$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

In other words, the solution will exist (iteration will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX=B$ are greater than the sum of absolute values of the other coefficients of that row. The condition is *sufficient* but not *necessary*.

4·8 Jacobi method of iteration or Gauss-Jacobi method

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations,

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \quad ...(1)$$

Let us assume $|a_1| > |b_1| + |c_1|$
$$|b_2| > |a_2| + |c_2|$$
$$|c_3| > |a_3| + |b_3|$$

Then, iterative method can be used for the system (1). Solve for x, y, z (whose coefficients are the larger values) in terms of the other variables. That is,

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned} \quad ...(2)$$

If $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial values of x, y, z respectively, then

$$\begin{aligned} x^{(1)} &= \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)}) \\ y^{(1)} &= \frac{1}{b_2}(d_2 - a_2 x^{(0)} - c_2 z^{(0)}) \end{aligned} \quad ...(3)$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3\,x^{(0)} - b_3\,y^{(0)})$$

Again using these values $x^{(1)}, y^{(1)}, z^{(1)}$ in (2), we get

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1\,y^{(1)} - c_1\,z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2\,x^{(1)} - c_2\,z^{(1)}) \qquad \ldots(4)$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3\,x^{(1)} - b_3\,y^{(1)})$$

Proceeding in the same way, if the rth iterates are $x^{(r)}, y^{(r)}, z^{(r)}$, the interation scheme reduces to

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1\,y^{(r)} - c_1\,z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2\,x^{(r)} - c_2\,z^{(r)}) \qquad \ldots(5)$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3\,x^{(r)} - b_3\,y^{(r)})$$

The procedure is continued till the convergence is assured (correct to required decimals).

Note 1. To get the $(r+1)$th iterates, we use the values of the rth iterates in the scheme (5).

2. In the absence of the initial values of x, y, z, we take, usually, (0, 0, 0) as the initial estimate.

4.9 Gauss-Seidel method of iteration

This is only a refinement of Gauss-Jacobi method. As before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad \ldots(6)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values $y^{(0)}, z^{(0)}$ for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1\,y^{(0)} - c_1\,z^{(0)})$$

While using the second equation, we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$ as in the Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ and $y^{(1)}$, use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued until the convergence is assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss-Seidel method is very fast when compared to Gauss-Jacobi method. The rate of convergence in Gauss-Seidel method is roughly two times than that of Gauss-Jacobi method. As we saw the sufficient conditions already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, *the method of iteration will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.* (The largest coefficients must be the coefficients for different unknowns).

Note 1. For all systems of equations, this method will not work (since convergence is not assured). It converges only for special systems of equations.

2. Iteration method is self-correcting method. That is, any error made in computation, is corrected in the subsequent iterations.

3. The iteration is stopped when the values of x, y, z start repeating with the required degree of accuracy.

Example 1. *Solve the following system by Gauss-Jacobi and Gauss-Seidel methods :*

$$10x - 5y - 2z = 3 \; ; \; 4x - 10y + 3z = -3 \; ; \; x + 6y + 10z = -3.$$

[*MS. Ap.* 1992]

Solution. Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is diagonally dominant, since

$|10| > |-5| + |-2|$, $|-10| > |4| + |3|$ and $|10| > |1| + |6|$

Gauss-Jacobi method. Solving for x, y, z, we have

$$x = \frac{1}{10}(3 + 5y + 2z) \quad \text{...(1)}$$

$$y = \frac{1}{10}(3 + 4x + 3z) \quad \text{...(2)}$$

$$z = \frac{1}{10}(-3 - x - 6y) \quad \text{...(3)}$$

First iteration : Let the initial values be (0, 0, 0).

Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0{\cdot}3$$

$$y^{(1)} = \frac{1}{10}[3 + 4(0) + 3(0)] = 0{\cdot}3$$

$$z^{(1)} = \frac{1}{10}[-3 - (0) - 6(0)] - 0{\cdot}3$$

Second iteration : Using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}[3 + 5(0{\cdot}3) + 2(-0{\cdot}3)] = 0{\cdot}39$$

$$y^{(2)} = \frac{1}{10}[3 + 4(0{\cdot}3) + 3(-0{\cdot}3)] = 0{\cdot}33$$

$$z^{(2)} = \frac{1}{10}[-3 - (0{\cdot}3) - 6(0{\cdot}3)] = -0{\cdot}51$$

Third iteration: Using the values of $x^{(2)}, y^{(2)}, z^{(2)}$ in (1), (2), (3) we, get

$$x^{(3)} = \frac{1}{10}[3 + 5(0{\cdot}33) + 2(-0{\cdot}51)] = 0\ 363$$

$$y^{(3)} = \frac{1}{10}[3 + 4(0{\cdot}39) + 3(-0{\cdot}51)] = 0{\cdot}303$$

$$z^{(3)} = \frac{1}{10}[-3 - (0{\cdot}39) - 6(0{\cdot}33)] = -0{\cdot}537$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}[3 + 5(0{\cdot}303) + 2(-0{\cdot}537)] = 0{\cdot}3441$$

$$y^{(4)} = \frac{1}{10}[3 + 4(0{\cdot}363) + 3(-0{\cdot}537)] = 0{\cdot}2841$$

$$z^{(4)} = \frac{1}{10}[-3 - 0.363 - 6\,(0.303)] = -\,0.5181$$

Fifth interation :

$$x^{(5)} = \frac{1}{10}[3 + 5\,(0.2841) + 2\,(-\,0.5181)] = 0.33843$$

$$y^{(5)} = \frac{1}{10}[3 + 4\,(0.3441) + 3\,(-\,0.5181)] = 0.2822$$

$$z^{(5)} = \frac{1}{10}[-3 - (0.3441) - 6\,(0.2841)] = -\,0.50487$$

Sixth interation :

$$x^{(6)} = \frac{1}{10}[3 + 5\,(0.2822) + 2\,(-\,0.50487)] = 0.340126$$

$$y^{(6)} = \frac{1}{10}[3 + 4\,(0.33843) + 3\,(-\,0.50487)] = 0.283911$$

$$z^{(6)} = \frac{1}{10}[-3 - (0.33843) - 6\,(0.2822)] = -\,0.503163$$

Seventh iteration :

$$x^{(7)} = \frac{1}{10}[3 + 5\,(0.283911) + 2\,(-\,0.503163)] = 0.3413229$$

$$y^{(7)} = \frac{1}{10}[3 + 4\,(0.340126) + 3\,(-\,0.503163)] = 0.2851015$$

$$z^{(7)} = \frac{1}{10}[-3 - (0.340126) - 6\,(0.283911)] = -\,0.5043592$$

Eighth iteration :

$$x^{(8)} = \frac{1}{10}[3 + 5\,(0.2851015) + 2\,(-\,0.5043592)] = 0.34167891$$

$$y^{(8)} = \frac{1}{10}[3 + 4\,(0.3413229) + 3\,(-\,0.5043592)] = 0.2852214$$

$$z^{(8)} = \frac{1}{10}[-3 - (0.3413229) - 6\,(0.2851015)] = -\,0.50519319$$

Nineth iteration :

$$x^{(9)} = \frac{1}{10}[3 + 5\,(0.2852214) + 2\,(-\,0.50519319)] = 0.341572062$$

$$y^{(9)} = \frac{1}{10}[3 + 4\,(0.34167891) + 3\,(-\,0.50519319)] = 0.285113607$$

$$z^{(9)} = \frac{1}{10}[-3 - (0.34167891) - 6\,(0.2852214)] = -\,0.505300731$$

Hence correct to 3 decimal places, the values are

$x = 0.342,\ y = 0.285,\ z = -\,0.505$

Gauss-Seidel method : Initial values : $y = 0,\ z = 0.$

First iteration :

