substitution method, we ...

$x_n, x_{n-1}, x_{n-2}, \cdots x_2, x_1$.

Note. This method of making the matrix A as upper triangular matrix had been taught in lower classes while finding the rank of the matrix A.

4·2·1 Gauss-Jordan elimination method (Direct method)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX = B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making all elements above the leading diagonal of A also as zeros. By this way, the system $AX = B$ will reduce to the form.

$$\left(\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & 0 & b_1 \\ 0 & b_{22} & 0 & 0 & 0 & c_2 \\ . & . & . & . & . & d_3 \\ 0 & 0 & 0 & 0 & \alpha_{nn} & K_n \end{array}\right) \quad \ldots(7)$$

From (7)

$$x_n = \frac{K_n}{\alpha_{nn}}, \ldots, x_2 = \frac{c_2}{b_{22}}, x_1 = \frac{b_1}{a_{11}}$$

Note. By this method, the values of $x_1, x_2, \ldots, x_n$ are got immediately without using the process of back substitution.

Example 1. *Solve the system of equations by (i) Gauss elimination method (ii) Gauss-Jordan method.*

$x + 2y + z = 3, \quad 2x + 3y + 3z = 10; \quad 3x - y + 2z = 13.$ [*MKU 198*

Solution. (By Gauss method)

The given system is equivalent to

$$\underset{A}{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} \underset{=}{=} \underset{B}{\begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}}$$

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

Now, we will make the matrix A upper triangular.

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \quad \begin{array}{ll} R_2 + (-2) R_1 & \text{i.e., } R_{21}(-2) \\ R_3 + (-3) R_1 & \text{i.e., } R_{31}(-3) \end{array}$$

Now take $b_{22} = -1$ as the pivot and make b_{32} as zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_{32}(-7) \qquad \ldots(2)$$

From this, we get

$$x + 2y + z = 3$$
$$-y + z = 4$$
$$-8z = -24$$

$\therefore$ $z = 3, y = -1, x = 2$ by back substitution.

i.e., $x = 2, y = -1, z = 3$

Solution. (Gauss-Jordan method)

In stage 2, make the element, in the position (1, 2) , also zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_{12}(2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right) \quad R_3\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right) \quad R_{13}(3), R_{23}(1)$$

i.e., $x = 2, -y = 1, -z = -3$

i.e., $x = 2, y = -1, z = 3$

Example 2. *Solve the system by Gauss-Elimination method* $2x + 3y - z = 5;\quad 4x + 4y - 3z = 3$ *and* $2x - 3y + 2z = 2.$ [*MKU 1980*]

Solution. The system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \qquad X = B$$

$$\therefore \quad (A, B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

Step 1. Taking $a_{11} = 2$ as the pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \quad R_{21}(-2), R_{31}(-1)$$

Step 2. Taking the element -2 in the position (2, 2) as pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) \quad R_{32}(-3)$$

Hence $\quad 2x + 3y - z = 5$

$$-2y - z = -7$$

$$6z = 18$$

$\therefore \; z = 3, y = 2, x = 1.$ by back substitution.

Example 3. *Solve the following system by Gauss-Jordan method:*

$$5x_1 + x_2 + x_3 + x_4 = 4; \quad x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5; \quad x_1 + x_2 + x_3 + 4x_4 = -6$$

Solution. Interchange the first and the last equation, so that the coefficient of x_1 in the first equation is 1. Then we have

$$(A, B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{6} & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) \quad R_{21}(-1), R_{31}(-1), R_{41}(-5)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{1} & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) \quad R_2\left(\frac{1}{6}\right) \text{ to make the pivot as 1}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{5} & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_{12}(-1),\ R_{42}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{1} & -0.6 & 0.2 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_3\left(\frac{1}{5}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 46.8 \end{array}\right) \quad R_{13}(-1),\ R_{43}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right). \quad R_4\left(\frac{1}{23.4}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \quad R_{34}\left(-\frac{3}{5}\right),\ R_{24}\left(-\frac{1}{2}\right),\ R_{14}(5.1)$$

$$x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2.$$

Example 4. *Solve the system of equations by Gauss-Jordan method :*

$$x + y + z + w = 2$$
$$2x - y + 2z - w = -5$$
$$3x + 2y + 3z + 4w = 7$$
$$x - 2y - 3z + 2w = 5$$

Solution. $(A, B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right)$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & \boxed{1} & 0 & 1 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad R_2\left(-\frac{1}{3}\right)$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right) \quad \begin{array}{l} R_1+(-1)R_2 \\ R_3+R_2 \\ R_4+3R_2 \end{array}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & -3 \end{array}\right) \quad \begin{array}{l} R_3\left(\frac{1}{2}\right) \\ R_4\left(-\frac{1}{4}\right) \end{array}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{c} \text{Interchanging} \\ R_3 \text{ and } R_4 \end{array}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_1+(-1)R_3$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1+(-1)R_4 \\ R_2+(-1)R_4 \\ R_3+R_4 \end{array}$$

$\therefore\ x=0,\ y=1,\ z=-1,\ w=2.$

Example 5. *Apply Gauss-Jordan method to find the solution of the following system :*

$10x+y+z=12;\ 2x+10y+z=13;\ x+y+5z=7.$ *[MS 1991]*

Solution. Since the coefficient of x in the last equation is unity, we rewrite the equations interchanging the first and the last. Hence the augmented matrix is

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad \begin{array}{l} R_2 + (-2) R_1 \\ R_3 + (-10) R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{473}{8} & -\frac{473}{8} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_3\left(-\frac{8}{473}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{49}{8} & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 + (-1)R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} R_2 + \left(\frac{9}{8}\right)R_3 \\ R_1 + \left(-\frac{49}{8}\right)R_3 \end{array}$$

$\therefore \quad x = 1, y = 1, z = 1.$

Example 6. *Using Gauss-Elimination method, solve the system:*

$$3{\cdot}15x - 1{\cdot}96y + 3{\cdot}85z = 12{\cdot}95$$

$$2{\cdot}13x + 5{\cdot}12y - 2{\cdot}89z = -8{\cdot}61$$

$$5{\cdot}92x + 3{\cdot}05y + 2{\cdot}15z = 6{\cdot}88 \qquad \text{[MKU 1981]}$$

