Chapter 6

Analytic Matrices and Power Series Solutions of Systems of Differential Equations

Objectives for this Chapter

The main objectives of this chapter is to gain an understanding of the following concepts regarding power series solutions of differential equations:

- analytic matrices;
- power series expansions of analytic functions;
- power series solutions for $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$;
- the exponential of a matrix.

Outcomes of this Chapter

After studying this chapter the learner should be able to:

- determine whether a matrix is analytic or not;
- solve the system $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$ using power series methods.

6.1 INTRODUCTION

The system $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, with $\mathbf{A}$ an $n \times n$ matrix with constant entries, is a special case of the more general system

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}, \quad a < t < b, \tag{6.1}$$

with $\mathbf{A}(t) = [a_{ij}(t)]$ a matrix whose entries are functions of time and which we will call continuous and real-valued if all the functions a_{ij} are continuous and real-valued. Everything that holds for the system $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$, is therefore applicable to the case where $\mathbf{A}$ is constant. The converse, is, however, not easy.

Consequently

$$\mathbf{U} = \begin{bmatrix} \frac{t}{2} \\ \frac{e^{-3t}}{3} + \frac{e^{-2t}}{4} \end{bmatrix}$$

and

$$\Phi(t)\,\mathbf{U}(t) = \begin{bmatrix} 2e^t & 0 \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} \frac{t}{2} \\ \frac{e^{-3t}}{3} + \frac{e^{-2t}}{4} \end{bmatrix} = \begin{bmatrix} te^t \\ \left(\frac{t}{2}+\frac{1}{4}\right)e^t - \frac{1}{3} \end{bmatrix}$$

is a solution of the given problem.

Example 4.3

Use Theorem 4.11 to find the general solution of the inhomogeneous system

$$\dot{\mathbf{X}} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 1 \\ \cot t \end{bmatrix}$$

where $t_0 = \frac{\pi}{2}$.

Solution:

The characteristic equation $C(\lambda) = 0$ has roots $\lambda = \pm i$. An eigenvector corresponding to $\lambda = i$ is

$$\mathbf{U} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}.$$

From this we obtain, with the aid of Theorem 2.7, the two real solutions

$$\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} \text{ and } \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}.$$

A fundamental matrix is therefore

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix}.$$

Now

$$\Phi^{-1}(t) = \begin{bmatrix} \cos t + \sin t & -\sin t \\ \sin t - \cos t & \cos t \end{bmatrix}.$$

Therefore, by choosing $t_0 = \frac{\pi}{2}$, we have from Theorem 4.11

$$\mathbf{X}_p(t) = \Phi(t) \int_{\frac{\pi}{2}}^{t} \begin{bmatrix} \cos s + \sin s & -\sin s \\ \sin s - \cos s & \cos s \end{bmatrix} \begin{bmatrix} 1 \\ \cot s \end{bmatrix} ds$$

$$= \Phi(t)\,\mathbf{J}$$

where

$$\mathbf{J} = \int_{\frac{\pi}{2}}^{t} \begin{bmatrix} \sin s \\ -\cos s + \operatorname{cosec} s \end{bmatrix} ds$$

$$\left(\text{recall } \frac{\cos^2 s}{\sin s} = \frac{1 - \sin^2 s}{\sin s} = \operatorname{cosec} s - \sin s\right)$$

$$= \begin{bmatrix} -\cos s \\ -\sin s - \ln|\operatorname{cosec} s + \cot s| \end{bmatrix}_{\frac{\pi}{2}}^{t}$$

$$= \begin{bmatrix} -\cos t \\ -\sin t - \ln|\operatorname{cosec} t + \cot t| + 1 \end{bmatrix}.$$

This yields

$$\mathbf{X}_p(t) = \boldsymbol{\Phi}(t)\,\mathbf{J}$$

$$= \begin{bmatrix} -1 + \sin t\,(1 - \ln|\operatorname{cosec} t + \cot t|) \\ -1 + (\cos t + \sin t)\,(1 - \ln|\operatorname{cosec} t + \cot t|) \end{bmatrix}.$$

Remark:

The inverse $\boldsymbol{\Phi}^{-1}(t)$ of the matrix

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix}$$

could quite easily be determined by recalling that the inverse of the matrix

$$\mathbf{S} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is given by

$$\mathbf{S}^{-1} = \frac{1}{AD - BC}\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}, \text{ provided that } AD - BC \neq 0.$$

If $\boldsymbol{\Phi}$ is a normalised fundamental matrix of $\dot{\mathbf{X}} = \mathbf{AX}$ at $t = 0$, the formula in Theorem 4.11 can be chang[illegible] slightly by applying Corollary 4.9.

Corollary 4.12 *If $\boldsymbol{\Phi}$ is a normalised fundamental matrix of $\dot{\mathbf{X}} = \mathbf{AX}$ at $t = 0$, then*

$$\mathbf{X}_p(t) = \int_0^t \boldsymbol{\Phi}(t - s)\,\mathbf{F}(s)\,ds \qquad (4[illegible]$$

is a solution of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$.

Remark:

Note that the solution $\mathbf{X}_p(t)$ of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$, given by (4.7) or (4.9) vanishes at t_0, since the integ[illegible] becomes zero at $t_0 = 0$.

Example 4.4

Use Corollary 4.12 to solve the inhomogeneous system

$$\dot{\mathbf{X}} = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 4e^t \cos t \\ 0 \end{bmatrix}$$

by taking $t_0 = 0$.

Solution:

The corresponding homogeneous system has the characteristic equation

$$C(\lambda) = \lambda^2 - 2\lambda + 2 = 0$$

which yields the roots $\lambda = 1 \pm i$.

i : Solve

$$\begin{bmatrix} 3-i & 5 \\ -2 & -3-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\begin{aligned} (3-i)\, v_1 + \quad 5v_2 &= 0 \qquad (1) \\ -2v_1 + (3+i)\, v_2 &= 0. \qquad (2) \end{aligned}$$

(Note that the second equation is identical to the first one, i.e. $-(3-i)/2 \times (2) = (1)$.) Choose, for example, $v_1 = 5$, then $v_2 = -3+i$, so that an eigenvector corresponding to $1+i$ is

$$\mathbf{V} = \begin{bmatrix} 5 \\ -3+i \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus we get the two real solutions of the corresponding homogeneous problem

$$\begin{aligned} \mathbf{X}_1(t) &= e^t \left(\begin{bmatrix} 5 \\ -3 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) \\ &= e^t \begin{bmatrix} 5\cos t \\ -3\cos t - \sin t \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{X}_2(t) &= e^t \left(\begin{bmatrix} 5 \\ -3 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t \right) \\ &= e^t \begin{bmatrix} 5\sin t \\ -3\sin t + \cos t \end{bmatrix}. \end{aligned}$$

A fundamental matrix is therefore

$$\mathbf{\Phi}(t) = e^t \begin{bmatrix} 5\cos t & 5\sin t \\ -3\cos t - \sin t & -3\sin t + \cos t \end{bmatrix}.$$

Hence

$$\mathbf{\Phi}(0) = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$$

and thus

$$\mathbf{\Phi}^{-1}(0) = \frac{1}{5} \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$$

so that the *normalized* fundamental matrix is given by

$$\mathbf{\Psi}(t) = \mathbf{\Phi}(t)\, \mathbf{\Phi}^{-1}(0) = e^t \begin{bmatrix} \cos t + 3\sin t & 5\sin t \\ -2\sin t & -3\sin t + \cos t \end{bmatrix}.$$

