}$$

ii) $$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1x_2+y_1y_2+i\cdot(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{1}{(x_2^2+y_2^2)^2}\cdot\left[(x_1x_2+y_1y_2)^2+(x_2y_1-x_1y_2)^2\right]$$

$$= \frac{x_1^2x_2^2+y_1^2y_2^2+x_2^2y_1^2+x_1^2y_2^2}{(x_2^2+y_2^2)^2}$$

$$= \frac{x_1^2\cdot(x_2^2+y_2^2)+y_1^2\cdot(x_2^2+y_2^2)}{(x_2^2+y_2^2)^2} = \frac{x_1^2+y_1^2}{x_2^2+y_2^2} = \frac{|z_1|}{|z_2|}$$

13/9. α) $$\frac{z_1}{z_2z_3} = \frac{r_1\cdot(\cos\theta_1+i\sin\theta_1)}{r_2r_3\cdot\left[\cos(\theta_1+\theta_2)+i\cdot\sin(\theta_1+\theta_2)\right]}$$

$$= \frac{r_1}{r_2r_3}\cdot\left[\cos(\theta_1-\theta_2-\theta_3)+i\cdot\sin(\theta_1-\theta_2-\theta_3)\right]$$

$$\overline{\left(\frac{z_1}{z_2z_3}\right)} = \frac{r_1}{r_2r_3}\left[\cos(\theta_1-\theta_2-\theta_3)-i\cdot\sin(\theta_1-\theta_2-\theta_3)\right]$$

$$= \frac{r_1}{r_2r_3}\cdot\left[\cos(-\theta_1+\theta_2+\theta_3)+i\cdot\sin(-\theta_1+\theta_2+\theta_3)\right]$$

$$\frac{\bar{z}_1}{\bar{z}_2\cdot\bar{z}_3} = \frac{r_1\cdot\left[\cos(-\theta_1)+i\cdot\sin(-\theta_1)\right]}{r_2\cdot r_3\cdot\left[\cos(-\theta_2-\theta_3)+i\sin(-\theta_2-\theta_3)\right]}$$

$$= \frac{r_1}{r_2r_3}\cdot\left[\cos(-\theta_1+\theta_2+\theta_3)+i\cdot\sin(-\theta_1+\theta_2+\theta_3)\right] \quad \text{it is true}$$

B) $$\frac{z_1}{z_2 z_3} = \frac{r_1}{r_2 r_3} \cdot \left[\cos(\theta_1 - \theta_2 - \theta_3) + i \cdot \sin(\theta_1 - \theta_2 - \theta_3)\right]$$

$$\left|\frac{z_1}{z_2 z_3}\right| = \frac{r_1}{r_2 \cdot r_3}$$

$$\frac{|z_1|}{|z_2| \cdot |z_3|} = \frac{|r_1 \cdot (\cos\theta_1 + i \sin\theta_1)|}{|r_2 \cdot (\cos\theta_2 + i \sin\theta_2)| \cdot |r_3 \cdot (\cos\theta_3 + i \sin\theta_3)|}$$

$$= \frac{r_1}{r_2 \cdot r_3} \Rightarrow$$

$$\left|\frac{z_1}{z_2 \cdot z_3}\right| = \frac{|z_1|}{|z_2| \cdot |z_3|}$$

13/10

$$|z_1 - z_2| \geqslant \big||z_1| - |z_2|\big|$$

It is true that $|z|^2 = z \cdot \bar{z}$ so we have

$$|z_1 - z_2|^2 = (z_1 - z_2)\cdot(\overline{z_1 - z_2}) = (z_1 - z_2)\cdot(\bar{z}_1 - \bar{z}_2)$$
$$= z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$$
$$= |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + z_2\bar{z}_1)$$

We have $z_1\bar{z}_2 = \overline{\bar{z}_1 z_2}$ and $z + \bar{z} = 2\operatorname{Re}(z)$

so $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\operatorname{Re}(z_1\bar{z}_2)$

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$$
$$= (|z_1| - |z_2|)^2 + 2|z_1|\cdot|z_2| - 2\operatorname{Re}(z_1\bar{z}_2)$$

$$|z_1 - z_2|^2 - \big||z_1| - |z_2|\big|^2 = 2|z_1|\cdot|z_2| - 2\operatorname{Re}(z_1\bar{z}_2)$$

$$|z_2| = |\bar{z}_2| \Rightarrow |z_1|\cdot|z_2| = |z_1|\cdot|\bar{z}_2| = |z_1\bar{z}_2|$$

$$\underbrace{\Big(|z_1 - z_2| - \big||z_1| - |z_2|\big|\Big)}_{\alpha}\cdot\underbrace{\Big(|z_1 - z_2| + \big||z_1| - |z_2|\big|\Big)}_{\beta > 0} = 2|z_1\bar{z}_2| - 2\operatorname{Re}(z_1\bar{z}_2)$$

It is always $|z_1\bar{z}_2| \geqslant \operatorname{Re}(z_1\bar{z}_2) \Rightarrow 2|z_1\bar{z}_2| - 2\operatorname{Re}(z_1\bar{z}_2) \geqslant 0$
and since $\beta > 0$, $(z_1 \neq z_2)$ we obtain that

$$|z_1 - z_2| - \big||z_1| - |z_2|\big| \geqslant 0 \Rightarrow |z_1 - z_2| \geqslant \big||z_1| - |z_2|\big|$$

ό.ε.δ.

13/11. $|z_2| \neq |z_3|$

$$\left|\frac{z_1}{z_2+z_3}\right| = \frac{|z_1|}{|z_2+z_3|}$$

$$|z_2 - z_3| \geq \big||z_2|-|z_3|\big| \quad , \quad z_3 \to -z_3$$

$$|z_2+z_3| \geq \big||z_2|-|-z_3|\big| \quad , \quad |z_3| = |-z_3|$$

$$|z_2+z_3| \geq \big||z_2|-|z_3|\big|$$

$$\frac{1}{|z_2+z_3|} \leq \frac{1}{\big||z_2|-|z_3|\big|} \qquad |z_1|>0$$

$$\frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|}$$

13/12. $|z|\cdot\sqrt{2} \geq |Re(z)|+|Im(z)|$ $\quad z = x+iy \Rightarrow |z| = (x^2+y^2)^{1/2}$

$Re(z)=x \Rightarrow |Re(z)|=|x|$

$Im(z)=y \Rightarrow |Im(z)|=|y|$

$$(x^2+y^2)^{1/2} \geq \frac{|x|+|y|}{\sqrt{2}}$$

$2\cdot(x^2+y^2) \geq |x|^2+|y|^2 + 2|xy|$

$x^2+y^2-2|xy| \geq 0$

$(|x|-|y|)^2 \geq 0$

$$(|x|^2+|y|^2)^2 = \left[|x|^2+|y|^2+2|x||y|-2|x|\cdot|y|\right]^{1/2} =$$

$$= \left[(|x|+|y|)^2 - 2|x||y|\right]^{1/2} \geq \left[(|x|+|y|)^2\right]^{1/2}$$

$$\geq |x|+|y|$$

It is true that $|x|+|y| \geq \frac{|x|+|y|}{\sqrt{2} > 1}$ so it is true the first.

13/13.

$$\left.\begin{array}{l} 2\cdot Re(z) = z+\bar{z} \\ z_1\bar{z}_2 = \overline{\bar{z}_1\cdot z_2} \\ \bar{z}_1 z_2 = \overline{z_1\bar{z}_2} \end{array}\right\} \Rightarrow \quad 2Re(z_1\bar{z}_2) = z_1\bar{z}_2 + \bar{z}_1 z_2$$

$$z_1\bar{z}_2 = r_1\cdot(\cos\theta_1 + i\cdot\sin\theta_1)\cdot r_2\cdot[\cos(-\theta_2) + i\cdot\sin(-\theta_2)]$$

$$z_1\bar{z}_2 = r_1 r_2\cdot[\cos(\theta_1-\theta_2) + i\cdot\sin(\theta_1-\theta_2)]$$

$$\bar{z}_1 z_2 = \overline{z_1\bar{z}_2} = r_1 r_2\cdot[\cos(\theta_1-\theta_2) - i\cdot\sin(\theta_1-\theta_2)]$$

$$Re(z_1\bar{z}_2) = \frac{1}{2}\cdot[r_1 r_2\cdot\cos(\theta_1-\theta_2) + r_1 r_2\cdot\cos(\theta_1-\theta_2)]$$

$$= r_1\cdot r_2\cdot\cos(\theta_1-\theta_2)$$

$$= |z_1|\cdot|z_2|\cdot\cos(\theta_1-\theta_2)$$

$$\cos(\theta_2-\theta_1) = \cos(\theta_1-\theta_2) = 1 = \cos(2n\pi)$$

$$\theta_2-\theta_1 = 2n\pi \quad \Rightarrow \quad \theta_2 = \theta_1 + 2n\pi$$

$$\arg z_2 = \arg z_1 + 2n\pi \quad , \quad n=0,1,2\ldots$$

13/14. $$|z_1+z_2|^2 = (z_1+z_2)\cdot(\overline{z_1+z_2})$$
$$= (z_1+z_2)\cdot(\bar{z}_1+\bar{z}_2)$$
$$= z_1\bar{z}_1 + z_2\bar{z}_2 + (z_1\bar{z}_2 + \bar{z}_1 z_2)$$

$$z_1\bar{z}_2 = \bar{z}_1 z_2 \Rightarrow$$

$$z_1\cdot\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\,Re(z_1\bar{z}_2)$$

$$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2| - 2|z_1|\cdot|z_2| + 2Re(z_1\bar{z}_2)$$

$$|z_1+z_2|^2 - (|z_1|+|z_2|)^2 = -2|z_1|\cdot|z_2| + 2Re(z_1\bar{z}_2)$$
$$= -2|z_1||\bar{z}_2| + 2Re(z_1\bar{z}_2)$$

From Exerc. 13 we have $Re(z_1\bar{z}_2) = |z_1|\cdot|z_2|$

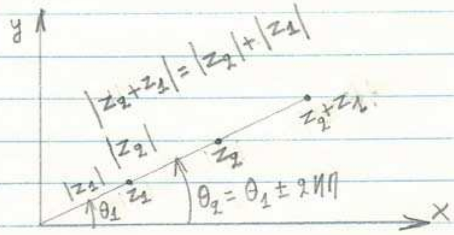
So $-2|z_1|\cdot|z_2| + 2Re(z_1\bar{z}_2) = 0$

or

$$\left[|z_1+z_2| - (|z_1|+|z_2|)\right]\cdot\underbrace{\left[|z_1+z_2| + |z_1| + |z_2|\right]}_{\beta>0 \text{ because } z_1z_2\neq 0} = 0$$

So $|z_1+z_2| = |z_1| + |z_2|$

if and only if $\arg z_2 = \arg z_1 + 2n\pi$



13/15. $$|z_1 - z_2|^2 = (z_1 - z_2)\cdot(\overline{z_1 - z_2}) = (z_1 - z_2)\cdot(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$$

$$z_1\bar{z}_2 = \overline{\bar{z}_1 z_2}\ , \quad \bar{z}_1 z_2 = \overline{z_1\bar{z}_2} \Rightarrow$$
$$z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\cdot Re\cdot(z_1\bar{z}_2)\ , \qquad |z|^2 = z\cdot\bar{z}$$

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\cdot|z_2| + 2|z_1|\cdot|z_2| - 2Re(z_1\bar{z}_2)$$

$$|z_1 - z_2|^2 - \left[\,|z_1| - |z_2|\,\right]^2 = 2|z_1|\cdot|z_2| - 2Re(z_1\bar{z}_2) = 0$$

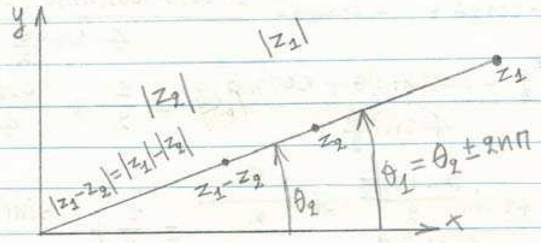
because of exerc. 13

$$|z_1 - z_2|^2 - \big|\,|z_1| - |z_2|\,\big|^2 = 0$$

$$\left[\,|z_1 - z_2| - \big|\,|z_1| - |z_2|\,\big|\,\right]\cdot\underbrace{\left[\,|z_1 - z_2| + \big|\,|z_1| - |z_2|\,\big|\,\right]}_{\text{b>0 because } z_1 z_2 \neq 0} = 0$$

So
$$|z_1 - z_2| = \big|\,|z_1| - |z_2|\,\big|$$

if and only if $\quad \arg z_2 = \arg z_1 + 2n\pi$



13/16. $$1+z+z^2+z^3+\dots+z^n = \frac{1-z^{n+1}}{1-z}$$

$$z = \cos\theta + i\sin\theta$$
$$z^2 = \cos 2\theta + i\sin 2\theta$$
$$z^3 = \cos 3\theta + i\sin 3\theta$$
$$\dots\dots\dots\dots$$
$$z^n = \cos n\theta + i\sin n\theta$$

$$1+z+z^2+z^3+\dots+z^n = 1+\cos\theta+\cos 2\theta+\cos 3\theta+\dots+\cos n\theta + i(\sin\theta+\sin 2\theta+\sin 3\theta+\dots+\sin n\theta)$$

$$\frac{1-z^{n+1}}{1-z} = \frac{[1-\cos(n+1)\theta] - i\sin(n+1)\theta}{[1-\cos\theta] - i\sin\theta} =$$

$$= \frac{[1-\cos(n+1)\theta]\cdot[1-\cos\theta] + \sin(n+1)\theta\cdot\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} +$$

$$+ i\,\frac{[1-\cos(n+1)\theta]\cdot\sin\theta - [1-\cos\theta]\cdot\sin(n+1)\theta}{(1-\cos\theta)^2+\sin^2\theta}$$

$$(1-\cos\theta)^2+\sin\theta = 1-2\cos\theta+\cos^2\theta+\sin^2\theta = 2\cdot(1-\cos\theta) = 4\sin^2\frac{\theta}{2}$$

α) $$1+\cos\theta+\cos 2\theta+\dots+\cos n\theta = \frac{1-\cos\theta-\cos(n+1)\theta+\cos\theta\cdot\cos(n+1)\theta+\sin\theta\cdot\sin(n+1)\theta}{4\cdot\sin^2\frac{\theta}{2}}$$

$$= \frac{2\cdot\sin^2\frac{\theta}{2} - \cos(n+1)\theta + \cos n\theta}{4\sin^2\frac{\theta}{2}} = \frac{1}{2} + \frac{\cos n\theta - \cos(n+1)\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{2} + \frac{+2\sin\frac{(2n+1)\theta}{2}\cdot\sin\frac{+\theta}{2}}{4\sin^2\frac{\theta}{2}} = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\cdot\sin^2\frac{\theta}{2}}$$ ο.ε.δ.

β) $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta =$

$$= \frac{\sin\theta - \sin\theta\cdot\cos(n+1)\theta - \sin(n+1)\theta + \cos\theta\cdot\sin(n+1)\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{\sin\theta - \sin(n+1)\theta + \cos\theta\cdot\sin(n+1)\theta - \sin\theta\cdot\cos(n+1)\theta}{2(1-\cos\theta)}$$

$$= \frac{2\cdot\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} + \frac{\sin(n\theta) - \sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{2}\cdot\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + \frac{2\cdot\cos\frac{2n+1}{2}\theta\cdot\sin\frac{n-n-1}{2}\theta}{4\cdot\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{2}\cdot\cot\frac{\theta}{2} - \frac{\cos\left(n+\frac{1}{2}\right)\theta}{2\cdot\sin\frac{\theta}{2}}$$

$$= \frac{1}{2}\cdot\cot\frac{\theta}{2} - \frac{\cos\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\cdot\sin\frac{\theta}{2}}$$

18/

EXERCISES

1. Find all values of each of the following roots. Check graphically.

(a) $(2i)^{\frac{1}{2}}$; (b) $(-i)^{\frac{1}{3}}$; (c) $(-1)^{\frac{1}{3}}$; (d) $8^{\frac{1}{6}}$.

Ans. (a) $\pm(1+i)$; (b) $i, (\pm\sqrt{3}-i)/2$;
(d) $\pm\sqrt{2}, (\pm 1 \pm i\sqrt{3})/\sqrt{2}$.

2. Find all values of

(a) $(-1+i\sqrt{3})^{\frac{3}{2}}$; (b) $(-1)^{-\frac{1}{4}}$. *Ans.* (a) $\pm 2\sqrt{2}$.

3. Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.
Ans. $(z^2 + 2z + 2)(z^2 - 2z + 2)$.

4. From the formula for the sum of a finite geometric series (Exercise 16, Sec. 7) show that, if w is any imaginary nth root of unity, then

$$1 + w + w^2 + \cdots + w^{n-1} = 0$$

5. Prove that the usual quadratic formula solves the quadratic equation $az^2 + bz + c = 0$ when the coefficients a, b, and c are complex numbers.

6. If m and n are positive integers, show (a) that

$$(z_1 z_2)^m = z_1^m z_2^m;$$ Προσοχή

(b) that the two sets of numbers $(z_1 z_2)^{1/n}$ and $z_1^{1/n} z_2^{1/n}$ are the same; and hence (c) that the two sets $(z_1 z_2)^{m/n}$ and $z_1^{m/n} z_2^{m/n}$ are the same.

7. Describe geometrically the region determined by each of the following conditions. Also, classify the region with the aid of the terms defined in Sec. 9.

(a) $|\mathcal{R}(z)| < 2$; (b) $|z-4| > 3$; (c) $|z - 1 + 3i| \leqq 1$;
(d) $|\mathcal{I}(z)| > 1$; (e) $\mathcal{R}(z) > 0$; (f) $0 \leqq \arg z \leqq \pi/4, z \neq 0$.

Ans. (a), (b), (e) unbounded domain; (c) closed region, the closure of a bounded domain; (d) unbounded open region, not connected.

8. Describe each of these regions geometrically:

(a) $-\pi < \arg z < \pi, |z| > 2$; (b) $1 < |z - 2i| < 2$;
(c) $|2z + 3| > 4$; (d) $\mathcal{I}(z^2) > 0$;
(e) $\mathcal{R}\left(\frac{1}{z}\right) < \frac{1}{2}$; (f) $|z - 4| > |z|$.

Exercises

18/1. (α) $(2i)^{\frac{1}{2}} = \sqrt{2i}$ $\qquad z = 2i \qquad n = 2$

$$\sqrt[n]{z} = \sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i \cdot \sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

$n = 2 \qquad k = 0, 1. \qquad z = 2i \Rightarrow |z| = \sqrt{4} = 2$

$$\left.\begin{array}{l} z = r\cdot(\cos\theta + i\cdot\sin\theta) \\ z = 2\cdot(0 + i\cdot 1) \end{array}\right\} \Rightarrow \cos\theta = 0,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\sqrt{2i} = \sqrt{2}\cdot\left[\cos\left(\frac{\pi}{2\cdot 2} + \frac{2k\pi}{2}\right) + i\cdot\sin\left(\frac{\pi}{2\cdot 2} + \frac{2k\pi}{2}\right)\right]$$

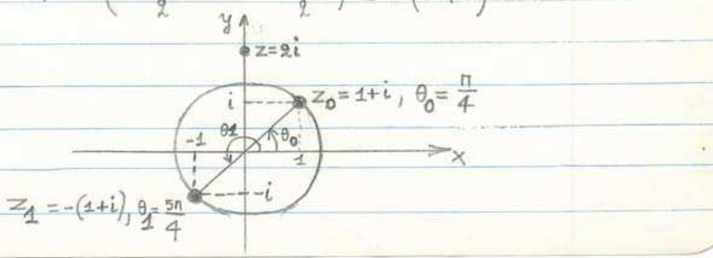
$$= \sqrt{2}\cdot\left[\cos\left(\frac{\pi}{4} + k\pi\right) + i\cdot\sin\left(\frac{\pi}{4} + k\pi\right)\right]$$

$k=0 \qquad z_0 = \sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right) \Rightarrow \theta_0 = \frac{\pi}{4}$

$$= \sqrt{2}\cdot\left(\frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}\right) = 1 + i$$

$k=1 \qquad z_1 = \sqrt{2}\cdot\left[\cos\left(\frac{\pi}{4} + \pi\right) + i\cdot\sin\left(\frac{\pi}{4} + \pi\right)\right] \Rightarrow \theta_1 = \frac{5\pi}{4}$

$$= \sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2} - i\cdot\frac{\sqrt{2}}{2}\right) = -(1+i)$$



b) $(-i)^{1/3} = \sqrt[3]{-i}$ $\qquad z=-i \;, \; n=3$

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\cdot\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \quad , \; k=0,1,2,\cdots(n-1)$$

Here: $n=3 \Rightarrow k=0,1,2.$

$z=-i \Rightarrow r=|z|=\sqrt{(-1)^2}=\sqrt{1}=1>0$ because $r>0$

$z=1(0-1\cdot i) \Longrightarrow \cos\theta=0 \quad \sin\theta=-1 \Rightarrow \theta=\frac{3\pi}{2}$

$$\sqrt[3]{-i} = \sqrt[3]{1}\cdot\left[\cos\left(\frac{3\pi}{2\cdot 3}+\frac{2k\pi}{3}\right)+i\cdot\sin\left(\frac{3\pi}{2\cdot 3}+\frac{2k\pi}{3}\right)\right]$$

$$= 1\cdot\left[\cos\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right)+i\,\sin\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right)\right] \qquad \text{So:}$$

$$\sqrt[3]{-i} = \cos\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right)+i\cdot\sin\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right) \quad , \; k=0,1,2$$

$k=0 \qquad z_0=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2} \Rightarrow \theta_1=\frac{\pi}{2}$

$\qquad = 0+i\cdot 1 = i$

$k=1 \qquad z_1=\cos\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)$

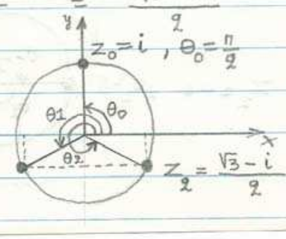
$\qquad = \cos\frac{7\pi}{6}+i\cdot\sin\frac{7\pi}{6} \Longrightarrow \theta_2=\frac{7\pi}{6}$

$\qquad = -\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2} = \frac{-\sqrt{3}-i}{2}$

$k=2 \qquad z_2=\cos\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)+i\cdot\sin\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)$

$\qquad = \cos\frac{11\pi}{6}+i\cdot\sin\frac{11\pi}{6} \Rightarrow \theta_3=\frac{11\pi}{6}$

$\qquad = \frac{\sqrt{3}}{2}-i\cdot\frac{1}{2} = \frac{\sqrt{3}-i}{2}$



$z_1=\frac{-\sqrt{3}-i}{2} \;,\; \theta_1=\frac{7\pi}{6}$ $\qquad z_2=\frac{\sqrt{3}-i}{2} \;,\; \theta_2=\frac{11\pi}{6}$

c) $(-1)^{\frac{1}{3}} = \sqrt[3]{-1}$ $\qquad z=-1 \;,\; n=3$

$$\sqrt[n]{z} = \sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i \cdot \sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \;,\; k = 0, 1, \cdots n-1$$

Here it is : $\quad z=-1 \quad n=3 \quad k=0,1,2$

$$|z| = r = 1 > 0 \quad (r > 0)$$

$$z = 1 \cdot (-1 + i \cdot 0) \Rightarrow \cos\theta = -1 \;,\; \sin\theta = 0 \Rightarrow \theta = \pi$$

$$\sqrt[3]{-1} = \sqrt[3]{1} \cdot \left[\cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right]$$

$$= \cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) \qquad k=0,1,2$$

$k=0 \quad z_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \quad \Rightarrow \theta_0 = \frac{\pi}{3}$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i = \frac{1 + i \cdot \sqrt{3}}{2}$$

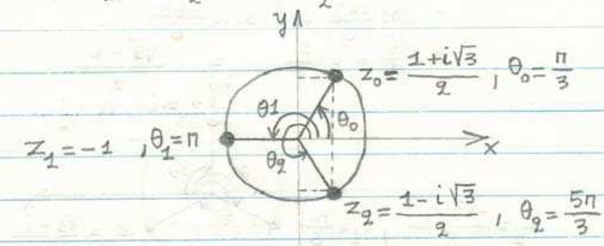
$k=1 \quad z_1 = \cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) \quad \Rightarrow \theta_1 = \pi$

$$= \cos\pi + i \cdot \sin\pi = -1$$

$k=2 \quad z_2 = \cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) \Rightarrow \theta_2 = \frac{5\pi}{3}$

$$= \cos\frac{5\pi}{3} + i \cdot \sin\frac{5\pi}{3}$$

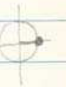
$$= \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} = \frac{1 - i \cdot \sqrt{3}}{2}$$



d) $8^{\frac{1}{6}} = \sqrt[6]{8}$ $\qquad z = 8 \quad n = 6$

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \qquad k=0,1,2,\cdots(n-1)$$

Here: $n=6 \quad k=0,1,2,3,4,5$



$z=8, \; |z| = r = \sqrt{8^2} = 8 > 0 \quad (r>0)$

$z = 8\cdot(1+i\cdot 0) \Rightarrow \cos\theta = 1 \quad \sin\theta = 0 \Rightarrow \theta = 0$

$$\sqrt[6]{8} = \sqrt[6]{8}\cdot\left[\cos\left(0+\frac{2k\pi}{6}\right)+i\sin\left(0+\frac{2k\pi}{6}\right)\right] \quad , \quad \sqrt[6]{8} = \sqrt[2\cdot 3]{2^3} = \sqrt[2]{2} = \sqrt{2}$$

$$= \sqrt{2}\cdot\left(\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}\right) \qquad k=0,1,2,3,4,5$$

$k=0 \qquad z_0 = \sqrt{2}\cdot(\cos 0 + i\sin 0) = \sqrt{2} \qquad \Rightarrow \quad \theta_0 = 0$

$k=1 \qquad z_1 = \sqrt{2}\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \qquad \Rightarrow \quad \theta_1 = \frac{\pi}{3}$

$\qquad = \sqrt{2}\cdot\left(\frac{1}{2}+i\sqrt{3}/2\right) = \frac{1+i\sqrt{3}}{\sqrt{2}}$

$k=2 \qquad z_2 = \sqrt{2}\cdot\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) \qquad \Rightarrow \quad \theta_2 = \frac{2\pi}{3}$

$\qquad = \sqrt{2}\cdot\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \frac{-1+i\sqrt{3}}{\sqrt{2}}$

$k=3 \qquad z_3 = \sqrt{2}\cdot(\cos\pi + i\sin\pi) \qquad \Rightarrow \quad \theta_3 = \pi$

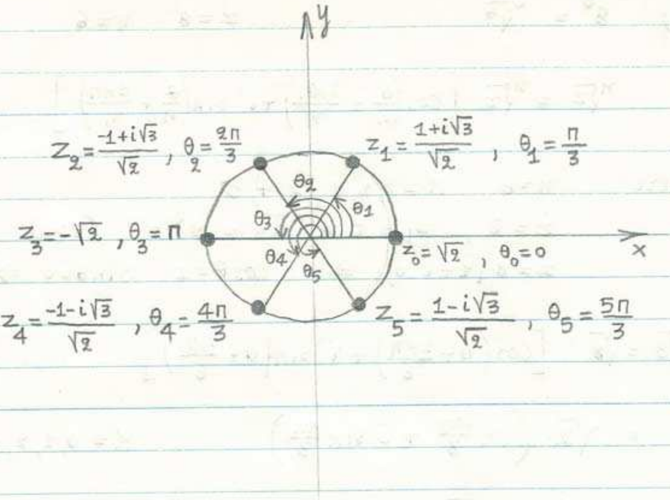
$\qquad = \sqrt{2}\,(-1+0) = -\sqrt{2}$

$k=4 \qquad z_4 = \sqrt{2}\cdot\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right) \qquad \Rightarrow \quad \theta_4 = \frac{4\pi}{3}$

$\qquad = \sqrt{2}\cdot\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = \frac{-1-i\sqrt{3}}{\sqrt{2}}$

$k=5 \qquad z_5 = \sqrt{2}\cdot\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right) \qquad \Rightarrow \quad \theta_5 = \frac{5\pi}{3}$

$\qquad = \sqrt{2}\cdot\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = \frac{1-i\sqrt{3}}{\sqrt{2}}$



y
x
$z_2 = \frac{-1+i\sqrt{3}}{\sqrt{2}}$, $\theta_2 = \frac{2\pi}{3}$
$z_1 = \frac{1+i\sqrt{3}}{\sqrt{2}}$, $\theta_1 = \frac{\pi}{3}$
$z_3 = -\sqrt{2}$, $\theta_3 = \pi$
$z_0 = \sqrt{2}$, $\theta_0 = 0$
$z_4 = \frac{-1-i\sqrt{3}}{\sqrt{2}}$, $\theta_4 = \frac{4\pi}{3}$
$z_5 = \frac{1-i\sqrt{3}}{\sqrt{2}}$, $\theta_5 = \frac{5\pi}{3}$
θ_1 θ_2 θ_3 θ_4 θ_5

18/2. α) $\left(-1+i\sqrt{3}\right)^{\frac{3}{2}} = \sqrt{\left(-1+i\sqrt{3}\right)^3}$ $\qquad z=-1+i\sqrt{3} \quad m=3,\ n=2$

$$z^{\frac{m}{n}} = \sqrt[n]{z^m} = \sqrt[n]{r^m}\cdot\left[\cos\left(\frac{m\theta}{n}+\frac{2K\pi\cdot m}{n}\right)+i\cdot\sin\left(\frac{m\theta}{n}+\frac{2K\pi}{n}\right)\right]$$

$K=0,1,2,\cdots,n-1$

Here: $m=3 \quad n=2 \quad K=0,1$

$z=-1+i\sqrt{3} \Rightarrow |z|=r=\sqrt{1+3}=2$

$z=2\cdot\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \Rightarrow \cos\theta=-\frac{1}{2} \quad \sin\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{2\pi}{3}$

$$\left(-1+i\sqrt{3}\right)^{\frac{3}{2}} = \sqrt[2]{2^3}\cdot\left[\cos\left(\frac{3\cdot 2\pi}{3\cdot 2}+\frac{2K\pi}{2}\right)+i\sin\left(\frac{3\cdot 2\pi}{3\cdot 2}+\frac{2K\pi}{2}\right)\right]$$

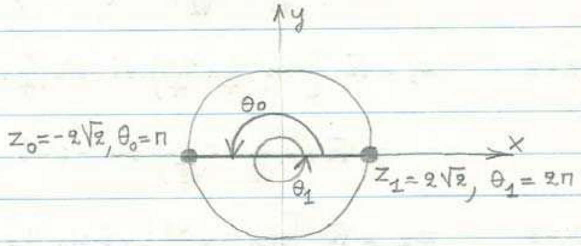
$$= 2\sqrt{2}\cdot\left[\cos(\pi+K\pi)+i\cdot\sin(\pi+K\pi)\right]$$

$K=0 \qquad z_0 = 2\sqrt{2}\ (\cos\pi + i\cdot\sin\pi) \qquad \Rightarrow \theta_0=\pi$

$\qquad = 2\sqrt{2}\cdot(-1+0) = -2\sqrt{2}$

$K=1 \qquad z_1 = 2\sqrt{2}\cdot(\cos 2\pi + i\sin 2\pi) \qquad \Rightarrow \theta_1=2\pi$

$\qquad = 2\sqrt{2}\cdot(1+0) = 2\sqrt{2}$



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18/3. $z^4+4=0 \qquad z^4=-4 \qquad z=\sqrt[4]{-4}$

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right]$$

Here $z=-4 \qquad n=4 \qquad k=0,1,2,3$

$|z|=\sqrt{(-4)^2}=4=r>0$

$z=4(-1+0\cdot i) \qquad \cos\theta=-1 \quad \sin\theta=0 \Rightarrow \theta=\pi$

$$\sqrt[4]{-4}=\sqrt[4]{4}\cdot\left[\cos\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)\right]$$

$k=0 \qquad z_0=\sqrt[4]{4}\cdot\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) \Rightarrow \theta_0=\frac{\pi}{4}$

$=\sqrt[4]{4}\cdot\left(\frac{1}{\sqrt{2}}+i\cdot\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{\sqrt{4}}}{\sqrt{2}}(1+i)=\frac{\sqrt{2}}{\sqrt{2}}(1+i)$

$=1+i$

$k=1 \qquad z_1=\sqrt{2}\cdot\left[\cos\left(\frac{\pi}{4}+\frac{2\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi}{4}\right)\right] \Rightarrow \theta_1=\frac{3\pi}{4}$

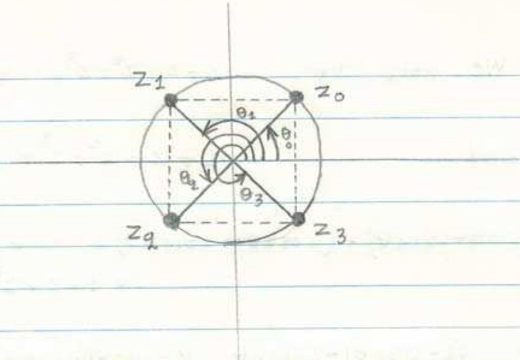
$=\sqrt{2}\cdot\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=(-1+i)$

$k=2 \qquad z_2=\sqrt{2}\cdot\left[\cos\left(\frac{\pi}{4}+\frac{4\pi}{4}\right)+i\cdot\sin\left(\frac{\pi}{4}+\frac{4\pi}{4}\right)\right] \Rightarrow \theta_2=\frac{5\pi}{4}$

$=\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{i\cdot 1}{\sqrt{2}}\right)=-1-i$

$k=3 \qquad z_3=\sqrt{2}\cdot\left[\cos\left(\frac{\pi}{4}+\frac{6\pi}{4}\right)+i\cdot\sin\left(\frac{\pi}{4}+\frac{6\pi}{4}\right)\right] \Rightarrow \theta_3=\frac{7\pi}{4}$

$=\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}-i\cdot\frac{1}{\sqrt{2}}\right)=1-i$



$z^4 + 4 = 0$

$z^4 + 4 = (z - z_0)(z - z_1)\cdot(z - z_2)(z - z_3)$

$= [z - (1+i)][z - (-1+i)][z - (-1-i)][z - (1-i)]$

$= [z - (1+i)][z + (1+i)]\cdot[z + (1-i)]\cdot[z - (1-i)]$

$= [z - (1+i)]\cdot[z - (1-i)]\cdot[z + (1+i)]\cdot[z + (1-i)]$

$= [z^2 - z(1+i) - z(1-i) + (1+i)(1-i)]\cdot$
$[z^2 + z(1+i) + z(1-i) + (1+i)(1-i)]$

$= (z^2 - z\cdot 2 + 2)\cdot(z^2 + 2\cdot z + 2)$ So we have

$z^4 + 4 = (z^2 + 2z + 2)(z^2 - 2z + 2).$

18/4. We have that $1+z+z^2+z^3+\dots+z^n = \frac{1-z^{n+1}}{1-z}$

and $1+z+z^2+z^3+\dots+z^{n-1} = \frac{1-z^n}{1-z}$

$$1+(\cos\theta+i\sin\theta)+(\cos2\theta+i\sin2\theta)+\dots+[\cos(n-1)\theta+i\sin(n-1)\theta]$$
$$= 1+z+z^2+\dots+z^{n-1} = \frac{1-z^n}{1-z}$$

$$\frac{1-z^n}{1-z} = \frac{(1-\cos n\theta)-i\sin n\theta}{(1-\cos\theta)-i\sin\theta} = \frac{(1-\cos n\theta)(1-\cos\theta)+\sin\theta\cdot\sin n\theta}{(1-\cos\theta)^2+\sin^2\theta} +$$
$$+i\,\frac{(1-\cos n\theta)\cdot\sin\theta-(1-\cos\theta)\cdot\sin n\theta}{(1-\cos\theta)^2+\sin^2\theta}$$
$$= \frac{1}{2-2\cos\theta}\cdot\Big[(1-\cos n\theta-\cos\theta+\cos\theta\cdot\cos n\theta+\sin\theta\cdot\sin n\theta)+$$
$$+i(\sin\theta-\sin\theta\cdot\cos n\theta-\sin n\theta+\cos\theta\cdot\sin n\theta)\Big]$$

α) $1+\cos\theta+\cos2\theta+\dots+\cos(n-1)\theta = \frac{1}{2(1-\cos\theta)}\cdot(1-\cos\theta-\cos n\theta+$

$+\cos\theta\cdot\sin n\theta+\sin\theta\cdot\sin n\theta) =$

$$= \frac{1}{2\cdot2\cdot\sin^2\frac{\theta}{2}}\cdot\left(2\sin^2\frac{\theta}{2}-\cos n\theta+\cos(n-1)\theta\right)$$
$$= \frac{1}{2}+\frac{1}{2\cdot\sin^2\frac{\theta}{2}}\cdot\left(\cos(n-1)\theta-\cos n\theta\right) = \frac{1}{2}+\frac{1}{2\cdot\sin^2\frac{\theta}{2}}\cdot2\sin\frac{(2n-1)}{2}\theta\cdot\sin\frac{-\theta}{2}$$
$$= \frac{1}{2}+\frac{1}{2}\cdot\frac{\sin(n-\frac{1}{2})\theta}{\sin\frac{\theta}{2}}, \quad \text{καθόσον } \cos\alpha-\cos\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta-\alpha}{2}$$

β) $\sin\theta+\sin2\theta+\dots+\sin(n-1)\theta = \frac{1}{2\cdot2\cdot\sin^2\frac{\theta}{2}}(\sin\theta-\sin\theta\cdot\cos n\theta-\sin n\theta+\cos\theta\cdot\sin n\theta) =$

$$= \frac{1}{2\cdot 2\cdot \sin^2\frac{\theta}{2}} \cdot \left[2\cdot \sin\frac{\theta}{2}\cdot \cos\frac{\theta}{2} - \sin n\theta + \sin(n-1)\theta\right]$$

$$= \frac{\cos\frac{\theta}{2}}{2\cdot \sin\frac{\theta}{2}} + \frac{1}{2\cdot 2\cdot \sin^2\frac{\theta}{2}} \cdot 2\cdot \cos\frac{(2n-1)\theta}{2} \cdot \sin\frac{-\theta}{2} \cdot$$

$$= \frac{\cos\frac{\theta}{2}}{2\cdot \sin\frac{\theta}{2}} - \frac{1}{2\cdot \sin\frac{\theta}{2}} \cdot \cos\left(n-\frac{1}{2}\right)\theta \cdot$$

$$= \frac{1}{2}\cdot \cot\frac{\theta}{2} - \frac{\cos\left(n-\frac{1}{2}\right)\theta}{2\cdot \sin\frac{\theta}{2}}$$

$$W = \sqrt[n]{1} = \sqrt[n]{|1|}\cdot\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = \cos\frac{2k\pi}{n} + i\cdot \sin\frac{2k\pi}{n}$$

From all above for $\theta = \frac{2k\pi}{n}$ we have

$$1 + W + W^2 + W^3 + \cdots W^{n-1} = \frac{1-W^n}{1-W} =$$

$$1 + \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) + \left(\cos 2\cdot\frac{2k\pi}{n} + i\cdot \sin 2\cdot \frac{2k\pi}{n}\right) + \cdots + \left(\cos\frac{(n-1)2k\pi}{n} + i\sin\frac{(n-1)2k\pi}{n}\right)$$

we have that :

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta = \frac{1}{2} + \frac{1}{2}\,\frac{\sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} \quad \text{and so}$$

α) $$1 + \cos\frac{2k\pi}{n} + \cos 2\cdot\frac{2k\pi}{n} + \cdots + \cos(n-1)\cdot\frac{2k\pi}{n} = \frac{1}{2} + \frac{1}{2}\cdot \frac{\sin\left(n-\frac{1}{2}\right)\cdot\frac{2k\pi}{n}}{\sin\frac{2k\pi}{2n}}$$

$$= \frac{1}{2} + \frac{1}{2}\cdot \frac{\sin\left(2k\pi - \frac{k\pi}{n}\right)}{\sin\frac{2k\pi}{n}} = \frac{1}{2} + \frac{\sin\left(-\frac{k\pi}{n}\right)}{2\cdot 2\cdot \sin\frac{k\pi}{n}\cdot \cos\frac{k\pi}{n}} = \frac{1}{2} - \frac{1}{4\cdot \cos\frac{k\pi}{n}}$$

b) $\sin\theta + \sin 2\theta + \dots + \sin(n-1)\theta = \frac{1}{2}\cot\theta - \frac{\cos\left(n-\frac{1}{2}\right)\theta}{2\cdot \sin\frac{\theta}{2}}$ or

$\sin\frac{2k\pi}{n} + \sin 2\cdot\frac{2k\pi}{n} + \dots + \sin(n-1)\cdot\frac{2k\pi}{n} =$

$$= \frac{1}{2}\cdot\cot\frac{2k\pi}{n} - \frac{1}{2}\cdot\frac{\cos\left(n-\frac{1}{2}\right)\frac{2k\pi}{n}}{\sin\frac{2k\pi}{n}}$$

$$= \frac{1}{2}\cdot\frac{\cos 2k\pi/n}{\sin 2k\pi/n} - \frac{1}{2}\cdot\frac{\cos\left(2k\pi - \frac{k\pi}{n}\right)}{2\sin\frac{k\pi}{n}\cdot\cos\frac{k\pi}{n}}$$

$$= \frac{\cos\frac{2k\pi}{n}}{4\cdot\sin\frac{k\pi}{n}\cdot\cos\frac{k\pi}{n}} - \frac{\cos\frac{k\pi}{n}}{4\cdot\sin\frac{k\pi}{n}\cdot\cos\frac{k\pi}{n}}$$

$$= -\frac{1}{4\sin\frac{k\pi}{n}} + \frac{\cos\frac{2k\pi}{n}}{4\cdot\sin\frac{k\pi}{n}\cdot\cos\frac{k\pi}{n}}$$

18/5. $az^2 + bz + c = 0$

$$(a_1 + ia_2)z^2 + (b_1 + ib_2)z + c_1 + ic_2 = 0$$

$$(a_1 z^2 + b_1 z + c_1) + i\cdot(a_2 z^2 + b_2 z + c_2) = 0 \qquad z_1, z_2$$

$$(a_1 z_1^2 + b_1 z_1 + c_1) + i\cdot(a_2 z_1^2 + b_2 z_1 + c_2) = 0$$
$$(a_1 z_2^2 + b_1 z_2 + c_1) + i\cdot(a_2 \cdot z_2^2 + b_2 z_2 + c_2) = 0$$

So we must have:

$$\left.\begin{array}{l} a_1 z_1^2 + b_1 z_1 + c_1 = 0 \\ a_1 z_2^2 + b_1 z_2 + c_1 = 0 \\ a_2 z_1^2 + b_2 z_1 + c_2 = 0 \\ a_2 z_2^2 + b_2 z_2 + c_2 = 0 \end{array}\right\} \quad \text{From these we can have :}$$

$$z_1 + z_2 = -\frac{b_1}{a_1} = -\frac{b_2}{a_2} \quad , \quad z_1 z_2 = \frac{c_1}{a_1} = \frac{c_2}{a_2}$$

but $$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{ib_2}{ia_2} = \frac{b_1 + ib_2}{a_1 + ia_2} = \frac{b}{a}$$

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{ic_2}{ia_2} = \frac{c_1 + ic_2}{a_1 + ia_2} = \frac{c}{a} \qquad \text{So :}$$

$$\left.\begin{array}{l} z_1 + z_2 = -\dfrac{b_1}{a_1} = -\dfrac{b_2}{a_2} = -\dfrac{b}{a} \\ z_1 z_2 = \dfrac{c_1}{a_1} = \dfrac{c_2}{a_2} = \dfrac{c}{a} \end{array}\right\} \quad \text{From these we see that we can calculate the } z$$

in the same way as the a, b, c were real.

18/6. α) $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \left[\cos(\theta_1+\theta_2) + i \cdot \sin(\theta_1+\theta_2)\right]$

$$(z_1 \cdot z_2)^m = r_1^m \cdot r_2^m \cdot \left[\cos m(\theta_1+\theta_2) + i \cdot \sin m(\theta_1+\theta_2)\right]$$

$$z_1^m \cdot z_2^m = r_1^m \cdot r_2^m \cdot \left[\cos(m\theta_1 + m\theta_2) + i \cdot \sin(m\theta_1 + m\theta_2)\right]$$

$$= r_1^m \cdot r_2^m \left[\cos m(\theta_1+\theta_2) + i \sin m(\theta_1+\theta_2)\right]$$

B) $(z_1 z_2)^{1/n} = \sqrt[n]{r_1 r_2} \cdot \left[\cos\left(\frac{\theta_1+\theta_2}{n} + \frac{2K\pi}{n}\right) + i \sin\left(\frac{\theta_1+\theta_2}{n} + \frac{2K\pi}{n}\right)\right]$

$$z_1^{1/n} \cdot z_2^{1/n} = \sqrt[n]{r_1 r_2} \cdot \left[\cos\left(\frac{\theta_1+\theta_2}{n} + \frac{2(K_1+K_2)\pi}{n}\right) + i \cdot \sin\left(\frac{\theta_1+\theta_2}{n} + \frac{2(K_1+K_2)\pi}{n}\right)\right]$$

It must be $\frac{2K\pi}{n} = \frac{2(K_1+K_2)}{n}\pi \pm 2p\pi$ or

$$K - (K_1+K_2) = \pm np$$

K, K_1, K_2, n, p : integers (positive)

$n = 1, 2, 3, \dots$

$K, K_1, K_2 = 0, 1, 2, \dots, (n-1)$

$p = 0, 1, 2, 3, \dots$

c) $(z_1 z_2)^{m/n} = (r_1 r_2)^{m/n} \cdot \left[\cos\left(\frac{(\theta_1+\theta_2)\cdot m}{n} + \frac{2K\pi}{n}\right) + i \sin\left(\frac{(\theta_1+\theta_2)m}{n} + \frac{2K\pi}{n}\right)\right]$

$$z_1^{m/n} \cdot z_2^{m/n} = (r_1 r_2)^{m/n} \left[\cos\left(\frac{m\theta_1 + m\theta_2}{n} + \frac{2\cdot(K_1+K_2)\pi}{n}\right) + i \cdot \sin \cdots\right]$$

we must have : $\frac{2(K_1+K_2)}{n} = \frac{2K\pi}{n} \mp 2p\pi$

or $K - (K_1+K_2) = \pm np$ in all cases.

