Interpolation

FRAME 1

Introduction and Linear Interpolation

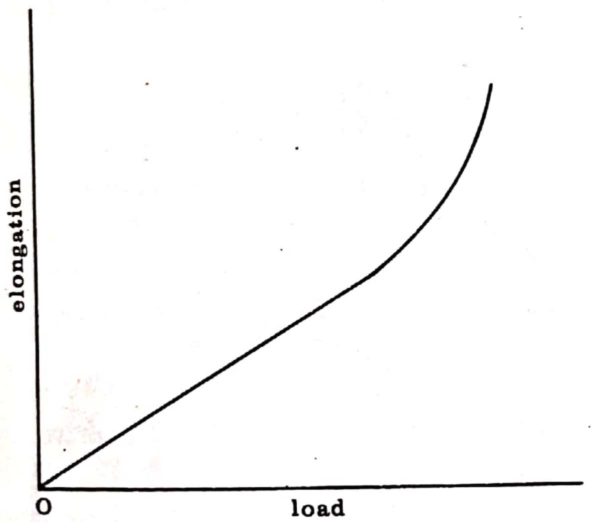
You may already have met the idea of interpolation. In case you haven't, just a few words as to its meaning. Suppose the length of a wire has been measured for various loads suspended from it. The following table might result:

Load (kg)	0	1	2	3	4	5
Length (mm)	2027·1	2029·4	2031·8	2034·1	2036·5	2039·0

One could then ask – what would be the length of the wire if the load is 2·7, 4·1 or 6·3 kg? The first two of these loads lie within the range of the table. Finding the length of the wire for either of these loads, or for any other non-tabulated load between 0 and 5 kg, from the information given is known as INTERPOLATION. Finding the extension for a load outside the range of the table is known as EXTRAPOLATION. Unfortunately, extrapolation can be a dangerous process. Do you know, or if not, can you guess, why?

****

1A



The law connecting the variables may only apply within certain limits. For example, Hooke's law that the elongation of a wire is proportional to the load to which it is subjected only applies provided the elastic limit has not been reached. After that the material begins to flow and the graph of elongation against load curves sharply upwards, as is shown in the diagram.

FRAME 2

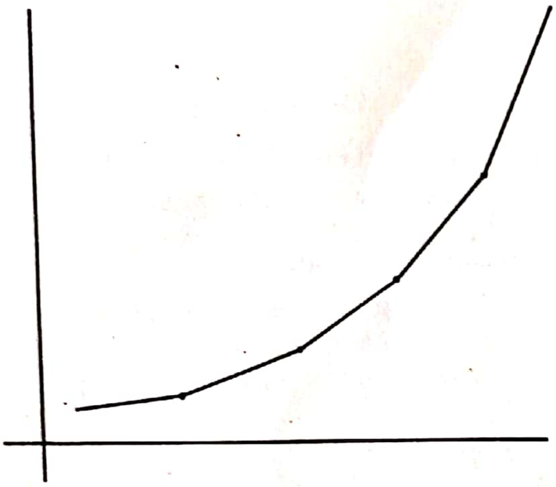
When a set of numerical data is given as in the last frame, it may be possible to find an analytical law fitting the data reasonably well – we have already seen how to do this by least squares. This law can then be used to estimate the value of the dependent variable for a value of the independent variable not given in the table. However, in this programme we shall be concerned with the problem of interpolating when the form of the curve passing through a series of points is not known. But, for illustration purposes, some examples will be taken where the analytical formula is known. Doing this will enable us to compare the results obtained by purely numerical methods with those given by analytical means.

In pre-computer days interpolation was also useful for filling in intermediate values for tables of functions whose analytical forms were known. This use became of relatively little importance in early computer days when memory facilities were very limited. Then the values of

FRAME 2 (continued)

functions were more often calculated directly. Now this trend is being reversed with the advent of larger and less expensive computer memories.

FRAME 3



The simplest method of interpolation is that known as LINEAR INTERPOLATION. This simply joins any two consecutive points in a table by a straight line. If a series of points are given, then a series of straight lines are used, one such line joining each pair of consecutive points, as shown in the diagram.

The process can be illustrated by taking, for example, the figures given on page 1:2 in our Volume 1 of "Mathematics for Engineers and Scientists" for the cooling of a particular hot body, i.e.,

t	0	120	240	360	480	600	720
θ	100	86	74	64	56	49	44

where t, the time, is in seconds and θ, the temperature, in degrees Celsius. From these figures, we can find the values of θ at, say, $t = 144, 168, 192$ and 216, as follows:

During the 120 second interval from $t = 120$ to $t = 240$, θ falls by 12°. In 24 seconds it therefore falls by $2{\cdot}4$ degrees. The temperatures at the required times are thus $83{\cdot}6$, $81{\cdot}2$, $78{\cdot}8$ and $76{\cdot}4$.

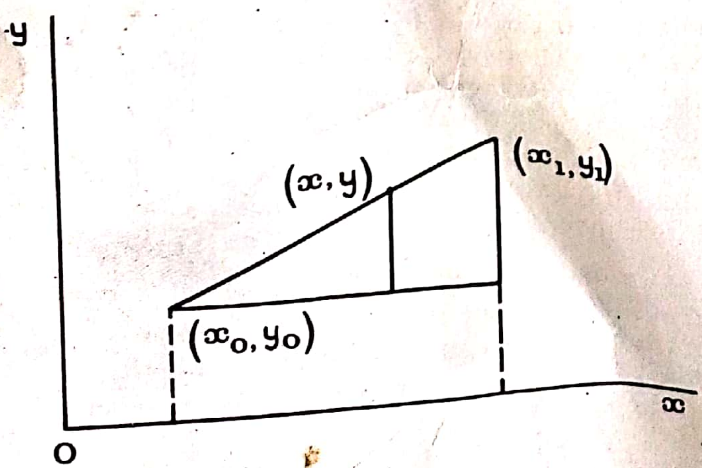
Using this method, what will be the values of θ at $t = 50, 75$, and at $t = 400, 420$? **

3A

Between $t = 0$ *and* $t = 120$, θ *falls by* 14°. *In* 50 s *it falls by* $\frac{50}{120} \times 14 = 5\frac{5}{6}$ *and in* 75 s *by* $8\frac{3}{4}$. *Required values are* $94\frac{1}{6}$, $91\frac{1}{4}$.

Similarly, when $t = 400$, $\theta = 61\frac{1}{3}$ *and when* $t = 420$, $\theta = 60$.

FRAME 4



If the two given points are (x_0, y_0) and (x_1, y_1) it is a simple matter to express the connection between x and y by a mathematical formula. This connection is obviously the equation of the straight line joining (x_0, y_0) and (x_1, y_1), i.e.,

FRAME 4 (continued)

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} \quad \text{or} \quad y = y_0 + \frac{x - x_0}{x_1 - x_0}(y_1 - y_0)$$

FRAME 5

For our present purposes, it will be instructive to link up this formula with the notation that was used in the previous programme. There, the general value of x was denoted by x_p and so, using y_p instead of y,

$$y_p = y_0 + \frac{x_p - x_0}{x_1 - x_0}(y_1 - y_0) \qquad (5.1)$$

Also we wrote $x_p = x_0 + ph$ where $h = x_1 - x_0$, and $y_1 - y_0 = \Delta y_0$.

With this notation (5.1) becomes

$$y_p = y_0 + p\Delta y_0 \qquad (5.2)$$

As, in FRAME 4, x was taken as lying between x_0 and x_1, p will lie between 0 and 1.

Now you remember that, when working with differences, you can choose any value of x in the table to be labelled x_0. Returning now to the example you did at the end of FRAME 3, what values of t would you take to be t_0 and what would be the corresponding values of $\Delta\theta_0$ and p when finding θ for $t = 50, 75, 400, 420$? Keep p to be within the range 0 to 1.

****

5A

Here, $t_p = t_0 + ph$ *and* $h = 120$.

For $t = 50$, *take* $t_0 = 0$ *and then* $\Delta\theta_0 = -14$, $p = \dfrac{t_p - t_0}{h} = \dfrac{5}{12}$.

For $t = 75$, *take* $t_0 = 0$ *and then* $\Delta\theta_0 = -14$, $p = \dfrac{5}{8}$.

For $t = 400$, *take* $t_0 = 360$ *and then* $\Delta\theta_0 = -8$, $p = \dfrac{1}{3}$.

