FRAME 1

Introduction

At the beginning of the first programme in Unit 1 of this book, you saw some of the difficulties that you can meet if your knowledge of mathematics is limited to analytical methods. In the other programmes in that first Unit we saw how some problems, for example non-linear equations, simultaneous linear equations and certain properties of matrices, can be tackled by numerical methods.

In the first programme in this Unit, the problem of finding an analytical expression to fit as best as possible given numerical data was considered. Even so the method used there was based on a knowledge of the form of the law that was to be expected. The result was a straight line, or curve, which did not usually pass through any of the given points, when these were plotted. Having found this straight line, or curve, it could then be used to estimate the value of the function at intermediate points and also, as it was an analytical formula, the values of derivatives and integrals.

FRAME 2

In this and the following programmes, we shall be looking at these problems from a different viewpoint. Now given a set of points, there are obviously many curves which do actually pass through all of them and our object this time will effectively be to find the simplest curve which does just this. By the term 'simplest curve' is meant the curve with the simplest equation. Having found this curve it can be used for estimating the value of the function at other points, and for differentiating and integrating. However, there is one major difference in technique between what we shall be doing now and least squares.

When finding a least squares law, once the form of the law has been decided, the tabulated values are used to give us that law and are then effectively forgotten. Now the various formulae that we shall get will be expressed in terms of the tabular values and these values (or their differences) will be used only after the formulae have been obtained. If you have met Simpson's rule for the area under a curve, you are already familiar with one particular integration formula which works in this way. In the next Unit, the ideas involved in integration will be extended to differential equations.

Many of the methods used will be based on what are known as finite differences and in this programme we shall look at these and related ideas.

FRAME 3

Finite Differences

You have already been introduced to the idea of a difference table in the first Unit in this book. There the table shown on page 180 was given in which each number in any column to the right of the first is formed by subtracting the two adjacent numbers in the column immediately to its left.

Thus the number enclosed in $\bigcirc$ is obtained by subtracting the number in $\square$ from that in $\diamond$. Apart from the entries in the left hand

FRAME 3 (continued)

column [which are the values of $x^3/10$ for $x = 0(1)10$, i.e., x increasing by steps of 1 from 0 to 10], each subsequent entry is a FINITE DIFFERENCE. The second column of the table is the first column of differences, known as the FIRST DIFFERENCES, and so on. In this particular case, the analytical formula for the entries in the left hand column is known. Usually this will not be so.

0·0				
	0·1			
0·1		0·6		
	0·7		0·6	
0·8		1·2		0
	1·9		0·6	
2·7		1·8		0
	3·7		0·6	
6·4		2·4		0
	6·1		0·6	
12·5		3·0		0
	$\boxed{9{\cdot}1}$		0·6	
21·6		3·6		0
	12·7		0·6	
34·3		4·2		0
	16·9		0·6	
51·2		4·8		0
	21·7		0·6	
72·9		5·4		
	27·1			
100·0				

Now form a difference table, following the same layout as above, for the values 437 166 47 8 1 2 11 52 173 446 up to the fifth differences. **

3A

437					
	-271				
166		152			
	-119		-72		
47		80		24	
	-39		-48		0
8		32		24	
	-7		-24		0
1		8		24	
	1		0		0
2		8		24	
	9		24		0
11		32		24	
	41		48		0
52		80		24	
	121		72		
173		152			
	273				
446					

FRAME 4

The values in the left hand column of the table are those of $y = f(x)$ where $f(x) = x^4 - 2x^3 + 3x^2 - x + 1$ and x takes, in turn, the values $-4(1)5$. You have found, in this example, that the 4th differences are all the same (24) and the 5th (and consequently the higher) differences are all zero. Similarly, you will notice that for the cubic in the first example, the 3rd differences are constant while the 4th and higher differences are zero. In like manner, if y is an nth degree polynomial, the nth differences will all be the same and all higher differences will be zero provided that the values of y are calculated at equally spaced values of x. As the values of x are important they are usually also shown in a difference table, as follows:

x	f(x)	1st diffs	2nd diffs	3rd diffs	4th diffs	5th diffs
-4	437					
		-271				
-3	166		152			
		-119		-72		
-2	47		80		24	
		-39		-48		0
-1	8		32		24	
		-7		-24		
0	1		8			
		1				
1	2					

This is a part of the table in 3A and here the various columns have been labelled.

FRAME 5

The Link between Differencing and Differentiation

As you know, one way in which the formula for the derivative can be written is

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

and at the point x_0

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If you think of the process involved when first differences are being found, you will realise that all you are doing is finding the numerator of such an expression as that above, i.e., $f(x_0 + h) - f(x_0)$ where h is the difference between consecutive values of x. Differencing is thus the beginning of the differentiation process and finds the actual change in y between consecutive values of x instead of the limit of the rate of change. Also h remains finite instead of becoming infinitesimal and so we now have the CALCULUS OF FINITE DIFFERENCES as against the infinitesimal calculus.

If $f(x) = x^n$, where n is a positive integer, then

$$f(x + h) - f(x) = (x + h)^n - x^n$$
$$= nx^{n-1}h + \text{terms in lower powers of } x \qquad (5.1)$$

Because of this, if $f(x)$ is a polynomial of degree n in x, then the first

FRAME 5 (continued)

differences are the values of a polynomial of degree $n - 1$. Similarly the second differences are the values of a polynomial of degree $n - 2$. Continuing in this way the nth differences are the values of a polynomial of degree zero, i.e., are all the same. The next and subsequent differences are thus zero.

Making use of (5.1) and the paragraph that follows it, find the values of the nth differences of

$$a_0x^n + a_1x^{n-1} + \dots\dots\dots + a_{n-1}x + a_n \qquad (5.2)$$

in terms of n and h. Check that the difference tables you have already met in this programme agree with your result.

****

5A

As differencing takes place n times, the contributions of all terms other than the first will be zero.

The leading term in the polynomial giving the 1st differences is $a_0nx^{n-1}h$. *The formula giving the differences of this is* $a_0n(x + h)^{n-1}h - a_0nx^{n-1}h$, *and the leading term of this polynomial is* $a_0n(n - 1)x^{n-2}h^2$. *(Note that this result can alternatively be obtained by applying* $a_0nx^{n-1}h$ *to itself, so to speak.) This is the leading term in the formula for the 2nd differences of (5.2). Similarly the leading term in the formula for the third differences is* $a_0n(n - 1)(n - 2)x^{n-3}h^3$ *and so on. The leading term (the only term in fact) in the formula for the nth differences is*

$$a_0n(n - 1)(n - 2) \dots\dots 1\, x^0h^n = a_0n!h^n$$

FRAME 6

The fact that the nth differences of a polynomial of degree n are constant enables us to extend a difference table to find further values of the polynomial, simply by additions instead of a series of additions and multiplications. To illustrate the process, let us find the next entry after 446 in the table in 3A. Repeating the bottom part only of this table, we have

11		32		24
	41		48	
52		80		24
	121		72	
173		152		
	273			
446				

The next entry in the last column will again be 24. Then, as this is the difference of two entries in the preceding column, the next entry in that column is the sum of 72 and 24. The calculation then proceeds as shown on the next page.