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4\,(0.3) + 3(0)] = 0.42$$

$$z^{(1)} = \frac{1}{10}[-3 - (0.3) - 6\,(0.42)] = -0.582$$

Second iteration :

$$x^{(2)} = \frac{1}{10}[3 + 5\,(0.42) + 2\,(-0.582)] = 0.3936$$

$$y^{(2)} = \frac{1}{10}[3 + 4\,(0.3936) + 3\,(-0.582)] = 0.28284$$

$$z^{(2)} = \frac{1}{10}[-3 - (0.3936) - 6\,(0.28284)] = -0.509064$$

Third iteration :

$$x^{(3)} = \frac{1}{10}[3 + 5\,(0.28284) + 2\,(-0.509064)] = 0.3396072$$

$$y^{(3)} = \frac{1}{10}[3 + 4\,(0.3396072) + 3\,(-0.509064)] = 0.28312368$$

$$z^{(3)} = \frac{1}{10}[-3 - (0.3396072) - 6\,(0.28312368)] = -0.503834928$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}[3 + 5\,(0.28312368) + 2\,(-0.503834928)] = 0.34079485$$

$$y^{(4)} = \frac{1}{10}[3 + 4\,(0.34079485) + 3\,(-0.50383492)] = 0.285167464$$

$$z^{(4)} = \frac{1}{10}[-3 - (0.34079485) - 6\,(0.28516746)] = -0.50517996$$

Fifth iteration :

$$x^{(5)} = \frac{1}{10}[3 + 5\,(0.28516746) + 2\,(-0.50517996)] = 0.34155477$$

$$y^{(5)} = \frac{1}{10}[3 + 4\,(0.34155477) + 3\,(-0.50517996)] = 0.28506792$$

$$z^{(5)} = \frac{1}{10}[-3 - (0.34155477) - 6\,(0.28506792)] = -0.505196229$$

Sixth iteration :

$$x^{(6)} = \frac{1}{10}[3 + 5\,(0.28506792) + 2\,(-0.505196229)] = 0.341494714$$

$$y^{(6)} = \frac{1}{10}[3 + 4(0.341494714) + 3(-0.505196229)] = 0.285039017$$

$$z^{(6)} = \frac{1}{10}[-3 - (0.341494714) - 6(0.285039017)] = -0.5051728$$

Seventh iteration :

$$x^{(7)} = \frac{1}{10}[3 + 5(0.285039017) + 2(-0.5051728)] = 0.3414849$$

$$y^{(7)} = \frac{1}{10}[3 + 4(0.3414849) + 3(-0.5051728)] = 0.28504212$$

$$z^{(7)} = \frac{1}{10}[-3 - (0.3414849) - 6(0.28504212)] = -0.5051737$$

The values at each iteration by both methods are tabulated below:

Iteration	Gauss-Jacobi method			Gauss-Seidel method		
	x	y	z	x	y	z
1	0·3	0·3	−0·3	0·3	0·42	−0·582
2	0·39	0·33	−0·51	0·3936	0·28284	−0·509064
3	0·363	0·303	−0·537	0·3396072	0·28312364	−0·503834928
4	0·3441	0·2841	−0·5181	0·34079485	0·28516746	−0·50517996
5	0·33843	0·2822	−0·50487	0·3415547	0·28506792	−0·505196229
6	0·340126	0·283911	−0·503163	0·3414947	0·2850390	−0·5051728
7	0·3413229	0·2851015	−0·5043592	0·3414849	0·28504212	−0·5051737
8	0·34167891	0·2852214	−0·50519319			
9	0·341572062	0·285113607	−0·505300731			

The values correct to 3 decimal places are

$$x = 0.342,\ y = 0.285,\ z = -0.505$$

Note. After getting the values of the unknowns, substitute these values in the given equations, and check the correctness of the results.