For $t = 420$, *take* $t_0 = 360$ *and then* $\Delta\theta_0 = -8$, $p = \dfrac{1}{2}$.

FRAME 6

The Newton-Gregory Forward Difference Interpolation Formula

The method of interpolation adopted in the last few frames assumed that two adjacent points in a table were joined, when plotted, by a straight line. In practice this is very seldom the case and to obtain a more accurate estimate of y for intermediate values of x, it is necessary to take into account the curvature of the graph between the two points. But this immediately raises the difficulty that we have, as a rule in this sort of work, no information about this curvature. For example, suppose we are given a table of four pairs of values of x and y, which, when plotted, give Fig. (i). Fig. (ii) shows just three of the many curves on which the four points might lie. So, in order to proceed, it is necessary to guess which is the most likely form. Fortunately, in practice, things usually behave fairly well unless some upset occurs.

Fig (i)

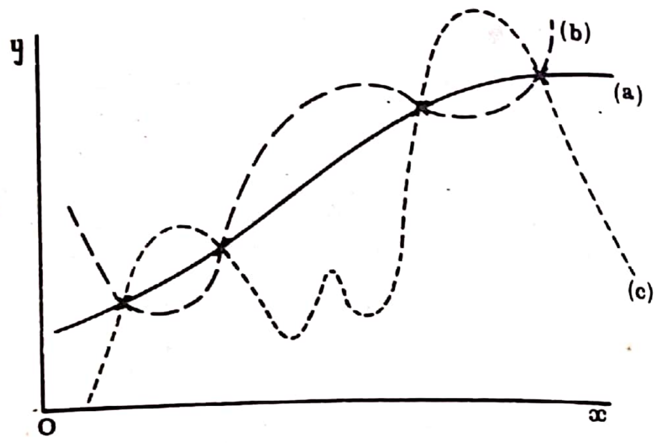
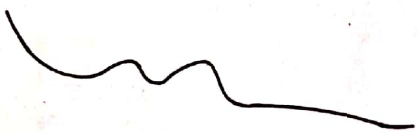


Fig (ii)

For example, in the case of the cooling of a hot body, cooling takes place continuously until the temperature of the surroundings is reached, and the rate of cooling decreases as room temperature is approached. This means that the temperature-time graph is of the form

and unless something else occurs, will not look like



or

We thus assume that, in the absence of any more information, (a) in Fig. (ii) is more likely than (b) or (c).

FRAME 7

Many formulae which take into account the curvature of the graph have been evolved. One of the simplest of these is the NEWTON-GREGORY (sometimes, alternatively, Gregory-Newton, or even just Newton) FORWARD DIFFERENCE FORMULA. It makes no assumptions as to the actual form of the curve, but does assume that there is a polynomial function whose graph is not too far removed from the true curve.

In FRAME 36, page 198, it was seen that f_p, the value of a function f when x takes the value $x_0 + ph$, can be written as $E^p f_0$. Also, in FRAME 28, page 195, E was found to be equivalent to $1 + \Delta$.

Thus $f_p = E^p f_0 = (1 + \Delta)^p f_0$.

The next step is to expand $(1 + \Delta)^p$ by the binomial theorem, giving

$$(1 + \Delta)^p = 1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\ldots\ldots\ldots$$

or, more shortly,

$$(1 + \Delta)^p = 1 + \binom{p}{1}\Delta + \binom{p}{2}\Delta^2 + \binom{p}{3}\Delta^3 + \ldots\ldots\ldots\ldots\ldots\ldots$$

FRAME 7 (continued)

Then, $f_p = \{1 + \binom{p}{1}\Delta + \binom{p}{2}\Delta^2 + \binom{p}{3}\Delta^3 + \ldots\ldots\ldots\ldots\}f_0$

i.e. $$f_p = f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ldots\ldots\ldots\ldots \quad (7.1)$$

and this is the required formula. You will notice that the first two terms on the R.H.S. are equivalent to the linear interpolation formula (5.2). Further, if the formula is truncated or, as may happen, automatically stops, at any other point, a polynomial in p results, e.g., if the last term included is $\binom{p}{3}\Delta^3 f_0$, we have a polynomial of degree 3. [If you haven't met the notation $\binom{p}{r}$ before, this simply stands for $\frac{p(p-1)(p-2)\ldots\ldots\ldots(p-r+1)}{r!}$. This is the same formula that you have for ${}^pC_r$. But when dealing with permutations and combinations, ${}^pC_r$ only makes sense if p is a positive integer. But, whatever the value of p, it is still possible to evaluate $\frac{p(p-1)(p-2)\ldots.(p-r+1)}{r!}$ and it is convenient to have a symbol for this expression. Note that, in it, the number of terms is the same in both numerator and denominator.]

FRAME 8

As a very simple example of this formula, let us find f(2·31) from the table

x	0	1	2	3	4	5	6
f(x)	2	3	10	29	66	127	218

Start by suggesting what value for x_0 will make $0 \leqslant p < 1$, stating the value of p and forming a difference table.

****

8A

$x_0 = 2, \quad p = 0{\cdot}31$

x_{-2}	0	2					
			1				
				6			
x_{-1}	1	3			6		
			7			0	
				12			
x_0	2	$\underline{10}$			6		
			$\underline{19}$			0	
				$\underline{18}$			
x_1	3	29			$\underline{6}$		
			37			$\underline{0}$	
				24			
x_2	4	66			6		
			61				
				30			
x_3	5	127					
			91				
x_4	6	218					

(The reason for the underlining is given in the next frame.)

FRAME 9

If you compare the table in 8A with the first one in FRAME 24, page 192, you will see that $f_0 = 10$, $\Delta f_0 = 19$, $\Delta^2 f_0 = 18$, $\Delta^3 f_0 = 6$, $\Delta^4 f_0 = 0$. The higher differences cannot be stated from the table, but you will

FRAME 9 (continued)

recognise the table as that of a cubic, for which all these higher differences are zero. The values just stated are the quantities underlined in 8A. It is a good idea to indicate in this way which entries in your table you are going to use. Now, substituting into (7.1) gives

$$f(2\cdot31) = 10 + \binom{0\cdot31}{1}19 + \binom{0\cdot31}{2}18 + \binom{0\cdot31}{3}6$$

$$= 10 + \frac{0\cdot31}{1}\times 19 + \frac{0\cdot31\times(-0\cdot69)}{2}\times 18 + \frac{0\cdot31\times(-0\cdot69)\times(-1\cdot69)}{6}\times 6$$

$$= 10 + 5\cdot89 - 1\cdot9251 + 0\cdot361\,491 = 14\cdot326\,391$$

FRAME 10

The actual cubic used here was $f(x) = x^3 + 2$ and, when $x = 2\cdot31$, this gives $14\cdot326\,391$ and so, in this case, the value obtained from the formula (7.1) is exact. This is because no term which does not give a non-zero contribution has been omitted.

Now, by the same method, find $f(1\cdot6)$ and check that your result is the same as $1\cdot6^3 + 2$.

****

10A

Taking x_0 as 1 and $p = 0\cdot6$ gives

$$f(1\cdot6) = 3 + \binom{0\cdot6}{1}7 + \binom{0\cdot6}{2}12 + \binom{0\cdot6}{3}6$$

$$= 3 + 0\cdot6\times7 + 0\cdot6\times(-0\cdot4)\times6 + 0\cdot6\times(-0\cdot4)\times(-1\cdot4) = 6\cdot096$$

which agrees with $1\cdot6^3 + 2$.

FRAME 11

Taking now $x_0 = 1$, what will be the expression giving f_p for any value of p?

****

11A

$$3 + \binom{p}{1}7 + \binom{p}{2}12 + \binom{p}{3}6$$

FRAME 12

This can also be written as

$$3 + 7p + 6p(p - 1) + p(p - 1)(p - 2) \qquad (12.1)$$

and so is a cubic in p. But $x_p = x_0 + ph$ and so, here, $p = x_p - 1$ as $x_0 = 1$ and $h = 1$. (12.1) can thus be written as

$$3 + 7(x_p - 1) + 6(x_p - 1)(x_p - 2) + (x_p - 1)(x_p - 2)(x_p - 3) = x_p^3 + 2$$

which is the function from which the original table was formed. This means that, for this example, the correct result will always be obtained whatever value is given to p. It is thus not necessary in this case to restrict p to the range 0 to 1 although in practice this is usually done.