Now find (i) the next entry after 967 in the table on the next page, and (ii) the next two entries in the list -205 -35 7 17 assuming that these are values of a polynomial of degree 3 calculated at equal intervals of x.

FRAME 6 (continued)

11		32		24
	41		48	
52		80		24
	121		72	
173		152		24
	273		96	
446		248		
	521			
967				

****

6A

(i) 1856; (ii) 91, 325

(The figures −205, −35, 7, 17 are the values of $2x^3 - 4x^2 + 5x + 7$ at $x = -4(2)2$.)

FRAME 7

Decimals in a Difference Table

Construct a difference table for $f(x) = 2x^3 - 4x^2 + 5x + 7$, $x = 0(0\cdot1)0\cdot6$.

****

7A

x	$f(x)$	1st diffs	2nd diffs	3rd diffs
0	7·000			
		0·462		
0·1	7·462		−0·068	
		0·394		0·012*
0·2	7·856		−0·056	
		0·338		0·012
0·3	8·194		−0·044	
		0·294		0·012
0·4	8·488		−0·032	
		0·262		0·012
0·5	8·750		−0·020	
		0·242		
0·6	8·992			

As $f(x)$ is a polynomial of degree 3, the third differences are constant. The portion above the dashed line must be found in the usual way. That below the line can be found by the method of FRAME 6. The figure starred should be checked against the formula $a_0 n! h^n$ before using it for further calculations.

FRAME 8

When a difference table contains decimals, the difference columns themselves are often written as whole numbers. If you do this, you must

FRAME 8 (continued)

remember that these numbers are actually decimals and must be treated as such in any subsequent calculation with them. Using this idea, a slightly extended version of the table in FRAME 7 would appear as:

0	7·000			
		462		
0·1	7·462		-68	
		394		12
0·2	7·856		-56	
		338		12
0·3	8·194		-44	
		294		12
0·4	8·488		-32	
		262		12
0·5	8·750		-20	
		242		12
0·6	8·992		-8	
		234		12
0·7	9·226		4	
		238		
0·8	9·464			

FRAME 9

The Build-up of Errors in a Difference Table due to Errors in the Functional Values

Replace each of the values of f(x) in the table in FRAME 8 by that obtained after rounding to 2 decimal places. Then form the difference table for these rounded values, going as far as the 7th differences.

****

9A

x	$f(x)$	1st diffs	2nd diffs	3rd diffs	4th diffs	5th diffs	6th diffs	7th diffs
0	7·00							
		46						
0·1	7·46		-6					
		40		-1				
0·2	7·86		-7		5			
		33		4		-10		
0·3	8·19		-3		-5		18	
		30		-1		8		-29
0·4	8·49		-4		3		-11	
		26		2		-3		11
0·5	8·75		-2		0		0	
		24		2		-3		
0·6	8·99		0		-3			
		24		-1				
0·7	9·23		-1					
		23						
0·8	9·46							

FRAME 10

The effect of rounding off the functional values to 2 decimal places is to introduce slight errors into them. Examine the difference table you have just formed and see how these errors have affected it. For a true cubic (as in FRAME 8) the first differences are relatively quite large, the second differences are smaller, the third differences are all the same and the fourth and higher differences are zero. But in 9A, although the 1st and 2nd differences behave reasonably well, the 3rd differences are certainly not constant and the rest are certainly not all zero. They do, in fact, start to increase and before long get quite large. Small errors are thus seen to mask completely the true values of the higher differences and consequently the columns on the right of such a difference table are extremely unreliable. Generally speaking, one would not use columns in a table like this after the magnitudes of the entries start increasing.

FRAME 11

All the difference tables so far constructed have been for polynomials. They can equally well be formed for other functions which may or, as is very often the case in practice, may not be known analytically. Construct the difference table as far as the 7th differences for sin x, $x = 0(0{\cdot}05)0{\cdot}5$, working to 4 decimal places. Again notice how the higher differences misbehave (i.e., increase in magnitude), due to presence of round-off errors in the functional values.

****

11A

					Differences			
x	*sin x*	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*	*7th*
0	0							
		500						
0·05	0·0500		−2					
		498		0				
0·10	0·0998		−2		−1			
		496		−1		−2		
0·15	0·1494		−3		−3		12	
		493		−4		10		−37
0·20	0·1987		−7		7		−25	
		486		3		−15		55
0·25	0·2473		−4		−8		30	
		482		−5		15		−57
0·30	0·2955		−9		7		−27	
		473		2		−12		45
0·35	0·3428		−7		−5		18	
		466		−3		6		
0·40	0·3894		−10		1			
		456		−2				
0·45	0·4350		−12					
		444						
0·50	0·4794							

FRAME 12

To examine further how small errors can affect differences let us make up a difference table of errors from functional values, just one of the functional values having an error of amount ε in it. As all other

If the functional values are f_0, f_1, f_2, f_3 etc., say, and just one of them is in error, then the errors in the readings can be listed readings are to be assumed correct, the error in each of these will be zero.

0, 0, 0, 0, 0, 0, ε, 0, 0, 0, 0, 0, 0 $\quad$ ε being the one error.

Form a difference table of these errors, placing the errors in the functional values column and going as far as the 6th differences.

****

12A

0						
	0					
0		0				
	0		0			
0		0		0		
	0		0		0	
0		0		0		ε
	0		0		ε	
0		0		ε		-6ε
	0		ε		-5ε	
0		ε		-4ε		15ε
	ε		-3ε		10ε	
ε		-2ε		6ε		-20ε*
	$-\varepsilon$		3ε		-10ε	
0		ε		-4ε		15ε
	0		$-\varepsilon$		5ε	
0		0		ε		-6ε
	0		0		$-\varepsilon$	
0		0		0		ε
	0		0		0	
0		0		0		
	0		0			
0		0				
	0					
0						

FRAME 13

From the table, you will see that

i) the effect of a single error spreads out fanwise from that error, shown by the dashed lines.

ii) each column of differences has one more entry affected by the error than the preceding column.

iii) individual readings in the column are affected by different amounts, those in the centre of the fan being affected more than those near the edge. For example, the magnitude of the effect on the entry starred is 20 times the original error.

iv) the multiples of ε in the nth difference column are the coefficients in the binomial expansion of $(1 - a)^n$.

It doesn't take much imagination to sense how much a difference table is affected when there are several readings containing errors.

FRAME 14

If the only errors involved in a table are round-off errors, the worst situation that can arise, as far as a difference table is concerned, is when each reading is in error by $\pm\frac{1}{2}$ in the last decimal place, the signs alternating from term to term. Such an extreme situation is very unlikely to occur in practice, but in order to see the worst possible effect, form a difference table for the values $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$ going as far as the sixth differences.

****

14A

$\frac{1}{2}$						
	-1					
$-\frac{1}{2}$		2				
	1		-4			
$\frac{1}{2}$		-2		8		
	-1		4		-16	
$-\frac{1}{2}$		2		-8		32
	1		-4		16	
$\frac{1}{2}$		-2		8		-32
	-1		4		-16	
$-\frac{1}{2}$		2		-8		32
	1		-4		16	
$\frac{1}{2}$		-2		8		-32
	-1		4		-16	
$-\frac{1}{2}$		2		-8		32
	1		-4		16	
$\frac{1}{2}$		-2		8		
	-1		4			
$-\frac{1}{2}$		2				
	1					
$\frac{1}{2}$						

You will see that, with this set-up, the error in, for example, the 6th differences is 64 times the error in each individual reading and is of magnitude 32 in the last decimal place. Generally, in the nth difference column, the magnitude of the error is 2^{n-1}.