Example 2. *Solve the following system of equations by using Gauss-Jacobi and Gauss-Seidel methods (correct to 3 decimal places) :*

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35. \quad \text{[BR. Ap. '94]}$$

Solution. Since the diagonal elements are dominant in the coefficient matrix, we write x, y, z as follows

$$x = \frac{1}{8}[20 + 3y - 2z] \quad ...(1)$$

$$y = \frac{1}{11}[33 - 4x + z] \quad ...(2)$$

$$z = \frac{1}{12}[35 - 6x - 3y] \quad ...(3)$$

Gauss-Jacobi method :

First iteration: Let the initial values be $x = 0,\ y = 0,\ z = 0$

Using the values $x = 0,\ y = 0,\ z = 0$ in (1), (2), (3) we get,

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)] = 2 \cdot 5$$

$$y^{(1)} = \frac{1}{11}[33 - 4(0) + 0] = 3 \cdot 0$$

$$z^{(1)} = \frac{1}{12}[35 - 6(0) - 3(0)] = 2 \cdot 916666$$

Second iteration : Using these values $x^{(1)}, y^{(1)}, z^{(1)}$ again in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{8}[20 + 3\,(3 \cdot 0) - 2\,(2 \cdot 916666)] = 2 \cdot 895833$$

$$y^{(2)} = \frac{1}{11}[33 - 4\,(2 \cdot 5) + (2 \cdot 916666)] = 2 \cdot 356060$$

$$z^{(2)} = \frac{1}{12}[35 - 6\,(2 \cdot 5) - 3\,(3 \cdot 0)] = 0 \cdot 916666$$

Third interation :

$$x^{(3)} = \frac{1}{8}[20 + 3\,(2 \cdot 356060) - 2\,(0 \cdot 916666)] = 3 \cdot 154356$$

$$y^{(3)} = \frac{1}{11}[33 - 4\,(2 \cdot 895833) + (0 \cdot 916666)] = 2 \cdot 030303$$

$$z^{(3)} = \frac{1}{12}[35 - 6\,(2 \cdot 895833) - 3\,(2 \cdot 356060)] = 0 \cdot 879735$$

Fourth iteration :

$$x^{(4)} = \frac{1}{8}[20 + 3\,(2 \cdot 030303) - 2\,(0 \cdot 879735)] = 3 \cdot 041430$$

$$y^{(4)} = \frac{1}{11}[33 - 4\,(3 \cdot 154356) + (0 \cdot 879735)] = 1 \cdot 932937$$

$$z^{(4)} = \frac{1}{12}[35 - 6\,(3 \cdot 154356) - 3\,(2 \cdot 030303)] = 0 \cdot 831913$$

Fifth iteration :

$$x^{(5)} = \frac{1}{8}[20 + 3\,(1 \cdot 932937) - 2\,(0 \cdot 831913)] = 3 \cdot 016873$$

$$y^{(5)} = \frac{1}{11}[33 - 4\,(3 \cdot 041430) + (0 \cdot 831913)] = 1 \cdot 969654$$

$$z^{(5)} = \frac{1}{12}[35 - 6\,(3 \cdot 041430) - 3\,(1 \cdot 932937)] = 0 \cdot 912717$$

Sixth iteration :

$$x^{(6)} = \frac{1}{8}[20 + 3\,(1 \cdot 969654) - 2\,(0 \cdot 912717)] = 3 \cdot 010441$$

$$y^{(6)} = \frac{1}{11}[33 - 4\,(3{\cdot}016873) + (0{\cdot}912717)] = 1{\cdot}985930$$

$$z^{(6)} = \frac{1}{12}[35 - 6\,(3{\cdot}016873) - 3\,(1{\cdot}969654)] = 0{\cdot}915817$$