FRAME 13

If a difference table is formed from any other polynomial, it will be found that this interpolation formula, i.e., (7·1) is always exact provided that all the decimal places are kept. Furthermore, when a process similar to that in FRAME 12 is done, the original polynomial will always be recovered.

FRAME 14

More often than not, when a difference table is being formed, the values of f will not be those of a polynomial. The table in 11A, page 185, is an example. From the values in this table, find f(0·12) using the Newton-Gregory formula. In doing so, remember what was said in FRAME 10, page 185, about the use of the higher differences when there are round-off errors present. Give your answer to 4 decimal places.

****

14A

Taking $x_0 = 0{\cdot}10$, $p = \dfrac{0{\cdot}12 - 0{\cdot}10}{0{\cdot}05} = 0{\cdot}4$ *as* $h = 0{\cdot}05$.

$$f(0{\cdot}12) = 0{\cdot}0998 + \binom{0{\cdot}4}{1}\times 0{\cdot}0496 + \binom{0{\cdot}4}{2}\times(-0{\cdot}0003) + \binom{0{\cdot}4}{3}\times(-0{\cdot}0004) + \ldots$$

$$\simeq 0{\cdot}0998 + 0{\cdot}019\,84 + 0{\cdot}000\,036 - 0{\cdot}000\,0256 = 0{\cdot}119\,6504 = 0{\cdot}1197$$

Obviously one could take more terms in the interpolation formula but as the higher differences are unreliable, it is not only pointless to do so, but can also lead to an answer that is less accurate than that quoted.

FRAME 15

The differences used in the Newton-Gregory forward interpolation formula lie on a diagonal line sloping downwards to the right. Can you suggest when this would be inconvenient?

****

15A

If x_0 *is near the end of the table, the differences required will not be available.*

FRAME 16

The Newton-Gregory Backward Difference Formula

The difficulty in FRAME 15 would be overcome if it could be arranged that the differences required were on a diagonal line sloping upwards to the right. This is accomplished by the use of the NEWTON-GREGORY BACKWARD DIFFERENCE FORMULA. As before $f_p = E^p f_0$ but now E is eliminated by using the formula $E = (1 - \nabla)^{-1}$. (Derived in 31A, page 196.)

Thus $f_p = (1 - \nabla)^{-p} f_0$.

Expand the R.H.S. of this as far as the term in ∇^3.

****

16A

$$\{1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\ldots\ldots\ldots\}f_0$$

$$= f_0 + p\nabla f_0 + \frac{(p+1)p}{2!}\nabla^2 f_0 + \frac{(p+2)(p+1)p}{3!}\nabla^3 f_0 + \ldots\ldots\ldots\ldots$$

FRAME 17

The formula can thus be written as

$$f_p = f_0 + \binom{p}{1}\nabla f_0 + \binom{p+1}{2}\nabla^2 f_0 + \binom{p+2}{3}\nabla^3 f_0 + \ldots\ldots\ldots$$

If only the first few terms in this are used, it, like the forward difference formula, becomes a polynomial in p.

As an example, let us find f(4·2) from the table in 8A. x_0 can be taken as 4, then p = 0·2 and the entries used will be

6

18

37

66

(Note: backward differences of f_0 lie on a diagonal line sloping upwards to the right.)

and so $f(4\cdot 2) = 66 + \binom{0\cdot 2}{1}37 + \binom{1\cdot 2}{2}18 + \binom{2\cdot 2}{3}6$

$= 66 + 7\cdot 4 + 2\cdot 16 + 0\cdot 528 = 76\cdot 088$

which agrees with $(4\cdot 2)^3 + 2$.

Now use this formula to find f(0·43) from the table of values in 11A, page 185.

****

17A

$h = 0\cdot 05, \quad p = 0\cdot 6$

$f(0\cdot 43) = 0\cdot 3894 + \binom{0\cdot 6}{1} \times 0\cdot 0466 + \binom{1\cdot 6}{2} \times (-0\cdot 0007) + \binom{2\cdot 6}{3} \times 0\cdot 0002$

$= 0\cdot 3894 + 0\cdot 027\,96 - 0\cdot 000\,34 + 0\cdot 000\,08 \simeq 0\cdot 4171$

FRAME 18

Other Finite Difference Interpolation Formulae

There are other formulae besides the two already dealt with. We shall not go into the derivation of these here but simply quote some of them and do some examples. Notice that, in each of them, if the formula is truncated, a polynomial in p results.

BESSEL'S FORMULA is

$$f_p = \mu f_{\frac{1}{2}} + (p - \tfrac{1}{2})\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\mu\delta^2 f_{\frac{1}{2}} + \frac{p(p-1)(p-\frac{1}{2})}{3!}\delta^3 f_{\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 f_{\frac{1}{2}} + \ldots\ldots\ldots\ldots$$

STIRLING'S FORMULA is

$$f_p = f_0 + p\mu\delta f_0 + \frac{p^2}{2!}\delta^2 f_0 + \frac{p(p^2-1)}{3!}\mu\delta^3 f_0 + \frac{p^2(p^2-1)}{4!}\delta^4 f_0 + \ldots\ldots$$

EVERETT'S FORMULA is

$$f_p = \binom{q}{1}f_0 + \binom{q+1}{3}\delta^2 f_0 + \binom{q+2}{5}\delta^4 f_0 + \ldots\ldots\ldots\ldots\ldots + \binom{p}{1}f_1 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1 + \ldots\ldots\ldots\ldots\ldots$$

where q = 1 − p. Also, as $\binom{q+1}{3} = \frac{(q+1)q(q-1)}{3!} = \frac{q(q^2-1)}{3!}$ and

FRAME 18 (continued)

similarly $\binom{q+2}{5} = \dfrac{q(q^2-1)(q^2-4)}{5!}$, the formula can be expressed in the alternative form

$$f_p = qf_0 + \frac{q(q^2-1)}{3!}\delta^2 f_0 + \frac{q(q^2-1)(q^2-4)}{5!}\delta^4 f_0 + \ldots\ldots\ldots\ldots$$
$$+ pf_1 + \frac{p(p^2-1)}{3!}\delta^2 f_1 + \frac{p(p^2-1)(p^2-4)}{5!}\delta^4 f_1 + \ldots\ldots\ldots\ldots$$

which you may prefer.

All of these formulae are central difference formulae as they are expressed in terms of δ rather than Δ or ∇, although it is possible to use Δ or ∇. We don't suggest that you try to learn these formulae. You will also find that they are not always quoted by different authors in exactly the same form.

FRAME 19

The main point about using any of these formulae is that you should be able to interpret the various δ expressions in terms of the entries in your difference table. For your convenience we repeat here the table in FRAME 34 on page 197.

x_{-2}	f_{-2}				
		$\delta f_{-1\frac{1}{2}}$			
			$\delta^2 f_{-1}$		
x_{-1}	f_{-1}			$\delta^3 f_{-\frac{1}{2}}$	
		$\delta f_{-\frac{1}{2}}$			$\delta^4 f_0$
x_0	f_0		$\delta^2 f_0$		
		$\delta f_{\frac{1}{2}}$		$\delta^3 f_{\frac{1}{2}}$	
			$\delta^2 f_1$		$\delta^4 f_1$
x_1	f_1				
		$\delta f_{1\frac{1}{2}}$		$\delta^3 f_{1\frac{1}{2}}$	
			$\delta^2 f_2$		$\delta^4 f_2$
x_2	f_2				
		$\delta f_{2\frac{1}{2}}$		$\delta^3 f_{2\frac{1}{2}}$	
			$\delta^2 f_3$		
x_3	f_3				
		$\delta f_{3\frac{1}{2}}$			
x_4	f_4				

As an example on the formulae in the last frame, let us find $f(2{\cdot}36)$ from the following table:

x	1·6	1·8	2·0	2·2	2·4	2·6	2·8	3·0
$f(x)$	0·0495	0·0605	0·0739	0·0903	0·1102	0·1346	0·1644	0·2009

Which value of x would you label x_0 and what will then be the values of p and q?

** 19A

$x_0 = 2{\cdot}2$, $p = 0{\cdot}8$, $q = 0{\cdot}2$ (Note: $h = 0{\cdot}2$)

FRAME 20

Now form the difference table for the values given in the last frame and then state the values of the various δ quantities required for each of the three formulae in FRAME 18. State also which is the last column of differences you would use.