FRAME 15

So far, we have mainly concentrated on the effect of round-off errors in a difference table, these being unavoidable and a great nuisance. But other errors can occur and the build-up of these can be of assistance in that this build-up enables us to spot where such errors have occurred.

The entries in the left hand column of the following table (on page 188) have allegedly been formed from a cubic. If all entries are perfectly correct, then, as you know, there should be a constant column of differences and the presence of the figures 46, -114, 126, -34 in a column of sixes means that something has gone wrong here. Now the effect of an error spreads out fanwise from the reading that is in error and the construction of a fan backwards from the group 46 to -34 points to 2995 as being the guilty party.

Can you suggest what this reading should be and a probable cause of the error? Having changed this value accordingly, reconstruct the difference table and check it for reasonableness.

<u>FRAME 15</u> (continued)

1111			
	353		
		68	
1464			
			6
	421		
		74	
1885			
			46
	495		
		120	
2380			
			-114
	615		
		6	
2995			
			126
	621		
		132	
3616			
			-34
	753		
		98	
4369			
			6
	851		
		104	
5220			
			6
	955		
		110	
6175			
			6
	1065		
		116	
7240			
	1181		
8421			

****

<u>15A</u>

Changing 46 into 6 and working backwards replaces 2995 by 2955. This suggests that the wrong digit was repeated when copying. The revised difference table is as shown.

1111			
	353		
1464		*68*	
	421		*6*
1885		*74*	
	495		*6*
2380		*80*	
	575		*6*
2955		*86*	
	661		*6*
3616		*92*	
	753		*6*
4369		*98*	
	851		*6*
5220		*104*	
	955		*6*
6175		*110*	
	1065		
7240			

<u>FRAME 16</u>

Some sets of figures may have more than one error in them.

Form a difference table for the figures -48, -75, -132, -147, 0, 447, 1550, 3333, 6288, 10 725, 17 052, 25 725, 37 248, 52 173 and suggest which figures are in error.

****

−48				
	−27			
−75		−30		
	−57		72	
−132		42		48
	−15		120	
−147		162		18
	147		138	
0		300		218
	447		356	
447		656		−332
	1103		24	
1550		680		468
	1783		492	
3333		1172		−182
	2955		310	
6288		1482		98
	4437		408	
10 725		1890		48
	6327		456	
17 052		2346		48
	8673		504	
25 725		2850		48
	11 523		552	
37 248		3402		
	14 925			
52 173				

The fan suggests that 447 and 1550 are both in error.

FRAME 17

Now suggest possible corrections. Then construct a fresh difference table to see if an improvement in the table results.

****

17A

The 48's in the last column give the clue.

447 should be 477 and 1550 should be 1500. Your difference table should then have a constant column of 48.

FRAME 18

A more awkward situation arises when a table contains round-off errors and one or more blunders into the bargain. Construct a difference table for the following set of values. Then examine it and see if you can make any suggestions as to a possible unintentional error.

0·0000, 0·0500, 0·1002, 0·1506, 0·2013, 0·2526, 0·3054, 0·3572, 0·4108, 0·4653, 0·5211.

****

18A

0·0000		
	500	
0·0500		*2*
	502	
0·1002		*2*
	504	
0·1506		*3*
	507	
0·2013		*6*
	513	
0·2526		*15*
	528	
0·3054		*−10*
	518	
0·3572		*18*
	536	
0·4108		*9*
	545	
0·4653		*13*
	558	
0·5211		

The pattern of differences suggests an error in 0·3054. No difference column gives an exact amendment to be made to this figure, as, due to the presence of round-off, no one column follows a set, well-defined sequence. The sudden jump from 6 to 15 suggests an error of about 8 or 9 in the last decimal place. A difference of 9 would be obtained from a reversal of the last two digits. This would replace 0·3054 by 0·3045 and if this is done, the table becomes much smoother.

FRAME 19

Now try this example:

The following figures are rounded to five places of decimals and also contain two copying errors. Locate these two errors and suggest possible corrections.

0·000 00, 0·087 49, 0·167 33, 0·267 95, 0·363 97, 0·466 31, 0·577 35, 0·700 21, 0·839 10, 1·100 00, 1·191 75, 1·428 15, 1·732 05.

****

19A

A difference table locates the errors at 0·167 33, 1·100 00.

Changes in these to 0·176 33 and 1·000 00 produce a much more likely difference table.

FRAME 20

Finite Difference Notations

Due to the use made of the differential coefficient in various formulae, it is necessary to use a symbol to denote it. Similarly, a notation is necessary when formulae involving finite differences are used. But, as you might suspect, mathematicians are not content with just one notation. This is not due to awkwardness on their part, though, as it is found that while one notation is best for one particular problem, another is better in other circumstances.

The essential feature of any notation is that it should enable us to locate immediately each and every entry in a difference table. A start is made by choosing, at will, one entry in the x column and labelling it x_0. If h is the difference between consecutive values of x quoted in the table, then the other values of x will be

FRAME 20 (continued)

$\ldots, x_0 - 3h, x_0 - 2h, x_0 - h$ before x_0 and $x_0 + h, x_0 + 2h, x_0 + 3h, \ldots$ after it.

The complete set in the table will then be

$\ldots, x_0 - 3h, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h, \ldots\ldots$

For brevity, these values are usually denoted by

$$\ldots\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots\ldots$$

and so x_p is used for $x_0 + ph$.

Although the only values which will actually appear in a table are those for which p is an integer, it is very valuable, as you will see later, to extend this notation to other values of p. Then, in general,

$$x_p = x_0 + ph$$

where p can take any value.

FRAME 21

In a similar way, the functional values are denoted by

$$\ldots\ldots, f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3, \ldots\ldots$$

f_0 being that value which corresponds to x_0, etc. y or any other convenient letter may, of course, be used instead of f. Again, only the values just listed will actually appear in a difference table, but the notation is extended so that f_p indicates the value of the function when $x = x_0 + ph$, whatever the value of p.

FRAME 22

The Forward Difference Operator Δ

There are three notations in common use for the actual difference columns themselves. One of these involves what are known as FORWARD DIFFERENCES and uses the symbol Δ as the FORWARD DIFFERENCE OPERATOR. Δf_n is then defined by the equation

$$\Delta f_n = f_{n+1} - f_n$$

where here n is an integer. Then

$\Delta f_0 = f_1 - f_0$, $\quad\Delta f_2 = f_3 - f_2$, $\quad\Delta f_{-3} = f_{-2} - f_{-3}$, etc.