Solution. $(A, B) = \left(\begin{array}{ccc|c} 3{\cdot}15 & -1{\cdot}96 & 3{\cdot}85 & 12{\cdot}95 \\ 2{\cdot}13 & 5{\cdot}12 & -2{\cdot}89 & -8{\cdot}61 \\ 5{\cdot}92 & 3{\cdot}05 & 2{\cdot}15 & 6{\cdot}88 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} \boxed{3.15} & -1.96 & 3.85 & 12.95 \\ 0 & \boxed{6.4453} & -5.4933 & -17.3666 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array}\right) \begin{array}{l} R_2+\left(-\frac{2.13}{3.15}\right)R_1 \\ R_3+\left(\frac{-5.92}{3.15}\right)R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & \boxed{6.4453} & -5.4933 & -17.3666 \\ 0 & 0 & 0.6534 & 0.6853 \end{array}\right)$$

$$\therefore \quad 3.15x - 1.96y + 3.85z = 12.95$$

$$6.4453y - 5.4933z = -17.3666$$

$$0.6534z = 0.6853$$

$$\therefore \quad z = \frac{0.6853}{0.6534} = 1.0488$$

$$y = \frac{5.4933 \times 1.0488 - 17.3666}{6.4453} = -1.8005$$

$$x = \frac{1.96 \times (-1.8005) - 3.85\,(1.0488) + 12.95}{3.15} = 1.7089$$

$\therefore \; x = 1.7089, y = -1.8005, z = 1.0488.$

Example 7. *Solve by Gauss-Elemination method:*

$3x+4y+5z=18,\;\; 2x-y+8z=13,\;\; 5x-2y+7z=20.$

Solution. $(A, B) = \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -\frac{11}{3} & \frac{14}{3} & 1 \\ 0 & -\frac{26}{3} & -\frac{4}{3} & -10 \end{array}\right) \begin{array}{l} R_2-\frac{2}{3}R_1 \\ R_3-\frac{5}{3}R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 13 & 2 & 15 \end{array}\right) \begin{array}{l} R_2\,(3) \\ R_3\left(-\frac{3}{2}\right) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & \frac{204}{11} & \frac{204}{11} \end{array}\right) \; R_3+\frac{13}{11}R_2$$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \; R_3\left(\frac{11}{204}\right)$$

$\therefore \qquad z = 1, \; -11y + 14z = 3, \; 3x + 4y + 5z = 18$

Hence, $\qquad z = 1, \; y = \dfrac{3 - 14z}{-11} = 1, \; x = \dfrac{18 - 4y - 5z}{3} = 3$

$\therefore \qquad x = 3, \; y = 1, z = 1.$

EXERCISE 4.1

Solve the following systems by (*i*) Gauss-Elimination (*ii*) Gauss-Jordan methods:

1. $2x + y = 3, \quad 7x - 3y = 4$
2. $11x + 3y = 17, \quad 2x + 7y = 16$
3. $4x - 3y = 11, \quad 3x + 2y = 4$
4. $x - y + z = 1, \quad -3x + 2y - 3z = -6, \quad 2x - 5y + 4z = 5$
5. $x + 3y + 10z = 24, \quad 2x + 17y + 4z = 35, \quad 28x + 4y - z = 32$ [*MS Ap 1992*]
6. $x - 3y - z = -30, \quad 2x - y - 3z = 5, \quad 5x - y - 2z = 142$
7. $5x - 9y - 2z + 4w = 7, \quad 3x + y + 4z + 11w = 2,$
   $10x - 7y + 3z + 5w = 6, \quad -6x + 8y - z - 4w = 5$
8. $10x + y + z = 12, \quad x + 10y + z = 12, \quad x + y + 10z = 12$
9. $10x + y + z = 18{\cdot}141, \; x + 10y + z = 28{\cdot}140, \; x + y + 10z = 38{\cdot}139$ [*MS 1991*]
10. $3x + y - z = 3, \; 2x - 8y + z = -5, \; x - 2y + 9z = 8$
11. $3x - y + 2z = 12, \; x + 2y + 3z = 11, \; 2x - 2y - z = 2$
12. $2x - 3y + z = -1, \; x + 4y + 5z = 25, \; 3x - 4y + z = 2$
13. $x + 2y + 3z = 6, \quad 2x + 4y + z = 7, \quad 3x + 2y + 9z = 14$
14. $2x - y + 3z + w = 9, \quad 3x + y - 4z + 3w = 3,$
    $5x - 4y + 3z - 6w = 2, \quad x - 2y - z + 2w = -2$
15. $4x + y + 3z = 11, \quad 3x + 4y + 2z = 11, \quad 2x + 3y + z = 7$
16. $x + y + 2z = 4, \quad 3x + y - 3z = -4, \quad 2x - 3y - 5z = -5$ [*MS Ap 87*]
17. $2x + 6y - z = -12, \quad 5x - y + z = 11, \quad 4x - y + 3z = 10$
18. $x + 2y + z - w = -2, \quad 2x + 3y - z + 2w = 7$ [*MS Nov 86*]
    $x + y + 3z - 2w = -6, \quad x + y + z + w = 2$
19. $4{\cdot}12x - 9{\cdot}68y + 2{\cdot}01z = 4{\cdot}93$
    $1{\cdot}88x - 4{\cdot}62y + 5{\cdot}50z = 3{\cdot}11$
    $1{\cdot}10x - 0{\cdot}96y + 2{\cdot}72z = 4{\cdot}02$
20. $6x - y + z = 13, \quad x + y + z = 9, \quad 10x + y - z = 19$
21. $x + 2y - 12z + 8w = 27, \quad 5x + 4y + 7z - 2w = 4,$
    $6x - 12y - 8z + 3w = 49, \quad 3x - 7y - 9z - 5w = -11$
22. $x + 0{\cdot}5y + 0{\cdot}33z = 1, \quad 0{\cdot}33x + 0{\cdot}25y + 0{\cdot}2z = 0, \quad 0{\cdot}5x + 0{\cdot}33y + 0{\cdot}25z = 0$
23. $2x + 4y + z = 3, \quad 3x + 2y - 2z = -2, \quad x - y + z = 6$
24. $x + y + z - w = 2, \quad 7x + y + 3z + w = 12,$
    $8x - y + z - 3w = 5, \quad 10x + 5y + 3z + 2w = 20$
25. $2x + 4y + 8z = 41, \quad 4x + 6y + 10z = 56, \quad 6x + 8y + 10z = 64$
26. $2x + 2y - z + w = 4, \quad 4x + 3y - z + 2w = 6,$
    $8x + 5y - 2z + 4w = 12, \quad 3x + 3y - 2z + 2w = 6$

4.3 Inversion of a matrix using Gauss-Elimination method

Let us find the inversion of a non-singular square matrix A of order three. If X is the inverse of A, then $AX = I$ where I is the unix matrix of order three. Now, we have to find the elements of X.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

Therefore, $AX = I$ reduces to

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ...(1)$$

This equation is equivalent to the three equations give below:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad ...(2)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad ...(3)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ...(4)$$

From equations (2), (3), (4), we can solve for the vectors $\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$, $\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}$ and $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$ by using Gaussian elimination method or even by Gauss-Jordan method. The solution set of each system (2), (3), (4) will be the corresponding column of the inverse matrix X.