****

	1·6	0·0495						
			110					
	1·8	0·0605		24				
			134		6			
	2·0	0·0739		30		−1		
			164		5		6	
x_0	2·2	0·0903		35		5		−12
			199		10		−6	
x_1	2·4	0·1102		45		−1		11
			244		9		5	
	2·6	0·1346		54		4		
			298		13			
	2·8	0·1644		67				
			365					
	3·0	0·2009						

Bessel: $\mu f_{\frac{1}{2}} = 0{\cdot}100\,25$, $\delta f_{\frac{1}{2}} = 0{\cdot}0199$, $\mu\delta^2 f_{\frac{1}{2}} = 0{\cdot}0040$, $\delta^3 f_{\frac{1}{2}} = 0{\cdot}0010$, $\mu\delta^4 f_{\frac{1}{2}} = 0{\cdot}0002$

Stirling: $f_0 = 0{\cdot}0903$, $\mu\delta f_0 = 0{\cdot}018\,15$, $\delta^2 f_0 = 0{\cdot}0035$, $\mu\delta^3 f_0 = 0{\cdot}000\,75$, $\delta^4 f_0 = 0{\cdot}0005$

Everett: $f_0 = 0{\cdot}0903$, $\delta^2 f_0 = 0{\cdot}0035$, $\delta^4 f_0 = 0{\cdot}0005$
$f_1 = 0{\cdot}1102$, $\delta^2 f_1 = 0{\cdot}0045$, $\delta^4 f_1 = -0{\cdot}0001$

(Remember that all the entries in the difference columns have to be multiplied by 10^{-4}.)

Up to the 4th differences (no more can be relied upon).

FRAME 21

Bessel's formula now gives

$$0{\cdot}100\,25 + 0{\cdot}3 \times 0{\cdot}0199 + \frac{0{\cdot}8 \times (-0{\cdot}2)}{2} \times 0{\cdot}0040 + \frac{0{\cdot}8 \times (-0{\cdot}2) \times 0{\cdot}3}{6} \times 0{\cdot}0010$$
$$+ \frac{1{\cdot}8 \times 0{\cdot}8 \times (-0{\cdot}2) \times (-1{\cdot}2)}{24} \times 0{\cdot}0002 \simeq 0{\cdot}1059$$

Try working out the value obtained by either of the other two formulae.

****

21A

Stirling's formula gives:

$$0{\cdot}0903 + 0{\cdot}8 \times 0{\cdot}018\,15 + \frac{0{\cdot}8^2}{2} \times 0{\cdot}0035$$
$$+ \frac{0{\cdot}8(0{\cdot}8^2 - 1)}{6} \times 0{\cdot}000\,75 + \frac{0{\cdot}8^2(0{\cdot}8^2 - 1)}{24} \times 0{\cdot}0005 \simeq 0{\cdot}1059$$

Everett's formula gives:

$$0{\cdot}2 \times 0{\cdot}0903 + \frac{1{\cdot}2 \times 0{\cdot}2 \times (-0{\cdot}8)}{6} \times 0{\cdot}0035$$
$$+ \frac{2{\cdot}2 \times 1{\cdot}2 \times 0{\cdot}2 \times (-0{\cdot}8) \times (-1{\cdot}8)}{120} \times 0{\cdot}0005$$

21A (continued)

$$+\ 0{\cdot}8\times 0{\cdot}1102 + \frac{1{\cdot}8\times 0{\cdot}8\times(-0{\cdot}2)}{6}\times 0{\cdot}0045$$
$$+\ \frac{2{\cdot}8\times 1{\cdot}8\times 0{\cdot}8\times(-0{\cdot}2)\times(-1{\cdot}2)}{120}\times(-0{\cdot}0001) \simeq 0{\cdot}1059$$

FRAME 22

You have seen that several formulae are available for interpolation, all doing more or less the same job. Which one is it best to use in any particular case? In FRAME 15, you saw that if it is necessary to interpolate with x_0 near the end of a table, then the Newton-Gregory forward difference formula is not suitable as the required differences are not available. Similarly the Newton-Gregory backward difference formula would not be used near the beginning of a table, nor would central difference formulae be used near the beginning or end of a table.

If the table is sufficiently long, there may be an overlap where more than one type of formula can be used. In such cases it is usually better to use one of the central difference formulae as these are likely to converge more quickly than the others. There are sometimes reasons for choosing one central difference formula in preference to another but we shall not go into these here. Because of its symmetry, Everett's is sometimes a popular choice. However, when $p = \frac{1}{2}$, one form is particularly suitable. Can you suggest which one it is?

****

22A

Bessel's is the obvious choice as all terms involving a factor $p - \frac{1}{2}$ will become zero.

Everett's is another reasonable choice, for, if $p = \frac{1}{2}$, then $q = p$, and the coefficients in the two rows become the same.

FRAME 23

One other point in favour of central difference formulae is that as they tend to converge more quickly, their use does not involve such high order differences as forward or backward difference formulae. Consequently they are not so susceptible to the effects of any round-off errors that may exist in the tabulated values.

FRAME 24

In working the example quoted in FRAME 19, only differences up to the fourth order were used, higher differences being too unreliable. You also found that some of the terms in the formulae were so small when you calculated their values as to be negligible. Generally, when using an interpolation formula, only the first few terms in it are necessary, the rest being either so small as to be negligible to the degree of accuracy to which you are working or unreliable because of round-off errors in the difference table. It is because successive terms in a formula become smaller in magnitude and eventually negligible, i.e., the series in the formula converges, that one can use the binomial theorem in FRAME 7.

FRAME 25

Unequal Intervals of Tabulation

All the work on interpolation done so far has depended on a function being tabulated at equal intervals of x. But this might not happen and so the question immediately arises: Can anything be done in this case?

First of all, let us look at some of the places where our work will break down. Right at the start, our table of finite differences will no longer behave nicely as it did before. To see this try forming a difference table for

x	0	1	3	5	8	9	11	15	18	20
f(x)	-4	-3	5	21	60	77	117	221	320	396

going as far as the fourth differences.

****

25A

		★			
0	*-4*				
		1			
1	*-3*		*7*		
		8		*1*	
3	*5*		*8*		*14*
		16		*15*	
5	*21*		*23*		*-60*
		39		*-45*	
8	*60*		*-22*		*90*
		17		*45*	
9	*77*		*23*		*-4*
		40		*41*	
11	*117*		*64*		*-110*
		104		*-69*	
15	*221*		*-5*		*51*
		99		*-18*	
18	*320*		*-23*		
		76			
20	*396*				

The equation connecting x and $f(x)$ was $f(x) = x^2 - 4$. But the column of second differences is not now constant, neither are all higher differences zero.

FRAME 26

Divided Differences

Another trouble that arises is that h is no longer constant, and this leads to difficulty with p.

Returning to the question of differences, you remember that forming a first difference is equivalent to the first part of the process of differentiation from first principles. (See FRAME 5, page 181 if you have forgotten this.) The next step in differentiation from first principles is to divide $f(x_0 + h) - f(x_0)$ by h. The corresponding process here is to divide each first difference by the difference between the two corresponding values of x. The result of this division is called a DIVIDED DIFFERENCE. Thus, going down the column headed by ★ in 25A, 1 is divided by 1 − 0, 8 by 3 − 1, 16 by 5 − 3, 39 by 8 − 5 and so on. The results form the first column of a table of divided differences. Just as the differences in the first column in an ordinary

FRAME 26 (continued)

table are known as first differences, so these values are called FIRST DIVIDED DIFFERENCES.

Work out this column for the function f(x) in FRAME 25.

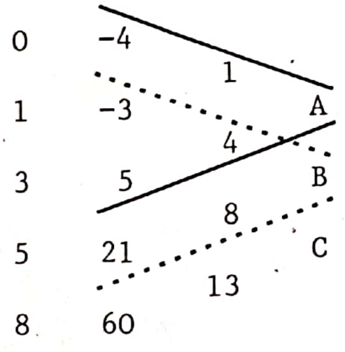
****

26A

0	*-4*	
		1
1	*-3*	
		4
3	*5*	
		8
5	*21*	
		13
8	*60*	
		17
9	*77*	
		20
11	*117*	
		26
15	*221*	
		33
18	*320*	
		38
20	*396*	

FRAME 27

The SECOND DIVIDED DIFFERENCES are formed in a somewhat similar manner.