Using the symbols so far defined, the beginning of a difference table can be expressed symbolically as, for example,

$$\begin{array}{ccc}
x_{-2} & f_{-2} & \\
 & & \Delta f_{-2} \\
x_{-1} & f_{-1} & \\
 & & \Delta f_{-1} \\
x_0 & f_0 & \\
 & & \Delta f_0 \\
x_1 & f_1 & \\
 & & \Delta f_1 \\
x_2 & f_2 & \\
 & & \Delta f_2 \\
x_3 & f_3 &
\end{array}$$

FRAME 23

You have already met the idea of an operator in mathematics. For example, D is sometimes used to denote the operation of differentiation and i is used in complex numbers to denote the operation of rotating a vector 90° anti-clockwise. Repeated differentiation is then denoted by D^2, D^3, etc., and repeated vector rotation by i^2, i^3, etc. The power of i need not be an integer as it is possible to rotate vectors through angles other than multiples of 90°, but powers of D must be integral as one cannot differentiate, say, $2\frac{1}{2}$ times.

In a similar way, $\Delta^2 f_n$ for example is used to denote $\Delta(\Delta f_n)$ and is an instruction to take the difference of a difference, which is a second difference. How this works can be seen by denoting Δf_n by u_n, say, then

$$\Delta(\Delta f_n) = \Delta u_n = u_{n+1} - u_n = \Delta f_{n+1} - \Delta f_n$$

Thus, for example,

$$\Delta^2 f_3 = \Delta f_4 - \Delta f_3, \qquad \Delta^2 f_{-1} = \Delta f_0 - \Delta f_{-1}$$

We can now go one stage further in the difference table in the last frame and write

x_{-2}	f_{-2}		
		Δf_{-2}	
x_{-1}	f_{-1}		$\Delta^2 f_{-2}$
		Δf_{-1}	
x_0	f_0		$\Delta^2 f_{-1}$
		Δf_0	
x_1	f_1		$\Delta^2 f_0$
		Δf_1	
x_2	f_2		$\Delta^2 f_1$
		Δf_2	
x_3	f_3		

FRAME 24

The extension of this will be obvious, Δ^3 being used for 3rd differences and so on. In general, $\Delta^r f_n$ is defined by the equation

$$\Delta^r f_n = \Delta^{r-1} f_{n+1} - \Delta^{r-1} f_n$$

and the table from x_{-2} to x_4 can be exhibited as

x_{-2}	f_{-2}				
		Δf_{-2}			
x_{-1}	f_{-1}		$\Delta^2 f_{-2}$		
		Δf_{-1}		$\Delta^3 f_{-2}$	
x_0	f_0		$\Delta^2 f_{-1}$		$\Delta^4 f_{-2}$
		Δf_0		$\Delta^3 f_{-1}$	
x_1	f_1		$\Delta^2 f_0$		$\Delta^4 f_{-1}$
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

and extended upwards, downwards and to the right as necessary.

FRAME 24 (continued)

You will notice that all the entries having a common power of Δ appear in the same vertical column and all entries having a common f suffix appear on a downwards sloping diagonal.

To relate these symbol forms with a specific set of figures, suppose we have the table

x	f(x)						
-6	0·250 00						
		-8333					
-4	0·166 67		4166				
		-4167		-2499			
-2	0·125 00		1667		1665		
		-2500		-834		-1187	
0	0·100 00		833		478		885
		-1667		-356		-302	
2	0·083 33		477		176		208
		-1190		-180		-94	
4	0·071 43		297		82		49
		-893		-98		-45	
6	0·062 50		199		37		32
		-694		-61		-13	
8	0·055 56		138		24		
		-556		-37			
10	0·050 00		101				
		-455					
12	0·045 45						

If 4 is labelled x_0 then $f_0 = 0{\cdot}071\,43$, $f_{-2} = 0{\cdot}100\,00$, Δf_3 indicates the entry -455, $\Delta^2 f_{-1}$, 297 and $\Delta^4 f_{-2}$, 82. (Don't forget that -455 really means $-0{\cdot}004\,55$, etc.)

Now, for the above table:

1. What are the entries corresponding to

   i) $\Delta^2 f_2$, $\Delta^6 f_1$, $\Delta^5 f_4$, if $x_0 = -6$,
   ii) Δf_{-3}, $\Delta^3 f_2$, $\Delta^4 f_{-1}$ if $x_0 = 0$?

2. What are the symbols for

   i) -1190, 82, -356, if $x_0 = -4$,
   ii) 199, 885, -45 if $x_0 = 8$?

