Solution of Non-Linear Equations

Introduction

FRAME 1

It is by no means uncommon for non-linear and transcendental equations to occur in practice. By the term 'non-linear equations' is meant those equations that involve powers other than the first, for example, $x^4 - x^3 + 5x - 7 = 0$. This is a quartic equation as it involves x^4. Transcendental equations are those which involve functions other than just powers of x, for example $e^x = \cos x + x^2$.

There are a few non-linear and transcendental equations that are readily solvable but these are definitely in the minority. Some examples of such equations are $2x^2 - 3x - 5 = 0$, $\sin x = 1/2$ and $e^x = 1$. Where no simple method exists for solving equations like these (only rather more complicated) numerical methods are frequently employed and it is the purpose of this programme to investigate some of these numerical techniques. Before doing so, however, let us have a look at a few practical examples where some equations requiring these techniques arise.

FRAME 2

If you have read the programme "Partial Differential Equations for Technologists" in our book, "Mathematics for Engineers and Scientists, Vol. 2" in this series, you will have come across the equation

$m\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = 0$ for the transverse oscillations of a thin rod.

Under certain conditions, i.e., when the rod is clamped horizontally at both ends, it is found that vibrations take place only if the equation $\cos n\ell \cosh n\ell = 1$ is satisfied. In a similar way, if the beam is pinned at the end where $x = \ell$ instead of being clamped there, then the corresponding equation is $\tan n\ell = \tanh n\ell$.

In a certain crank mechanism, Freudenstein's equation $R_1 \cos\theta - R_2 \cos\phi + R_3 - \cos(\theta - \phi) = 0$, giving the relation between θ and ϕ, the input and output crank angles, must be satisfied. Although it is possible to find ϕ as a function of θ directly from this equation, a numerical method of solution is sometimes preferred.

Taking off now into space, so to speak, it is found that if a missile is subjected to a thrust, then the equation $x^2(1 - \cos x \cosh x) - \gamma \sin x \sinh x = 0$ is associated with its flexural vibrations. γ is a constant which would be fixed in particular circumstances, and the solution for x required.

As a final example, if two space vehicles are describing the same orbit about the earth and a link-up between them is required, it is necessary to solve for e an equation of the form

$$\frac{\phi}{2} = \left(\frac{1 + e\cos\theta}{e^2 - 1}\right)^{3/2}\left[\frac{e\sqrt{e^2 - 1}}{1 + e\cos\theta}\sin\theta - \ln\left(\frac{\sqrt{e+1} + \sqrt{e-1}\tan\frac{1}{2}\theta}{\sqrt{e+1} - \sqrt{e-1}\tan\frac{1}{2}\theta}\right)\right]$$

e being the eccentricity of what is known as the transfer orbit. θ and ϕ are fixed in any particular problem.

FRAME 3

Numerical techniques often make use of a process known as ITERATION. Before actually discussing non-linear equations themselves, we will have a look at what is meant by this process.

FRAME 4

Iteration

First let us have a look at the formula $y = \frac{1}{2}\left(x + \frac{N}{x}\right)$ (4.1)

with $N = 1973$ and find the value of y when $x = 40$. It gives

$$y = \frac{1}{2}\left[40 + \frac{1973}{40}\right] \simeq \frac{1}{2}(40 + 49{\cdot}3) \simeq 44{\cdot}6$$

Now suppose $44{\cdot}6$ is substituted for x instead of 40. The result this time is

$$y = \frac{1}{2}\left[44{\cdot}6 + \frac{1973}{44{\cdot}6}\right] \simeq \frac{1}{2}(44{\cdot}6 + 44{\cdot}24) \simeq 44{\cdot}42$$

If now x is put equal to $44{\cdot}42$, the value of y is found to be $\frac{1}{2}(44{\cdot}42 + 44{\cdot}416\,92) \simeq 44{\cdot}418\,46$. Putting $x = 44{\cdot}418\,46$ then gives $y = \frac{1}{2}(44{\cdot}418\,46 + 44{\cdot}418\,47) \simeq 44{\cdot}418\,46$.

You will notice that x and N/x have been getting closer and closer to each other as we have proceeded, and at the last stage they were equal to one another to 6 significant figures. If the process is continued, you will find that they become equal to an even higher degree of accuracy. It is because of this gradual approach to equality that the number of significant figures has been gradually increased at each stage.

What can you say is the relationship between x and N when the difference between x and N/x is negligible?

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4A

$x = \frac{N}{x}$ *gives* $x = \sqrt{N}$.

FRAME 5

Now see what happens if you take $N = 57{\cdot}46$ and use the formula (4.1) as in the last frame, starting with $x = 8$. What is value of y when you have used it 3 times? Give your answer to 5 significant figures if possible, otherwise stop at 4 or even 3.

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5A

Successive values of y are $7{\cdot}59$, $7{\cdot}5802$, $7{\cdot}5802$. The fact that the last two are the same indicates that you have arrived at the value you are looking for.

FRAME 6

$7{\cdot}5802$ is the square root of $57{\cdot}46$ correct to 5 significant figures. Continuing the process in this case very quickly leads to the more accurate result $7{\cdot}580\,237\,46$.

As this is a repetitive process, it is very convenient for programming on to a computer. A computer print out for this particular problem would give the values of successive calculations as

FRAME 6 (continued)

VALUE OF X USED	VALUE OF CORRESPONDING Y
	8.00000000
1ST	7.59125000
2ND	7.58024545
3RD	7.58023746
4TH	7.58023746

You will notice that all these are given to the same number of decimal places. It is easier to let the computer do this than to program it to increase the number of decimal places as you go along. The computer would be told to stop when two successive calculations give the same result to a pre-specified degree of accuracy.

The square root of any positive number can be found in this way. It is necessary to make an initial guess as to the first value of x to take, but this is not difficult. It helps if you take it reasonably close to the actual value.

FRAME 7

The process of finding successive approximations to a quantity, as was done in FRAMES 4 and 5, is called an ITERATIVE PROCESS. Each use of the particular formula ((4.1) in our examples) is an ITERATION and each successive approximation is an ITERATE.

In FRAME 4, the first value of y was taken as the second value of x, the second value of y as the third value of x and so on. In practice, however, a different notation is used, in order to avoid possible confusion. The initial guess is often labelled x_0, the first value of y, i.e., the first iterate, x_1, the second x_2 and so on. Then (4.1) would lead to the successive equations

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) \qquad x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right) \qquad x_3 = \frac{1}{2}\left(x_2 + \frac{N}{x_2}\right) \qquad \text{etc.}$$

These can all be combined into the single equation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \qquad (7.1)$$

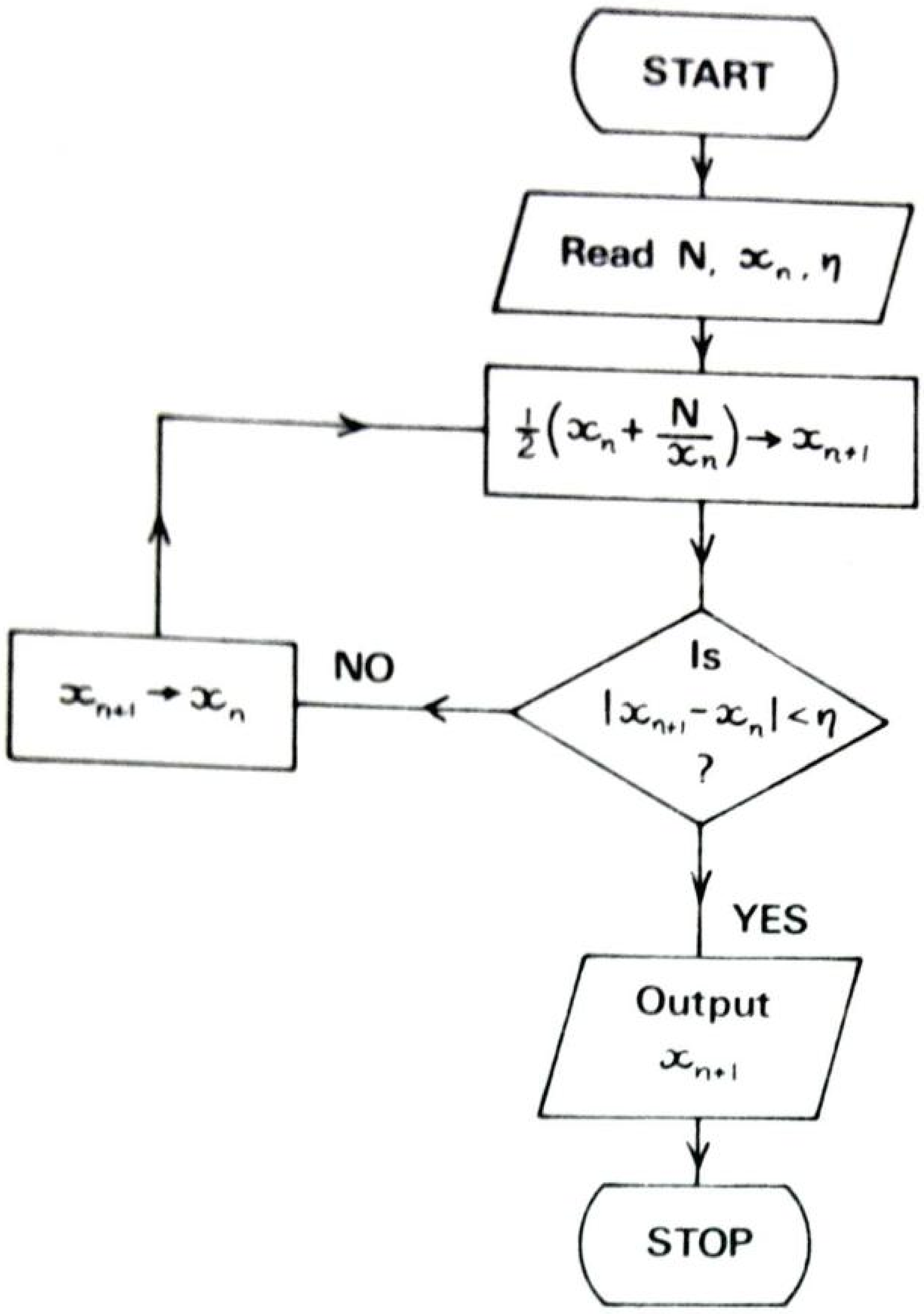
where n takes the values 0, 1, 2 .... in turn.
This notation is used in the flow diagram (illustrated on page 31) for the process of finding a square root by this method.

FRAME 8

The equation corresponding to (7.1) for finding a cube root is

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^{\,2}}\right)$$

[Don't worry for the time being how the formulae $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ and $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^{\,2}}\right)$ are found. You will discover this later.]



The value of x_n here will be x_0, the initial approximation to $\sqrt{N}$.

η is the pre-specified degree of accuracy mentioned in FRAME 6. As the computer may be working to more decimal places than shown in the print-out, η could be set at, say, 0·000 000 001. The computer would then cease the calculation when two successive iterates differ by less than this amount.

Flow Diagram for FRAME 7.

(Programs for finding the square root of a number may be found in references (1) and (3).)

FRAME 8 (continued)

By taking a suitable value for x_0, use this formula to find $\sqrt[3]{70\cdot13}$ to 5 significant figures. (Again, stop at fewer significant figures if you can't manage 5.)

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8A

As $4^3 = 64$ *and* $5^3 = 125$, $x_0 = 4$ *is a reasonable initial guess. Successive iterates are*

$$x_1 = 4\cdot13, \qquad x_2 = 4\cdot1238, \qquad x_3 = 4\cdot1238.$$

FRAME 9

In the examples that have been taken so far, the approximations have got better and better as the number of iterations has increased. When this happens, the process is said to be CONVERGENT. In the cases taken, this convergence has been quite rapid. That this was to be expected can easily be shown theoretically. (If a process does not converge, it will be of no use, and if it converges only slowly, the numerical working involved is going to be considerably increased.)

Taking $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$, let us assume that x_n is reasonably close to $\sqrt{N}$.

Then we can write $\sqrt{N} - x_n = \varepsilon_n$, and so $x_n = \sqrt{N} - \varepsilon_n$.

$$\therefore \quad x_{n+1} = \frac{1}{2}\left(\sqrt{N} - \varepsilon_n + \frac{N}{\sqrt{N} - \varepsilon_n}\right) = \frac{1}{2}\left\{\sqrt{N} - \varepsilon_n + \frac{N}{\sqrt{N}\left(1 - \frac{\varepsilon_n}{\sqrt{N}}\right)}\right\}$$

Now, as x_n is assumed to be near $\sqrt{N}$, ε_n will be small compared with $\sqrt{N}$ and so $\left|\frac{\varepsilon_n}{\sqrt{N}}\right| < 1$.

By writing $\dfrac{1}{1 - \frac{\varepsilon_n}{\sqrt{N}}}$ as $\left(1 - \frac{\varepsilon_n}{\sqrt{N}}\right)^{-1}$ and expanding, find x_{n+1} as a series in ε_n as far as the term in $\varepsilon_n^{\,3}$.