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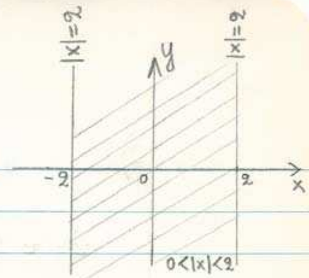
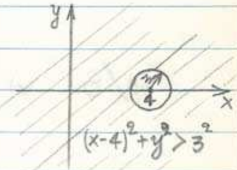
18/7. a) $|Re(z)| = |Re(x+iy)| = |x| < 2 \quad , |x|>0$

$|x|^2 < 4 \Rightarrow (x-2)(x+2) < 0$



b) $|z-4| > 3 \qquad |x-4+iy| > 3$

$\left[(x-4)^2 + y^2\right]^{1/2} > 3 \Rightarrow (x-4)^2 + y^2 > 3^2$



all the points interior $\Rightarrow (x-4)^2 + y^2 < 3^2$

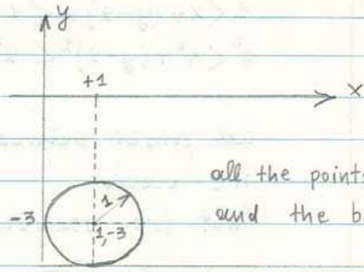
" " " on the bountary $\Rightarrow (x-4)^2 + y^2 = 3^2$

" the othe points $\Rightarrow (x-4)^2 + y^2 > 3^2$

c) $|z-1+3i| \leq 1$

$|(x-1)+(y+3)i|^2 \leq 1^2$

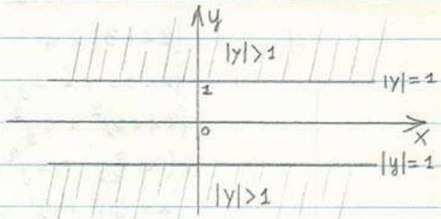
$(x-1)^2 + (y+3)^2 \leq 1^2$



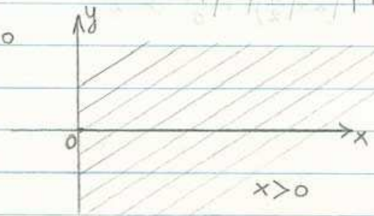
all the points interior and the boundary line

d) $|Im(z)| > 1$

$|Im(z)| = |x+iy| = |y| > 1$



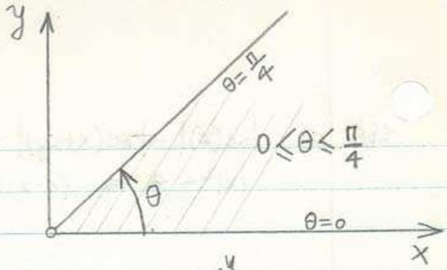
e) $Re(z) > 0 \quad x > 0$



f) $0 \leq \arg z \leq \frac{\pi}{4}$

$z \neq 0$

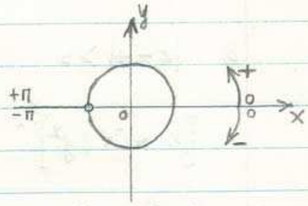
$z = r \cdot (\cos\theta + i \sin\theta)$



18/8. a) $-\pi < \arg z < \pi$, $|z| > 2$

$x^2 + y^2 > 2^2$

all points out of the boundary line (not the boundary line and the axis $\arg z = \pm\pi$)

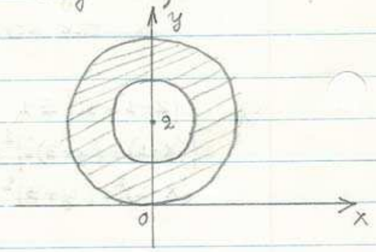


b) $1 < z - 2i < 2$

$1 < x + (y-2)i < 2$

$1^2 < x^2 + (y-2)^2 < 2^2$

all points between the circles (not the boundary lines)



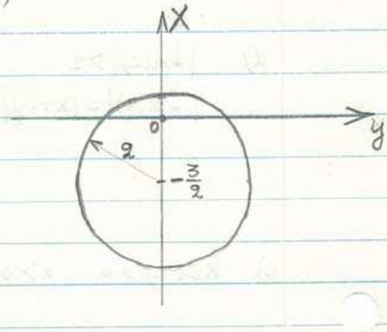
c) $|2z+3| > 4$

$|2x + 3 + i \cdot 2y| > 4$

$(2x+3)^2 + 4y^2 > 4^2$

$4(x + \frac{3}{2})^2 + 4y^2 > 4^2$

$(x + \frac{3}{2})^2 + y^2 > 2^2$



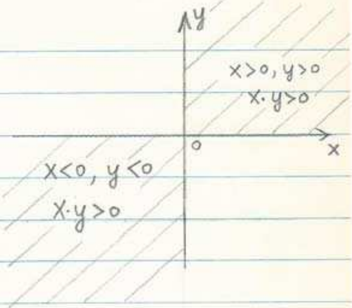
d) $Im(z^2) > 0$

$z = x + iy$

$z^2 = x^2 + y^2 + 2ixy$

$Im(z^2) = 2xy > 0$

(not the axis where $x=0$ or $y=0$)

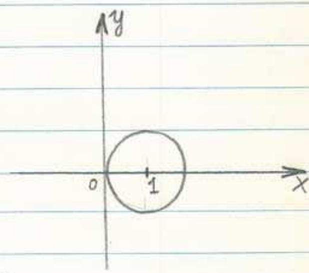


e) $Re\left(\frac{1}{z}\right) < \frac{1}{2}$

$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$\frac{1}{z} = \frac{x}{x^2+y^2} - i \cdot \frac{y}{x^2+y^2}$

$Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2} < \frac{1}{2}$



$2x < x^2 + y^2$, $x^2 - 2x + y^2 + 1 - 1 > 0$

$(x-1)^2 + y^2 > 1^2$

(all points out of the circle, not the boundary line)

f) $|z-4| > |z|$

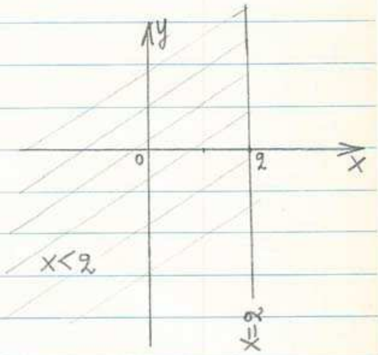
$|x - 4 + iy| > |x + iy|$

$(x-4)^2 + y^2 > x^2 + y^2$

$x^2 - 8x + 16 + y^2 - x^2 - y^2 > 0$

$-8x + 16 > 0$

$x < 2$



Elementary functions of complex variables

a. <u>Power function</u> : $f(z) = z^n \quad , \quad (n = 0, 1, 2, \cdots)$

$$f(z) = (x + iy)^n$$
$$= r^n \cdot (\cos n\theta + i \sin n\theta)$$

(1) Polynomial : $f(z) = \sum_{n=0}^{N} A_n \cdot z^n$

$A_n = A$ sub n

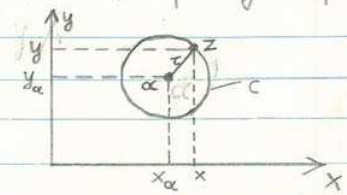
(2) Power series : $f(z) = \sum_{n=0}^{\infty} A_n \cdot (z - \alpha)^n$

where α = real or complex
z = complex variable

Ratio test : $r = \lim_{n=\infty} \left| \dfrac{A_n}{A_{n+1}} \right|$ when $|z-\alpha| < r$

r = radius of convergence (page. 6. A.1)
c = circle " "

because of $|z - \alpha| < r \Rightarrow |x + iy - \alpha| < r$



Theorem :

Any linear combination of elementary functions is an elementary function.

For example the function :

$$W = A \cdot \log z + B \cdot \sin z$$

is an elementary function.

21 The exponential function

We define that : $\exp z = e^x \cdot (\cos y + i \sin y) = e^z$

where $z = x + iy$ and y in radians

If $y = 0 \Rightarrow z = x \Rightarrow \exp z = e^x \cdot 1 = e^x$ real expon.
and so we have $\exp x = e^x$ as in real numbers

If $x = 0 \Rightarrow z = iy \Rightarrow \exp z = \exp(iy) = e^0 \cdot (\cos y + i \sin y)$
$\exp(iy) = \cos y + i \cdot \sin y = e^{iy}$

$$f(z) = f(0) + f'(0) \cdot z + f''(0) \cdot \frac{z^2}{2!} + \dots + f^{(n)}(0) \cdot \frac{z^n}{n!}$$

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + f''(z_0) \cdot \frac{(z-z_0)^2}{2!} + \dots + f^{(n)}(z_0) \cdot \frac{(z-z_0)^n}{n!}$$

α) $f(z) = e^z$

$f(0) = e^0 = 1$

$f'(z) = (e^z)' = e^z \Rightarrow f'(0) = e^0 = 1$

$f''(z) = (e^z)' = e^z \Rightarrow f''(0) = e^0 = 1$

$f'''(z) = f^{IV}(z) = \dots = f^{(n)}(z) = e^z$

$f'''(0) = f^{IV}(z) = \dots = f^{(n)}(0) = 1$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^{n-1}}{(n-1)!} + \frac{z^n}{n!} + \frac{z^{n+1}}{(n+1)!}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^n = \sum_{n=0}^{\infty} A_n \cdot z^n$$

$$r = \lim_{n=\infty} \left| \frac{A_n}{A_{n+1}} \right| = \lim_{n=\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \lim_{n=\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n=\infty} \left| \frac{1\cdot 2\cdot 3 \cdots n\cdot(n+1)}{1\cdot 2\cdot 3\cdots n\cdot} \right| = \lim_{n=\infty} |n+1| = \infty$$

then e^z exists everywhere (Integral function)

B) $f(z) = iy$ since $z = iy$

$f(0) = 0 \qquad f'(iy) = 1 \qquad f'(0) = 1$

$$\boxed{0! = 1}$$

$$\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}\cdot y^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}\cdot y^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{y^{2n}}{(2n)!} + i\cdot \sum (-1)^n \cdot \frac{y^{2n+1}}{(2n+1)!}$$

22. **Other properties of exp z**

$e^x > 0$ for each real number x

$$e^z = \exp z = e^x \cdot (\cos y + i \cdot \sin y)$$
$$= p \cdot (\cos\theta + i \cdot \sin\theta) \quad \text{where } p = e^x,\ \theta = y$$

$$\theta = \arg \exp z = \arg e^z = y \quad \text{and} \quad |\exp z| = |e^z| = e^x = p$$

$$e^x = p \Rightarrow x = \log p$$

$$z_1 = x_1 + i \cdot y_1 = r_1 \cdot (\cos\theta_1 + i \cdot \sin\theta_1)$$
$$z_2 = x_2 + i\, y_2 = r_2 \cdot (\cos\theta_2 + i \cdot \sin\theta_2)$$

α) $$\left.\begin{aligned} \exp z_1 = e^{z_1} &= e^{x_1} \cdot (\cos y_1 + i \cdot \sin y_1) \\ \exp z_2 = e^{z_2} &= e^{x_2} \cdot (\cos y_2 + i\, \sin y_2) \end{aligned}\right\}$$

$$(\exp z_1) \cdot (\exp z_2) = e^{z_1} \cdot e^{z_2} = e^{x_1} \cdot e^{x_2} \cdot \left[\cos(y_1 + y_2) + i \cdot \sin(y_1 + y_2)\right]$$

$$e^{z_1 + z_2} = e^{x_1 + x_2} \cdot \left[\cos(y_1 + y_2) + i \cdot \sin(y_1 + y_2)\right]$$

So we have:

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$$

or $$(\exp z_1) \cdot (\exp z_2) = \exp(z_1 + z_2)$$

β) $$\frac{\exp z_1}{\exp z_2} = \frac{e^{x_1} \cdot (\cos y_1 + i \sin y_1)}{e^{x_2} \cdot (\cos y_1 + i \sin y_2)} = e^{x_1 - x_2} \cdot \left[\cos(y_1 - y_2) + i\, \sin(y_1 - y_2)\right]$$

$$\frac{\exp z_1}{\exp z_2} = \exp(z_1 - z_2) \quad \text{or} \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

$$\frac{1}{\exp z} = \exp(-z) \quad \text{or} \quad \frac{1}{e^z} = e^{-z}$$

$$\exp z_1 \cdot \exp z_2 \cdot \dots \cdot \exp z_m = \exp(z_1 + z_2 + \dots + z_m)$$

$$(\exp z)^m = \exp(mz) \quad \text{or} \quad (e^z)^m = e^{mz}$$

$$(\exp z)^{1/n} = \exp\frac{z}{n} \quad \text{or} \quad (e^z)^{1/n} = e^{\frac{z}{n}}$$

$$(\exp z)^{\frac{m}{n}} = \exp\frac{mz}{n} \quad \text{or} \quad (e^z)^{\frac{m}{n}} = e^{z\cdot\frac{m}{n}}$$

$$(\exp z)^{\frac{m}{n}} = \exp\left[\frac{m}{n}\cdot(z + 2\pi k i)\right], \; k = 0,1,2 \cdots n-1$$

$$\exp(2\pi i) = e^0 \cdot (\cos 2\pi + i \sin 2\pi) = 1$$
$$\exp(z + 2\pi i) = \exp z \cdot \exp 2\pi i = \exp z \cdot 1 = \exp z .$$

$$\exp\bar{z} = e^x \cdot (\cos y - i \sin y) = \overline{\exp z}$$

$$\left.\begin{aligned} \exp(i\theta) &= \cos\theta + i\cdot\sin\theta = e^{i\theta} \\ z &= r\cdot(\cos\theta + i\cdot\sin\theta) \end{aligned}\right\} \Rightarrow \quad z = r\cdot e^{i\theta}$$

$$\bar{z} = r\cdot e^{-i\theta}$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\theta_1 + \theta_2)} = r_1 r_2 \cdot \exp\left[i(\theta_1 + \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot e^{i(\theta_1 - \theta_2)} = \frac{r_1}{r_2}\cdot \exp\left[i\cdot(\theta_1 - \theta_2)\right]$$

$r_2 \neq 0$

Euler's Formula

$$e^z = \exp z = e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot (\cos y + i \cdot \sin y)$$

$$e^{i\theta} = \exp(i\theta) = \cos\theta + i \cdot \sin\theta \qquad z = r \cdot (\cos\theta + i \sin\theta)$$

$$z = r \cdot e^{i\theta} \qquad z^n = r^n \cdot e^{in\theta}$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i \cdot (\theta_1 + \theta_2)} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$$

example:

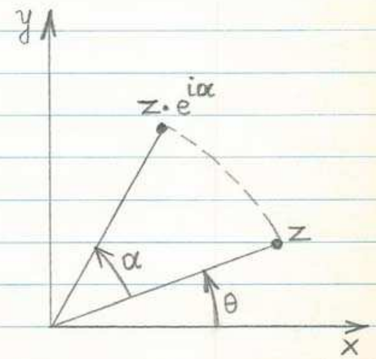
$z \cdot e^{i\alpha}$ where α = real number

$e^{i\alpha} = \cos\alpha + i \cdot \sin\alpha$ therefore $|e^{i\alpha}| = 1$

$z = r \cdot (\cos\theta + i \cdot \sin\theta)$

$$z \cdot e^{i\alpha} = r \cdot (\cos\theta + i \cdot \sin\theta) \cdot (\cos\alpha + i \cdot \sin\alpha)$$
$$= r \cdot [\cos(\theta + \alpha) + i \cdot \sin(\theta + \alpha)] = r \cdot e^{i \cdot (\theta + \alpha)}$$

$$\arg(z e^{i\alpha}) = \theta + \alpha$$
$$|z| = |z \cdot e^{i\alpha}| = r$$



23. The Trigonometric or Circular functions

we have $e^{z} = e^{x+iy} = e^{x}\cdot e^{iy} = e^{x}\cdot(\cos y + i\sin y)$

that is, $e^{iy} = \cos y + i\cdot\sin y$

and : $e^{-iy} = \cos y - i\cdot\sin y$

So it is : $\cos y = \dfrac{e^{iy}+e^{iy}}{2}$, $\sin = \dfrac{e^{iy}-e^{-iy}}{2i}$

and : $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, $\sin = \dfrac{e^{iz}-e^{-iz}}{2i}$

$\tan z = \dfrac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

The Maclaurin Series is :

$$f(z) = f(0) + f'(0)\cdot z + f''(0)\cdot\frac{z^2}{2!} + \dots + f^{(n)}(0)\cdot\frac{z^n}{n!}$$

α) $\sin z = f(z)$ $\quad \sin(0) = 0 \quad = f(0)$

$(\sin z)' = \cos z \Rightarrow (\sin 0)' = \cos 0 = 1 \quad = f'(0)$

$(\sin z)'' = -\sin z \Rightarrow (\sin 0)'' = -\sin 0 = 0 \quad = f''(0)$

$(\sin z)''' = -\cos z \Rightarrow (\sin 0)''' = -\cos 0 = -1 \quad = f'''(0)$

$(\sin z)^{IV} = \sin z \Rightarrow (\sin 0)^{IV} = \sin 0 = 0 \quad = f^{IV}(0)$

$(\sin z)^{V} = \cos z \Rightarrow (\sin 0)^{V} = \cos 0 = 1 \quad = f^{V}(0)$

$$f(z) = 0 + 1\cdot z + 0\cdot\frac{z^2}{2!} - 1\cdot\frac{z^3}{3!} + 0\cdot\frac{z^4}{4!} + 1\cdot\frac{z^5}{5!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^n\cdot\frac{z^{2n+1}}{(2n+1)!}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

In like manner we find that:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \cdot \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$$

$\cos z + i \sin z = e^{iz}$ and so we have

$$e^{iz} = \exp(iz) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\cos z &= \cos(x+iy) = \cos x \cdot \cos iy - \sin x \cdot \sin iy \\
&= \frac{1}{4}\left\{ (e^{ix} + e^{-ix}) \cdot (e^{-y} + e^{+y}) + (e^{ix} - e^{-ix}) \cdot (e^{-y} - e^{y}) \right\} \\
&= \frac{1}{4}\{ e^{xi} \cdot e^{iyi} + e^{xi} \cdot e^{-yii} + e^{iyi} \cdot e^{-xi} + e^{-xi} \cdot e^{-iyi} + \\
&\qquad e^{xi} \cdot e^{iyi} - e^{xi} \cdot e^{-iyi} - e^{-xi} \cdot e^{iyi} + e^{-xi} \cdot e^{-iyi} \} \\
&= \frac{1}{2} \cdot (e^{ix} \cdot e^{iyi} + e^{-xi} \cdot e^{-iyi}) = \frac{1}{2}(e^{ix-y} + e^{-ix+y})
\end{aligned}$$

$$\cos z = \cos(x+iy) = \frac{1}{2} \cdot (e^{ix+iyi} + e^{-ix-iyi}) = \frac{1}{2}(e^{ix-y} + e^{-ix+y})$$

$$\cos z = \cos(x+iy) = \frac{1}{2}\cdot\left(e^{ix-y} + e^{-ix+y}\right)$$
$$= \frac{1}{2}\cdot e^{-y}\cdot e^{ix} + \frac{1}{2} e^{-xi}\cdot e^{y}$$
$$= \frac{1}{2} e^{-y}\cdot(\cos x + i\sin x) + \frac{1}{2} e^{y}\cdot(\cos x - i\cdot\sin x)$$
$$= \frac{1}{2}\{e^{-y}\cdot\cos x + e^{y}\cdot\cos x\} + i\cdot\frac{1}{2}\cdot\{e^{-y}\cdot\sin x - e^{y}\cdot\sin x\}$$

$$= \frac{1}{2}\cdot(e^{y}+e^{-y})\cdot\cos x - i\,\frac{1}{2}\cdot(e^{y}-e^{-y})\,\sin x$$

$$= \frac{e^{y}+e^{-y}}{2}\cdot\cos x - i\cdot\frac{e^{y}-e^{-y}}{2}\cdot\sin x$$ So

$$\cos z = \cosh y\cdot\cos x - i\cdot\sinh y\cdot\sin x$$

In the same manner we find that

$$\sin z = \sin(x+iy) = \sin x\cdot\cosh y + i\cdot\cos x\cdot\sinh y$$

$$\sin(iy) = i\cdot\sinh y \quad , \quad \cos(iy) = \cosh y$$

24. Further properties of trigonometric functions

$\sin^2 x + \cos^2 x = 1$

$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \cdot \sin z_2$

$\cos(z_1 + z_2) = \cos z_1 \cdot \cos z_2 - \sin z_1 \cdot \sin z_2$

$\sin(-z_1) = -\sin z_1$

$\cos(-z_1) = \cos z_2$

$\sin\left(\frac{\pi}{2} - z_1\right) = \cos z_1$

$\sin 2z = 2 \cdot \sin z \cdot \cos z$

$\cos 2z = \cos^2 z - \sin^2 z$

25. Hyperbolic functions

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cdot \cosh z_2 + \cosh_1 z_1 \cdot \sinh z_2$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$$

$$\sinh 2z = 2 \sinh z \cdot \cosh z$$

$$\sinh(-z) = -\sinh z$$

$$\cosh(-z) = \cosh z$$

$$\sin(iy) = i \cdot \sinh y$$

$$\cos(iy) = \cosh y$$

$$\sin(iz) = i \cdot \sinh z \qquad \cos(iz) = \cosh z$$

$$\sinh(iz) = i \cdot \sin z \qquad \cosh(iz) = \cos z$$

$$\sinh z = \sinh(x+iy) = \sinh x \cdot \cos y + i \cdot \cosh x \cdot \sin y$$

$$\cosh z = \cos(x+iy) = \cosh x \cdot \cos y + i \cdot \sinh x \cdot \sin y$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

<u>Example</u>

$\sin(1-i)$ $\qquad\qquad z = 1-i$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin(1-i) = \frac{e^{(1-i)i} - e^{-i(1-i)}}{2i} = \frac{1}{2i}\cdot\left\{ e^{i+1} - e^{-i-1} \right\}$$

$$= \frac{1}{2i}\left\{ e^{i}\cdot e^{1} - e^{-i}\cdot e^{-1} \right\}$$

$$= \frac{1}{2i}\, e^{1}\cdot\left\{ \cos(1) + i\sin(1) \right\} - \frac{1}{2i}\, e^{-1}\cdot\left\{ \cos(-1) + i\sin(-1) \right\}$$

$$= \frac{1}{2i}\left[\cos(1)\cdot e^{1} + i\cdot e^{1}\cdot\sin(1) - e^{-1}\cdot\cos(1) + i\cdot e^{-1}\cdot\sin(1) \right]$$

$$= \frac{1}{2i}\left[\cos(1)\cdot\left\{ e^{1} - e^{-1} \right\} + i\cdot\sin(1)\cdot\left\{ e^{1} + e^{-1} \right\} \right]$$

$$= \frac{e^{1} - e^{-1}}{2i}\cdot\cos(1) + i\cdot\frac{e^{1} + e^{-1}}{2i}\cdot\sin(1)$$

$$= \frac{1}{i}\cdot\cos(1)\cdot\sinh(1) + \sin(1)\cdot\cosh(1)$$

26 The Logarithmic Function. Branches

$z = e^{w}$ then $\log z = w$

If $w = u + iv$

$z = e^{u+iv} = e^{u} \cdot e^{iv} = e^{u} \cdot (\cos v + i \sin v) = x + iy$

$x = e^{u} \cdot \cos v \qquad y = e^{u} \cdot \sin v$

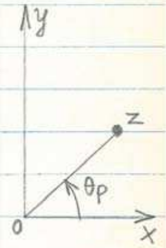
$$x^2 + y^2 = (e^{u})^2 = r^2 \quad \Rightarrow \quad |e^{u}| = r = |z|$$

$$e^{u} = |z| = r \quad \Rightarrow \quad \log r = \log|z| = u$$

$$v = \theta \quad \Rightarrow \quad \theta = \theta_p + 2K\pi \quad , \quad K = 0, \pm 1, \pm 2, \dots$$
$$0 \leq \theta_p \leq 2\pi$$

$$\log z = \log r + i \cdot (\theta_p + 2K\pi) \quad , \quad z = r \cdot (\cos\theta_p + i \cdot \sin\theta_p)$$
$$= \log|z| + i \cdot (\theta_p + 2K\pi)$$

$$\left.\begin{aligned} K=0 &\Rightarrow (\log z)_0 = \log r + i \cdot \theta_p \\ K=1 &\Rightarrow (\log z)_1 = \log r + i(\theta_p + 2\pi) \\ K=2 &\Rightarrow (\log z)_2 = \log r + i \cdot (\theta_p + 2 \cdot 2\pi) \\ &\vdots \\ K=K &\Rightarrow (\log z)_K = \log r + i \cdot (\theta_p + 2K\pi) \end{aligned}\right\} \text{Branches}$$



Each logarithm has infinitely branches. $\left\{ \begin{array}{l} \text{When } z=0 \text{ this is} \\ \text{Branch point.} \end{array} \right.$

example : Compute the $\log z$ when $z = i$

we have : $z = i \Rightarrow |z| = 1 = r$

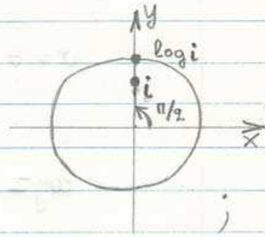
so we can write : $\log z = \log r + i(\theta_p + 2k\pi)$

$$\log i = \log 1 + i \cdot (\theta_p + 2k\pi).$$

$$z = i = 1 \cdot (0 + i \cdot 1) \Rightarrow \cos\theta_p = 0 \text{ , } \sin\theta_p = 1 \Rightarrow \theta_p = \frac{\pi}{2}$$

$$\log i = \log 1 + i \cdot \left(\frac{\pi}{2} + 2k\pi\right) = 0 + i \cdot \left(\frac{\pi}{2} + 2k\pi\right)$$

$$\log i = i \cdot \left(\frac{\pi}{2} + 2k\pi\right)$$



For $k = 0 \Rightarrow \log i = i \cdot \frac{\pi}{2}$

27. **Properties of Logarithms.**

$z = e^w \Rightarrow \log z = w$, $e^z = \exp z = e^x \cdot \{\cos y + i \sin y\}$

$e^w = \exp w$

$\log z = \log r + i(\theta_p + 2k\pi)$, $(\theta = \theta_p + 2k\pi)$

$$e^w = \exp w = \exp(\log r + i\theta) = \exp(\log r) + \exp(i\theta) =$$
$$= r \cdot e^{i\theta} = z$$

$$\exp(\log z) = z$$

$$z = e^w \Rightarrow \log z = w \Rightarrow \exp(\log z) = \exp w = e^w = z$$

exp. and log are inverses

$$z_1 = x_1 + iy_1 = r_1 \cdot (\cos\theta_1 + i\sin\theta_1) = r_1 \cdot e^{i\theta_1} = r_1 \cdot \exp(i\theta_1)$$

$$z_2 = x_2 + iy_2 = r_2 \cdot (\cos\theta_2 + i\sin\theta_2) = r_2 \cdot e^{i\theta_2} = r_2 \cdot \exp(i\theta_2)$$

$\log z = \log r + i \cdot \theta$ So we have:

$$\log z_1 + \log z_2 = \log r_1 + \log r_2 + i\theta_1 + i\theta_2 = \log(r_1 r_2) + i(\theta_1 + \theta_2)$$

$\log z_1 + \log z_2 = \log(z_1 \cdot z_2)$ and similarly

$$\log z_1 - \log z_2 = \log\left(\frac{z_1}{z_2}\right)$$

example: when $z_1 = z_2 = e^{\pi i} = e^0 \cdot (\cos\pi + i\sin\pi) = -1$

then $|z_1| = |z_2| = r_1 = r_2 = 1$

$\log z_1 = \log z_2 = i \cdot \pi$, $\log z_1 + \log z_2 = 2\pi i$, $z_1 \cdot z_2 = 1$

$\log 1 = \log z_1 z_2 = 2\pi i$

Now let m, n denote two fixed positive integers.

$$z = x + iy = \tau \cdot (\cos\theta + i\sin\theta) = \tau \cdot e^{i\theta} = \tau \cdot \exp(i\theta)$$

where $\tau > 0$ $\quad -\pi < \theta \leq \pi$

If p, p' take the successive values $0, 1, 2, 3, \ldots$

we can write : $\quad z = e^{w} \Rightarrow \log z = w$

$$\log z = \log\tau + i(\theta_p + 2p\pi)$$
$$m\cdot\log z = m\log\tau + i\cdot m(\theta_p + 2p\pi)$$
$$\log z^{m} = \log\tau^{m} + i\,(m\theta_p + 2\pi\cdot mp)$$
$$\log z^{m} = \log\tau^{m} + i\,(m\theta_p + 2\pi p') \quad , \Big\} \boxed{\log z^{m} = m\cdot\log z}$$

$$mp = p'$$

$$n\log z^{1/n} = n\log\left[\tau \cdot \exp i\theta\right]^{1/n} = n\log\left[\tau^{1/n}\cdot\exp\left(i\frac{\theta+2\pi q}{n}\right)\right]$$

$$= \log\left[\tau\cdot\exp\{i(\theta+2\pi q) + npi\cdot 2\pi\}\right]$$

$$= \log\tau + i\cdot\left[\theta + 2\pi(q + p\cdot n)\right]$$

$$\boxed{\log\cdot z^{1/n} = \frac{1}{n}\log z}$$

$$z = e^{W} \Rightarrow \log z = w \qquad e^{w} = \exp w = z$$
$$m\cdot\log z = m\cdot w \quad\Rightarrow\quad \exp(m\log z) = e^{mw}$$
$$\log z^{m} = m\log z \quad\Rightarrow\quad \exp(\log z^{m}) = e^{mw}.$$

$$z^{\frac{m}{n}} = \exp\left(\frac{m}{n}\cdot\log z\right)$$

28. ## Complex exponents

$K = \text{real} \Rightarrow z^K = \exp(K \cdot \log z)$, $(K = \frac{m}{n}, \; z \neq 0)$

$c = \text{complex} \Rightarrow z^c = \exp(c \cdot \log z)$, $z = \text{complex} \neq 0$

example: $i^{-2i} = \exp \cdot (-2i \log i)$
$= \exp \cdot \left[-2i\left(\frac{1}{2}\pi \pm 2\pi n\right)i\right]$ $\quad n = 0, 1, 2, \dots$

$= \exp\left[+1 \cdot (\pi \pm 4n\pi)\right] = \exp(\pi \pm 4n\pi)$

$$\boxed{z^c = \exp(c \cdot \log z)}$$

$$\boxed{z^K = \exp(K \cdot \log z)}$$

$z^c = \exp(\log z^c) = \exp(c \cdot \log z) = \exp(cw) = e^{cw}$
$z = e^w \Rightarrow \log z = w$

$z^c = \log(\exp z^c) = \log\left(\exp e^{cw}\right) = \log[e]^{[e]^{cw}} = [e]^{cw}$

$1 = \log(\exp) = \exp(\log) = 1.$

29. <u>Inverse Trigonometric Functions</u>

$$z = x+iy \Rightarrow \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$z = \sin w \Rightarrow z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$w = \sin^{-1} z$ So we have : $z = \dfrac{e^{iw} - e^{-iw}}{2i}$

$$e^{iw} - \frac{1}{e^{iw}} - z \cdot 2i = 0 \qquad e^{iw} \neq 0$$

$$\left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0$$

$$e^{iw} = iz \pm \sqrt{-z^2+1} \Rightarrow e^{iw} = iz \pm (1-z^2)^{1/2}$$

$$e^{iw} > 0 \Rightarrow e^{iw} = iz + (1-z^2)^{1/2}$$

We may write :

$$\log e^{iw} = \log\left[iz + (1-z^2)^{1/2}\right]$$

$$iw = \log\left[iz + (1-z^2)^{1/2}\right]$$

$$i^2 \cdot w = i \cdot \log\left[iz + (1-z^2)^{1/2}\right]$$

$$\boxed{\sin^{-1} z = w = -i \cdot \log\left[iz + (1-z^2)^{1/2}\right]}$$

Similarly we have:

$$\cos w = z \Rightarrow w = \cos^{-1} z$$

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2} \Rightarrow e^{iw} + e^{-iw} - 2z = 0$$

$$e^{iw} + \frac{1}{e^{iw}} - 2z = 0 \Rightarrow \left(e^{iw}\right)^2 - 2z \cdot e^{iw} + 1 = 0$$

$$e^{iw} = z \pm \sqrt{z^2 - 1} \Rightarrow \log e^{iw} = \log\left[z \pm \sqrt{z^2-1}\right]$$

$$iw = \log\left[z \pm \sqrt{z^2 - 1}\right]$$

$$\boxed{w = \cos^{-1} z = -i \cdot \log\left[z \pm (z^2-1)^{1/2}\right]}$$

Example: Calculate the $\cos^{-1} z$ if $z = \frac{1}{2}$

$$\cos^{-1} z = -i \cdot \log\left[\frac{1}{2} \pm \left(\frac{1}{4} - 1\right)^{1/2}\right] = -i \cdot \log\left[\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right]$$

Now we must calculate the $\log z$ when:

(α) $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

(β) $z_2 = \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$

(α) $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i \Rightarrow |z| = \left(\frac{1}{4} + \frac{3}{4}\right)^{1/2} = r_1 = 1 \Rightarrow \cos\theta_{P_1} = \frac{1}{2} \quad \sin\theta_{P_2} = \frac{\sqrt{3}}{2}$

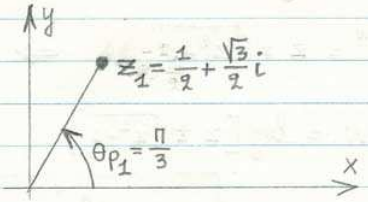
$$\theta_{P_1} = \frac{\pi}{3}$$

$\theta_1 = \theta_{p_1} + 2K_1\pi \qquad , \quad K = 0, 1, 2, \cdots$

$\log z_1 = \log r_1 + i\cdot\left(\theta_{p_1} + 2K_1\pi\right) = \log 1 + i\cdot\left(\frac{\pi}{3} + 2K_1\pi\right)$

$\log z_1 = i\cdot\left(\frac{\pi}{3} + 2K_1\pi\right) \Rightarrow \left[\cos^{-1}\frac{1}{2}\right]_1 = -i\cdot i\left(\frac{\pi}{3} + 2K_1\pi\right)$

$\left[\cos^{-1}\frac{1}{2}\right]_1 = \frac{\pi}{3} + 2K_1\pi \qquad K_1 = 0, 1, 2, \cdots$



β) $z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}\cdot i \Rightarrow |z_2| = r_2 = 1 \Rightarrow \cos\theta_{p_2} = \frac{1}{2}, \ \sin\theta_{p_2} = -\frac{\sqrt{3}}{2}$

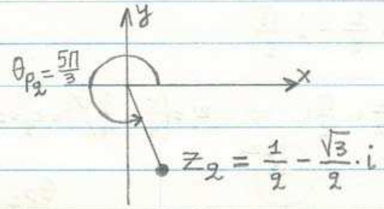
$\theta_{p_2} = \frac{5\pi}{3}$

$\theta_2 = \theta_{p_2} + 2K_2\pi \qquad K_2 = 0, 1, 2, \cdots$

$\log z_2 = \log r_2 + i\left(\theta_{p_2} + 2K_2\pi\right) = \log 1 + i\cdot\left(\frac{5\pi}{3} + 2K_2\pi\right)$

$\log z_2 = i\left(\frac{5\pi}{3} + 2K_2\pi\right) \Rightarrow \left[\cos^{-1}\frac{1}{2}\right]_2 = -i\cdot i\left(\frac{5\pi}{3} + 2K_2\pi\right)$

$\left[\cos^{-1}\frac{1}{2}\right]_2 = \frac{5\pi}{3} + 2K_2\pi \qquad K_2 = 0, 1, 2, \cdots$



III. 8.14.

$$\sinh^{-1} z = \log\left[z + (z^2+1)^{\frac{1}{2}}\right]$$

$$\cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right]$$

$$\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}$$

49/

1. Show that

(a) $\exp 0 = 1$; (b) $\exp(2 \pm 3\pi i) = -e^2$;

(c) $\exp\left(\frac{\pi}{2} i\right) = i$; (d) $\exp \frac{2 + \pi i}{4} = \sqrt{e}\,\frac{1 + i}{\sqrt{2}}$.

2. Show that

(a) $\exp(z + \pi i) = -\exp z$;

(b) $\exp(-nz) = \dfrac{1}{(\exp z)^n}$ $(n = 1, 2, \ldots)$.

3. When z has the polar representation $z = r \exp(i\theta)$, show that

(a) $\bar{z} = r \exp(-i\theta)$; (b) $\exp(\log r + i\theta) = z$.

4. Find all values of z such that

(a) $\exp z = -2$; (b) $\exp z = 1 + i\sqrt{3}$; (c) $\exp(2z - 1) = 1$.

Ans. (a) $z = \log 2 \pm (2n + 1)\pi i$; (c) $z = \frac{1}{2} \pm n\pi i$ $(n = 0, 1, 2, \ldots)$.

5. Derive the exponential laws (6) and (7), with the aid of formulas in Sec. 7.

6. Derive the exponential law (8), with the aid of formulas in Sec. 8.

7. Show that $\exp(i\bar{z}) \neq \overline{\exp(iz)}$ unless $z = \pm n\pi$, where $n = 0, 1, 2, \ldots$.

8. Simplify $|\exp(2z + i)|$ and $|\exp(iz^2)|$ and show that

$$|\exp(2z + i) + \exp(iz^2)| \leqq e^{2x} + e^{-2xy}.$$

9. Prove that $|\exp(-2z)| < 1$ whenever the point z lies in the half plane $x > 0$, and only for such values of z.

50/

10. (a) If $\exp z$ is real, show that $\mathcal{I}(z) = \pm n\pi$ $(n = 0, 1, 2, \ldots)$.
(b) For what set of values of z does $\exp z$ have pure imaginary values?

11. Examine the behavior (a) of $\exp(x + iy)$ as $x \to -\infty$; (b) of $\exp(2 + iy)$ as $y \to \infty$.

12. State why the function $2z^2 - 3 - ze^z + e^{-z}$ is entire.

13. Prove that $\exp \bar{z}$ is nowhere an analytic function of z.

14. Show in two ways that the function $\exp(z^2)$ is entire. What is its derivative? *Ans.* $2z \exp(z^2)$.

15. Simplify $\mathcal{R}[\exp(1/z)]$. Why must this be a harmonic function of x and y in every domain that does not contain the origin?

16. If $u + iv$ is an analytic function of z in a domain D, show that the functions U and V, where

$$U(x,y) = \exp[u(x,y)] \cos[v(x,y)],$$
$$V(x,y) = \exp[u(x,y)] \sin[v(x,y)],$$

must be harmonic in D, in fact, that they are conjugate harmonic functions.

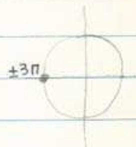
Exercises

49/1. a) $\exp 0 = 1$

$$\exp z = e^x \cdot (\cos y + i \cdot \sin y) \qquad z = 0 \Rightarrow x = y = 0$$

$$\exp 0 = e^0 \cdot (\cos 0 + 0) \quad = 1$$

b) $$\exp(2 \pm 3\pi i) = e^2 \cdot [\cos(\pm 3\pi) + i \cdot \sin(\pm 3\pi)]$$
$$= e^2 \cdot (-1 + 0) \quad = -e^2$$



c) $$\exp\left(\frac{\pi}{2} i\right) = e^0 \cdot \left(\cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2}\right)$$
$$= 1 \cdot (0 + i \cdot 1)$$
$$= i$$

d) $$\exp \frac{2 + \pi \cdot i}{4} = \exp\left(\frac{1}{2} + \frac{\pi}{4} \cdot i\right) = e^{\frac{1}{2}} \cdot \left(\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}\right)$$
$$= \sqrt{e} \cdot \left(\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) = \sqrt{e} \cdot \frac{1 + i}{\sqrt{2}}$$

49/2. a) $$\exp(z + \pi i) = \exp z \cdot \exp(\pi i) = \exp z \cdot e^0 \cdot (\cos\pi + i \sin\pi)$$
$$= \exp z \cdot 1 \cdot (-1 + 0) \quad = -\exp z$$

b) $$\exp(-nz) = \frac{1}{(\exp z)^n}$$
$$\exp(-nz) = e^{-nz} = (e^z)^{-n} = \frac{1}{(e^z)^n} = \frac{1}{(\exp z)^n}$$

49/3. b) $\exp(\log r + i\theta) = z$

since $\log(\exp A) = \exp(\log A) = A$ we have

$\log[\exp(\log r + i\theta)] = \log z$ or

$\log r + i\theta = \log z$ that is true

a) $\bar{z} = r \cdot \exp(-i\theta)$

$z = r \cdot (\cos\theta + i\sin\theta) = r \cdot e^{i\theta} = r \cdot \exp(i\theta)$

$\bar{z} = r \cdot (\cos\theta - i\sin\theta)$

$= r \cdot [\cos(-\theta) + i\sin(-\theta)] = r \cdot e^{-i\theta} = r \cdot \exp(-i\theta)$

49/4. a) $\exp z = -2 \Rightarrow e^{z} = -2$

$\log(\exp z) = \log(-2) \Rightarrow z = \log(-2)$

$z = \log[(-1)\cdot 2] = \log \frac{2}{(-1)^2}$

(i) $z = \log[(-1)\cdot 2] = \log 2 + \log(-1) = \log 2 + \log i^2 = \log 2 + 2\log i$

$\log z = \log r + i(\theta_p + 2n\pi) \Rightarrow \log i = \log 1 + i\cdot(\frac{\pi}{2} + 2n\pi)$

$z = i \Rightarrow |z| = 1 \quad \theta_p = \frac{\pi}{2}$

$\log i = i(\frac{\pi}{2} + 2n\pi) = \frac{1}{2} i \cdot (1+4n)\pi \Rightarrow 2\log i = i(1+4n)\pi$

$z = \log 2 + i(1+4n)\pi$

(ii) $z = \log 2 - 2\log i = \log 2 - i(1+4n)\pi$

b) $\exp z = 1 + i\cdot\sqrt{3} \Rightarrow \log(\exp z) = z = \log(1+i\sqrt{3})$

or $e^z = 1+i\sqrt{3} \Rightarrow z = \log(1+i\sqrt{3})$

$\log z = \log r + i\cdot(\theta_p + 2n\pi)$ $\qquad z = 1+i\sqrt{3} \quad |z| = \sqrt{1+3} = 2 > 0$

$, z = 2\left(\frac{1}{2} + i\sqrt{3}/2\right) \Rightarrow \theta_p = \frac{\pi}{3}$

$\log(1+i\sqrt{3}) = \log 2 + i\cdot\left(\frac{\pi}{3} + 2n\pi\right)$

$= \log 2 + i\left(\frac{1}{3} + 2n\right)\pi$

c) $\exp(2z-1) = 1 \Rightarrow$ α) $e^{2z-1} = 1 \Rightarrow e^{2z}\cdot e^{-1} = 1 \Rightarrow \frac{e^{2z}}{e^1} = 1$

$e^{2z} = e^1 \Rightarrow 2z = 1$ (ελλειπώς)

β) $\log \exp(2z-1) = \log 1$

$2z - 1 = \log 1 \Rightarrow 2z = 1 + \log 1$

$\log z = \log r + i\cdot(\theta_p \pm 2n\pi)$ $\qquad z = 1, \ |z| = r = 1, \ \theta_p = 0$

$\log 1 = \log|1| + i(0 \pm 2n\pi)$

$\log 1 = \pm i\cdot 2n\pi \Rightarrow 2z - 1 = \pm i\cdot 2n\pi$

$2z = 1 \pm i\cdot 2n\pi$

$z = \frac{1}{2} \pm i\cdot n\pi \qquad n = 0, 1, 2, \cdots$

49/5. a) $\frac{\exp z_1}{\exp z_2} = \exp(z_1 - z_2)$

$$\exp z_1 = e^{z_1} = e^{r_1\cdot(\cos\theta_1 + i\sin\theta_1)} =$$

$$= e^{r_1\cdot\cos\theta_1} \cdot e^{ir_1\sin\theta_1}$$

$$\exp z_2 = e^{z_2} = e^{r_2\cdot(\cos\theta_2 + i\sin\theta_2)}$$

$$= e^{r_2\cdot\cos\theta_2} \cdot e^{i\cdot r_2\cdot\sin\theta_2}$$

$$\frac{\exp z_1}{\exp z_2} = \frac{e^{r_1\cdot(\cos\theta_1 + i\sin\theta_1)}}{e^{r_2\cdot(\cos\theta_2 + i\sin\theta_2)}} = \frac{e^{r_1\cos\theta_1}\cdot e^{ir_1\sin\theta_1}}{e^{r_2\cdot\cos\theta_2}\cdot e^{ir_2\sin\theta_2}}$$

$$= e^{r_1\cdot\cos\theta_1 - r_2\cdot\cos\theta_2} \cdot \left[e^{r_1\sin\theta_1 - r_2\sin\theta_2}\right]^i$$

$$= e^{r_1\cdot\cos\theta_1 - r_2\cos\theta_2} \cdot e^{ir_1\sin\theta_1 - ir_2\sin\theta_2}$$

$$= e^{r_1\cos\theta_1 + ir_1\sin\theta_1 - r_2\cdot\cos\theta_2 - ir_2\cdot\sin\theta_2}$$

$$= e^{r_1\cdot(\cos\theta_1 + i\cdot\sin\theta_1) - r_2\cdot(\cos\theta_2 + i\sin\theta_2)}$$

$$= e^{z_1 - z_2} = \exp(z_1 - z_2)$$

b) $(\exp z)^n = \exp(nz)$

$$\exp z = e^z = e^{r(\cos\theta + i\sin\theta)} = e^{r\cdot\cos\theta} \cdot e^{ri\sin\theta}$$

$$(\exp z)^n = \left[e^{r\cos\theta}\right]^n \cdot \left[e^{ir\sin\theta}\right]^n = e^{nr\cos\theta} \cdot e^{inr\sin\theta}$$

$$= e^{nr(\cos\theta + i\sin\theta)} = e^{nz}$$

$$= \exp(nz)$$

49/6 m, n = positive integers , $K = 0, 1, 2, \cdots, (n-1)$

$$(\exp z)^{\frac{m}{n}} = \exp\cdot\left[\frac{m}{n}(z + 2K\pi i)\right]$$

$$= \exp\cdot\left[\frac{m}{n}(x + yi + 2K\pi i)\right]$$

$$= \exp\left[\frac{m}{n}\cdot x + \frac{m}{n}\cdot i\cdot(y + 2K\pi)\right]$$

$$= e^{\frac{m}{n}\cdot x}\cdot\left[\cos\frac{m(y+2K\pi)}{n} + i\sin\frac{m(y+2K\pi)}{n}\right]$$

But

we have : $\exp z = e^x\cdot(\cos y + i\sin y)$

$$(\exp z)^{\frac{m}{n}} = e^{x\cdot\frac{m}{n}}\cdot(\cos y + i\cdot\sin y)^{\frac{m}{n}}$$

and :

$$\left.\begin{array}{l} z = r\cdot(\cos\theta + i\sin\theta) \\ z_0 = r_0\cdot(\cos\theta_0 + i\sin\theta_0) \\ z = z_0^n \Rightarrow z^{1/n} = z_0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r\cdot(\cos\theta + i\sin\theta) = r_0^n\cdot(\cos n\theta_0 + i\sin n\theta_0) \\ \quad r = r_0^n \Rightarrow r_0^{1/n} \end{array}\right.$$

$$n\theta_0 = \theta + 2K\pi \Rightarrow \theta_0 = \frac{\theta + 2K\pi}{n}$$

$$z^{1/n} = r^{1/n}\cdot\left(\cos\frac{\theta+2K\pi}{n} + i\cdot\sin\frac{\theta+2K\pi}{n}\right), \quad K = 0, 1, \cdots n-1$$

$$z^{\frac{m}{n}} = r^{\frac{m}{n}}\cdot\left(\cos\frac{m(\theta+2K\pi)}{n} + i\cdot\sin\frac{m(\theta+2K\pi)}{n}\right)$$

So we have : $(\exp z)^{\frac{m}{n}} = e^{x\cdot\frac{m}{n}}\cdot\left[\cos\frac{m(y+2K\pi)}{n} + i\cdot\sin\frac{m\cdot(y+2K\pi)}{n}\right]$

which is the truth of above

49/7. $\exp(i\bar{z}) = e^{i\bar{z}}$ $\qquad \exp(iz) = e^{iz}$

$i = 1(0+i\cdot 1) = \cos\frac{\pi}{2} + i\cdot\sin\frac{\pi}{2}$ $\qquad z = r\cdot(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos\theta - i\sin\theta)$ $\qquad \bar{z} = r\cdot[\cos(-\theta) + i\sin(-\theta)]$

$$iz = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot r\cdot(\cos\theta + i\cdot\sin\theta) = r\cdot\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\cdot\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

$$i\bar{z} = \left(\cos\frac{\pi}{2} + i\cdot\sin\frac{\pi}{2}\right) r\left[\cos(-\theta) + i\sin(-\theta)\right] = r\cdot\left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]$$

it is $\exp(iz) = \exp(i\bar{z})$ or

$\log[\exp(iz)] = \log[\exp(i\bar{z})]$ or

$iz = i\bar{z}$ if and only if :

$$r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right] = r\left[\cos\left(\frac{\pi}{2}-\theta\right) + i\cdot\sin\left(\frac{\pi}{2}-\theta\right)\right]$$

or $\frac{\pi}{2} + \theta = \frac{\pi}{2} - \theta \pm 2n\pi$ or $\theta = \pm n\pi$, $n = 0, 1, 2, \dots$

or $z = r\cdot(\cos\theta \pm i\sin\theta) = r\cdot\cos\theta$.