3. What is the value of h?

****

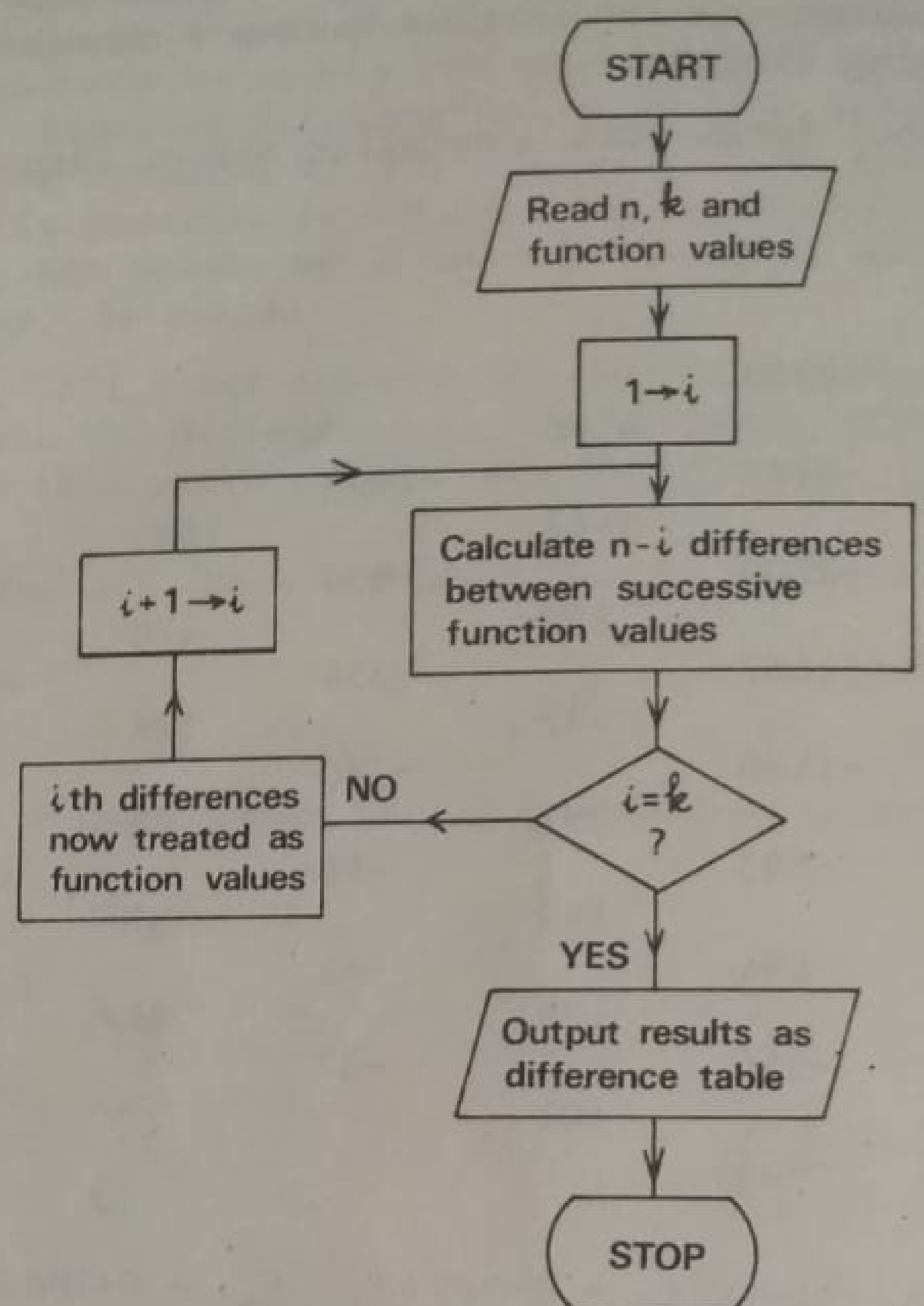
24A

1. *i)* *833, 208, -13* *ii)* *-8333, -61, 176*
2. *i)* Δf_3, $\Delta^4 f_2$, $\Delta^3 f_1$ *ii)* $\Delta^2 f_{-2}$, $\Delta^6 f_{-7}$, $\Delta^5 f_{-4}$
3. $h = 2$

FRAME 25

The following figure is a flow diagram for forming a difference table from n function values as far as the kth differences $(n > k)$.

FRAME 25 (continued)



This will happen automatically at the next step, when i is incremented, if the notation below is used.

This forms the column of ith differences. In hand calculations they would be written down at this stage but with a computer they would go into store* to await the print-out of the whole table at the end.

*This can be achieved using a double suffix, e.g., f_{im} to denote $\Delta^i f_m$. The difference formula $\Delta^i f_m = \Delta^{i-1} f_{m+1} - \Delta^{i-1} f_m$ then becomes $f_{im} = f_{i-1,\,m+1} - f_{i-1,\,m}$ and the column of ith differences is obtained by applying this formula for n - i consecutive values of m. Incrementing i by 1 when this is completed will make the differences just found into the values which have to be differenced the next time round. The values of f_{im} will be stored as an array.

[A program for forming a difference table can be found in reference (5).]

FRAME 26

Just as Δf_2, for example, can be expressed in terms of functional values, i.e., $f_3 - f_2$, so also can any of the higher differences. Later, in for example, integration and the solution of differential equations, you will find that this is done quite frequently.

As an example,

$$\Delta^2 f_3 = \Delta f_4 - \Delta f_3 = (f_5 - f_4) - (f_4 - f_3) = f_5 - 2f_4 + f_3$$

Similarly, $\Delta^2 f_{-2} = f_0 - 2f_{-1} + f_{-2}$

By a similar method, other differences can be expressed in terms of functional values. However, for the higher differences it is easier to use another operator - the shift operator E.

FRAME 27

The Shift Operator E

The shift operator E simply has the effect of taking you forwards from one reading in a column to the next. Thus

$$Ef_2 = f_3, \quad E\Delta^2 f_{-1} = \Delta^2 f_0, \quad E\Delta^5 f_0 = \Delta^5 f_1.$$

What do you think will be the effect of i) E^2, ii) E^5, iii) E^{-1}?

****

27A

i) To go forwards two readings in a column.
ii) To go forwards five readings in a column.
iii) To go backwards one reading in a column. Remember the use of i^{-1} for the rotation of a complex number vector through 90° clockwise.

FRAME 28

Thus, $E^3 f_3 = f_6$, $E^4\Delta^3 f_{-1} = \Delta^3 f_3$ and $E^{-2}\Delta^4 f_0 = \Delta^4 f_{-2}$.

Now $\Delta f_n = f_{n+1} - f_n = Ef_n - f_n = (E - 1)f_n$

and so Δ is symbolically equivalent to $E - 1$. Alternatively, E is symbolically equivalent to $1 + \Delta$.

Can you suggest, without introducing Δ, an interpretation for $(E - 1)^2 f_3$ in terms of functional values? Is your result the same as for $\Delta^2 f_3$ as found in FRAME 26?

****

28A

$(E - 1)^2 f_3 = (E^2 - 2E + 1)f_3 = f_5 - 2f_4 + f_3$ *Yes*

FRAME 29

Any higher difference can similarly be expressed in terms of functional values. Thus, for example,

$$\begin{aligned}\Delta^4 f_{-2} = (E - 1)^4 f_{-2} &= (E^4 - 4E^3 + 6E^2 - 4E + 1)f_{-2} \\ &= f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}\end{aligned}$$

What will $\Delta^5 f_{-1}$ be when expressed in terms of functional values?

****

29A

$f_4 - 5f_3 + 10f_2 - 10f_1 + 5f_0 - f_{-1}$

FRAME 30

The Backward Difference Operator ∇

$f_{n+1} - f_n$ was expressed as Δf_n in FRAME 22, i.e., Δf_n was expressed in terms of f_n and the functional value one step forwards, that is, f_{n+1}. Sometimes it is desirable to express a difference using a functional value one step backwards instead of one step forwards. To accomplish this the entry in the table in the same position as Δf_n is also written as ∇f_{n+1}. Then

$$\nabla f_{n+1} = f_{n+1} - f_n$$

and so $\nabla f_2 = f_2 - f_1$, $\nabla f_0 = f_0 - f_{-1}$, $\nabla f_n = f_n - f_{n-1}$.

FRAME 30 (continued)

It is very important to note that in a numerical table Δf_n and ∇f_{n+1} are exactly the same entry. Δf_n and ∇f_{n+1} are only two different symbols for the same thing. Continuing, second differences follow in a manner similar to that obtaining when forward differences are used. For example,

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1} = (f_n - f_{n-1}) - (f_{n-1} - f_{n-2}) = f_n - 2f_{n-1} + f_{n-2}$$

Using backward differences, the symbolic table in FRAME 24 would appear as

$$\begin{array}{llllll}
x_{-2} & f_{-2} & & & & \\
 & & \nabla f_{-1} & & & \\
x_{-1} & f_{-1} & & \nabla^2 f_0 & & \\
 & & \nabla f_0 & & \nabla^3 f_1 & \\
x_0 & f_0 & & \nabla^2 f_1 & & \nabla^4 f_2 \\
 & & \nabla f_1 & & \nabla^3 f_2 & \\
x_1 & f_1 & & \nabla^2 f_2 & & \nabla^4 f_3 \\
 & & \nabla f_2 & & \nabla^3 f_3 & \\
x_2 & f_2 & & \nabla^2 f_3 & & \nabla^4 f_4 \\
 & & \nabla f_3 & & \nabla^3 f_4 & \\
x_3 & f_3 & & \nabla^2 f_4 & & \\
 & & \nabla f_4 & & & \\
x_4 & f_4 & & & &
\end{array}$$

You will notice that this time all entries having the same f suffix appear on an upwards sloping diagonal.

Using the numerical table in FRAME 24, what will be

i) the entries corresponding to ∇f_1, $\nabla^2 f_3$ and $\nabla^3 f_5$ if $x_0 = -6$,
ii) the entries corresponding to $\nabla^4 f_{-2}$ and $\nabla^5 f_0$ if $x_0 = 6$,
iii) the backward difference symbols for -1190, 82, -356 if $x_0 = -4$,
iv) the backward difference symbols for 199, 885, -45 if $x_0 = 8$?