Note. Since the coefficient matrix is same in all equations (2), (3), (4), we can solve all of them simultaneously.

Example 1. *Find, by Gaussian elimination method, the inverse of*

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}.$$

Solution. *Step 1.* We write down the augmented system (A, I). That is,

$$(A, I) = \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \quad \ldots(1)$$

Step 2. Our aim is to reduce the matrix A to an upper trianguiar matrix. Now we will reduce all elements below a_{11} to zero.

System (1) becomes

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{5}{3} & 0 & 1 \end{array}\right) \begin{array}{l} R_2 + 5R_1 \\ R_3 + \left(-\frac{5}{3}\right)R_1 \end{array} \quad \ldots(2)$$

Note. When we reduce the elements below a_{11} in A to zero, only the first column of I is changed while the second and third column remain unchanged.

Step 3. Now, we will reduce the elements below the position (2, 2) to zero.

Now the system (1), reduces to

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right) R_3 + \left(\frac{1}{3}\right)R_2 \quad \ldots(3)$$

Note. When the elements below the position (2, 2) are reduced to zero, only the second column of I is changed whereas the third column of I is unchanged.

Step 4. Now the system is equivalent to the three systems,

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & \frac{1}{3} & 0 \end{array}\right) \quad \ldots(4)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{array}\right) \quad \ldots(5)$$

and

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \end{array}\right) \quad \ldots(6)$$

That is ,

$$\left.\begin{aligned} 3x_{11} - x_{21} + x_{31} &= 1 \\ x_{21} &= 5 \\ \frac{1}{3}x_{31} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{31} &= 0 \\ x_{21} &= 5 \\ x_{11} &= 2 \end{aligned}$$

$$\left.\begin{aligned} 3x_{12} - x_{22} + x_{32} &= 0 \\ x_{22} &= 1 \\ \frac{1}{3}x_{32} &= \frac{1}{3} \end{aligned}\right\} \quad \begin{aligned} x_{32} &= 1 \\ x_{22} &= 1 \\ x_{12} &= 0 \end{aligned}$$

and

$$\left.\begin{aligned} 3x_{13} - x_{23} + x_{33} &= 0 \\ x_{23} &= 0 \\ \frac{1}{3}x_{33} &= 1 \end{aligned}\right\} \quad \begin{aligned} x_{33} &= 3 \\ x_{23} &= 0 \\ x_{13} &= -1 \end{aligned}$$

Hence $$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Example 2. *By Gaussian elimination, find the inverse of*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{pmatrix}.$$

Solution. The augmented system (A, I) is

$$(A, I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right) \qquad ...(1)$$

Since the element $a_{11} = 0$, we will interchange the first and second row. The reduced system is

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right). \qquad ...(2)$$

By performing $R_3 + (-3)R_1$, we get

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & 0 & -3 & 1 \end{array}\right) \qquad ...(3)$$

Performing $R_3 + 7R_2$,

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right)$$

Thus,

$$\left.\begin{aligned} x_{11} + 2x_{21} &= 0 \\ x_{21} + x_{31} &= 1 \\ 3x_{31} &= 7 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{31} &= \tfrac{7}{3} \\ x_{21} &= -\tfrac{4}{3} \\ x_{11} &= \tfrac{8}{3} \end{aligned}$$

$$\left.\begin{aligned} x_{12} + 2x_{22} &= 1 \\ x_{22} + x_{32} &= 0 \\ 3x_{32} &= -3 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{32} &= -1 \\ x_{22} &= 1 \\ x_{12} &= -1 \end{aligned}$$

$$\left.\begin{aligned} x_{13} + 2x_{23} &= 0 \\ x_{23} + x_{33} &= 0 \\ 3x_{33} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{33} &= \tfrac{1}{3} \\ x_{23} &= -\tfrac{1}{3} \\ x_{13} &= \tfrac{2}{3} \end{aligned}$$

Hence $A^{-1} = \begin{pmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ -\frac{4}{3} & 1 & -\frac{1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{pmatrix}$

Example 3. *By Gaussian elimination, find A^{-1} if*

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}.$$

Solution. $(A, I) \rightarrow \left(\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$ ...(1)

Stage 1. Perform $R_2 + \left(-\dfrac{1}{2}\right)R_1$ and $R_3 + \left(-\dfrac{1}{4}\right)R_1$. Then (1) reduces to

$$\left(\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 1 \end{array}\right)$$

Stage 2. Again taking $\frac{5}{2}$ as the pivot, reduce the position (3, 2) to zero. Perform $R_3 + \left(\frac{9}{10}\right)R_2$. Then augmented system reduces to

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$$\left(\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{3}{10} & -\frac{14}{20} & \frac{9}{10} & 1 \end{array}\right)$$

$$\therefore \quad \left.\begin{aligned} 4x+y+2z &= 1 \\ \tfrac{5}{2}y-2z &= -\tfrac{1}{2} \\ -\tfrac{3}{10}z &= -\tfrac{14}{20} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} z &= \tfrac{7}{3} \\ y &= \tfrac{5}{3} \\ x &= -\tfrac{4}{3} \end{aligned}$$

Again
$$\left.\begin{aligned} 4x+y+2z &= 0 \\ \tfrac{5}{2}y-2z &= 1 \\ -\tfrac{3}{10}z &= \tfrac{9}{10} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} z &= -3 \\ y &= -2 \\ x &= 2 \end{aligned}$$