49/8. $|\exp(2z+i)|$

$$\exp(2z+i) = e^{2z+i} = e^{2x+i(1+2y)} = e^{x}\cdot e^{i(1+2y)}$$

$$= e^{x}\cdot\left[\cos(1+2y) + i\sin(1+2y)\right]$$

$$|\exp(2z+i)| = e^{x}$$

$$\exp(iz^2) = e^{iz^2} = e^{i(x^2-y^2+2xyi)} = e^{i(x^2-y^2)-2xy}$$

$$= e^{-2xy}\cdot\left[\cos(x^2-y^2) + i\sin(x^2-y^2)\right] \Rightarrow |\exp(iz^2)| = e^{-2xy}$$

From above we have:

$$|\exp(2z+i)| = e^{x} \qquad\qquad |\exp(iz^2)| = e^{-2xy}$$

$$\exp(2z+i)+\exp(iz^2) = e^{i(x^2-y^2)-2xy} + e^{x+i(1+2y)}$$

$$= e^{x}\cdot\cos(1+2y) + e^{x}\cdot i\cdot\sin(1+2y) + e^{-2xy}\cdot\cos(x^2-y^2) + i\cdot e^{-2xy}\cdot\sin(x^2-y^2)$$

$$= \left[e^{x}\cdot\cos(1+2y) + e^{-2yx}\cdot\cos(x^2-y^2)\right] + i\left[e^{x}\cdot\sin(1+2y) + e^{-2xy}\cdot\sin(x^2-y^2)\right]$$

$$\left|\exp(2z+i)+\exp(iz^2)\right|^2 =$$

$$e^{2x}\cdot\cos^2(1+2y) + e^{-4xy}\cdot\cos^2(x^2-y^2) + 2e^{x-2xy}\cdot\cos(1+2y)\cdot\cos(x^2-y^2)$$

$$+ e^{2x}\cdot\sin^2(1+2y) + e^{-4xy}\cdot\sin^2(x^2-y^2) + 2\cdot e^{x-2xy}\cdot\sin(1+2y)\cdot\sin(x^2-y^2)$$

$$= e^{2x} + e^{-4xy} + 2\cdot e^{x}\cdot e^{-2xy}\cdot\left[\cos(1+2y)\cdot\cos(x^2-y^2)+\sin(1+2y)\cdot\sin(x^2-y^2)\right]$$

Since: $\cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta = \cos(\alpha-\beta)$

$$= e^{2x} + e^{-4xy} + 2\cdot e^{x}\cdot e^{-2xy}\cdot\cos(1+2y-x^2+y^2)$$

$$= e^{2x} + e^{-4xy} + 2e^{x}\cdot e^{-2xy}\cdot\cos\left[(1+y)^2-x^2\right] = A$$

1) If $-\frac{\pi}{2} \leq (1+y)^2 - x^2 \leq \frac{\pi}{2} \Rightarrow 1 \geqslant \cos\left[(1+y)^2-x^2\right] \geqslant 0$

and so $2\cdot e^{x}\cdot e^{-2xy}\cdot\cos\left[(1+y)^2-x^2\right] \leq 2e^{x}\cdot e^{-2xy}$

2) If $\frac{\pi}{2} < (1+y)^2 - x^2 < \frac{3\pi}{2} \Rightarrow -1 < \cos\left[(1+y)^2-x^2\right] < 0$

and so $2e^{x}\cdot e^{-2xy}\cdot\cos\left[(1+y)^2-x^2\right] < 0 < 2\cdot e^{x}\cdot e^{-2xy}$

Therefore in any case we can write

$$\left|\exp(2z+i)+\exp(iz^2)\right|^2 =$$

$$= e^{2x} + e^{-4xy} + 2e^{x}\cdot e^{-2xy}\cdot \cos\left[(1+y)^2 - x^2\right] \leq$$

$$\leq e^{2x} + e^{-4xy} + 2e^{x}\cdot e^{-2xy}$$

or

$$\left|\exp(2z+i)+\exp(iz^2)\right|^2 \leq \left(e^{x}+e^{-2xy}\right)^2$$

$$\left|\exp(2z+i)+\exp(iz^2)\right| \leq e^{x}+e^{-2xy}$$

49/9 $\exp(-2z) = e^{-2z} = e^{-2x-2iy} = e^{-2x}\cdot\left[\cos(-2y) + i\cdot\sin(-2y)\right]$

$|\exp(-2z)| = e^{-2x}$

$x>0 \Rightarrow 2x>0 \qquad e^{-2x} = \frac{1}{e^{2x}}$

$e^{2x}>1 \qquad e^{0}=1$

$\frac{1}{e^{2x}}<1$ and so $e^{-2x}<1$ or

$|\exp(-2z)|<1$

50/10 α) $\exp z = \text{real}$

$\exp z = e^{z} = e^{x+iy} = e^{x}\cdot(\cos y + i\cdot\sin y)$

$= e^{x}\cdot\cos y + i\cdot e^{x}\cdot\sin y = \text{real}$

So: $e^{x}\cdot\sin y = 0$ or $\sin y = 0 \Rightarrow y = \pm n\pi$

$z = x+iy \Rightarrow Im(z) = y = \pm n\pi \quad , (n=0,1,2\cdots)$

β) $\exp z = \text{Imaginary}$

$\exp z = e^{x}\cdot\cos y + i\,e^{x}\cdot\sin y = \text{Imaginary}$

$e^{x}\cdot\cos y = 0 \Rightarrow \cos y = 0 \Rightarrow \cos y = \cos\frac{\pi}{2} = \cos\left(3\frac{\pi}{2}\right)$

$y_1 = \frac{\pi}{2} + 2K\pi \qquad y_2 = \frac{3\pi}{2} + 2K\pi.$

therefore $\therefore$ $\left.\begin{matrix} y = \frac{\pi}{2} + 2K\pi \\ y = \frac{\pi}{2} + (2K+1)\pi \end{matrix}\right\}$ or $\left\{ y = \frac{\pi}{2} \pm 2K\pi \right.$

50/11 α) $\exp(x+iy)$ as $x \longrightarrow -\infty$

$$\exp(x+iy) = e^{x} \cdot e^{+iy} = e^{x} \cdot (\cos y + i \cdot \sin y)$$
$$= e^{x} \cdot \cos y + i \cdot e^{x} \cdot \sin y$$
$$= e^{-\infty} \cdot \cos y + i e^{-\infty} \cdot \sin y$$
$$= \frac{1}{e^{\infty}} \cdot \cos y + i \cdot \frac{1}{e^{\infty}} \cdot \sin y = \frac{1}{\infty} \cdot \cos y + i \cdot \frac{1}{\infty} \cdot \sin y$$

$$y = 0 \Rightarrow \exp z = \frac{1}{\infty} \cdot e^{i \cdot 0} = \frac{1}{\infty} \cdot e^{0} = \frac{1}{\infty} \cdot 1 = 0$$
$$y = \frac{\pi}{2} \Rightarrow \exp z = \frac{1}{\infty} \cdot e^{i \cdot \frac{\pi}{2}} = \quad = 0$$
$$y = \pi \Rightarrow \exp z = \frac{1}{\infty} \cdot e^{i\pi} = \quad = 0$$

In any case $\exp z = 0$ when $x \to -\infty$

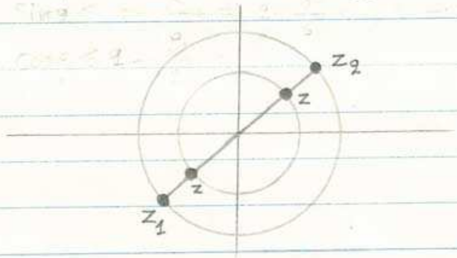
B) $\exp(2+iy)$ as $y \to +\infty$

$$\exp(2+iy) = e^{2} \cdot e^{iy}$$
$$= e^{2} \cdot (\cos y + i \cdot \sin y)$$

$-1 \leq \cos y \leq 1$, $-1 \leq \sin y \leq 1$

$-i \leq i \cdot \sin y \leq i$

$$-1 - i \leq \cos y + i \sin y \leq 1 + i$$
$$e^{2}(-1-i) \leq e^{2}(\cos y + i \cdot \sin y) \leq e^{2}(1+i)$$
$$z_1 = e^{2} \cdot \sqrt{2}\left(\cos\frac{\pi 5}{4} + i \sin\frac{5\pi}{4}\right) \leq e^{2}(\underbrace{\cos y + i \cdot \sin y}_{z}) \leq e^{2} \cdot \sqrt{2} \cdot \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) = z_2$$



?

III. A. 6

50

50/12

$$f(z) = 2z^2 - 3 - z \cdot e^{z} + e^{-z}$$
$$= 2(x+iy)^2 - 3 - (x+iy) \cdot e^{x} \cdot e^{iy} + e^{-x} \cdot e^{-iy}$$
$$= 2x^2 - 2y^2 + 4xyi - 3 - (x+iy)e^{x} \cdot (\cos y + i \sin y) +$$
$$+ e^{-x} \cdot (\cos y - i \cdot \sin y)$$ is real single valued.

$$f(z) = 2x^2 - 2y^2 - 3 - e^{x} \cdot x \cdot \cos y + e^{x} \cdot y \cdot \sin y + e^{-x} \cdot \cos y +$$
$$+ i\left[4xy - e^{x} \cdot x \cdot \sin y - e^{x} \cdot y \cdot \cos y - e^{-x} \cdot \sin y\right]$$

1) $u = 2x^2 - 2y^2 - 3 - e^{x} \cdot x \cdot \cos y + e^{x} \cdot y \cdot \sin y + e^{-x} \cdot \cos y$

$v = 4xy - e^{x} \cdot x \cdot \sin y - e^{x} \cdot y \cdot \cos y - e^{-x} \cdot \sin y$ exist

2) $\dfrac{\partial u}{\partial x} = 4x - e^{x} \cdot x \cdot \cos y - e^{x} \cdot \cos y + e^{x} \cdot y \cdot \sin y - e^{-x} \cdot \cos y$

$\dfrac{\partial v}{\partial y} = 4x - e^{x} \cdot x \cdot \cos y + e^{x} \cdot y \cdot \sin y - e^{x} \cdot \cos y - e^{-x} \cdot \cos y$

$\dfrac{\partial u}{\partial y} = -4y + e^{x} \cdot x \cdot \sin y + e^{x} \cdot \sin y + e^{x} \cdot y \cdot \cos y - e^{-x} \cdot \sin y$

$\dfrac{\partial v}{\partial x} = 4y - e^{x} \cdot x \cdot \sin y - e^{x} \cdot \sin y - e^{x} \cdot y \cdot \cos y + e^{-x} \cdot \sin y$

3) All the above are continuous ($u, v, \frac{\partial u}{\partial x}$ etc.)

4) From 2 we see that the Cauchy-Riemann equations are satisfied: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

then:

$f'(z)$ exists everywhere and so

$f(z)$ is analytic everywhere or

$f(z)$ is an entire function.

or is sum of entire functions

A). 1) $2z^2 = 2(x^2-y^2)+i\cdot 4xy = f_1(z) = u_1+iv_1$ real single valued

$u_1 = 2x^2-2y^2 \quad v_1 = 4xy$

2) $\frac{\partial u_1}{\partial x} = 4x = \frac{\partial v_1}{\partial y} \qquad \frac{\partial u_1}{\partial y} = -4y = -\frac{\partial v_1}{\partial x}$ exists

3) $u_1, v_1, \frac{\partial u_1}{\partial x}$ etc are continuous

4) From 2 we have that the Cauchy-Riemann equations are satisfied.

then :

$f_1'(z)$ exists everywhere or $f_1(z)$ is analytic everywhere or $f_1(z)$ entire. $f_1'(z) = \frac{\partial u_1}{\partial x} + i\frac{\partial v_1}{\partial x} = 4x + i\cdot 4y$

B) $f_2(z) = z\cdot e^z = (x+iy)e^x\cdot(\cos y + i\sin y)$

$= e^x\cdot(x\cdot\cos y - y\cdot\sin y) + i\cdot e^x(x\cdot\sin y + y\cdot\cos y)$

$= (e^x\cdot x\cdot\cos y - e^x\cdot y\cdot\sin y) + i\cdot(e^x\cdot x\cdot\sin y + e^x\cdot y\cdot\cos y)$

$= u_2 + i\cdot v_2$ real single valued

1) $u_2 = e^x\cdot x\cdot\cos y - e^x\cdot y\cdot\sin y$

$v_2 = e^x\cdot x\cdot\sin y + e^x\cdot y\cdot\cos y$

2) $\frac{\partial u_2}{\partial x} = e^x\cdot x\cdot\cos y + e^x\cdot\cos y - e^x\cdot y\cdot\sin y = \frac{\partial v_2}{\partial y}$

$\frac{\partial u_2}{\partial y} = -e^x\cdot x\cdot\sin y - e^x\sin y - e^x\cdot y\cdot\cos y = -\frac{\partial v_2}{\partial x}$ exist

3) all the above are continuous ($u_2, v_2, \frac{\partial u_2}{\partial x}$ etc.)

4) From 2 we see that Cauchy-Riemann equations are Satisfied

then :

$f_2'(z)$ exists everywhere $f_2'(z) = \frac{\partial u_2}{\partial x} + i\frac{\partial v_2}{\partial x}$

$f_2'(z) = (e^x\cdot x\cdot\cos y + e^x\cdot\cos y - e^x\cdot y\cdot\sin y) + i(e^x\cdot x\cdot\sin y + e^x\sin y + e^x\cdot y\cdot\cos y)$

or $f_2(z)$ is analytic everywhere or $f_2(z)$ entire

c) $f_3(z) = \bar{e}^{z} = \bar{e}^{x}\cdot(\cos y - i\sin y) = \bar{e}^{x}\cdot\cos y - i\cdot\bar{e}^{x}\cdot\sin y$

1) $u_3 = \bar{e}^{x}\cdot\cos y \qquad v_3 = -\bar{e}^{x}\cdot\sin y$ real - single valued

2) $\frac{\partial u_3}{\partial x} = -\bar{e}^{x}\cdot\cos y = \frac{\partial v_3}{\partial y}, \qquad \frac{\partial u_3}{\partial y} = -\bar{e}^{x}\cdot\sin y = -\frac{\partial v_3}{\partial x}$

3) All the above are continuous ($u_3, v_3, \frac{\partial u_3}{\partial x}$ etc.)

4) From 2 we have that the Cauchy - Riemann equations are satisfied, then:

$f_3'(z)$ exists everywhere $f_3'(z) = \frac{\partial u_3}{\partial x} + i\frac{\partial v_3}{\partial x}$

$f_3'(z) = -\bar{e}^{x}\cdot\cos y + i\cdot\bar{e}^{x}\cdot\sin y$

or $f_3(z)$ is analytic everywhere

or $f_3(z)$ is entire

From A, B, C we have that:

$$F'(z) = (2z^2 - 3 - z e^{z} + \bar{e}^{z})' = f_1'(z) - f_2'(z) + f_3'(z) =$$

$$= (4x - e^{x}\cdot x\cdot\cos y - e^{x}\cdot\cos y + e^{x}\cdot y\cdot\sin y - \bar{e}^{x}\cdot\cos y) +$$

$$i\cdot(4y - e^{x}\cdot x\cdot\sin y - e^{x}\cdot\sin y - e^{x}\cdot y\cdot\cos y + \bar{e}^{x}\cdot\sin y)$$

$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ exists everywhere or

$F(z)$ is analytic everywhere or $F(z)$ entire

etc. as exercise 45/16

50/13. $\exp \bar{z} = e^{\bar{z}} = e^{x-iy} = e^{x} \cdot e^{-iy} = e^{x} \cdot [\cos(-y) + i \cdot \sin(-y)]$

$= e^{x}(\cos y - i \cdot \sin y)$

$= e^{x} \cdot \cos y - i \cdot e^{x} \cdot \sin y$

1) $u = e^{x} \cdot \cos y \quad v = -e^{x} \cdot \sin y$ real single valued

2) $\frac{\partial u}{\partial x} = e^{x} \cdot \cos y \quad , \quad \frac{\partial u}{\partial y} = -e^{x} \cdot \sin y$

$\frac{\partial v}{\partial y} = -e^{x} \cdot \cos y \quad , \quad \frac{\partial v}{\partial x} = -e^{x} \cdot \sin y$

3) all the above are continuous ($u, v, \frac{\partial u}{\partial x}$ etc.)

4) Cauchy-Riemann equations are not satisfied because $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

then:

$f'(z)$ exists nowhere or

$f(z)$ is analytic nowhere

50/14. 1st Way $\quad \exp(z^2) = \exp[(x+iy)^2] = \exp(x^2 - y^2 + i \cdot 2xy)$

$= e^{x^2-y^2} \cdot e^{i2xy}$

$= e^{x^2} \cdot e^{-y^2} \cdot (\cos 2xy + i \cdot \sin 2xy)$

$= e^{x^2} \cdot e^{-y^2} \cdot \cos 2xy + i \cdot e^{x^2} \cdot e^{-y^2} \cdot \sin 2xy$

1) $u = e^{x^2} \cdot e^{-y^2} \cdot \cos 2xy \quad , \quad v = e^{x^2} \cdot e^{-y^2} \cdot \sin 2xy$ real single valued

2) $\frac{\partial u}{\partial x} = 2x \cdot e^{x^2} \cdot e^{-y^2} \cdot \cos 2xy - e^{x^2} \cdot e^{-y^2} \cdot 2y \cdot \sin 2xy = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -2y \cdot e^{x^2} \cdot e^{-y^2} \cdot \cos 2xy - 2x \cdot e^{x^2} \cdot e^{-y^2} \cdot \sin 2xy = -\frac{\partial v}{\partial x}$

3) all the above ($u, v, \frac{\partial u}{\partial x}$ etc.) are continuous

4) From 2 we have that Cauchy-Riemann equations are satisfied, then :

$f'(z)$ exists everywhere or
$f(z)$ is analytic everywhere or
$f(z)$ is entire

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} =$$

$$= 2x \cdot e^{x^2-y^2} \cdot \cos 2xy - 2y \cdot e^{x^2-y^2} \cdot \sin 2xy +$$
$$i \cdot \left(2y \cdot e^{x^2-y^2} \cdot \cos 2xy + 2x \cdot e^{x^2-y^2} \cdot \sin 2xy \right)$$

$$= 2x \cdot e^{x^2-y^2} \cdot (\cos 2xy + i \cdot \sin 2xy) + i2y \cdot e^{x^2-y^2} \cdot (\cos 2y + i \cdot \sin 2y)$$

$$= 2x \cdot e^{x^2-y^2} \cdot e^{i2xy} + i \cdot 2y \cdot e^{x^2-y^2} \cdot e^{i2xy}$$
$$= 2x \cdot e^{(x+iy)^2} + i\,2y \cdot e^{(x+iy)^2} = 2x \cdot e^{z^2} + i\,2y \cdot e^{z^2}$$
$$= e^{z^2}(x+iy)\,2 = 2 \cdot z \cdot \exp(z^2)$$

<u>2^{nd} Way</u> $\quad \exp(z^2) = \exp\left[\left(r \cdot e^{i\theta}\right)^2\right]$

50/15 $u = Re\left[\exp\frac{1}{z}\right]$

A function u is harmonic when satisfies the Laplace's equation $\nabla^2 u = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = Re\left[\exp\frac{1}{x+iy}\right] = Re\left[\exp\frac{x-iy}{x^2+y^2}\right] = Re\cdot\left[e^{\frac{x-iy}{x^2+y^2}}\right]$$

$$= Re\left[e^{\frac{x}{x^2+y^2}}\cdot e^{-i\cdot\frac{y}{x^2+y^2}}\right] = Re\left[e^{\frac{x}{x^2+y^2}}\cdot\left(\cos\frac{y}{x^2+y^2} - i\cdot\sin\frac{y}{x^2+y^2}\right)\right]$$

$$= e^{\frac{x}{x^2+y^2}}\cdot\cos\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \left\{\begin{array}{l}\left(\frac{x}{x^2+y^2}\right)'_x\cdot e^{\frac{x}{x^2+y^2}}\cdot\cos\frac{y}{x^2+y^2} \\ - e^{\frac{x}{x^2+y^2}}\cdot\sin\frac{y}{x^2+y^2}\cdot\left(\frac{y}{x^2+y^2}\right)'_x\end{array}\right\} = \begin{array}{l}\frac{y^2-x^2}{(x^2+y^2)^2}\cdot e^{\frac{x}{x^2+y^2}}\cdot\cos\frac{y}{x^2+y^2} \\ +\frac{2xy}{(x^2+y^2)^2}\cdot e^{\frac{x}{x^2+y^2}}\cdot\sin\frac{y}{x^2+y^2}\end{array}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2x^5-4x^3y^2-6xy^4}{(x^2+y^2)^4}\cdot e^{\frac{x}{x^2+y^2}}\cdot\cos\frac{y}{x^2+y^2}$$

$$+\frac{y^2-x^2}{(x^2+y^2)^2}\cdot\frac{y^2-x^2}{(x^2+y^2)^2}\cdot e^{\frac{x}{x^2+y^2}}\cdot\cos\frac{y}{x^2+y^2}$$

$$+\frac{y^2-x^2}{(x^2+y^2)^2}\cdot e^{\frac{x}{x^2+y^2}}\cdot\sin\frac{y}{x^2+y^2}\cdot\frac{2xy}{(x^2+y^2)^2}$$

$$+\frac{2y^5-4y^3x^2-6yx^4}{(x^2+y^2)^4}\cdot e^{\frac{x}{x^2+y^2}}\cdot\sin\frac{y}{x^2+y^2} +$$

$$+\frac{2xy}{(x^2+y^2)^2}\cdot\frac{y^2-x^2}{(x^2+y^2)^2}\cdot e^{\frac{x}{x^2+y^2}}\cdot\sin\frac{y}{x^2+y^2} +$$

$$+\frac{2xy}{(x^2+y^2)^2} \cdot e^{\frac{x}{x^2+y^2}} \cdot \cos\frac{y}{x^2+y^2} \cdot \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{(x^2+y^2)^2} \cdot e^{\frac{x}{x^2+y^2}} \cdot \left[\left(\cos\frac{y}{x^2+y^2}\right)\left(2x^5+x^4-4x^3y^2+2x^2y^2-6xy^4+y^4\right) + \left(\sin\frac{y}{x^2+y^2}\right)\left(2y^5+4y^3\cdot x-4y^3\cdot x^2-4yx^3-6yx^4\right)\right]$$

Similarly we can find that $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$

and so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

for every $z(x,y)$ except $z=0$ or $x^2+y^2=0$ or $x=0,\ y=0$ that is origin.

50/16 $f(z) = u + iv$ analytic in domain D.

a) $\nabla^2 U(x,y) = 0$

$U(x,y) = \exp[u(x,y)] \cdot \cos[v(x,y)] = e^u \cdot \cos v$

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial x} \cdot e^u \cdot \sin v$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \cdot e^u \cdot \cos v + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} \cdot e^u \cdot \cos v -$$

$$- \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \cdot e^u \cdot \sin v - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x} \cdot e^u \cdot \sin v -$$

$$- \frac{\partial^2 v}{\partial x^2} \cdot e^u \cdot \sin v - \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial x} \cdot e^u \cdot \cos v$$

$$\frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial y} \cdot e^u \cdot \sin v$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} \cdot e^u \cdot \cos v + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot e^u \cdot \cos v -$$

$$- \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \cdot e^u \cdot \sin v - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \cdot e^u \cdot \sin v -$$

$$- \frac{\partial^2 v}{\partial y^2} \cdot e^u \cdot \sin v - \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial y} \cdot e^u \cdot \cos v$$

Since $u + iv$ = analytic we have $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous and that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \nabla^2 u = \nabla^2 v = 0$$

b) $V(x,y) = \exp[u(x,y)] \cdot \sin[v(x,y)] = e^u \cdot \sin v$

$$\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} \cdot e^u \cdot \sin v + \frac{\partial v}{\partial x} \cdot e^u \cdot \cos v$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \cdot e^u \cdot \sin v + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} \cdot e^u \cdot \sin v + \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \cdot e^u \cdot \cos v$$

$$+ \frac{\partial^2 v}{\partial x^2} \cdot e^u \cdot \cos v + \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial x} \cdot e^u \cdot \sin v$$

$$\frac{\partial V}{\partial y} = \frac{\partial u}{\partial y} \cdot e^u \cdot \sin v + \frac{\partial v}{\partial y} \cdot e^u \cdot \cos v$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} \cdot e^u \cdot \sin v + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot e^u \cdot \sin v + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \cdot e^u \cdot \cos v$$

$$+ \frac{\partial^2 v}{\partial y^2} \cdot e^u \cdot \cos v + \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial y} \cdot e^u \cdot \sin v$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = e^u \cdot \sin v \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + e^u \cdot \cos v \cdot \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) +$$

$$+ e^u \cdot \sin v \cdot \left[\left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 \right]$$

$$+ e^u \cdot \cos v \cdot \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = e^u \cdot \sin v \cdot 0 + e^u \cdot \cos v \cdot 0 +$$
$$+ e^u \cdot \sin v \cdot 0 + e^u \cdot \cos v \cdot 0 = 0$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = e^u \cdot \cos v \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + e^u \cdot \cos v \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y} - \right.$$

$$\left. - \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial y} \right) - e^u \cdot \sin v \cdot \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \right.$$

$$\left. + \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \right) -$$

$$- e^u \cdot \sin v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = e^u \cdot \cos v \cdot \nabla^2 u - e^u \cdot \sin v \cdot \nabla^2 v +$$

$$+ e^u \cdot \cos v \cdot \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 \right]$$

$$- e^u \cdot \sin v \cdot 2 \cdot \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \right)$$

We have: $\nabla^2 u = 0$, $\nabla^2 v = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2, \quad \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} = - \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} \qquad \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2,$$

and So:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = e^u \cdot \cos v \cdot 0 - e^u \cdot \sin v \cdot 0 +$$

$$e^u \cdot \cos v \cdot 0 - e^u \cdot \sin v \cdot 0 = 0 \Rightarrow \nabla^2 U = 0$$

Similarly we have $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \nabla^2 V = 0$

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} \cdot e^{u} \cdot \cos v - \frac{\partial v}{\partial x} \cdot e^{u} \cdot \sin v =$$

$$= \frac{\partial v}{\partial y} \cdot e^{u} \cdot \cos v + \frac{\partial u}{\partial y} \cdot e^{u} \cdot \sin v = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} \cdot e^{u} \cdot \cos v - \frac{\partial v}{\partial y} \cdot e^{u} \cdot \sin v =$$

$$= -\frac{\partial v}{\partial x} \cdot e^{u} \cdot \cos v - \frac{\partial u}{\partial x} \cdot e^{u} \cdot \sin v = -\frac{\partial V}{\partial x}$$

From the above we have that U, V satisfy the cauchy - Riemann equations and so : they are conjugate harmonic functions or $F(x,y) = U + i \cdot V$ is analytic.

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53/

1. Establish the differentiation formulas (5), Sec. 23.
2. Derive formulas (7) and (8) in Sec. 23.
3. Derive formula (1) above; then show that

$$|\sinh y| \leqq |\sin z| \leqq \cosh y.$$

4. Derive formula (2); then show that $|\sinh y| \leqq |\cos z| \leqq \cosh y$.
5. Show that $|\sin z| \geqq |\sin x|$ and $|\cos z| \geqq |\cos x|$.
6. Establish identities (3) and (4) of this section.
7. Prove that (a) $1 + \tan^2 z = \sec^2 z$; (b) $1 + \cot^2 z = \csc^2 z$.
8. Establish the identities

(a) $2 \sin (z_1 + z_2) \sin (z_1 - z_2) = \cos 2z_2 - \cos 2z_1$;
(b) $2 \cos (z_1 + z_2) \sin (z_1 - z_2) = \sin 2z_1 - \sin 2z_2$.

9. Show that $\cos (i\bar{z}) = \overline{\cos (iz)}$ for all z, and that $\sin (i\bar{z}) \neq \overline{\sin (iz)}$ unless $z = \pm n\pi i$, where $n = 0, 1, 2, \ldots$.
10. Prove statement (10) of this section.
11. With the aid of the identities in Exercise 8 show that (a) if $\cos z_1 = \cos z_2$, then $z_2 = \pm z_1 \pm 2n\pi$; (b) if $\sin z_1 = \sin z_2$, then either $z_2 = z_1 \pm 2n\pi$ or $z_2 = -z_1 \pm (2n + 1)\pi$, where $n = 0, 1, 2, \ldots$.
12. Find all roots of the equation $\cos z = 2$.
Ans. $z = \pm 2n\pi + i \cosh^{-1} 2 = \pm 2n\pi \pm i \log (2 + \sqrt{3})$ $(n = 0, 1, 2, \ldots)$.
13. Find all roots of the equation $\sin z = \cosh 4$.
Ans. $z = (\pm 2n + \frac{1}{2})\pi \pm 4i$ $(n = 0, 1, 2, \ldots)$.
14. Show in two ways that each of these functions is everywhere harmonic:

(a) $\sin x \sinh y$; (b) $\cos 2x \sinh 2y$.

15. If w is an analytic function of z in some domain, state why $\sin w$ and $\cos w$ are analytic functions of z in that domain, with derivatives $\cos w\, dw/dz$ and $-\sin w\, dw/dz$, respectively.
16. Show that neither (a) $\sin \bar{z}$ nor (b) $\cos \bar{z}$ is an analytic function of z anywhere.

55/

1. Derive the differentiation formulas (2) and (4).
2. Prove the identities (5) and (7).
3. Show how formulas (12) and (13) follow from identities (6), (7), and (10).
4. Derive formula (15); then show that $\sinh |x| \leqq |\cosh z| \leqq \cosh x$.
5. Show that $\sinh (z + \pi i) = -\sinh z$ and $\cosh (z + \pi i) = -\cosh z$, and hence that $\tanh (z + \pi i) = \tanh z$.
6. Find all the zeros of (a) $\sinh z$; (b) $\cosh z$.
7. Find all the roots of the equations

(a) $\cosh z = \frac{1}{2}$; (b) $\sinh z = i$; (c) $\cosh z = -2$.
Ans. (a) $(\pm\frac{1}{3} \pm 2n)\pi i$; (b) $(\frac{1}{2} \pm 2n)\pi i$ $(n = 0, 1, 2, \ldots)$.

8. Why is the function $\sinh (e^z)$ entire? Write its real component as a function of x and y and state why that component must be a harmonic function everywhere.

Exercises

53/1 α) $\frac{d}{dz}\tanh z = \operatorname{sech}^2 z$

$$\frac{d}{dz}\frac{\sinh z}{\cosh z} = \frac{d}{dz}\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}} = \frac{(e^{z}+e^{-z})(e^{z}-e^{-z})' - (e^{z}+e^{-z})'(e^{z}-e^{-z})}{(e^{z}+e^{-z})^2}$$

$$= \frac{(e^{z}+e^{-z})^2-(e^{z}-e^{-z})^2}{(e^{z}+e^{-z})^2}$$

$$= \frac{(e^{z}+e^{-z}-e^{z}+e^{-z})(e^{z}+e^{-z}+e^{z}-e^{-z})}{(e^{z}+e^{-z})^2}$$

$$= \frac{2\cdot e^{-z}\cdot e^{z}\cdot 2}{(e^{z}+e^{-z})^2} = \frac{1}{\left(\frac{e^{z}+e^{-z}}{2}\right)^2}$$

$$= \frac{1}{(\cosh z)^2} = (\operatorname{sech} z)^2$$

β) $$\frac{d}{dz}\cot z = \frac{d}{dz}\frac{\cos z}{\sin z} = \frac{d}{dz}\frac{e^{iz}+e^{-iz}}{\frac{e^{iz}-e^{-iz}}{i}}$$

$$= i\cdot\frac{(e^{iz}-e^{-iz})\cdot(i\cdot e^{iz}-i\cdot e^{-iz})-(e^{iz}+e^{-iz})(i\cdot e^{iz}+i e^{-iz})}{(e^{iz}-e^{-iz})^2}$$

$$= -\frac{(e^{iz}-e^{-iz})^2-(e^{iz}+e^{-iz})^2}{(e^{iz}-e^{-iz})^2}$$

$$= -\frac{(e^{iz}-e^{-iz}-e^{iz}-e^{-iz})(e^{iz}-e^{-iz}+e^{iz}+e^{-iz})}{(e^{iz}-e^{-iz})^2}$$

$$= -\frac{-2\cdot e^{-iz}\cdot 2\cdot e^{iz}}{(e^{iz}-e^{-iz})^2} = \frac{-4\cdot}{(e^{iz}-e^{-iz})^2} = \frac{1}{\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2} =$$

$$= \frac{1}{(\sin z)^2} = (\csc z)^2$$

53/2. $\cos z = \cos(x+iy) = \frac{1}{2}\left(e^{ix+iiy} + e^{-ix-iiy}\right) = \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$

$= \frac{1}{2}\cdot e^{-y}\cdot(\cos x + i\cdot\sin x) + \frac{1}{2}\cdot e^{y}\cdot(\cos x - i\cdot\sin x)$

$= \frac{1}{2}(e^{y}+e^{-y})\cdot\cos x - i\cdot\frac{1}{2}(e^{y}-e^{-y})\cdot\sin x$

$= \cos x\cdot\cosh y - i\cdot\sin x\cdot\sinh y$

$\sin z = \sin(x+iy) = \frac{1}{2i}\left(e^{ix-y} - e^{-xi+y}\right)$

$= \frac{1}{2i}e^{-y}\cdot(\cos x + i\sin x) + \frac{1}{2i}\cdot e^{y}\cdot(-\cos x + i\sin x)$

$= -\frac{1}{2i}(e^{y}-e^{-y})\cos x + \frac{1}{2i}i(e^{y}+e^{-y})\cdot\sin x$

$= -\frac{1}{i}\cdot\sinh y\cdot\cos x + \sin x\cdot\cosh y$ $\left(-\frac{1}{i} = -\frac{i}{i^2} = -\frac{i}{-1} = i\right.$

$= \sin x\cdot\cosh y + i\cdot\cos x\cdot\sinh y$

$\sin(iy) = \frac{1}{2i}\left[e^{iiy} - e^{-iiy}\right] = \frac{1}{2i}\left(e^{-y}-e^{y}\right) = \frac{i}{2i^2}\left(e^{-y}-e^{y}\right)$

$= \frac{-i}{2}\left(e^{-y}-e^{y}\right) = \frac{1}{2}\left(e^{y}-e^{-y}\right) = \sinh y$

$\cos(iy) = \frac{1}{2}\left(e^{iiy}+e^{-iiy}\right) = \frac{1}{2}\left(e^{-y}+e^{y}\right) = \cosh y.$

53/3. $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$\sin z = \sin x \cdot \cosh y + i \cdot \cos x \cdot \sinh y$

$$\cosh y = \frac{1}{2}(e^{y} + e^{-y}) = \frac{1}{2}\left(e^{-(-1)y} + e^{(-1)y}\right) = \frac{1}{2}\left(e^{-iiy} + e^{iiy}\right)$$

$$= \frac{1}{2}\left(e^{iiy} + e^{-iiy}\right) = \cos(iy)$$

$$\sinh y = \frac{1}{2}(e^{y} - e^{-y}) = \frac{1}{2}\left(e^{-iiy} - e^{iiy}\right) = -\frac{1\cdot i}{2\cdot i}\left(e^{iiy} - e^{-iiy}\right)$$

$$= -i \cdot \sin(iy)$$

$\sin z = \sin x \cdot \cosh y + i \cos x \cdot \sinh y$

$$|\sin z|^2 = \sin^2 x \cdot \cosh^2 y + \cos^2 x \cdot \sinh^2 y$$

$$= \sin^2 x \cdot \cosh^2 y + (1 - \sin^2 x) \cdot \sinh^2 y$$

$$\cosh^2 y - \sinh^2 y = \frac{1}{4}\left\{ e^{2y} + e^{-2y} + 2e^{y}e^{-y} - e^{2y} - e^{-2y} + 2e^{y}e^{-y} \right\} = 1$$

$$|\sin z|^2 = \sin^2 x\,(1 + \sinh^2 y) + (1 - \sin^2 x) \cdot \sinh^2 y$$
$$= \sin^2 x + \sinh^2 y = 1 - \cos^2 x + 1 + \cosh^2 y$$

From this we have $|\sin z|^2 \geq \sinh^2 y = |\sinh^2 y| = |\sinh y|^2$

$$|\sinh y| \leq |\sin z|$$

It is $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$$\sin^2 x + \sinh^2 y = \frac{e^{2ix} + e^{-2ix} - 2}{-4} + \frac{e^{2y} + e^{-2y} - 2}{4}$$

$$= \frac{2 - e^{2xi} - e^{-2xi}}{4} + \frac{e^{2y} + e^{-2y} - 2}{4}$$

$$= \frac{1}{4}\left\{ 2 - e^{2xi} - \frac{1}{e^{2xi}} + e^{2y} + \frac{1}{e^{2y}} - 2 \right\}$$

$$= \frac{1}{4}\left\{ e^{2y} + \frac{1}{e^{2y}} - e^{2xi} - \frac{1}{e^{2xi}} \right\}$$

$$= \frac{1}{4 \cdot e^{2y} \cdot e^{2xi}} \left\{ e^{4y} \cdot e^{2xi} - e^{2y} - e^{4xi} \cdot e^{2y} - e^{2xi} \right\}$$

$$= \frac{1}{4e^{2y}} \left\{ e^{4y} - \frac{e^{2y}}{e^{2xi}} - e^{2xi} \cdot e^{2y} - 1 \right\}$$

$$|\sin z| = \frac{1}{2e^{y}} \cdot \sqrt{e^{4y} - 1 - \frac{e^{2y}}{e^{2xi}} - e^{2xi} \cdot e^{2y}}$$

$$\cosh y = \frac{e^{y} + e^{-y}}{2} = \frac{e^{2y} + 1}{2e^{y}} = \frac{1}{2e^{y}} \sqrt{e^{4y} + 1 + 2e^{2y}}$$

$$e^{4y} + 2e^{2y} + 1 > \underset{>0}{e^{4y}} - 1 - \frac{e^{2y}}{\underset{>0}{e^{2xi}}} - \underset{>0}{e^{2xi}} \cdot \underset{>0}{e^{2y}}$$ and so

$$\cosh y \geq |\sin z|$$

or : $|\sin z|^2 = \sin^2 x + \cosh^2 y - 1 = \underset{\leq 0}{(\sin^2 x - 1)} + \cosh^2 y \leq \cosh^2 y$

and so $|\sin z| \leq \cosh y$.

53/4. $\cos z = \cos(x+iy) = \cos x \cdot \cosh y - i \cdot \sin x \cdot \sinh y$

$$|\cos z|^2 = \cos^2 x \cdot \cosh^2 y + \sin^2 x \sinh^2 y$$

$$= \cos^2 x \cdot (1 + \sinh^2 y) + (1 - \cos^2 x) \sinh^2 y$$

$$= \cos^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \cosh^2 y - 1 = \underbrace{(\cos^2 x - 1)}_{\leqq 0} + \cosh^2 y$$

$$\leqq \cosh^2 y$$

$|\cos z| \leq \cosh y$ since. $\cosh y = \dfrac{e^y + e^{-y}}{2} \geq 1$

53/ 5. a) $|\sin z| \geq |\sin x|$

$$|\sin z|^2 = \sin^2 x + \underbrace{\sinh^2 y}_{>0} \geq \sin^2 x = |\sin x|^2$$

$$|\sin z| \geq |\sin x|$$

b) $|\cos z|^2 = \cos^2 x + \underbrace{\sinh^2 y}_{\geq 0} \geq \cos^2 x = |\cos x|^2$

$$|\cos z| \geq |\cos x|$$

$$\sinh^2 y = \underbrace{\left[\frac{e^y - e^{-y}}{2}\right]^2}_{\geq 0}$$

53/⑥ α) $\sin^2 z + \cos^2 z = 1$

$$2\sin z = e^{z} - e^{-z} \Rightarrow 4\cdot \sin^2 z = -e^{2z} + e^{-2z} + 2$$

$$2\cos z = e^{z} + e^{-z} \Rightarrow 4\cdot \cos^2 z = e^{2z} + e^{-2z} + 2$$

$$4\cdot(\sin^2 z + \cos^2 z) = 4 \Rightarrow \sin^2 z + \cos^2 z = 1$$

β) $\sin(z_1+z_2) = \sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2$

$$\sin z_1 = \frac{e^{iz_1} - e^{-iz_1}}{2i}, \quad \cos z_2 = \frac{e^{iz_2} + e^{-iz_2}}{2}$$

$$\sin z_2 = \frac{e^{iz_2} - e^{-iz_2}}{2i}, \quad \cos z_1 = \frac{e^{iz_1} + e^{-iz_1}}{2}$$

$$\sin z_1 \cdot \cos z_2 = \frac{1}{4i}\left(e^{iz_1}\cdot e^{iz_2} - \cancel{e^{-iz_1}\cdot e^{iz_2}} + \cancel{e^{iz_1}\cdot e^{-iz_2}} - e^{-iz_1}\cdot e^{-iz_2}\right)$$

$$\sin z_2 \cdot \cos z_1 = \frac{1}{4i}\left(e^{iz_1}\cdot e^{iz_2} + \cancel{e^{-iz_1}\cdot e^{iz_2}} - \cancel{e^{iz_1}\cdot e^{-iz_2}} - e^{-iz_1}\cdot e^{-iz_2}\right)$$

$$\sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2 = \frac{1}{2i}\cdot\left(e^{iz_1+iz_2} - e^{-iz_1-iz_2}\right)$$

but: $\sin(z_1+z_2) = \frac{1}{2i}\left(e^{iz_1+iz_2} - e^{-iz_1-iz_2}\right)$

and so :

$$\sin(z_1+z_2) = \sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2 .$$

53/7. α) $1+\tan^2 z = \sec^2 z$

$$1+\tan^2 z = 1+\frac{\sin^2 z}{\cos^2 z} = \frac{\cos^2 z+\sin^2 z}{\cos^2 z} =$$
$$= \frac{1}{\cos^2 z} = \sec^2 z$$

B) $1+\cot^2 z = \csc^2 z$

$$1+\cot^2 z = 1+\frac{\cos^2 z}{\sin^2 z} = \frac{\cos^2 z+\sin^2 z}{\sin^2 z} =$$
$$= \frac{1}{\sin^2 z} = \csc^2 z$$

53/8 α) $2\sin(z_1+z_2)\cdot\sin(z_1-z_2) =$

$$= 2\cdot\frac{(e^{iz_1+iz_2}-e^{-iz_1-iz_2})}{2i}\cdot\frac{e^{iz_1-iz_2}-e^{-iz_1+iz_2}}{2i}$$
$$= \frac{1}{2ii}\cdot\left(e^{2iz_1}-e^{-2iz_2}-e^{2iz_2}+e^{-2iz_1}\right)$$
$$= -\frac{e^{i2z_1}+e^{-i2z_1}}{2i}+\frac{e^{i2z_2}+e^{-i2z_2}}{2i} = \cos 2z_2-\cos 2z_1$$

B) $2\cos(z_1+z_2)\cdot\sin(z_1-z_2) =$

$$= \frac{2}{2i\cdot 2}\cdot\left(e^{iz_1+iz_2}+e^{-iz_1-iz_2}\right)\cdot\left(e^{iz_1-iz_2}-e^{-iz_1+iz_2}\right)$$
$$= \frac{1}{2i}\cdot\left(e^{i2z_1}+e^{-i2z_2}-e^{i2z_2}-e^{-i2z_1}\right)$$
$$= \frac{e^{i2z_1}-e^{-i2z_1}}{2i}-\frac{e^{i2z_2}-e^{-i2z_2}}{2i} = \sin 2z_1-\sin 2z_2$$

53/9 α)

$$\cos(i\bar{z}) = \overline{\cos(iz)}$$

$$\cos(i\bar{z}) = \cos(ix - iiy) =$$

$$= \cos(y+ix) = \cos y \cdot \cosh x - i \sin y \cdot \sinh x$$

$$\cos(iz) = \cos(ix + iiy) =$$

$$= \cos(-y+ix) = \cos y \cdot \cosh x + i \sin y \cdot \sinh x$$

$$\overline{\cos(iz)} = \cos y \cdot \cosh x - i \cdot \sin y \cdot \sinh x = \cos(i\bar{z})$$

β)

$$\sin(i\bar{z}) = \sin(ix - iiy) =$$

$$= \sin(y+ix) = \sin y \cdot \cosh x + i \cdot \cos y \cdot \sinh x$$

$$\sin(iz) = \sin(ix + iiy) =$$

$$= \sin(-y+ix) = -\sin y \cdot \cosh x + i \cos y \cdot \sinh x$$

$$\overline{\sin(iz)} = -\sin y \cdot \cosh x - i \cos y \cdot \sinh x$$

$$= -\sin(y+ix) = -\sin(i\bar{z}) \neq \sin(i\bar{z})$$

We will have $\sin(i\bar{z}) = \overline{\sin(iz)}$

when : $2(\sin y \cdot \cosh x + i \cdot \cos y \cdot \sinh x) = 0$

or $\begin{cases} \sin y \cdot \cosh x = 0 \Rightarrow y = \pm n\pi & (n = 0, 1, 2, 3, \dots) \\ \cos y \cdot \sinh x = 0 \Rightarrow x = 0 & , \left\{ \frac{e^x - e^{-x}}{2} = 0 \Rightarrow x = 0 \right\} \end{cases}$

$\cosh x = \frac{1}{2}(e^x + e^{-x}) > 0$

or $z = x + iy = 0 \pm i \cdot n \cdot \pi$ $\quad z = \pm n\pi i$

53/10 If $\cos z = 0 \Rightarrow z = \pm \frac{2n-1}{2}\cdot\pi$, $(n=1,2,\dots)$

$\cos z = \cos(x+iy) = \cos x\cdot\cosh y - i\cdot\sin x\cdot\sinh y = 0$

We must have:

$\cos x\cdot\cosh y = \cos x\cdot\frac{e^{y}+e^{-y}}{2} = 0$

and $\sin x\cdot\sinh y = \sin y\cdot\frac{e^{y}-e^{-y}}{2} = 0$

We always have $\frac{e^{y}+e^{-y}}{2} > 0 \Rightarrow$

$\cos x = 0 \Rightarrow x = \pm(2n-1)\frac{\pi}{2}$

and $\sinh y = 0 \Rightarrow e^{y}-e^{-y}=0$ or $e^{2y}=1$ or $y=0$

So: $z = x+iy = \pm(2n-1)\frac{\pi}{2} + i\cdot 0 \Rightarrow z = \pm\frac{2n-1}{2}\cdot\pi$.