To find the entry labelled A, a fan is taken from A back to the tabulated values. A is now the ordinary difference 4 - 1 divided by the difference of the two values of x corresponding to the front of the fan i.e. 3 - 0. Thus $A = \dfrac{4-1}{3-0} = 1$. Similarly $B = \dfrac{8-4}{5-1} = 1$ and $C = \dfrac{13-8}{8-3} = 1$. Continue in this way and so find the remainder of the entries in the second divided differences column.

****

27A

All entries in this column are 1.

FRAME 28

We now have the table shown on page 218. Third and higher divided differences are found in a similar way. Thus to find D a fan is constructed backwards to the functional values and then $D = \dfrac{1-1}{5-0} = 0$. Similarly $E = \dfrac{1-1}{8-1} = 0$ and so on. It is easy to see that in this example all the third and consequently higher divided differences are zero.

FRAME 28 (continued)

0	-4			
		1		
1	-3		1	D
		4		
3	5		1	E
		8		
5	21		1	
		13		
8	60		1	
		17		
9	77		1	
		20		
11	117		1	
		26		
15	221		1	
		33		
18	320		1	
		38		
20	396			

Now, form a divided differences table for

x	-2	-1	3	5	8	9	14	15	18	24
f(x)	-26	-14	34	226	994	1426	5446	6706	11 614	27 586

****

28A

-2	*-26*			
		12		
-1	*-14*		*0*	
		12		*2*
3	*34*		*14*	
		96		*2*
5	*226*		*32*	
		256		*2*
8	*994*		*44*	
		432		*2*
9	*1426*		*62*	
		804		*2*
14	*5446*		*76*	
		1260		*2*
15	*6706*		*94*	
		1636		*2*
18	*11 614*		*114*	
		2662		
24	*27 586*			

and all higher divided differences are zero.

FRAME 29

The table in the last frame was formed from the cubic $f(x) = 2x^3 - 2x - 14$ and you will have noticed that the <u>third</u> divided differences are constant. In the quadratic in FRAME 25, the <u>second</u> divided differences were constant. In a similar way, the <u>fourth</u> divided differences of a quartic are constant and so on. All differences after a constant column will, of course, be zero.

FRAME 30

When forming divided differences, it is not even necessary for the x's to be in numerical order. Take the order $x = 0, 3, 9, 5, 11, 18, 8, 20, 1, 15$ and construct the new divided differences table for the function given by the table in FRAME 25.

****

30A

0	-4		
		3	
3	5		1
		12	
9	77		1
		14	
5	21		1
		16	
11	117		1
		29	
18	320		1
		26	
8	60		1
		28	
20	396		1
		21	
1	-3		1
		16	
5	221		

FRAME 31

When we were considering ordinary differences, it was found useful to have a notation (actually three notations were used) for them. In a similar way it is useful to have a notation for divided differences. The following table shows one notation:

x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$		
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		$f(x_0, x_1, x_2, x_3, x_4)$
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$		$f(x_1, x_2, x_3, x_4, x_5)$
		$f(x_3, x_4)$		$f(x_2, x_3, x_4, x_5)$	
x_4	$f(x_4)$		$f(x_3, x_4, x_5)$		$f(x_2, x_3, x_4, x_5, x_6)$
		$f(x_4, x_5)$		$f(x_3, x_4, x_5, x_6)$	
x_5	$f(x_5)$		$f(x_4, x_5, x_6)$		
		$f(x_5, x_6)$			
x_6	$f(x_6)$				

As you know, ordinary differences can be expressed in terms of the functional values. For example, $\Delta f_2 = f_3 - f_2$ and $\delta^2 f_0 = f_1 - 2f_0 + f_{-1}$. Divided differences can be expressed in terms of functional values and the values of x. Thus, remembering the way in which divided differences are formed,

FRAME 31 (continued)

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (31.1)$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (31.2)$$

and so on for the other first divided differences.

What will be $f(x_3, x_4)$ and $f(x_5, x_6)$?

****

31A

$$f(x_3, x_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} \qquad f(x_5, x_6) = \frac{f(x_6) - f(x_5)}{x_6 - x_5}$$

FRAME 32

A little later on, you will see a pattern emerging for the values of divided differences. To fit in with this pattern, (31.1) can be written

as $f(x_0, x_1) = \dfrac{f(x_0)}{x_0 - x_1} + \dfrac{f(x_1)}{x_1 - x_0}$ and (31.2) as

$$f(x_1 , x_2) = \frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}$$

and similarly for the other first differences.

Turning now to second differences,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \quad (32.1)$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \quad \text{etc.}$$

What will be the formula for $f(x_2, x_3, x_4)$?

****

32A

$$\frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2}$$

FRAME 33

In (32.1), the formulae (31.2) and (31.1) can be used for the terms in the numerator. Doing this gives

$$f(x_0, x_1, x_2) = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

which can be arranged as

FRAME 33 (continued)

$$f(x_0, x_1, x_2) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \qquad (33.1)$$

Similarly,

$$f(x_1, x_2, x_3) = \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)}$$

FRAME 34

As an example of a third divided difference,

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

Substitute for the expressions in the numerator and obtain this in a form corresponding to that of (33.1) for $f(x_0, x_1, x_2)$.

****

34A

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$
$$+ \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

You should now be able to see the pattern that is emerging and how the first divided differences fit into it.

FRAME 35

What do you think will be the form for $f(x_0, x_1, x_2, \ldots\ldots, x_n)$?

****

35A

$$\frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ \ldots\ (x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ \ldots\ (x_1 - x_n)}$$
$$+ \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\ \ldots\ (x_2 - x_n)} + \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ \ldots\ (x_n - x_{n-1})} \qquad (35A.1)$$

FRAME 36

Lagrange's Interpolation Formula

It was pointed out in FRAME 6 that many curves can be drawn through a number of points. It was also suggested that in practical situations, where the points represent the results of an experiment say, they are more likely to lie on a simple curve than one that jumps about all over the place.

FRAME 36 (continued)

Now, if two points only are given, the algebraically simplest curve passing through them is a straight line, the equation of which is linear and contains two constants. Taking the equation as $f(x) = a_0 + a_1x$, these constants are a_0 and a_1. What do you think will be the algebraically simplest curve passing through three given points, assuming that the three points are not collinear?

****

36A

A parabola, whose equation is quadratic and is of the form $f(x) = a_0 + a_1x + a_2x^2$, *containing three constants* a_0, a_1, a_2.

FRAME 37

Similarly, if four points are given which do not lie on either a straight line or parabola, the simplest curve passing through them is a cubic whose equation is of the form $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, containing four constants. This argument can obviously be extended to $n + 1$ given points, in which case the simplest equation would be of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_nx^n$, containing $n + 1$ constants. The Lagrange interpolation formula enables us to write down this equation immediately, although not quite in this form.

FRAME 38

In FRAMES 28-29 it was seen that the second divided differences of a quadratic are constant and the third are zero, also that for a cubic the third divided differences are constant and the fourth zero. For a polynomial of degree n, the nth divided differences are constant and the $(n + 1)$th are zero. Thus the $(n + 1)$th divided differences of $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$ are zero.

What is the least number of functional values you must have in order to be able to find one $(n + 1)$th divided difference?

****

38A

$n + 2$

If you are given two functional values you can obtain one first difference, if you are given three values you can obtain two first differences and one second difference, and so on.

FRAME 39

Now a polynomial of degree n can be made to pass through $n + 1$ points. But for an $(n + 1)$th difference we require $n + 2$ points. For this extra point we take an unspecified value which we can call x and then $f(x)$, which is unknown and is what we are seeking, is the corresponding functional value. As the $(n + 1)$th divided difference is zero,

$$f(x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n, x) = 0 \qquad (39.1)$$

where $x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n$ are the x-values of the given $n + 1$ points.

What will (39.1) become when the L.H.S. is written in a form similar to (35A.1)?

****

$$\frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \;\ldots\; (x_0 - x_n)(x_0 - x)}$$

$$+ \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \;\ldots\; (x_1 - x_n)(x_1 - x)} + \ldots\ldots\ldots\ldots$$

$$+ \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \;\ldots\; (x_n - x_{n-1})(x_n - x)}$$

$$+ \frac{f(x)}{(x - x_0)(x - x_1) \;\ldots\; (x - x_n)} = 0$$

FRAME 40

This can be rewritten as

$$\frac{f(x)}{(x - x_0)(x - x_1) \;\ldots\; (x - x_n)} = \frac{f(x_0)}{(x - x_0)(x_0 - x_1)(x_0 - x_2) \;\ldots\; (x_0 - x_n)}$$

$$+ \frac{f(x_1)}{(x - x_1)(x_1 - x_0)(x_1 - x_2) \;\ldots\; (x_1 - x_n)} + \ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{f(x_n)}{(x - x_n)(x_n - x_0)(x_n - x_1) \;\ldots\; (x_n - x_{n-1})}$$

and this is the LAGRÀNGE INTERPOLATION FORMULA. x can now be given any desired value. Notice that although this formula has been derived to deal with the case of a function tabulated at unequal intervals of x, it can still be used even if the function is tabulated at equal intervals, as the result depends only on the values of x and $f(x)$ at the tabulated points.