Seventh iteration :

$$x^{(7)} = \frac{1}{8}[20 + 3\,(1{\cdot}985930) - 2\,(0{\cdot}915817)] = 3{\cdot}015770$$

$$y^{(7)} = \frac{1}{11}[33 - 4\,(3{\cdot}010441) + (0{\cdot}915817)] = 1{\cdot}988550$$

$$z^{(7)} = \frac{1}{12}[35 - 6\,(3{\cdot}010441) - 3\,(1{\cdot}985930)] = 0{\cdot}914964$$

Eigth iteration :

$$x^{(8)} = \frac{1}{8}[20 + 3\,(1{\cdot}988550) - 2\,(0{\cdot}914964)] = 3{\cdot}016946$$

$$y^{(8)} = \frac{1}{11}[33 - 4\,(3{\cdot}015770) + (0{\cdot}914964)] = 1{\cdot}986535$$

$$z^{(8)} = \frac{1}{12}[35 - 6\,(3{\cdot}015770) - 3\,(1{\cdot}988550)] = 0{\cdot}911644$$

Ninth iteration :

$$x^{(9)} = \frac{1}{8}[20 + 3\,(1{\cdot}986535) - 2\,(0{\cdot}911644)] = 3{\cdot}017039$$

$$y^{(9)} = \frac{1}{11}[33 - 4\,(3{\cdot}016946) + (0{\cdot}911644)] = 1{\cdot}985805$$

$$z^{(9)} = \frac{1}{12}[35 - 6\,(3{\cdot}016946) - 3\,(1{\cdot}986535)] = 0{\cdot}911560$$

Tenth iteration :

$$x^{(10)} = \frac{1}{8}[20 + 3\,(1{\cdot}985805) - 2\,(0{\cdot}911560)] = 3{\cdot}016786$$

$$y^{(10)} = \frac{1}{11}[33 - 4\,(3{\cdot}017039) + (0{\cdot}911560)] = 1{\cdot}985764$$

$$z^{(10)} = \frac{1}{12}[35 - 6\,(3{\cdot}017039) - 3\,(1{\cdot}985805)] = 0{\cdot}911696$$

In 8th, 9th and 10th iterations the values of x, y, z are same correct to 3 decimal places. Hence we stop at this level.

Gauss-Seidel method :

We take the initial values as $y = 0, z = 0$ and use equations (1)

First iteration :

$$x^{(1)} = \frac{1}{8}[20 + 3\,(0) - 2\,(0)] = 2{\cdot}5$$

$$y^{(1)} = \frac{1}{11}[33 - 4(2\cdot5) + 0] = 2\cdot090909$$

$$z^{(1)} = \frac{1}{12}[35 - 6(2\cdot5) - 3(2\cdot090909)] = 1\cdot143939$$

Second iteration :

$$x^{(2)} = \frac{1}{8}[20 + 3(2\cdot090909) - 2(1\cdot143939)] = 2\cdot998106$$

$$y^{(2)} = \frac{1}{11}[33 - 4(2\cdot998106) + (1\cdot143939)] = 2\cdot013774$$

$$z^{(2)} = \frac{1}{12}[35 - 6(2\cdot998106) - 3(2\cdot013774)] = 0\cdot914170$$

Third iteration :

$$x^{(3)} = \frac{1}{8}[20 + 3(2\cdot013774) - 2(0\cdot914170)] = 3\cdot026623$$

$$y^{(3)} = \frac{1}{11}[33 - 4(3\cdot026623) + (0\cdot914170)] = 1\cdot982516$$

$$z^{(3)} = \frac{1}{12}[35 - 6(3\cdot026623) - 3(1\cdot982516)] = 0\cdot907726$$

Fourth iteration :

$$x^{(4)} = \frac{1}{8}[20 + 3(1\cdot982516) - 2(0\cdot907726)] = 3\cdot016512$$

$$y^{(4)} = \frac{1}{11}[33 - 4(3\cdot016512) + (0\cdot907726)] = 1\cdot985607$$

$$z^{(4)} = \frac{1}{12}[35 - 6(3\cdot016512) - 3(1\cdot985607)] = 0\cdot912009$$