From Corollary 4.12 it therefore follows that

$$
\begin{aligned}
\mathbf{X}_p(t) &= \int_0^t \Psi(t-s)\,\mathbf{F}(s)\,ds \\
&= \int_0^t e^{t-s}\begin{bmatrix} \cos(t-s)+3\sin(t-s) & 5\sin(t-s) \\ -2\sin(t-s) & -3\sin(t-s)+\cos(t-s) \end{bmatrix}\begin{bmatrix} 4e^s\cos s \\ 0 \end{bmatrix} ds \\
&= 4e^t\int_0^t \begin{bmatrix} \cos(t-s)\cos s + 3\sin(t-s)\cos s \\ -2\sin(t-s)\cos \end{bmatrix} ds \\
&= 2e^t\int_0^t \begin{bmatrix} \cos t + \cos(t-2s) + 3\sin t + 3\sin(t-2s) \\ -2\sin t - 2\sin(t-2s) \end{bmatrix} ds \\
&= 2e^t\begin{bmatrix} s\cos t + 3s\sin t - \frac{1}{2}\sin(t-2s) + \frac{3}{2}\cos(t-2s) \\ -2s\sin t - \cos(t-2s) \end{bmatrix}_0^t \\
&= 2e^t\begin{bmatrix} t\cos t + 3t\sin t + \sin t \\ -2t\sin t \end{bmatrix}.
\end{aligned}
$$

Remark:

Note that the fundamental matrix $\Phi(t)$ in Corollary 4.12 is *normalized* while the one in Theorem 4.11 is *not necessarily* normalized. Also, in Theorem 4.11 the point t_0 is arbitrary, while in Corollary 4.12, $t_0 = 0$.

Exercise 4.10

1. Find a particular solution of the problem in Example 4.2 by using formula (4.7). Show that the answer obtained and the solution already found, differ by $\left[0, \frac{e^{3t}}{12}\right]^T$ — a solution of the complementary system $\dot{\mathbf{X}} = \mathbf{AX}$.

2. Prove that the difference between any two particular solutions of a non–homogeneous system, is a solution of the complementary homogeneous system.

From this it follows that the solutions of the non–homogeneous system $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$ may be expressed as $\mathbf{X}(t) = \mathbf{X}_c(t) + \mathbf{X}_p(t)$ with $\mathbf{X}_p(t)$ a particular solution of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$ and $\mathbf{X}_c(t)$ a general solution of $\dot{\mathbf{X}} = \mathbf{AX}$. As $\mathbf{X}_c(t)$ assumes all solutions of $\dot{\mathbf{X}} = \mathbf{AX}$, it follows that $\mathbf{X}(t)$ assumes all solutions of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$. As a result of Definition 4.4, we define a general solution of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$ as follows:

Definition 4.5 Suppose that $\mathbf{X}_p(t)$ is any particular solution of the non–homogeneous system $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$ and Φ a fundamental matrix of the complementary system $\dot{\mathbf{X}} = \mathbf{AX}$. Then the vector function

$$\mathbf{X}(t) = \mathbf{X}_p(t) + \Phi(t)\,\mathbf{K} \tag{4.10}$$

with $\mathbf{K}$ a constant vector, is called a *general solution* of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$.

The fact proved in Exercise 4.9, no. 2, together with the uniqueness theorem for homogeneous initial value problems, enables us to prove a uniqueness theorem for the corresponding non–homogeneous initial value

Theorem 4.12 (Uniqueness Theorem for the Non–Homogeneous Initial Value Problem $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F},\ \mathbf{X}(t_0) = \mathbf{X}_0$) *The problem*

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F},\ \ \mathbf{X}(t_0) = \mathbf{X}_0 \tag{4.11}$$

has one and only one solution

$$\mathbf{X}(t) = \mathbf{X}_p(t) + \mathbf{\Phi}(t)\,\mathbf{X}_0 \tag{4.12}$$

with $\mathbf{\Phi}$ *a normalised fundamental matrix of the complementary system* $\dot{\mathbf{X}} = \mathbf{AX}$ *at* t_0 *and*

$$\mathbf{X}_p(t) = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\,\mathbf{F}(s)\,ds.$$

Proof: By differentiation it follows that the function given by equation (4.12), is a solution of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$, which also satisfies the initial condition. Suppose $\mathbf{Y}(t)$ is any solution of (4.11). We then have to prove that $\mathbf{X}(t) \equiv \mathbf{Y}(t)$. Define $\mathbf{Z}(t) = \mathbf{X}(t) - \mathbf{Y}(t)$. Then

$$\begin{aligned}
\dot{\mathbf{Z}}(t) &= \dot{\mathbf{X}}(t) - \dot{\mathbf{Y}}(t) \\
&= \mathbf{AX}(t) + \mathbf{F}(t) - \mathbf{AY}(t) - \mathbf{F}(t) \\
&= \mathbf{A}(\mathbf{X}(t) - \mathbf{Y}(t)) \\
&= \mathbf{AZ}(t)
\end{aligned}$$

and

$$\mathbf{Z}(t_0) = \mathbf{X}(t_0) - \mathbf{Y}(t_0) = \mathbf{0}.$$

The vector function $\mathbf{Z}(t)$ is, therefore, a solution of a homogeneous system which vanishes at t_0. From Corollary 4.7, $\mathbf{Z}(t)$ is identically zero, so that $\mathbf{X}(t) \equiv \mathbf{Y}(t)$. This completes the proof. ■

Example 4.5

Find a general solution of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix},\quad \mathbf{F}(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}.$$

Solution:

Take $t_0 = 0$.

A general solution of $\dot{\mathbf{X}} = \mathbf{AX}$ is

$$\mathbf{X}(t) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t},$$

with c_1, c_2 arbitrary constants. A fundamental matrix is therefore given by

$$\mathbf{\Phi}(t) = \begin{bmatrix} 2e^t & 0 \\ e^t & e^{3t} \end{bmatrix}.$$

We find a particular solution by using Corollary 4.12. A normalized fundamental matrix at $t_0 = 0$ for the homogeneous system is given by

$$\begin{aligned}\Psi(t) &= \Phi(t)\,\Phi^{-1}(0)\\ &= \begin{bmatrix} 2e^t & 0 \\ e^t & e^{3t}\end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2\end{bmatrix}\\ &= \begin{bmatrix} e^t & 0 \\ \dfrac{e^t - e^{3t}}{2} & e^{3t}\end{bmatrix}.\end{aligned}$$

From equation (4.9), a particular solution is

$$\begin{aligned}\mathbf{X}_p(t) &= \int_0^t \Psi(t-s)\,\mathbf{F}(s)\,ds\\ &= \int_0^t \begin{bmatrix} e^{t-s} & 0 \\ \frac{e^{t-s}-e^{3(t-s)}}{2} & e^{3(t-s)}\end{bmatrix}\begin{bmatrix} e^s \\ 1\end{bmatrix} ds\\ &= \int_0^t \begin{bmatrix} e^t \\ \frac{e^t - e^{3t-2s}}{2} + e^{3(t-s)}\end{bmatrix} ds\\ &= \begin{bmatrix} se^t \\ \frac{se^t}{2} - \frac{e^{3t-2s}}{-4} + \frac{e^{3(t-s)}}{-3}\end{bmatrix}_0^t\end{aligned}$$

by using the fact that the integral of a matrix function $\mathbf{A}(t) = [a_{ij}(t)]$ is the matrix function with entries $\int a_{ij}(t)\,dt$ (see p. 46 of the prescribed book).