The formula can alternatively be written as

$$f(x) = \frac{(x - x_1)(x - x_2) \;\ldots\ldots\; (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \;\ldots\; (x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2) \;\ldots\ldots\; (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \;\ldots\; (x_1 - x_n)} f(x_1) + \ldots\ldots\ldots\ldots$$

$$+ \frac{(x - x_0)(x - x_1) \;\ldots\ldots\; (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \;\ldots\; (x_n - x_{n-1})} f(x_n)$$

and in this form it is obvious that it is a polynomial of degree n.

FRAME 41

As an example, let us find the polynomial that will fit $(0, -4)$, $(1, -3)$ and $(3, 5)$, three points which do not lie on a straight line. As we are given three points, $n + 1 = 3$, so $n = 2$ and the polynomial, which will be of degree 2, is given by

FRAME 41 (continued)

$$\frac{f(x)}{(x-0)(x-1)(x-3)} = \frac{-4}{(x-0)(0-1)(0-3)} + \frac{-3}{(x-1)(1-0)(1-3)} + \frac{5}{(x-3)(3-0)(3-1)}$$

As you can verify, this reduces to $f(x) = x^2 - 4$. The points were actually the first three from the table in FRAME 25.

Now find the polynomial of degree three that will fit the points $(-1, -3)$, $(0, -1)$, $(1, 1)$ and $(3, 29)$.

****

41A

$$\frac{f(x)}{(x+1)(x-0)(x-1)(x-3)} = \frac{-3}{(x+1)(-1-0)(-1-1)(-1-3)} +$$

$$\frac{-1}{(x-0)(0+1)(0-1)(0-3)} + \frac{1}{(x-1)(1+1)(1-0)(1-3)} +$$

$$\frac{29}{(x-3)(3+1)(3-0)(3-1)}$$

which reduces to $f(x) = x^3 + x - 1$.

FRAME 42

If the value of $f(x)$ is required when $x = 2$, say, this can now be found immediately. If this only is required (i.e., the algebraic formula is not asked for first) then $f(2)$ can be found directly from

$$\frac{f(2)}{(2+1)(2-0)(2-1)(2-3)} = \frac{-3}{(2+1)(-1-0)(-1-1)(-1-3)} +$$

$$\frac{-1}{(2-0)(0+1)(0-1)(0-3)} + \frac{1}{(2-1)(1+1)(1-0)(1-3)} +$$

$$\frac{29}{(2-3)(3+1)(3-0)(3-1)}$$

giving $f(2) = 9$.

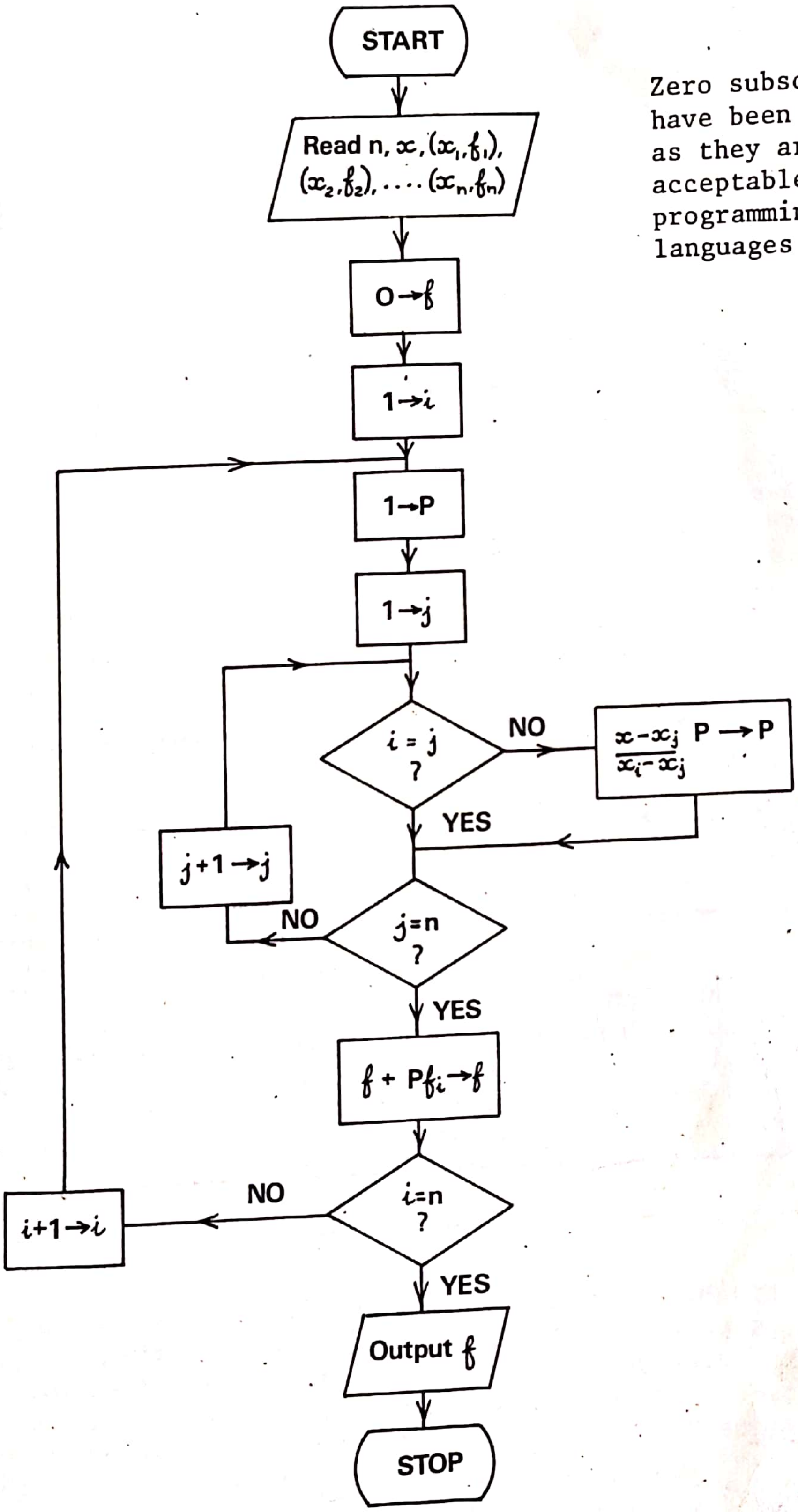
FRAME 43

A flow diagram for using Lagrangian Interpolation to find the value of a function $f(x)$ at a given value of x is shown on page 225.

FRAME 44

One other thing remains to be said about Lagrange's formula. Suppose the whole of the table in FRAME 25 were given. As there are 10 points, it is possible that the polynomial sought is of degree 9, although actually we know that it is only of degree 2. If you wish, you can proceed directly, using the formula with all 10 points, but this will entail an unnecessary amount of work. As there is always the possibility that a set of n points may lie on a curve of degree less than $n - 1$, it may be as well to test first to see if this is so. The method of testing is to construct a divided differences table. If this gives a constant column, then the order of this column is the degree of the polynomial sought. For example, the table in 28A indicates that the relevant polynomial is of

Flow diagram for FRAME 43.



Zero subscripts have been avoided as they are not acceptable in some programming languages.

[Programs using Lagrangian interpolation can be found in references (2), (3), (5), (7) and (9).]

FRAME 44 (continued)

degree 3 and so only the first, or any, four points need to be used when finding f(x).