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9A

$$x_{n+1} = \frac{1}{2}\left\{\sqrt{N} - \varepsilon_n + \sqrt{N}\left(1 + \frac{\varepsilon_n}{\sqrt{N}} + \frac{\varepsilon_n^{\,2}}{N} + \frac{\varepsilon_n^{\,3}}{N\sqrt{N}} + \ldots\ldots\ldots\ldots\right)\right\}$$

$$= \sqrt{N} + \frac{1}{2}\,\frac{\varepsilon_n^{\,2}}{\sqrt{N}} + \frac{1}{2}\,\frac{\varepsilon_n^{\,3}}{N} + \ldots\ldots\ldots\ldots$$

FRAME 10

Thus $\sqrt{N} - x_{n+1} \simeq -\frac{1}{2}\,\frac{\varepsilon_n^{\,2}}{\sqrt{N}}$. This means that if the error in the nth iterate is of magnitude $|\varepsilon_n|$, then $|\varepsilon_{n+1}|$, the magnitude of the error in the (n+1)th iterate is of magnitude $\frac{1}{2}\,\frac{\varepsilon_n^{\,2}}{\sqrt{N}}$. As $\left|\frac{\varepsilon_n}{\sqrt{N}}\right| < 1$, this error is smaller in magnitude than the previous one. Similarly the error in the next iterate will be smaller than this one, and so on. A process in which the error at one stage is approximately a multiple of the square of the error at the previous stage is said to be a SECOND ORDER PROCESS.

FRAME 10 (continued)

Now take the formula $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$ and proceed similarly to show that the method based on this formula for finding $N^{1/3}$ is also a second order process.

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10A

Let $N^{1/3} - x_n = \varepsilon_n$ *then* $x_n = N^{1/3} - \varepsilon_n$ *and*

$$x_{n+1} = \frac{1}{3}\left\{2(N^{1/3} - \varepsilon_n) + \frac{N}{(N^{1/3} - \varepsilon_n)^2}\right\} = \frac{1}{3}\left\{2(N^{1/3} - \varepsilon_n) + N^{1/3}\left(1 - \frac{\varepsilon_n}{N^{1/3}}\right)^{-2}\right\}$$

$$= \frac{1}{3}\left\{2(N^{1/3} - \varepsilon_n) + N^{1/3}\left(1 + \frac{2\varepsilon_n}{N^{1/3}} + \frac{3\varepsilon_n^2}{N^{2/3}} + \ldots\right)\right\} \simeq N^{1/3} + \frac{\varepsilon_n^2}{N^{1/3}}$$

$$\text{i.e.} \quad N^{1/3} - x_{n+1} \simeq -\frac{\varepsilon_n^2}{N^{1/3}}$$

FRAME 11

Solution of Non-Linear Equations by means of the Iteration Formula $x_{n+1} = F(x_n)$

Having seen the basic ideas behind the process of iteration, the next thing is to consider how this can be applied to find the solution of an equation for which we do not know a simple analytical technique.

For a first example, let us take the equation

$$x^3 - 4x^2 + x - 10 = 0$$

Now the two formulae that were used earlier were

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \qquad \text{and} \qquad x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$$

i.e. they were of the form $x_{n+1} = F(x_n)$. Furthermore, in the later stages, x_{n+1} was very nearly equal to x_n. This suggests that it might be a help to rearrange $x^3 - 4x^2 + x - 10 = 0$ so that it is in the form $x = F(x)$. There is obviously more than one way in which this can be done. Some possibilities are

$$x = 10 - x^3 + 4x^2, \quad x = \tfrac{1}{2}\sqrt{x^3 + x - 10} \quad (\text{or } -\tfrac{1}{2}\sqrt{x^3 + x - 10}),$$

$$x = \frac{4x^2 - x + 10}{x^2}, \quad x = \frac{x^3 + x - 10}{4x}, \quad x = \sqrt[3]{4x^2 - x + 10}$$

Taking the first of these would suggest $x_{n+1} = 10 - x_n^3 + 4x_n^2$, the second $x_{n+1} = \frac{1}{2}\sqrt{x_n^3 + x_n - 10}$ and so on.

Now with an iteration process, it is necessary to find, by some other means, an initial starting point. Can you suggest any ways in which this can be done?

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11A

Various methods are possible, such as:

i) *A rough sketch of the curve* $y = x^3 - 4x^2 + x - 10$. x_0 *would be taken close to where this curve crosses the x-axis.*

ii) *Rough sketches, on the same axes, of* $y = x^3$ *and* $y = 4x^2 - x + 10$. x_0 *would be taken close to the value of* x *where they cross.*

iii) *A table of values of* x *and* y *where* $y = x^3 - 4x^2 + x - 10$. *At some point in the table the sign of* y *will change.* x_0 *would be taken close to where this happens.*

You should have met at least some of these methods at school.

FRAME 12

Using any one of the methods in 11A, see if you can find a suitable value for x_0. (You should find the required value lies between -2 and 6.)

****

12A

$x_0 = 4$

(iii) *A table of values of* y *against* x *is*

x	-2	-1	0	1	2	3	4	5	6
y	-36	-16	-10	-12	-16	-16	-6	20	68

(i) *The table in (iii) gives rise to the sketch in Figure (i).*

(ii) *New sets of values would be required and Figure (ii) would result.*

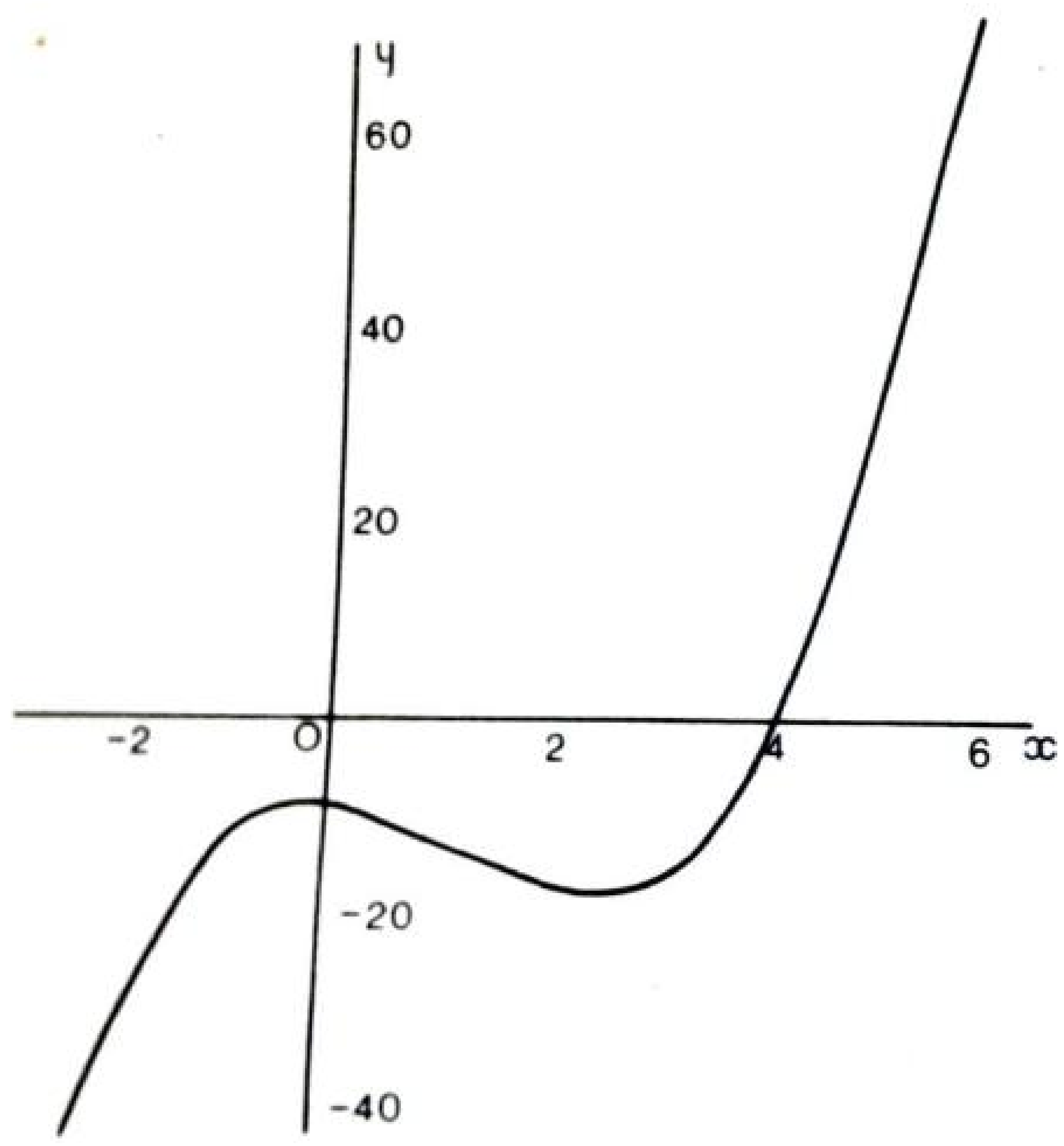

Fig (i)

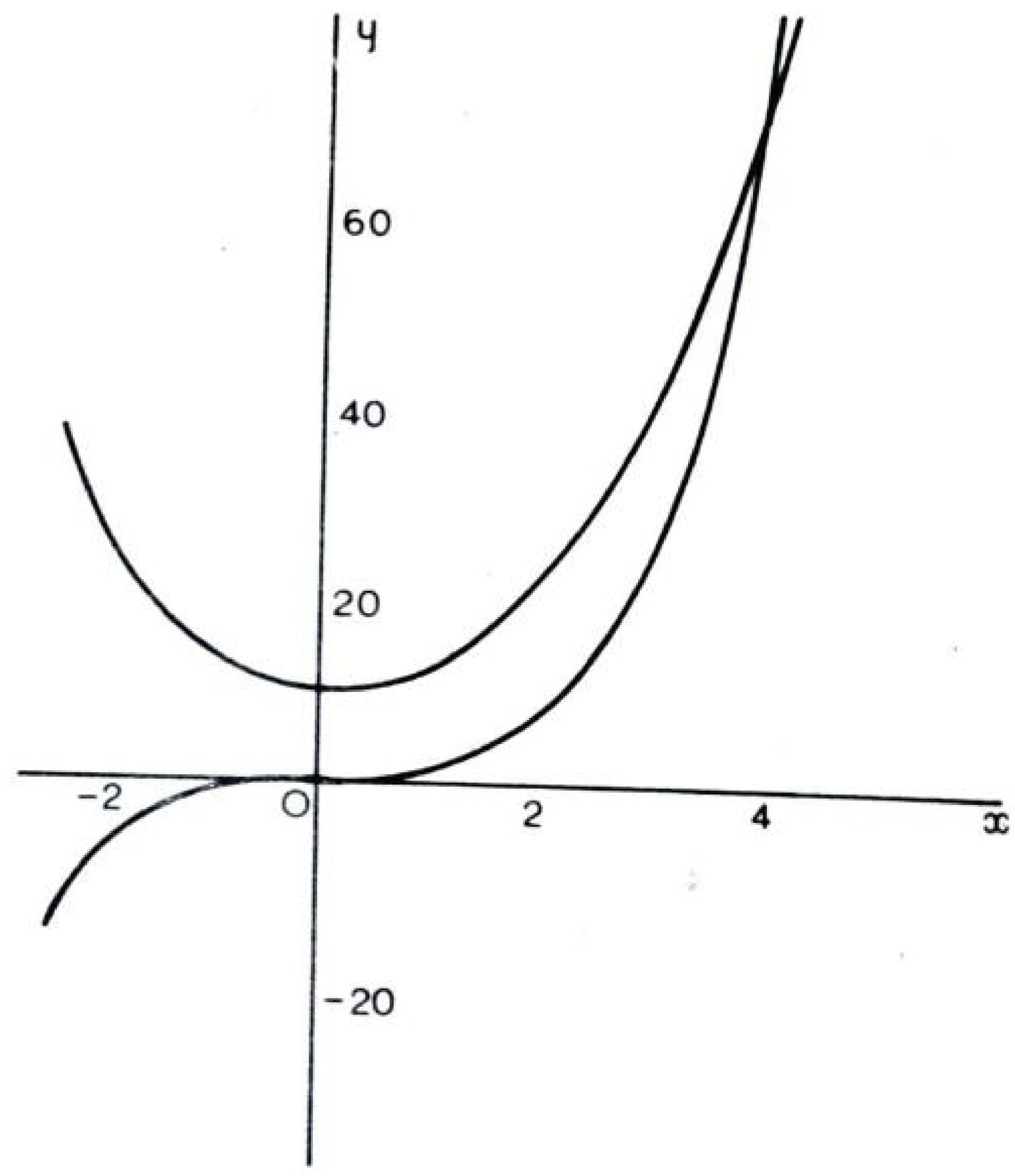

Fig (ii)

Note that this cubic equation only has one real root. For the time being, equations having more than one real root will not be introduced.

FRAME 13

Now, taking $x_{n+1} = 10 - x_n^{\,3} + 4x_n^{\,2}$ and $x_0 = 4$ gives

$$x_1 = 10\,, \qquad x_2 = -590\,, \qquad x_3 = -203\,986\,590\,,$$

and the whole process is obviously going wild.