We therefore have the particular solution

$$\mathbf{X}_p(t) = \begin{bmatrix} te^t \\ \frac{te^t}{2} + \frac{e^t}{4} - \frac{1}{3} - \frac{e^{3t}}{4} + \frac{e^{3t}}{3}\end{bmatrix} = \begin{bmatrix} te^t \\ \left(\frac{t}{2} + \frac{1}{4}\right)e^t + \left(\frac{e^{3t}}{12} - \frac{1}{3}\right)\end{bmatrix}.$$

A general solution is, therefore,

$$\mathbf{X}(t) = \begin{bmatrix} te^t \\ \left(\frac{t}{2} + \frac{1}{4}\right)e^t + \left(\frac{e^{3t}}{12} - \frac{1}{3}\right)\end{bmatrix} + c_1\begin{bmatrix} 2 \\ 1\end{bmatrix}e^t + c_2\begin{bmatrix} 0 \\ 1\end{bmatrix}e^{3t},$$

with c_1, c_2 arbitrary constants. Special additional methods still exist for determining a general solution of the non-homogeneous system $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}$ (see Section 4.5 of the prescribed book.)

Exercise 4.11

Exercise 4.4, p. 144–145 of the prescribed book.

4.5 THE INEQUALITY OF GRONWALL

In applications of differential equations, it is not always possible to obtain an exact solution of the differential equation. In such cases it is sometimes necessary to find, instead, an estimate for, or an upperbound of, the so-called *norm of the solution*. The latter quantity, as we will see below, measures the "size" of the solution.

In this section we treat the *inequality of Gronwall*, which is a useful tool in obtaining estimates for the norm of a solution of a differential equation of the form $\dot{\mathbf{X}} = \mathbf{AX}$.

Firstly we recall that if $\mathbf{X}(t)$ is a vector function, e.g.

$$\mathbf{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T,$$

then

$$\begin{aligned} ||\mathbf{X}(t)|| &= \sqrt{x_1^2(t) + x_2^2(t) + \ldots + x_n^2(t)} \\ &= \left(\sum_{i=1}^{n} x_i^2(t)\right)^{\frac{1}{2}}. \end{aligned}$$

Next we recall that for an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ the number $||\mathbf{A}||$ is defined [2] by

$$||\mathbf{A}|| = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}.$$

We show that

$$||\mathbf{A}|| = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}$$

is indeed a norm. The system of all $n \times n$ matrices is then a normed linear space. If $\mathbf{A}$ denotes $[a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, we have

(i) $||\mathbf{A}|| = 0 \Longleftrightarrow a_{ij} = 0 \quad \forall i = 1, \ldots, n; \quad j = 1, \ldots, n$
$\Longleftrightarrow$ $\mathbf{A}$ is the zero matrix.

(ii) $$\begin{aligned} ||\lambda\mathbf{A}|| &= \left(\sum_{i,j=1}^{n} (\lambda a_{ij})^2\right)^{\frac{1}{2}} \\ &= |\lambda|\,||\mathbf{A}|| \text{ for any scalar } \lambda. \end{aligned}$$

[2] The norm of $\mathbf{A}$ may also be defined in another way, as long as the properties of a norm are satisfied — see (i)–(iii) further

(iii) $$\begin{aligned} \|\mathbf{A}+\mathbf{B}\| &= \left(\sum_{i,j=1}^{n} (a_{ij}+b_{ij})^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{i,j=1}^{n} a_{i,j}^2\right)^{\frac{1}{2}} + \left(\sum_{i,j=1}^{n} b_{ij}^2\right)^{\frac{1}{2}} \\ &= \|\mathbf{A}\| + \|\mathbf{B}\| \end{aligned}$$

where in the second last step we used the inequality of Minkowski.

Before formulating the inequality of Gronwall, we prove the following auxiliary result, which will be used in proving Gronwall's inequality.

Lemma 4.1 *Suppose that f and g are continuous, real–valued functions in $a \le t \le b$ and that $\dot{f}$ exists in this interval. If, in addition, $\dot{f}(t) \le f(t)\, g(t)$ in $a \le t \le b$, then*

$$f(t) \le f(a) \exp\left[\int_a^t g(s)\, ds\right] \quad \forall t \in [a,b]. \tag{4.13}$$

Exercise 4.12

Prove Lemma 4.1.

Hint: See p. 151 of the prescribed book. The only difficulty in the proof is to determine $\dot{p}(t)$. Once more you have to use the differentiation formula (4.8), from which it follows that if

$$p(t) = \exp\left(-\int_a^t g(s)\, ds\right),$$

then

$$\begin{aligned} \dot{p}(t) &= \exp\left(-\int_a^t g(s)\, ds\right) \cdot \frac{d}{dt}\left(-\int_a^t g(s)\, ds\right) \\ &= \exp\left(-\int_a^t g(s)\, ds\right)[-g(t)] \\ &= -p(t)\, g(t). \end{aligned}$$

We now proceed to

Theorem 4.13 (The Inequality of Gronwall) *If f and g $(g \ge 0)$ are continuous, real-valued functions in $a \le t \le b$, and K a real constant such that*

$$f(t) \le K + \int_a^t f(s)\, g(s)\, ds, \quad a \le t \le b, \tag{4.14}$$

then

$$f(t) \le K \exp\left[\int_a^t g(s)\, ds\right], \quad a \le t \le b. \tag{4.15}$$

Proof: Put

$$G(t) = K + \int_a^t f(s)\, g(s)\, ds.$$

Then $f(t) \leq G(t)$ in $[a,b]$. G is now differentiable in $[a,b]$. Indeed $G'(t) = f(t)\, g(t)$. Since $g \geq 0$ and $f \leq G$, we have $fg \leq Gg$; therefore $G' \leq Gg$. All the requirements of Lemma 4.1 are satisfied with the role of f assumed by G. Consequently we have, according to inequality (4.13),

$$G(t) \leq G(a) \exp\left(\int_a^t g(s)\, ds\right), \quad a \leq t \leq b.$$

Now

$$G(a) = K + \int_a^a f(s)\, g(s)\, ds = K.$$

Therefore

$$G(t) \leq K \exp\left[\int_a^t g(s)\, ds\right], \quad a \leq t \leq b.$$

The result now follows since $f(t) \leq G(t)$ in $[a,b]$. ■

Exercise 4.13

Study Examples 4.6.1 and 4.6.2, p. 152 of the prescribed book, in which the inequality of Gronwall is applied.

The technique used in these examples is applied to systems of differential equations in the following section. As in the case of the one–dimensional equation, the procedure is to replace the differentiable equation by an equation containing an integral, and then to take the norm.

4.6 THE GROWTH OF SOLUTIONS

The Gronwall Inequality may be used to obtain an *estimate of the norm of solutions* of the system $\dot{\mathbf{X}} = \mathbf{AX}$, where $\mathbf{A}$ is a constant matrix.

Theorem 4.14 *If* $\mathbf{X}(t)$ *is any solution of* $\dot{\mathbf{X}} = \mathbf{AX}$, *then the following inequality holds for all* t *and* t_0:

$$\|\mathbf{X}(t)\| \leq \|\mathbf{X}(t_0)\| \exp\left(\|\mathbf{A}\|\, |t - t_0|\right) \tag{4.16}$$

with equality for $t = t_0$.

(See Problem 2.4.II:3, p. 71 of the prescribed book for an interpretation of $\exp(t\mathbf{A})$.)