Also
$$\left.\begin{aligned} 4x+y+2z &= 0 \\ \tfrac{5}{2}y-2z &= 0 \\ -\tfrac{3}{10}z &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} z &= -\tfrac{10}{3} \\ y &= -\tfrac{8}{3} \\ x &= \tfrac{7}{3} \end{aligned}$$

$$\therefore \quad A^{-1} = \begin{pmatrix} -\frac{4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & -\frac{8}{3} \\ \frac{7}{3} & -3 & -\frac{10}{3} \end{pmatrix}$$

EXERCISE 4.2

Find, by Gaussian elimination, the inverses of the following matrices:

1. $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ 2. $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ 3. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$ 5. $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ 6. $P = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$

7. $A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$ 8. $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{pmatrix}$ 9. $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

4.4 Method of Triangularization (Or Method of factorization) (*Direct method*)

This method is also called as *decomposition* method. In this method, the coefficient matrix A of the system $AX = B$, is decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. We will explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad ...(1)$$

This system is equivalent to $AX = B$ ...(2)

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

and an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \text{ so that}$$

$$LUX = B \quad ...(3)$$

Let $UX = Y$ ...(4)

and hence $LY = B$ ...(5)

That is, $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad ...(6)$$

$\therefore\ y_1 = b,\ l_{21}y_1 + y_2 = b_2,\ l_{31}y_1 + l_{32}y_2 + y_3 = b_3$

By forward substitution, y_1, y_2, y_3 can be found out if L is known.

From (4),

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

i.e.,

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}$$

From these, x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known.

Now L and U can be found from

$$LU = A$$

i.e., $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

i.e., $$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3l's and 6 u's.

That is, L and U are known. Hence X is found out. Going into details, we get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first row of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$, $l_{21}u_{12} + u_{22} = a_{22}$, $l_{21}u_{13} + u_{23} = a_{23}$

$$\therefore \quad l_{21} = \frac{a_{21}}{a_{11}}, \quad u_{22} = a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12} \text{ and } \quad u_{23} = a_{23} - \frac{a_{21}}{a_{11}} \cdot a_{13}$$

Again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12} + l_{32}u_{22} = a_{32}$ and

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$$

Solving, $$l_{31} = \frac{a_{31}}{a_{11}}, \quad l_{32} = \frac{a_{32} - \dfrac{a_{31}}{a_{11}} \cdot a_{12}}{a_{22} - \dfrac{a_{21}}{a_{11}} \cdot a_{12}},$$

$$u_{33} = a_{33} - \frac{a_{31}}{a_{11}} \cdot a_{13} - \left(\frac{a_{32} - \dfrac{a_{31}}{a_{11}} \cdot a_{12}}{a_{22} - \dfrac{a_{21}}{a_{11}} \cdot a_{12}}\right)\left(a_{23} - \frac{a_{21}}{a_{11}} \cdot a_{13}\right)$$

Therefore L and U are known.

Note. In selecting L and U we can also take as

$$= \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \text{ and } U = = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

so that $A = LU$.

Example 1. *By the method of triangularization, solve the following system:*

$$5x - 2y + z = 4, \; 7x + y - 5z = 8; \; 3x + 7y + 4z = 10.$$

Solution. The system is equivalent to

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix} \quad ...(1)$$

i.e.,

$$A \cdot X = B$$

Now, let $$LU = A$$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$

Multiplying and equating coefficients,

$$u_{11} = 5,\ u_{12} = -2,\ u_{13} = 1$$

$$l_{21}\, u_{11} = 7,\ l_{21}\, u_{12} + u_{22} = 1,\ l_{21}\, u_{13} + u_{23} = -5$$

Hence $l_{21} = \frac{7}{5},\ u_{22} = 1 - \frac{7}{5}(-2) = \frac{19}{5}$

$$u_{23} = -5 - \frac{7}{5} \times 1 = -\frac{32}{5}$$

Again equating elements in the third row,

$$l_{31}\, u_{11} = 3,\ l_{31}\, u_{12} + l_{32}\, u_{22} = 7,\ l_{31}\, u_{13} + l_{32}\, u_{23} + u_{33} = 4$$

$\therefore$ $l_{31} = \frac{3}{5},\ l_{32} = \dfrac{7 - \frac{3}{5}(-2)}{\frac{19}{5}} = \frac{41}{19}$

$$u_{33} = 4 - \frac{3}{5}(1) - \frac{41}{19}\left(-\frac{32}{5}\right)$$

$$= 4 - \frac{3}{5} + \frac{1312}{95}$$

$$= \frac{1635}{95} = \frac{327}{19}$$

Now L and U are known.

Since $LUX = B$

i.e., $LY = B$ where $UX = Y$.

From $LY = B$,

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$\therefore$ $y_1 = 4,\ \frac{7}{5} y_1 + y_2 = 8,\ \frac{3}{5} y_1 + \frac{41}{19} y_2 + y_3 = 10$

$\therefore$ $y_2 = 8 - \frac{28}{5} = \frac{12}{5}$

$$y_3 = 10 - \frac{12}{5} - \frac{41}{19} \times \frac{12}{5} = 10 - \frac{12}{5} - \frac{492}{95} = \frac{46}{19}$$

$$UX = Y \text{ gives } \begin{pmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{pmatrix}$$

$$\therefore \quad 5x - 2y + z = 4$$

$$\frac{19}{5}y - \frac{32}{5}z = \frac{12}{5}$$

$$\frac{327}{19}z = \frac{46}{19};$$ By back substitution,

$$\therefore \quad z = \frac{46}{327}$$

$$\frac{19y}{5} = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right)$$

$$\therefore \quad y = \frac{284}{327}$$

$$5x = 4 + 2y - z = 4 + \frac{568}{327} - \frac{46}{327}$$

$$\therefore \quad x = \frac{366}{327}$$

$$\therefore \quad x = \frac{366}{327}, \; y = \frac{284}{327}, \; z = \frac{46}{327}$$

Example 2. *Solve, by Triangularization method, the following system:* $x + 5y + z = 14$, $2x + y + 3z = 13$, $3x + y + 4z = 17$.