Find f(2) for the simplest curve satisfying the points

x	0	1	-2	3	-3	6	10
f(x)	3	0	21	6	36	45	153

****

44A

The divided difference table

0	3		
		−3	
1	0		2
		−7	
−2	21		2
		−3	
3	6		2
		−5	
−3	36		2
		1	
6	45		2
		27	
10	153		

indicates that $f(x)$ *is of degree 2. Consequently only three points need be used, say* $(0, 3)$, $(1, 0)$ *and* $(3, 6)$. *Then*

$$\frac{f(2)}{2\times 1\times(-1)} = \frac{3}{2\times(-1)\times(-3)} + \frac{0}{1\times 1\times(-2)} + \frac{6}{-1\times 3\times 2}$$

from which $f(2) = 1$.

FRAME 45

Certain modifications to the Lagrange method have been suggested, for example, that due to Aitken. This will not be considered here, but, if you are interested, you will find it dealt with in several of the books on numerical work. One such book is "Computers and Computing" by J.M. Rushforth and J.Ll. Morris (John Wiley & Sons).

FRAME 46

A Word of Warning

In all our work on interpolation it has been assumed that, if y is a function of x, then a polynomial f(x) exists which is such that y = f(x) is an adequate representation of the curve on which the points lie. This may not be so. For example, there is no polynomial in x which adequately represents f(x) as given in the following table:

x	0	0·5	1	1·5	2	2·5	3·0
f(x)	0	0·8409	1	1·1066	1·1892	1·2574	1·3161

Form a difference table from these values and see what happens when you do so.

****

0	0					
		8409				
0·5	0·8409		−6818			
		1591		6293		
1·0	1·0000		−525		−6008	
		1066		285		5819
1·5	1·1066		−240		−189	
		826		96		142
2·0	1·1892		−144		−47	
		682		49		
2·5	1·2574		−95			
		587				
3·0	1·3161					

You will notice that there is no column containing only small differences.

FRAME 47

A difference table like this would give us the clue that it is not reasonable to try and proceed further. The values tabulated above were taken from the function $f(x) = x^{\frac{1}{4}}$ and here we have a situation somewhat analogous to that in analytical work if an attempt is made to obtain a Maclaurin series for $x^{\frac{1}{4}}$. If Lagrange's method is used there is no built in early warning system, because, although this method is based on divided differences, these do not have to be found.

FRAME 48

Inverse Interpolation

The process we have concentrated on so far is: Given a table of values of x and f(x), what is the value of f(x) at some other value of x? The inverse problem is: Given a table of values of x and f(x), what value of x will correspond to some other value of f(x)? The problem of finding such an x is known as INVERSE INTERPOLATION. As an example of this, one might be asked: From the table in FRAME 3, at what time would the temperature have been 70° C? As you might suspect from previous work, there are certain methods of dealing with a problem of this sort when the original table is given at unequal intervals of the independent variable and others which can only be used if the original table is an equal interval one.

FRAME 49

To take an example, suppose a curve passes through the points (0, −4), (0·6, −3·64), (1, −3) and it is necessary to find the value of x when y = −3·5.

You will notice that the intervals between successive values of x are unequal. One way in which we can proceed is to find y using the Lagrangian interpolation formula in FRAME 40, put the result equal to −3·5 and solve the equation so formed for x.

What will you get if you do it this way?

****

49A

$$\frac{y}{x(x-0\cdot6)(x-1)} = \frac{-4}{x(-0\cdot6)(-1)} + \frac{-3\cdot64}{(x-0\cdot6)\,0\cdot6\,(-0\cdot4)} + \frac{-3}{(x-1)0\cdot4},$$

i.e., $y = x^2 - 4$ (49A.1)

$x^2 - 4 = -3\cdot5$ *gives* $x \simeq \pm0\cdot71$. *To keep* x *within the range* 0 *to* 1, *i.e., within the range of the given three points, take* $x = 0\cdot71$.

FRAME 50

If f(x) is of higher degree than two, then the Newton-Raphson process can be used to solve the equation obtained for x.

A second method is to interchange the roles of x and y. This means that an equation is found for x as a polynomial function of y. Once this has been done, the given value of y can be inserted into it to give the required value of x. These two steps can be combined into one as in FRAME 42, but in order that you can see fully what is involved, this will not be done in this example.

Use the three points (0, -4), (0•6, -3•64), (1, -3) to obtain a Lagrangian interpolation formula for x in terms of y.

****

50A

$$\frac{x}{(y+4)(y+3\cdot64)(y+3)} = \frac{0}{(y+4)(-0\cdot36)(-1)} + \frac{0\cdot6}{(y+3\cdot64)0\cdot36(-0\cdot64)} + \frac{1}{(y+3)0\cdot64}$$

$x = -1\cdot041\,67y^2 - 6\cdot291\,67y - 8\cdot5$ (50A.1)

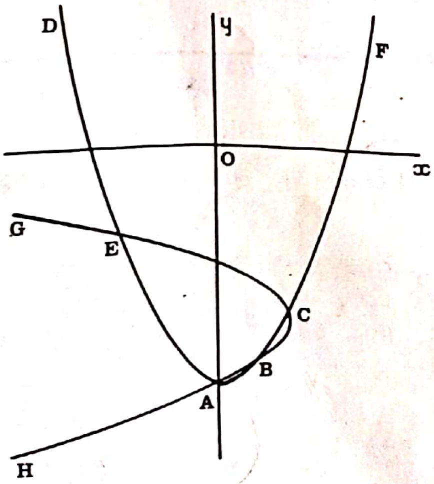
FRAME 51

When y = -3•5, this gives x ≃ 0•76, which is not the same as that obtained before. So the question now is: Why the difference?

Can you spot the reason? [Don't worry if you can't, it is a bit tricky. As a hint, it is suggested that you consider sketches of the two curves represented by (49A.1) and (50A.1).]

****

51A



In both 49A and 50A you found a parabola passing through the three given points, A, B and C. However the one found in 49A had its axis parallel to Oy (the parabola DEABCF) while that found in 50A had its axis parallel to Ox (GECBAH), and these two parabolas are not the same. For an intermediate value of y, they give different values for x.

FRAME 52

As only three points are known, and there is actually no information about what happens in between these points, one cannot be definite and say, with no shadow of doubt, which, if indeed either, should be taken. As we have based our direct interpolation on a polynomial of the form $y = f(x)$, it would seem logical to continue to use one of this form for inverse interpolation and so quote the answer as approximately $0{\cdot}71$.

Sometimes an answer can be immediately discarded. For example, in 41A you found a polynomial to fit the points $(-1, -3)$, $(0, -1)$, $(1, 1)$ and $(3, 29)$. Suppose you were required to find the value of x when $y = 20$, knowing in addition that, over the range given, y is increasing as x increases. The value you would find by the second method is approximately $6{\cdot}4$ and so you would immediately reject it under the conditions given.

FRAME 53

When the original table is given at equal intervals of x, both of the methods just given can be used, care being taken to reject any ridiculous answers. Alternatively, finite differences can now be used, the method, however, being basically similar to that in FRAME 49 and 49A. To see how it works, let us return to the difference table in 20A and seek the value of x for which $f(x) = 0{\cdot}08$. Obviously the value required will lie between $2{\cdot}0$ and $2{\cdot}2$. For this calculation, x_0 is now put equal to $2{\cdot}0$ and x_1, $2{\cdot}2$. Any convenient interpolation formula is now used, putting in it $f_p = 0{\cdot}08$, $f_0 = 0{\cdot}0739$, etc., but leaving p as p.

Suppose we use Bessel's formula. What equation will you get when you make these substitutions?

****

53A

$$0{\cdot}08 = 0{\cdot}0821 + (p - \tfrac{1}{2}) \times 0{\cdot}0164 + \frac{p(p-1)}{2} \times 0{\cdot}003\,25$$
$$+ \frac{p(p-1)(p-\frac{1}{2})}{6} \times 0{\cdot}0005 + \frac{(p+1)p(p-1)(p-2)}{24} \times 0{\cdot}0002,$$

stopping at the fourth differences as no more are reliable.