Proof: If $t = t_0$, the assertion holds. If $t > t_0$, we have $|t - t_0| = t - t_0$, so that the inequality (4.16) reduces to

$$\|\mathbf{X}(t)\| \leq \|\mathbf{X}(t_0)\| \exp\left[\|\mathbf{A}\|\, (t - t_0)\right].$$

By integrating $\dot{\mathbf{X}}(t) = \mathbf{AX}(t)$, we obtain

$$\mathbf{X}(t) - \mathbf{X}(t_0) = \int_{t_0}^t \mathbf{AX}(s)\, ds.$$

From the properties of norms (see also Theorem 1.10.2 and Problem 1.9I:5 of the prescribed book), we have

$$||\mathbf{X}(t)|| \leq ||\mathbf{X}(t_0)|| + \int_{t_0}^{t} ||\mathbf{A}|| \, ||\mathbf{X}(s)|| \, ds.$$

By putting $f(t) = ||\mathbf{X}(t)||$, $g(t) = ||\mathbf{A}||$ and $K = ||\mathbf{X}(t_0)||$, it follows from Theorem 4.13 that

$$||\mathbf{X}(t)|| \leq ||\mathbf{X}(t_0)|| \exp\left(\int_{t_0}^{t} ||\mathbf{A}|| \, ds\right) = ||\mathbf{X}(t_0)|| \exp\left[||\mathbf{A}|| (t - t_0)\right].$$

We still have to prove the theorem for $t < t_0$. In this case, put $\mathbf{Y}(t) = \mathbf{X}(-t)$. If we assume that $\mathbf{X}(t)$ is a solution of $\dot{\mathbf{X}} = \mathbf{AX}$, it follows that

$$\dot{\mathbf{Y}}(t) = \dot{\mathbf{X}}(-t) = -\mathbf{AY}(t).$$

Therefore $\mathbf{Y}(t)$ is a solution of $\dot{\mathbf{Y}} = -\mathbf{AY}$. Moreover $-t > -t_0$. Therefore, by applying (4.17) to the system $\dot{\mathbf{Y}} = -\mathbf{AY}$, we obtain

$$\begin{aligned} ||\mathbf{X}(t)|| &= ||\mathbf{Y}(-t)|| \\ &\leq ||\mathbf{Y}(-t_0)|| \exp\left[||-\mathbf{A}|| (-t + t_0)\right] \\ &= ||\mathbf{X}(t_0)|| \exp\left[||\mathbf{A}|| (-t + t_0)\right] \\ &= ||\mathbf{X}(t_0)|| \exp\left[||\mathbf{A}|| \, |t - t_0|\right], \end{aligned}$$

since if $t < t_0$, then $|t - t_0| = -t + t_0$. This completes the proof. ∎

Remark:

If $t - t_0 > 0$, the inequality (4.16) shows that $||\mathbf{X}(t)||$ does not increase more rapidly than a certain exponential function. Such functions $\mathbf{X}(t)$, are known as functions of exponential type.

Example 4.6

The norms of a vector

$$\mathbf{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$$

and an $n \times n$ matrix $\mathbf{A} = [a_{i,j}]$ were previously defined by

$$||\mathbf{X}|| = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \text{and} \quad ||\mathbf{A}|| = \sqrt{\sum_{i,j}^{n} a_{ij}^2}.$$

Alternatively we can define

$$||\mathbf{X}|| = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad ||\mathbf{A}|| = \max_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}|.$$

The latter definition means that the norm of a matrix $\mathbf{A}$ is defined as the maximum of the sums obtained by addition of the absolute values of the entries in each column of $\mathbf{A}$.

Now consider the system

$$\begin{aligned} \dot{x}_1 &= -(\sin t) x_2 + 4 \\ \dot{x}_2 &= -x_1 + 2t x_2 - x_3 + e^t \\ \dot{x}_3 &= 3(\cos t) x_1 + x_2 + \frac{1}{t} x_3 - 5t^2. \end{aligned}$$

Assume that there exists solutions $\mathbf{X}$ and $\mathbf{Y}$ on the interval $(1,3)$ with

$$\mathbf{X}(2) = [7;\ 3;\ -2]^T \quad \text{and} \quad \mathbf{Y}(2) = [6.2;\ 3.2;\ -1,9]^T.$$

Use the alternative definitions given above to estimate the error

$$\|\mathbf{X}(t) - \mathbf{Y}(t)\| \quad \text{for } 1 < t < 3.$$

Solution:

For the given system we have

$$\mathbf{A}(t) = \begin{bmatrix} 0 & -\sin t & 0 \\ -1 & 2t & -1 \\ 3\cos t & 1 & \frac{1}{t} \end{bmatrix}$$

so that

$$\begin{aligned} \|\mathbf{A}(t)\| &= \max\left\{1 + 3|\cos t|,\ |\sin t| + 2t + 1,\ 1 + \frac{1}{t}\right\} \\ &\le \max\{4,\ 2 + 2t,\ 2\} \\ &= 2 + 2t \ \text{ since } 1 < t < 3. \end{aligned}$$

By the inequality of Gronwall we have

$$\|\mathbf{X}(t) - \mathbf{Y}(t)\| \le \|\mathbf{X}(2) - \mathbf{Y}(2)\| \exp\left|\int_2^t \|\mathbf{A}(s)\|\, ds\right|^*$$

i.e.

$$\begin{aligned} \|\mathbf{X}(t) - \mathbf{Y}(t)\| &\le \|\mathbf{X}(2) - \mathbf{Y}(2)\|\, e^{\left|\int_2^t (2+2s)ds\right|} \\ &< (0.3 + 0.2 + 0.1)\, e^7 \\ &= 1.1e^7. \end{aligned}$$

The term e^7 is obtained by computing the integral and bearing in mind that $t < 3$. (The proof of $*$ is similar to the proof of (4.14).)

A less sharp estimate can be obtained by using

$$\|\mathbf{A}(t)\| \le 8 \ \text{ on } 1 < t < 3.$$

In this case it follows that

$$\begin{aligned} \|\mathbf{X}(t) - \mathbf{Y}(t)\| &\le \|\mathbf{X}(2) - \mathbf{Y}(2)\|\, e^{8|t-2|} \\ &< 1.1e^8. \end{aligned}$$

Exercise 4.14

1. Show that no solution of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ can satisfy $\|\mathbf{X}(t)\| = e^{t^2}$.
2. [illegible] Examples 4.7.1 and 4.7.2, p. 54 of the prescribed book. These illustrate that inequality (4.16) could yield an over–estimate, but also a precise estimation.

Chapter 5

Higher Order One–dimensional Equations as Systems of First Order Equations

Objectives of this Chapter

The main objective of this chapter is to gain an understanding of the following concepts regarding higher order one–dimensional equations:

- the link between higher order one–dimensional equations and linear systems of first order differential equations;
- companion matrix;
- companion system for the n–th order equation with constant coefficients as given by (5.1).

Outcomes of this Chapter

After studying this chapter the learner should be able to

- rewrite the n–th order equation (5.1) as a linear system of first order differential equations;
- solve the corresponding linear system of first order differential equations and interpret the solution in terms of the original higher order one dimensional equation.

5.1 INTRODUCTION

In Chapter 5 of the prescribed book the link that exists between higher order one–dimensional linear differential equations and linear systems of first order differential equations is explored. The study of Sections 5.1 through 5.5 from this chapter in the prescribed book is set as an assignment for the student, as it is very important for students to get practise in studying from text books.

Being able to "read" a variety of books on the particular topic that you are studying, should be one of one's ultimate goals when studying at a university.

We shall therefore only discuss the concept of companion systems and illustrate by means of an example

The rest is left to the student. Mark theorems which look interesting, eg. Theorem 5.3.3 and Theorem 5.5.3, which deal with the relationship between the auxiliary equation of the general n–th order one–dimensional equation and the characteristic equation of the companion system. In this way your understanding of n–th order one–dimensional linear differential equations will be deepened. Work through the proofs of the theorems (these are not required for examination purposes) and select problems from Exercise 5.2 through 5.5.