****

30A

i) -8333, 1667, -356
ii) 1665, -302
iii) ∇f_4, $\nabla^4 f_6$, $\nabla^3 f_4$
iv) $\nabla^2 f_0$, $\nabla^6 f_{-1}$, $\nabla^5 f_1$

FRAME 31

Can you now express ∇ in terms of E and show that $\nabla = E^{-1}\Delta$?

****

31A

$\nabla f_n = f_n - f_{n-1} = f_n - E^{-1} f_n = (1 - E^{-1}) f_n \qquad \therefore \ \nabla = 1 - E^{-1}$

Also $\quad 1 - E^{-1} = E^{-1}(E - 1) \qquad \therefore \ \nabla = E^{-1}\Delta$

These two results can also be written as $E = (1 - \nabla)^{-1}$ *and* $\Delta = E\nabla$ *respectively.*

FRAME 32

In 24A, question No. 2, you expressed certain entries in the numerical table in FRAME 24 in terms of Δ. Take these results and use $\Delta = E\nabla$ to express them in terms of ∇. Then check that your answers agree with those you obtained in 30A.

****

32A

$\Delta f_3 = (E\nabla)f_3 = \nabla f_4$ $\qquad$ *$\Delta^4 f_2 = (E\nabla)^4 f_2 = \nabla^4 f_6$*

The others follow similarly.

FRAME 33

The Central Difference Operator δ

In the case of the forward difference operator, all entries with the same f suffix appear on a downwards sloping diagonal. When using the backward difference operator they all appear on an upwards sloping diagonal. The CENTRAL DIFFERENCE OPERATOR δ is defined in such a way that entries with the same f suffix all appear on a horizontal line. This immediately leads to a snag - the odd differences are not placed on the same level as any of the functional values. Omitting these differences for a moment, the even differences can be relabelled as shown below - and remember, it is only a relabelling. The numerical values are the same as before.

x_{-2}	f_{-2}				
		·			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		·		·	
x_0	f_0		$\delta^2 f_0$		$\delta^4 f_0$
		⊙		·	
x_1	f_1		$\delta^2 f_1$		$\delta^4 f_1$
		·		·	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		·		·	
x_3	f_3		$\delta^2 f_3$		
		·			
x_4	f_4				

The dots indicate the awkward ones.

FRAME 34

Now each entry shown by a dot is written on a level half way between two functional values. Thus, for example, the ringed dot is on a level half way between f_0 and f_1. It is therefore assumed to lie on the level of $f_{\frac{1}{2}}$, although this entry does not, of course, exist in the table. Using this idea the ringed entry is labelled $\delta f_{\frac{1}{2}}$. Extending this notation to the other dots, the table now appears as

x_{-2}	f_{-2}				
		$\delta f_{-1\frac{1}{2}}$			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\delta f_{-\frac{1}{2}}$		$\delta^3 f_{-\frac{1}{2}}$	
x_0	f_0		$\delta^2 f_0$		$\delta^4 f_0$
		$\delta f_{\frac{1}{2}}$		$\delta^3 f_{\frac{1}{2}}$	
x_1	f_1		$\delta^2 f_1$		$\delta^4 f_1$
		$\delta f_{1\frac{1}{2}}$		$\delta^3 f_{1\frac{1}{2}}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{2\frac{1}{2}}$		$\delta^3 f_{2\frac{1}{2}}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{3\frac{1}{2}}$			
x_4	f_4				

Notice that even powers of δ are always associated with integral suffixes of f and odd powers with fractional suffixes.

Returning now to the numerical table in FRAME 24, what will be

i) the entries corresponding to $\delta^2 f_2$ and $\delta^5 f_{\frac{1}{2}}$ if $x_0 = -2$,

ii) the entries corresponding to $\delta f_{1\frac{1}{2}}$ and $\delta^3 f_{-2\frac{1}{2}}$ if $x_0 = 4$,

iii) the central difference symbols for -1190, 82, -356 if $x_0 = -4$,

iv) the central difference symbols for 199, 885, -45 if $x_0 = 8$?

****

34A

i) 477, -1187 *ii)* -694, -834

iii) $\delta f_{3\frac{1}{2}}$, $\delta^4 f_4$, $\delta^3 f_{2\frac{1}{2}}$ *iv)* $\delta^2 f_{-1}$, $\delta^6 f_{-4}$, $\delta^5 f_{-1\frac{1}{2}}$

FRAME 35

The formula for any central difference can be expressed as

$$\delta^r f_n = \delta^{r-1} f_{n+\frac{1}{2}} - \delta^{r-1} f_{n-\frac{1}{2}}$$

provided n and r satisfy the association noted under the table in the last frame.

A few special cases of this are

$$\delta f_{1\frac{1}{2}} = f_2 - f_1, \quad \delta^4 f_2 = \delta^3 f_{2\frac{1}{2}} - \delta^3 f_{1\frac{1}{2}}, \quad \delta^3 f_{-\frac{1}{2}} = \delta^2 f_0 - \delta^2 f_1$$

FRAME 36

In order to express δ in terms of E, it is necessary to give a meaning to fractional powers of E.

In FRAME 20, we wrote x_p to denote the value of $x_0 + ph$, where p could take any value, and, in FRAME 21, we used f_p to denote the corresponding value of the function. The symbol E^p is used in a similar sense and indicates that one goes forwards p entries in the table, even if p is fractional, thus landing one on an "entry" that isn't really there. So f_p can also be written as $E^p f_0$. Using this notation, $f_{\frac{1}{2}}$ is written as $E^{\frac{1}{2}} f_0$, $f_{-\frac{1}{2}}$ as $E^{-\frac{1}{2}} f_0$ and, extending it, we can also write $E^{1\frac{1}{2}} f_1 = f_{2\frac{1}{2}}$, $E^{-2\frac{1}{2}} f_{-1\frac{1}{2}} = f_{-4}$, $E^{-\frac{1}{2}} f_2 = f_{1\frac{1}{2}}$, etc.

Now, as $\delta f_{\frac{1}{2}} = f_1 - f_0$ and as f_1 can be written as $E^{\frac{1}{2}} f_{\frac{1}{2}}$ and f_0 as $E^{-\frac{1}{2}} f_{\frac{1}{2}}$, $\delta f_{\frac{1}{2}} = E^{\frac{1}{2}} f_{\frac{1}{2}} - E^{-\frac{1}{2}} f_{\frac{1}{2}} = (E^{\frac{1}{2}} - E^{-\frac{1}{2}}) f_{\frac{1}{2}}$.

So here, δ is equivalent to $E^{\frac{1}{2}} - E^{-\frac{1}{2}}$. That it is true generally can be seen if the front δ is replaced by this in, say, $\delta(\delta^{r-1} f_n)$. Then

$$\delta(\delta^{r-1} f_n) = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})(\delta^{r-1} f_n) = E^{\frac{1}{2}}\delta^{r-1} f_n - E^{-\frac{1}{2}}\delta^{r-1} f_n$$

$$= \delta^{r-1} E^{\frac{1}{2}} f_n - \delta^{r-1} E^{-\frac{1}{2}} f_n = \delta^{r-1} f_{n+\frac{1}{2}} - \delta^{r-1} f_{n-\frac{1}{2}}$$

FRAME 36 (continued)

But $\delta(\delta^{r-1}f_n) = \delta^r f_n$, and, as was seen in FRAME 35, this is equal to $\delta^{r-1}f_{n+\frac{1}{2}} - \delta^{r-1}f_{n-\frac{1}{2}}$.