53/11 α) $\cos z_1 = \cos z_2 \Rightarrow \cos 2z_1 = \cos 2z_2$

From exercise 8 we have:

$2\cdot\sin(z_1+z_2)\cdot\sin(z_1-z_2) = \cos 2z_2 - \cos 2z_1 = 0$

i) $\sin(z_1+z_2)=0 \Rightarrow z_1+z_2 = \pm n\pi$

$\Rightarrow z_2 = -z_1 \pm n\pi$

or ii) $\sin(z_1-z_2)=0 \Rightarrow z_1-z_2 = \pm n\pi$

$\Rightarrow z_2 = +z_1 \pm n\pi$

$$z_2 = \pm z_1 \pm n\pi.$$

β) $\sin 2z_1 = \sin 2z_2 \Rightarrow \cos(z_1+z_2)\cdot\sin(z_1-z_2) = 0$

i) $\cos(z_1+z_2)=0 \Rightarrow z_1+z_2 = \pm(2n+1)\frac{\pi}{2} \Rightarrow z_2 = -z_1 \pm \frac{2n+1}{2}\cdot\pi$

$n=0,1,2,\cdots$

or ii) $\sin(z_1-z_2)=0 \Rightarrow z_1-z_2 = \pm n\pi \Rightarrow z_2 = z_1 \pm n\pi$, 53/12 $\quad \cos z = 2 \Rightarrow \dfrac{e^{iz} + e^{-iz}}{2} = 2 \Rightarrow e^{iz} + e^{-iz} - 4 = 0$

$$e^{2iz} - 4 \cdot e^{iz} + 1 = 0$$

$$e^{iz} = 2 + \sqrt{3} \Rightarrow iz = \log(2+\sqrt{3})$$

$$z_0 = -i \cdot \log(2+\sqrt{3})$$

$$\cos z = \cos z_0$$

$$z = z_0 \pm 2n\pi \Rightarrow z = \pm 2n\pi - i \cdot \log(2+\sqrt{3})$$

$$\cos z = \cos(-z_0)$$

$$z = -z_0 \pm 2n\pi \Rightarrow z = \pm 2n\pi + i \cdot \log(2+\sqrt{3})$$

$$z = \pm 2n\pi \pm i \cdot \log(2+\sqrt{3}) \quad , \quad (n = 0, 1, 2, \cdots)$$

$$2 = \cos^{-1} 2$$

53/13 $\quad \sin z = \cosh 4 = \dfrac{e^{4} + e^{-4}}{2} = \dfrac{e^{-ii4} + e^{ii4}}{2} = \dfrac{e^{i(i4)} + e^{-i(i4)}}{2} = \cos(i4)$

$$= \sin\left(\frac{\pi}{2} - 4i\right)$$

$$= \sin z_0$$

$$z = \pm 2n\pi + z_0 = 2n\pi + \frac{\pi}{2} - 4i = \left(2n + \frac{1}{2}\right)\pi - 4i$$

$$z = \pm(2n+1)\pi - z_0 = \pm(2n+1)\pi - \frac{\pi}{2} + 4i$$

$$= \left(2n + \frac{1}{2}\right)\pi + 4i$$

$$z = \left(2n + \frac{1}{2}\right)\pi \pm 4i$$

53/14 a) $u = \sin x \cdot \sinh y = (e^{ix} - e^{-ix})(e^{y} - e^{-y}) \frac{1}{4i}$

$$= \frac{1}{4i}\left(e^{y+ix} - e^{y-ix} - e^{-y+ix} + e^{-y-ix}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{4i}\left[i \cdot e^{y+ix} + i \cdot e^{y-ix} - i \cdot e^{-y+ix} - i\, e^{-y-ix}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4i}\left[-e^{y+ix} + e^{y-ix} + e^{-y+ix} - e^{-y-ix}\right]$$

$$\frac{\partial y}{\partial y} = \frac{1}{4i}\left[e^{y+ix} - e^{y-ix} - e^{-y+ix} + e^{-y-ix}\right] = \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

therefore this is harmonic.

Stage 1

53/15 We have that w is analytic so:

$$W = u + iv$$

1) u, v are single-real-valued

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist

3) all the above are continuous ($u, v, \frac{\partial u}{\partial x}$ etc.)

4) The Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and the derivative exists and is equal $W' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Stage 2

For this function we have:

$$f(z) = \exp(w) = \exp(u+iv) = \exp u \cdot \exp(iv)$$
$$= e^u \cdot e^{iv} = e^u \cdot (\cos v + i \cdot \sin v)$$
$$= e^u \cdot \cos v + i \cdot e^u \cdot \sin v$$
$$= A + i \cdot B$$

1) $A = e^u \cdot \cos v$, $B = e^u \cdot \sin v$ real-single valued.

2) $$\frac{\partial A}{\partial x} = \frac{\partial u}{\partial x} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial x} \cdot e^u \cdot \sin v$$

$$= \frac{\partial v}{\partial y} \cdot e^u \cdot \cos v + \frac{\partial u}{\partial y} \cdot e^u \cdot \sin v = \frac{\partial B}{\partial y}$$

$$\frac{\partial A}{\partial y} = \frac{\partial u}{\partial y} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial y} \cdot e^u \cdot \sin v$$

$$= -\frac{\partial v}{\partial x} \cdot e^u \cdot \cos v - \frac{\partial u}{\partial x} \cdot e^u \cdot \sin v = -\frac{\partial B}{\partial x}, \text{ exist}$$

3) all the above are continuous $\left(A, B, \frac{\partial A}{\partial x} \text{ etc.}\right)$

4) Cauchy-Riemann equations are satisfied:

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y} \quad , \quad \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$$

then:

$[f'(z)]_z$ exists everywhere in the Domain or $f(z) = \exp(w)$ is analytic if w is analytic.

$$\left[f'(z)\right]_z = \left[\exp(w)\right]'_z = \frac{\partial A}{\partial x} + i \frac{\partial B}{\partial x}$$

$$= \frac{\partial u}{\partial x} \cdot e^u \cdot \cos v - \frac{\partial v}{\partial x} \cdot e^u \cdot \sin v$$

$$+ i \cdot \frac{\partial v}{\partial x} \cdot e^u \cdot \cos v + i \frac{\partial u}{\partial x} \cdot e^u \cdot \sin v$$

$$= e^u \cdot \cos v \cdot \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) + i \cdot e^u \cdot \sin v \cdot \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) \cdot e^u \cdot (\cos v + i \sin v) =$$

$$= \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}\right) \cdot \exp W = [\exp W] \cdot \frac{dW}{dz}$$

Stage 3

Now we have that $F(z) = \sin W$

$$\sin w = \sin(u+iv) = \sin u \cdot \cosh v + i \cdot \cos u \cdot \sinh v$$

$$= \frac{e^{iu} - e^{-ui}}{2i} \cdot \frac{e^{v} + e^{-v}}{2} + i \cdot \frac{e^{iu} + e^{-iu}}{2} \cdot \frac{e^{v} - e^{-v}}{2}$$

$$= \frac{1}{4i} \cdot \left(e^{iu} - e^{-iu}\right) \cdot \left(e^{v} + e^{-v}\right) + \frac{i}{4} \cdot \left(e^{iu} + e^{-iu}\right) \cdot \left(e^{v} - e^{-v}\right)$$

$$= -\frac{i}{4} \cdot \left[\exp(iu) - \exp(-iu)\right] \cdot \left[\exp(v) + \exp(-v)\right] +$$

$$+ \frac{i}{4} \cdot \left[\exp(iu) + \exp(-iu)\right] \cdot \left[\exp(v) - \exp(-v)\right]$$

$$\begin{aligned}\sin w = \frac{i}{4} \cdot \Big[& -\exp(iu) \cdot \exp v - \exp(iu) \cdot \exp(-v) + \\ & + \exp(-iu) \cdot \exp v + \exp(-iu) \cdot \exp(-v) + \\ & + \exp(iu) \cdot \exp v - \exp(iu) \cdot \exp(-v) + \\ & + \exp(-iu) \cdot \exp v - \exp(-iu) \cdot \exp(-v) \Big]\end{aligned}$$

$$\sin w = \frac{i}{4} \cdot 2 \cdot \left[\exp(-iu) \cdot \exp v - \exp(iu) \cdot \exp(-v)\right]$$

$$\sin w = \frac{i}{2} \cdot \left[\exp(-iu) \cdot \exp v - \exp(iu) \cdot \exp(-v)\right]$$

This function is a Subtraction of products of the analytic functions $\exp(-iu)$, $\exp(iu)$, $\exp(v)$, $\exp(-v)$, that is, analytic because: {exercise 16, Sect. 20 and page 50 Sect. 23) and page 41.}

Since : $\exp(-iu)$, $\exp(iu)$, $\exp(v)$, $\exp(-v)$

are analytic functions

we have : $\exp(-iu)\cdot\exp(v)$, $\exp(iu)\cdot\exp(-v)$

analytic functions

and : $\exp(-iu)\cdot\exp(v) - \exp(iu)\cdot\exp(-v)$

is analytic

that is :

$$\sin W = \frac{i}{2}\cdot\left[\exp(-iu)\cdot\exp(v) - \exp(iu)\cdot\exp(-v)\right]$$

From page 32 we have : (or from above)

$$[\sin W]'_z = [\sin W]'_W \cdot [W]'_z$$

$$= \cos W \cdot \frac{dW}{dz}$$

In the same way we have for $\cos W$:

$$[\cos W]'_z = (\cos W)'_W \cdot [W]'_z$$

$$= -\sin W \cdot \frac{dW}{dz}.$$

53/15.
$$\sin w = \frac{i}{2}\cdot\left[\exp(-iu)\cdot\exp v - \exp(iu)\cdot\exp(-v)\right]$$

$$= \frac{i}{2}\left(e^{-iu}\cdot e^{v} - e^{iu}\cdot e^{-v}\right) = -\frac{1}{2i}\left(e^{-i\cdot u - i^2\cdot v} - e^{iu+i^2\cdot v}\right)$$

$$= -\frac{1}{2i}\left(e^{-i(u+iv)} - e^{i(u+iv)}\right) = \frac{e^{iw} - e^{-iw}}{2i}$$

But this we have as follows:

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{i}{2}\left(e^{-iw} - e^{iw}\right)$$

$$= \frac{i}{2}\cdot\left(e^{-iu-i^2\cdot v} - e^{iu+i^2 v}\right) = \frac{i}{2}\cdot\left(e^{-iu}\cdot e^{v} - e^{iu}\cdot e^{-v}\right)$$

$$\sin W = \frac{i}{2}\cdot\left[\exp(-iu)\cdot\exp(v) - \exp(iu)\cdot\exp(-v)\right]$$

Similarly:

$$\cos W = \frac{e^{iw}+e^{-iw}}{2} = \frac{1}{2}\cdot\left(e^{iw} + e^{-iw}\right)$$

$$= \frac{1}{2}\left(e^{iu-v} - e^{-iu+v}\right) = \frac{1}{2}\cdot\left(e^{iu}\cdot e^{-v} - e^{-iu}\cdot e^{v}\right)$$

$$\cos W = -\frac{1}{2}\cdot\left[\exp(-iu)\cdot\exp v - \exp(iu)\cdot\exp(-v)\right]$$

53/16 $\quad \sin \bar{z} = \sin(x-iy)$

$$= \frac{1}{2i} \cdot \left(e^{\bar{z}} - e^{-\bar{z}} \right)$$

First we must have that $w = x - iy = \bar{z}$ is analytic

1) $u = x \qquad v = -y \qquad$ real - single valued

2) $\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial v}{\partial x} = 0$

3) all above are continuous $\left(u, v, \frac{\partial u}{\partial x} \text{ etc.} \right)$

4) Cauchy-Riemann equations are not satisfied :

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

and then:

$w' = (\bar{z})'$ does not exist anywhere

$\left\{ \text{See page : } \frac{\text{II.A.8}}{34/7} \right\}$

or : $\quad w = \bar{z}$ is nowhere analytic

and so $\quad f(\bar{z}) = \sin \bar{z}$ or

$f(\bar{z}) = \cos \bar{z}$ is nowhere analytic.

Exercises

55/1 α) $\frac{d}{dz}\sinh z = \cosh z$

$$\frac{d}{dz}\left(\frac{e^{z}-e^{-z}}{2}\right) = \frac{\frac{de^{z}}{dz}-\frac{de^{-z}}{dz}}{2} = \frac{e^{z}+e^{-z}}{2} = \cosh z$$

β) $\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \cdot \tanh z$

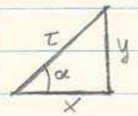
$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z}+e^{-z}}$$

$$\frac{d}{dz}\operatorname{sech} z = \frac{d}{dz}\left(\frac{1}{\cosh z}\right) = \frac{1}{dz}\left(\frac{2}{e^{z}+e^{-z}}\right)$$

$$= -\frac{2(e^{z}-e^{-z})}{(e^{z}+e^{-z})^2} = -\frac{2}{e^{z}+e^{-z}}\cdot\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$$

$$= -\frac{1}{\frac{e^{z}+e^{-z}}{2}}\cdot\frac{\frac{e^{z}-e^{-z}}{2}}{\frac{e^{z}+e^{-z}}{2}} = -\frac{1}{\cosh z}\cdot\frac{\sinh z}{\cosh z}$$

$$= -\operatorname{sech} z \cdot \tanh z$$



Sine α = $\sin\alpha = \frac{y}{r}$

Cosine α = $\cos\alpha = \frac{x}{r}$

Tangent α = $\tan\alpha = \frac{y}{x} = \frac{\sin\alpha}{\cos\alpha}$

Cotangent α = $\cot\alpha = \frac{x}{y}$

Secant α = $\sec\alpha = \frac{r}{x} = \frac{1}{\cos\alpha}$

Cosecant α = $\csc\alpha = \frac{r}{y} = \frac{1}{\sin\alpha}$

Versine α = $\operatorname{vers}.\alpha = \frac{r-x}{r} = 1-\frac{x}{r} = 1-\cos\alpha$

Coversine α = $\operatorname{Covers}.\alpha = \frac{r-y}{r} = 1-\frac{y}{r} = 1-\sin\alpha$

Haversine α = $\operatorname{hav}.\alpha = \frac{r-x}{2r} = \frac{1}{2}\left(1-\frac{x}{r}\right) = \frac{1}{2}\cdot\operatorname{vers}.\alpha.$

Exercises

55/②. α) $\cosh^2 z - \sinh^2 z = 1$ $\qquad \cosh z = \dfrac{e^z + e^{-z}}{2}, \quad \sinh z = \dfrac{e^z - e^{-z}}{2}$

$$\cosh^2 z = \frac{(e^z + e^{-z})^2}{4}$$

$$\sinh^2 z = \frac{(e^z - e^{-z})^2}{4}$$

$$\cosh^2 z - \sinh^2 z = \frac{1}{4}\left\{(e^z + e^{-z})^2 - (e^z - e^{-z})^2\right\}$$

$$= \frac{1}{4}\cdot(e^z + e^{-z} - e^z + e^{-z})\cdot(e^z + e^{-z} + e^z - e^{-z})$$

$$= \frac{1}{4}\, 2\cdot e^{-z}\cdot 2\cdot e^z = e^0 = 1$$

β) $\cosh(z_1 + z_2) = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$

$$\cosh z_1 = \frac{1}{2}(e^{z_1} + e^{-z_1}) \qquad \cosh z_2 = \frac{1}{2}(e^{z_2} + e^{-z_2})$$

$$\sinh z_1 = \frac{1}{2}(e^{z_1} - e^{-z_1}) \qquad \sinh z_2 = \frac{1}{2}(e^{z_2} - e^{-z_2})$$

$$\cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2 = \frac{1}{4}\left\{(e^{z_1} + e^{-z_1})\cdot(e^{z_2} + e^{-z_2})\right\} +$$
$$+ \frac{1}{4}\left\{(e^{z_1} - e^{-z_1})\cdot(e^{z_2} - e^{-z_2})\right\} = \frac{1}{4}\left\{e^{z_1}\cdot e^{z_2} + e^{z_1}\cdot e^{-z_2} + e^{-z_1}\cdot e^{z_2} + e^{-z_1}\cdot e^{-z_2} + \right.$$
$$\left. + e^{z_1}\cdot e^{z_2} - e^{z_1}\cdot e^{-z_2} - e^{-z_1}\cdot e^{z_2} + e^{-z_1}\cdot e^{-z_2}\right\}$$
$$= \frac{1}{2}\left\{e^{z_1}\cdot e^{z_2} + e^{-z_1}\cdot e^{-z_2}\right\} = \frac{1}{2}\left\{e^{z_1+z_2} + e^{-z_1-z_2}\right\} = \cosh(z_1 + z_2)$$

55/3. α) $\sinh z = \sinh(x+iy) = \sinh x \cdot \cos y + i \cdot \cosh x \cdot \sin y$

(i) if $z_1 = x$, $z_2 = iy \Rightarrow$ we have the above

(ii) $\sinh(x+iy) = \frac{1}{2}\left(e^{x+iy} - e^{-x-iy}\right)$

$= \frac{1}{2} e^{x} \cdot (\cos y + i \sin y) + e^{-x} \cdot (-\cos y + i \sin y) \cdot \frac{1}{2}$

$= \frac{1}{2}(e^{x} + e^{-x}) \cdot i \cdot \sin y + \frac{1}{2} \cdot (e^{x} - e^{-x}) \cdot \cos y$

$= \cosh x \cdot i \cdot \sin y + \sinh x \cdot \cos y$

$= \sinh x \cdot \cos y + i \cdot \cosh x \cdot \sin y$

And Similarly we can have

β) $\cosh(x+iy) = \cosh x \cdot \cos y + i \cdot \sinh x \cdot \sin y$

55/4. $|\cosh z|^2 = \cosh^2 x \cdot \cos^2 y + \sinh^2 x \cdot \sin^2 y$

$= (1 + \sinh^2 x) \cos^2 y + \sinh^2 x \cdot (1 - \cos^2 y)$

$= \cos^2 y + \cos^2 y \cdot \sinh^2 x + \sinh^2 x - \cos^2 y \cdot \sinh^2 x$

$= \sinh^2 x + \cos^2 y$

$|\cosh z|^2 = \cosh^2 x - 1 + \cos^2 y = \cosh^2 x + \underbrace{(\cos^2 y - 1)}_{\leq 0} \leq |\cosh^2 x|$

$|\cosh z| \leq |\cosh x|$

$|\cosh z|^2 = \underbrace{\cos^2 y}_{\geq 0} + \sinh^2 x \geq \sinh^2 x = |\sinh x|^2 = \sinh^2 |x|$

$|\cosh z| \geq \sinh |x|$

$\sinh|x| \leq |\cosh z| \leq \cosh x$

55/5 α) $\sinh(z+\pi i) = -\sinh z$

$$\sinh[x+i(\pi+y)] = \sinh x\cdot\cos(\pi+y) + i\cdot\cosh x\cdot\sin(\pi+y)$$
$$= -\sinh x\cdot\cos y - i\cdot\cosh x\cdot\sin y$$
$$= -\sinh z$$

β) $\cosh(z+\pi i) = -\cos z$

$$\cosh[x+i(\pi+y)] = \cosh x\cdot\cos(\pi+y) + i\cdot\sinh x\cdot\sin(\pi+y)$$
$$= -\cosh x\cdot\cos y - i\sinh x\cdot\sin y$$
$$= -\cos z$$

γ) $\tanh(z+\pi i) = \dfrac{\sinh(z+\pi i)}{\cosh(z+\pi i)} = \dfrac{-\sin z}{-\cos z} = \tan z$

55/6. β) $\cosh z = 0 = \cosh x\cdot\cos y + i\cdot\sinh x\cdot\sin y =$

$$= \frac{e^x+e^{-x}}{2}\cdot\frac{e^{iy}+e^{-iy}}{2} + i\cdot\frac{e^x-e^{-x}}{2}\cdot\frac{e^{iy}-e^{-iy}}{2i}$$

$$(e^x+e^{-x})(e^{iy}+e^{-iy}) + (e^x-e^{-x})(e^{iy}-e^{-iy})$$
$$e^x e^{iy} + e^{-x}\cdot e^{iy} + e^x\cdot e^{-iy} + e^{-x}\cdot e^{-iy} +$$
$$e^x e^{iy} - e^{-x}\cdot e^{iy} - e^x\cdot e^{-iy} + e^{-x}\cdot e^{-iy}$$
$$= 2(e^x\cdot e^{iy} + e^{-x}e^{-iy}) = 2\left(e^{2(x+iy)}+1\right)\cdot\frac{1}{e^{x+iy}} = 0$$

if $e^{x+iy} \neq 0$

$$e^{2x+2iy} = -1 \Rightarrow 2(x+iy) = \log(-1)$$
$$x+iy = z = \frac{1}{2}\log(-1)$$

$-1 = 1(-1+i0) \Rightarrow \theta_p = \pm\pi$

$$\log(-1) = \mathrm{Log}|1| + i(\theta_p \pm 2n\pi) = \pm(1+2n)\pi i$$
$$x+iy = \pm\left(\frac{1}{2}+n\right)\pi\cdot i \Rightarrow x=0,\ y=\pm\left(\frac{1}{2}+n\right)\pi$$

$n = 0, 1, 2, \ldots$

α) $\sinh z = \sinh(x+iy) = \sinh x \cdot \cos y + i \cdot \cosh x \cdot \sin y =$

$$= \frac{e^{x} - e^{-x}}{2} \cdot \frac{e^{iy}+e^{-iy}}{2} + i\,\frac{e^{x}+e^{-x}}{2} \cdot \frac{e^{iy}-e^{-iy}}{2i} = 0$$

$$e^{x+iy} + \cancel{e^{x-iy}} - \cancel{e^{-x+iy}} - e^{-x-iy} +$$

$$e^{x+iy} - \cancel{e^{x-iy}} + \cancel{e^{-x+iy}} - e^{-x-iy} = 0$$

$$e^{x+iy} - e^{-x-iy} = e^{x+iy} - \frac{1}{e^{x+iy}} = \frac{1}{e^{x+iy}}\left(e^{2x+2iy} - 1\right) = 0$$

$$e^{x+iy} \neq 0 \Rightarrow e^{2(x+iy)} = 1 \Rightarrow 2(x+iy) = \log 1$$

$$\log 1 = \log|1| + i(\theta_p \pm 2\pi n) \quad , \quad 1 = 1(1+0i),\ \theta_p = 0$$

$$\log 1 = \pm 2n\pi i$$

$$x+iy = \frac{1}{2}(\pm 2n\pi i) \Rightarrow \quad x+iy = \pm n\pi i$$

$$x = 0 \quad , \quad y = \pm n\pi$$

55/7. α) $\cosh z = \frac{1}{2}$

From above we have: $\cosh z = \frac{1}{2}\left(e^{x+iy} + e^{-x-iy}\right) = \frac{1}{2}$

$$e^{x+iy} + e^{-x-iy} = 1$$

$$e^{x+iy} + \frac{1}{e^{x+iy}} - 1 = 0 \Rightarrow e^{2x+2iy} - e^{x+iy} + 1 = 0$$

$$e^{x+iy} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

(i) $e^{x+iy} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow x+iy = \log\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \quad , \theta_p = \frac{\pi}{3}$

$$= \log 1 + i\left(\frac{\pi}{3} \pm 2n\pi\right) = \left(\frac{1}{3} \pm 2n\right)\pi i$$

ii $e^{x+iy} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow x+iy = \log\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \quad , \theta_p = -\frac{\pi}{3}$

$$= \log 1 + i\left(-\frac{\pi}{3} \pm 2n\pi\right) = \left(-\frac{1}{3} \pm 2n\right)\pi i$$

b) $\sinh z = i$

$\sinh z = \sinh(x+iy) = \sinh x \cdot \cos y + i \cdot \cosh x \cdot \sin y$

$= \frac{1}{4}\left(e^{x}-e^{-x}\right)\left(e^{iy}+e^{-iy}\right) + \frac{1}{4}\left(e^{x}+e^{-x}\right)\cdot\left(e^{iy}-e^{-iy}\right)$

$= \frac{1}{4}\left(2e^{x+iy} - 2e^{-x-iy}\right) = \frac{1}{2}\left(e^{x+iy} - \frac{1}{e^{x+iy}}\right) = i$

$e^{x+iy} - \frac{1}{e^{x+iy}} - 2i = 0$

$e^{2(x+iy)} - 2i \cdot e^{x+iy} - 1 = 0$

$e^{x+iy} = i \pm \sqrt{i^2+1} = i \quad \Rightarrow \quad x+iy = \log i$

$x+iy = \log i = \log 1 + i\left(\frac{\pi}{2} \pm 2n\pi\right) = \left(\frac{1}{2} \pm 2n\right)\pi i$

c) $\cosh z = -2 \quad \Rightarrow \quad \frac{1}{2}\left(e^{x+iy} + e^{-x-iy}\right) = -2$

$e^{x+iy} + \frac{1}{e^{x+iy}} + 4 = 0 \quad \Rightarrow \quad e^{2x+2iy} + 4e^{x+iy} + 1 = 0$

$e^{x+iy} = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}$

(i) $e^{x+iy} = 2+\sqrt{3} \quad \Rightarrow \quad x+iy = \log(2+\sqrt{3}) = \mathrm{Log}\sqrt{7} + i(\theta_p \pm 2n\pi)$

$= \frac{1}{2}\mathrm{Log}\,7 + i\cdot\left[\arctan\frac{\sqrt{3}}{2} \pm 2n\pi\right]$

(ii) $e^{x+iy} = 2-\sqrt{3} \quad \Rightarrow \quad x+iy = \log(2-\sqrt{3}) = \log\sqrt{7} + i(\theta_p' \pm 2n\pi)$

$= \frac{1}{2}\mathrm{Log}\,7 + i\left[\arctan\frac{-\sqrt{3}}{2} \pm 2n\pi\right]$

55/8. α) $f(z) = \sinh(e^z)$

$= \sinh(e^x \cdot e^{iy})$

$= \sinh(e^x \cdot \cos y + i \cdot e^x \cdot \sin y)$

$= \sinh(e^x \cdot \cos y) \cdot \cos(e^x \cdot \sin y) +$
$i \cdot \cosh(e^x \cdot \cos y) \cdot \sin(e^x \cdot \sin y)$

1)

$$\left.\begin{array}{l} u = \sinh(e^x \cdot \cos y) \cdot \cos(e^x \cdot \sin y) \\ v = \cosh(e^x \cdot \cos y) \cdot \sin(e^x \cdot \sin y) \end{array}\right\} \text{real-single valued.}$$

2) $$\frac{\partial u}{\partial x} = \cosh(e^x \cdot \cos y) \cdot e^x \cdot \cos y \cdot \cos(e^x \cdot \sin y) - \sin(e^x \cdot \sin y) \cdot e^x \cdot \sin y \cdot \sinh(e^x \cdot \cos y)$$

$$\frac{\partial v}{\partial y} = -\sinh(e^x \cdot \cos y) \cdot e^x \cdot \sin y \cdot \sin(e^x \cdot \sin y) + \cosh(e^x \cdot \cos y) \cdot e^x \cdot \cos y \cdot \cos(e^x \cdot \sin y)$$

$$\frac{\partial u}{\partial y} = -\cosh(e^x \cdot \cos y) \cdot e^x \cdot \sin y \cdot \cos(e^x \cdot \sin y) - \sinh(e^x \cdot \cos y) \cdot e^x \cdot \cos y \cdot \sin(e^x \cdot \sin y) = -\frac{\partial v}{\partial x}$$

3) all the above are continuous $(u, v, \frac{\partial u}{\partial x}$ etc.)

4) Cauchy – Riemann equations are satisfied.

then:

$f'(z)$ exists everywhere or

$f(z)$ is analytic everywhere or

$f(z)$ is entire.

β) $f(z) = \sinh(e^z)$

Is entire because it is entire function sinh of the entire function e^z or:

$\sinh e^z = \frac{1}{2}\left(e^{e^z} - e^{-e^z}\right) = \frac{1}{2}\left(\exp e^z - \exp(-e^z)\right) =$ entire.

$$\mathrm{Re}\left[\sinh(e^{z})\right] = u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \cdot \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \cdot \partial y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \nabla^2 u .$$

The above from the relations of α.

59/1. α) $\log 1 = \pm 2n\pi i$

$\log z = \log r + i\cdot(\theta_p \pm 2n\pi)$ $\quad n=0,1,2,\dots$

$z=1 \quad |z| = r = 1>0 \ , \ z = 1\cdot(1+0\cdot i) \Rightarrow \theta_p = 0$

$\log 1 = \log|1| + i\cdot(0 \pm 2n\pi)$

$\log 1 = 0 + i(\pm 2n\pi) \Rightarrow \log 1 = \pm 2n\pi i$

β) $\log(-1) = \pm(2n+1)\pi$

$\log z = \log r + i(\theta_p \pm 2n\pi)$ $\quad n=0,1,2,\dots$

$z=-i \Rightarrow |z| = r = 1>0 \ , \ z = 1(-1+0i) \Rightarrow \theta_p = \pm\pi$

$\log(-1) = \log|1| + i(\pm\pi \pm 2n\pi)$

$\log(-1) = 0 + i(\pm\pi \pm 2n\pi) \Rightarrow \log(-1) = \pm(2n+1)\pi i$

c) $\log i = \frac{1}{2}\pi i \pm 2n\pi i$

$\log z = \log r + i(\theta_p \pm 2\pi n)$ $\quad n=0,1,2,3,\dots$

$z=i \quad |z| = r = 1>0 \quad z = 1(0+i\cdot 1) \Rightarrow \theta_p = \frac{\pi}{2}$

$\log i = \log|1| + i\left(\frac{\pi}{2} \pm 2n\pi\right)$

$\log i = 0 + i\left(\frac{\pi}{2} \pm 2n\pi\right) \Rightarrow \log i = \frac{1}{2}\pi i \pm 2n\pi i$

EXERCISES

59/ **1.** When $n = 0, 1, 2, \ldots$, show that

(a) $\log 1 = \pm 2n\pi i$; (b) $\log(-1) = \pm(2n+1)\pi i$;
(c) $\log i = \frac{1}{2}\pi i \pm 2n\pi i$; (d) $\log(i^{\frac{1}{2}}) = \frac{1}{4}\pi i \pm n\pi i$.

60/ **2.** Show that

(a) $\operatorname{Log}(-ei) = 1 - \frac{1}{2}\pi i$; (b) $\operatorname{Log}(1 - i) = \frac{1}{2}\operatorname{Log} 2 - \frac{1}{4}\pi i$.

3. Find all roots of the equation $\log z = \frac{1}{2}\pi i$. *Ans.* $z = i$.

4. Find all roots of the equation $e^z = -3$.

Ans. $z = \operatorname{Log} 3 \pm (2n + 1)\pi i$.

5. Establish formula (5) of this section.

6. For all points z in the right half plane $x > 0$ show that

$$\operatorname{Log} z = \frac{1}{2}\operatorname{Log}(x^2 + y^2) + i \arctan\frac{y}{x},$$

where the inverse tangent has the principal value used in calculus, that is, $-\pi/2 < \arctan t < \pi/2$. Use this representation together with Theorem 1, Sec. 18, to give another proof that the principal branch $\operatorname{Log} z$ is analytic in the domain $x > 0$ and that formula (5), Sec. 26, is true there. But note that some complications arise with the inverse ? tangent and its differentiation in the remaining part of the full domain of analyticity, $r > 0$, $-\pi < \arg z < \pi$, of $\operatorname{Log} z$, especially on the line $x = 0$.

7. Show in two ways that the function $\operatorname{Log}(x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

8. Write $z = r\exp(i\theta)$ and $z - 1 = \rho\exp(i\phi)$ and show that

$$\mathcal{R}[\log(z - 1)] = \frac{1}{2}\operatorname{Log}(1 + r^2 - 2r\cos\theta) \qquad (z \neq 1)$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

exercises

59/① ⓓ $\log(i^{1/2}) = \frac{1}{4}\pi i \pm n\cdot\pi\cdot i$ $\qquad \log i^{1/2} = \frac{1}{2}\cdot\log i$

$\log z = \log r + i(\theta_p \pm 2n\pi) \qquad n = 0, 1, 2, \dots$

$z = i \Rightarrow |z| = r = 1 \Rightarrow z = 1(0 + i\cdot 1) \Rightarrow$

$\cos\theta_p = 0, \ \sin\theta_p = 1 \Rightarrow \theta_p = \frac{\pi}{2}$

$\log i = \log 1 + i\left(\frac{\pi}{2} \pm 2\pi n\right) = 0 + i\left(\frac{\pi}{2} \pm 2n\pi\right)$

$\log(i^{1/2}) = \frac{1}{2}\log i = \frac{\pi}{4}\cdot i \pm n\pi i \quad , \quad n = 0, 1, 2, \dots$

60/2. α) $\log(-ei) = 1 - \frac{1}{2}\cdot\pi\cdot i$

$\operatorname{Log}_e z = \log_e r + i\cdot\theta_p \quad , \quad \log_{10} z = \operatorname{Log}_e r + i(\theta_p \pm 2n\pi)$

$z = -ei, \ r = |z| = e > 0 \qquad z = e\cdot(0 - 1\cdot i) \Rightarrow \theta_p = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

$\operatorname{Log}_e z = \operatorname{Log}_e e + i\theta_p = 1 + i\frac{3\pi}{2}$ or

$= 1 - i\cdot\frac{\pi}{2}$

β) $\operatorname{Log}_e(1-i) = \frac{1}{2}\operatorname{Log}_e 2 - \frac{1}{4}\pi i$

$\operatorname{Log}_e(1-i) = \operatorname{Log}_e r + i\,\theta_p \qquad z = 1 - i \quad |z| = r = \sqrt{2} = 2^{1/2}$

$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \Rightarrow \theta_p = \frac{7\pi}{4}, -\frac{\pi}{4}$

$\operatorname{Log}(1-i) = \operatorname{Log} 2^{1/2} + i\,\theta_p = \frac{1}{2}\operatorname{Log} 2 - \frac{1}{4}\pi i$ or

$= \frac{1}{2}\operatorname{Log} 2 + \frac{7}{4}\pi i$

60/3. $\log z = \frac{1}{2}\pi i \quad \exp\log z = \exp\frac{1}{2}\pi i \Rightarrow z = e^{\frac{1}{2}\pi i}$

or :

$\log z = w \Rightarrow z = e^{w}$

$z = e^{\frac{1}{2}\pi i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i\cdot 1 \Rightarrow z = i$

60/4. $e^{z} = -3 \Rightarrow \exp z = -3 \Rightarrow \log\exp z = \log(-3)$

$z = \log(-3)$

$\log z = \operatorname{Log} r + i(\theta_p \pm 2n\pi)$

$z = -3 \quad |z| = 3 = r > 0 \quad z = 3(-1 + 0\cdot i) \Rightarrow \theta_p = \pm\pi$

$\log(-3) = \operatorname{Log}3 \pm i(\pm\pi \pm 2n\pi)$

$z = \log(-3) = \operatorname{Log}3 \pm (2n+1)\pi i$

60/5. $\log z_1 - \log z_2 = \log\frac{z_1}{z_2} \qquad z = e^{w} \Rightarrow \log z = w$

(i) it is true that : $e^{w_1}\cdot e^{w_2} = e^{w_1+w_2}, \quad \frac{e^{w_1}}{e^{w_2}} = e^{w_1-w_2}$

$\log\left[e^{w_1}\cdot e^{w_2}\right] = \log e^{w_1} + \log e^{w_2} = w_1 + w_2 = \log z_1 + \log z_2$

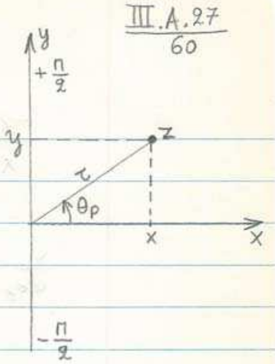
$\log e^{w_1+w_2} = \qquad = w_1 + w_2 = \log z_1 + \log z_2$

(ii)

$\left.\begin{array}{l}\log z_1 = \operatorname{Log} r_1 + i\theta_1 \\ \log z_2 = \operatorname{Log} r_2 + i\theta_2\end{array}\right\} \Rightarrow \log z_1 - \log z_2 = \operatorname{Log} r_1 - \operatorname{Log} r_2 + i(\theta_1 - \theta_2)$

$= \operatorname{Log}\frac{r_1}{r_2} + i\cdot(\theta_1 - \theta_2)$

$= \log\frac{z_1}{z_2}$



60/6. Show: $\text{Log}\, z = \frac{1}{2}\text{Log}(x^2+y^2) + i \arctan\frac{y}{x}$

when: $-\frac{\pi}{2} < \arctan\frac{y}{x} < \frac{\pi}{2}$.

We have that $\text{Log}\, z = \text{Log}\, r + i\cdot\theta_p$
when $-\pi < \theta_p < \pi$

Since $x > 0$ then $-\frac{\pi}{2} < \theta_p < \frac{\pi}{2}$

and so $\text{Log}\, z = \text{Log}\, r + i\cdot\theta_p$ with $-\frac{\pi}{2} < \theta_p < \frac{\pi}{2}$

But $r^2 = x^2+y^2$

$r = (x^2+y^2)^{1/2}$ and $\tan\theta_p = \frac{y}{x}$ or $\theta_p = \arctan\frac{y}{x}$

So we can write

$$\text{Log}\, z = \text{Log}(x^2+y^2)^{1/2} + i\cdot\arctan\frac{y}{x} \quad \text{when } -\frac{\pi}{2} < \theta_p < \frac{\pi}{2}$$

or $\text{Log}\, z = \frac{1}{2}\cdot\text{Log}(x^2+y^2) + i\cdot \text{arctag}\frac{y}{x}$ when $-\frac{\pi}{2} < \arctan\frac{y}{x} < \frac{\pi}{2}$

Now, we must show that $\text{Log}\, z$ is analytic:

$f(z) = \text{Log}\, z = \frac{1}{2}\cdot\text{Log}(x^2+y^2) + i\cdot\arctan\frac{y}{x}$ real-single valued

1) $u = \frac{1}{2}\cdot\text{Log}(x^2+y^2)$ $\quad v = \arctan\frac{y}{x}$

2) $\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(x^2+y^2)'_x = \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(x^2+y^2)'_y = \frac{y}{x^2+y^2}$

$v = \arctan\frac{y}{x} = \theta_p \Rightarrow \frac{y}{x} = \tan\theta_p$

$\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{d}{d\theta_p}(\tan\theta_p)\cdot\frac{\partial\theta_p}{\partial x}$

$$\frac{-y}{x^2} = \frac{d}{d\theta_p}(\tan\theta_p) \cdot \frac{\partial\theta_p}{\partial x}$$

$$\frac{d}{d\theta_p}(\tan\theta_p) = \frac{d}{d\theta_p}\left(\frac{\sin\theta_p}{\cos\theta_p}\right) = \frac{\cos\theta_p\cdot(\sin\theta_p)'_{\theta_p} - \sin\theta_p\cdot(\cos\theta_p)'_{\theta_p}}{(\cos\theta_p)^2}$$

$$= \frac{\cos^2\theta_p + \sin^2\theta_p}{\cos^2\theta_p} = \frac{1}{\cos^2\theta_p}$$

So:

$$\frac{-y}{x^2} = \frac{1}{\cos^2\theta_p} \cdot \frac{\partial\theta_p}{\partial x}$$

$$\tan^2\theta_p = \frac{\sin^2\theta_p}{\cos^2\theta_p} = \frac{1-\cos^2\theta_p}{\cos^2\theta_p} = \frac{1}{\cos^2\theta_p} - 1$$

$$\cos^2\theta_p = \frac{1}{1+\tan^2\theta_p}$$

$$\frac{-y}{x^2} = (1+\tan^2\theta_p)\cdot\frac{\partial\theta_p}{\partial x}$$

$$= \left(1+\frac{y^2}{x^2}\right)\cdot\frac{\partial\theta_p}{\partial x} = \frac{x^2+y^2}{x^2}\cdot\frac{\partial\theta_p}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial\theta_p}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\partial u}{\partial y}$$

Similarly: $\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{d}{d\theta_p}(\tan\theta_p)\cdot\frac{\partial\theta_p}{\partial y}$

$$\frac{1}{x} = \frac{x^2+y^2}{x^2}\cdot\frac{\partial\theta_p}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial\theta_p}{\partial y} = \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x}$$

3) all above are continuous ($u, v, \frac{\partial u}{\partial x}$ etc.)
because $r \neq 0 \Rightarrow x^2+y^2 \neq 0$

4) From 2 we see that Cauchy-Riemann equations are Satisfied

then :

$f'(z)$ exists everywhere in this domain $r>0$, $-\frac{\pi}{2} < \theta_p < \frac{\pi}{2}$ and so $f(z)$ is analytic there

and $f'(z) = (\text{Log}\,z)'$

Formula 5 sect. 26 $\Rightarrow$ $\frac{d}{dz}\text{Log}\,z = \frac{1}{z}$

We have $\frac{d}{dz}\text{Log}\,z = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2-i^2y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

Note : $z = \sin w$ then $w = \sin^{-1} z$

$= \text{inverse } \sin z$

$= \text{arc. } \sin z$

60/7 **1st Way** $\quad \text{Log}(x^2+y^2) = u$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} = \dfrac{-2x^2+2y^2}{(x^2+y^2)^2} \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = \dfrac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2} \end{array}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0$$

that is, u is harmonic

2nd Way We have that $\text{Log}\, z = \frac{1}{2}\text{Log}(x^2+y^2) + i \tan^{-1}\frac{y}{x}$ in $x^2+y^2 \neq 0$ and $-\frac{\pi}{2} < \tan^{-1}\frac{y}{x} < \frac{\pi}{2}$

So $\frac{1}{2}\text{Log}(x^2+y^2)$ and $\tan^{-1}\frac{y}{x}$ are

conjugate harmonic functions because $\text{Log}\, z$ is analytic is that domain (precending exercise)

So for $\text{Log}(x^2+y^2)$ we have

$$\frac{\partial^2}{\partial x^2}\left[\text{Log}(x^2+y^2)\right] + \frac{\partial^2}{\partial y^2}\left[\text{Log}(x^2+y^2)\right] = 0$$

$$\nabla^2 \text{Log}(x^2+y^2) = 0$$

60/8. a) $z = r \cdot \exp(i\theta)$, $z-1 = \rho \cdot \exp(i\phi)$

$$Re\left[\log(z-1)\right] = \frac{1}{2} \operatorname{Log}(1 + r^2 - 2r\cos\theta) \qquad z \neq 1$$

$\log(z-1) = \log(\rho \cdot \exp(i\phi)) = \log\rho + i\phi$
$Re[\log(z-1)] = \log\rho$

$z = r \cdot (\cos\theta + i \cdot \sin\theta) = r\cos\theta + i\, r \cdot \sin\theta$
$z - 1 = r \cdot \cos\theta - 1 + i \cdot r \cdot \sin\theta$
$\log(z-1) = \operatorname{Log}|z-1| + i(\theta_p \pm 2n\pi)$

$$|z-1| = \left[(r \cdot \cos\theta - 1)^2 + r^2 \sin^2\theta\right]^{1/2} =$$

$$= \left(r^2 \cdot \cos^2\theta + 1 - 2r\cos\theta + r^2 \cdot \sin^2\theta\right)^{1/2}$$
$$= \left[r^2(\cos^2\theta + \sin^2\theta) + 1 - 2 \cdot r \cdot \cos\theta\right]^{1/2}$$
$$= \left(r^2 + 1 - 2r\cos\theta\right)^{1/2}$$

$$\log(z-1) = \frac{1}{2}\operatorname{Log}(1 + r^2 - 2r\cos\theta) + i \cdot (\theta_p \pm 2n\pi)$$

$$Re\left[\log(z-1)\right] = \frac{1}{2} \cdot \operatorname{Log}(1 + r^2 - 2r\cos\theta)$$

b) We found that :
$$\log(z-1) = \frac{1}{2} \cdot \operatorname{Log}(1 + r^2 - 2r \cdot \cos\theta) + i \cdot (\theta_p \pm 2n\pi)$$
From page 57 we have that $\log z$ is analytic in the domain $r > 0$ and $\theta_p < \theta < \theta_p + 2\pi$

So $\log(z-1)$ is analytic function of the analytic function $z-1$ everywhere except at the point $z-1=0$ in which log has no meaning.