5.2 COMPANION SYSTEMS FOR HIGHER–ORDER ONE–DIMENSIONAL DIFFERENTIAL EQUATIONS

The n–th order equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y = f(t) \tag{5.1}$$

has been treated in MAT216–W. This equation can be reduced to a system of n first order equations. We illustrate by means of an example:

Consider the equation

$$y^{(3)} - 6y^{(2)} + 5y^{(1)} = \sin t \tag{5.2}$$

with initial conditions $y(0) = y'(0) = y''(0) = 0$.

If $y(t)$ represents the distance of a body from a fixed point on a line, this equation describes the movement of that body along the line, subject to different forces. Since y measures distance, $y^{(1)}$ measures velocity and $y^{(2)}$ acceleration. By putting $y^{(1)} = v$ and $y^{(2)} = a$, equation (5.2) may be expressed as

$$\begin{aligned} y^{(1)} &= v \\ y^{(2)} &= a \\ y^{(3)} &= -5v + 6a + \sin t. \end{aligned} \tag{5.3}$$

This, in turn, may be expressed as

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{y} \\ \dot{v} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & 6 \end{bmatrix} \begin{bmatrix} y \\ v \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix} \tag{5.4}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & 6 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}. \tag{5.5}$$

Equation (5.5) is now merely a special case of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}(t)$ with the initial condition

$$\mathbf{X}(0) = \begin{bmatrix} y(0) \\ v(0) \\ a(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y^{(1)}(0) \\ y^{(2)}(0) \end{bmatrix} = 0.$$

More generally, the third order equation

$$y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y = f(t) \tag{5.6}$$

from which we obtain

$$y^{(3)} = -a_0 y - a_1 y^{(1)} - a_2 y^{(2)} + f(t),$$

can be expressed in matrix form as three first order equations as follows:

Let

$$\mathbf{X} = \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \end{bmatrix}.$$

Then

$$\dot{\mathbf{X}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix}$$

so that

$$\begin{aligned} \dot{\mathbf{X}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} &= \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ -a_0 y - a_1 y^{(1)} - a_2 y^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \mathbf{X} + f(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \mathbf{A}\mathbf{X} + \mathbf{F}(t) \end{aligned} \quad (5.7)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_2 & -a_3 \end{bmatrix}$$

and

$$\mathbf{F}(t) = f(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The system (5.7) is known as the *companion system* for the third order equation (5.6). The companion system for the n–th order equation (5.1) may be defined in a similar way.

Definition 5.1 A matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

is known as a *companion matrix*.

Definition 5.2 The system

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{X} + f(t) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5.8}$$

is known as the *companion system* for the n–th order equation (5.1).

Theorem 5.1 *If y is a solution of equation (5.1), then the vector*

$$\mathbf{X} = \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

is a solution of the companion system (5.8). Conversely: if

$$\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is a solution of the system (5.8), then x_1, the first component of $\mathbf{X}$, is a solution (5.1) and

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_1^{(1)} \\ \vdots \\ x_1^{(n-1)} \end{bmatrix},$$

i.e.

$$x_2 = x_1^{(1)}, \quad x_3 = x_1^{(2)}, \ldots, x_n = x_1^{(n-1)}.$$

Exercise 5.1

1. Prove Theorem 5.1.
2. Prove, by applying Theorem 5.1, that the initial value problem

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y = f(t),$$

$$y(t_0) = y_1, \; y^{(t)}(t_0) = y_2, \ldots, y^{(n-1)}(t_0) = y_n,$$

[illegible]olution.

We summarize: *Equations of order n in one dimension may be solved by determining a general solution*

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

of the companion system. The function $x_1(t)$ is then a solution of the given n-th order equation.

As the companion system is but a special case of the non-homogeneous linear system with constant coefficients, one needs no new techniques.

Example 5.1

Determine the general solution of

$$\frac{d^2y}{dt^2} + 4y = \sin 3t \tag{5.9}$$

by using the companion system.

Solution:

The companion system is found as follows:

Rewrite (5.9) as

$$y^{(2)} = -4y + \sin 3t.$$

Let

$$\mathbf{X} = \begin{bmatrix} y \\ y^{(1)} \end{bmatrix},$$

then

$$\dot{\mathbf{X}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}$$

so that

$$\begin{aligned} \dot{\mathbf{X}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} y \\ y^{(1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \sin 3t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} y \\ y^{(1)} \end{bmatrix} + \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \mathbf{AX} + \mathbf{F}(t). \end{aligned} \tag{5.10}$$

We first determine a general solution of $\dot{\mathbf{X}} = \mathbf{AX}$. Now

$$\det \begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda = \pm 2i.$$

Choose $\lambda = 2i$. (Remember $\lambda = -2i$ yields identical solutions up to a constant). Now put

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then

$$-2iu_1 + u_2 = 0$$
$$-4u_1 - 2iu_2 = 0.$$

These two equations are identical. Choose $u_1 = 1$, then $u_2 = 2i$. The vector

$$\mathbf{U} \equiv \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

is then an eigenvector corresponding to the eigenvalue $\lambda = 2i$. Consequently

$$\mathbf{X}_1(t) = \left\{ \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} = \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix}$$

and

$$\mathbf{X}_2(t) = \left\{ \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} = \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

are solutions of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ and a general solution is

$$\mathbf{X}(t) = c_1 \mathbf{X}_1(t) + c_2 \mathbf{X}_2(t)$$

with c_1 and c_2 arbitrary constants.

A fundamental matrix of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ is therefore

$$\mathbf{\Phi}(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix}$$

and a normalised fundamental matrix of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ at $t_0 = 0$ is

$$\mathbf{\Psi}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0) = \frac{1}{2}\begin{bmatrix} 2\cos 2t & \sin 2t \\ -4\sin 2t & 2\cos 2t \end{bmatrix}.$$

Therefore, a particular solution of the system (5.10) is

$$\begin{aligned}
\mathbf{X}_p(t) &= \frac{1}{2}\int_0^t \left\{ \begin{bmatrix} 2\cos 2(t-s) & \sin 2(t-s) \\ -4\sin 2(t-s) & 2\cos 2(t-s) \end{bmatrix} \begin{bmatrix} 0 \\ \sin 3s \end{bmatrix} \right\} ds \\
&= \frac{1}{2}\int_0^t \begin{bmatrix} \sin 2(t-s)\sin 3s \\ 2\cos 2(t-s)\sin 3s \end{bmatrix} ds \\
&= \frac{1}{2}\int_0^t \begin{bmatrix} \frac{1}{2}\cos(2t-5s) - \frac{1}{2}\cos(2t+s) \\ \sin(2t+s) + \sin(5s-2t) \end{bmatrix} ds \\
&= \frac{1}{2}\begin{bmatrix} \left(\frac{-\sin(2t-5s)}{10} - \frac{\sin(2t+s)}{2}\right) \\ \left(-\cos(2t+s) - \frac{\cos(5s-2t)}{5}\right) \end{bmatrix}_0^t \\
&= \frac{1}{2}\begin{bmatrix} \left(-\frac{2\sin 3t}{5} + \frac{3\sin 2t}{5}\right) \\ \left(-\frac{6\cos 3t}{5} + \frac{6\cos 2t}{5}\right) \end{bmatrix}.
\end{aligned}$$

Consequently the general solution of $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}(t)$ is

$$\mathbf{X}(t) = c_1 \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix} + \frac{1}{10} \begin{bmatrix} -2\sin 3t + 3\sin 2t \\ -6\cos 3t + 6\cos 2t \end{bmatrix}.$$

According to Theorem 5.1, a general solution of (5.9) is given by the first component of $\mathbf{X}(t)$. There...