Fifth iteration :

$$x^{(5)} = \frac{1}{8}[20 + 3(1\cdot985607) - 2(0\cdot912009)] = 3\cdot016600$$

$$y^{(5)} = \frac{1}{11}[33 - 4(3\cdot016600) + (0\cdot912009)] = 1\cdot985964$$

$$z^{(5)} = \frac{1}{12}[35 - 6(3\cdot016600) - 3(1\cdot985964)] = 0\cdot911876$$

Sixth iteration :

$$x^{(6)} = \frac{1}{8}[20 + 3(1\cdot985964) - 2(0\cdot911876)] = 3\cdot016767$$

$$y^{(6)} = \frac{1}{11}[33 - 4(3\cdot016767) + (0\cdot911876)] = 1\cdot985892$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3\cdot016767) - 3(1\cdot985892)] = 0\cdot911810$$

The values of x, y, z got by Jacobi method correct to 3 decimal places are got even in the 6th iteration by Gauss-Seidel method.)

Seventh iteration :

$$x^{(7)} = \frac{1}{8}[20 + 3(1{\cdot}985892) - 2(0{\cdot}911810)] = 3{\cdot}016757$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3{\cdot}016757) + (0{\cdot}911810)] = 1{\cdot}985889$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3{\cdot}016757) - 3(1{\cdot}985889)] = 0{\cdot}911816$$

Since the seventh and eighth iterations give the same values for x, y, z correct to 4 decimal places, we stop here.

$\therefore \; x = 3{\cdot}0168, \; y = 1{\cdot}9859, \; z = 0{\cdot}9118$

The values of x, y, z by both methods at each iteration are tabulated below:

Iteration	*Gauss-Jacobi method*			*Gauss-Seidel method*		
	x	y	z	x	y	z
1	2·5	3·0	2·916666	2·5	2·090909	1·143939
2	2·895833	2·356060	0·916666	2·998106	2·013774	0·914170
3	3·154356	2·030303	0·879735	3·026623	1·982516	0·907726
4	3·041430	1·932937	0·831913	3·016512	1·985607	0·912009
5	3·016873	1·969654	0·912717	3·016600	1·985964	0·911876
6	3·010441	1·985930	0·915817	3·016767	1·985892	0·911810
7	3·015770	1·988550	0·914964	3·016757	1·985889	0·911816
8	3·016946	1·986535	0·911644			
9	3·017039	1·985805	0·911560			
10	3·016786	1·985764	0·911696			

This shows that the convergence is rapid in Gauss-Seidel method when compared to Gauss-Jacobi method. We see that 10 iterations are necessary in Jacobi method to get the same accuracy as got by 7 iterations in Gauss-Seidel method.

Example 3. *Solve the following system of equations by Gauss-Jacobi and Gauss-Seidel method correct to three decimal places :*

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

Solution. As the coefficient matrix is not diagonally dominant as it is, we rewrite the equation, as noted below, so that the coefficient matrix becomes diagonally dominant

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Solving for x, y, z, we get

$$x=\frac{1}{27}[85-6y+z] \quad ...(1)$$

$$y=\frac{1}{15}[72-6x-2z] \quad ...(2)$$

$$z=\frac{1}{54}[110-x-y] \quad ...(3)$$

Starting with the initial value $x=0, y=0, z=0$ and using (1), (2), (3) and repeating the process we get the values of x, y, z as the tabulated by both methods. (Gauss-Jacobi and Gauss-Seidel)

Iteration	*Gauss-Jacobi method*			*Gauss-Seidel method*		
	x	y	z	x	y	z
1	3·14815	4·8	2·03704	3·14815	3·54074	1·91317
2	2·15693	3·26913	1·88985	2·43218	3·57204	1·92585
3	2·49167	3·68525	1·93655	2·42569	3·57294	1·92595
4	2·40093	3·54513	1·92265	2·42549	3·57301	1·92595
5	2·43155	3·58327	1·92692	2·42548	3·57301	1·92595
6	2·42323	3·57046	1·92565	2·42548	3·57301	1·92595
7	2·42603	3·57395	1·92604			
8	2·42527	3·57278	1·92593			