So $\frac{1}{2}\operatorname{Log}(1+r^2-2r\cdot\cos\theta)$, $\theta_p \pm 2n\pi$

are conjugate harmonic functions or they satisfy the Laplace equation

that is the:

$$\operatorname{Re}\left[\log(z-1)\right] = \frac{1}{2}\operatorname{Log}\left(r^2+1-2r\cdot\cos\theta\right)$$

must satisfy Laplace's equation when $z \neq 1$

EXERCISES

63/ **1.** When $n = 0, 1, 2, \ldots$, show that

(a) $(1 + i)^i = \exp\left(-\frac{1}{4}\pi \pm 2n\pi\right) \exp\left(\frac{1}{2} i \operatorname{Log} 2\right)$;
(b) $(-1)^{1/\pi} = \exp\left[\pm(2n + 1)i\right]$.

2. Find the principal value of

(a) i^i; (b) $\left[\frac{1}{2}e(-1 - i\sqrt{3})\right]^{3\pi i}$; (c) $(1 - i)^{4i}$.
Ans. (a) $\exp\left(-\frac{1}{2}\pi\right)$; (b) $-\exp\left(2\pi^2\right)$.

3. Show that, if $z \neq 0$,

(a) $z^0 = 1$;
(b) $|z^k| = \exp\left(k \operatorname{Log} |z|\right) = |z|^k$ when k is real.

4. Let b, c, and z denote complex numbers, where $z \neq 0$. If all powers here are principal values, prove that

(a) $z^{-c} = \dfrac{1}{z^c}$;
(b) $(z^c)^n = z^{nc}$ $(n = 1, 2, \ldots)$;
(c) $z^b z^c = z^{b+c}$; (d) $\dfrac{z^b}{z^c} = z^{b-c}$.

5. Use the principal values of z^i to write the conjugate harmonic functions $u(r,\theta)$ and $v(r,\theta)$ when $z^i = u + iv$.

6. Derive formula (8), Sec. 28; also, the formula for the derivative with respect to z of c^w, where $w'(z)$ exists.

7. Find the values of

(a) $\tan^{-1}(2i)$; (b) $\tan^{-1}(1 + i)$; (c) $\cosh^{-1}(-1)$; (d) $\tanh^{-1} 0$.
Ans. (a) $\pm\left(n + \frac{1}{2}\right)\pi + \frac{1}{4} i \operatorname{Log} 3$; (d) $\pm n\pi i$ $(n = 0, 1, 2, \ldots)$.

64/ **8.** Solve the equation $\sin z = 2$ for z (a) by identifying real and imaginary components of its members; (b) by using formula (1).
Ans. $z = \frac{1}{2}\pi(1 \pm 4n) \pm i \operatorname{Log}\left(2 + \sqrt{3}\right)$ $(n = 0, 1, 2, \ldots)$.

9. Solve the equation $\cos z = \sqrt{2}$ for z.

10. Derive formulas (2) and (5) of this section.

11. Derive formulas (3) and (4) of this section.

12. Derive formulas (6) and (8) of this section.

Exercises

63/1. a)

$$(1+i)^i = \exp\left(-\tfrac{1}{4}\pi \pm 2n\pi\right)\cdot\exp\left(\tfrac{1}{2}i\cdot \mathrm{Log}\,2\right)$$

$$z^c = \exp[c\cdot \log z] \qquad z = 1+i \quad , \; c = i$$

$$(1+i)^i = \exp\left[i\cdot \log(1+i)\right]$$

$$\log(1+i) = \mathrm{Log}|1+i| + i(\theta_p \pm 2n\pi), \left\{ \begin{array}{l} z = 1+i \qquad |z| = r = \sqrt{2} = 2^{1/2} \\ z = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ \qquad\qquad\qquad \theta_p = \frac{\pi}{4} \end{array}\right.$$

$$\log(1+i) = \mathrm{Log}\,2^{1/2} + i\left(\tfrac{\pi}{4} \pm 2n\pi\right)$$

$$= \tfrac{1}{2}\mathrm{Log}\,2 + i\left(\tfrac{\pi}{4} \pm 2n\pi\right)$$

$$i\cdot\log(1+i) = \tfrac{1}{2}i\cdot\mathrm{Log}\,2 + \left(-\tfrac{\pi}{4} \pm 2n\pi\right)$$

$$\exp\left(i\log(1+i)\right) = \exp\left[\tfrac{1}{2}i\,\mathrm{Log}\,2 + \left(-\tfrac{\pi}{4} \pm 2n\pi\right)\right]$$

$$= \exp\left(-\tfrac{\pi}{4} \pm 2n\pi\right)\cdot\exp\left(\tfrac{1}{2}i\,\mathrm{Log}\,2\right)$$

b) $$(-1)^{1/\pi} = \exp\left[\pm(2n+1)i\right]$$

$$\log(-1) = \mathrm{Log}\,1 + i(\pm\pi \pm 2n\pi)$$
$$\log(-1) = \pm i(2n+1)\pi$$

$$(-1)^{1/\pi} = \exp\left[\tfrac{1}{\pi}\log(-1)\right]$$

$$= \exp\left[\tfrac{1}{\pi}\cdot\left[\pm i(2n+1)\right]\pi\right]$$

$$= \exp\left[\pm(2n+1)i\right]$$

63/2. α) $i^i = \exp(i \operatorname{Log} i)$ $\qquad \operatorname{Log} i = \operatorname{Log} 1 + i\theta_p \qquad i = 1(0+i1) \Rightarrow \theta_p = \frac{\pi}{2}$

$\operatorname{Log} i = 0 + i\frac{\pi}{2} \Rightarrow \operatorname{Log} i = i\frac{\pi}{2}$

$$i^i = \exp\left(i \cdot i \cdot \frac{1}{2} \cdot \pi\right)$$
$$= \exp\left(-\frac{1}{2}\pi\right)$$

Principal value of $z^c = \exp(c \cdot \operatorname{Log} z)$

$$\operatorname{Log} z = \operatorname{Log}|z| + i \cdot \theta_p$$

β) $\left[\frac{1}{2} e \cdot (-1 - i\sqrt{3})\right]^{3\pi i} = \exp\left[3\pi i \cdot \operatorname{Log}\left(\frac{1}{2} e(-1-i\sqrt{3})\right)\right]$

$$\operatorname{Log}\left[\frac{1}{2} e(-1-i\sqrt{3})\right] = \operatorname{Log}\frac{1}{2} + \operatorname{Log} e + \operatorname{Log}(-1-i\sqrt{3})$$

$\operatorname{Log}\frac{1}{2} = \operatorname{Log}\left|\frac{1}{2}\right| + i\theta_{p_1} = \operatorname{Log}\left|\frac{1}{2}\right| + i \cdot 0$ $\qquad \left(\frac{1}{2} = \frac{1}{2}(1+i0) \Rightarrow \theta_{p_1} = 0\right.$

$$= \operatorname{Log}|1| - \operatorname{Log} 2 = -\operatorname{Log} 2$$

$\operatorname{Log} e = \operatorname{Log}|e| + i\theta_{p_2}$ $\qquad e = |e| \cdot (1 + i \cdot 0) \Rightarrow \theta_{p_2} = 0$

$$= 1 = \operatorname{Log} e$$

$, \quad |-1-\sqrt{3}i| = \sqrt{1+3} = 2$

$\operatorname{Log}(-1-i\sqrt{3}) = \operatorname{Log} r + i\theta_{p_3}$ $\quad , \quad -1-\sqrt{3} \cdot i = 2 \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \quad \theta_{p_3} = \frac{4\pi}{3}$

$$= \operatorname{Log} 2 + i \cdot \frac{4\pi}{3}$$

$$\operatorname{Log}\left[\frac{1}{2} \cdot e \cdot (-1 - i \cdot \sqrt{3})\right] = -\operatorname{Log}|2| + \operatorname{Log}|e| + \operatorname{Log} 2 + i\frac{4\pi}{3}$$

$$\exp\left[3\pi i \cdot \operatorname{Log}\left[\frac{e}{2}(-1-i\sqrt{3})\right]\right] = \exp\left[3\pi i \cdot \left(1 + i \cdot \frac{4\pi}{3}\right)\right]$$

$$= \exp(3\pi i - 4\pi^2)$$

;

c) $(1-i)^{4i} = \exp\left[4i \log(1-i)\right]$

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

$\theta_p = -\frac{\pi}{4}$

$$\begin{aligned} \mathrm{Log}(1-i) &= \mathrm{Log}\sqrt{2} + i\cdot\frac{7\pi}{4} \\ &= \mathrm{Log}\sqrt{2} - i\cdot\frac{\pi}{4} \quad \text{or} \\ &= \frac{1}{2}\mathrm{Log}\,2 + i\,\frac{7\pi}{4} \\ &= \frac{1}{2}\mathrm{Log}\,2 - i\frac{\pi}{4}. \end{aligned}$$

$$(1-i)^{4i} = \exp\left[4i\left(\frac{1}{2}\mathrm{Log}\,2 + i\,\frac{7\pi}{4}\right)\right] = \exp\left(2i\,\mathrm{Log}\,2\right) + \exp(-7\pi)$$

or

$$(1-i)^{4i} = \exp\left[4i\cdot\left(\frac{1}{2}\mathrm{Log}\,2 - i\frac{\pi}{4}\right)\right] = \exp\left(2i\,\mathrm{Log}\,2\right) + \exp(\pi)$$

63/3. a) $z^0 = 1$ $\qquad z \neq 0$

$$z^0 = \exp\left[0\cdot \log z\right] = \exp 0 = e^0 = 1$$

b) $z^K = \exp(K\cdot \log z)$ $\qquad K = \text{real}$

$$\begin{aligned} &= \exp\left[K\cdot \mathrm{Log}|z| + \left(K\cdot\theta_p \pm 2nK\pi\right)i\right] \\ &= e^{K\cdot \mathrm{Log}|z|} \cdot e^{i\cdot K\cdot(\theta_p \pm 2n\pi)} \\ &= e^{K\cdot\mathrm{Log}|z|}\cdot\left(\cos(\theta_p \pm 2n\pi)K + i\cdot\sin(\theta_p \pm 2n\pi)K\right) \end{aligned}$$

$$|z^K| = e^{K\cdot \mathrm{Log}|z|} = \exp\left(K\cdot\mathrm{Log}|z|\right)$$

$$\begin{aligned} z = \exp(\log z) &= \exp\left[\mathrm{Log}|z| + i(\theta_p \pm 2n\pi)\right] \\ &= e^{\mathrm{Log}|z|}\cdot e^{i(\theta_p \pm 2n\pi)} = e^{\mathrm{Log}|z|}\left[\cos(\theta_p \pm 2n\pi) + i\cdot\sin(\theta_p \pm 2n\pi)\right] \end{aligned}$$

$$|z| = e^{\mathrm{Log}|z|}$$

$$|z|^K = \left[e^{\mathrm{Log}|z|}\right]^K = e^{K\,\mathrm{Log}|z|} = \exp\left(K\cdot\mathrm{Log}|z|\right)$$

So we have : $\quad |z^K| = \exp\left(K\cdot\mathrm{Log}|z|\right) = |z|^K$

SOME PAGES ARE MISSING

Exercises

63/(12) α) $\sinh^{-1} z = \log\left[z+(z^2+1)^{1/2}\right]$

$$\sinh^{-1} z = w \Rightarrow z = \sinh w = \frac{e^{w} - e^{-w}}{2}$$

$$2z = e^{w} - e^{-w} \Rightarrow e^{w} - \frac{1}{e^{w}} - 2z = 0$$

$$(e^{w})^2 - 2z \cdot e^{w} - 1 = 0 \Rightarrow e^{w} = z \pm \sqrt{z^2+1}$$

$e^{w} = z - \sqrt{z^2+1}$ we have that $z - \sqrt{z^2+1} < 0$
So $e^{w} < 0$. But it is true that we have always $e^{w} > 0$. So we shall have only:

$$e^{w} = z + \sqrt{z^2+1} = z + (z^2+1)^{1/2}$$

$$\log e^{w} = \log\left[z+(z^2+1)^{1/2}\right] \Rightarrow w = \log e^{w} = \sinh^{-1} z$$

$$\sinh^{-1} z = \log\left[z+(z^2+1)^{1/2}\right]$$

B) $\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}$

$$\tanh^{-1} z = w \Rightarrow z = \tanh w = \frac{\sinh w}{\cosh w} = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$$

$$z \cdot (e^{w} + e^{-w}) - (e^{w} - e^{-w}) = 0 \rightarrow z\left(e^{w} + \frac{1}{e^{w}}\right) - \left(e^{w} - \frac{1}{e^{w}}\right) = 0$$

$$\left[z \cdot (e^{w})^2 + z\right] \cdot \frac{1}{e^{w}} - \frac{1}{e^{w}} \cdot \left[(e^{w})^2 - 1\right] = 0$$

$$z \cdot (e^{w})^2 + z - (e^{w})^2 + 1 = 0 \Rightarrow (z-1) \cdot [e^{w}]^2 = -1 - z$$

$$(1-z) \cdot (e^{w})^2 = 1 + z \Rightarrow (e^{w})^2 = \frac{1+z}{1-z}$$

$$e^w = \left(\frac{1+z}{1-z}\right)^{1/2}$$

$$\log e^w = \log\left(\frac{1+z}{1-z}\right)^{1/2} = \frac{1}{2}\log\frac{1+z}{1-z}$$

$$w = \log e^w = \tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$$

T. Costopoulos

(1) a

$f(z)$ is analytic if $f'(z)$ exists in D and single valued there

A function $W = u + iv$ in order to be analytic at z_0 must satisfy the following:

If:

1) its components u, v ~~must be~~ are single-real-valued.

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at point z_0

3) all the above $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ ~~must be~~ are continuous at that point z_0

4) Its derivatives ~~must~~ satisfy the Cauchy - Riemann equations (conditions) that is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

prove this

then:

$f'(z)$ at z_0 exists and

$f(z)$ is analytic at that point.

A function $f(z) = W = u + iv$ is entire when is analytic at every point of z-plane, so a polynomial is entire:

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n .$$

If a function is not analytic at a point z_0 but is analytic at a neighborhood of z_0

then we say that the point z_0 is a singular point of $f(z)$ or a singularity of $f(z)$, and especially then this point is an isolated singular point.

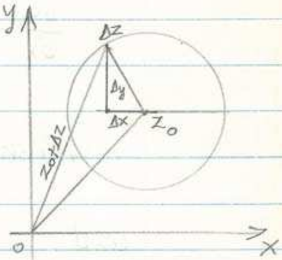
T. Costopoulos.

(1) β. A function $f(z)$ is analytic in a domain D if it is ~~real~~ - single valued in D and if its derivative $f'(z)$ exists at any point of D.

Let w be an analytic function in D, then $w' = \frac{dw}{dz}$ exists at any point of D. Let z_0 a point in D. $\quad w = u + iv$

$$\Delta w = \Delta u + i\Delta v \qquad \frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz} = w' = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$



$$\frac{\Delta w}{\Delta z} = \frac{\Delta u \cdot \Delta x + \Delta v \cdot \Delta y + i(\Delta u \Delta y + \Delta v \cdot \Delta x)}{\Delta x^2 + \Delta y^2}$$

$$Re\left(\frac{\Delta w}{\Delta z}\right) = \frac{\Delta u \Delta x + \Delta v \Delta y}{\Delta x^2 + \Delta y^2} = \alpha$$

$$Im\left(\frac{\Delta w}{\Delta z}\right) = \frac{\Delta v \cdot \Delta x - \Delta u \cdot \Delta y}{\Delta x^2 + \Delta y^2} = \beta$$

1) if $\Delta x = 0 \Rightarrow \quad \alpha = \frac{\Delta v}{\Delta y} \Rightarrow \lim_{\Delta z \to 0} \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y} = \alpha$

$$\beta = -\frac{\Delta u}{\Delta y} \Rightarrow \lim_{\Delta z \to 0} -\frac{\Delta u}{\Delta y} = -\frac{\partial u}{\partial y} = \beta$$

2) if $\Delta y = 0 \Rightarrow \alpha = \frac{\Delta u}{\Delta x} \Rightarrow \lim_{\Delta z \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} = \alpha$

$$\beta = \frac{\Delta v}{\Delta x} \Rightarrow \lim_{\Delta z \to 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x} = \beta$$

So we have:

$$\alpha = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\beta = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad \text{or}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are the Cauchy - Riemann equations

and $W' = \alpha + i\beta = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$.

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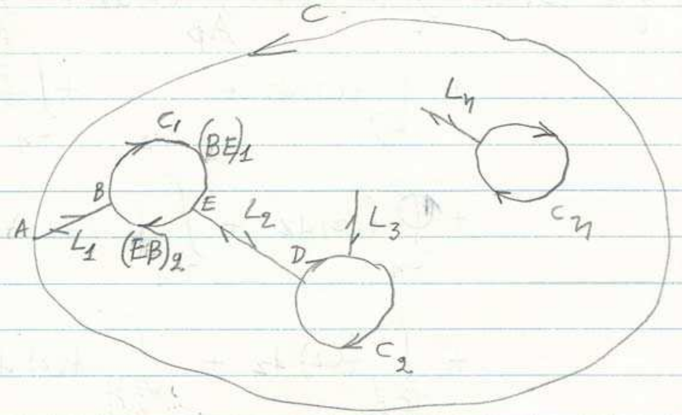
(2) Let C be a closed contour and $c_1, c_2, \dots c_j$ closed contours Inside c as in the figure.



Let R be a region which is consisted of points interior of c and exterior of the curves $c_1, c_2 \dots c_n$ and B the oriented path which has the Region R on the left (positive sense) then the path B states a simply connected region for which we can write the Cauchy's Integral theorem:

$$\oint_B f(z)\, dz = 0$$ ✓

where:

$f(z)$ is a function analytic in R

which R has also the boundaries of the curves $C, C_1, C_2 \dots C_n$. Then

$$\oint_B f(z)\,dz = \oint_C f(z)\,dz + \cancel{\int_{AB} f(z)\,dz} + \cancel{\int_{(BE)_1} f(z)\,dz} +$$

$$+ \cancel{\int_{ED} f(z)\,dz} + \dots + \cancel{\int_{L_n} f(z)\,dz} +$$

$$+ \oint_{C_n} f(z)\,dz + \cancel{\int_{-L_n} f(z)\,dz} + \cancel{\int_{DE} f(z)\,dz}$$

$$+ \cancel{\int_{(EB)_2} f(z)\,dz} + \cancel{\int_{BA} f(z)\,dz}$$

$$\oint_B f(z)\,dz = \oint_C f(z)\,dz + \oint_{C_n} f(z)\,dz + \oint_{(EB+BE)} f(z)\,dz$$

$$+ \dots + \dots = 0$$

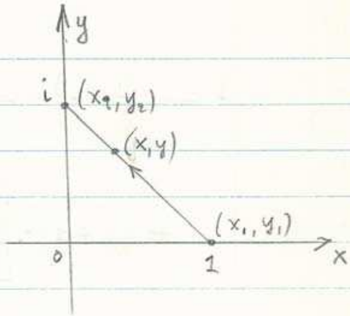
or $$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \dots = 0$$

or $$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \dots + \oint_{C_n} f(z)\,dz$$

that is: the integral on C is equal to the sum of integrals on all interior C_j circles (All the integralls with the positive sense)
[Positive sense ⇒ counterclockwis ↺]

T. Costopoulos

(3) $$\int_1^i (x^2 + iy^3)\, dz$$



This line has the equation:

$$y - y_1 = (x - x_1)\,\frac{y_2 - y_1}{x_2 - x_1}$$

$, 1 \Rightarrow x_1 = 1, \; y_1 = 0$

$, 2 \Rightarrow x_2 = 0, \; y_2 = 1$

$$y - 0 = (x-1)\cdot\frac{1-0}{0-1} = \frac{x-1}{-1}$$

$$y = 1 - x \;\Rightarrow\; \begin{aligned} z &= x + iy \\ &= x + i - ix \\ dz &= dx\cdot(1-i) . \end{aligned}$$

And so:

$$I = \int_1^0 \left[x^2 + i\cdot(1-x)^3\right]\cdot(1-i)\cdot dx$$

$$= \int_1^0 \left[x^2 + i\cdot(1 - 3x + 3x^2 - x^3)\right](1-i)\, dx$$

$$= \int_1^0 (1-i)\cdot\left[x^2 + i\left(-x^3 + 3x^2 - 3x + 1\right)\right] dx$$

$$= (1-i)\cdot\int_1^0 x^2\, dx \quad + i\cdot(1-i)\int_1^0 (-x^3 + 3x^2 - 3x + 1)\, dx$$

$$I = (1-i)\cdot \frac{x^3}{3}\Big|_1^0 +$$

$$+ (i+1)\cdot\left[\frac{-x^4}{4} + x^3 - 3\frac{x^2}{2} + x\right]_1^0$$

$$= (1-i)\left(-\frac{1}{3}\right) + (i+1)\cdot\left[-\left[\frac{-1}{4} + 1 - \frac{3}{2} + 1\right]\right]$$

$$= \frac{i-1}{3} + (i+1)\cdot\left(\frac{1}{4} - 2 + \frac{3}{2}\right)$$

$$= \frac{i-1}{3} + (i+1)\cdot\left(\frac{1}{4} - \frac{8}{4} + \frac{6}{4}\right)$$

$$= \frac{i-1}{3} + (i+1)\cdot\frac{-1}{4} = \frac{4(i-1) - 3(i+1)}{12}$$

$$= \frac{4i - 4 - 3i - 3}{12} = \frac{(4-3)i - 4 - 3}{12}$$

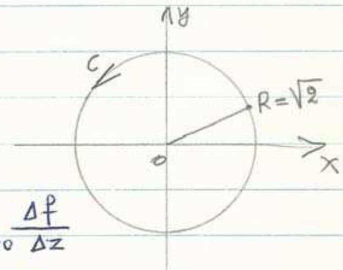
$$= \frac{1}{12}(-7+i)$$

or

T. Costopoulos.

4 $\quad C: \ |z| = \sqrt{2} \Rightarrow (x^2+y^2)^{1/2} = \sqrt{2}$

$$x^2+y^2=(\sqrt{2})^2.$$



$$\int_C (z^2-2z+1)\cdot dz = I.$$

$f(z) = z^2-2z+1, \quad f'(z) = \lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z}$

$$f'(z) = \lim_{\Delta z\to 0} \frac{\left[(z+\Delta z)^2-2(z+\Delta z)+1\right]-(z^2-2z+1)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\cancel{z^2}+\Delta z^2+2z\Delta z-\cancel{2z}-2\Delta z+\cancel{1}-\cancel{z^2}+\cancel{2z}-\cancel{1}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\Delta z^2+2z\Delta z-2\Delta z}{\Delta z} = \lim_{\Delta z\to 0}(\Delta z+2z-2)$$

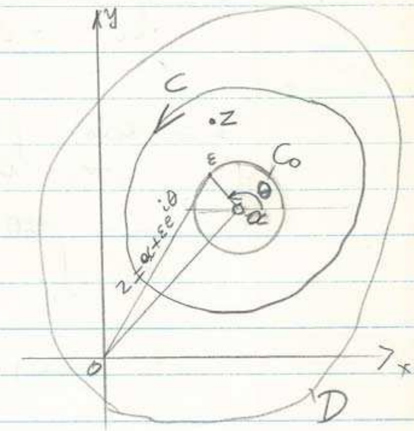
$$= 2z-2$$

for every z. So f(z) is an entire function and so it is analytic in the circle C. From Cauchy's Integral theorem we have:

$$\oint_C (z^2-2z+1)\,dz = 0.$$

I. Costopoulos

(5)



We have that:
$f(z)$ is analytic in D.
So if around α we write a circle C_0 with radius ε then for the region between C and C_0 we can aplly the Cauchy Integral theorem because this region is doubly connected and the function $\frac{f(z)}{z-\alpha}$ is analytic in this doubly connected region since $\frac{f(z)}{z-\alpha}$ is not analytic only at the point $z=\alpha$ which is out of this doubly connected region. So we have

$$\oint_C \frac{f(z)}{z-\alpha}\,dz + \oint_{C_0} \frac{f(z)}{z-\alpha}\,dz = 0$$

or $I = \oint_C \frac{f(z)}{z-\alpha}\,dz = \oint_{C_0} \frac{f(z)\,dz}{z-\alpha}$

but $z = \alpha + \varepsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$

$dz = \varepsilon \cdot i e^{i\theta} \cdot d\theta$ and so:

$$I = \oint_{C_0} \frac{f(a+\varepsilon \cdot e^{i\vartheta})}{\varepsilon \cdot e^{i\vartheta}} \cdot \varepsilon i e^{i\vartheta} d\vartheta \quad \text{for } \varepsilon \to 0$$

$$I = \lim_{\varepsilon \to 0} \int_0^{2\pi} f(a+\varepsilon \cdot e^{i\vartheta}) \cdot i \, d\vartheta$$

$$= \int_0^{2\pi} f(a) \cdot i \, d\vartheta$$

$$= f(a) \cdot \int_0^{2\pi} i \, d\theta = f(a) \cdot i \cdot (\theta]_0^{2\pi}$$

$$= f(a) \cdot i \, 2\pi = 2\pi i \, f(a)$$

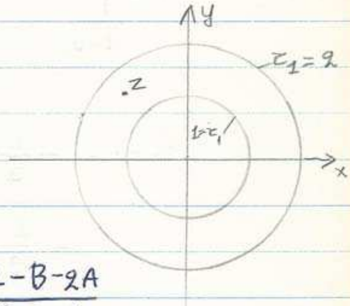
And from the above:

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

T. Costopoulos

(6) $1<|z|<2 \Rightarrow 1^2 < x^2+y^2 < 2^2$

$$f(z) = \frac{1}{(z-1)(z-2)}$$



$$= \frac{A}{z-1} + \frac{B}{z-2}$$

$$= \frac{Az-2A+Bz-B}{(z-1)(z-2)} = \frac{(A+B)z-B-2A}{(z-1)(z-2)}$$

$$\left.\begin{array}{ll} A+B=0, & -B-2A=1 \\ \text{or } A=-B & A-2A=1 \end{array}\right\} \Rightarrow A=-1, \quad B=1$$

and: $f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$

Since $1<|z|<2 \Rightarrow \frac{1}{|z|}<1, \quad \frac{|z|}{2}<1$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\cdot\frac{1}{\left(1-\frac{z}{2}\right)}$$

Now we can use the expess:

$$\frac{1}{1-a} = 1+a+a^2+\cdots+a^{n-1}+\frac{a^n}{1-a} \quad \text{with } a^n \to \infty \text{ when } n\to\infty$$

Since $\alpha < 1$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \dots + \alpha^n + \dots$$

$$f(z) = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots \dots\right)$$

$$-\frac{1}{2}\cdot\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \dots\right).$$

$$= -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right)$$

$$-\left(\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \dots\right)$$

$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

T. Costopoulos

(7) $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z\cdot(z-2)}$

At $z=0$ $f(z)$ has a simple pole.

At $z=2$ $f(z)$ has a simple pole.

So:

$$\operatorname{Res} f(z)\Big|_{z=0} = \lim (z-z_0)\cdot f(z)\Big|_{z=z_0}$$

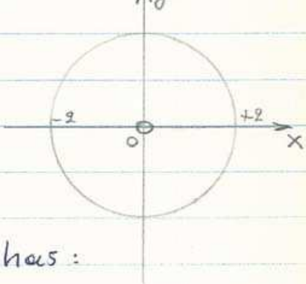
$$= \lim \cancel{z}\cdot\frac{z+1}{\cancel{z}(z-2)}\Big|_{z=0} = \frac{0+1}{0-2} = -\frac{1}{2}$$

$$\operatorname{Res} f(z)\Big|_{z=2} = \lim \cancel{(z-2)}\cdot\frac{z+1}{\cancel{(z-2)}\cdot z}\Big|_{z=2}$$

$$= \frac{2+1}{2} = \frac{3}{2}$$

T. Costopoulos.

(8) $\oint \dfrac{dz}{z^2\cdot(z+4)}$.



$C: |z| = 2 \Rightarrow x^2 + y^2 = 2^2$

The function $f(z) = \dfrac{1}{z^2\cdot(z+4)}$ has:

at $z=0$ a pole of double order $(N=2)$ and at $z=-4$ a simple pole.

From these singularities of the function $f(z)$, only $z=0$ lies inside the domain of definition of $f(z)$ and so only for this we shall calculate the Residue.

$$\oint_c = \frac{\frac{1}{z+4}}{z^2}\,dz = 2\pi i\cdot K_1$$

$$K_1 = \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}(z-z_0)^N\cdot f(z)\Big|_{z=z_0} . \text{ Here } N=2$$

$$\operatorname{Res}f(z)\Big|_{z=0} = K_1 = \frac{1}{(2-1)!}\cdot\frac{d}{dz}(z-0)^2\cdot\frac{1}{z^2\cdot(z+4)}\Big|_{z=0}$$

$$= \frac{1}{1}\cdot\frac{d}{dz}\cancel{z^2}\,\frac{1}{\cancel{z^2}(z+4)}\Big|_{z=0}$$

95

$$= \frac{-(z+4)'}{(z+4)^2}\Big|_{z=0} = \frac{-\cancel{4}\,1}{(0+4)^2} = -\frac{1}{\cancel{4}\,16}$$

and so: $K_1 = -\dfrac{1}{\cancel{4}\,16}$ or $\oint_c \dfrac{dz}{z^2(z+4)} = -\dfrac{\pi i}{\cancel{2}\,8}$

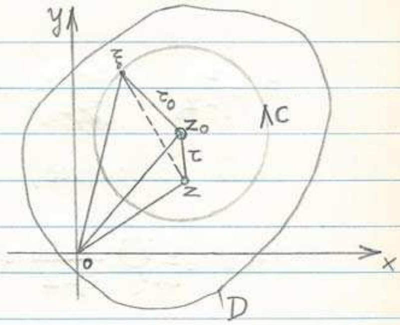
56. Taylor's Series.

Theorem : Let $f(z)$ be analytic in D, and C be a circle in D having a center at z_0 and radius r_0. Then at each point z inside C

$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}\cdot(z-z_0)^n + \dots$$

that is, the infinite series converges to $f(z)$.

Solution :



Since $f(z)$ is analytic in D we can write the Cauchy's Integral Formula for the closed contour C :

$$f(z) = \frac{1}{2\pi i}\cdot \oint_C \frac{f(\xi)}{\xi - z}\cdot d\xi$$

For real number α we have :

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha} \quad \text{because}$$

$$\frac{1-\alpha^n}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1}$$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{\frac{1}{\xi - z_0}}{1 - \frac{z - z_0}{\xi - z_0}}$$

$$= \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$$

$$= \frac{1}{\xi - z_0} \cdot \left[1 + \frac{z - z_0}{\xi - z_0} + \left(\frac{z - z_0}{\xi - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{\xi - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\xi - z_0}\right)^n}{1 - \frac{z - z_0}{\xi - z_0}} \right]$$

$$= \frac{1}{\xi - z_0} \cdot \left[1 + \frac{z - z_0}{\xi - z_0} + \left(\frac{z - z_0}{\xi - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{\xi - z_0}\right)^{n-1} + \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \cdot \left(\frac{z - z_0}{\xi - z_0}\right)^n \right]$$

$$= \frac{1}{\xi - z_0} + \frac{z - z_0}{(\xi - z_0)^2} + \frac{(z - z_0)^2}{(\xi - z_0)^3} + \cdots + \frac{(z - z_0)^{n-1}}{(\xi - z_0)^n} + \frac{1}{\xi - z} \cdot \left(\frac{z - z_0}{\xi - z_0}\right)^n$$

and therefore,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} \cdot d\xi = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} d\xi + \frac{z - z_0}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^2} \cdot d\xi +$$

$$+ \frac{(z - z_0)^2}{2\pi i} \oint \frac{f(\xi)}{(\xi - z_0)^3} \cdot d\xi + \cdots +$$

$$+ \frac{(z - z_0)^{n-1}}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^n} d\xi + \frac{(z - z_0)^n}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)^n} \cdot d\xi$$

But : $\quad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \cdot d\xi$

$$f'(z_0) = \frac{1}{2\pi i} \cdot \oint_C \frac{f(\xi)}{(\xi - z_0)^2} d\xi$$

$$f''(z_0) = \frac{2!}{2\pi i} \cdot \oint_C \frac{f(\xi)}{(\xi - z_0)^3} \cdot d\xi$$

$$\vdots \qquad f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i} \cdot \oint_C \frac{f(\xi)}{(\xi - z_0)^n} \cdot d\xi$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \cdot \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot d\xi$$

and so :

$$f(z) = f(z_0) + (z - z_0) \cdot f'(z_0) + (z - z_0)^2 \cdot \frac{f''(z_0)}{2!} + \ldots +$$

$$+ (z - z_0)^{n-1} \cdot \frac{f^{(n-1)}(z_0)}{(n-1)!} + R_n$$

where $R_n = \dfrac{(z - z_0)^n}{2\pi i} \displaystyle\oint_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)^n} \cdot d\xi$

We shall show that $\lim_{n \to \infty} R_n = 0$

Let M be the maximum value of $|f(\xi)|$ on C

$|\xi - z_0| = r_0$, $|z - z_0| = r$

$$|\xi - z| = |\xi - z_0 - (z - z_0)| \geqslant |\xi - z_0| - |z - z_0| = r_0 - r$$

$$|\xi - z| \geqslant r_0 - r$$

$$|R_n| = \frac{|z-z_0|^n}{2\pi\cdot|i|}\cdot\left|\oint_C \frac{f(\xi)}{(\xi-z)(\xi-z_0)^n}\,dz\right| \leq$$

$$\leq \frac{r^n}{2\pi}\cdot\oint_C \frac{|f(\xi)|}{|\xi-z|\cdot|\xi-z_0|^n}\cdot|d\xi|$$

$$\leq \frac{r^n}{2\pi}\cdot M\cdot\frac{1}{(r_0-r)\cdot r_0^n}\cdot\oint_C |d\xi|$$

$$\leq \frac{M}{r_0-r}\cdot\frac{1}{2\pi}\cdot\frac{r^n}{r_0^n}\cdot 2\pi r_0$$

$$\leq \frac{M\cdot r_0}{r_0-r}\cdot\left(\frac{r}{r_0}\right)^n$$

Since $r<r_0$ or $\frac{r}{r_0}<1$ we have:

$\lim\limits_{n\to\infty} R_n = 0$ for every z interior of C

and so $f(z)$ becomes:

$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{f''(z_0)}{2!}\cdot(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}\cdot(z-z_0)^n + \dots$$

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}\cdot(z-z_0)^n$$

Taylor's Series.

Let $z_0 = 0 =$ origin then :

$$f(z) = f(0) + f'(0)\cdot z + \frac{f''(0)}{2!}\cdot z^2 + \frac{f'''(0)}{3!}\cdot z^3 + \dots + \frac{f^{(n)}(0)}{n!}\cdot z^n + \dots$$

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot z^n$$

Maclaurin's Series.

57. Observations and Examples.

1) $$f(z) = \frac{1+2z}{z^2+z^3} = \frac{2z+1}{z^3+z^2}$$

$$= \frac{1}{z^2}\cdot\left(\frac{2z+1}{1+z}\right) = \frac{1}{z^2}\left(2 - \frac{1}{1+z}\right) = \frac{1}{z^2}\left(2 - \frac{1}{1-(-z)}\right)$$

a) $$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}$$

$\alpha = -z$

$$f(z) = \frac{1}{z^2}\left[2 - \left(1 - z + z^2 - z^3 + z^4 - \dots\right)\right]$$

$$= \frac{1}{z^2}\cdot\left(1 + z - z^2 + z^3 - z^4 + \dots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + z^3 - \dots$$

b)

$$\begin{array}{l} 1 + 2z \\ \underline{-1 - z} \\ \quad z \\ \quad \underline{-z - z^2} \\ \qquad -z^2 \\ \qquad \underline{+z^2 + z^3} \\ \qquad\quad z^3 \\ \qquad\quad \underline{-z^3 - z^4} \\ \qquad\qquad -z^4 \text{ etc.} \end{array}
\qquad
\begin{array}{|l} z^2 + z^3 \\ \hline \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \text{etc.} \end{array}$$

c) $f(z) = \frac{1}{z^2}\left(2 - \frac{1}{1+z}\right)$

$$\begin{array}{l} 1 \\ \underline{-1 - z} \\ \quad -z \\ \quad \underline{+z + z^2} \\ \qquad z^2 \\ \qquad \underline{-z^2 - z^3} \\ \qquad\quad -z^3 \text{ etc.} \end{array}
\qquad
\begin{array}{|l} 1 + z \\ \hline 1 - z + z^2 - z^3 + \cdots \text{ etc.} \end{array}$$

$$f(z) = \frac{1}{z^2}\left(2 - 1 + z - z^2 + z^3 - z^4 + \ldots\right)$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + z^3 - \text{ etc.}$$

58. Laurent's Series

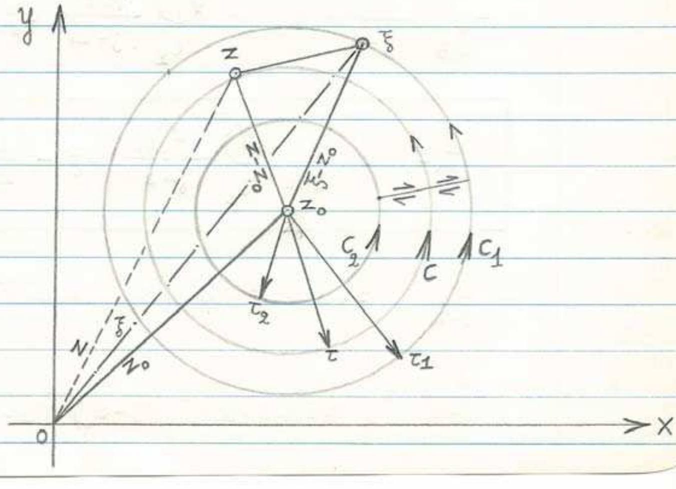
Theorem: If the function $f(z)$ is analytic on the circles C_1 and C_2, and throughout the region between them, then at each point z of this region, $f(z)$ is represented as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where:

$$a_n = \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot d\xi \qquad , (n = 0, 1, 2, \cdots)$$

$$a_{-n} = b_n = \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}} \cdot d\xi \qquad , (n = 1, 2, 3, \cdots)$$

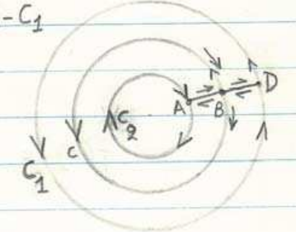


Solution :

$$\oint_{B} f(\xi)\cdot d\xi = 0$$

and
$$f(z) = \frac{1}{2\pi i}\oint_{B}\frac{f(\xi)}{\xi - z}\cdot d\xi$$

$$B = AB + C + BD + C_2 + DB - C + BA - C_1$$



$$2\pi i\cdot f(z) = \int_{AB}\frac{f(\xi)}{\xi - z}\,d\xi + \oint_{C}\frac{f(\xi)}{\xi - z}\,d\xi +$$

$$+\int_{BD}\frac{f(\xi)}{\xi - z}\,d\xi + \oint_{C_1}\frac{f(\xi)}{\xi - z}\cdot d\xi$$

$$-\int_{BD}\frac{f(\xi)}{\xi - z}\,d\xi - \oint_{C}\frac{f(\xi)}{\xi - z}\cdot d\xi$$

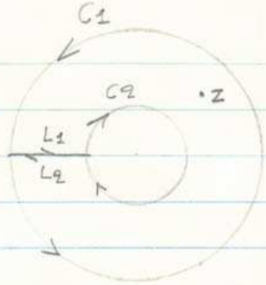
$$-\int_{AB}\frac{f(\xi)}{\xi - z}\cdot d\xi - \oint_{C_2}\frac{f(\xi)}{\xi - z}\cdot d\xi$$

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(\xi)}{\xi - z}\cdot d\xi - \frac{1}{2\pi i}\cdot\oint_{C_2}\frac{f(\xi)}{\xi - z}\cdot d\xi}$$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}\,, \quad \alpha < 1$$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0}\cdot\frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$$

OR.