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{\sin 3t}{5} + \frac{3\sin 2t}{10}.$$

Note that the second component of $\mathbf{X}(t)$ is the derivative of $y(t)$.

Remark:

This solution can be obtained more readily by applying the standard techniques for solving higher ord... dimensional equations: For the equation $(D^2 + 4)\, y = \sin 3t$, we have the auxiliary equation $m^2 + 4 =$ ... roots $\pm 2i$. Hence

$$y_{\text{C.F.}}(t) = d_1 \cos 2t + d_2 \sin 2t.$$

Furthermore

$$y_{\text{P.I.}}(t) = \frac{\sin 3t}{-3^2 + 4} = -\frac{\sin 3t}{5}.$$

Hence

$$y(t) = d_1 \cos 2t + d_2 \sin 2t - \frac{\sin 3t}{5}.$$

This is equivalent to the solution (5.11).

Exercise 5.2

Exercise 5.2, p. 162 of the prescribed book.

Whereas we have to our disposal standard techniques for finding n linearly independent solutions of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, each of which is usually a linear combination of known functions, while we also know that n linearly independent solutions exist, the construction of only a single solution of $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$, entails an enormous amount of work. Moreover, it requires advanced mathematics to prove theorems that n linearly independent solutions always exist.

A discussion of this theory, which is outside the scope of this course, is found, amongst others, in "Lectures on Ordinary Differential Equations" by W. Hurewicz[1]. It is sufficient to say that, as in the case of equation $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, fundamental and normalised fundamental matrices $\mathbf{\Phi}(t)$ of $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$, are used in the development of this theory.

Whereas the solution of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ are usually linear combinations of known functions such as $\sin t$ and ..., such linear combinations are rarely solutions of $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$. Indeed, systems $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$ are often used to define and to study new functions. For instance, the functions of Bessel, Legendre and Laguerre and Hermite Functions (some of which you have encountered in MAT216–W), are examples of functions defined as solutions of certain differential equations.

The main result of this chapter is that, under certain conditions on the matrix $\mathbf{A}(t)$, viz that it is possible to write the entries of $\mathbf{A}(t)$ as power series, a solution of $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$ can be expressed as a power series

$$\mathbf{X}(t) = \mathbf{U}_0 + \mathbf{U}_1(t - t_0) + \ldots + \mathbf{U}_k(t - t_0)^k + \ldots \tag{6.2}$$

about t_0. If we wish to find a power series solution of $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$, the vectors $\mathbf{U}_0, \mathbf{U}_1, \ldots$ have to be determined, while we also have to show that the infinite sum (6.2) converges to a solution of $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$. The property which $\mathbf{A}(t)$ must satisfy, is that of *analyticity*, a powerful property, since continuity and even the existence of derivatives of all orders, are not sufficient to ensure this property. However, in most cases the coefficient matrices of differential equations, as well as of systems that are important for all practical purposes, do have the property of analyticity.

6.2 POWER SERIES EXPANSIONS OF ANALYTIC FUNCTIONS

Although the property of analyticity is usually associated with functions of a complex variable, we ... define the property of analyticity for real functions, in view of the fact that a function of a real variable is only a special case of a function of a complex variable.

Definition 6.1 A function F of a real variable is *analytic at a real number* t_0 if a positive number r and a sequence of numbers $a_0, a_1, a_2, \ldots$ exist such that

$$F(t) = \lim_{N \to \infty} \sum_{k=0}^{N} a_k (t - t_0)^k = \sum_{k=0}^{\infty} a_k (t - t_0)^k \quad \text{for } |t - t_0| < r. \tag{6.3}$$

The following theorem deals with the differentiation of power series. *This is no trivial matter, since infinite sums are involved.*

[1] ... ctures on Ordinary Differential Equations, The Technology Press, Cambridge, Mass., 1958.

Theorem 6.1 (Taylor's Theorem) *If F is analytic at t_0, and has series expansion*

$$F(t)=\sum_{k=0}^{\infty}a_k(t-t_0)^k \quad \text{for } |t-t_0|<r,$$

then

$$F^{(j)}(t)=\sum_{k=0}^{\infty}\frac{(k+j)!}{k!}a_{k+j}(t-t_0)^k \quad \text{for } |t-t_0|<r,$$

for every positive integer j.

From the above it is clear that the j-th derivative $F^{(j)}$ is also analytic at t_0 and $F^{(j)}(t_0)=j!a_j$. The constants

$$a_k=\frac{1}{k!}F^{(k)}(t_0)$$

are known as the *Taylor coefficients* of F at t_0, and the expansion

$$\sum_{k=0}^{\infty}a_k(t-t_0)^k=\sum_{k=0}^{\infty}\frac{1}{k!}F^{(k)}(t_0)(t-t_0)^k$$

is the *Taylor series of F about t_0*.

Remark:

(1) It is worthwhile selecting analytic functions to form a class of functions for the following reasons:

(a) It is possible that a function may not possess derivatives of all orders. (Can you think of a good example?)

(b) Functions F exist which have derivatives of all orders, but of which the Taylor series

$$\sum_{k=0}^{\infty}\frac{1}{k!}F^{(k)}(t_0)(t-t_0)^k$$

converges only at $t=t_0$. It may happen that a function F has a Taylor series expansion

$$\sum_{k=0}^{\infty}\frac{1}{k!}F^{(k)}(t_0)(t-t_0)^k$$

converging for $|t-t_0|<r$ for some $r>0$, but not to $F(t)$, i.e.

$$F(t)\neq\sum_{k=0}^{\infty}\frac{1}{k!}F^{(k)}(t_0)(t-t_0)^k,$$

unless $t=t_0$.

(2) If F is a function of the complex variable z and F is analytic at $z=z_0$ (with the concept of analyticity defined as in Complex Analysis), a power series expansion of the form

$$\sum_{k=0}^{\infty}a_k(z-z_0)^k$$

exists for $F(z)$, viz the Laurent Series

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} b_k (z-z_0)^{-k},$$

with all the coefficients b_k of the "principal part"

$$\sum_{k=0}^{\infty} b_k (z-z_0)^{-k}$$

equal to zero.

In this case the coefficients a_k $(k = 0, 1, 2, \ldots)$ are given in terms of contour integrals by means of the formula

$$a_k = \frac{1}{2\pi i} \oint_C \frac{F(z)}{(z-z_0)^{k+1}} dz$$

with C a circle with centre z_0. This is, of course, a result from Complex Analysis, quoted here only for the sake of interest (as this result is dealt with in the module on Complex Analysis).

A list of Taylor series for some elementary functions is supplied on p. 196 of the prescribed book.

The following theorem deals with addition, subtraction and multiplication of power series:

Theorem 6.2 *If*

$$F(t) = \sum_{k=0}^{\infty} F_k (t-t_0)^k, \quad -r < t - t_0 < r,$$

$$G(t) = \sum_{k=0}^{\infty} G_k (t-t_0)^k, \quad -r < t - t_0 < r$$

with F_k and G_k the Taylor Coefficients of F and G, then

$$F(t) \pm G(t) = \sum_{k=0}^{\infty} (F_k \pm G_k)(t-t_0)^k$$

and

$$F(t) G(t) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} F_i G_{k-i} (t-t_0)^k \quad \text{for } |t-t_0| < r.$$

The following theorem is often used.

Theorem 6.3 *If*

$$F(t) = \sum_{k=0}^{\infty} F_k (t-t_0)^k \equiv 0, \quad -r < t - t_0 < r,$$

then $F_0 = F_1 = \ldots = 0$ with F_k the Taylor coefficients of F.