But if we take $x_{n+1} = \dfrac{4x_n^{\,2} - x_n + 10}{x_n^{\,2}}$ and $x_0 = 4$, then

$x_1 = 4{\cdot}4$, $x_2 = 4{\cdot}29$, $x_3 = 4{\cdot}310$, $x_4 = 4{\cdot}3063$, $x_5 = 4{\cdot}3070$,
$x_6 = 4{\cdot}3069$, $x_7 = 4{\cdot}3069$

The fact that the last two results are the same indicates that the solution has now been obtained correct to 5 significant figures.

Two points are of interest here:

1) Why did the first formula for x_{n+1} lead to a ridiculous result, or rather, to no result?

2) Rather more steps had to be taken (up to x_7) than when square and cube roots were being obtained, for a similar degree of accuracy.

However, before going on to consider these points, you might like to find the root of $x^3 - 5x^2 - 29 = 0$, correct to 4 significant figures.

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13A

A table of values gives, if $y = x^3 - 5x^2 - 29$,

x	0	1	2	3	4	5	6	7
y	−29	−33	−41	−47	−45	−29	7	69

and this suggests $x_0 = 6$.

From $x^3 - 5x^2 - 29 = 0$, $\quad x = \dfrac{5x^2 + 29}{x^2}$

and so one formula for x_{n+1} *is* $x_{n+1} = \dfrac{5x_n^{\,2} + 29}{x_n^{\,2}}$

If you tried $x = \dfrac{x^3 - 29}{5x}$, *you will have found that this will not give a solution.*

Successive approximations are:

$x_1 = 5{\cdot}8$, $x_2 = 5{\cdot}86$, $x_3 = 5{\cdot}845$, $x_4 = 5{\cdot}849$, $x_5 = 5{\cdot}848$, $x_6 = 5{\cdot}848$.

5·848 is the required root.

FRAME 14

To find the answer to point number 1) in the last frame, a method is adopted which is somewhat similar to that used in FRAME 9.

Let the equation to be solved be $f(x) = 0$, and assume that this has been written in the form $x = F(x)$ so that the iteration formula $x_{n+1} = F(x_n)$ is used. Now let $x = a$ be the actual value of the root sought. Then $f(a) = 0$ and also $a = F(a)$. If x_n is an approximation for a, then we can write

$$a - x_n = \varepsilon_n$$

FRAME 14 (continued)

where, if x_n is a reasonably good approximation, ε_n is small. Thus $x_n = a - \varepsilon_n$ and so $\quad x_{n+1} = F(a - \varepsilon_n)$

The R.H.S. of this equation can now be expanded by Taylor's series in powers of ε_n. By using this expansion as far as the term in ε_n, find an expression for ε_{n+1}, where $\varepsilon_{n+1} = a - x_{n+1}$.

****

14A

$$a - \varepsilon_{n+1} \simeq F(a) - \varepsilon_n F'(a) \qquad (14A.1)$$

$$\varepsilon_{n+1} \simeq \varepsilon_n F'(a) \quad \text{as} \quad a = F(a)$$

FRAME 15

The iteration process is based on the assumption that the further we go, the less becomes the magnitude of the difference between a and each successive iterate. This means that $|\varepsilon_{n+1}| < |\varepsilon_n|$. Your answer in 14A shows that this can be expected if $|F'(a)| < 1$. The word 'expected' has been used here because (14A.1) is not exact as the powers of ε_n have been neglected. When testing $F(x)$ to see whether $|F'(a)| < 1$ or not, you will realise that $F'(a)$ cannot be found until a is known. But in order to find whether the iterations will converge to a the value of $F'(a)$ is required! To avoid this difficulty $F'(x)$ is tested in the neighbourhood of the root and if $|F'(x)| < 1$ for values of x in this neighbourhood then the iteration process can be expected to converge.

Returning to the example in FRAME 11, the first expression used in FRAME 13 for $F(x)$ was $10 - x^3 + 4x^2$. For this $F'(x) = -3x^2 + 8x$ which, when $x = 4$, becomes -16. ($x = 4$ was the first approximation used for the root). Thus $|F'(4)| = 16$ which is certainly not < 1. As you saw in FRAME 13, the iterations using this formula for $F(x)$ did not converge. Even if you take a value of x closer to the actual root than 4, $|F'(x)|$ will still not become < 1.

Using $F(x) = \dfrac{4x^2 - x + 10}{x^2}$, $F'(x) = \dfrac{1}{x^2} - \dfrac{20}{x^3}$ and now $F'(4) = -\dfrac{1}{4}$, so $|F'(4)| < 1$. As you will remember, the process worked for this $F(x)$.

Three other possibilities for $F(x)$ were given in FRAME 11. Would any of them be suitable for finding the root of the equation $x^3 - 4x^2 + x - 10 = 0$?

****

15A

Yes: $\sqrt[3]{4x^2 - x + 10}$. *If this is used however, the convergence is not so rapid as for* $\dfrac{4x^2 - x + 10}{x^2}$. *This is because* $|F'(x)|$, *although* < 1, *is larger for* $\sqrt[3]{4x^2 - x + 10}$ *than it is for* $\dfrac{4x^2 - x + 10}{x^2}$.

FRAME 16

What is an estimate for the value of $|F'(x)|$ near the root of $x^3 - 5x^2 - 29 = 0$ when $F(x) = \frac{5x^2 + 29}{x^2}$? (You found the root of $x^3 - 5x^2 - 29 = 0$ in 13A.)

****

16A

$F(x) = -\frac{58}{x^3}$, $|F'(6)| = \frac{58}{216}$

FRAME 17

In FRAME 13 the remark was made that in the example worked there it was necessary to take more steps than was the case with the square and cube roots evaluated earlier, for a similar degree of accuracy. From the work that has been done, can you suggest any theoretical reason why this might be so? (This is not an easy question so - HINT - examine very closely the first paragraph of FRAME 10 and the working in 14A for a clue.)

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17A

In the case in FRAME 10, the error at one stage was approximately a multiple of the square of the error at the previous stage. In 14A, you found that for the process in FRAME 13, the error at one stage was approximately a multiple of the actual error at the previous stage, not the square of the error. For an error numerically less than one, the square of the error will be smaller than the actual error itself and so the convergence will be more rapid. You will remember that the process in FRAME 10 is called a second order process. Similarly one where ε_{n+1} is a multiple of ε_n is called a FIRST ORDER PROCESS.

FRAME 18

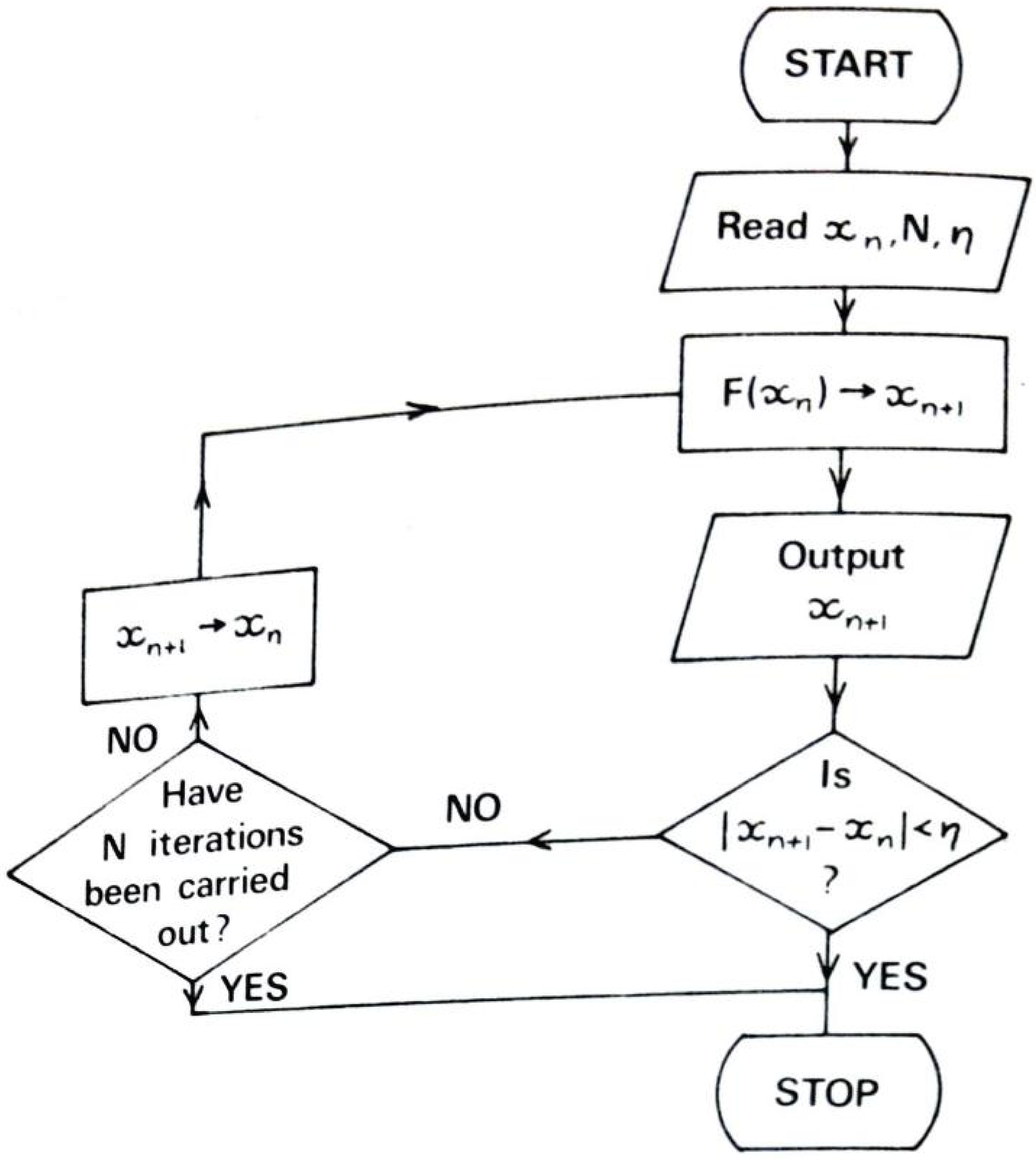
A simple flow chart for finding the root of an equation that has been put in the form $x = F(x)$ is shown on page 38.

FRAME 19

The Newton-Raphson Iteration Formula

You have seen that, given an equation $f(x) = 0$, it is possible to find a solution if it can be put in the form $x = F(x)$, where $F(x)$ is such that $|F'(x)| < 1$ in the neighbourhood of the root. Further, the smaller is $F'(x)$, the quicker will the solution be obtained.

Now let us return to the equation $x^2 = N$. (Effectively this was the equation that was being solved at the beginning of the programme when $\sqrt{N}$ was being found.) Can this be written in the form $x = F(x)$ in such a way that $|F'(x)| < 1$? The only obvious ways of rewriting $x^2 = N$ are $x = \sqrt{N}$ and $x = \frac{N}{x}$. The former isn't much help where iteration is concerned as $F(x)$ must really contain x, not just be a constant. If you try the latter, the iteration formula would be



Flow Diagram for FRAME 18.

Note:- The purpose of N is to stop the computer carrying on for evermore should it meet a situation like that at the beginning of FRAME 13. If the root, to the accuracy specified by η, has not been reached after N steps, the process is terminated.

(A program using the basic iteration method may be found in reference (6).)

FRAME 19 (continued)

$$x_{n+1} = \frac{N}{x_n}$$

and using the same figures as in FRAME 4, i.e. $N = 1973$ and $x_0 = 40$,

$$x_1 = \frac{1973}{40} \simeq 49{\cdot}3 \qquad x_2 = \frac{1973}{49{\cdot}3} \simeq 40$$

and all that will happen is an oscillation between 40 and 49·3, which isn't very helpful. Now although it isn't obvious why this should be

20A

$$OR = OP - RP$$
$$= OP - PQ \cot P\hat{R}Q$$
$$= OP - \frac{PQ}{\tan P\hat{R}Q}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

FRAME 21

In a similar way, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ and generally $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

What will this last formula give if

(i) $f(x) \equiv x^2 - N$ (ii) $f(x) \equiv x^3 - N$?

****

21A

(i) $x_{n+1} = x_n - \dfrac{x_n{}^2 - N}{2x_n}$.

$$= \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

(ii) $x_{n+1} = x_n - \dfrac{x_n{}^3 - N}{3x_n{}^2}$.

$$= \frac{1}{3}\left(2x_n + \frac{N}{x_n{}^2}\right)$$

These are the formulae used earlier when finding the square and cube roots of N.

FRAME 22

In the case illustrated in FRAME 20, the sequence of numbers x_0, x_1, x_2 ..... was decreasing and tending to the actual solution. Now take x_0 on the other side of A, make a similar construction to that made at P previously and see what happens to x_1, x_2 etc.