FRAME 54

This is now a quartic equation in p, but instead of solving it, as is possible, by Newton-Raphson, an alternative iterative technique is often used at this stage. By solving for the p in the term $(p - \frac{1}{2}) \times 0{\cdot}0164$, the quartic can be rewritten as

$$p = 0{\cdot}371\,95 - 0{\cdot}099\,09p(p-1) - 0{\cdot}005\,08p(p-1)(p-\tfrac{1}{2}) \quad (54.1)$$
$$- 0{\cdot}000\,51(p+1)p(p-1)(p-2)$$

A first estimate of p can be taken as $p_0 = 0{\cdot}372$, say (i.e. the constant term on the R.H.S.). A second estimate, p_1, is obtained by substituting this value of p into the R.H.S. Thus

$$\begin{aligned} p_1 &= 0{\cdot}371\,95 - 0{\cdot}099\,09 \times 0{\cdot}372 \times (-0{\cdot}628) \\ &\quad - 0{\cdot}005\,08 \times 0{\cdot}372 \times (-0{\cdot}628) \times (-0{\cdot}128) \\ &\quad - 0{\cdot}000\,51 \times 1{\cdot}372 \times 0{\cdot}372 \times (-0{\cdot}628) \times (-1{\cdot}628) \\ &= 0{\cdot}371\,95 + 0{\cdot}023\,15 - 0{\cdot}000\,15 - 0{\cdot}000\,27 = 0{\cdot}394\,68 \end{aligned}$$

FRAME 54 (continued)

A third estimate, p_2, is obtained by using this value of p_1 in the R.H.S. of (54.1). What do you get when you do this?

****

54A

0·395 23

FRAME 55

Using this value in the R.H.S. of (54.1) produces no further change in the estimate of p and so this gives the value we want.

Then, as $x = x_0 + ph$, $x = 2\cdot0 + 0\cdot395\,23 \times 0\cdot2 = 2\cdot079\,046$.

There is no reason why it has to be Bessel's formula which is used for this process. Repeat this example using Everett's formula (FRAME 18). (q must be replaced immediately by $1 - p$).

****

55A

$$0\cdot08 = (1-p)\times0\cdot0739 + \frac{(2-p)(1-p)(-p)}{6}\times 0\cdot0030$$
$$+\frac{(3-p)(2-p)(1-p)(-p)(-1-p)}{24}\times(-0\cdot0001) + p\times0\cdot0903$$
$$+\frac{(p+1)p(p-1)}{6}\times0\cdot0035 + \frac{(p+2)(p+1)p(p-1)(p-2)}{24}\times0\cdot0005$$

Solving for p from the two linear terms,

$$p = 0\cdot371\,95 + p(p-1)(p-2)0\cdot030\,49 - p(p^2-1)(p-2)(p-3)0\cdot0003$$
$$- p(p^2-1)0\cdot035\,57 - p(p^2-1)(p^2-4)0\cdot0013$$

Successive estimates for p are $0\cdot372$, $0\cdot394\,83$, $0\cdot395\,37$, $0\cdot395\,38$.
x is then $2\cdot079\,076$.

FRAME 56

In the last frame, the two values of p do not agree beyond the third decimal place. The main cause of this was the division by 0·0164, the first difference coefficient of p. (In 53A, 0·0164 appeared directly; in 55A it comes from 0·0903 − 0·0739.) As this division is equivalent to multiplying by approximately 61, it can cause quite a difference to the final answer.

FRAME 57

Summary of Interpolation Formulae

For your convenience, the various interpolation formulae mentioned in this programme are listed below:

Newton-Gregory Forward

$$f_p = f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ldots\ldots\ldots\ldots\ldots$$

Newton-Gregory Backward

$$f_p = f_0 + p\nabla f_0 + \binom{p+1}{2}\nabla^2 f_0 + \binom{p+2}{3}\nabla^3 f_0 + \ldots\ldots\ldots\ldots\ldots$$

FRAME 57 (continued)

Bessel

$$f_p = \mu f_{\frac{1}{2}} + (p - \tfrac{1}{2})\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\mu\delta^2 f_{\frac{1}{2}} + \frac{p(p-1)(p-\frac{1}{2})}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 f_{\frac{1}{2}} + \ldots\ldots\ldots\ldots$$

Stirling

$$f_p = f_0 + p\mu\delta f_0 + \frac{p^2}{2!}\delta^2 f_0 + \frac{p(p^2-1)}{3!}\mu\delta^3 f_0 + \frac{p^2(p^2-1)}{4!}\delta^4 f_0 + \ldots\ldots$$

Everett

$$f_p = \binom{q}{1} f_0 + \binom{q+1}{3}\delta^2 f_0 + \binom{q+2}{5}\delta^4 f_0 + \ldots\ldots\ldots\ldots\ldots \qquad (q = 1 - p)$$
$$+ \binom{p}{1} f_1 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1 + \ldots\ldots\ldots\ldots\ldots$$

Lagrange

$$\frac{f(x)}{(x-x_0)(x-x_1)\ \ldots\ (x-x_n)} = \frac{f(x_0)}{(x-x_0)(x_0-x_1)(x_0-x_2)\ \ldots\ (x_0-x_n)}$$
$$+ \frac{f(x_1)}{(x-x_1)(x_1-x_0)(x_1-x_2)\ \ldots\ldots\ (x_1-x_n)} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{f(x_n)}{(x-x_n)(x_n-x_0)(x_n-x_1)\ \ldots\ldots\ (x_n-x_{n-1})}$$

FRAME 58

Miscellaneous Examples

In this frame a collection of miscellaneous examples is given for you to try. Answers are provided in FRAME 59, together with such working as is considered helpful.

Note: The data from some of these questions will be used in subsequent programmes. You will find it helpful then to be able to refer quickly to the difference tables that you will now be forming.

1. The following readings of current, i, against deflection, θ, were obtained for a certain galvanometer.

θ	0·40	0·45	0·50	0·55	0·60	0·65
i	1·268	1·449	1·639	1·839	2·052	2·281

What will be the current when $\theta = 0{\cdot}536$?

2. i) A polynomial $f(x)$ of low degree is tabulated as follows:

x	−3	−2	−1	0	1	2	3	4	5	6
$f(x)$	−14	−2	2	4	10	25	59	112	194	310

Errors in two values of $f(x)$ are suspected. Locate and correct them.

ii) Derive the Newton interpolation formulae:

a) $f_p = f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ldots\ldots\ldots\ldots$

b) $f_p = f_0 + \binom{p}{1}\nabla f_0 + \binom{p+1}{2}\nabla^2 f_0 + \binom{p+2}{3}\nabla^3 f_0 + \ldots\ldots\ldots\ldots$

iii) Obtain the values of $f(x)$ as given in (i) after correction when $x = -4, -2{\cdot}5, 5{\cdot}5$. (L.U.)

3. A fourth degree polynomial is tabulated as follows:

x	0	0·1	0·2	0·3	0·4
y	1·0000	0·9208	0·6928	0·3448	-0·0752
x	0·5	0·6	0·7	0·8	0·9
y	-0·5000	-0·8452	-0·9992	-0·8432	-0·2312

Show from a difference table that there is an error and use the corrected table with the Stirling interpolation formula

$$f_p = f_0 + \tfrac{1}{2}p(\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}}) + \tfrac{1}{2}p^2\delta^2 f_0 + \frac{p(p^2-1)}{2(3!)}(\delta^3 f_{\frac{1}{2}} + \delta^3 f_{-\frac{1}{2}})$$
$$+ \frac{p^2(p^2-1)}{4!}\delta^4 f_0 + \ldots\ldots\ldots\ldots$$

to find the value of y when $x = 0{\cdot}45$. (L.U.)

(It was pointed out in FRAME 18 that interpolation formulae are not always quoted in exactly the same form. You should check that the form given in this question agrees with that in FRAME 18.)

4. The following table gives readings of the temperature (θ^{o}) recorded at given times (t):

t	0	1	2	3	4	5	6	7	8	9	10
θ	80·00	70·48	61·87	54·08	47·03	40·65	34·88	29·66	24·93	20·66	16·79

Using (i) Bessel's formula, (ii) Everett's formula, find θ at $t = 4{\cdot}3$. At what time would you expect θ to be 50?

5. The deflection, y, measured at various distances, x, from one end of a cantilever is given by

x	0·0	0·2	0·4	0·6	0·8	1·0
y	0·0000	0·0347	0·1173	0·2160	0·2987	0·3333

For what value of x is $y = 0{\cdot}2$?

6. Write down the Lagrange interpolation polynomial which fits the points $(0, y_0)$, $(1, y_1)$, $(2, y_2)$. Express y_1 and y_2 in terms of y_0 and its differences and hence obtain the Gregory-Newton polynomial.

Write down the Lagrange polynomial which fits the points (1, 4), (3, 7), (4, 8), (6, 11) and use it to interpolate values of y at $x = 2$ and $x = 5$. Check these values by differencing. (L.U.)

7. The following table gives values of the current i in a certain L,C,R circuit at various times t.