Proof: The result follows immediately by recalling that if

$$F(t) \equiv 0, \text{ i.e. } F(t) = 0 \textit{ for all } t,$$

then the derivatives of all orders are also zero. ■

Remark:

$F(0) = 0$ does not imply $F'(0) = 0$! For example

$$F(x) = x \text{ implies } F(0) = 0, \text{ but } F'(0) = 1.$$

We now define analytic matrices:

Definition 6.2 The $n \times m$ matrix $A(t) = [a_{ij}(t)]$ is said to be *analytic at* $t = t_0$ if *every entry*

$$a_{ij}(t), \quad i = 1, \ldots, n, j = 1, \ldots, m$$

is analytic at $t = t_0$.

In this definition it may happen that the intervals of convergence of the expansions for the separate entries differ from one another. The interval of convergence of $\mathbf{A}(t)$ is then taken as that interval in which all the entries converge.

The preceding theorems may now be reformulated for analytic $\mathbf{A}(t)$ by simply applying the relevant conditions to each entry of $\mathbf{A}(t)$ and then making the conclusion in a similar way.

The notation

$$\mathbf{A}(t) = \sum_{k=0}^{\infty} \mathbf{A}_k (t - t_0)^k,$$

with

$$\mathbf{A}_k = \frac{1}{k!} \mathbf{A}^{(k)}(t_0), \quad |t - t_0| < r,$$

is, therefore, just an abbreviated notation for $n \times m$ expansions — one for each entry of $\mathbf{A}(t)$.

Example 6.1

Determine $\mathbf{A}_k$ such that

$$\begin{bmatrix} \cos t & \arctan t \\ e^t & \dfrac{1}{1-t} \end{bmatrix} = \sum_{k=0}^{\infty} \mathbf{A}_k t^k \equiv \mathbf{A}(t).$$

Solution:

Note that $t_0 = 0$. Since $\cos t$ is an even function and $\arctan t$ is an odd function, we rewrite the series expansion

$$\sum_{k=0}^{\infty} \mathbf{A}_k t^k$$

as the sum of its even and odd expansions as follows:

$$\sum_{k=0}^{\infty} \mathbf{A}_k t^k = \sum_{i=0}^{\infty} \mathbf{A}_{2i} t^{2i} + \sum_{i=0}^{\infty} \mathbf{A}_{2i+1} t^{2i+1}.$$

Then

$$\mathbf{A}_{2i} = \begin{bmatrix} \dfrac{(-1)^{2i}}{(2i)!} & 0 \\ \dfrac{1}{(2i)!} & 1 \end{bmatrix}$$

and

$$\mathbf{A}_{2i+1} = \begin{bmatrix} 0 & \dfrac{(-1)^{2i}}{(2i+1)!} \\ \dfrac{1}{(2i+1)!} & 1 \end{bmatrix}$$

for $i = 1, 2, \ldots$ by using the expansions for $\cos t$, e^t and $\arctan t$ given on page 96 of the textbook. Th… common interval of convergence is $-1 < t < 1$, so that $\mathbf{A}(t)$ is analytic at $t_0 = 0$.

Exercise 6.1

1. Study Example 6.2.4, p. 199 of the prescribed book.
2. Exercises 6.2.I:4, II:4,5,6 of the prescribed book.

6.3 SERIES SOLUTIONS FOR $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$

Consider the initial value problem

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{A}(t)\mathbf{X}, \\ \mathbf{X}(t_0) &= \mathbf{X}_0. \end{aligned}$$

Assume that $\mathbf{A}(t)$ is analytic at $t = t_0$.

According to a result which we do not prove, *a unique solution* $\mathbf{X}(t)$, analytic at t_0, exists for the ini… value problem (6.4), i.e. $\mathbf{X}(t)$ has series expansion

$$\mathbf{X}(t) = \sum_{k=0}^{\infty} \mathbf{X}_k (t - t_0)^k .$$

If we want to find a solution for (6.4) in the form of the expansion (6.5), we have to determine formul… for $\mathbf{X}_k$. Since $\mathbf{A}(t)$ and $\mathbf{X}(t_0) = \mathbf{X}_0$ are known, we therefore determine the relation between the const… vectors $\mathbf{X}_k$, and $\mathbf{A}(t)$ and $\mathbf{X}_0$.

Since

$$\mathbf{X}_k = \frac{\mathbf{X}^{(k)}(t_0)}{k!}, \quad k = 0, 1, 2, \ldots$$

we must differentiate (6.5) $(k-1)$ times, and determine the value of the derivative at $t = t_0$ in each ca… In order to do this, we use Leibniz's Rule, according to which

$$\frac{d^k}{dt^k}\left[\mathbf{A}(t)\mathbf{X}(t)\right] = \sum_{i=0}^{k} \binom{k}{i} \mathbf{A}^{(i)}(t)\,\mathbf{X}^{(k-i)}(t)$$

with

$$\binom{k}{i} = \frac{k!}{i!\,(k-i)!}.$$

From (6.6) we have

$$\mathbf{X}_{k+1} = \frac{\mathbf{X}^{(k+1)}(t_0)}{(k+1)!} = \frac{1}{(k+1)!}\left[\frac{d^k}{dt^k}\dot{\mathbf{X}}(t)\right]_{t=t_0}$$

$$= \frac{1}{(k+1)!}\left[\frac{d^k}{dt^k}(\mathbf{A}(t)\mathbf{X}(t))\right]_{t=t_0}$$

$$= \frac{1}{(k+1)!}\sum_{i=0}^{k}\binom{k}{i}\mathbf{A}^{(i)}(t_0)\,\mathbf{X}^{(k-i)}(t_0)$$

from (6.7).

Since

$$\mathbf{A}(t) = \sum_{k=0}^{\infty}\frac{\mathbf{A}^{(k)}(t_0)}{k!}(t-t_0)^k$$

from Taylor's Theorem, we can, by putting

$$\mathbf{A}_k = \frac{\mathbf{A}^{(k)}(t_0)}{k!},$$

and using (6.6), write

$$\mathbf{X}_{k+1} = \frac{k!}{(k+1)!}\sum_{i=0}^{k}\mathbf{A}_i\mathbf{X}_{k-i},$$

i.e.

$$\mathbf{X}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k}\mathbf{A}_i\mathbf{X}_{k-i} \tag{6.8}$$

for $k = 0, 1, \ldots$ and $\mathbf{X}_0 = \mathbf{X}(t_0)$.

Since a unique solution of (6.4), analytic at t_0, exists, as is proved in Theorem 7.1, we have the following theorem:

Theorem 6.4 *Suppose that $\mathbf{A}(t)$ is analytic at t_0 with a power series expansion*

$$\mathbf{A}(t) = \sum_{k=0}^{\infty}\mathbf{A}_k(t-t_0)^k, \quad |t-t_0| < r.$$

Let $\mathbf{X}_0$ be an arbitrary vector. Then

$$\mathbf{X}(t) = \sum_{k=0}^{\infty}\mathbf{X}_k(t-t_0)^k, \quad |t-t_0| < r$$

is a solution of (6.4), analytic at t_0, with

$$\mathbf{X}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k}\mathbf{A}_i\mathbf{X}_{k-i},$$

i.e.