FRAME 37

In 31A you worked out the relation between ∇ and Δ, i.e., $\nabla = E^{-1}\Delta$. See if you can now show that $\delta = E^{-\frac{1}{2}}\Delta$ and $\delta = E^{\frac{1}{2}}\nabla$.

****

37A

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = E^{-\frac{1}{2}}(E - 1) = E^{-\frac{1}{2}}\Delta$$
$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = E^{\frac{1}{2}}(1 - E) = E^{\frac{1}{2}}\nabla$$

FRAME 38

You may have noticed, as you have been reading this programme that we have sometimes reversed the order of two operators. For example, in FRAME 36, $E^{\frac{1}{2}}\delta^{r-1}$ was assumed to be equivalent to $\delta^{r-1}E^{\frac{1}{2}}$. This sort of reversal has not been justified but if you think about it you will see that it is quite reasonable. Thus, $E^2\Delta^4 f_0$ tells us to start from f_0, go four columns diagonally downwards to the right and then two rows down vertically. $\Delta^4 E^2 f_0$ tells us to go two rows down vertically and then four columns diagonally downwards to the right. The net effect is to arrive at the same place. Similarly, three columns to the right of f_{-1} and then $4\frac{1}{2}$ rows down (i.e. $E^{4\frac{1}{2}}\delta^3 f_{-1}$) gets to the same place as $4\frac{1}{2}$ rows down from f_{-1} and then three columns to the right $(\delta^3 E^{4\frac{1}{2}} f_{-1})$.

(You have met similar ideas in analytical work, for example, $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ and $i^2 i^{1\frac{1}{2}} = i^{1\frac{1}{2}} i^2$. The first of these is saying that the order of differentiation doesn't matter and the second that a complex number vector rotated through 135° and then 180° arrives in the same position as if it is rotated first through 180° and then 135°.)

FRAME 39

The Averaging Operator μ

There is one further operator that will be required in later work. This is the MEAN or AVERAGING OPERATOR μ. It is used to denote the mean or average of two adjacent readings in a column. Thus

$$\mu f_{\frac{1}{2}} = \tfrac{1}{2}(f_1 + f_0), \quad \mu\delta^2 f_{-\frac{1}{2}} = \tfrac{1}{2}(\delta^2 f_0 + \delta^2 f_{-1}), \quad \text{and so on.}$$

Be careful not to confuse $f_{\frac{1}{2}}$ with $\mu f_{\frac{1}{2}}$, etc. The difference in meaning is easily seen by means of a simple example.

If $f(x) = x^2$, the following table of values can be formed:

x	0	0·5	1·0	1·5	2·0	2·5	3·0
$f(x)$	0	0·25	1·00	2·25	4·00	6·25	9·00

Now suppose x_0 is chosen to be 1·0. $h = 0{\cdot}5$ and so $x_1 = 1{\cdot}5$, $f_0 = 1{\cdot}00$ and $f_1 = 2{\cdot}25$. Then $x_{\frac{1}{2}} = 1{\cdot}25$ and so $f_{\frac{1}{2}} = 1{\cdot}25^2 = 1{\cdot}5625$, but $\mu f_{\frac{1}{2}} = \frac{1}{2}(f_1 + f_0) = \frac{1}{2}(2{\cdot}25 + 1{\cdot}00) = 1{\cdot}625$, which is not the same

FRAME 39 (continued)

value.

Referring to the numerical table in FRAME 24, what will be the numbers corresponding to $\mu f_{1\frac{1}{2}}$, $\mu\delta^2 f_{-1\frac{1}{2}}$ and $\mu\delta f_2$ if $x_0 = 4$? Can you also find an expression for μ in terms of E?

****

39A

0·059 03, 655, -625.

$\mu = (E^{\frac{1}{2}} + E^{-\frac{1}{2}})/2$

FRAME 40

Having defined Δ, E, ∇ and δ, we have seen how to interpret operators such as Δ^2, E^3, δ^4, etc. Is it possible then, in a similar way, to attach a meaning to μ^2?

To investigate, let us see if we can interpret, say, $\mu^2 f_0$.

As $\Delta^2 f_0 = \Delta(\Delta f_0)$, it seems reasonable to assume at the outset that $\mu^2 f_0 = \mu(\mu f_0)$. Now μf_0 doesn't really mean anything from the way in which μ has been defined, but as $\mu = (E^{\frac{1}{2}} + E^{-\frac{1}{2}})/2$, let us agree that $\mu f_0 = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})f_0$, i.e., $\frac{1}{2}(f_{\frac{1}{2}} + f_{-\frac{1}{2}})$.

Then $\mu(\mu f_0) = \mu(f_{\frac{1}{2}} + f_{-\frac{1}{2}})/2 = (\mu f_{\frac{1}{2}} + \mu f_{-\frac{1}{2}})/2$ and both of these are defined.

Applying the requisite formulae for $\mu f_{\frac{1}{2}}$ and $\mu f_{-\frac{1}{2}}$, find an interpretation for $\mu^2 f_0$.

****

40A

$\mu^2 f_0 = \{\frac{1}{2}(f_1 + f_0) + \frac{1}{2}(f_0 + f_{-1})\}/2 = (f_1 + 2f_0 + f_{-1})/4$

FRAME 41

Now, using $\mu = (E^{\frac{1}{2}} + E^{-\frac{1}{2}})/2$, express μ^2 in terms of E and see whether the result, when applied to f_0, agrees with what you have just found.

****

41A

$\mu^2 = \{(E^{\frac{1}{2}} + E^{-\frac{1}{2}})/2\}^2 = (E + 2 + E^{-1})/4$

$\mu^2 f_0 = \{(E + 2 + E^{-1})/4\}f_0 = (f_1 + 2f_0 + f_{-1})/4$

Other Operational Formulae

FRAME 42

There are many other formulae that have been developed for work in connection with finite differences. Apart from showing you a couple of examples by way of illustration, it is not proposed to go into these in detail. Any that are required for later work can be obtained when they are needed.

For the first of these examples, take $1 + \frac{1}{4}\delta^2$, express δ in terms of E, expand and simplify.

****

42A

$$1 + \tfrac{1}{4}\delta^2 = 1 + \tfrac{1}{4}(E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2 = 1 + (E - 2 + E^{-1})/4 = (E + 2 + E^{-1})/4$$

FRAME 43

Comparing this result with that obtained in 41A, you will see that $\mu^2 = 1 + \tfrac{1}{4}\delta^2$

For a second example, let us see if there is a result in this work analogous to the derivative of a product. Thus, $\Delta(f_n g_n)$ might be required.