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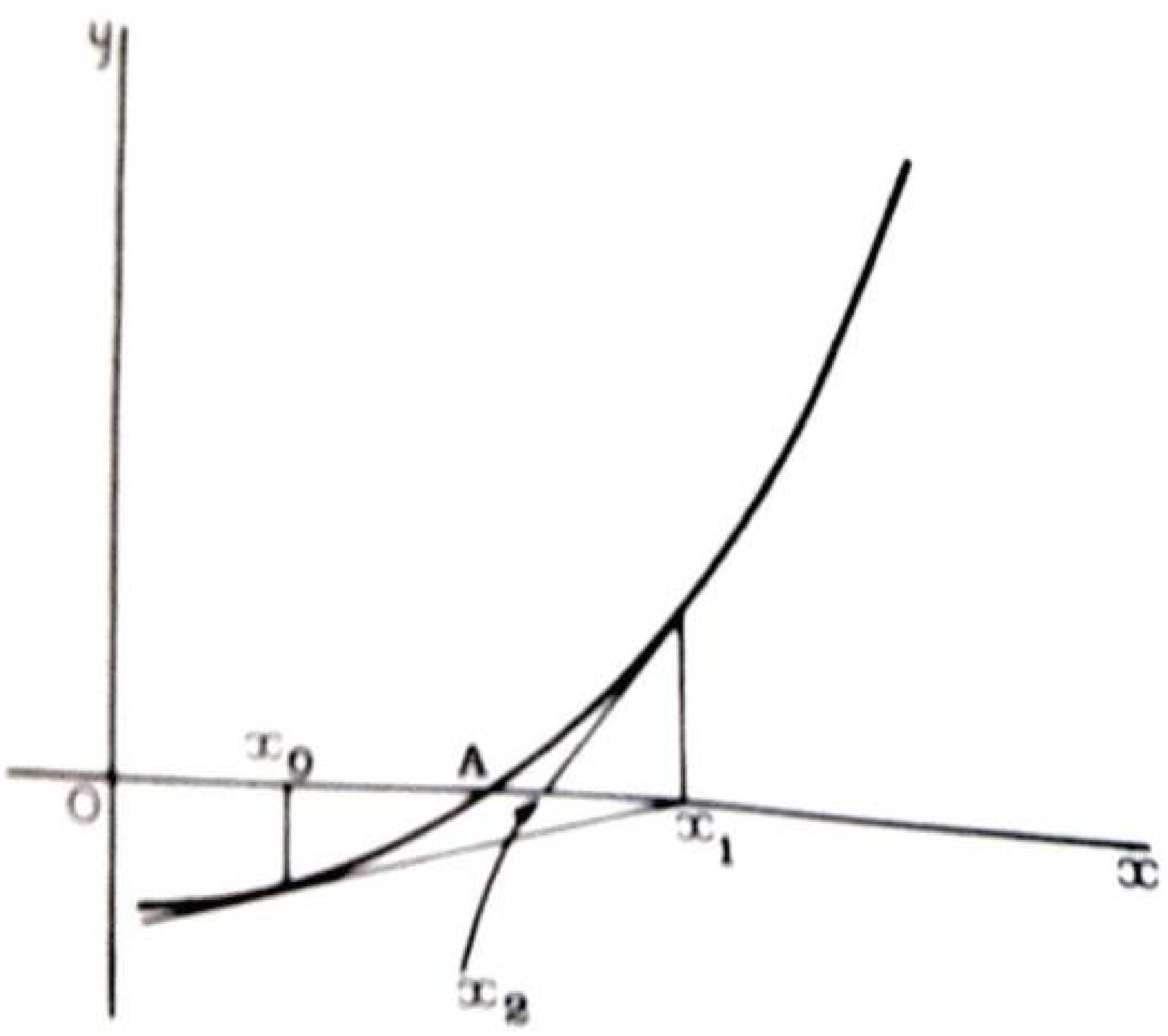
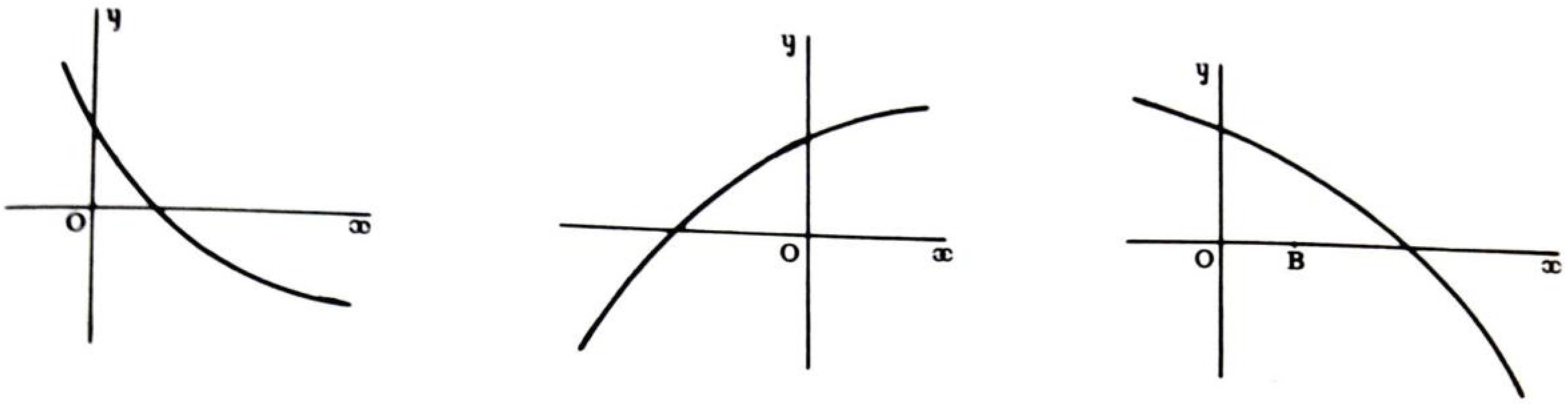
22A

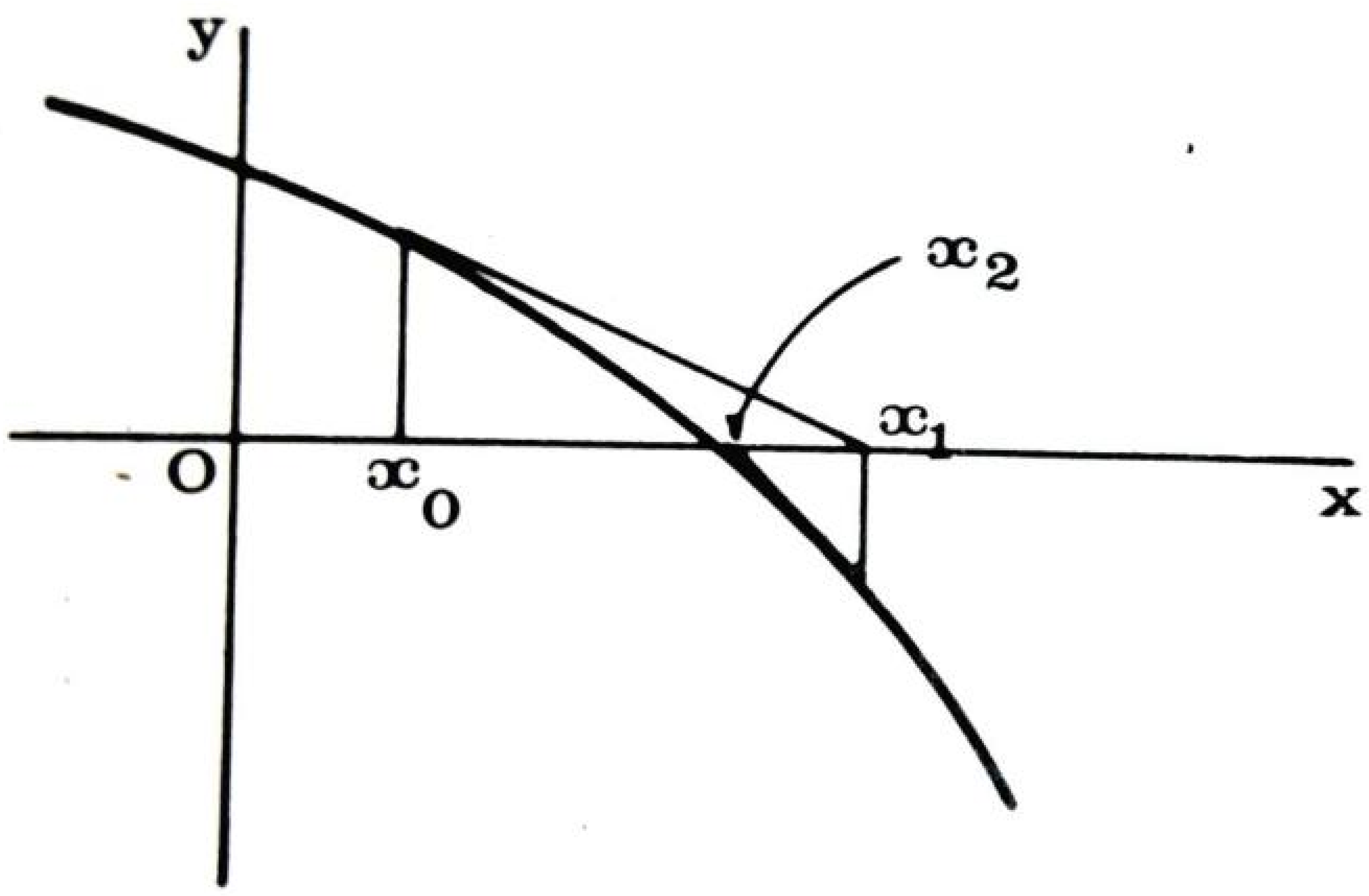


x_1 is on the opposite side of A from x_0. It may even be further away from A than x_0 is itself. After x_1 has been obtained, the ordinary process, described in FRAME 20, takes place.

FRAME 23

In a similar way, the formula will give a method of finding the root of $f(x) = 0$ when $y = f(x)$ takes on a form such as one of the following:





You can easily verify by construction that the desired result is going to be obtained in any of these cases. Thus taking x_0 at B in the last figure, x_1, x_2 etc., are as shown.

It may seem that these diagrams cover all conceivable cases. However, as you will see later, difficulties with this method can occur, but for the time being, no awkward examples will be taken.

FRAME 24

The solution of $x^3 - 4x^2 + x - 10 = 0$ was found in FRAMES 11-13 by rewriting this equation in the form $x = F(x)$, it being arranged so that $|F'(x)| < 1$ in the neighbourhood of the root. This equation will now be used to illustrate the Newton-Raphson process. Taking $f(x)$ as $x^3 - 4x^2 + x - 10$, what will be the formula for x_{n+1} in terms of x_n?

****

24A

$f'(x) = 3x^2 - 8x + 1$

$$x_{n+1} = x_n - \frac{x_n^3 - 4x_n^2 + x_n - 10}{3x_n^2 - 8x_n + 1} \qquad (24A.1)$$

FRAME 25

As in FRAME 11, it is necessary to find a value for x_0 by some other means. Taking, as there, x_0 to be 4, we have

FRAME 25 (continued)

$$x_1 = 4 - \frac{4^3 - 4\times 4^2 + 4 - 10}{3\times 4^2 - 8\times 4 + 1} = 4 + 0\cdot 35 = 4\cdot 35$$

$x_2 = 4\cdot 35 - 0\cdot 0424 = 4\cdot 3076$

$x_3 = 4\cdot 3076 - 0\cdot 0007 = 4\cdot 3069$

$x_4 = 4\cdot 3069 - 0\cdot 0000 = 4\cdot 3069$

Note that, as was suggested earlier in a similar situation, the number of figures in the working is increased as we proceed. The equality of x_3 and x_4 indicates that the solution has been obtained accurate to 4 decimal places.

FRAME 26

There are a few points to notice at this stage about the solution.

Firstly, the R.H.S. of (24A.1) may either be left as it is, or put as a single fraction, i.e., $\dfrac{2x_n^3 - 4x_n^2 + 10}{3x_n^2 - 8x_n + 1}$. Whether or not this is done, both numerator and denominator can be nested for the evaluation of the polynomials. If the R.H.S. of (24A.1) is not combined into a single fraction, the numerator of the last term on the R.H.S. will tend to zero as the root is approached. This is because it is of the same form as the left hand side of the original equation, which is zero at the root.

Secondly, you will notice that on the first application of the process we jumped across the root from 4 to 4·35. The situation is similar to that illustrated in 22A.

Thirdly, the fact that x_4 and x_3 are the same indicates that the root has been obtained to the required degree of accuracy. More about this later, however.

Fourthly, the root has been obtained after fewer iterations than were necessary in FRAME 13.

By now you will have realised that iterative processes are simply a series of repetitions. As such they are very convenient for programming on to a computer. A set of instructions to perform operations which lead to the solution of a problem is called an ALGORITHM.

Now use this process to solve, correct to 4 significant figures,

(i) $x^3 - 5x^2 - 29 = 0$ (ii) $e^x = 10 - x$

The first of these you have already met and the second was mentioned in the previous programme.

****

26A

i) $x_{n+1} = x_n - \dfrac{x_n^3 - 5x_n^2 - 29}{3x_n^2 - 10x_n}$. *Taking, as before,* $x_0 = 6$, *leads to* $x_1 = 5\cdot 85$, $x_2 = 5\cdot 848$, $x_3 = 5\cdot 848$.