Solution. This is equivalent to

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

i.e.
$$A \; X = B$$

Let $$LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

By seeing, we can write $u_{11} = 1$, $u_{12} = 5$, $u_{13} = 1$.

$$\therefore \quad \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

Hence, $l_{21} = 2$; $5l_{21} + u_{22} = 1$, $l_{21} + u_{23} = 3$

$$\therefore \quad l_{21} = 2, \; u_{22} = -9, \; u_{23} = 1$$

Again, $l_{31} = 3$; $5l_{31} + l_{32}u_{22} = 1$; $l_{31} + l_{32}u_{23} + u_{33} = 4$

$\therefore \quad l_{32} = \dfrac{1-15}{-9} = \dfrac{14}{9}$; $u_{33} = 4 - 3 - \dfrac{14}{9} = -\dfrac{5}{9}$

$LUX = B$ implies $\quad LY = B$ where $UX = Y$

$LY = B$ gives,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

i.e., $\quad y_1 = 14,\ 2y_1 + y_2 = 13,\ 3y_1 + \dfrac{14}{9} y_2 + y_3 = 17$

$\therefore \quad y_1 = 14,\ y_2 = -15,\ y_3 = -\dfrac{5}{3}$

$UX = Y$ implies,

$$\begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{pmatrix}$$

i.e.,

$$x + 5y + z = 14$$
$$-9y + z = -15$$
$$-\frac{5}{9} z = -\frac{5}{3}$$

$\therefore \quad z = 3,\ y = 2,\ x = 1.$

Example 3. *Solve the following system by triangularization method:*

$$x + y + z = 1,\quad 4x + 3y - z = 6,\quad 3x + 5y + 3z = 4.$$

Solution. Here $A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$

$$\therefore \quad LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$$

$\therefore \quad u_{11} = u_{12} = u_{13} = 1.$

$l_{21} u_{11} = 4,\ l_{21} u_{12} + u_{22} = 3,\ l_{21} u_{13} + u_{23} = -1$

$\therefore \quad l_{21} = 4,\ u_{22} = -1,\ u_{23} = -5$

$l_{31} = 3,\ l_{31} + l_{32} u_{22} = 5,\ l_{31} + l_{32} u_{23} + u_{33} = 3$

$l_{32} = -2,\ u_{33} = -10$

Now, $LUX = B$ implies $\quad LY = B$ where $UX = Y$

As in the previous article, we want to solve the system

$$AX = B$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ ...(1)

Suppose we decompose $A = LU$

where $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and $U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$ ...(2)

Since $AX = B$, $LUX = B$

$\therefore$ $LY = B$ where $UX = Y$ ...(3)

$LU = A$ reduces to

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad ...(4)$$

i.e., $\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Equating coefficients and simplifying as in the previous article, we have

$$l_{11} = a_{11},\ l_{21} = a_{21},\ l_{31} = a_{31}$$

$$u_{12} = \frac{a_{12}}{a_{11}},\ u_{13} = \frac{a_{13}}{a_{11}}$$

$$l_{22} = a_{22} - l_{21}u_{12},\ l_{32} = a_{32} - l_{31}u_{12}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}},\ l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Now L and U are known.

Since $LY = B$, we get

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Multiplying and equating coefficients,

$l_{11}y_1 = b_1$

$l_{21}y_1 + l_{22}y_2 = b_2$

Therefore

$$y_1 = \frac{b_1}{a_{11}}$$

$$l_{31} y_1 + l_{32} y_2 + l_{33} y_3 = b_3 \qquad y_2 = \frac{b_2 - l_{21} y_1}{l_{22}}$$

$$y_3 = \frac{b_3 - l_{31} y_1 - l_{32} y_2}{l_{33}}$$

Knowing $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, L and U,

X can be found out from $UX = Y$.

Note. *Computation scheme by Crout's method* : We write down the 12 unknowns $l_{11}, l_{21}, l_{22}, l_{31}, l_{32}, l_{33}, u_{12}, u_{13}, u_{23}, y_1, y_2, y_3$ as a matrix below, called, auxiliary matrix or derived matrix.

$$\text{derived matrix} = \begin{pmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{pmatrix}$$

If we know the derived matrix, we can write L, U and Y. The derived matrix is got as explained below, using the augmented matrix (A, B).

Step 1. The first column of D.M. (derived matrix) is the same as the first column of A.

Step 2. The remaining elements of first row of D.M. Each element of the first row of D.M. (except the first element l_{11}) is got by dividing the corresponding element in (A, B) by the leading diagonal element of that row.

Step 3. Remaining elements of second column of D.M.

Since $l_{22} = a_{22} - l_{21} u_{12}$; $l_{32} = a_{32} - l_{31} u_{12}$

Each element of second column except the top element = Corresponding element in (A, B) minus the product of the first element in that row and in that column.

Step 4. Remaining elements of second row.

Each element = Corresponding element in (A, B) minus sum of the inner products of the previously calculated elements in the same row and column divided by diagonal element in that row.

Step 5. Remaining element of third column.

$$l_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23}$$

The element = Corresponding element of (A, B) – (sum of the inner products of the previously calculated elements in the same row and column).

Step 6. Remaining element of third row.

$$y_3 = \frac{b_3 - (l_{31}\, y_1 + l_{32}\, y_2)}{l_{33}}$$

The element = Corresponding element of (A, B) – Sum of the inner products of the previously calculated elements in the same row and column divided by the diagonal element in that row.

Example 1. *By Crout's method, solve the system:*

$$2x + 3y + z = -1, \quad 5x + y + z = 9; \quad 3x + 2y + 4z = 11.$$

Solution. Augmented matrix $= (A, B) = \left[\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 5 & 1 & 1 & 9 \\ 3 & 2 & 4 & 11 \end{array}\right]$

Let the derived matrix be D.M $= \begin{pmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{pmatrix}$

Step 1. Elements of first column of D.M. are $\begin{bmatrix} 2 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \end{bmatrix}$

Step 2. Elements of first row.