By definition $\quad \Delta(f_n g_n) = f_{n+1}g_{n+1} - f_n g_n \qquad (43.1)$

But $\quad \Delta g_n = g_{n+1} - g_n \quad$ and so $\quad g_{n+1} = g_n + \Delta g_n$

$$\therefore \quad \Delta(f_n g_n) = f_{n+1}(g_n + \Delta g_n) - f_n g_n = (f_{n+1} - f_n)g_n + f_{n+1}\Delta g_n$$
$$= g_n \Delta f_n + f_{n+1}\Delta g_n$$

Show that (43.1) can also be expressed as $f_n \Delta g_n + g_{n+1}\Delta f_n$.

****

43A

$$\Delta f_n = f_{n+1} - f_n \qquad \therefore \quad f_{n+1} = f_n + \Delta f_n$$

$$\therefore \quad \Delta(f_n g_n) = (f_n + \Delta f_n)g_{n+1} - f_n g_n = f_n(g_{n+1} - g_n) + g_{n+1}\Delta f_n$$
$$= f_n \Delta g_n + g_{n+1}\Delta f_n$$

FRAME 44

Some of the ideas developed in this programme may, at this stage, seem somewhat unusual. But when you get used to working with them, you will find that they are really no queerer than the ideas you came across in differentiation and integration, for example. The various techniques that you learn in mathematics were originally invented to perform certain functions and the methods of finite differences are just some of these techniques.

FRAME 45

<u>Summary</u>

In this frame, some of the definitions and formulae are listed for your convenience.

$$\Delta f_n = f_{n+1} - f_n$$
$$\Delta^r f_n = \Delta^{r-1} f_{n+1} - \Delta^{r-1} f_n$$
$$\nabla f_n = f_n - f_{n-1}$$
$$\nabla^r f_n = \nabla^{r-1} f_n - \nabla^{r-1} f_{n-1}$$
$$\delta f_{n+\frac{1}{2}} = f_{n+1} - f_n$$
$$\delta^r f_n = \delta^{r-1} f_{n+\frac{1}{2}} - \delta^{r-1} f_{n-\frac{1}{2}}$$

$$E f_n = f_{n+1}$$
$$E^r f_n = f_{n+r}$$
$$\mu f_{n+\frac{1}{2}} = (f_{n+1} + f_n)/2$$
$$\Delta = E - 1$$
$$\nabla = 1 - E^{-1}$$
$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$
$$E = 1 + \Delta$$

FRAME 45 (continued)

$E = (1 - \nabla)^{-1}$

$\Delta = E\nabla$

$\nabla = E^{-1}\Delta$

$\delta = E^{-\frac{1}{2}}\Delta = E^{\frac{1}{2}}\nabla$

$\mu = (E^{\frac{1}{2}} + E^{-\frac{1}{2}})/2$

$\mu^2 = 1 + \frac{1}{4}\delta^2$

FRAME 46

Miscellaneous Examples

In this frame a collection of miscellaneous examples is given for you to try. Answers are provided in FRAME 47, together with such working as is considered helpful.

1. $-7, -6, -1, 2, 21$ are five consecutive entries in the tabulation of a certain quartic. Find the two entries preceding -7 and the two following 21.

2. A polynomial $f(x)$ of low degree is tabulated as follows:

x	-3	-2	-1	0	1	2	3	4	5	6
f(x)	-30	-12	0	3	4	5	9	19	38	69

   Two errors in the values of $f(x)$ are suspected. Locate and correct them.

3. There are some copying errors in the following table. Find them and suggest probable corrections.

   0·000 00, 0·036 59, 0·072 32, 0·107 23, 0·143 18, 0·174 80,
   0·207 53, 0·239 61, 0·271 07, 0·301 93, 0·333 32, 0·361 98,
   0·391 21, 0·419 95, 0·448 52, 0·476 03.

4. Obtain the results

   i) $\Delta\left(\frac{f_n}{g_n}\right) = \frac{g_n\,\Delta f_n - f_n\,\Delta g_n}{g_n g_{n+1}}$ ii) $\Delta\left(\frac{1}{g_n}\right) = \frac{-\Delta g_n}{g_n g_{n+1}}$

   iii) $\Delta(\log f_n) = \log(f_{n+1}/f_n)$

5. Show that

   i) $\Delta - \nabla = \Delta\nabla$ ii) $\delta^2 = \Delta - \nabla$

   iii) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ iv) $\mu + \frac{1}{2}\delta = E^{\frac{1}{2}}$

Answers to Miscellaneous Examples

FRAME 47

1. 147, 26; 98, 299

2. The last four 3rd differences suggest a polynomial of degree 3. Taking all the third differences as 3 leads to tabular entries of -11 and -1 instead of -12 and 0.

3.

0·000 00			
	3659		
0·036 59		$\boxed{-86}$	
	3573		$\boxed{4}$
0·072 32		$\boxed{-82}$	
	3491		186
0·107 23		104	
	3595		-537
0·143 18		-433	
	3162		544
0·174 80		111	
	3273		-176
0·207 53		$\boxed{-65}$	
	3208		$\boxed{3}$
0·239 61		$\boxed{-62}$	
	3146		$\boxed{2}$
0·271 07		$\boxed{-60}$	
	3086		113
0·301 93		53	
	3139		-326
0·333 32		-273	
	2866		330
0·361 98		57	
	2923		-106
0·391 21		$\boxed{-49}$	
	2874		32
0·419 95		-17	
	2857		-89
0·448 52		-106	
	2751		
0·476 03			

The differences enclosed in rectangles seem to be following a reasonable pattern and fans would suggest errors in 0·143 18, 0·333 32 and 0·448 52. There are insufficient differences possible at the end of the table to enable us to be reasonably sure at the moment, one way or the other, about 0·476 03.

The change from -82 to 104 in the 2nd differences column suggests that 0·143 18 is in error by something in the region of 180 and the reversal of the two digits 31 to 13 will make a change of this magnitude. 0·141 38 instead of 0·143 18 gives reasonable differences in the top fan.

The change from -60 to 53 in the 2nd differences column suggests that 0·333 32 is in error by about 110. Changing it to 0·332 22 then produces reasonable differences in the second fan.

The change from -49 to -17 suggests an error of about 30 in 0·448 52. 0·448 25 and 0·448 22 are possible corrections and the latter gives better differences at the end of the table. Finally, no change is indicated in 0·476 03.

4. i) $$\Delta\left(\frac{f_n}{g_n}\right) = \frac{f_{n+1}}{g_{n+1}} - \frac{f_n}{g_n} = \frac{f_{n+1}g_n - f_ng_{n+1}}{g_ng_{n+1}}$$

$$= \frac{(f_{n+1} - f_n)g_n - f_n(g_{n+1} - g_n)}{g_ng_{n+1}} = \frac{g_n\,\Delta f_n - f_n\,\Delta g_n}{g_ng_{n+1}}$$

ii) Putting $f_n = 1$ and consequently $\Delta f_n = 0$ in (i) gives result.

iii) $\Delta(\log f_n) = \log f_{n+1} - \log f_n = \log(f_{n+1}/f_n)$

5. i) $\Delta - \nabla = (E - 1) - (1 - E^{-1}) = E - 2 + E^{-1}$

$\Delta\nabla = (E - 1)(1 - E^{-1}) = E - 2 + E^{-1}$

Hence result

ii) $\delta^2 = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2 = E - 2 + E^{-1}$ and result follows from (i)