Notice that, once again, this process gives the result more quickly than that used in 13A.

26A (continued)

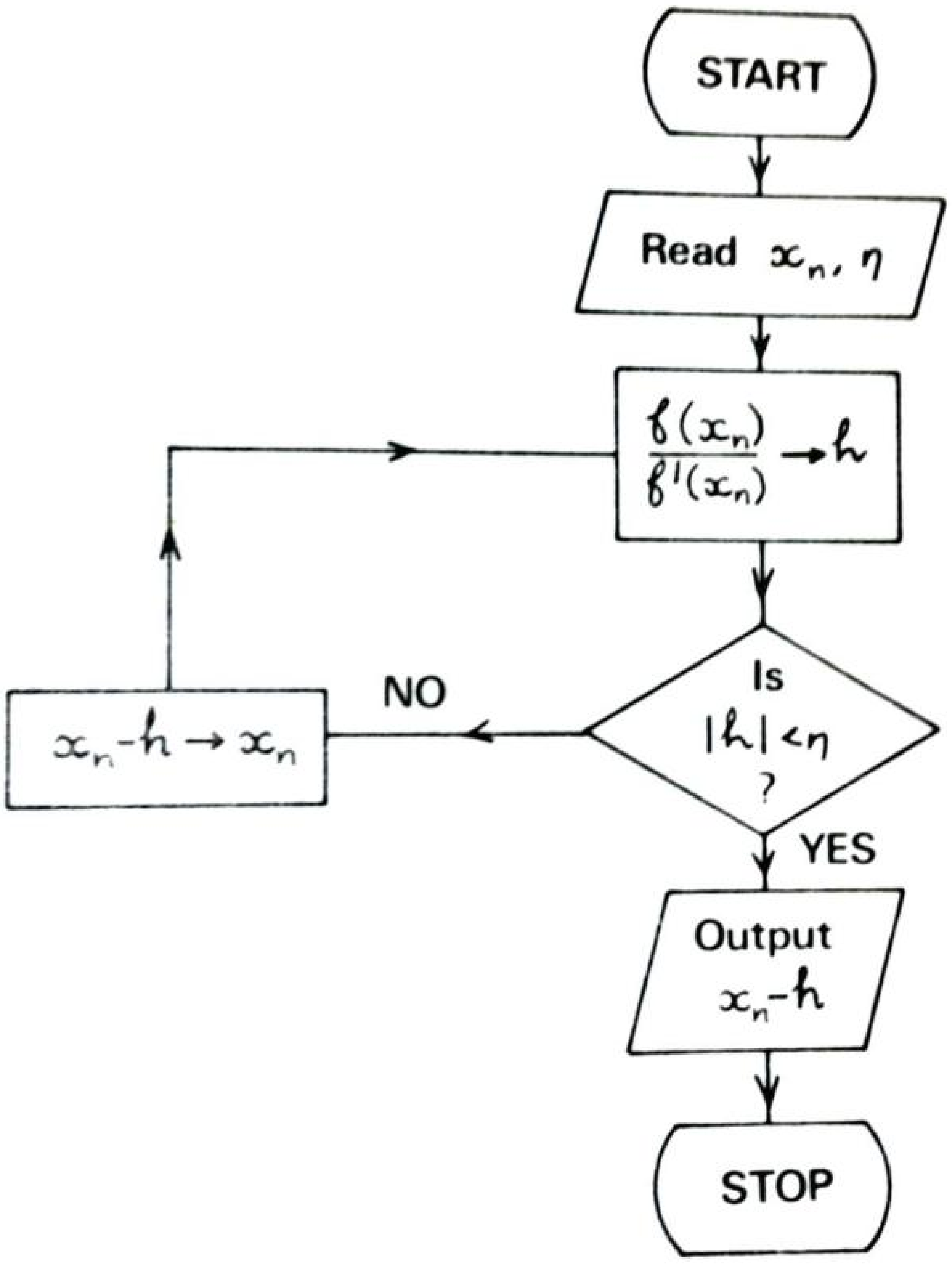
ii) $f(x) = e^x + x - 10$, $f'(x) = e^x + 1$, $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n - 10}{e^{x_n} + 1}$

x	0	1	2	3
$f(x)$	-9	$-6{\cdot}28$	$-0{\cdot}61$	$13{\cdot}09$

The table shows $x_0 = 2$ *to be a suitable starting point.*
$x_1 = 2{\cdot}073$, $x_2 = 2{\cdot}071$, $x_3 = 2{\cdot}071$

FRAME 27

The following is a flow diagram for finding a root of $f(x) = 0$ by the Newton-Raphson method.



[Programs using the Newton-Raphson method may be found in references (1), (3), (4), (5), (6), (7) and (9).]

FRAME 28

In the examples taken so far, the iterations have converged quite rapidly to the root. Is this coincidence, or is it to be expected? To investigate this, a process similar to that in FRAME 14 is used. With the same notation as adopted there,

FRAME 28 (continued)

$$x_{n+1} = a - \varepsilon_n - \frac{f(a - \varepsilon_n)}{f'(a - \varepsilon_n)}$$

Assuming that ε_n is small compared with a and remembering that $f(a) = 0$, show, by using Taylor's series to expand both $f(a - \varepsilon_n)$ and $f'(a - \varepsilon_n)$, and then the binomial series, that this gives

$$x_{n+1} \simeq a + \frac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)} \qquad \text{Assume } f'(a) \neq 0.$$

****

28A

$$x_{n+1} = a - \varepsilon_n - \frac{f(a) - \varepsilon_n f'(a) + \frac{1}{2}\varepsilon_n^2 f''(a) \ldots\ldots\ldots}{f'(a) - \varepsilon_n f''(a) + \frac{1}{2}\varepsilon_n^2 f'''(a) \ldots\ldots\ldots}$$

$$= a - \varepsilon_n + \frac{\varepsilon_n \left\{1 - \frac{1}{2}\varepsilon_n \frac{f''(a)}{f'(a)} \ldots\ldots\ldots\ldots\right\}}{1 - \varepsilon_n \frac{f''(a)}{f'(a)} + \frac{1}{2}\varepsilon_n^2 \frac{f'''(a)}{f'(a)} \ldots\ldots}$$

$$= a - \varepsilon_n + \varepsilon_n \left\{1 - \tfrac{1}{2}\varepsilon_n \frac{f''(a)}{f'(a)} \ldots\ldots\right\}\left\{1 + \varepsilon_n \frac{f''(a)}{f'(a)} \ldots\ldots\right\}$$

$$= a - \varepsilon_n + \varepsilon_n \left\{1 - \tfrac{1}{2}\varepsilon_n \frac{f''(a)}{f'(a)} + \varepsilon_n \frac{f''(a)}{f'(a)} \ldots\ldots\right\}$$

$$\simeq a + \tfrac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)}$$

FRAME 29

$$\text{Therefore} \quad a - x_{n+1} = -\tfrac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)} \quad \text{or} \quad \varepsilon_{n+1} = -\tfrac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)} \qquad (29.1)$$

Thus, the Newton-Raphson iteration formula is a second order process where the error at one stage is a multiple of the square of the error at the previous stage. Unless the multiplying factor is high (which will occur if $f'(a)$ is small and/or $f''(a)$ is large) the convergence will therefore be quite rapid.

This means that very often one can tell that the root of an equation has been reached without getting two equal iterates. To illustrate this, let us return to the equation $x^3 - 4x^2 + x - 10 = 0$, solved in FRAME 25.

The changes from x_0 to x_1, x_1 to x_2, etc., are 0·35, −0·0424, −0·0007, −0·0000. Taking into account magnitude only the second is about one third of the square of the first and the third, one third of the square of the second. The fourth can therefore be expected to be in the region of $\frac{1}{3} \times 0{\cdot}0007^2$, which to four decimal places, is negligible.

Now examine the results obtained in 26A for the two equations you solved there and see whether you would have been justified in stopping at x_2 in each case.

****

29A

i) The changes are $-0{\cdot}15$, $-0{\cdot}002$, $0{\cdot}000$.
Second $\simeq \frac{1}{10} \times$ *square of first. Third can be expected to be in the region of* $0{\cdot}002^2/10$, *i.e., negligible to 3 decimal places.*

ii) The changes are $0{\cdot}073$, $-0{\cdot}002$, $0{\cdot}000$.
Second $\simeq \frac{2}{5} \times$ *square of first. Third can be expected to be in the region of* $\frac{2}{5} \times 0{\cdot}002^2$ *which is negligible to 3 decimal places.*

Hence in each case, a stop could have been made at x_2. *But this idea is only useful when you are performing the calculations manually, i.e. with tables or on a desk machine. When a computer is being used it is easier to let it carry on until it gets two equal iterates than to try and incorporate this more sophisticated test into a program.*

FRAME 30

In practice, you will see that this means that if the changes are going down rapidly, you can stop when a change becomes very small, having regard to the accuracy to which you are working.

Now try the following two problems, which are examples of some applications of this work.

1. The equation $x - 0{\cdot}2 \sin x = 0{\cdot}5$ is a particular case of Kepler's equation which is used in computing the orbits of satellites. Find the solution correct to 4 decimal places.

2. In the design of high voltage tubular electrical insulators, the equation $Q = \dfrac{\pi q^2 (x^2 - 1)}{(\ln x)^2}$ occurs. Assuming q to be a constant, find the value of x (to 4 decimal places) for which Q is a minimum. (Note: $x = 1$ obviously makes the numerator of $\frac{dQ}{dx}$ zero but then the denominator is zero also. The value of x required is the other one that makes the numerator of $\frac{dQ}{dx}$ zero.)

****

30A

1. *0•6155 (0, 1 or anything in between is a reasonable starting point.)*
2. *2•2185 (2 is a reasonable starting point.)*

FRAME 31

So far, only examples that have behaved themselves quite well have been taken to illustrate the use of the Newton-Raphson method. Also, the first approximation has been taken reasonably close to the root and there has only been one real root. The next examples to have a look at are some which have more than one real root, do not behave quite so nicely or where difficulties can arise if care is not taken.

FRAME 32

The Newton-Raphson Formula applied to Examples with more than One Real Root

Some equations, of course, are satisfied by more than one value of x, for example, $x^4 - 6x^2 - 13x + 1 = 0$. To find first approximations, a table of values of $y = x^4 - 6x^2 - 13x + 1$ gives

x	-2	-1	0	1	2	3	4
y	19	9	1	-17	-33	-11	109

This table shows that there are roots near to 0 and 3. To find these roots more accurately, each of these values in turn is used as an x_0 in the Newton-Raphson formula, i.e., in

$$x_{n+1} = x_n - \frac{x_n^{\,4} - 6x_n^{\,2} - 13x_n + 1}{4x_n^{\,3} - 12x_n - 13}$$

Using, first, $x_0 = 0$, successive iterations give $x_1 = 0{\cdot}077$, $x_2 = 0{\cdot}074\,38$, $x_3 = 0{\cdot}074\,37$, and so the root near to 0 is $0{\cdot}074\,37$.

In a similar way, taking $x_0 = 3$ leads to the root $3{\cdot}163\,73$.

Now find the roots of $x^4 - 1{\cdot}12x^3 - 2{\cdot}01x^2 - 3{\cdot}87x - 30{\cdot}72 = 0$, correct to 2 decimal places.

****

32A

A table of values shows that there are two roots, at approximately -2 and 3. Taking x_0 as -2 leads to the root -2·13 and as 3 to 3·14.

FRAME 33

However, sometimes when solving polynomial equations, it is not necessary to use the Newton-Raphson method for all the roots. As an example, let us take the equation $x^3 - 4x^2 + 6 = 0$. Start by finding the largest root correct to 4 decimal places by the Newton-Raphson formula.

****

33A

3·5141.

FRAME 34

The fact that there is a root 3·5141 means that the left hand side has a factor $x - 3{\cdot}5141$. When this factor is removed, we are left with the equation $x^2 - 0{\cdot}4859x - 1{\cdot}7075 = 0$. This is now an ordinary quadratic which can be solved in the usual way to give roots 1·5720 and -1·0862.

From what has been said here, you will realise that an alternative way of solving the quartic in FRAME 32 is to find one root by Newton-Raphson and then take out the corresponding factor from the left hand side. This will leave a cubic equation to which Newton-Raphson can then be applied.