From Cauchy's Integral Formula:

$$f(z) = \oint_B \frac{f(\xi)}{\xi - z}\, d\xi$$

$$= \oint_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi + \oint_{C_2} \frac{f(\xi)\, d\xi}{\xi - z}$$

$$B = C_1 + L_1 + C_2 + L_2$$
$$= C_1 + L_1 - L_1 + C_2$$
$$= C_1 + C_2$$
$$= C_1 - C_2$$

$$= \oint_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi - \oint_{C_2} \frac{f(\xi)\, d\xi}{\xi - z}$$

ςo :

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \cdot \left[1 + \frac{z-z_0}{\xi - z_0} + \left(\frac{z-z_0}{\xi - z_0}\right)^2 + \dots + \left(\frac{z-z_0}{\xi - z_0}\right)^{n-1} + \right.$$

$$\left. + \frac{1}{1 - \frac{z-z_0}{\xi - z_0}} \cdot \left(\frac{z-z_0}{\xi - z_0}\right)^n \right]$$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \cdot \left[1 + \frac{z-z_0}{\xi - z_0} + \left(\frac{z-z_0}{\xi - z_0}\right)^2 + \dots + \left(\frac{z-z_0}{\xi - z_0}\right)^{n-1} \right]$$

$$+ \frac{1}{\xi - z} \cdot \left(\frac{z-z_0}{\xi - z_0}\right)^n$$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \cdot \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^n} + Q_n$$

where $\quad Q_n = \frac{1}{\xi - z} \cdot \left(\frac{z-z_0}{\xi - z_0}\right)^n \quad$ (and ξ on C_1)

$$\left.\begin{array}{l} \xi - z_0 = \tau_1 \\ z - z_0 = \tau \end{array}\right\} , \quad |\xi - z| = \left|(\xi - z_0) - (z - z_0)\right| \geq |\xi - z_0| - |z - z_0| = \tau_1 - \tau$$

$$|Q_n| = \frac{1}{|\xi - z|} \cdot \left(\frac{|z-z_0|}{|\xi - z_0|}\right)^n = \frac{1}{\tau_1 - \tau} \cdot \left(\frac{\tau}{\tau_1}\right)^n$$

Since $\quad \tau < \tau_1 \Rightarrow \frac{\tau}{\tau_1} < 1$

and ςo : $\quad \lim_{n \to \infty} Q_n = 0$

Now we can write :

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \cdot \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^n}$$

and :

$$f(z) = \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{\xi - z_0} \cdot \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi-z_0)^n} \cdot d\xi$$

$$- \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} \cdot d\xi = I_1 - I_2$$

$$I_1 = \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{\xi - z} \cdot d\xi$$

$$= \frac{1}{2\pi i} \cdot \oint_{C_1} \sum_{n=0}^{\infty} \frac{f(\xi) \cdot (z-z_0)^n}{(\xi - z_0)^{n+1}} \cdot d\xi$$

$$= \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} (z-z_0)^n \cdot \oint_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot d\xi$$

$$= \sum_{n=0}^{\infty} \left[(z-z_0)^n \cdot \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi \right] \quad , (n = 0, 1, 2 \ldots)$$

$$= \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n \qquad \text{where} \quad \alpha_n = \frac{1}{2\pi i} \cdot \oint \frac{f(\xi)}{(\xi-z_0)^{n+1}} \cdot d\xi$$

Similarly we have:

$$\frac{1}{1-\alpha} = 1+\alpha+\alpha^2+\dots+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}\quad,\quad \alpha<1$$

$$\frac{1}{\xi-z} = \frac{1}{\xi-z_0+z_0-z} = \frac{1}{(\xi-z_0)-(z-z_0)} = \frac{1}{-(z-z_0)}\cdot\frac{1}{1-\frac{\xi-z_0}{z-z_0}}$$

$$= -\frac{1}{z-z_0}\cdot\left[1+\frac{\xi-z_0}{z-z_0}+\left(\frac{\xi-z_0}{z-z_0}\right)^2+\dots+\left(\frac{\xi-z_0}{z-z_0}\right)^{n-1}+\right.$$

$$\left.+\frac{1}{1-\frac{\xi-z_0}{z-z_0}}\cdot\left(\frac{\xi-z_0}{z-z_0}\right)^n\right]$$

$$= -\frac{1}{z-z_0}\cdot\sum_{n=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^n - Q'_n$$

$$Q'_n = \frac{1}{z-\xi}\cdot\left(\frac{\xi-z_0}{z-z_0}\right)^n .$$

Here ξ is on circle C_2 So:

$$|\xi-z_0| = r_2 \qquad |z-z_0| = r \qquad r_2<r \qquad \frac{r_2}{r}<1$$

$$|Q'_n| = \frac{1}{|z-\xi|}\cdot\frac{|\xi-z_0|^n}{|z-z_0|^n}\quad,\quad |z-\xi| = |z-z_0+z_0-\xi|$$

$$\geq |z-z_0|-|\xi-z_0|$$

$$\leq \frac{1}{r-r_2}\cdot\left(\frac{r_2}{r}\right)^n \qquad\qquad \geq r-r_2$$

$\frac{r_2}{r}<1 \Rightarrow \left(\frac{r_2}{r}\right)^n \to 0$ when $n\to\infty$ or $\lim\limits_{n\to\infty} Q'_n = 0$

Now we can write :

$$\frac{1}{\xi - z} = -\frac{1}{z-z_0} \cdot \sum_{n=0}^{\infty} \frac{(\xi - z_0)^n}{(z-z_0)^n} = -\frac{1}{z-z_0} \cdot \sum_{n=1}^{\infty} \frac{(\xi - z_0)^{n-1}}{(z-z_0)^{n-1}}$$

and :

$$f(z) = \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{\xi - z_0} \cdot \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^n} \cdot d\xi$$

$$+ \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{-(z-z_0)} \cdot \sum_{n=0}^{\infty} \frac{(\xi - z_0)^n}{(z-z_0)^n} \cdot d\xi \quad = I_1 - I_2$$

$$I_2 = \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{\xi - z} \cdot d\xi$$

$$= \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{-(z-z_0)} \cdot \sum_{n=1}^{\infty} \frac{(\xi - z_0)^{n-1}}{(z-z_0)^{n-1}} \cdot d\xi$$

$$= -\frac{1}{2\pi i} \cdot \oint_{C_2} \sum_{n=1}^{\infty} \frac{f(\xi) \cdot (\xi - z_0)^{n-1}}{(z-z_0)^{n-1+1}} \cdot d\xi$$

$$= -\sum_{n=1}^{\infty} \left[\frac{1}{(z-z_0)^n} \cdot \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}} \cdot d\xi \right]$$

$$= -\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad \text{where} \quad b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}} \cdot d\xi$$

$$, (n = 1, 2, 3, \cdots)$$

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^{-n+1}}$$

$$= \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n + \sum_{n=-1}^{-\infty} b_n \cdot (z - z_0)^{n+1}$$

$$= \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n + \sum_{n=0}^{-\infty} b_n \cdot (z - z_0)^n$$

but then:

$$b_n = \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{(\xi - z_0)^n} \cdot d\xi \quad \text{where} \quad (n = 0, -1, -2, \cdots -\infty)$$

or

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n + \sum_{n=-\infty}^{0} b_n \cdot (z - z_0)^n$$

$$f(z) = \sum_{-\infty}^{+\infty} \alpha_n \cdot (z - z_0)^n \qquad \text{where:}$$

$$\alpha_n = \frac{1}{2\pi i} \oint_{C} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot d\xi \quad , \text{ over } C.$$

$$f(z) = \sum_{-\infty}^{\infty} b_n \cdot (z - z_0)^n$$

Examples

a) Find the Laurent's series expansion of the $\frac{e^z}{z^2}$ about $z_0 = 0$

We have $\frac{e^z}{z^2} = \sum_{-\infty}^{\infty} b_n \cdot z^n$, $z_0 = 0$

$$e^z = f(0) + f'(0) \cdot z + \frac{f'(0)}{2!} z^2 + \dots + \frac{f^{(n)}(0)}{n!} \cdot z^n + \dots$$

$$f'(z) = f''(z) = f'''(z) = \dots = f^{(n)}(z) = e^z$$
$$f'(0) = f''(0) = \dots = f^{(n)}(0) = f(0) = e^0 = 1$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots + \frac{z^n}{n!} + \frac{z^{n+1}}{(n+1)!} + \frac{z^{n+2}}{(n+2)!} + \dots$$

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots + \frac{z^{n-2}}{n!} + \frac{z^{n-1}}{(n+1)!} + \frac{z^n}{(n+2)!} +$$

$$= \frac{z^{-2}}{0!} + \frac{z^{-1}}{1!} + \frac{z^0}{2!} + \frac{z}{3!} + \dots + \frac{z^{n-2}}{n!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^{n-2}$$

$$= \sum_{n=-2}^{\infty} \frac{1}{(n+2)!} \cdot z^n \qquad \text{etc.} \qquad \text{Since } \boxed{0! = 1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \cdot z^n + \sum_{n=1}^{2} \frac{1}{z^n}$$

β) Find the Laurent's Series expansion of the:

$$f(z) = \frac{1}{z^2-3z+2} \quad \text{if} \quad 1<|z|<2 .$$

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)\cdot(z-2)} = \frac{1}{z-1}\cdot\frac{1}{z-2}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} = \frac{(A+B)z-2A-B}{(z-1)\cdot(z-2)}$$

$A+B=0 \qquad B+2A=-1 \quad \Rightarrow \quad A=-1 \;,\; B=1$

$A=-B$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= -\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} \quad , \begin{pmatrix} |z|<2 \Rightarrow \frac{2}{z}<1 \\ |z|>1 \end{pmatrix}$$

$$f(z) = -\frac{1}{2}\cdot\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots+\left(\frac{z}{2}\right)^n+\dots\right] -$$

$$-\frac{1}{z}\cdot\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots+\frac{1}{z^n}+\dots\right]$$

$$f(z) = -\left(\frac{z^0}{2}+\frac{z^1}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\dots+\frac{z^n}{2^{n+1}}+\dots\right)$$

$$-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\dots+\frac{1}{z^{n+1}}+\dots\right)$$

$$f(z) = \frac{1}{(z-2)(z-1)} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty}\frac{1}{z^n}$$

1. Evaluate the following integrals: see (V.A.24.)

(a) $\int_c \frac{\cos Z}{z}\, dz$ ans: $2\pi i$ where c is a closed path formed by lines $x = \mp 1$, $y = \mp 1$.

(b) $\int_c \frac{e^z}{z-\left(\frac{1}{2}\right) i}\, dz$ c?

(c) $\oint_c \frac{3z^2+z}{z^2-1}\, dz$ where c is the circle $|z| = 2$

<u>Hint:</u> Decompose the integrand into partial fractions.

2. (a) Expand $f(z)=\log z$ about $z = 1$ and determine the radius of convergence,

ans: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$, $R = 1$.

(b) Obtain the Maclaurin series expansion of $f(z) = e^z$.

ans: $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $R = \infty$.

3. Show that $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ for $|z+1| < 1$.

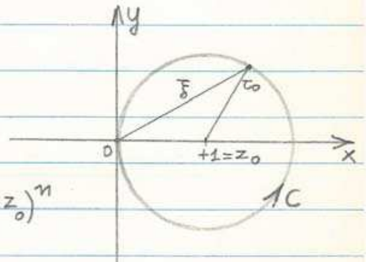
4. Prove that for $0 < |z| < 4$

$$\frac{1}{4z-z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}} .$$

Exercises

2. (a) Expand $f(z) = \log z$ about $z = 1$ and determine the radius of convergence.

In general case we have:



$$f(z) = f(z_0) + f'(z_0)\cdot(z - z_0) + \\ + f''(z_0)\cdot(z-z_0)^2 \cdot \frac{1}{2!} + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$f(z) = \log z \qquad z_0 = 1 \qquad f(z_0) = \log 1 = 0$

$f'(z) = \frac{1}{z} \qquad f'(z_0) = 1$

$f''(z) = -\frac{1}{z^2} \quad , \quad f''(z_0) = -1$

$f'''(z) = \frac{2z}{z^4} = \frac{2!}{z^3} \quad , \quad f'''(z_0) = 2!$

$f''''(z) = -\frac{1\cdot 2\cdot 3}{z^4} \quad , \quad f''''(z_0) = -3!$

$f^{(5)}(z) = \frac{4!}{z^5} \quad , \quad f^{(5)}(z_0) = -4!$

$\vdots$

$f^{(n)}(z) = \frac{(n-1)!}{z^n} \quad , \quad f^{(n)}(z_0) = (-1)^{n-1}\cdot(n-1)!$ or

$$f(z) = 0 + 1\cdot(z-1) - \frac{1}{2!}(z-1)^2 + \frac{2!}{3!}(z-1)^3 - \cdots + (-1)^{n-1}\cdot\frac{(n-1)!(z-1)^n}{n!}$$

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \dots + (-1)^{n-1} \cdot \frac{(z-1)^n}{n}$$

$$\log z = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(z-1)^n}{n}$$

Radius of convergence. Is r_0, that is the radius of c in all points of which $f(z) = \log z$ must be analytic.
But we know that $\log 0$ has no meaning
so: $\max r_0 = 1 \Rightarrow R = 1$.

(b) Obtain the Maclaurin series expansion of e^z.

$$f(z) = f(0) + \frac{f'(0)}{1} \cdot z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \dots + \frac{f^{(n)}(0)}{n!} \cdot z^n$$

$f(z) = e^z \qquad f(0) = e^0 = 1$
$f'(z) = e^z \qquad f'(0) = e^0 = 1$
$f''(z) = f'''(z) = \dots f^{(n)}(z) = e^z \Rightarrow f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$

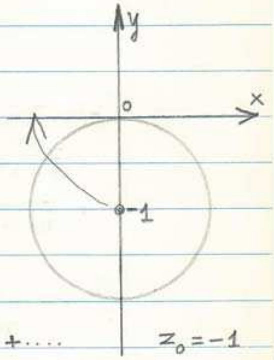
$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} \quad \text{or}$$

$$e^z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \quad \text{and} \quad R = \infty .$$

③ Show that $\dfrac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)\cdot(z+1)^n$ for $|z+1|<1$

$|z+1|<1$

$|x+iy+1|<1 \Rightarrow (x+1)^2 + y^2 < 1$



$z_0 = -1$

$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{f''(z_0)}{2!}\cdot(z-z_0)^2 + \frac{f'''(z_0)}{3!}\cdot(z-z_0)^3 + \dots + \frac{f^{(n)}}{n!}\cdot(z-z_0)^n + \dots$$

$f(z) = \dfrac{1}{z^2}$ $\qquad f(z_0) = \dfrac{1}{1} = 1$

$f'(z) = \dfrac{-2}{z^3}$ $\qquad f'(z_0) = \dfrac{-2}{-1} = 2 = 2!$

$f''(z) = \dfrac{6}{z^4}$ $\qquad f''(z_0) = 6 = 1\cdot 2\cdot 3 = 3!$

$f'''(z) = \dfrac{-24}{z^5}$ $\qquad f'''(z_0) = 24 = 1\cdot 2\cdot 3\cdot 4 = 4!$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$f^{(n)}(z) = \dfrac{(-1)^n\cdot(n+1)!}{z^{n+2}}$ $\qquad f^{(n)}(z_0) = (n+1)!$

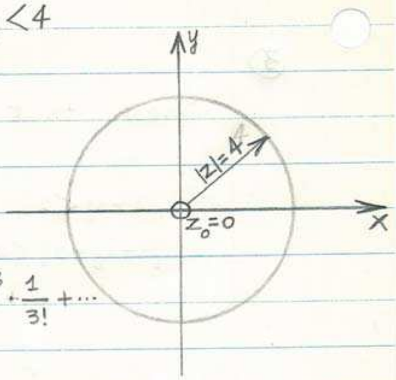
$$\frac{1}{z^2} = 1 + 2!\cdot z + \frac{3!}{2!}\cdot z^2 + \frac{4!}{3!}\cdot z^3 + \dots + \frac{(n+1)!}{n!}\cdot(z+1)^n + \dots$$

$$\frac{1}{z^2} = 1 + 2z + 3\cdot z^2 + 4\cdot z^3 + \dots + (n+1)\cdot z^n = 1 + \sum_{n=1}^{\infty}(1+n)\cdot(z+1)^n$$

④ Prove that for $0 < |z| < 4$

$z_0 = 0$

$f(z) = \dfrac{1}{4z - z^2}$



$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{1}{2!} f''(z_0)\cdot(z-z_0)^2 + f'''(z_0)\cdot(z-z_0)^3 \cdot \frac{1}{3!} + \dots + \frac{1}{n!}\cdot f^{(n)}(z_0)\cdot(z-z_0)^n + \dots$$

$$f(z) = \frac{1}{4z-z^2} = \frac{1}{4z\cdot\left(1-\frac{z}{4}\right)} = \frac{1}{4z}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \dots + \left(\frac{z}{4}\right)^n\right]$$

$\frac{z}{4} < 1 \implies$

$$= \frac{1}{4z} + \frac{1}{4} + \frac{z}{4^3} + \frac{z^2}{4^4} + \dots + \frac{z^{n-1}}{4^{n+1}} + \dots$$

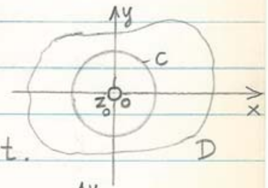
$$\frac{1}{4z-z^2} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

66. Residues - Poles

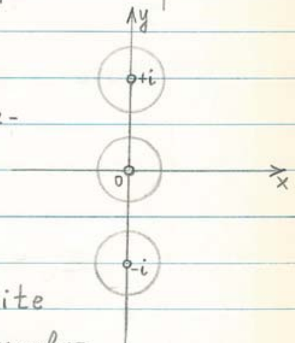
Singular points of single valued functions $f(z)$.
Points at which $f(z)$ is not analytic are called singular points of $f(z)$ or singularities of $f(z)$

A. Isolated singular point or isolated singularity of a function $f(z)$. If $f(z)$ is analytic throughout a neighborhood of z_0 except at z_0 itself then $f(z)$ has an isolated singularity at z_0.

examples: a) $f(z) = \frac{1}{z}$ is analytic everywhere except at $z_0 = 0$ which is an isolated point.



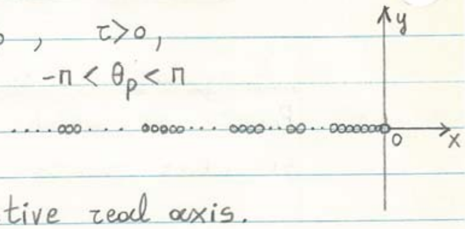
b) $f(z) = \frac{z+1}{z^3(z^2+1)}$ has three isolated points or three isolated singularities, namely,
$z = 0$, $z = +i$, $z = -i$



c) $f(z) = \frac{1}{\sin\frac{\pi}{z}}$ has an infinite number of singular points at $\sin\frac{\pi}{z} = 0$ or
$\frac{\pi}{z} = \pm n\pi$ or $z = \pm\frac{1}{n}$
$n = 1, 2, 3, \cdots$
$z = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$

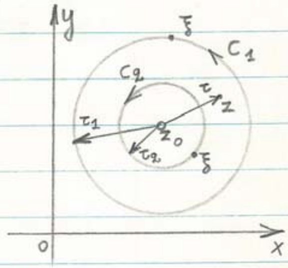
d) $f(z) = \mathrm{Log}\,z = \mathrm{Log}\,r + i\theta_p$, $r > 0$, $-\pi < \theta_p < \pi$

Logz is not analytic at the points of the negative real axis.



If z_0 is an isolated singularity of $f(z)$ or an isolated singular point of $f(z)$, then Laurent series expansion can be written :

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$



where : $\alpha_n = \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot d\xi$

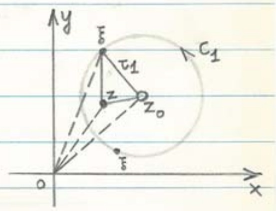
with $n = 0, 1, 2, \dots$

$$\alpha_{-n} = b_n = \frac{1}{2\pi i} \cdot \oint_{C_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}}$$

with $n = 1, 2, 3, \dots$

$r_2 < r < r_1$ $\quad r = |z - z_0|$, $r_1 = |\xi - z_0|_{C_1}$, $r_2 = |\xi - z_0|_{C_2}$

$$r_2 < |z - z_0| < r_1$$



But if r_2 is very-very small
$r_2 \to 0$ then :

$$0 < |z - z_0| < r_1$$

and we never have $z = z_0$
So we can write :

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

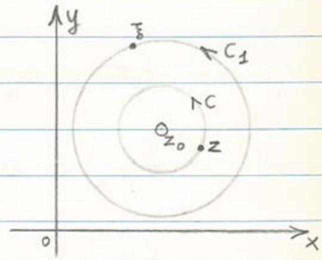
$$= \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

where :

$$b_1 = \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{(\xi - z_0)^{-1+1}} d\xi$$

$$= \frac{1}{2\pi i} \cdot \oint_{C_1} \frac{f(\xi)}{(\xi - z_0)^0} d\xi$$

$$= \frac{1}{2\pi i} \cdot \oint_{C_1} f(\xi) \cdot d\xi$$



Since $f(\xi)$ is analytic inside C_1 except at z_0 the region inside C_1 is a doubly connected region. So from Cauchy's Integral theorem for these regions we have

$$\oint_{C_1} f(\xi)\, d\xi = \oint_{C} f(z) \cdot dz \qquad \text{where } c \text{ is arbitrary}$$

So, $\boxed{b_1 = \frac{1}{2\pi i} \cdot \oint_C f(z) \cdot dz}$ = Residue of $f(z)$.

Case I

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

If in $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ all b_n are equal to zero for every n then

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = 0 \quad \text{and so :}$$

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n \qquad \text{Taylor's series .}$$

$$f(z) = \alpha_0 + \alpha_1 (z-z_0) + \alpha_1 \cdot (z-z_0)^2 + \dots$$

$f(z_0) = \alpha_0$ = constant that is if

$f(z)$ is analytic in C_1 except at z_0 and if $f(z)$ becomes analytic at z_0 by $f(z_0) = \alpha_0$ = constant then this point

B. z_0 is called removable singularity or removable singular point of $f(z)$.

<u>example</u> : $f(z) = \frac{\sin z}{z}$

$$\sin z = \sin z_0 + \sin' z_0 (z-z_0) + \frac{\sin'' z_0}{2!} \cdot (z-z_0)^2 + \frac{\sin''' z_0}{3!} \cdot (z-z_0)^3 + \dots$$

$\sin z_0 = \sin 0 = 0$

$\sin' z = \cos z \qquad \cos z_0 = 1 = \sin' z_0$

$\sin'' z = -\sin z \qquad \sin'' z_0 = -\sin 0 = 0$

$\sin'''(z) = -\cos z \qquad \sin''' z_0 = -\cos 0 = -1$ etc.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$f(0) = 1 = \alpha_0 = \text{const.}$ and so this is the removable singular point or removable singularity of this function $f(z)$.

Case II $\qquad f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

If a finite number N terms of negative powers of $f(z)$ exists in the Laurent series then we have for domain $0<|z-z_0|<r_1$

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0) + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_N}{(z-z_0)^N} .$$

then the series of all negative powers is called principal part of $f(z)$ about z_0 and has a major influence upon the behavior of $f(z)$ near the singular point z_0.

For this isolated singularity z_0 or for this domain $0<|z-z_0|<r_1$ and for these N negative powers we say that : This isolated singular point z_0 is called a pole of order N of the function

If $N=1$ then z_0 is called : pole of order 1 or pole of first order or simple pole.

Consider the function : $\phi(z) = (z-z_0)^N \cdot f(z)$

If the largest power of $f(z)$ is N,

and if $\phi(z) = (z-z_0)^N \cdot f(z)$ is analytic and is not zero at z_0 then the number N is called the order of the pole at z_0 and $f(z)$ has a pole of order N at z_0.

example : $$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \frac{b_1}{z-z_0} + \dots + \frac{b_{N'}}{(z-z_0)^{N'}}$$

$$\phi(z) = (z-z_0)^N \cdot f(z) =$$

$$= (z-z_0)^N \cdot \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + b_1 \cdot (z-z_0)^{N-1} + \dots + b_{N'} \cdot (z-z_0)^{N-N'}$$

in order to be analytic the $\phi(z)$, we must have no denominators, so $N = N'$, that is $f(z)$ has N negative powers or z_0 is a pole of order N.

Case III $\quad f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots$

$$\cdots + \frac{b_N}{(z-z_0)^N} \quad , \quad N \to \infty$$

If N (of Case II) is infinite that is if the principal part of the function about z_0 has an infinite number of terms then the point z_0 is called an essential singular point or an essential singularity of $f(z)$, and we say that $f(z)$ has an essential singularity at z_0.

examples a) $\quad f(z) = \frac{z^2}{(z-1)\cdot(z^2+1)\cdot(z+1)^3}$

at point	$z=1$	$f(z)$ has	one	pole	of	order	first
»	$z=-1$	»	one	»	of	»	third
»	$z=+i$	»	one	»	»	»	first
»	$z=-i$	»	one	»	»	»	first
or »	$z=\pm i$	»	two	poles	»	»	first.

b) $\quad f(z) = e^{\frac{1}{z}}$

$\varphi(z) = e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

and : $\qquad z \to \frac{1}{z}$

$f(z) = e^{\frac{1}{z}} = + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 \cdot 3!} + \cdots$

$f(z)$ has an essential Singularity at $z=0$

<u>Residue</u>. $$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where :

$$\alpha_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \quad , n = 0,1,2 \cdots$$

$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}} d\xi \quad , n = 1,2,\cdots$$

If $C_2 \Rightarrow 0$ then $C_2 \Rightarrow z_0$ and $f(z)$ will be analytic everywhere in C_1 except at z_0 and so :

$$0 < |z - z_0| < r_1$$

for which :

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b}{(z-z_0)^n} + \cdots$$

The coefficient $b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz = \frac{1}{2\pi i} \oint_{C_2} f(\xi)\, d\xi$

is called <u>Residue</u> of $f(z)$ at $z = z_0$

Evaluation of Residue

(I) At simple poles or at poles of order $N=1$

then we have : $f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \frac{b_1}{z-z_0}$

$$b_1 = f(z)(z-z_0) - \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^{n+1}$$

$$b_1 = \lim_{z \to z_0} \left[(z-z_0) \cdot f(z) - \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^{n+1} \right]$$

$$b_1 = \lim_{z \to z_0} (z-z_0) \cdot f(z)$$

$$= \text{Res. } f(z) \Big|_{z=z_0}$$

Example : $f(z) = \frac{z+1}{z^2+1} = \frac{z+1}{(z+i)(z-i)}$ at $z = \pm i$

a) At $z_0 = i$

$$b_1 = \lim_{z \to i} (z-i) \cdot f(z)$$

$$= \lim_{z \to i} (z-i) \cdot \frac{z+1}{(z+i)(z-i)} = \frac{z+1}{z+i} \Big|_{z=i} = \frac{i+1}{2i}$$

b) At $z_0 = -i$

$$b_1 = \lim_{z_0 = -i} (z+i) \cdot \frac{z+1}{(z+i)(z-i)} = \frac{z+1}{z-i} \Big|_{z=-i} = \frac{i-1}{2i}$$

and $\sum \text{Res} f(z) = \frac{i+1}{2i} + \frac{i-1}{2i} = \frac{2i}{2i} = 1$

(II) At poles of $f(z) = \frac{F(z)}{G(z)}$

Obviously simple zeros of $G(z)$ are the simple poles of $f(z)$ and if z_0 is one of these simple zeros of $G(z)$ then:

$$b_1 = \lim_{z \to z_0} (z - z_0) \cdot f(z) = \lim_{z \to 0} \frac{(z-z_0) \cdot F(z)}{G(z)} = \frac{0}{G(z_0)} = \frac{0}{0}$$

So:

$$b_1 = \lim_{z \to z_0} \frac{[(z-z_0) \cdot F(z)]'}{[G(z)]'} = \lim_{z \to z_0} \frac{F(z) + (z-z_0) \cdot F'(z)}{G'(z)}$$

$$b_1 = \frac{F(z_0) + 0}{G'(z_0)} = \frac{F(z_0)}{G'(z_0)} = \text{Res.} \left. \frac{F(z)}{G(z)} \right|_{z=z_0}$$

<u>example</u>: $f(z) = \frac{e^{iz}}{z^2 + \alpha^2}$ at $z = -i\alpha$

$$b_1 = \lim_{z \to -i\alpha} \frac{F(z)}{G'(z)} = \frac{e^{-ii\alpha}}{2 \cdot (-i\alpha)} = \frac{e^{\alpha}}{-2i\alpha} = \text{Res} \left. f(z) \right|_{z=-i\alpha}$$

If $G(z)$ has a zero of the second order so that $f(z)$ has a pole of the second order at that point z_0 we have that:

$$\text{Res} \left. f(z) \right|_{z=z_0} = \frac{6 \cdot F'(z_0) \cdot G''(z_0) - 2 \cdot F(z_0) \cdot G'''(z_0)}{3 \cdot [G''(z_0)]^2}$$

III At poles of order N. (When $N > 1$)

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_N}{(z-z_0)^N}$$

Since $f(z)$ has a pole of N order:

$\phi(z) = (z-z_0)^N \cdot f(z)$ and so

$$(z-z_0)^N \cdot f(z) = (z-z_0)^N \cdot \sum_{n=0}^{\infty} \alpha_n \cdot (z-z_0)^n + b_1 \cdot (z-z_0)^{N-1} +$$

$$+ b_2 \cdot (z-z_0)^{N-2} + \dots + b_{N-1} \cdot (z-z_0) + b_N$$

but:

$$\phi(z) = \phi(z_0) + \phi'(z_0) \cdot (z-z_0) + \dots + \frac{\phi^{(N-1)}(z_0)}{(N-1)!} \cdot (z-z_0)^{N-1} + \dots$$

but: $\phi(z) = (z-z_0)^N \cdot f(z)$ and so we shall have:

$$b_1 \cdot (z-z_0)^{N-1} = \frac{\phi^{(N-1)}(z_0)}{(N-1)!} \cdot (z-z_0)^{N-1}$$

or $$b_1 = \frac{\phi^{(N-1)}(z_0)}{(N-1)!}$$

or $$b_1 = \frac{1}{(N-1)!} \cdot \frac{d^{N-1}}{dz^{N-1}} (z-z_0)^N \cdot f(z) \Big|_{z=z_0}$$

examples: a) $f(z) = \frac{\sin z}{z^4}$ at $z=0$

$f(z)$ has a pole of order $N=3$.

$$\operatorname{Res} f(z)\Big|_{z=0} = b_1 = \frac{1}{(N-1)!}\cdot \frac{d^{N-1}}{dz^{N-1}} (z-z_0)^N \cdot f(z)\Big|_{z=0}$$

$$\phi(z) = (z-z_0)^3 \cdot f(z) = z^3 \cdot \frac{\sin z}{z^4} = \frac{\sin z}{z}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$$\operatorname{Res} f(z)\Big|_{z=0} = \frac{d^2}{dz^2} z^3 \cdot \frac{\sin z}{z^4} \cdot \frac{1}{2!} = \frac{1}{2}\cdot \frac{d^2}{dz^2} \frac{\sin z}{z}$$

$$= \frac{1}{2}\cdot\left(-\frac{2}{3!}\right) = -\frac{1}{3!}$$ but we have

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{z\cdot 3!} + \frac{z}{5!} - \dots$$ and so $b_1 = -\frac{1}{3!}$ is true.

b) $f(z) = \frac{z\cdot e^z}{(z-\alpha)^3}$

$$(z-\alpha)^N f(z) = (z-\alpha)^3 \cdot \frac{z\cdot e^z}{(z-\alpha)^3} = z\cdot e^z = (z-\alpha)^3 \cdot f(z)$$

for $z=\alpha$ $\phi(z) = (z-\alpha)^N \cdot f(z)$ is analytic and not zero because $\phi(\alpha) = \alpha \cdot e^{\alpha} \neq 0$ if $\alpha \neq 0$

Therefore $N = 3$ and

$$\operatorname{Res} f(z)\Big|_{z=\alpha} = \frac{1}{(N-1)!} \cdot \frac{d^{N-1}}{dz^{N-1}} (z-z_0)^N \cdot f(z)\Big|_{z=\alpha}$$

$$= \frac{1}{2} \cdot \frac{d^2}{dz^2} \cdot (z-z_0)^3 \cdot \frac{z \cdot e^z}{(z-z_0)^3}\Big|_{z=\alpha}$$

$$= \frac{1}{2} \cdot \frac{d^2}{dz^2} z e^z \Big|_{z=\alpha} = \frac{1}{2} \cdot (z \cdot e^z + e^z)'\Big|_{z=\alpha}$$

$$= \frac{1}{2} \cdot (z e^z + e^z + e^z)\Big|_{z=\alpha} = e^{\alpha} \cdot \left(1 + \frac{\alpha}{2}\right)$$

c) $f(z) = \dfrac{1}{z^3 - z^2} = \dfrac{1}{z^2 \cdot (z-1)}$

at $z = 0$ one pole of order 2
$z = 1$ one pole of order 1

$z = 0 \Rightarrow$

$$\operatorname{Res} f(z)\Big|_{z=0} = \frac{1}{1!} \frac{d}{dz} z^2 \frac{1}{z^2(z-1)}\Big|_{z=0}$$

$$= \frac{1}{1} \cdot \frac{-1}{(z-1)^2}\Big|_{z=0} = -1$$

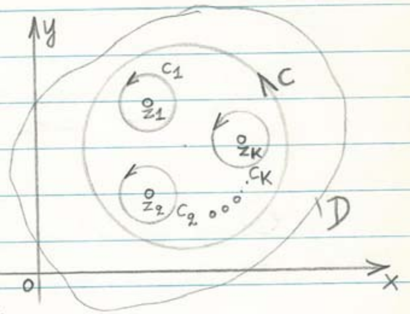
$z = 1 \Rightarrow$

$$\operatorname{Res} f(z) = \lim_{z \to 1} (z-1) \cdot \frac{1}{z^2 \cdot (z-1)} = 1$$

67. Cauchy's Residue theorem

The integral of $f(z)$ over a closed curve C which includes a finite number of isolated singularities of $f(z)$ is equal to the sum of Residues of these points as follows:

$$\oint_C f(z)\,dz = 2\pi i \cdot (K_1 + K_2 + \cdots + K_K)$$



Let $z_1, z_2, \ldots, z_K$ be isolated singularities of $f(z)$.
Let $C_1, C_2, \ldots, C_K$ be the curves enclosing these singularities
Let $K_1, K_2, \ldots, K_K$ be the Residues at the singularities.

From Cauchy's Integral theorem we have (multiply connected)

$$\oint_C f(z)dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_K} f(z)\,dz$$

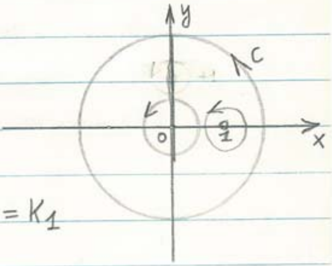
But from definition : $\quad K_i = \dfrac{1}{2\pi i} \cdot \oint_{C_i} f(z)\,dz \quad , i = 1, 2, \cdots K$

and so :

$$\oint_C f(z)\,dz = 2\pi i \cdot (K_1 + K_2 + \cdots + K_K) = 2\pi i \cdot \left[\sum_{j=1}^{K} (b_1)_j \right] = 2\pi i \cdot \operatorname{Res} f(z) \Big|_{z=z_j}$$

Evaluate the Integral:

$$\oint_C \frac{z-2}{z\cdot(z-1)}\,dz \quad , \text{ where } \quad C: \ |z| = 2$$



At $z=0$ one pole of order one

At $z=1$ one pole of first order

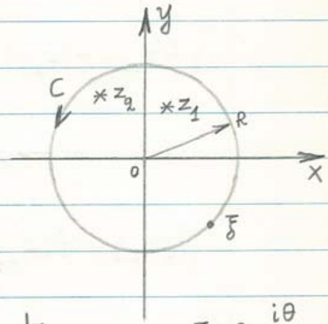
$$\operatorname{Res} f(z)\Big|_{z=0} = \lim_{z\to 0} (z)\cdot\frac{z-2}{z\cdot(z-1)} = 2 = K_1$$

$$\operatorname{Res} f(z)\Big|_{z=1} = \lim_{z\to 1} (z-1)\cdot\frac{z-2}{z\cdot(z-1)} = -1 = K_2$$

$$\oint_{C_1} f(z)\,dz = 2\pi i\cdot(K_1+K_2) = 2\pi i\cdot(2-1) = 2\pi i$$

Liouville's Theorem (page 125)

Let $f(z)$ be analytic at all finite points and at $z=\infty$ ($f(z)$ entire), then, if $|f(z)|<M$, $f(z)$ must be a constant.



Proof α. $f'(z_1) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_1)^2} d\xi$, $\xi = R \cdot e^{i\theta}$, $|\xi| = R$

$$|f'(z_0)| = \frac{1}{2\pi |i|} \cdot \oint_C \frac{|f(\xi)|}{|\xi - z_0|^2} \cdot |d\xi|$$

$$\leq \frac{1}{2\pi} \cdot \frac{M \cdot 2\pi R}{R^2} = \frac{M}{R} \longrightarrow 0 \text{ when } R \to \infty$$

So $f'(z_0) = 0$ or $f(z) = \text{const.}$

Proof β. $$f(z_1) = \frac{1}{2\pi i} \cdot \oint_C \frac{f(\xi)}{\xi - z_1} d\xi$$

$$f(z_2) = \frac{1}{2\pi i} \cdot \oint \frac{f(\xi)}{\xi - z_2} \cdot d\xi$$

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \cdot \oint_C \left(\frac{1}{\xi - z_1} - \frac{1}{\xi - z_2} \right) \cdot f(\xi) \cdot d\xi$$

$$|f(z_1) - f(z_2)| = \frac{1}{2\pi |i|} \cdot \oint_C \frac{|f(\xi)| \cdot |z_1 - z_2|}{|\xi - z_1| \, |\xi - z_2|} |d\xi|$$

$$\leq \frac{1}{2\pi} \cdot M \cdot 2\pi R \cdot |z_1 - z_2| \cdot \frac{1}{(|\xi| - |z_1|)(|\xi| - |z_2|)}$$

$$|\zeta - z_1| > |\zeta| - |z_1| \qquad , \quad |\zeta| = R$$

$$|f(z_1) - f(z_2)| \leq \frac{M \cdot |z_1 - z_2| \cdot R}{R^2} \cdot \frac{1}{\left(1 - \frac{|z_1|}{R}\right)\left(1 - \frac{|z_2|}{R}\right)}$$

If $R \to 0$

$$|f(z_1) - f(z_2)| \Rightarrow 0 \qquad \text{or}$$

$$f(z_1) = f(z_2) = \ldots = \text{Const.}$$

The Fundamental theorem of Algebra.

Every Polynomial of degree $n \geq 1$ have at least one root in the field of complex numbers.

$$P(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n \cdot z^n \quad , \quad n = 1, 2, \cdots$$

If does not have a root then: $\alpha_n \neq 0$

$f(z) = \dfrac{1}{P(z)}$ will be entire and as $|z| \to \infty$, $|P(z)| \to \infty$, $|f(z)| \to 0$

and $|P(z)|$ has a minimum value or
$|f(z)|$ is bounded for all z
$|f(z)| \leq M$ So from Liouville's theorem we have that $f(z) = \text{const}$ or
$P(z) = \text{const}$. But $P(z)$ is not a const.
and so $P(z)$ has at least one root.

(1) Locate and classify the singularities of the following functions:

(a) $\frac{Z}{Z^2+1}$ (b) $\tan Z$ (c) $\frac{Z^3-2Z+1}{Z^5+2Z^3+Z}$

ans: (a) simple poles at $Z = \mp i$

(b) simple poles at $Z = (2k+1)\,\pi/2$

(c) simple pole at $Z = 0$, double poles at $Z = \mp i$

(2) Calculate residues of the following functions at each of their finite poles and in each case find the sum of these residues.

(a) $\frac{e^Z}{Z^2+a^2}$ (b) $\frac{1}{Z^4-a^4}$ (c) $\frac{\sin Z}{Z^3}$ (d) $\frac{1}{(Z^2+a^2)^2}$

ans: (a) $\text{Res } f(Z)\big|_{Z=\mp ai} = \mp e^{\mp ai}/2ai$, Σ Res = Sin a/a.

(b) $\text{Res } f(Z)\big|_{Z=\mp a} = \mp \frac{1}{4a^3}$, $\text{Res } f(Z)\big|_{Z=\mp ai} = \mp \frac{i}{4a^3}$,

Σ Res = 0

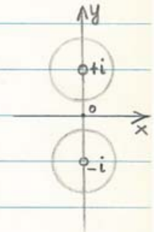
(c) $\text{Res } f(Z)\big|_{Z=0} = 0$

(d) $\text{Res } f(Z)\big|_{Z=\mp ai} = \mp \frac{1}{4ia^3}$, Σ Res = 0.

Exercises

① Locate and classify the singularities of the functions

a) $f(z) = \dfrac{z}{z^2+1} = \dfrac{z}{(z+i)(z-i)}$



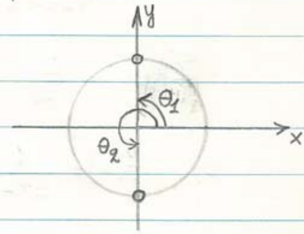
At $z=-i$ $f(z)$ has one simple pole

or " " " pole of first order

at $z=+i$ $f(z)$ has one simple pole

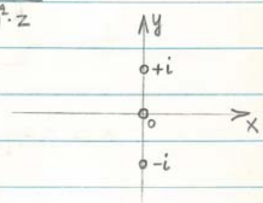
or at $z=\pm i$ $f(z)$ has simple poles

b) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$



$f(z)$ has poles at zeros of $\cos z$ that is at $(2K+1)\dfrac{\pi}{2}$ and they are simple poles.

c) (i) $f(z) = \dfrac{z^3-2z+1}{z^5+2z^4+z} = \dfrac{z^3-2z+1}{z\cdot(z^4+2z^2+1)} = \dfrac{z^3-2z+1}{(z^2+1)^2\cdot z}$

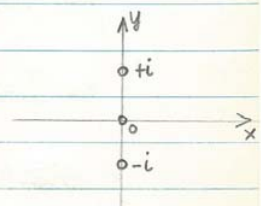


at $z=0$ simple pole

at $z=\pm i$ double poles

OR

? (ii) $f(z) = \dfrac{z^3-2z+1}{z^5+2z^4+z} = \dfrac{(z-1)(z^2+1)}{z\cdot(z^2+1)^2} = \dfrac{z-1}{z(z^2+1)}$



at $z=0$ simple pole

at $z=\pm i$ simple poles

② Calculate residues of the following functions at each of their finite poles and in each case find the sum of these residues.

a) $f(z) = \frac{e^z}{z^2+\alpha^2} = \frac{e^z}{(z+i\alpha)\cdot(z-i\alpha)}$, at $z = \pm i\alpha$ simple poles

$$\operatorname{Res} f(z)\Big|_{z=i\alpha} = \lim_{z=i\alpha} (z-i\alpha) \cdot \frac{e^{i\alpha}}{(z-i\alpha)(z+i\alpha)} = \frac{e^{i\alpha}}{i\cdot 2\alpha}$$

$$\operatorname{Res} f(z)\Big|_{z=-i\alpha} = \lim_{z=-i\alpha} (z+i\alpha) \cdot \frac{e^{-i\alpha}}{(z+i\alpha)(z-i\alpha)} = \frac{e^{-i\alpha}}{-2i\alpha} = \frac{-e^{-i\alpha}}{i\cdot 2\alpha}$$

$$\sum \operatorname{Res} = \frac{1}{\alpha} \cdot \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \frac{1}{\alpha} \cdot \sin\alpha$$

b) $f(z) = \frac{1}{z^4-\alpha^4} = \frac{1}{(z-\alpha)(z+\alpha)(z-i\alpha)(z+i\alpha)}$

at $z = \pm\alpha$ simple poles
at $z = \pm i\alpha$ simple poles

$$\operatorname{Res} f(z)\Big|_{z=\alpha} = \lim (z-\alpha)\cdot f(z)\Big|_{z=\alpha} = \frac{1}{4\alpha^3}$$

$$\operatorname{Res} f(z)\Big|_{z=-\alpha} = \lim (z+\alpha)\cdot f(z)\Big|_{z=-\alpha} = \frac{-1}{4\alpha^3}$$

$$\operatorname{Res} f(z)\Big|_{z=+i\alpha} = \lim (z-i\alpha)\cdot f(z)\Big|_{z=i\alpha} = \frac{-1}{4i\alpha^3}$$

$$\operatorname{Res} f(z)\Big|_{z=-i\alpha} = \lim (z+i\alpha)\cdot f(z)\Big|_{z=-i\alpha} = \frac{+1}{4i\alpha^3}$$

$$\sum \operatorname{Res} = 0$$

c) $f(z) = \dfrac{\sin z}{z^3}$

From Maclaurin's series :

$$\sin z = \sin 0 + \sin'(0)\cdot z + \frac{\sin'' 0}{2!}\cdot z^2 + \frac{\sin''' 0}{3!}\cdot z^3 + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

(i) at $z=0$ $f(z)$ has a pole of order three

$N=3$ so :

$$\operatorname{Res} f(z)\Big|_{z=0} = \frac{1}{(N-1)!}\,\frac{d^{(N-1)}}{dz^{N-1}}\, z^N\cdot f(z)\cdot\Big|_{z=0}$$

$$= \frac{1}{2}\cdot\frac{d^2}{dz^2}\, z^3\cdot\frac{\sin z}{z^3}\Big|_{z=0} = 0$$

(ii) we take $N=2$, then : , $\dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \dots$

$$\operatorname{Res} f(z)\Big|_{z=0} = \frac{1}{1!}\cdot\frac{d}{dz}\, z^2\cdot\frac{\sin z}{z^3}\Big|_{z=0}$$

$$= \frac{d}{dz}\,\frac{\sin z}{z}\Big|_{z=0} = \frac{d}{dz}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)\Big|_{z=0}$$

$$= -\frac{2z}{3!} + \frac{4z^3}{5!} - \dots\Big|_{z=0} = 0$$

ⓓ $f(z) = \dfrac{1}{(z^2+\alpha^2)^2} = \dfrac{1}{(z+i\alpha)^2\cdot(z-i\alpha)^2}$

at $z = i\alpha$ a pole of double order

» $z = -i\alpha$ a » of » »

$$\operatorname{Res} f(z)\Big|_{z=z_0} = \frac{1}{(N-1)!}\cdot\frac{d^{(N-1)}}{dz^{(N-1)}}\cdot(z-z_0)^N\cdot f(z)\Big|_{z=z_0}$$

$$\operatorname{Res} f(z)\Big|_{z=i\alpha} = \frac{1}{1!}\cdot\frac{d}{dz}(z-i\alpha)^2\cdot\frac{1}{(z-i\alpha)^2(z+i\alpha)^2}\Bigg|_{z=i\alpha}$$

$$= \frac{d}{dz}\frac{1}{(z+i\alpha)^2}\Bigg|_{z=i\alpha} = \frac{-2\cdot(z+i\alpha)}{(z+i\alpha)4}\Bigg|_{z=i\alpha}$$

$$= \frac{-2}{(z+i\alpha)^3}\Bigg|_{z=i\alpha} = \frac{-2}{8\cdot i^3\cdot\alpha^3} = \frac{1}{4i\alpha^3}$$

$$\operatorname{Res} f(z)\Big|_{z=-i\alpha} = \frac{1}{1!}\cdot\frac{d}{dz}\cdot(z+i\alpha)^2\cdot\frac{1}{(z+i\alpha)^2\cdot(z-i\alpha)^2}\Bigg|_{z=i\alpha}$$

$$= \frac{-2}{(z-i\alpha)^3}\Bigg|_{z=-i\alpha} = \frac{-2}{-8\cdot i^3\cdot\alpha^3} = \frac{-1}{4i\alpha^3}$$

$\Sigma \operatorname{Res} = 0$

EXERCISES

Establish the following integration formulas with the aid of residues:

167/

1. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$ 2. $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}.$

3. $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$ 4. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$

5. $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a} \quad (a \geqq 0).$ 6. $\displaystyle\int_0^\infty \frac{\cos x\,dx}{(x^2+1)^2} = \frac{\pi}{2e}.$

7. $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \qquad (a > b > 0).$

8. $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \qquad (a > 0,\, b > 0).$

9. $\displaystyle\int_{-\infty}^\infty \frac{x \sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a \qquad (a > 0).$

Use residues to find the values of the following integrals:

10. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}.$

11. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)}.$ *Ans.* $-\pi/5.$

12. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)^2}.$

13. $\displaystyle\int_0^\infty \frac{x \sin x\,dx}{(x^2+1)(x^2+4)}.$

14. $\displaystyle\int_{-\infty}^\infty \frac{\sin x\,dx}{x^2+4x+5}.$ *Ans.* $-(\pi/e)\sin 2.$

15. $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,dx}{(x+a)^2+b^2}.$

Exercises

167/1
$$I = \int_0^{\infty} \frac{x^2 \cdot dx}{(x^2+1)(x^2+4)} = \frac{1}{2} \cdot \int_{-\infty}^{+\infty} \frac{x^2 \cdot dx}{(x^2+1)(x^2+4)}$$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)}$$

$$K_1 = \operatorname{Res} f(z)\Big|_{z=i} = \lim (z-i)\cdot f(z)\Big|_{z=i}$$

$$= \frac{z^2}{(z+i)(z^2+4)}\Bigg|_{z=i} = \frac{-1}{2i\cdot 3} = -\frac{1}{6i}$$

$$K_2 = \operatorname{Res} f(z)\Big|_{z=2i} = \lim (z-2)\cdot f(z)\Big|_{z=2i}$$

$$= \frac{z^2}{(z^2+1)\cdot(z+i2)}\Bigg|_{z=2i} = \frac{-4}{-3\cdot 4i} = \frac{2}{6i}$$

$$\int_{-R}^{+R} f(x)\cdot dx + \int_{C_R} f(z)\cdot dz = 2\pi i\cdot(K_1+K_2)$$

$$= 2\pi i\cdot\frac{2-1}{6i} = \frac{2\pi i}{6i} = \frac{\pi}{3}$$

$$|f(z)| = \frac{|z|^2}{|z^2+1|\cdot|z^2+4|} \leq \frac{|z|^2}{(|z|^2-1)(|z|^2-4)} = \frac{R^2}{(R^2-1)(R^2-4)}$$

$$|dz| = \pi\cdot R$$

$$\left|\int_{C_R} f(z)\cdot dz\right| \leq \frac{R^3\cdot\pi}{(R^2-1)(R^2-4)} = \frac{\pi}{(1-1/R^2)(R-4/R)} \to 0 \quad \text{as } R\to\infty$$