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{3}{2}$$

$$u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{2}$$

$$y_1 = \frac{b_1}{l_{11}} = \frac{-1}{2}$$

$$\text{D.M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \end{bmatrix}$$

Step 3. Elements of second column.

$$l_{22} = a_{22} - u_{12}\, l_{21}$$

$$= 1 - 5 \times \frac{3}{2} = \frac{-13}{2}$$

$$l_{32} = a_{32} - l_{31}\, u_{12}$$

$$= 2 - 3 \times \frac{3}{2} = -\frac{5}{2}$$

$$\text{D.M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & -\frac{13}{2} & \cdot & \cdot \\ 3 & -\frac{5}{2} & \cdot & \cdot \end{bmatrix}$$

Step 4. Elements of 2nd row.

$$u_{23} = \frac{a_{23} - u_{13}\, l_{31}}{l_{22}}$$

$$= \frac{1-5\times\frac{1}{2}}{-13/2} = \frac{3}{13} \qquad \text{D.M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & \frac{-13}{2} & \frac{3}{13} & \frac{-23}{13} \\ 3 & \frac{-5}{2} & \cdot & \cdot \end{bmatrix}$$

$$y_2 = \frac{9-5(-1/2)}{\frac{-13}{2}} = \frac{-23}{13}$$

Step 5. $l_{33} = 4 - 3\left(\frac{1}{2}\right) - (-5/2)\left(\frac{3}{13}\right) \qquad \text{D.M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & \frac{-13}{2} & \frac{3}{13} & \frac{-23}{13} \\ 3 & \frac{-5}{2} & \frac{40}{13} & \cdot \end{bmatrix}$

$$= 4 - \frac{3}{2} + \frac{15}{26} = \frac{40}{13}$$

Step 6. $y_3 = \dfrac{11 - 3(-1/2) - (-5/2)\left(\frac{-23}{13}\right)}{\frac{40}{13}} = \dfrac{21}{8}$

$$\therefore \quad \text{D.M.} = \begin{bmatrix} 2 & 3/2 & 1/2 & -1/2 \\ 5 & \frac{-13}{2} & \frac{3}{13} & \frac{-23}{13} \\ 3 & \frac{-5}{2} & \frac{40}{13} & \frac{21}{8} \end{bmatrix}$$

The solution is got from $UX = Y$

i.e.,
$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{13} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{23}{13} \\ \frac{21}{8} \end{bmatrix}$$

$$\therefore \quad z = \frac{21}{8};\ y + \frac{3}{13}z = \frac{-23}{13};\ x + \frac{3y}{2} + \frac{1}{z} = -1/2$$

$$\therefore \quad y = \frac{-23}{13} - \frac{3}{13}\left(\frac{21}{8}\right) = \frac{-19}{8}$$

$$x = -\frac{3}{2}\left(-\frac{19}{8}\right) - \frac{1}{2}\left(\frac{21}{8}\right) - \frac{1}{2} = \frac{7}{4}$$

$$\therefore \quad x = \frac{7}{4},\ y = -\frac{19}{8},\ z = \frac{21}{8}$$

Example 2. *Solve, by Crout's method, the following:*
$x+y+z=3,\ 2x-y+3z=16,\ 3x+y-z=-3.$ [MKU 1981]

Solution. Here, $(A, B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 16 \\ 3 & 1 & -1 & -3 \end{array}\right]$

Let the derived matrix be: D.M.$= \left[\begin{array}{ccc|c} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{array}\right]$

Step 1. Elements of first column of D.M. are $= \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \end{bmatrix}$

Step 2. Elements of first row of D.M.:

$$u_{12} = \frac{1}{1} = 1;\quad u_{13} = \frac{1}{1} = 1\ ;\quad y_1 = \frac{3}{1} = 3$$

$$\therefore \quad \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \end{bmatrix}$$

Step 3. Elements of second column:

$$l_{22} = a_{22} - u_{12}\, l_{21} = -1 - 2 = -3$$

$$l_{32} = a_{32} - u_{12}\, l_{31} = 1 - 1 \times 3 = -2$$

$$\therefore \quad \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & \cdot & \cdot \\ 3 & -2 & \cdot & \cdot \end{bmatrix}$$

Step 4. Elements of second row:

$$u_{23} = \frac{3 - 1\,(+2)}{-3} = \frac{-1}{3}$$

$$y_2 = \frac{16 - 3 \times 2}{-3} = \frac{-10}{3}$$

$$\therefore \quad \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -\frac{1}{3} & \frac{-10}{3} \\ 3 & -2 & \cdot & \cdot \end{bmatrix}$$

Step 5. Elements of third column:

$$l_{33} = -1 - 1\,(3) - (-1/3)\,(-2) = -\frac{14}{3}$$

where

Step 6. Elements of third row:

$$y_3 = \frac{-3-(3)(3)-(-2)\left(\frac{-10}{3}\right)}{-\frac{14}{3}} = 4$$

$$\text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1/3 & \frac{-10}{3} \\ 3 & -2 & \frac{-14}{3} & 4 \end{bmatrix}$$

The solution is got from $UX = Y$, *i.e.*,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10/3 \\ 4 \end{bmatrix}$$

$$x + y + z = 3$$

$$y - \frac{1}{3} z = \frac{-10}{3}$$

$$z = 4$$

By back substitution, $z = 4$, $y = -2$, $x = 1$.

Example 3. *By Crout's method, solve the system:*

$$x + 2y + 3z + 4w = 20$$
$$3x - 2y + 8z + 4w = 26$$
$$2x + y - 4z + 7w = 10$$
$$4x + 2y - 8z - 4w = 2$$

Solution. The augmented matrix $= (A, B) = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & -2 & 8 & 4 & 26 \\ 2 & 1 & -4 & 7 & 10 \\ 4 & 2 & -8 & -4 & 2 \end{array}\right]$

The derived matrix $= \text{D.M.} = \begin{bmatrix} l_{11} & u_{12} & u_{13} & u_{14} & y_1 \\ l_{21} & l_{22} & u_{23} & u_{24} & y_2 \\ l_{31} & l_{32} & l_{33} & u_{34} & y_3 \\ l_{41} & l_{42} & l_{43} & l_{44} & y_4 \end{bmatrix}$

Step 1. The first column of D.M. is $= \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$

Step 2. The elements of first row:

$$u_{12}=\frac{2}{1}=2;\; u_{13}=\frac{3}{1}=3;\; u_{14}=\frac{4}{1}=4;\; y_1=\frac{20}{1}=20$$

$$\text{D.M.}=\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

Step 3. Elements of second column:

$$l_{22}=-2-3\times 2=-8$$

$$l_{32}=1-4=-3$$

$$l_{42}=2-8=-6$$

$$\therefore \quad \text{D.M.}=\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & -8 & \cdot & \cdot & \cdot \\ 2 & -3 & \cdot & \cdot & \cdot \\ 4 & -6 & \cdot & \cdot & \cdot \end{array}\right]$$