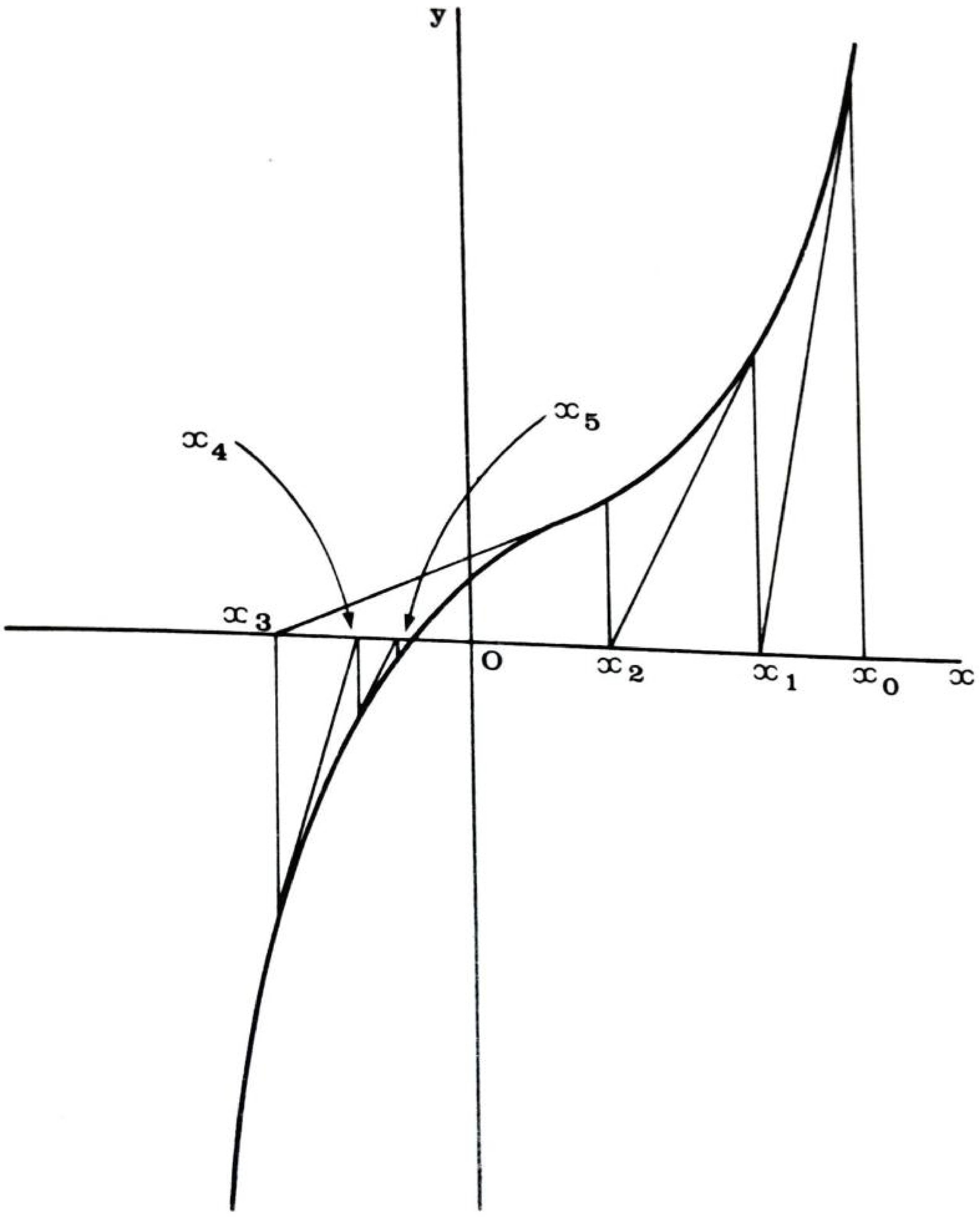
Now take the equation you solved in 32A, divide the left hand side by $x - 3{\cdot}14$ and apply Newton-Raphson to the resulting cubic. You will then be able to compare the amount of work involved by each method.

****

The cubic is $x^3 + 2 \cdot 02x^2 + 4 \cdot 33x + 9 \cdot 73 = 0$ *which with* $x_0 = -2$ *leads to the root* $-2 \cdot 13$. 34A

The Choice of x_0 for the Newton-Raphson Process

FRAME 35



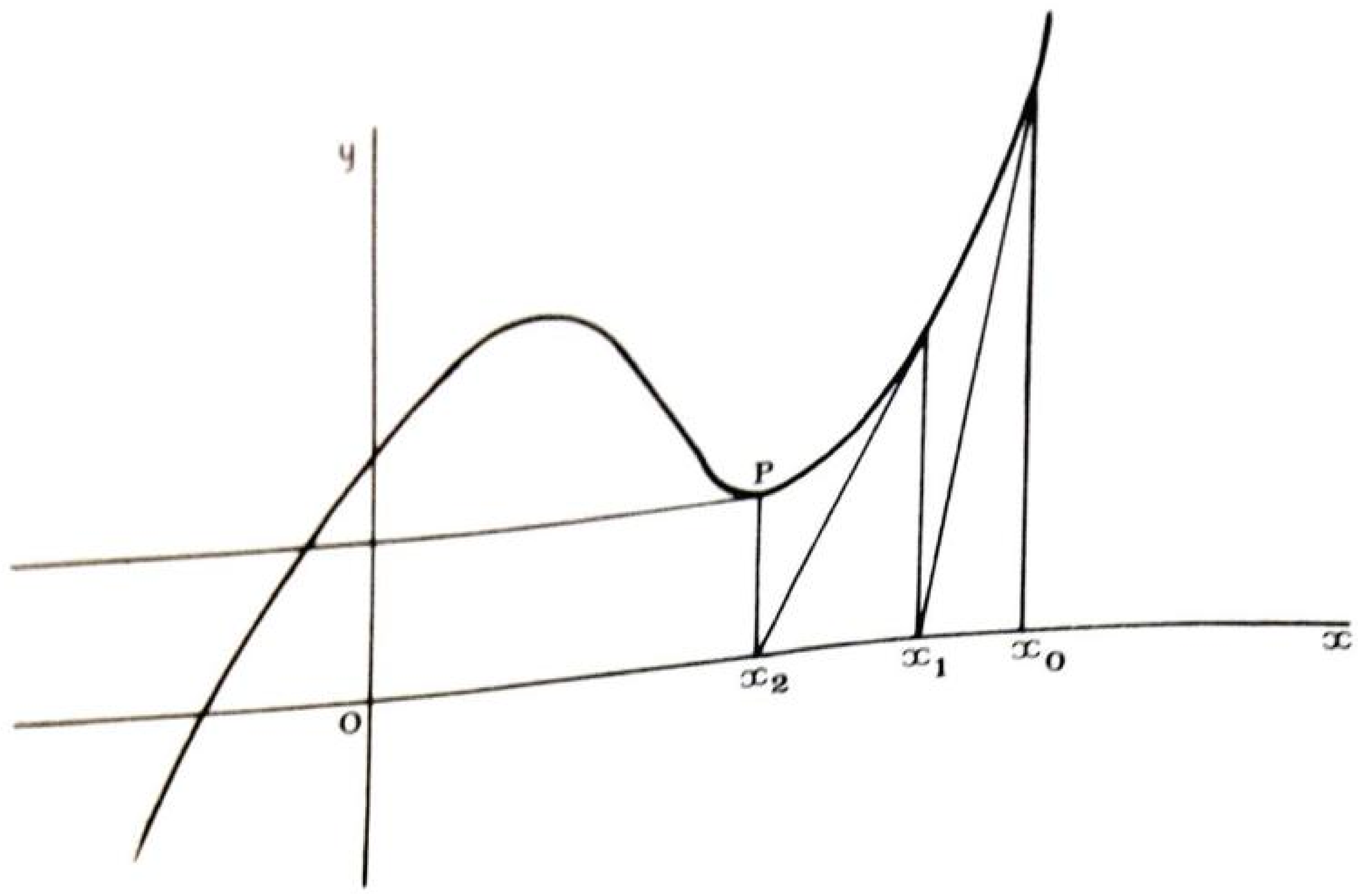
So far, not a great deal has been said about the value you take for x_0. It is more or less common sense to take it close to the root so that the number of iterations required to reach the root is relatively small. But what happens if you don't? Well, in some cases it doesn't make any difference apart from the number of iterations required. The diagram illustrates such a case.

You will notice that the second order property of the process does not hold until we are reasonably close to the root. You will realise that (29.1) is only an approximate formula, ignoring the higher powers of ε_n. Although this is permissible when ε_n is small, it certainly isn't when ε_n is not small.

FRAME 36

The diagram shown on the next page illustrates a case where a poor choice of x_0 is absolutely disastrous.

FRAME 36 (continued)



Here there is a minimum at P, $f'(x_2) = 0$ and so x_3 is now at infinity.

What would have happened if the ordinate to the curve at x_2 had met the curve very slightly to the left of P?

****

36A

x_3 would have been larger than x_2 and could even have been larger than x_0.

Thus, a small value (either +ve or -ve) of $f'(x_n)$ at any stage can seriously upset the convergence of the process.

FRAME 37

Returning for a moment to the example in FRAME 32 what would have happened if the larger root had been required and x_0 had been taken as zero, i.e. the simplest possible value?

****

37A

The iterations would have converged to the root 0·074 37, not to the one required.

This shows that, if an equation has more than one root, and a particular root is required, care must be taken that x_0 is taken near to that root, not to another one.

Errors in the Calculation

FRAME 38

Assuming now that x_0 has been well chosen and convergence is taking place, various things may happen if you make a mistake in your arithmetic at any point. What consequences can you think of? (Suppose the mistake occurs in calculating, say, x_3 from x_2.)

****

38A

Any of the following may occur:

1) *The incorrect value of* x_3 *may be nearer the root than the correct value. You will have been very lucky as the mistake will have helped you on your way.*

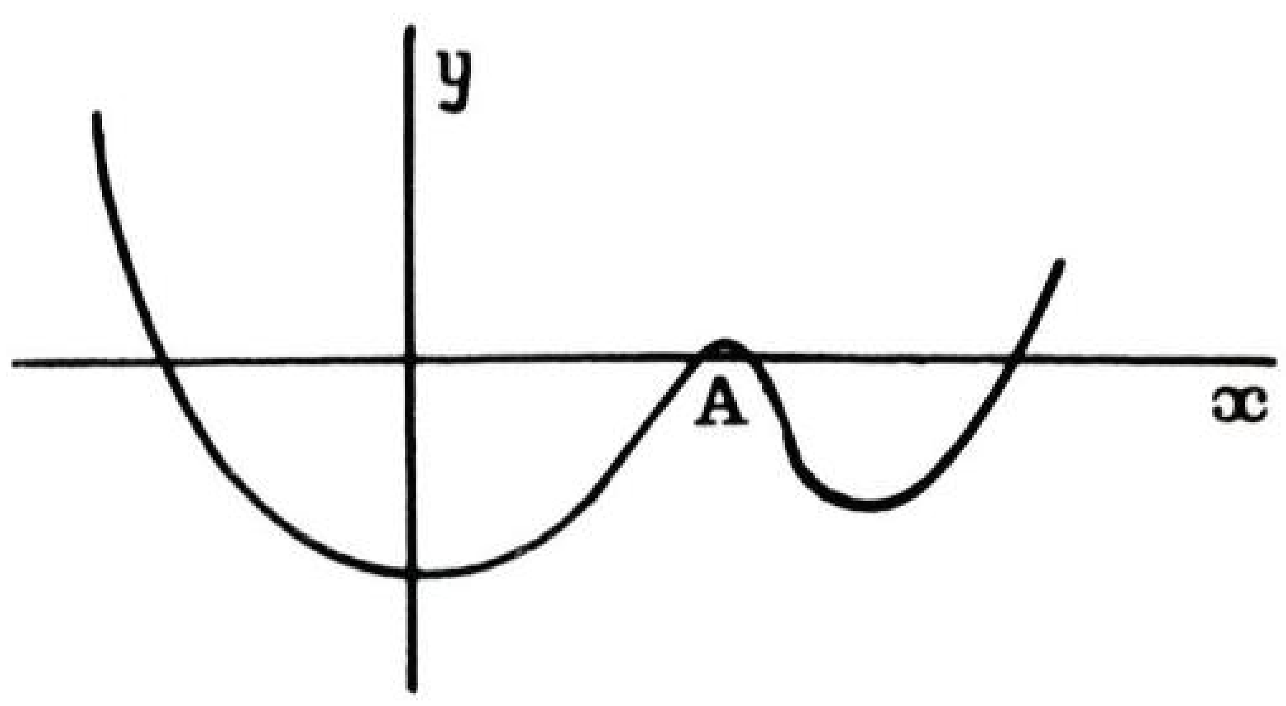
2) *The incorrect value of* x_3 *may be further from the root but not very much. Further iterations will then probably converge to the root, but more slowly than if* x_3 *had been correct. This is the type of error referred to in the previous programme as being automatically taken care of. It is often referred to as a self-correcting error.*

3) *The incorrect* x_3 *may be near to another root of the equation, so that further iterations converge to this other root instead of to the one you are seeking.*

4) *The incorrect* x_3 *may be in such a position that in the end you get nowhere. For example, a situation similar to that in FRAME 36 might now occur.*

FRAME 39

Equal or Nearly Equal Roots



If the graph of a function is of the form shown in the figure, there are two roots very close together in the neighbourhood of A. If the local maximum shown there is actually on the x - axis, then these two roots become coincident. Both cases lead to difficulty with the Newton-Raphson formula as dy/dx is small or zero in the neighbourhood of the roots. Roots such as these occur less frequently than those we have been finding and the method of obtaining them is dealt with in APPENDIX A, rather than in the main part of the programme.

FRAME 40

The Accuracy of the Result

As has been seen, the use of the Newton-Raphson formula enables a root to be obtained to a high degree of accuracy provided that the initial approximation is a reasonable one. Even if it isn't, in some cases no trouble will arise apart from the necessity of using the formula a greater number of times. However, we have so far assumed, in order not to introduce additional complications when dealing with the mechanics of the process, that the equations themselves are exact. This means that all the working has been done on the assumption that all numerical coefficients in the equation are exact. In practice, of course, such coefficients may be subject to round-off errors.