So,

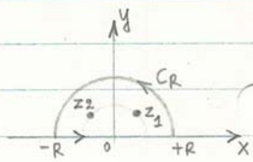
$$\lim_{R\to\infty}\int_{-R}^{+R} f(x)\,dx = \frac{\pi}{3}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \frac{\pi}{3}$$

or

$$\int_{0}^{\infty} \frac{x^2\cdot dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$$

167/2 $$\int_0^{\infty}\frac{dx}{x^4+1} = \frac{1}{2}\cdot\int_{-\infty}^{+\infty}\frac{dx}{x^4+1}$$



$$f(z) = \frac{1}{z^4+1} = \frac{1}{(z^2-i)(z^2+i)}$$

$$= \frac{1}{(z-\sqrt{i})(z+\sqrt{i})(z-\sqrt{-i})(z+\sqrt{-i})} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$z_1=\sqrt{i} \Rightarrow \quad i = z^2 \Rightarrow \quad 1\cdot(0+i\cdot 1) = r^2\cdot(\cos\theta+i\sin\theta)^2$

$r=1 \qquad \cos 2\theta = 0,\ \sin 2\theta = 1 \qquad 2\theta = \frac{\pi}{2},\ \theta = \frac{\pi}{4}$

$$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\pi/4}$$

or $\quad z^4 = -1 \quad\Rightarrow\quad z = \sqrt[4]{-1} \quad\Rightarrow$

$r_0^4\cdot(\cos\theta_0 + i\sin\theta_0)^4 = 1(\cos\pi + i\sin\pi) \quad , \quad r_0 = 1,\ \theta = \pi$

$4\theta_0 = \theta + 2k\pi \quad \rightarrow \quad \theta_0 = \frac{\theta}{4} + \frac{2k\pi}{4} = \frac{\pi}{4} + \frac{2k\pi}{4}$

$k = 0, 1, 2, 3$

$$z = \cos\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)$$

$K=0 \qquad z_1 = \cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4} = e^{i\frac{\pi}{4}}$

$K=1 \qquad z_2 = \cos\left(\frac{\pi}{4}+\frac{2\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{2\pi}{4}\right) = e^{i\cdot\frac{3\pi}{4}}$

$K=2 \qquad z_3 = \cos\left(\frac{\pi}{4}+\frac{4\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}+\frac{4\pi}{4}\right) = e^{i\frac{5\pi}{4}}$

$K=3 \qquad z_4 = \cos\left(\frac{\pi}{4}+\frac{6\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{6\pi}{4}\right) = e^{i\cdot\frac{7\pi}{4}}$

$$\int_{-R}^{+R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\,(K_1 + K_2)$$

$$K_1 = \operatorname{Res} f(z)\Big|_{z=z_1} = \lim \cancel{(z-z_1)}\cdot\frac{1}{\cancel{(z-z_1)}(z-z_2)(z-z_3)(z-z_4)}\Big|_{z=z_1}$$

$$= \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{1}{\left(e^{i\frac{\pi}{4}}-e^{i\frac{3\pi}{4}}\right)\left(e^{i\frac{\pi}{4}}-e^{i\cdot\frac{5\pi}{4}}\right)\left(e^{i\frac{\pi}{4}}-e^{i\frac{7\pi}{4}}\right)} = \frac{1}{e^{i\frac{3\pi}{4}}\cdot\left(1-e^{i\frac{\pi}{2}}\right)\left(1-e^{i\pi}\right)\left(1-e^{i\cdot\frac{3\pi}{2}}\right)}$$

$$= \frac{1}{e^{i\frac{3\pi}{4}}\cdot\left(1-\cos\frac{\pi}{2}-i\cdot\sin\frac{\pi}{2}\right)\left(1-\cos\pi-i\sin\pi\right)\left(1-\cos\frac{3\pi}{2}-i\cdot\sin\frac{3\pi}{2}\right)}$$

$$= \frac{1}{e^{i\frac{3\pi}{4}}\cdot(1-i)\cdot(1+1)(1+i)} = \frac{1}{2\cdot(1+1)\cdot\left(\cos\frac{3\pi}{4}+i\cdot\sin\frac{3\pi}{4}\right)} = \frac{1}{\frac{4}{2}\left(-\sqrt{2}+i\sqrt{2}\right)}$$

$$= \frac{1}{2\sqrt{2}\cdot(-1+i)} = \frac{1}{4\cdot z_2}$$

$$K_2 = \operatorname{Res} f(z)\Big|_{z=z_2} = \lim \cancel{(z-z_2)}\frac{1}{\cancel{(z-z_2)}(z-z_1)(z-z_3)(z-z_4)}\Big|_{z=z_2}$$

$$= \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{1}{\left(e^{\frac{3\pi i}{4}}-e^{\frac{\pi i}{4}}\right)\left(e^{\frac{3\pi i}{4}}-e^{\frac{5\pi i}{4}}\right)\left(e^{\frac{3\pi i}{4}}-e^{\frac{7\pi i}{4}}\right)}$$

$$= \frac{1}{e^{\frac{7\pi i}{4}}\left(e^{i\frac{\pi}{2}}-1\right)\left(1-e^{\frac{\pi i}{2}}\right)\left(1-e^{\pi i}\right)} = \frac{1}{-e^{\frac{7\pi i}{4}}\cdot\left(1-\cos\frac{\pi}{2}-i\cdot\sin\frac{\pi}{2}\right)^2\left(1-\cos\pi-i\sin\pi\right)}$$

$$= \frac{1}{-e^{\frac{7\pi i}{4}}\cdot(1-i)(1-i)(1+1)} = \frac{1}{-2\cdot e^{\frac{7\pi i}{4}}\cdot\left(1+(-i)^2-2i\right)} = \frac{1}{4i\cdot e^{\frac{7\pi i}{4}}}$$

$$= \frac{1}{4i\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)} = \frac{1}{4i\left(\frac{\sqrt{2}}{2}-i\cdot\frac{\sqrt{2}}{2}\right)} = \frac{1}{2\sqrt{2}\,i\,(1-i)}$$

$$= \frac{1}{2\sqrt{2}(i+1)}$$

$$|f(z)| = \frac{1}{|z^4+1|} \leq \frac{1}{|z|^4-1} = \frac{1}{R^4-1}$$

$|dz| = \pi\cdot R$ and so,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi\cdot R}{R^4-1} = \frac{\pi}{R^3-\frac{1}{R}} \longrightarrow 0 \quad \text{as} \quad R\to\infty$$

$$\lim_{R\to\infty}\int_{-R}^{+R}\frac{dx}{x^4+1} = 2\pi i\,(K_1+K_2) = 2\pi i\left(\frac{1}{2\sqrt{2}(i-1)}+\frac{1}{2\sqrt{2}(i+1)}\right)$$

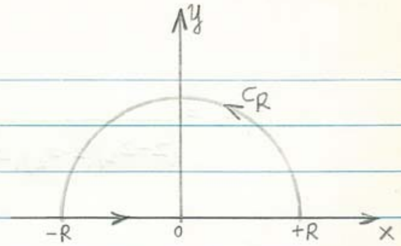
$$= 2\pi i\,\frac{1}{2\sqrt{2}}\cdot\frac{i+1+i-1}{-1-1} = \frac{2\pi i\cdot 2i}{-4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

or $$\int_{-\infty}^{+\infty}\frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2}$$

$$\int_0^{\infty}\frac{dx}{x^4+1} = \frac{1}{2}\cdot\int_{-\infty}^{+\infty}\frac{dx}{x^4+1} = \frac{\pi\cdot\sqrt{2}}{4}$$

167/3

$$\int_0^\infty \frac{x^2 \cdot dx}{x^6+1} = \frac{\pi}{6}$$



$$f(z) = \frac{z^2}{z^6+1}$$

$$\int_{-R}^{+R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum \overset{+}{\text{Res}}\, f(z)$$

$z^6 = -1 \qquad z = \sqrt[6]{-1}$

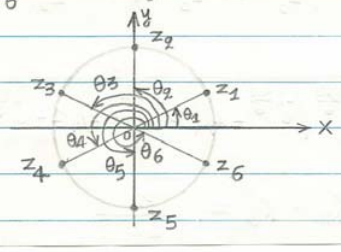
$r^6 \cdot (\cos 6\theta + i \cdot \sin 6\theta) = 1(\cos\pi + i \sin\pi) \qquad r = \sqrt[6]{1} = 1$

$$6\theta = \pi + 2K\pi$$

$$\theta = \frac{\pi}{6} + \frac{2K\pi}{6} \qquad K = 0, 1, 2, 3, 4, 5$$

$$z = \cos\left(\frac{\pi}{6} + \frac{2K\pi}{6}\right) + i \cdot \sin\left(\frac{\pi}{6} + \frac{2K\pi}{6}\right)$$

$K=0$	$\theta_1 = \frac{\pi}{6}$	$z_1 = e^{i\frac{\pi}{6}}$	only these are in the upper half plane
$K=1$	$\theta_2 = \frac{3\pi}{6}$	$z_2 = e^{i\frac{3\pi}{6}}$	
$K=2$	$\theta_3 = \frac{5\pi}{6}$	$z_3 = e^{i\frac{5\pi}{6}}$	
$K=3$	$\theta_4 = \frac{7\pi}{6}$	$z_4 = e^{i\frac{7\pi}{6}}$	
$K=4$	$\theta_5 = \frac{9\pi}{6}$	$z_5 = e^{i\cdot\frac{9\pi}{6}}$	
$K=5$	$\theta_6 = \frac{11\pi}{6}$	$z_6 = e^{i\cdot\frac{11\pi}{6}}$	



$$f(z) = \frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)}$$

$$K_1 = \frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_1-z_5)(z_1-z_6)} = \lim_{z=z_1} f(z)\cdot(z-z_1)$$

$$K_1 = \frac{1}{z_1^3\cdot\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_3}{z_1}\right)\left(1-\frac{z_4}{z_1}\right)\left(1-\frac{z_5}{z_1}\right)\left(1-\frac{z_6}{z_1}\right)}$$

$$= \frac{1}{+e^{i\frac{3\pi}{6}}\cdot\left(1-e^{i\frac{2\pi}{6}}\right)\left(1-e^{i\frac{4\pi}{6}}\right)\left(1-e^{i\pi}\right)\left(1-e^{i\frac{8\pi}{6}}\right)\left(1-e^{i\frac{10\pi}{6}}\right)} = \frac{1}{6i}$$

$$K_2 = \frac{1}{z_2^3\cdot\left(\frac{z_1}{z_2}-1\right)\left(1-\frac{z_3}{z_2}\right)\left(1-\frac{z_4}{z_2}\right)\left(1-\frac{z_5}{z_2}\right)\left(1-\frac{z_6}{z_2}\right)}$$

$$= \frac{1}{+e^{i\frac{9\pi}{6}}\cdot\left(e^{-i\cdot\frac{\pi}{3}}-1\right)\left(1-e^{i\frac{2\pi}{6}}\right)\left(1-e^{i\frac{4\pi}{6}}\right)\left(1-e^{i\frac{6\pi}{6}}\right)\left(1-e^{i\frac{8\pi}{6}}\right)} = \frac{1}{6i}$$

$$K_3 = \frac{1}{z_3^3\cdot\left(\frac{z_1}{z_3}-1\right)\left(\frac{z_2}{z_3}-1\right)\left(1-\frac{z_4}{z_3}\right)\left(1-\frac{z_5}{z_3}\right)\left(1-\frac{z_6}{z_3}\right)}$$

$$= \frac{1}{e^{i\cdot\frac{15\pi}{6}}\cdot\left(e^{-i\frac{4\pi}{6}}-1\right)\left(e^{-i\cdot\frac{2\pi}{6}}-1\right)\left(1-e^{i\frac{2\pi}{6}}\right)\left(1-e^{i\frac{4\pi}{6}}\right)\left(1-e^{i\pi}\right)} = \frac{1}{6i}$$

$$K_1 + K_2 + K_3 = \frac{3}{6i}$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R} |f(z)|\cdot|dz| = \int_{C_R} \frac{|z|^2}{|z^6+1|}\,|dz|$$

$$\leq \frac{R^2}{|R|^6-1}\cdot \pi R = \frac{\pi R^3}{R^6-1}$$

$$\leq \frac{\pi}{R^3-\frac{1}{R^3}} = \frac{\pi}{\infty-\frac{1}{\infty}} = 0 \quad \text{when} \quad R\to\infty$$

So,

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^6+1}\,dx = 2\pi i\cdot \sum \overset{+}{\text{Res}}\, f(z)$$

$$\int_{0}^{+\infty} \frac{x^2\,dx}{x^6+1} = \pi i\cdot \sum \overset{+}{\text{Res}}\, f(z)$$

167/4 $$\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$$

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2\cdot(z+i)^2}$$

at $z=i$, $f(z)$ has a pole of order $N=2$

$$K_1 = \operatorname{Res} f(z)\Big|_{z=i} = \frac{1}{(N-1)!}\,\frac{d^{(N-1)}}{dz^{N-1}}\,(z-i)^N\cdot f(z)\Big|_{z=i}$$

$$= \frac{d}{dz}\,(z-i)^2\cdot\frac{1}{(z-i)^2\cdot(z+i)^2}\Big|_{z=i}$$

$$= \frac{-2\cdot(z+i)\cdot 1}{(z+i)^4}\Big|_{z=i} = \frac{-2}{(z+i)^3}\Big|_{z=i} = \frac{-2}{2^3\cdot i^3}$$

$$= \frac{-2}{-8\cdot i} = \frac{1}{4i}$$

$$\int_{-R}^{+R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\cdot K_1$$
$$= 2\pi i\cdot\frac{1}{4i} = \frac{\pi}{2}$$

$$|f(z)| = \frac{1}{|z^2+1|^2} \leq \frac{1}{(|z|^2-1)^2} = \frac{1}{R^4-2R^2+1}\ , \quad |dz| = \pi R$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R}{R^4-2R^2+1} = \frac{\pi}{R^3-2R+\frac{1}{R}} \to 0 \quad \text{as } R\to\infty$$

and so,

$$\lim_{R\to\infty}\int_{-R}^{+R} f(x)\,dx = \frac{\pi}{2} \qquad \text{or}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} \quad , \quad \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2} \cdot \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

167/5 $$\int_0^{\infty} \frac{\cos \alpha x}{x^2+1} \cdot dx = \frac{1}{2} \cdot \int_{-\infty}^{+\infty} \frac{\cos \alpha x}{x^2+1} \cdot dx \quad , \quad \alpha \geq 0$$

we have : $$\int_{-\infty}^{+\infty} \frac{\cos \alpha x + i \cdot \sin \alpha x}{x^2+1} \cdot dx = \int_{-\infty}^{+\infty} \frac{e^{i\alpha x}}{x^2+1} \cdot dx$$

$$f(z) = \frac{e^{i\alpha z}}{z^2+1} = \frac{e^{i\alpha z}}{(z-i)(z+i)}$$

$$K_1 = \operatorname{Res} f(z)\Big|_{z=i} = \lim \cancel{(z-i)} \cdot \frac{e^{i\alpha z}}{\cancel{(z-i)}(z+i)}\Bigg|_{z=i} = \frac{e^{ii\alpha}}{2i}$$

$$= \frac{1}{2i \cdot e^{\alpha}}$$

$$\int_{-R}^{+R} \frac{e^{i\alpha x}}{x^2+1} dx + \int_{C_R} \frac{e^{i\alpha z}}{z^2+1} dz = 2\pi i \cdot K_1$$

$$= \frac{2i \cdot \pi}{2i \cdot e^{\alpha}} = \frac{\pi}{e^{\alpha}}$$

$$|f(z)| = \frac{|e^{i\alpha z}|}{|z^2+1|} = \frac{|e^{i\alpha x} \cdot e^{-\alpha y}|}{|z^2+1|} = \frac{|e^{-\alpha y}|}{|z^2+1|} \leq \frac{1}{(|z^2|-1)|e^{\alpha y}|}$$

$$\leq \frac{1}{(R^2-1)\,|e^{\alpha y}|} \leq \frac{1}{R^2-1} \quad , \quad |dz| = \pi \cdot R$$

As $\alpha > 1$ and $y \geq 0 \Rightarrow e^{\alpha y} \geq 1$ or $\frac{1}{e^{\alpha y}} \leq 1$

$$\left| \int_{C_R} f(z) \cdot dz \right| \leq \frac{\pi R}{R^2-1} = \frac{\pi}{R - \frac{1}{R}} \longrightarrow 0 \quad \text{as} \quad R \to \infty$$

Then,

$$\lim_{R\to\infty} \int_{-R}^{+R} \frac{e^{i\alpha x}}{x^2+1}\,dx = \pi \cdot e^{-\alpha}$$

or

$$\int_0^{\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx = \frac{1}{2}\cdot \int_{-\infty}^{+\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx = \frac{\pi}{2}\cdot e^{-\alpha}$$

or

$$\int_0^{\infty} \frac{\cos\alpha x}{x^2+1}\,dx + i \cdot \int_0^{\infty} \frac{\sin\alpha x}{x^2+1}\,dx = \frac{\pi}{2}\cdot e^{-\alpha} + i \cdot 0$$

or

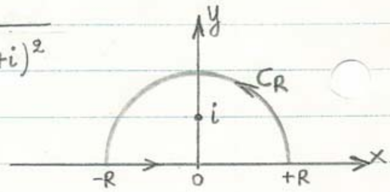
$$\int_0^{\infty} \frac{\cos\alpha x}{x^2+1}\,dx = \frac{\pi}{2}\cdot e^{-\alpha} \quad , \quad \int_0^{\infty} \frac{\sin\alpha x}{x^2+1}\,dx = 0$$

167/6 $$\int_0^{\infty} \frac{\cos x \cdot dx}{(x^2+1)^2} = \frac{\pi}{2e}$$

$$\int_0^{\infty} \frac{\cos x \; dx}{(x^2+1)^2} = \frac{1}{2} \cdot \int_{-\infty}^{+\infty} \frac{\cos x \cdot dx}{(x^2+1)^2}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ix} \cdot dx}{(x^2+1)^2} = \int_{-\infty}^{+\infty} \frac{\cos x \cdot dx}{(x^2+1)^2} + i \cdot \int_{-\infty}^{+\infty} \frac{\sin x \cdot dx}{(x^2+1)^2}$$

$$f(x) = \frac{e^{ix}}{(x^2+1)^2} = \frac{e^{ix}}{(x-i)^2 \cdot (x+i)^2}$$



$$f(z) = \frac{e^{iz}}{(z-i)^2 \cdot (z+i)^2}$$

$$\int_{-R}^{+R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \sum \overset{+}{\mathrm{Res}} f(z)$$

$$K_1 = \mathrm{Res} f(z)\Big|_{z=i} = \frac{1}{(N-1)!} \cdot \frac{d^{(N-1)}}{dz^{N-1}} (z-i)^N \cdot f(z)\Big|_{z=z_1} \quad , N=2$$

$$K_1 = \frac{d}{dz} \cancel{(z-i)^2} \frac{e^{iz}}{\cancel{(z-i)^2} \cdot (z+i)^2} = \frac{i \cdot e^{iz} \cdot (z+i)^2 - 2 \cdot (z+i) \cdot e^{iz}}{(z+i)^{\cancel{4}3}}\Bigg|_{z=i}$$

$$K_1 = \frac{(i \cdot 2i - 2)e^{ii}}{(2i)^3} = \frac{-4 \cdot e^{-1}}{8i \cdot i^2} = \frac{1}{2i \cdot e}$$

$$2\pi i \sum \overset{+}{\mathrm{Res}} f(z) = \frac{2i \cdot \pi}{2i \cdot e} = \frac{\pi}{e}$$

7.A.9.

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\cdot|dz| = \int_{C_R} \frac{|e^{iz}|}{|z^2+1|^2}|dz|$$

$$\le \int_{C_R} \frac{|e^{ix}|\cdot|e^{-y}|}{(|z|^2-1)^2}|dz| = \int_{C_R} \frac{|e^{-y}|\cdot|dz|}{(R^2-1)^2}$$

$e^{-y} = \frac{1}{e^y} \le 1$,
since $y \ge 0$,

$$\le \int_{C_R} \frac{|dz|}{(R^2-1)^2} = \frac{\pi R}{R^4-2R^2+1}$$

$$\le \frac{\pi}{R^3-2R+\frac{1}{R}} = 0 \quad \text{as} \quad R \to \infty$$

So,

$$\int_{-\infty}^{+\infty} f(x)\cdot dx = 2\pi i \sum^{+} \operatorname{Res} f(z) = \frac{\pi}{e}$$

$$\int_{-\infty}^{+\infty} \frac{\cos x\cdot dx}{(x^2+1)^2} + i\cdot\int_{-\infty}^{+\infty} \frac{\sin x\cdot dx}{(x^2+1)^2} = \frac{\pi}{e}$$

$$\int_0^{\infty} \frac{\cos x\cdot dx}{(x^2+1)^2} = \frac{1}{2}\cdot\int_{-\infty}^{+\infty} \frac{\cos x\cdot dx}{(x^2+1)^2} = \frac{\pi}{2e}$$

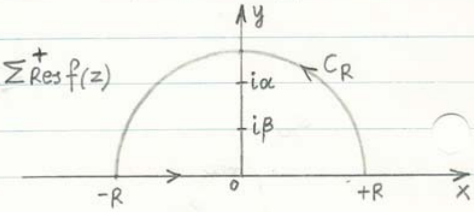
$$\int_0^{\infty} \frac{\sin x\cdot dx}{(x^2+1)^2} = 0$$

167/7 $$\int_{-\infty}^{+\infty} \frac{\cos x \cdot dx}{(x^2+\alpha^2)(x^2+\beta^2)} = \frac{\pi}{\alpha^2-b^2} \cdot \left(\frac{e^{-b}}{b} - \frac{e^{-\alpha}}{\alpha} \right) \quad , \quad \alpha > b > 0$$

$$\int_{-\infty}^{+\infty} \frac{e^{ix} \cdot dx}{(x^2+\alpha^2)(x^2+b^2)} = \int_{-\infty}^{+\infty} \frac{\cos x \cdot dx}{(x^2+\alpha^2)(x^2+\beta^2)} + i \cdot \int_{-\infty}^{+\infty} \frac{\sin x \cdot dx}{(x^2+\alpha^2)(x^2+\beta^2)}$$

$$f(z) = \frac{e^{iz}}{(z^2+\alpha^2)(z^2+b^2)} = \frac{e^{iz}}{(z-i\alpha)(z+i\alpha)(z-ib)(z+i\beta)}$$

$$\int_{-R}^{+R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \sum \overset{+}{\text{Res}}\, f(z)$$



$$K_1 = \text{Res} f(z)\Big|_{z=\alpha i} = \frac{e^{-\alpha}}{2i\alpha \cdot i \cdot i (\alpha^2-\beta^2)} = \frac{-e^{-\alpha}}{2i \cdot \alpha \cdot (\alpha^2-\beta^2)}$$

$$K_2 = \text{Res} f(z)\Big|_{z=i\beta} = \frac{e^{-\beta}}{2i\beta \cdot i \cdot i \cdot (\beta^2-\alpha^2)} = \frac{e^{-\beta}}{2i \cdot \beta \cdot (\alpha^2-\beta^2)}$$

$$K_1 + K_2 = \frac{1}{2i} \cdot \frac{1}{\alpha^2-\beta^2} \cdot \left(\frac{e^{-\beta}}{\beta} - \frac{e^{-\alpha}}{\alpha} \right)$$

$$2\pi i \sum \overset{+}{\text{Res}}\, f(z) = \frac{\pi}{\alpha^2-\beta^2} \cdot \left(\frac{e^{-\beta}}{\beta} - \frac{e^{-\alpha}}{\alpha} \right)$$

$$\left| \int_{C_R} f(z)\, dz \right| \leq \int_{C_R} \frac{|e^{iz}| \cdot |dz|}{|z^2+\alpha^2| \cdot |z^2+\beta^2|} \leq \int_{C_R} \frac{|e^{-y}| \cdot |e^{ix}| \cdot |dz|}{(|z|^2-\alpha^2)(|z|^2-\beta^2)}$$

$$\leq \int_{C_R} \frac{|dz|}{(R^2-\alpha^2)(R^2-\beta^2)} \quad , \; |e^{-y}| = \frac{1}{|e^{y}|} \leq 1 \; , \text{ when } y \geq 0$$

$$\leq \frac{\pi R}{R^2 \cdot R^2\left(1-\frac{\alpha^2}{R^2}\right)\left(1-\frac{\beta^2}{R^2}\right)} = \frac{\pi}{R^3 \cdot \left(1-\frac{\alpha^2}{R^2}\right)\left(1-\frac{\beta^2}{R^2}\right)}$$

$$\leq \frac{\pi}{\infty \cdot \left(1-\frac{1}{\infty}\right)\left(1-\frac{1}{\infty}\right)} = \frac{\pi}{\infty \cdot 1 \cdot 1} = 0 \quad \text{when } R \to \infty$$

So,

$$\int_{-\infty}^{+\infty} f(x)\, dx = \frac{\pi}{\alpha^2-\beta^2}\left(\frac{e^{-\beta}}{\beta} - \frac{e^{-\alpha}}{\alpha}\right)$$

or

$$\int_{-\infty}^{+\infty} \frac{\cos x \cdot dx}{(x^2+\alpha^2)(x^2+\beta^2)} = \frac{\pi}{\alpha^2-\beta^2} \cdot \left(\frac{e^{-\beta}}{\beta} - \frac{e^{-\alpha}}{\alpha}\right)$$

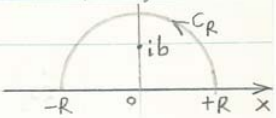
$$\int_{-\infty}^{+\infty} \frac{\sin x \cdot dx}{(x^2+\alpha^2)(x^2+\beta^2)} = 0$$

167/8 $$\int_0^{+\infty} \frac{\cos\alpha x\cdot dx}{(x^2+b^2)^2} = \frac{\pi}{4b^3}\cdot(1+\alpha b)\cdot e^{-\alpha b} \quad , \quad \alpha>0\,,\ b>0$$

$$\int_0^{\infty} \frac{\cos\alpha x\cdot dx}{(x^2+b^2)^2} = \frac{1}{2}\cdot\int_{-\infty}^{+\infty} \frac{\cos\alpha x\cdot dx}{(x^2+b^2)^2}$$

$$\int_{-\infty}^{+\infty} \frac{e^{i\alpha x}\,dx}{(x^2+b^2)^2} = \int_{-\infty}^{+\infty} \frac{\cos\alpha x\cdot dx}{(x^2+b^2)^2} + i\cdot\int_{-\infty}^{+\infty} \frac{\sin\alpha x\cdot dx}{(x^2+b^2)^2}$$

$$f(z) = \frac{e^{i\alpha z}}{(z^2+b^2)^2} = \frac{e^{i\alpha z}}{(z-ib)^2\cdot(z+ib)^2}$$



$$\int_{-R}^{+R} f(x)\,dx + \int_{C_R} f(z)\cdot dz = 2\pi i\sum^{+} \text{Res} f(z)$$

$$K_1 = \text{Res} f(z)\Big|_{z=i\beta} = \frac{1}{(N-1)!}\,\frac{d^{(N-1)}}{dz^{N-1}}(z-i\beta)^N\cdot f(z)\Big|_{z=i\beta} \quad , \ N=2 \quad , \ b=\beta$$

$$K_1 = \frac{d}{dz}\,\frac{e^{i\alpha z}}{(z+i\beta)^2}\Big|_{z=i\beta} = \frac{(z+i\beta)^{\cancel{2}}\cdot i\alpha\cdot e^{i\alpha z} - 2\cancel{(z+i\beta)}\cdot e^{i\alpha z}}{\cancel{(z+i\beta)}\cdot(z+i\beta)^3}\Big|_{z=i\beta}$$

$$= \frac{2i\beta\cdot i\alpha\cdot e^{i\alpha i\beta} - 2\cdot e^{i\alpha i\beta}}{(2i\beta)^3} = \frac{-2\cdot(1+\alpha\beta)\cdot e^{-\alpha b}}{-8\cdot i\cdot\beta^3}$$

$$2\pi i\sum^{+}\text{Res} f(z) = 2\pi i K_1 = \frac{2\pi i\cdot(1+\alpha\beta)e^{-\alpha b}}{4i\beta^3} = \frac{\pi}{2b^3}\cdot(1+\alpha b)\cdot e^{-\alpha b}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\,|dz| = \int_{C_R} \frac{|e^{i\alpha z}|}{|z^2+b^2|^2}\,|dz|$$

$$\le \int_{C_R} \frac{|e^{i\alpha x}|\cdot|e^{-\alpha y}|}{|z^2+b^2|^2}\,|dz| \qquad , |e^{i\alpha x}| = 1 \quad , |e^{-\alpha y}| = \frac{1}{|e^{\alpha y}|} \le 1$$

$$\le \int_{C_R} \frac{|dz|}{\left(|z|^2 - b^2\right)^2} \qquad , \text{when } \alpha y \ge 0$$

$$\le \frac{\pi R}{(R^2-b^2)^2} = \frac{\pi R}{R^4\left(1-\frac{b^2}{R^2}\right)^2}$$

$$\le \frac{\pi}{R^3\left(1-\frac{b^2}{R^2}\right)} = \frac{\pi}{\infty\cdot\left(1-\frac{b^2}{\infty}\right)} = 0 \quad \text{when } R\to\infty$$

So,

$$\lim_{R\to\infty}\int_{-R}^{+R} f(x)\cdot dx = \frac{\pi}{2b^3}\cdot(1+\alpha b)\cdot e^{-\alpha b}$$

or

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{e^{i\alpha x}\cdot dx}{(x^2+b^2)^2} =$$

$$= \int_{-\infty}^{+\infty} \frac{\cos\alpha x\cdot dx}{(x^2+b^2)^2} + i\cdot\int_{-\infty}^{+\infty} \frac{\sin\alpha x\cdot dx}{(x^2+b^2)^2}$$

$$\int_{-\infty}^{+\infty} \frac{\cos\alpha x \cdot dx}{(x^2+b^2)^2} = \frac{\pi}{2b^3}(1+\alpha b)\cdot e^{-\alpha b}$$

$$\int_{0}^{\infty} \frac{\cos\alpha x \cdot dx}{(x^2+b^2)^2} = \frac{1}{2}\cdot\int_{-\infty}^{+\infty} \frac{\cos\alpha x \cdot dx}{(x^2+b^2)^2} \qquad \text{oz}$$

$$\int_{0}^{\infty} \frac{\cos\alpha x \cdot dx}{(x^2+b^2)^2} = \frac{\pi}{4b^3}\cdot(1+\alpha b)\cdot e^{-\alpha b}$$

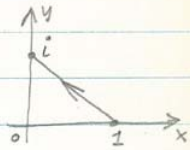
$$\int_{0}^{\infty} \frac{\sin\alpha x \cdot dx}{(x^2+b^2)^2} = 0$$

Examination March 13, 2 hours

1) Analytic functions, Cauchy-Riemann equations

2) evaluate the integral

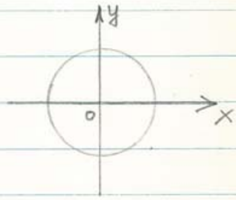
$$\int_1^i (x^2+iy^3)\,dz$$



3) Multiply Connected Domains.
Cauchy's Integral theorem for these domains

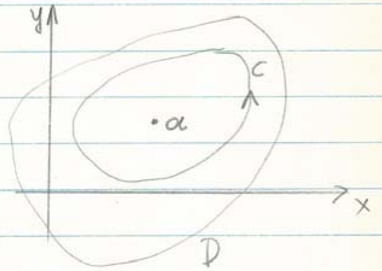
4). evaluate the integral

$$\int_C (z^2-2z+1)\,dz \qquad C:\ |z|=2$$



5) Prove the Cauchy Integral Formula

$$f(\alpha)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz$$



6) Classify the singularities of the function $\dfrac{z+1}{z\cdot(z-2)}$

and evaluate the Residues

7) Expand the function $f(z)=\dfrac{1}{(z-1)(z-2)}$ at $1<|z|<2$

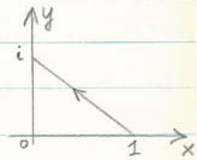
8) Evaluate the Integral with the Residues theorem

$$\oint_C \frac{dz}{z^2(z+4)} \qquad C:\ |z|=2$$

1) α) Sect 19 + 18

β) Sect. 17

2) $y - y_1 = (x - x_1) \cdot \frac{y_2 - y_1}{x_2 - x_1}$



$y - 0 = (x - 1) \cdot \frac{1 - 0}{0 - 1} \quad \Rightarrow \quad y = 1 - x$

$y^3 = (1-x)^3$

$\quad = 1 - 3x + 3x^2 - x^3$

$z = x + iy$

$z = x + i(1-x)$

$z = x(1-i)$

$dz = dx \cdot (1-i)$

$\int_1^0 \left[x^2 + i\left(1 - 3x + 3x^2 - x^3\right)\right](1-i)\, dx$

$= (1-i) \cdot \int_1^0 x^2 \cdot dx + i(1-i) \cdot \int_1^0 (1 - 3x + 3x^2 - x^3)\, dx$

$= (1-i) \cdot \left.\frac{x^3}{3}\right]_1^0 + (i+1) \cdot \left[x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4}\right]_1^0$

$= -(1-i) \cdot \frac{1}{3} - (i+1) \cdot \left(1 - \frac{3}{2} + 1 - \frac{1}{4}\right)$

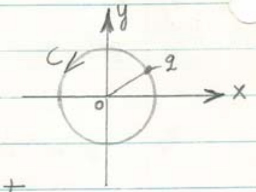
$= -\left[\frac{1-i}{3} + (i+1) \cdot \frac{8-7}{4}\right] = -\left[\frac{1-i}{3} + \frac{i+1}{4}\right]$

$= -\frac{4 - 4i + 3i + 3}{12}$

$= \frac{i - 7}{12}$

3) Sect. 49

4) $\oint_C (z^2 - 2z + 1)\, dz \qquad C: |z| = 2$



$f(z) = z^2 - 2z + 1$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - 2(z+\Delta z) + 1}{\Delta z} + \frac{-z^2 + 2z - 1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + \Delta z^2 + 2z\Delta z - 2z - 2\Delta z + 1 - z^2 + 2z - 1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z^2 + 2z\Delta z - 2\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (\Delta z + 2z - 2) = 2z - 2$$

So $f'(z)$ exists everywhere and the function $f(z)$ is analytic.

From Cauchy Integral theorem for simply connected region we can write:

$$\oint_C (z^2 - 2z + 1)\, dz = 0$$

5) Section 51.

6) $f(z) = \dfrac{z+1}{z\cdot(z-2)}$

at $z=0$ $f(z)$ has a simple pole
at $z=2$ $f(z)$ has a simple pole

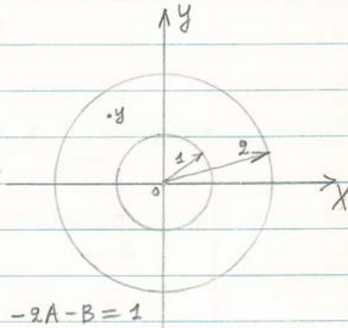
$$\operatorname{Res} f(z)\Big|_{z=0} = \lim (z-z_0)\cdot f(z)\Big|_{z=z_0}$$

$$= \lim z\cdot \frac{z+1}{z(z-2)}\Big|_{z=0} = -\frac{1}{2}$$

$$\operatorname{Res} f(z)\Big|_{z=2} = \lim \cancel{(z-2)} \frac{z+1}{\cancel{(z-2)}\cdot z}\Big|_{z=2} = \frac{3}{2}$$

7) $1<|z|<2$

$\dfrac{1}{|z|}<1$, $\dfrac{|z|}{2}<1$



$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$= \dfrac{Az-2A+Bz-B}{(z-1)(z-2)}$, $A+B=0$ $\quad -2A-B=1$

$A=-B$ $\quad -2A+A=1 \Rightarrow A=-1$

$B=1$

$$= \frac{-1}{z-1} + \frac{1}{z-2}$$

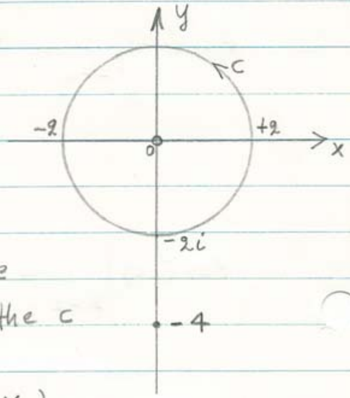
$$= -\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\cdot\left(1-\frac{z}{2}\right)} = -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{z}\cdot\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) - \frac{1}{2}\cdot\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\dots\right)$$

$$= -\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right) - \left(\frac{1}{2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\dots\right)$$

$$= -\sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

8) $f(z) = \dfrac{dz}{z^2\cdot(z+4)}$



at $z=0$ has a pole of double order

at $z=-4$ has a simple pole

But $z=-4$ is outside the c

and so:

$$\oint_c f(z)\,dz = 2\pi i(K_0)$$

$$\operatorname{Res} f(z)\Big|_{z=0} = K_0 = \frac{1}{1!}\frac{d}{dz}\cancel{z^2}\cdot\frac{1}{\cancel{z^2}\cdot(z+4)}\Bigg|_{z=0}$$

$$= \left(\frac{1}{z+4}\right)' = \frac{-1}{(z+4)^2} = -\frac{1}{16} \quad \text{and so:}$$

$$\oint_c f(z)\,dz = -2\pi i\cdot\frac{1}{16} = -\frac{\pi i}{8}$$

EXERCISES

161/ 1. If a function h is analytic at a point z_0 and $h(z_0) \neq 0$, show that z_0 is a simple pole of the function

$$f(z) = \frac{h(z)}{z - z_0}$$

and that $h(z_0)$ is the residue of f at that pole. Give examples.

2. If $h(z_0) = 0$ in Exercise 1, prove that z_0 is a removable singular point of f.

3. Show that all singular points of each of the following functions are poles. Determine the order m of each pole and the residue K of the function at the pole.

(a) $\dfrac{z+1}{z^2 - 2z}$; (b) $\tanh z$; (c) $\dfrac{1 - \exp(2z)}{z^4}$;

(d) $\dfrac{\exp(2z)}{(z-1)^2}$; (e) $\dfrac{z}{\cos z}$; (f) $\dfrac{\exp z}{z^2 + \pi^2}$.

Ans. (a) $m = 1, K = -\frac{1}{2}, \frac{3}{2}$; (b) $m = 1, K = \cancel{1}$ -2; (c) $m = 3, K = -\frac{4}{3}$.

162/ 4. Find the residue at $z = 0$ of the function

(a) $\csc^2 z$; (b) $z^{-3}\csc(z^2)$; (c) $z\cos\dfrac{1}{z}$. *Ans.* (a) 0; (b) $\frac{1}{6}$; (c) $-\frac{1}{2}$.

5. Find the value of the contour integral

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)}\,dz$$

taken counterclockwise around the circle (a) $|z - 2| = 2$; (b) $|z| = 4$.
Ans. (a) πi; (b) $6\pi i$.

6. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)}$$

taken counterclockwise around the circle (a) $|z| = 2$; (b) $|z + 2| = 3$.
Ans. (a) $\pi i/32$; (b) 0.

7. If C is the circle $|z| = 2$ described in the positive sense, evaluate the integral

(a) $\displaystyle\int_C \tan z\,dz$; (b) $\displaystyle\int_C \frac{dz}{\sinh 2z}$; (c) $\displaystyle\int_C \frac{\cosh \pi z\,dz}{z(z^2+1)}$.
Ans. (a) $-4\pi i$; (b) $-\pi i$.

8. Evaluate the integral of f in the positive sense around the unit circle about the origin, when $f(z)$ is

(a) $z^{-2}e^{-z}$; (b) $z^{-1}\csc z$; (c) $z^{-2}\csc z$; (d) $z\exp\dfrac{1}{z}$.
Ans. (a) $-2\pi i$; (b) 0; (d) πi.

9. If a function f is analytic at z_0 and if z_0 is a zero of order m of f, prove that the function $1/f$ has a pole of order m at z_0.

10. Let a function f be analytic throughout a domain D and let z_0 be the only zero of f in D. If C is a closed contour in D that encloses z_0, where C is described in the positive sense, prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = m,$$

where the positive integer m is the order of that zero. The quotient f'/f is known as the *logarithmic derivative* of f; it is the derivative of $\log f$.

Exercises

161/1 Since $h(z)$ is analytic at z_0, then its derivative exists there and $h(z)$ has no singular point there. So, Taylor series for $h(z)$ can be written

$$h(z) = h(z_0) + h'(z_0)\cdot(z-z_0) + \frac{h''(z_0)}{2!}(z-z_0)^2 + \dots$$

$$f(z) = \frac{h(z)}{z-z_0} = \frac{h(z_0)}{z-z_0} + h'(z_0) + \frac{h''(z_0)}{2!}(z-z_0) + \dots$$

that is $f(z) = \dfrac{h(z)}{z-z_0}$ has a pole of order

$N=1$ or a simple pole and it is obvious that $\operatorname{Res} f(z)\Big|_{z=z_0} = h(z_0)$

161/2 $\quad h(z_0) = 0$

$$f(z) = \frac{h(z_0)}{z-z_0} + h'(z_0) + \frac{h''(z_0)}{2!}(z-z_0) + \cdots$$

or

$$f(z) = h'(z_0) + \frac{h''(z_0)}{2!}(z-z_0) + \cdots$$

or

$$f(z_0) = h'(z_0) \quad .$$

We see that $f(z) = \frac{h(z_0)}{z-z_0} + h'(z_0) + \cdots$

is not analytic at $z = z_0$. But when we put $h(z_0) = 0$ then

$$f(z) = h'(z_0) + \frac{h''(z_0)}{2!}(z-z_0) + \cdots$$

$f(z)$ becomes an analytic function, that is, z_0 is a removable singular point.