Step 4. The elements in second row:

$$u_{23}=\frac{a_{23}-l_{21}\,u_{13}}{l_{22}}=\frac{8-9}{-8}=\frac{1}{8}$$

$$u_{24}=\frac{4-4\times 3}{-8}=1$$

$$y_2=\frac{26-20\times 3}{-8}=\frac{17}{4}$$

$$\therefore \quad \text{D.M.}=\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & -8 & \frac{1}{8} & 1 & \frac{17}{4} \\ 2 & -3 & \cdot & \cdot & \cdot \\ 4 & -6 & \cdot & \cdot & \cdot \end{array}\right]$$

Step 5. Elements of third column:

$$l_{33}=a_{33}-(l_{31}\,u_{13}+l_{32}\,u_{23})$$

$$=-4-(3\times 2)-(-3)\left(\frac{1}{8}\right)=-\frac{77}{8}$$

$$l_{43}=a_{43}-(l_{41}\,u_{13}+l_{42}\,u_{23})$$

$$=-8-4\times 3-(-6)\left(\frac{1}{8}\right)=-\frac{77}{4}$$

Ax = b
where...

Then D.M. is $=\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & -8 & \frac{1}{8} & 1 & \frac{17}{4} \\ 2 & -3 & \frac{-77}{8} & \cdot & \cdot \\ 4 & -6 & \frac{-77}{4} & \cdot & \cdot \end{array}\right]$

Step 6. Elements of third row:

$$u_{34}=\frac{a_{34}-(l_{31}\,u_{14}+l_{32}\,u_{24})}{l_{33}}$$

$$=\frac{7-4\times 2-(1)(-3)}{\frac{-77}{8}}=-\frac{16}{77}$$

$$y_3=\frac{b_3-(l_{31}\,y_1+l_{32}\,y_2)}{l_{33}}$$

$$=\left[10-20\times 2-(-3)\left(\frac{17}{4}\right)\right]\div\left(\frac{-77}{8}\right)=\frac{138}{77}$$

$$\text{D.M.}=\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & -8 & \frac{1}{8} & 1 & \frac{17}{4} \\ 2 & -3 & \frac{-77}{8} & \frac{-16}{77} & \frac{138}{77} \\ 4 & -6 & \frac{-77}{4} & & \end{array}\right]$$

Step 7. Elements of 4th column:

$$l_{44}=-4-4\times 4-1(-6)-\left(\frac{-16}{77}\right)\left(\frac{-77}{4}\right)=-18$$

Step 8. Elements of 4th row:

$$y_4=\frac{b_4-(l_{41}\,y_1+l_{42}\,y_2+l_{43}\,y_3)}{l_{44}}$$

$$=\frac{2-(4)(20)-(-6)\left(\frac{17}{4}\right)-\left(-\frac{77}{4}\right)\left(\frac{138}{77}\right)}{-18}$$

$$=\frac{2-80+25{\cdot}5+\frac{69}{2}}{-18}=1$$

$$\text{D.M.}=\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 20 \\ 3 & -8 & \frac{1}{8} & 1 & \frac{17}{4} \\ 2 & -3 & \frac{-77}{8} & \frac{-16}{77} & \frac{138}{77} \\ 4 & -6 & \frac{-77}{4} & -18 & 1 \end{array}\right]$$

The solution is got from $UX = Y$. That is,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{1}{8} & 1 \\ 0 & 0 & 1 & \frac{-16}{77} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 20 \\ \frac{17}{4} \\ \frac{138}{77} \\ 1 \end{bmatrix}$$

$$\therefore \qquad x + 2y + 3z + 4w = 20$$

$$y + \frac{1}{8}z + w = \frac{17}{4}$$

$$z - \frac{16}{77}w = \frac{138}{77}$$

$$w = 1$$

By back substitution, $\quad z = 2$

$$y + \frac{1}{4} + 1 = \frac{17}{4}; \quad \therefore \quad y = 3$$

$$\therefore \qquad x = -2y - 3z - 4w + 20 = 4$$

Hence, $x = 4, \; y = 3, \; z = 2, \; w = 1.$

EXERCISE 4.4

Using Crout's method, solve the following muster of equations:

1. $x + y + 2z = 7, \; 3x + 2y + 4z = 13, \; 4x + 3y + 2z = 8$
2. $2x + 4y + z = 5, \; 4x + 4y + 3z = 8, \; 4x + 8y + z = 9$
3. $2x - 6y + 8z = 24, \; 5x + 4y - 3z = 2, \; 3x + y + 2z = 16$
4. $2x + 3y + 2z = 2, \; 3x + 6y + z = -6, \; 10x + 3y + 4z = 16$
5. $x + y + z = 2, \; 2x + 3y - 2z = -4, \; x - 2y + 4z = 17$
6. $2x - 6y + 8z = 24, \; 5x + 4y - 3z = 2, \; 3x + y + 2z = 16$
7. $5x + 2y + z = -12, \; -x + 4y + 2z = 20, \; 2x - 3y + 10z = 3$
8. $10x + y + z = 12, \; 2x + 10y + z = 13, \; 2x + 2y + 10z = 14$
9. $x + 2y + 3z = 6, \; 2x + 4y + z = 7, \; 3x + 2y + 9z = 14$
10. $2x - y + 3z + w = 9, \; -x + 2y + z - 2w = 2, \; 3x + y - 4z + 3w = 3,$
    $5x - 4y + 3z - 6w = 2$
11. $10x - 7y + 3z + 5w = 6, \; 5x - 9y - 2z + 4w = 7,$
    $3x + y + 4z + 11w = 2, \; -6x + 8y - z - 4w = 5$
12. $x + y + z = 1; \; 3x + y - 3z = 5; \; x - 2y - 5z = 10$
13. $2x + y + 3z = 13, \; x + 5y + z = 14, \; 3x + y + 4z = 17$
14. $x + y + 2z = 4, \; 3x + y - 3z = -4, \; 2x - 3y - 5z = -5$
15. $x + y + z + w = 4. \; 2x + 3y + 4z + 5w = 14, \; 3x - y + z + w = 4, \; x - y + 3z + 5w = 8$

4·6. Crout's method for finding the inverse of matrix

Our aim is to find the inverse of a square matrix A. We have seen already, that A can be decomposed into $A = LU$ where L is lower triangular matrix and U is unit upper triangular matrix.

$$A^{-1} = (LU)^{-1} = U^{-1} L^{-1} \quad \ldots(1)$$

If L is lower triangular, then L^{-1} is also lower triangular. Also if U is upper triangular, then U^{-1} is also upper triangular.