Another source of possible error occurs when solving equations involving functions of x other than polynomials, for example, e^x, sin x, etc. If tables are used to find the values of such functions then each entry in the table is subject to round-off error and consequently this will

FRAME 40 (continued)

affect the accuracy to which the solution can be obtained.

FRAME 41

Although you are now aware of the problem, we think it is unlikely that you will normally have to worry about it. If you do, you will find it treated in APPENDIX B at the end of this programme.

FRAME 42

Complex Roots

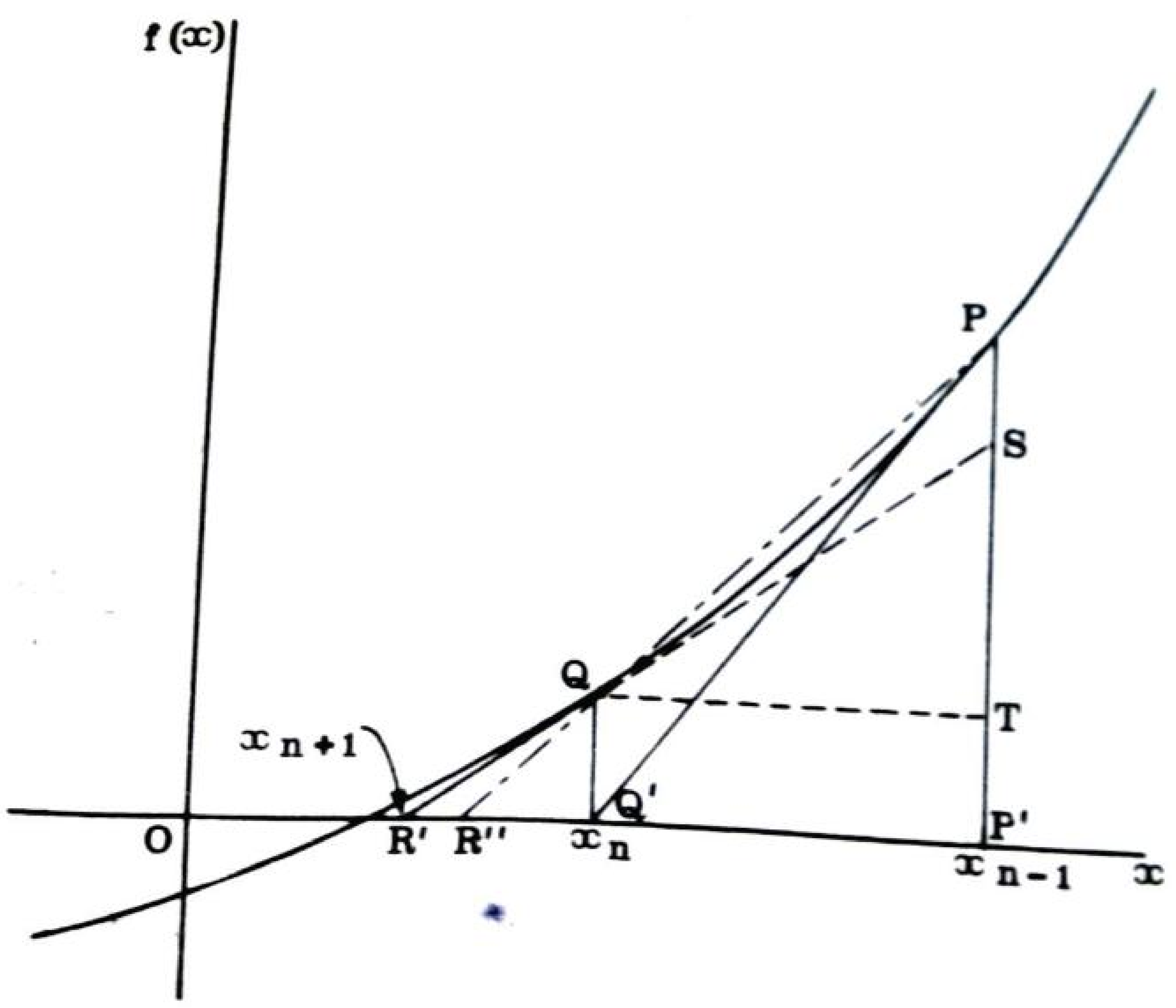
So far, no complex roots have been found, although they have existed in some of the equations solved. Thus, two roots of the quartic equation $x^4 - 6x^2 - 13x + 1 = 0$ were found in FRAME 32, but not the others. In that case they would have presented no difficulty because once two roots have been found, a quartic can be resolved into two quadratic factors (or two linear and one quadratic). However, some equations have no real roots, as, for example, $x^4 + 2x + 3 = 0$. This equation can be solved by a method similar to that which we have been using but the arithmetic does become somewhat worse. Alternatively, it is possible to resolve it directly into quadratic factors, from which the complex roots can be obtained.

Now you may be wondering why we should be interested in such roots. A practical case where complex roots occur is stability theory. There it is necessary to know where the complex roots of a polynomial forming the denominator of what is called the transfer function lie. The ideas behind this are given in the programme "Further Laplace Transforms" in our book "Mathematics for Engineers and Scientists, Vol. 2".

If you need to know about the techniques involved for finding complex roots, you will find them described in APPENDIX C at the end of this programme.

FRAME 43

The Secant Method and the Method of False Position



It has been seen that the advantage of the Newton-Raphson process over the straight-forward iteration considered in FRAMES 11-17 is its rapidity of convergence. However it requires the calculation of the derivative of the function. The SECANT METHOD dispenses with the calculation of derivatives and, although not quite so rapidly convergent as Newton-Raphson, is better in this respect

FRAME 43 (continued)

than the method of FRAMES 11-17.

The diagram shows the construction of x_n from x_{n-1} and of x_{n+1} from x_n when Newton-Raphson is used. As you know, x_{n+1} is given by $x_n - \dfrac{f(x_n)}{f'(x_n)}$ where $f'(x_n)$ is the slope of the line R'Q. If R'Q is produced to S, this slope is given by ST/QT. The secant method replaces this slope by that of the chord QP, i.e., by PT/QT, giving R" for x_{n+1} instead of R'. In the diagram as shown, there is a considerable difference between ST and PT, but the closer you get to the root, the less this difference becomes. Also, as you know, the slope of the chord joining two points on a curve tends to the slope of the tangent, i.e., the derivative, as the two points become closer and closer together. The net effect is that the convergence isn't quite so rapid as with Newton-Raphson, but the advantage of the method is that the calculation of the derivative, as remarked above, is no longer required.

What will the formula for x_{n+1} become when the secant method is used? Give the result in terms of x_n, x_{n-1}, $f(x_n)$, and $f(x_{n-1})$.

****

43A

$$OR'' = OQ' - \frac{OQ'}{\tan Q\hat{R}''x} \quad \text{and} \quad Q\hat{R}''x = P\hat{Q}T$$

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}} \qquad (43A.1)$$

$$= x_n - \frac{(x_{n-1} - x_n)\, f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x_{n-1}\, f(x_n) - x_n\, f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

FRAME 44

To reduce the amount of writing involved in formulae such as this, $f(x_n)$ is often denoted by f_n. With this notation, the secant formula for x_{n+1} can be written as

$$x_{n+1} = \frac{x_{n-1}\, f_n - x_n\, f_{n-1}}{f_n - f_{n-1}} \qquad (44.1)$$

You will notice that, unlike the Newton-Raphson formula, it contains <u>two</u> functional values. Thus, the calculation of x_{n+1} requires a knowledge of f_n <u>and</u> f_{n-1}. At the beginning, assuming $n = 1$,

$$x_2 = \frac{x_0\, f_1 - x_1\, f_0}{f_1 - f_0} \qquad (44.2)$$

x_0 and x_1 are taken as two values of x for which the function is numerically small. As you normally start a problem of this sort by

FRAME 44 (continued)

working out a table of values, two adjacent values in the table can usually be taken for x_0 and x_1.

To illustrate the method, examples will be taken that have previously been worked by other methods. You will then be able to compare the various processes.

The equation $x^3 - 4x^2 + x - 10 = 0$ has already been solved by two methods. In the first method the iteration formula

$$x_{n+1} = \frac{4x_n^{\ 2} - x_n + 10}{x_n^{\ 2}}$$

was used and the second method was by the use of the Newton-Raphson formula. From the table in 12A, you will see that we can take $x_0 = 4$ and $x_1 = 5$, for which $f_0 = -6$, $f_1 = 20$. (These two values of x enclose the root and so it is reasonable to start with them. An alternative selection would be 3 and 4 as these give rise to the two smallest values of $f(x)$ near the root.)

Then $$x_2 = \frac{4 \times 20 - 5 \times (-6)}{20 + 6} \qquad \text{from } (44.2)$$

$$\simeq 4{\cdot}23 \quad \text{and so} \quad f_2 \simeq -1{\cdot}65$$

Continuing $$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1} \qquad \text{from } (44.1) \text{ with } n = 2$$

$$= \frac{5 \times (-1{\cdot}65) - 4{\cdot}23 \times 20}{-1{\cdot}65 - 20} \qquad (44.3)$$

$$\simeq 4{\cdot}289 \qquad \text{and so} \qquad f_3 \simeq -0{\cdot}395$$

What will be the numerical expression giving x_4?

****

44A

$$x_4 = \frac{4{\cdot}23(-0{\cdot}395) - 4{\cdot}289(-1{\cdot}65)}{-0{\cdot}395 + 1{\cdot}65}$$

FRAME 45

From this $x_4 = 4{\cdot}3076$ and then $f_4 = 0{\cdot}015\,25$. Continuing gives $x_5 = 4{\cdot}3069$ and $f_5 = -0{\cdot}0003$. x_5 is the root correct to 4 decimal places. Although x_5 is the root, the iteration formula has only been used four times. The original method in FRAME 13 got the result after 6 iterations and Newton-Raphson after 3.

The working by this method can be exhibited in a table as follows:

4	-6
5	20
4•23	-1•65
4•289	-0•395
4•3076	0•015 25
4•3069	-0•0003

If you look at the working for any stage you will see how the table can be used. Thus, suppose you have progressed as far as

4	-6
5	20
4•23	-1•65

The numerator of (44.3) is the determinant $\begin{vmatrix} 5 & 20 \\ 4\cdot23 & -1\cdot65 \end{vmatrix}$ and the denominator is simply the difference $-1\cdot65 - 20$ of the elements in the second column of this determinant.

Now use this method for the equations (i) $x^3 - 5x^2 - 29 = 0$ and (ii) $e^x = 10 - x$, finding x to 4 significant figures. (See 13A and 26A for tables of values.)

****

45A

i) Taking x_0 as 5 and x_1 as 6 gives

5	*-29*
6	*7*
5•81	*-1•66*
5•846	*-0•087*
5•848	*0•0008*
5•848	

ii) Taking x_0 as 2 and x_1 as 3 gives

2	*-0•61*
3	*13•09*
2•04	*-0•27*
2•06	*-0•09*
2•070	*-0•0052*
2•071	*0•0038*
2•071	

In this example, as the root is obviously near to 2, 2 and 2•1 could be taken as starting values. The working would then be

2	*-0•61*
2•1	*0•27*
2•07	*-0•005*
2•071	*0•0038*
2•071	

giving the root that much quicker.