Hence $x=2{\cdot}425,\ y=3{\cdot}573$, and $z=1{\cdot}926$

(correct to 3 decimal places)

Example 4. *Solve, by Gauss-Seidel method, the following system:*

$$28x+4y-z=32$$

$$x+3y+10z=24$$

$$2x+17y+4z=35$$

Solution. Since the diagonal elements in the coefficient matrix are not dominant, we rearrange the equations, as follows, such that the elements in the coefficient matrix are dominant.

$$28x+4y-z=32$$

$$2x+17y+4z=35$$

$$x+3y+10z=24$$

Hence, $\quad x=\dfrac{1}{28}[32-4y+z]$

$$y=\frac{1}{17}[35-2x-4z]$$

$$z=\frac{1}{10}[24-x-3y]$$

Setting $y=0, z=0$, we get

First iteration :

$$x^{(1)} = \frac{1}{28}[32 - 4(0) + 0] = 1{\cdot}1429$$

$$y^{(1)} = \frac{1}{17}[35 - 2\,(1{\cdot}1429) - 4(0)] = 1{\cdot}9244$$

$$z^{(1)} = \frac{1}{10}[24 - 1{\cdot}1429 - 3\,(1{\cdot}9244)] = 1{\cdot}8084$$

Second iteration :

$$x^{(2)} = \frac{1}{28}[32 - 4\,(1{\cdot}9244) + 1{\cdot}8084] = 0{\cdot}9325$$

$$y^{(2)} = \frac{1}{17}[35 - 2\,(0{\cdot}9325) - 4\,(1{\cdot}8084)] = 1{\cdot}5236$$

$$z^{(3)} = \frac{1}{10}[24 - 0{\cdot}9325 - 3\,(1{\cdot}5236)] = 1{\cdot}8497$$

Third iteration :

$$x^{(3)} = \frac{1}{28}[32 - 4\,(1{\cdot}5236) + 1{\cdot}8497] = 0{\cdot}9913$$

$$y^{(3)} = \frac{1}{17}[35 - 2\,(0{\cdot}9913) - 4\,(1{\cdot}8497)] = 1{\cdot}5070$$

$$z^{(3)} = \frac{1}{10}[24 - 0{\cdot}9913 - 3\,(1{\cdot}5070)] = 1{\cdot}8488$$

Fourth iteration :

$$x^{(4)} = \frac{1}{28}[32 - 4\,(1{\cdot}5070) + 1{\cdot}8488] = 0{\cdot}9936$$

$$y^{(4)} = \frac{1}{17}[35 - 2\,(0{\cdot}9936) - 4\,(1{\cdot}8488)] = 1{\cdot}5069$$

$$z^{(4)} = \frac{1}{10}[24 - 0{\cdot}9936 - 3\,(1{\cdot}5069)] = 1{\cdot}8486$$

Fifth iteration :

$$x^{(5)} = \frac{1}{28}[32 - 4\,(1{\cdot}5069) + (1{\cdot}8486)] = 0{\cdot}9936$$

$$y^{(6)} = \frac{1}{17}[35 - 2\,(0{\cdot}9936) - 4\,(1{\cdot}8486)] = 1{\cdot}5069$$

$$z^{(6)} = \frac{1}{10}[24 - 0{\cdot}9936 - 3\,(1{\cdot}5069)] = 1{\cdot}8486$$

Since the values of x, y, z in the 4th and 5th iterations are same, we stop the process here.

Hence, $x = 0{\cdot}9936$, $y = 1{\cdot}5069$, $z = 1{\cdot}8486$