Since $LL^{-1} = I$ we can find the lower triangular matrix L^{-1} such that $LL^{-1} = I$ (since L is known).

Similarly, since $UU^{-1} = I$, we can find the upper triangular matrix U^{-1} such that $UU^{-1} = I$.

Having known, L^{-1} and U^{-1}, we get $A^{-1} = U^{-1} L^{-1}$.

Example 1. *Find the inverse of* $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$, *by Crout's method.*

Solution. Since $LU = A$, we have $A^{-1} = U^{-1} L^{-1} \quad \ldots(1)$

$LU = A$ implies

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

Using the method used in Crout's method, we have

$$l_{11} = 3, \; l_{21} = -15, \; l_{31} = 5$$

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{-1}{3}, \quad u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{3}$$

$$l_{22} = a_{22} - l_{21} u_{12} = 6 - (-15)\left(-\frac{1}{3}\right) = 1$$

$$l_{32} = a_{32} - l_{31} u_{12} = -2 - 5\left(-\frac{1}{3}\right) = -\frac{1}{3}$$

$$u_{23} = \frac{a_{23} - l_{21} u_{13}}{l_{22}} = \frac{-5 - (-15)\left(\frac{1}{3}\right)}{1} = 0$$

$$l_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23}$$

$$= 2 - 5\left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right)(0) = \frac{1}{3}$$

$$\therefore \quad L = \begin{pmatrix} 3 & 0 & 0 \\ -15 & 1 & 0 \\ 5 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We shall find L^{-1} and U^{-1}: Since $LL^{-1} = I$; and L^{-1} is also lower triangular,

$$\underset{L}{\begin{pmatrix} 3 & 0 & 0 \\ -15 & 1 & 0 \\ 5 & -1/3 & 1/3 \end{pmatrix}} \underset{L^{-1}}{\begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}} \underset{=}{=} \underset{I}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$3x_{11} = 1, \; -15\,x_{11} + x_{21} = 0, \; x_{22} = 1, \; 5x_{11} - \frac{1}{3}x_{21} + \frac{1}{3}x_{31} = 0$$

$$-\frac{1}{3}x_{22} + \frac{1}{3}x_{32} = 0, \; \frac{1}{3}x_{33} = 1$$

$$\therefore \quad x_{11} = \frac{1}{3}, \; x_{22} = 1, \; x_{33} = 3, \; x_{21} = 5, \; x_{31} = 0, \; x_{32} = 1$$

$$\therefore \qquad L^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Since, $UU^{-1} = I$, and U^{-1} is upper triangular, we have

$$\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_{11} = 1, \; c_{12} - \frac{1}{3}c_{22} = 0, \; c_{13} - \frac{1}{3}c_{23} + \frac{1}{3}c_{33} = 0$$

$$c_{22} = 1, \; c_{23} = 0, \; c_{33} = 1 \; \therefore \; c_{12} = \frac{1}{3}, \; c_{13} = -1/3$$

$$\therefore \; U^{-1} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note. This shows that if U is unit upper triangular, U^{-1} is also unit upper triangular.

$$\left[\text{Hence, we could have taken } U^{-1} = \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$A^{-1} = (LU)^{-1} = U^{-1} L^{-1}$$

$$= \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Example 2. *Find the inverse of* $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 0 \end{pmatrix}$, *by Crout's method.*

Solution. Let us decompose A as LU *i.e.*, $LU = A$ implies,

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 0 \end{pmatrix}$$

$$L \qquad\qquad U \qquad = \quad A$$

Equating elements or using the method employed in Crout's method,

$$l_{11} = 1,\ l_{21} = 0,\ l_{31} = -2,\ u_{12} = \frac{a_{12}}{l_{11}} = -2;\ u_{13} = \frac{a_{13}}{l_{11}} = 3$$

$$l_{22} = a_{22} - l_{21}\, u_{12} = -1 - 0 = -1;\ l_{32} = a_{32} - u_{12}\, l_{31} = 2 - 4 = -2$$

$$u_{23} = \frac{a_{23} - u_{13}\, l_{21}}{l_{22}} = \frac{4 - 0}{1} = -4$$

$$l_{33} = a_{33} - l_{31}\, u_{13} - l_{32}\, u_{23} = 0 + 6 - 8 = -2$$

$$\therefore\ L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & -2 \end{pmatrix},\ U = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

To find L^{-1}: $LL^{-1} = I$. Therefore

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L \qquad\qquad L^{-1} \qquad = \quad I$$

$\therefore\ x_{11} = 1,\ x_{22} = -1,\ x_{33} = -1/2$

$-x_{21} = 0;\ x_{11} + x_{21} + x_{31} = 0,\ x_{22} + x_{32} = 0$

$\therefore\ x_{31} = -1,\ x_{32} = 1$

$$\therefore \quad L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1/2 \end{pmatrix}$$

To find U^{-1}: we have $UU^{-1} = I$. Therefore

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equating elements,

$$c_{12} - 2 = 0, \; c_{13} - 2c_{23} + 3 = 0, \; c_{23} - 4 = 0$$

$$\therefore \quad c_{12} = 2, \; c_{23} = 4, \; c_{13} = 5$$

$$\therefore \quad U^{-1} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $A = LU$, $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$$\therefore \quad A^{-1} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1/2 \end{pmatrix} = \begin{pmatrix} -4 & 3 & -5/2 \\ -4 & 3 & -2 \\ -1 & 1 & -1/2 \end{pmatrix}$$

EXERCISE 4.5

Find the inverse of the following matrices using Crout's method:

1. $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$
2. $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{pmatrix}$
4. $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$
5. $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$
7. Find the inverses of the matrices given under Exercises 4·2 by using Crout's method.

4·7 Iterative methods

All the previous methods seen in solving the system of simultaneous algebraic linear equations are direct methods. Now we will see some *indirect methods* or *iterative methods*.

This iterative methods is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a ... ...dition will be satisfied if the ... ...atrix