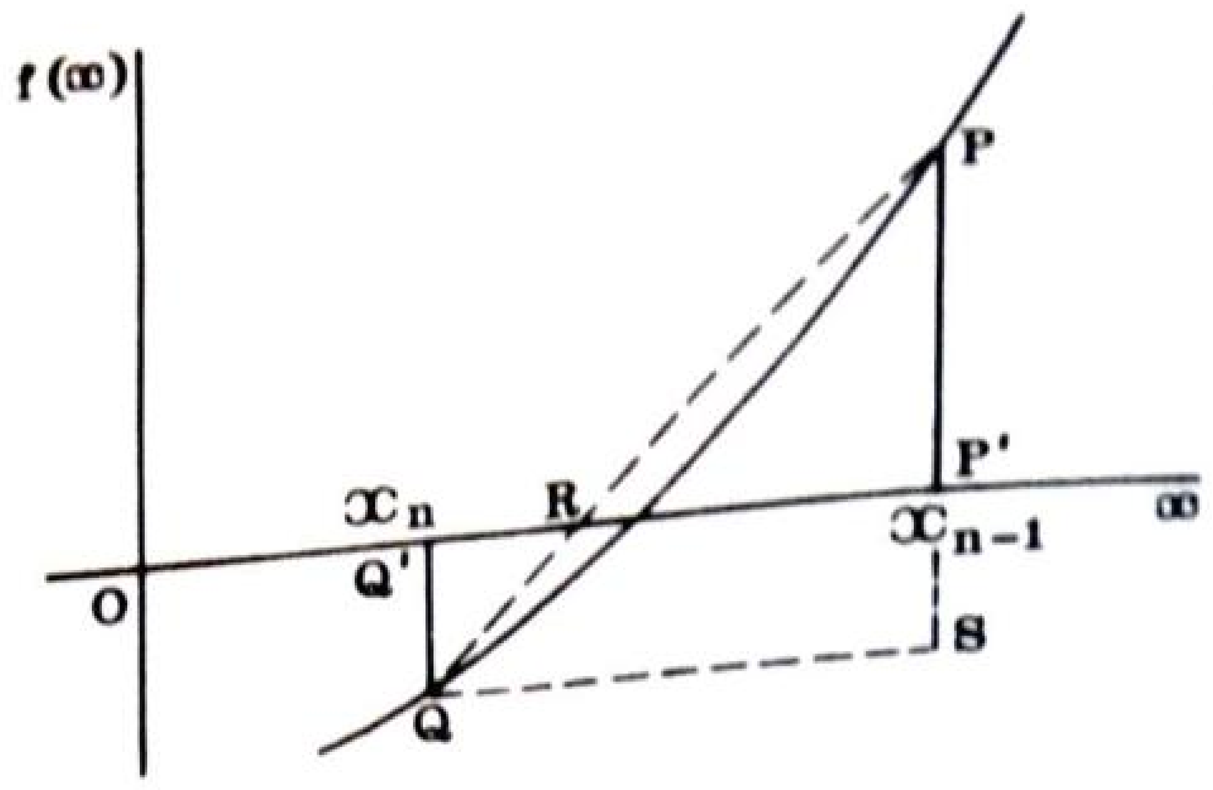
FRAME 46

A variation of this method retains the x_0 and f_0 in (44.2) in place of the x_{n-1} and f_{n-1} in (44.1), giving

$$x_{n+1} = \frac{x_0 f_n - x_n f_0}{f_n - f_0}$$

This means that the slope of the chord PQ is replaced by the slope of the chord joining Q to the top of the first ordinate used. In general, with this method, the iterations will converge to the root more slowly than previously.

FRAME 46 (continued)



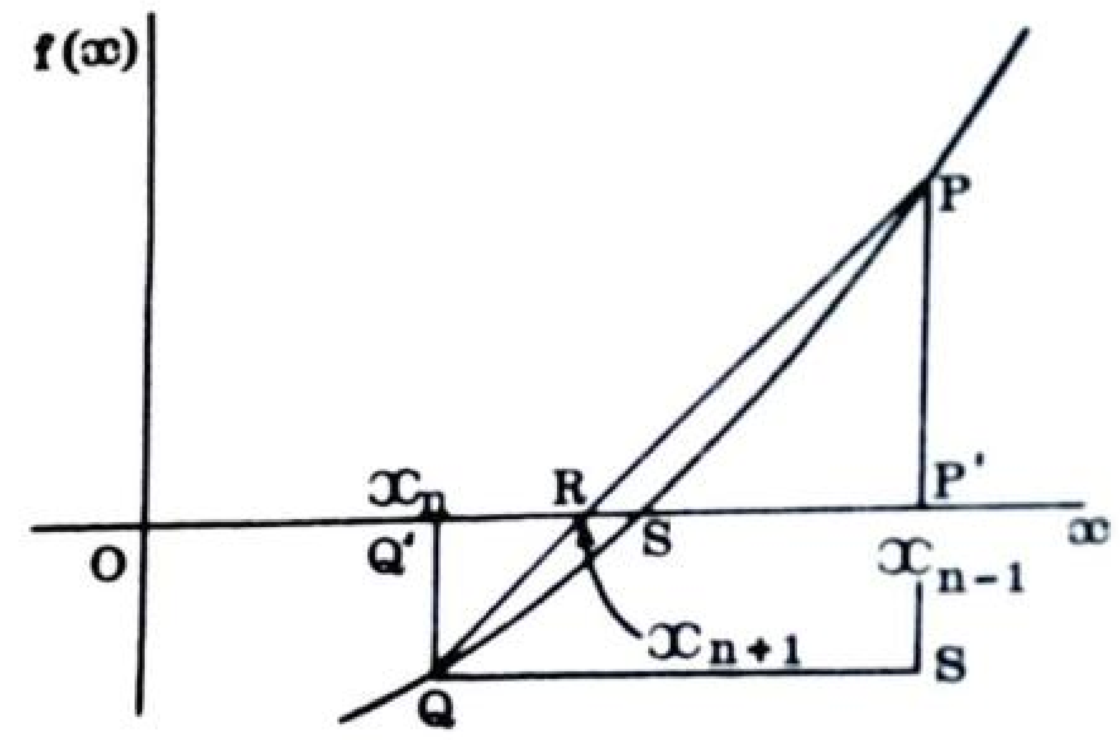
Now suppose the secant formula is used with x_{n-1} and x_n as shown here. Where will be x_{n+1} on this diagram? You can possibly state the answer immediately, but if not, obtain it geometrically by starting from the secant formula in the form (43A.1). If you do it this way, express the various terms in (43A.1) as distances in the figure, i.e. x_{n-1} = OP', $f(x_n)$ = -QQ', etc.

****

46A

In FRAME 43, R", the next approximation to the root, was the point where PQ produced met Ox. In the present situation PQ actually intersects Ox and x_{n+1} is given by R, this point of intersection.

Alternatively,



The right hand side of (43A.1) is equivalent to

$$OQ' - \frac{-QQ'}{PS/Q'P'}$$
$$= OQ' + \frac{QQ'}{PS/QS}$$
$$= OQ' + \frac{QQ'}{QQ'/Q'R}$$
$$= OQ' + Q'R = OR$$

FRAME 47

The only difference between this process and the secant method is that, here, P' and Q' are taken specifically on opposite sides of S. It follows that $f(x_{n-1})$ and $f(x_n)$ are of opposite sign. One starts by choosing x_0 and x_1 so that these two points enclose the root, and using the secant formula to find x_2. x_2 is then taken in conjunction with either x_0 or x_1 and the formula used to find x_3. The choice between x_0 or x_1 here is made by taking whichever gives a functional value opposite in sign to f_2.

Applying this process to the equation $x^3 - 4x^2 + x - 10 = 0$ and taking the same starting points as in FRAME 44, the first calculation is as before, giving the approximation 4·23. For this value of x, f(x) is negative (-1·65). As f(4) = -6 and f(5) = 20, 4·23 is now used in conjunction with 5, as f(4·23) and f(5) are opposite in sign. The

FRAME 47 (continued)

next approximation to the root is therefore

$$\frac{4 \cdot 23 \times 20 - 5 \times (-1 \cdot 65)}{20 - (-1 \cdot 65)} = 4 \cdot 289$$

Now $f(4 \cdot 289) = -0 \cdot 395$ and so using $4 \cdot 289$ in conjunction with a point previously used for which the functional value is positive, write down the numerical expression for the next approximation to the root.

****

47A

Using $4 \cdot 289$ *with* 5 *gives* $\dfrac{4 \cdot 289 \times 20 - 5 \times (-0 \cdot 395)}{20 - (-0 \cdot 395)}$.

This works out to be $4 \cdot 3028$.

FRAME 48

This gives a functional value of $-0 \cdot 09113$ and using this in conjunction with $(5, 20)$ gives $4 \cdot 3060$ as the next approximation. Continuing the process leads to $4 \cdot 3067$ and then $4 \cdot 3069$, the value previously obtained.

Any of the variations described in the last few frames can be called a METHOD of FALSE POSITION. Unfortunately, various authors are not unanimous as to which method they call the rule of false position, but the majority define this as being the last of the processes we have considered. The rule of false position is also known as REGULA FALSI.

A flow diagram for this rule is shown on page 56.

FRAME 49

Simultaneous Non-Linear Equations

This programme has concentrated on methods of finding solutions of single non-linear equations. Some of these methods can be extended to simultaneous non-linear equations and are considered in APPENDIX D at the end of this programme. If your course includes this topic, you should read this appendix. Otherwise you can omit it as to do so will not affect your understanding of the rest of the book.

Various other methods for tackling non-linear equations have been devised. These have not been considered here as this is by no means intended to be an exhaustive treatment of the subject. The idea has been to give you an insight into the sort of approach used so that you can proceed further with a study of the subject, if, at a later stage, this becomes necessary.

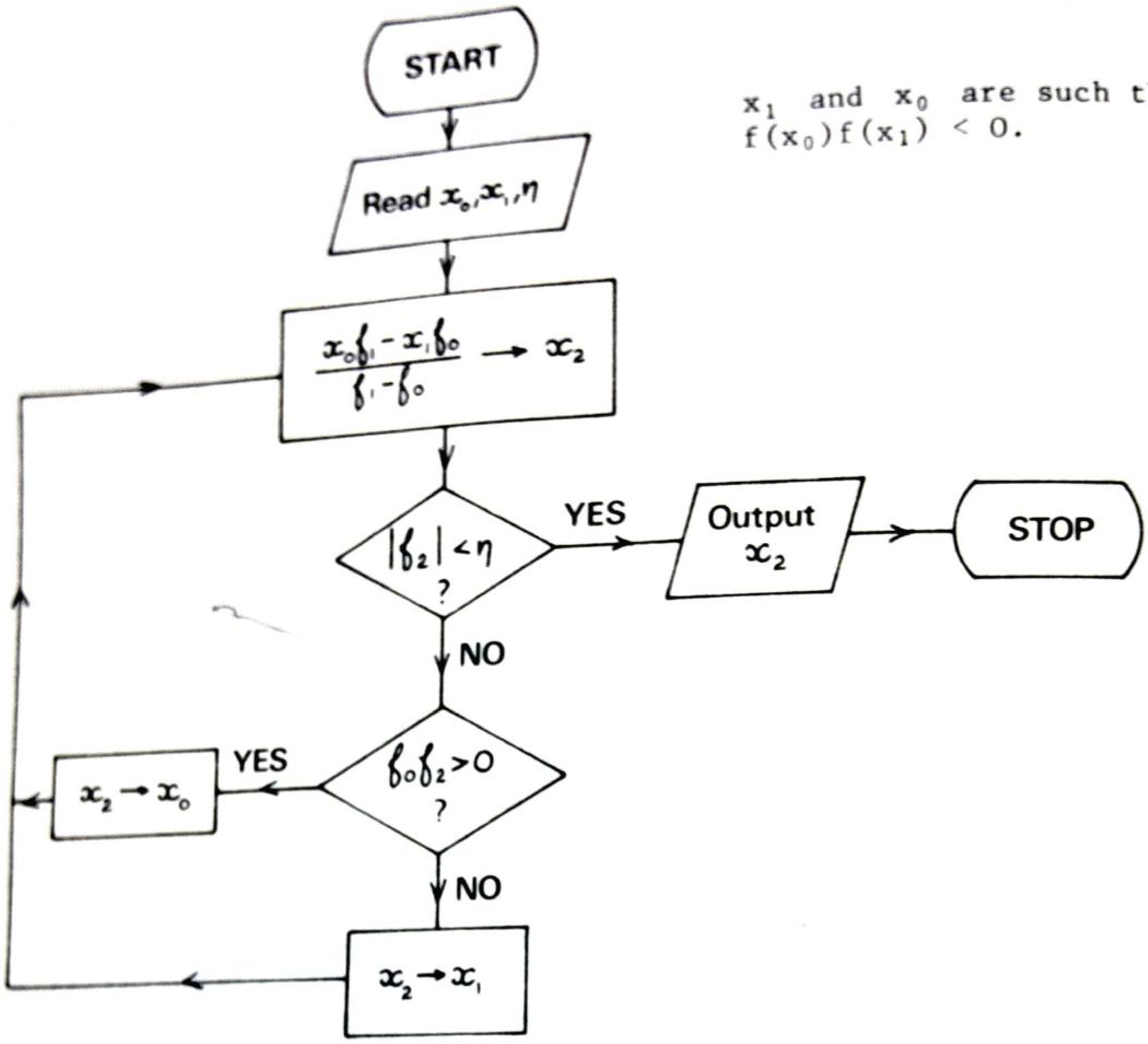
FRAME 50

Miscellaneous Examples

In this frame a collection of miscellaneous examples is given for you to try. Answers are provided in FRAME 51, together with such working as is considered helpful.

1. By taking logarithms, find the solution of $x^x = 10$, correct to 4 significant figures, by (i) Newton-Raphson, (ii) Secant Method.

2. Find, correct to 4 significant figures, the positive root of $4x^4 = x + 8$.



x_1 and x_0 are such that $f(x_0)f(x_1) < 0$.

Flow diagram for FRAME 48.

(Programs using the method of false position may be found in references (3) and (6).

Programs using the secant method (i.e. $f(x_0)$ and $f(x_1)$ not necessarily opposite in sign) may be found in references (6) and (9).)

FRAME 50 (continued)

3. The equation $\sin \omega t - e^{-at} = 0$ arises in the motion of a planetary gear system used in automatic transmission. Determine the smallest root correct to 3 decimal places when $\omega = 0{\cdot}573$ and $a = 0{\cdot}01$.

4. Find the diameter D of the pipe which satisfies the flow equation $8820D^5 - 2{\cdot}31D - 0{\cdot}6465 = 0$, correct to 3 decimal places.

5. The equation $\tan x - x = 0{\cdot}01$ arises in the motion of helical gears.