161/3 a) $f(z) = \dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z\cdot(z-2)}$

at $z=0$ $f(z)$ has a simple pole $m=1$
at $z=2$ $f(z)$ " " " " $m=1$

$K_1 = \operatorname{Res} f(z)\Big|_{z=0} = \lim z\cdot f(z)\Big|_{z=0} = \dfrac{0+1}{0-2} = -\dfrac{1}{2}$

$K_2 = \operatorname{Res} f(z)\Big|_{z=2} = \lim (z-2)\cdot f(z)\Big|_{z=2} = \dfrac{2+1}{2} = \dfrac{3}{2}$

b) $f(z) = \tanh z = \dfrac{\sinh z}{\cosh z} = \dfrac{e^{z}-e^{-z}}{e^{z}+e^{-z}} = \dfrac{e^{2z}-1}{e^{2z}+1}$

we have $e^{2z}+1=0$ when $e^{2z}=-1=e^{i\pi}$

or $z = \dfrac{i\pi}{2}$, that is,

at $z=\dfrac{i\pi}{2}$ $f(z)$ has a simple pole $m=1$

$K = \operatorname{Res} f(z)\Big|_{z=i\frac{\pi}{2}} = \lim (e^{2z}+1)\, f(z)\Big|_{z=\frac{i\pi}{2}} = e^{2z}-1 = -2$

c) $f(z) = \dfrac{1-e^{2z}}{z^4}$, With Maclaurin Series we have,

$e^{z} = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \cdots$

we put $2z$ in the position of z $\Rightarrow$

$$e^{2z} = 1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \frac{(2z)^5}{5!} + \frac{(2z)^6}{6!} + \cdots$$

$$e^{2z} = 1 + 2z + \frac{z^2 \cdot 2^2}{2!} + \frac{z^3 \cdot 2^3}{3!} + \frac{z^4 \cdot 2^4}{4!} + \frac{z^5 \cdot 2^5}{5!} + \frac{z^6 \cdot 2^6}{6!} + \cdots$$

$$-e^{2z} = -1 - 2z - \frac{z^2 \cdot 2^2}{2!} - \frac{z^3 \cdot 2^3}{3!} - \frac{z^4 \cdot 2^4}{4!} - \frac{z^5 \cdot 2^5}{5!} - \frac{z^6 \cdot 2^6}{6!} - \cdots$$

$$1 - e^{2z} = -2z - z^2 \cdot \frac{2^2}{2!} - \frac{z^3 \cdot 2^3}{3!} - \frac{z^4 \cdot 2^4}{4!} - \frac{z^5 \cdot 2^5}{5!} - \frac{z^6 \cdot 2^6}{6!} - \cdots$$

$$f(z) = \frac{1 - e^{2z}}{z^4} = \frac{-2}{z^3} - \frac{2^2}{2! \cdot z^2} - \frac{2^3}{3! \cdot z} - \frac{2^4}{4!} - \frac{z \cdot 2^5}{5!} - \frac{z^2 \cdot 2^6}{6!} - \cdots$$

$$\Downarrow$$

at $z = 0$ $f(z)$ has a pole of order $\boxed{m = 3}$

and

$$K = \operatorname{Res} f(z)\Big|_{z=0} = -\frac{2^3}{3!} = -\frac{8}{6} = -\frac{4}{3}$$

or otherwise,

$$K = \operatorname{Res} f(z)\Big|_{z=0} = \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} z^N \cdot f(z)\Big|_{z=0}, \quad N = 4$$

$$= \frac{1}{3!} \cdot \frac{d^{(3)}}{dz^3}\left(1 - e^{2z}\right)\Big|_{z=0} = \frac{1}{6} \cdot \left(-2 \cdot e^{2z}\right)''\Big|_{z=0}$$

$$= \frac{1}{6} \cdot \left(-4 \cdot e^{2z}\right)\Big|_{z=0} = \frac{1}{6} \cdot (-8) \cdot e^0 = -\frac{8}{6} = -\frac{4}{3}$$

162/4. a) $\csc^2 z = \dfrac{1}{\sin^2 z} = f(z)$

$$\operatorname{Res} f(z)\Big|_{z=0} = K$$

at $z=0$ $f(z)$ has a pole of order $m=2$

$$K = \frac{1}{(N-1)!} \cdot \frac{d^{(N-1)}}{dz^{N-1}} \sin^2 z \cdot \frac{1}{\sin^2 z}\Bigg|_{z=0} \quad , \; N=2=m$$

$$= \frac{1}{1!} \frac{d}{dz} 1 = 0$$

b) $f(z) = z^{-3} \cdot \csc(z^2) = \dfrac{1}{z^3 \cdot \sin z^2}$

at $z=0$ $f(z)$ has a pole of order $N=$

c) $f(z) = z \cdot \cos \frac{1}{z}$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

$$\cos \frac{1}{z} = 1 - \frac{1}{z^2 \cdot 2!} + \frac{1}{z^4 \cdot 4!} - \dots$$

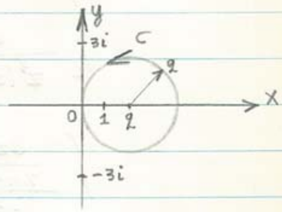
$$f(z) = z \cdot \cos \frac{1}{z} = z - \frac{1}{z \cdot 2!} + \frac{1}{z^3 \cdot 4!} - \dots$$

$$\Downarrow$$

$$K = -\frac{1}{2!} = -\frac{1}{2}$$

162/5 a) $|z-2| = 2$ $\qquad |(x-2)+iy| = 2 \Rightarrow (x-2)^2 + y^2 = 2^2$

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz =$$



$$= \int_C \frac{\frac{3z^3+2}{z^2+9} \cdot dz}{z-1} = 2\pi i \cdot K_1$$

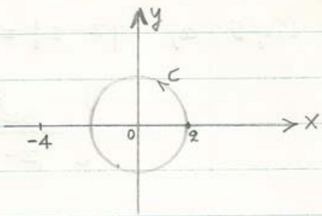
$$K_1 = \operatorname{Res} f(z)\Big|_{z=1} = \frac{3z^3+2}{z^2+9}\Big|_{z=1} = \frac{3+2}{1+9} = \frac{5}{10} = \frac{1}{2}$$

So,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz = 2\pi i\, \frac{1}{2} = \pi i$$

162/6 $\quad |z| = 2 \quad \text{or} \quad |z|^2 = 2^2$

$$\int_C \frac{dz}{z^3(z+4)} =$$



$$\int_C \frac{\frac{dz}{z+4}}{z^3} = 2\pi i \cdot K_1$$

$$K_1 = \operatorname{Res} f(z)\Big|_{z=0} = \frac{1}{(N-1)!} \cdot \frac{d^{N-1}}{dz^{N-1}} z^N \cdot f(z)\Big|_{z=0} \quad , N = 3$$

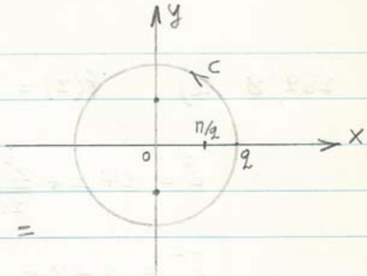
$$= \frac{1}{2!} \cdot \frac{d^{(2)}}{dz^2} \frac{1}{z+4}\Big|_{z=0} = \frac{1}{2} \cdot \left[(z+4)^{-1}\right]''$$

$$= \frac{1}{2} \cdot \left[-(z+4)^{-2}\right]' = \frac{1}{2} \cdot 2 \cdot (z+4)^{-3}\Big|_{z=0} = \frac{1}{4^3} = \frac{1}{64}$$

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \cdot \frac{1}{64} = \frac{\pi i}{32}$$

162/7 α) $|z| = 2$

$\pi = 3{,}14 \quad , \quad \frac{\pi}{2} = 1{,}57$



$$\int_C \tan z \cdot dz = \int_C \frac{\sin z}{\cos z} \cdot dz =$$

$$= 2\pi i \cdot \operatorname{Res} f(z)$$

$\sin z = (e^{iz} - e^{-iz})/2i$

$\cos z = (e^{iz} + e^{-iz})/2$

$$\int_C \tan z \cdot dz = \int_C \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} dz = \int_C \frac{(e^{iz})^2 - 1}{i\{(e^{iz})^2 + 1\}} dz$$

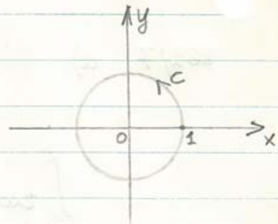
$$e^{iz \cdot 2} + 1 = 0 \Rightarrow e^{2iz} = -1 = e^{i\pi} \Rightarrow z = \frac{\pi}{2}$$

$$K_1 = \operatorname{Res} f(z)\Big|_{z=\frac{\pi}{2}} = \lim (e^{2iz} + 1) \cdot \frac{e^{2iz} - 1}{i(e^{2iz} + 1)}\Bigg|_{z = \frac{\pi}{2}} =$$

$$= \frac{e^{2i\frac{\pi}{2}} - 1}{i} = \frac{-1-1}{i} = \frac{-2}{i}$$

$$\int_C \tan z \cdot dz = 2\pi i \cdot \frac{-2}{i} = -4 .$$

162/8 α) $f(z) = z^{-2} \cdot e^{-z}$, $|z| = 1$



$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \dots$$

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \frac{z^2}{4!} - \dots$$

$$\Downarrow$$

$$K_1 = Res\left(z^{-2} \cdot e^{-z}\right)\Big|_{z=0} = -1$$

$$\int_C z^{-2} \cdot e^{-z} \cdot dz = 2\pi i \cdot K_1 = -2\pi i$$

162/9 Since z_0 is a zero of order m then $f(z_0)=f'(z_0)=f''(z_0)=f'''(z_0)=\dots=f^{(m-1)}(z_0)=0$ but $f^{(m)}(z_0)\neq 0$ and so,

$$f(z) = (z-z_0)^m \cdot \sum \alpha_{m+n} \cdot (z-z_0)^n \quad , n=0,1\dots$$

then,

$$f(z) = f^{(m)}(z_0) \cdot \frac{(z-z_0)^m}{m!} + \quad , \alpha_m \neq 0$$

$$+ \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \dots +$$

$$+ \dots + \frac{f^{(m+n)}(z_0)}{(m+n)!}(z-z_0)^{m+n} + \dots$$

for $n=0 \Rightarrow$

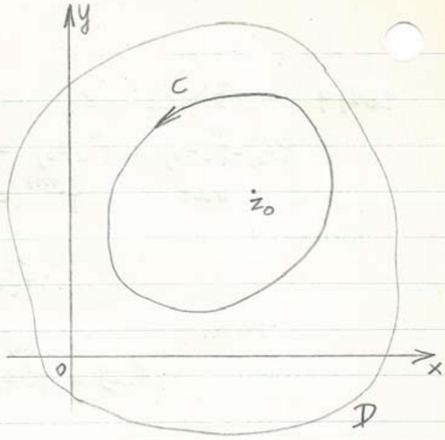
$$f(z) = f^{(m)}(z_0) \cdot \frac{1}{m!} \cdot (z-z_0)^m$$

$$\frac{1}{f(z)} = \frac{1}{f^{(m)}(z_0)} \cdot \frac{m!}{1} \cdot \frac{1}{(z-z_0)^m}$$ that is

$\frac{1}{f(z)}$ has a pole of order $N=m$ at $z=z_0$

162/10

$$I = \int_C \frac{f'(z)}{f(z)}\, dz = m$$



at z_0 $f(z)$ has a zero of order m

then,

$$f(z) = \alpha_m \cdot (z - z_0)^m$$

$$f'(z) = m \cdot \alpha_m \cdot (z - z_0)^{m-1}$$ and so

$$I = \int_C \frac{f'(z)}{f(z)}\, dz = \int_C \frac{\alpha_m \, m \cdot (z-z_0)^{m-1}}{\alpha_m \cdot (z-z_0)^m}\, dz$$

$$= \int_C m \cdot \frac{dz}{z - z_0}$$

and from Cauchy Integral formula

$$\int_C \frac{m}{z - z_0}\, dz = 2\pi i \cdot m$$ or

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \cdot dz = m$$

OR. z_0 is a zero of order m

so,

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

$$f^{(m)}(z_0) \neq 0$$

$$f(z) = \frac{f^{(m)}(z_0)}{m!} \cdot (z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!} + \cdots + \frac{f^{(m+n)}(z_0)}{(m+n)!}(z-z_0)^{m+n}$$

We call $\dfrac{f^{(m+n)}(z_0)}{(m+n)!} = \alpha_{m+n}$ $\quad n = 0, 1, 2 \cdots$

$$\alpha_m = \frac{f^{(m)}(z_0)}{m!} \neq 0$$

$$\alpha_{m+1} = \frac{f^{(m+1)}(z_0)}{(m+1)!}$$ etc. and so,

$$f(z) = \alpha_m \cdot (z-z_0)^m + \alpha_{m+1} \cdot (z-z_0)^{m+1} + \cdots + \alpha_{m+n} \cdot (z-z_0)^{m+n}$$

$$f(z) = (z-z_0)^m \cdot \left\{ \alpha_m + \alpha_{m+1} \cdot (z-z_0) + \cdots + \alpha_{m+n} \cdot (z-z_0)^n \right\}$$

$$f(z) = (z-z_0)^m \cdot \sum_{n=0}^{\infty} \alpha_{m+n} \cdot (z-z_0)^n$$

$$n = 0, 1, 2 \cdots \quad , \alpha_m \neq 0$$

$$f'(z) = m \cdot \alpha_m \cdot (z-z_0)^{m-1} + (m+1) \cdot \alpha_{m+1} \cdot (z-z_0)^m +$$

$$+ \alpha_{m+2} \cdot (m+2) \cdot (z-z_0)^{m+1} + \dots + \alpha_{m+n} \cdot (m+n) \cdot (z-z_0)^{m+n-1}$$

$$f'(z) = (z-z_0)^{m-1} \cdot \sum_{n=0}^{\infty} \alpha_{m+n} \cdot (m+n) \cdot (z-z_0)^n$$

$$\frac{f'(z)}{f(z)} = \frac{(z-z_0)^{m-1}}{(z-z_0)^m} \cdot \frac{\sum_{n=0}^{\infty} \alpha_{m+n} \cdot (m+n) \cdot (z-z_0)^n}{\sum_{n=0}^{\infty} \alpha_{m+n} \cdot (z-z_0)^n}$$

$$= \frac{1}{z-z_0} \cdot \sum \frac{\alpha_{m+n} \cdot (m+n) \cdot (z-z_0)^n}{\alpha_{m+n} \cdot (z-z_0)^n}$$

$$= \frac{1}{z-z_0} \cdot \frac{m \cdot \alpha_m + (m+1) \cdot \alpha_{m+1} \cdot (z-z_0) + \dots + (m+n) \alpha_{m+n} \cdot (z-z_0)^n +}{\alpha_m + \alpha_{m+1} \cdot (z-z_0) + \dots + \alpha_{m+n} \cdot (z-z_0)^n +}$$

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \cdot K_1 .$$

So $(z = z_0)$ is the simple pole of $\frac{f'(z)}{f(z)}$

$$K_1 = \lim \frac{m \cdot \alpha_m + (m+1) \alpha_{m+1} \cdot (z-z_0) + \dots}{\alpha_m + \alpha_{m+1} \cdot (z-z_0) + \dots} \Bigg|_{z=z_0} =$$

$$= \frac{m \cdot \cancel{\alpha_m}}{\cancel{\alpha_m}} = m$$

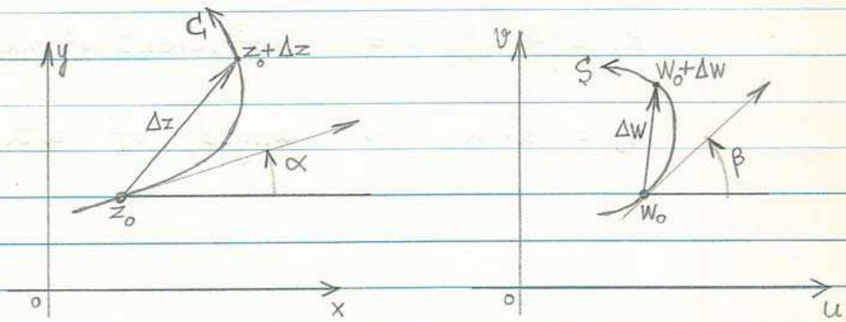
or

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \cdot dz = m$$

CONFORMAL MAPPING

75. Rotation of Tangents.

z_0 : a point of z-plane
$f(z)$: an analytic function
$f'(z_0)$: The derivative of $f(z)$ at z_0,
$f'(z_0) \neq 0$
$w = f(z)$



$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{z_0+\Delta z - z_0} = \lim_{\Delta z \to 0} \frac{W_0+\Delta W - W_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z}$$

$$f'(z_0) = |f'(z_0)| \cdot e^{i\psi_0} \qquad , \quad |f'(z_0)| = R_0$$

$f'(z_0) = R_0 \cdot e^{i\psi_0}$ So we have,

$$\lim_{\Delta z \to 0} \left| \frac{\Delta W}{\Delta z} \right| = R_0 \qquad \lim_{\Delta z \to 0} \left(\arg. \frac{\Delta W}{\Delta z} \right) = \psi_0$$

$$\Delta w = \Delta z \cdot \frac{\Delta w}{\Delta z} \quad \text{or}$$

$$\arg \Delta w = \arg \Delta z + \arg \frac{\Delta w}{\Delta z} \quad \text{or}$$

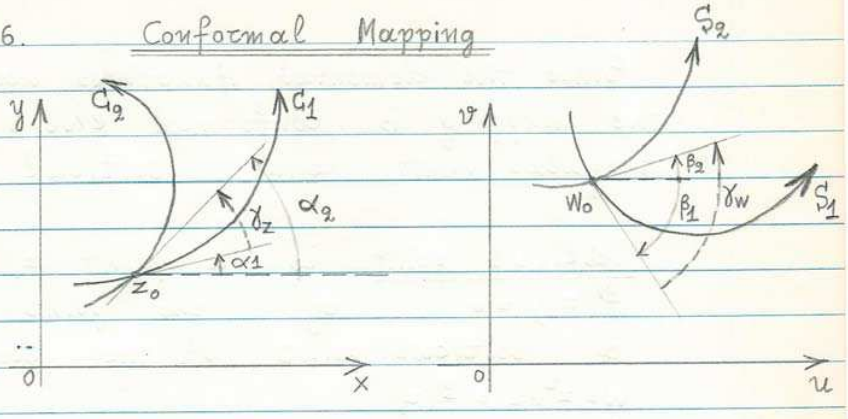
$$\lim_{\Delta z \to 0} (\arg \Delta w) = \lim_{\Delta z \to 0} \Delta z + \lim_{\Delta z \to 0} \left(\arg \frac{\Delta w}{\Delta z} \right) \quad \text{or}$$

$$\beta = \alpha + \psi_0$$

$R_0 = |f'(z_0)|$ = Coefficient of magnification

$\psi_0 = \beta - \alpha$ = angle of rotation.

76. Conformal Mapping



C_1 by $w \longrightarrow S_1 \Rightarrow \beta_1 = \alpha_1 + \psi_0$

C_2 by $w \longrightarrow S_2 \Rightarrow \beta_2 = \alpha_2 + \psi_0$

$$\beta_1 - \alpha_1 = \beta_2 - \alpha_2 \qquad \text{or}$$

$$\gamma_w = \beta_2 - \beta_1 = \alpha_2 - \alpha_1 = \gamma_z$$

$$\boxed{\measuredangle \gamma_w = \measuredangle \gamma_z}$$ the same in magnitude and sense

Mapping is Conformal if angles are preserved in magnitude and sense.

Since the elementary functions are analytic the mapping is conformal except at singular points and critical points.

Critical point is a point z_0 at which $f'(z_0) = 0$ e.g. the point $z=0$ is a critical point of the transformation
$W = z^2+1 = f(z)$
$f'(0) = 2z\Big|_{z=0} = 0$.

At each point of a domain where $f(z)$ is analytic and $f'(z_0) \neq 0$ the mapping $W = f(z)$ is conformal.

77. Examples.

Every conformal transformation must map orthogonal curves into orthogonal curves.

78. Conjugate Harmonic Functions

$u = xy$ $\qquad\qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = y = \frac{\partial v}{\partial y} \Rightarrow$

$v = \frac{1}{2} \cdot y^2 + \phi(x) \Rightarrow \frac{\partial v}{\partial x} = \phi'(x)$

$\frac{\partial u}{\partial y} = x = -\frac{\partial v}{\partial x} = -\phi'(x) \Rightarrow$

$\phi'(x) = -x \Rightarrow \phi(x) = -\frac{x^2}{2} + c$

$v = \frac{1}{2} y^2 - \frac{1}{2} x^2 + c$

$$f(z) = u + iv = xy + \frac{1}{2}(y^2 - x^2) i + ic$$

$$= -\frac{i}{2} \cdot 2ixy + \frac{i}{2}(y^2 - x^2) + i \cdot c$$

$$= -\frac{i}{2} \cdot (x^2 - y^2 + 2ixy) + i \cdot c$$

$$= -\frac{i}{2} \cdot (x + iy)^2 + i \cdot c$$

$$= -\frac{1}{2} \cdot i \cdot z^2 + i \cdot c .$$

79 Inverse Functions

Theorem. If $f(z)$ is an analytic function at a point z_0 and if $f'(z_0) \neq 0$ then exist a region in the neighborhood of z_0 which can be mapped by $w = f(z)$ to the neighborhood of w_0 and then mapping is conformal and sense preserving,

i) A unique inverse exists

$$u = u(x,y) \quad , \quad v = v(x,y)$$

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} dy \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

The Jacobian of the functions u, v

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

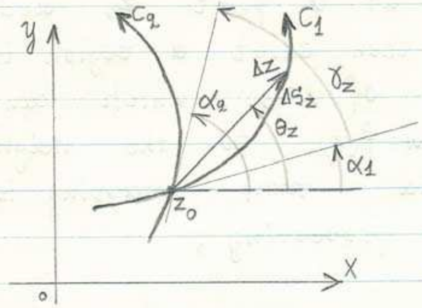
but $f(z)$ is analytic:

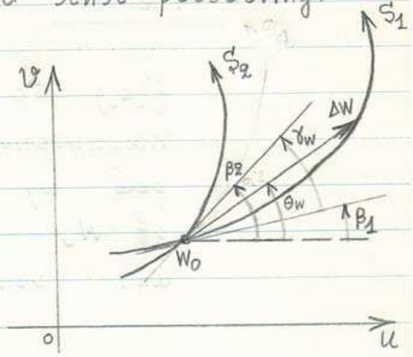
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \neq 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left| \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right|^2 = \left| f'(z) \right|^2 \neq 0$$

ii) Mapping is conformal and sense preserving.





$$\lim_{\Delta z \to 0} \frac{\Delta s_w}{\Delta s_z} = \lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|} = \lim_{\Delta z \to 0} \left| \frac{\Delta w}{\Delta z} \right| = \left| \frac{dw}{dz} \right|_{z_0} = \left| f(z_0) \right| \neq 0$$

$$\arg \frac{\Delta w}{\Delta z} = \arg \Delta w - \arg \Delta z$$

$$\lim_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z$$

or $\quad \arg \dfrac{dw}{dz} = \psi_0 = \beta_1 - \alpha_1 = \text{constant.}$

Riemann proved that :

"Every simply connected region can be mapped conformally onto the unit circle $|z| \leqslant 1$ in such a way that the previous boundary corresponds to the circular boundary $|z| \leq 1$ "

EXERCISES

178/ 1. Show that the transformation $w = z^2$ changes directions of curves at the point $z = 2 + i$ by the angle $\arctan \frac{1}{2}$. Illustrate this by using some particular curve. Show that the coefficient of magnification of distances at that point is $2\sqrt{5}$.

2. Show that the transformation $w = z^n$ changes directions at the point $z = r_0 \exp(i\theta_0)$ by the angle $(n - 1)\theta_0$, when $r_0 > 0$ and $n > 0$. What is the coefficient of magnification of distances at the point?

Ans. nr_0^{n-1}.

3. What change of directions is produced by the transformation $w = 1/z$ (*a*) at the point $z = 1$; (*b*) at the point $z = i$?

Ans. (*a*) A change in sense only; (*b*) none.

4. Under the transformation $w = 1/z$, show that the images of the lines $y = x - 1$ and $y = 0$ are the circle $u^2 + v^2 - u - v = 0$ and the line $v = 0$. Show those curves graphically, determine corresponding directions along them, and verify the conformality of the mapping at the point $z = 1$.

5. Show why the transformation $w = \exp z$ is everywhere conformal. Note that the mapping of directed line segments shown in Figs. 7 and 8 of Appendix 2 agrees with that conclusion.

6. Show that the transformation $w = \sin z$ is conformal at all points except $z = \pm\pi/2$, $z = \pm 3\pi/2$, ... Note that the mapping of directed line segments shown in Figs. 9, 10, and 11 of Appendix 2 agrees with that conclusion.

7. For a function g of Δz and a number $g_0 = R_0 \exp(i\psi_0)$, assume that

$$\lim_{\Delta z \to 0} g(\Delta z) = R_0 \exp(i\psi_0) \qquad (R_0 > 0).$$

Then to each positive number ϵ, where $\epsilon < R_0$, there corresponds a number δ such that the point $g(\Delta z)$ lies in a neighborhood of g_0 of radius ϵ whenever $|\Delta z| < \delta$. Draw a figure that shows such a neighborhood and the angle $\Delta\psi = \arg[g(\Delta z)] - \psi_0$, where $\psi_0 - \pi < \arg[g(\Delta z)] < \psi_0 + \pi$. Show that $|\sin \Delta\psi| < \epsilon/R_0$ when $|\Delta z| < \delta$ and hence that

$$\lim_{\Delta z \to 0} \arg[g(\Delta z)] = \psi_0;$$

thus justify formula (4), Sec. 75. Also justify formula (3), Sec. 75.

8. If a function f is analytic at a point z_0 and

$$f(z_0) = w_0, \qquad f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0,$$

while $f^{(m)}(z_0) \neq 0$, use Taylor's series to show that for some number r_1 the increment $\Delta w = f(z_0 + \Delta z) - f(z_0)$ has the form

$$\Delta w = \frac{1}{m!} f^{(m)}(z_0)(\Delta z)^m[1 + \Delta z\, h(\Delta z)] \qquad (|\Delta z| < r_1),$$

where h is continuous at $\Delta z = 0$. Under the transformation $w = f(z)$ of arc C into arc S shown in Fig. 48, show that the angles of inclination now satisfy the relation

$$\beta = m\alpha + \arg[f^{(m)}(z_0)]$$

and, if δ now denotes the angle from S_1 to S_2 in Fig. 49, that $\delta = m\gamma$. Thus the transformation is not conformal at z_0.

Exercises

178/1 a) $W = z^2$, $z = x+iy$, $W = u+iv$, $z_0 = 2+i$

$$\frac{dW}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z)$$

$$W = z^2 = (x+iy)^2 = x^2 - y^2 + i\cdot 2xy = u+iv$$

$$u = x^2 - y^2 \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{dW}{dz} = 2x + i\cdot 2y = f'(z)$$

by definition ψ_0 = change of direction =

$$= \arg\frac{dW}{dz} = \arg f'(z_0) = \arg\left(2x + i\cdot 2y\Big|_{\substack{x=2\\y=1}}\right) =$$

$$\psi_0 = \arg(4 + i\cdot 2)$$

$$\psi_0 = \arctan\frac{2}{4} = \arctan\frac{1}{2}$$

$$R_0 = |f'(z_0)| = \left|2x + i\cdot 2y\right|_{\substack{x=2\\y=1}} = |4+i\cdot 2| = \sqrt{16+4} = \sqrt{20} = 2\cdot\sqrt{5}\,.$$

b) $W = z^2 \Rightarrow W' = 2\cdot z = 2(x+iy) \Rightarrow R_0 = |W'|_{z_0} = 4\sqrt{5}$

$\psi_0 = \arg(W')_{z_0} = 2\cdot(2+i) = \arctan\frac{1}{2}$

178/2

$$W = z^n \qquad , z_0 = r_0 \cdot e^{i\theta_0}$$

$$\frac{dW}{dz} = n \cdot z^{n-1} \qquad , z = r \cdot e^{i\theta}$$

$$\left.\frac{dW}{dz}\right|_{z_0} = n \cdot z^{n-1}\Big|_{z_0} = \left. n \cdot r^{n-1} \cdot e^{i(n-1)\theta} \right|_{r_0, \theta_0}$$
$$= n \cdot r_0^{n-1} \cdot e^{i(n-1)\theta_0}$$

$$\psi_0 = \arg \left.\frac{dW}{dz}\right|_{z_0} = (n-1) \cdot \theta_0$$

$$R_0 = \left|\frac{dW}{dz}\right|_{z_0} = n \cdot r_0^{n-1}$$

178/3 α) $W = \frac{1}{z}$ $\qquad z_0 = 1 = 1 + i \cdot 0$

$$f'(z_0) = \left.\frac{dW}{dz}\right|_{z_0 = 1} = -\frac{1}{z^2} = -\frac{1}{1} = -1 = \cos\pi + i \cdot \sin\pi$$

$$\arg f'(z_0) = \pi$$

$R_0 = |f'(z_0)| = 1 \Rightarrow$ A change in sense only

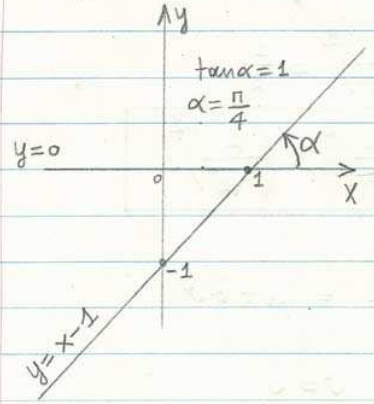
β) $W = \frac{1}{z}$ $\qquad z_0 = i$

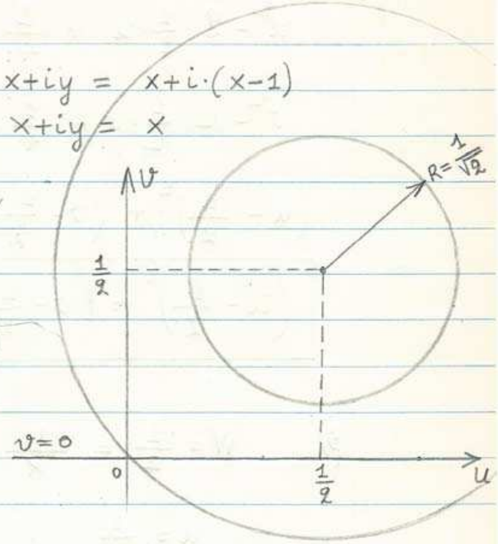
$$f'(z_0) = \left.\frac{dW}{dz}\right|_{z_0 = i} = -\left.\frac{1}{z^2}\right|_{z_0 = i} = -\frac{1}{i^2} = \frac{-1}{-1} = 1$$

$\arg f'(z_0) = 0$, $R_0 = 1$, No change.

178/4 α) $y = x-1 \Rightarrow z = x+iy = x+i\cdot(x-1)$

β) $y=0 \Rightarrow z = x+iy = x$





$$W = \frac{1}{z} \Rightarrow \quad \alpha) \quad W = \frac{1}{x+i(x-1)} = \frac{x-(x-1)i}{x^2+(x-1)^2} = u+iv$$

$$u = \frac{x}{x^2+(x-1)^2}, \quad v = \frac{1-x}{x^2+(x-1)^2}$$

$$u+v = \frac{1}{2x^2-2x+1} = \frac{2x^2-2x+1}{(2x^2-2x+1)^2} = \frac{(x^2-2x+1)+x^2}{(2x^2-2x+1)^2}$$

$$= \frac{(x-1)^2}{(2x^2-2x+1)^2} + \frac{x^2}{(2x^2-2x+1)^2} = v^2+u^2$$

$$\boxed{u^2+v^2-u-v=0}$$

$$u^2 - 2\cdot\frac{1}{2}\cdot u + \frac{1}{4} - \frac{1}{4} +$$

$$v^2 - 2\cdot\frac{1}{2}\cdot v + \frac{1}{4} - \frac{1}{4} = 0$$

$$\left(u-\frac{1}{2}\right)^2 + \left(v-\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\boxed{\left(u-\frac{1}{2}\right)^2 + \left(v-\frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2}$$

β) $W = \frac{1}{z} = \frac{1}{x} = u + i\cdot v$

$$u = \frac{1}{x}, \quad v = 0$$

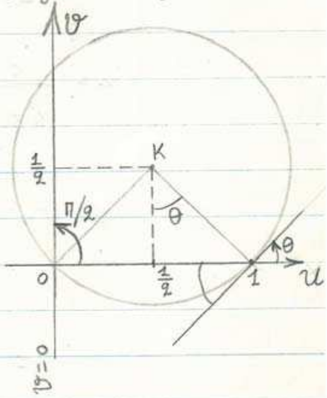
$$z_0 = 1 = 1 + i\cdot 0 \quad \Rightarrow \quad x=1,\ y=0$$

α) $u = \frac{1}{1+0} = 1, \quad v = 0$

$\beta = \theta, \quad \tan\theta = 1$

$\beta = \theta = \frac{\pi}{4}$

$\psi_0 = \arg f'(z_0=1) = \beta - \alpha = 0$



β) $\beta = \frac{\pi}{2}, \quad \alpha = 0 \Rightarrow \psi_0 = \frac{\pi}{2}$

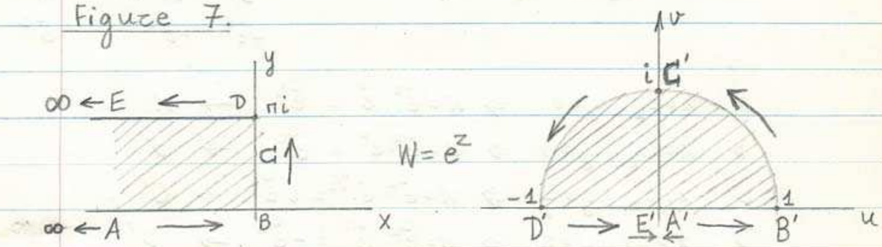
$W = e^z$

178/5 $\frac{dw}{dz} = f'(z) = e^z$ So,

the angles (magnitude and sense) are the same, that is, are preserved and so, the mapping conformal.

Appendix 2.

Figure 7.



D: $x=0,\ y=\pi \Rightarrow W = e^0 \cdot e^{i\pi} = -1 \rightarrow D'$

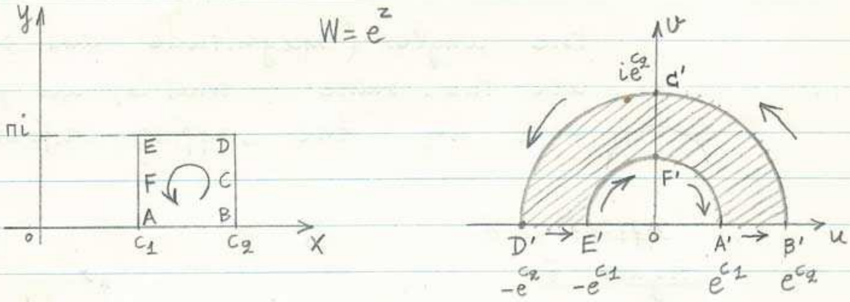
B: $x=0,\ y=0 \Rightarrow W = e^0 \cdot e^{i0} = 1 \rightarrow B'$

C: $x=0,\ y=\pi/2 \Rightarrow W = e^0 \cdot e^{i\pi/2} = i \rightarrow C'$

A: $x \to -\infty,\ y=0 \Rightarrow W = \frac{1}{e^\infty} \cdot e^{i0} = 0 \rightarrow A'$

E: $x \to -\infty,\ y=\pi \Rightarrow W = \frac{1}{e^\infty} \cdot e^{i\pi} = -0 \rightarrow E'$

Figure 8



A : $x = c_1,\ y = 0 \Rightarrow W = e^{c_1} \cdot e^{0} = e^{c_1} \rightarrow A'$

B : $x = c_2,\ y = 0 \Rightarrow W = e^{c_2} \cdot e^{0} = e^{c_2} \rightarrow B'$

C : $x = c_2,\ y = \frac{\pi}{2} \Rightarrow W = e^{c_2} \cdot e^{i\pi/2} = i \cdot e^{c_2} \rightarrow C'$

D : $x = c_2,\ y = \pi \Rightarrow W = e^{c_2} \cdot e^{i\pi} = -e^{c_2} \rightarrow D'$

E : $x = c_1,\ y = \pi \Rightarrow W = e^{c_1} \cdot e^{i\pi} = -e^{c_1} \rightarrow E'$

F : $x = c_1,\ y = \pi/2 \Rightarrow W = e^{c_1} \cdot e^{i\pi/2} = i \cdot e^{c_1} \rightarrow F'$

178/6

$w = \sin z \qquad \sin z = \sin x \cosh y + i \cos x \cdot \sinh y$

$W = u + i v$

$u = \sin x \cdot \cosh y \qquad \frac{\partial u}{\partial x} = \cos x \cdot \cosh y$

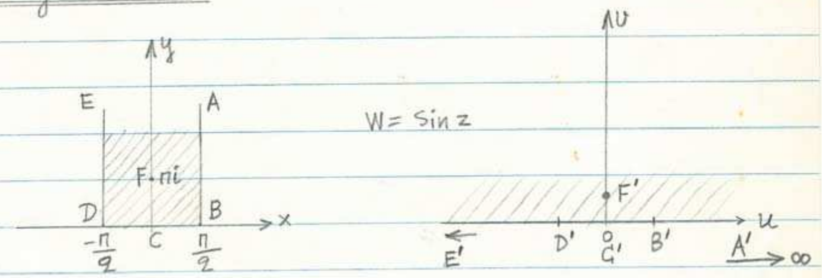
$v = \cos x \cdot \sinh y \qquad \frac{\partial v}{\partial x} = -\sin x \cdot \sinh y$

$$\frac{dW}{dz} = \cos z = \cos x \cdot \cosh y - i \sin x \cdot \sinh y$$

$\psi_0 = \operatorname{arc}\left[-\tan x \cdot \tanh y\right]$ at all points exept at $\pm(2K-1)\cdot\frac{\pi}{2}$

for which $\cos x = 0$ or $\tan x = \infty$

Figure 9.



$C: \quad x=0,\ y=0 \qquad W = 0 \Rightarrow G'$

$D: \quad x = -\frac{\pi}{2},\ y=0 \qquad W = -\sin\frac{\pi}{2} = -1 \rightarrow D'$

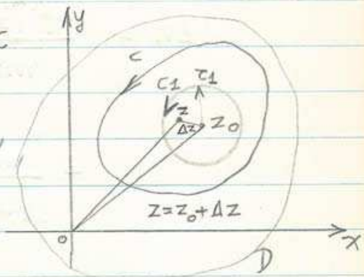
$B: \quad x = \frac{\pi}{2},\ y=0 \qquad W = \sin\frac{\pi}{2} = 1 \rightarrow B'$

$F: \quad x=0,\ y=\pi \qquad W = i\frac{e^{\pi}-e^{-\pi}}{2} = \frac{e^{\pi}-e^{-\pi}}{2}\cdot i \longrightarrow F'$

$A: \quad x = \frac{\pi}{2},\ y \to \infty \qquad W = \sin\frac{\pi}{2}\cdot\frac{1+\frac{1}{e^{\infty}}}{2\cdot\frac{1}{e^{\infty}}} = \infty \longrightarrow A'$

$E' \Rightarrow \quad x = -\frac{\pi}{2},\ y \to -\infty \qquad W = -1\cdot\frac{\frac{1}{e^{\infty}}+e^{\infty}}{2i} = -\infty \longrightarrow E'$

179/8 Since $f(z)$ is analytic in C it will be analytic in and on the circle c_1 and so, from Taylor Series we have:



$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{f''(z_0)}{2!}\cdot(z-z_0)^2 + \cdots$$

$$\cdots + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}}{(m+1)!}(z-z_0)^{m+1} +$$

$$+ \frac{f^{(m+2)}(z_0)}{(m+2)!}(z-z_0)^{m+2} + \cdots$$

but $f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$

so,

$$f(z) - f(z_0) = \frac{f^{(m)}(z_0)}{m!}\cdot(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} +$$

$$+ \frac{f^{(m+2)}(z_0)}{(m+2)!}(z-z_0)^{m+2} + \cdots$$

$z = z_0 + \Delta z$

$z - z_0 = \Delta z$

$$f(z_0+\Delta z) - f(z_0) = \frac{f^{(m)}(z_0)}{m!}\cdot(\Delta z)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}\cdot(\Delta z)^{m+1} +$$

$$+ \frac{f^{(m+2)}(z_0)}{(m+2)!}(\Delta z)^{m+2} + \cdots$$

$\Delta W = f(z_0+\Delta z) - f(z_0)$

$$\Delta W = \frac{f^{(m)}(z_0)}{m!}(\Delta z)^m \cdot \left\{ 1 + \frac{m!}{(m+1)!} \cdot \frac{f^{(m+1)}(z_0)}{f^{(m)}(z_0)} \cdot (z-z_0) + \right.$$

$$\left. + \frac{m!}{(m+2)!} \cdot \frac{f^{(m+2)}(z_0)}{f^{(m)}(z_0)} \cdot (z-z_0)^2 + \cdots \right\}$$

$$\Delta W = \frac{f^{(m)}(z_0)}{m!} \cdot (\Delta z)^m \cdot \left\{ 1 + (z-z_0) \cdot \left[\frac{1}{m+1} \cdot \frac{f^{(m+1)}(z_0)}{f^{(m)}(z_0)} + \right. \right.$$

$$\left. \left. + \frac{1}{(m+1)(m+2)} \cdot \frac{f^{(m+2)}(z_0)}{f^{(m)}(z_0)} (z-z_0) + \cdots \right] \right\}$$

$$\Delta W = \frac{f^{(m)}(z_0)}{m!} \cdot (\Delta z)^m \cdot \left\{ 1 + \Delta z \cdot h(\Delta z) \right\}$$

$$h(\Delta z) = \frac{1}{m+1} \cdot \frac{f^{(m+1)}(z_0)}{f^{(m)}(z_0)} + \frac{1}{(m+1)(m+2)} \cdot \frac{f^{(m+2)}(z_0)}{f^{(m)}(z_0)} \cdot \Delta z + \cdots$$

$$\cdots + \frac{1}{(m+1)(m+2)\cdots(m+n)} \cdot \frac{f^{(m+n)}(z_0)}{f^{(m)}(z_0)} \cdot (\Delta z)^{n-1} + \cdots$$

$h(\Delta z)$ is continuous $= \dfrac{1}{m+1} \cdot \dfrac{f^{(m+1)}(z_0)}{f^{(m)}(z_0}$ at $\Delta z = 0$

In this case we have:

$$\Delta W = (\Delta z)^m \cdot \frac{1}{m!} \cdot f^{(m)}(z_0) \cdot \left[1 + \Delta z \cdot h(\Delta z)\right]$$

or
$$\frac{\Delta W}{(\Delta z)^m} = \frac{1}{m!} \cdot f^{(m)}(z_0)$$

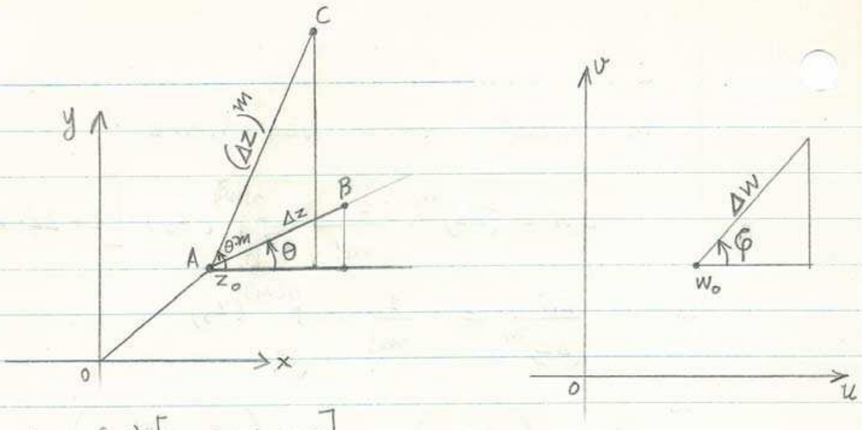
or
$$\arg\left(\frac{\Delta W}{(\Delta z)^m}\right) = \arg\left(\frac{1}{m!} \cdot f^{(m)}(z_0)\right) = \arg\left(f^{(m)}(z_0)\right) \cdot (1+\Delta z h)$$

or
$$\arg \Delta W - \arg\left((\Delta z)^m\right) = \arg\left(f^{(m)}(z_0)\left[1 + \Delta z \cdot h(\Delta z)\right]\right)$$

or $\arg \Delta W = m \cdot \arg \Delta z + \arg f^{m}(z_0) \cdot \left[1 + \Delta z \cdot h(\Delta z)\right]$

$$\lim_{\Delta z \to 0} \arg \Delta W = m \cdot \lim_{\Delta z \to 0} \arg \Delta z + \lim_{\Delta z \to 0} \arg f^{m}(z_0) \cdot \left[1 + \Delta z \cdot h(\Delta z)\right]$$

or
$$\beta = m \cdot \alpha + \arg f^{(m)}(z_0)$$



$$\Delta z = (AB)\cdot\left[\cos\theta + i\sin\theta\right]$$

$$(\Delta z)^m = (AB)^m\cdot\left[\cos\theta + i\sin\theta\right]^m = (AB)^m\cdot\left[\cos m\theta + i\cdot\sin m\theta\right]$$

So, $\boxed{\arg(\Delta z)^m = m\cdot\arg(\Delta z)}$

$$\lim_{\Delta z\to 0}\theta = \alpha \qquad \lim_{\Delta z\to 0}\varphi = \beta$$

187 /

EXERCISES

✓ **1.** The harmonic function

$$H = 2 - x + \frac{x}{x^2 + y^2}$$

assumes the value 2 on the circle $x^2 + y^2 = 1$. Under the change of variables $z = e^w$, find H as a function of u and v, and show directly that $H = 2$ on the image $u = 0$ of the circle, thus verifying one of the results of the preceding section for this special case.

2. The normal derivative of the harmonic function

$$H = e^{-x} \cos y$$

is zero along the line $y = 0$; that is, $\partial H/\partial y = 0$ on that line. Find H in terms of u and v under the change of variables $z = w^2$, and show directly that the normal derivatives of H along the images $u = 0$ and $v = 0$ of the line $y = 0$ also vanish.

3. The normal derivative of the harmonic function

$$H = 2y + e^{-x} \cos y$$

is constant, $\partial H/\partial y = 2$, along the line $y = 0$. Under the change of variables $z = w^2$, show that the normal derivative is not constant along the image of that line but that $\partial H/\partial u = 4v$ along $u = 0$ and $\partial H/\partial v = 4u$ along $v = 0$.

4. Let H be a harmonic function throughout a simply connected domain D of the xy plane. Then a function G exists such that $H + iG$ is an analytic function of z in D (Sec. 78). Deduce that the partial derivatives of H, of all orders, are continuous functions of x and y in D

5. If a function H is a solution of a Neumann problem (Sec. 80), show why $H + c$ is also a solution of that problem, where c is any constant.

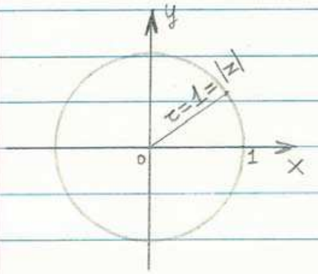
6. Use partial differentiation under a change of variables to show that

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \left(\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2}\right) \left|\frac{dw}{dz}\right|^2,$$

Exercises

187/1

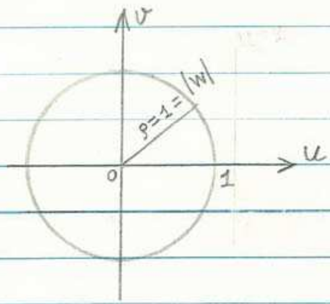
$$H = 2 - x + \frac{x}{x^2+y^2} \quad , \quad x = \cos\theta \ , \ y = \sin\theta$$



$$H = 2 - \cos\theta + \frac{\cos\theta}{\cos^2\theta + \sin^2\theta}$$

$$H = 2 - \cos\theta + \cos\theta$$

$$H = 2$$



$$z = e^{w} = e^{u} \cdot e^{iv}$$

$$= e^{u} \cos v + i e^{u} \cdot \sin v$$

$$x = e^{u} \cdot \cos v$$

$$y = e^{u} \cdot \sin v$$

$$H = 2 - e^{u} \cdot \cos v + \frac{e^{u} \cdot \cos v}{e^{2u} \cdot (\cos^2 v + \sin^2 v)}$$

$$H = 2 - e^{u} \cos v + \frac{\cos v}{e^{u}} \qquad \text{if } u = 0 \Rightarrow H = 2$$

then $\quad z = e^{iv} = \cos v + i \cdot \sin v$

13. If $\sin \alpha \neq 0$ in Exercise 12, show that the speed of the fluid along the line segment is infinite at the ends $z = \pm 2$ and equal to $A\,|\cos \alpha|$ at the mid-point.

14. For the sake of simplicity suppose that $0 < \alpha \leq \pi/2$ in Exercise 12. Then show that the velocity of the fluid along the upper side of the line segment representing the plate in Fig. 80 is zero at the point $x = 2 \cos \alpha$ and that the velocity along the lower side of the segment is zero at the point $x = -2 \cos \alpha$.

15. A circle with its center at a point x_0 on the x axis, where $0 < x_0 < 1$, and passing through the point $z = -1$, is subjected to the

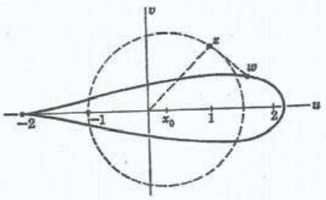


FIG. 81

transformation $w = z + 1/z$. Individual points $z = r \exp (i\theta)$ can be mapped geometrically by adding the vector $r^{-1} \exp (-i\theta)$ to the vector z. Indicate by mapping some points that the image of the circle is a profile of the type shown in Fig. 81 and that points exterior to the circle map into points exterior to the profile. This is a special case of the profile of a *Joukowski airfoil*. (See also Exercises 16 and 17 below.)

16. (*a*) Show that the mapping of the circle in Exercise 15 is con-