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{A}_0\mathbf{X}_0 \\
\mathbf{X}_2 &= \frac{1}{2}\{\mathbf{A}_0\mathbf{X}_1 + \mathbf{A}_1\mathbf{X}_0\} \\
&\vdots \\
\mathbf{X}_{k+1} &= \frac{1}{k+1}\{\mathbf{A}_0\mathbf{X}_k + \mathbf{A}_1\mathbf{X}_{k-1} + \ldots + \mathbf{A}_k\mathbf{X}_0\}
\end{aligned} \tag{6.9}$$

Example 6.2

Determine a series solution of

$$\dot{\mathbf{X}} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{X} \equiv \mathbf{A}\mathbf{X}.$$

Compare your answer with the solution obtained by the eigenvalue–eigenvector method.

Solution:

Since all entries are constants, the matrix $\mathbf{A}$ is analytic at $t = 0$. The sum

$$\sum_{k=0}^{\infty} \mathbf{A}_k (t - t_0)^k$$

can yield the constant matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

only if $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ are all the zero matrix. Hence, $\mathbf{A}_0 = \mathbf{A}$. From (6.9) we have

$$\begin{aligned}
\mathbf{X}_n &= \frac{1}{n}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{X}_{n-1} \\
&= \frac{1}{n(n-1)}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}^2\mathbf{X}_{n-2} \\
&= \ldots \\
&= \frac{1}{n!}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}^n\mathbf{X}_0.
\end{aligned}$$

It can be proved by induction that

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 0 & 2^{n-1} \\ 0 & 2^n \end{bmatrix}.$$

By putting

$$\mathbf{X}_0 = \begin{bmatrix} a \\ b \end{bmatrix},$$

we have

$$\mathbf{X}_n = \frac{1}{n!}\begin{bmatrix} 0 & 2^{n-1} \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{1}{n!}\begin{bmatrix} 2^{n-1}b \\ 2^n b \end{bmatrix}$$

$$= \frac{2^n}{n!}\begin{bmatrix} \frac{b}{2} \\ b \end{bmatrix}, \quad n \geq 1.$$

Therefore

$$\mathbf{X}(t) = \begin{bmatrix} a \\ b \end{bmatrix} + \sum_{n=1}^{\infty}\begin{bmatrix} \frac{b}{2} \\ b \end{bmatrix}\frac{2^n t^n}{n!}$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \frac{b}{2} \\ b \end{bmatrix}\sum_{n=1}^{\infty}\frac{(2t)^n}{n!}$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \frac{b}{2} \\ b \end{bmatrix}\left(e^{2t} - 1\right)$$

$$= \begin{bmatrix} a - \frac{b}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{b}{2} \\ b \end{bmatrix} e^{2t}.$$

(Recall that $e^{2t} - 1 = \sum_{n=1}^{\infty} \frac{(2t)^n}{n!}$.)

By putting $a - \frac{b}{2} = c_1$ and $\frac{b}{2} = c_2$, we obtain

$$\mathbf{X}(t) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}.$$

This solution is the same as that obtained by the method of eigenvalues and eigenvectors.

Example 6.3

Find a series solution for the second order initial value problem

$$\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Solution:

Let $\dot{x} = y$ so that the second order initial value problem becomes a system of two first order equations

$$\dot{x} = y$$
$$\dot{y} = -x$$

with $x(0) = 1$, $y(0) = 0$.

By putting $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ the problem can be rewritten as

$$\dot{\mathbf{X}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since all entries are constants, the matrix $\mathbf{A}$ is analytic at $t = 0$. Thus the sum

$$\sum_{k=0}^{\infty} \mathbf{A}_k (t - t_0)^k$$

can yield the constant matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

if and only if $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ are all the zero matrix. Hence, $\mathbf{A}_0 = \mathbf{A}$ so that

$$\begin{aligned} \mathbf{X}_n &= \frac{1}{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{X}_{n-1} \\ &= \frac{1}{n!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^n \mathbf{X}_0 . \end{aligned} \tag{1}$$

Now

$$\mathbf{X}(t) = \sum_{k=0}^{\infty} \mathbf{X}_k t^k$$

so that, from (1),

$$\begin{aligned} \mathbf{X}(t) &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k \mathbf{X}_0 \right) t^k \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{2k} \mathbf{X}_0 t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{2k+1} \mathbf{X}_0 t^{2k+1}. \end{aligned}$$

But for $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we have (check this for a few values of k)

$$\begin{aligned} \mathbf{A}^{2k} &= (-1)^k \mathbf{I} = \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \\ \mathbf{A}^{2k+1} &= (-1)^k \mathbf{A} = \begin{bmatrix} 0 & (-1)^k \\ (-1)^{k+1} & 0 \end{bmatrix} \end{aligned}$$

with $\mathbf{I}$ the identity matrix. This gives

$$\begin{aligned} \mathbf{X}(t) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{bmatrix} 0 & (-1)^k \\ (-1)^{k+1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{2k+1}. \end{aligned} \tag{2}$$

Remark:
If we used the eigenvalue–eigenvector method to solve this problem, we would have found the exact solution

$$x(t) = \cos t, \quad y(t) = -\sin t.$$

This result can easily be obtained from equation (2) by making use of standard series expansions of $\sin t$ and $\cos t$ (see page 196 of the textbook); we find

$$\mathbf{X}(t) = \begin{bmatrix} \cos t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}.$$

Exercise 6.2

1. Study Exercise 6.3.2 and 6.3.3, pp. 204–206 of the prescribed book.
2. Exercise 6.3.I of the prescribed book.
3. Show that

$$\mathbf{X}(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \mathbf{A}^k \mathbf{X}_0$$

   is the solution of

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \quad \mathbf{X}(t_0) = \mathbf{X}_0.$$

Remark:
We have restricted ourselves to series solutions about "ordinary" points, i.e. points where $\mathbf{A}(t)$ is analytic. Series solutions about certain types of singular points are also possible, but for the purpose of this course, we shall not deal with them. This subject is treated in Section 6.6 of the prescribed book, whilst the equation of Legendre, to which we have referred earlier in this chapter, is treated in Section 6.4.

6.4 THE EXPONENTIAL OF A MATRIX

Our object is this section is to generalize the exponential function e^{ta}, where a is a constant. We wish to attach meaning to the symbol $e^{t\mathbf{A}}$ when $\mathbf{A}$ is a constant matrix, say, an $n \times n$ matrix. That this has significance in the process of a study of the system $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, and the initial value problem $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, $\mathbf{X}(t_0) = \mathbf{X}_0$, can be "suspected" from the fact that $e^{ta}y$ is the (unique) solution of $\dot{x} = ax$, $x(0) = y$.

In view of our previous work, it is possible to define in either of the following two ways:

Definition 6.3 (1) By keeping in mind that

$$\mathbf{X}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k \mathbf{X}_0$$

is the unique solution of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, $\mathbf{X}(t_0) = \mathbf{X}_0$, it makes sense to define

$$e^{t\mathbf{A}}\mathbf{U} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k \mathbf{U} \tag{6.10}$$

with $\mathbf{U}$ a constant n–dimensional vector. With this definition

$$e^{t\mathbf{A}}\mathbf{X}_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k \mathbf{X}_0 = \mathbf{X}(t)$$

so that the "function" $e^{t\mathbf{A}}$ corresponds to e^{ta}, the solution of $\dot{x} = ax$.

(2) By recalling that the solution $\mathbf{X}(t)$ of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, $\mathbf{X}(t_0) = \mathbf{X}_0$, may also be expressed in terms of the normalised fundamental matrix $\mathbf{\Phi}(t)$ of $\mathbf{A}$ at $t = 0$, viz as $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{X}_0$, we can also define

$$e^{t\mathbf{A}} = \mathbf{\Phi}(t). \tag{6.11}$$

Definition (6.11) is interpreted as

$$e^{t\mathbf{A}}\mathbf{U} = \mathbf{\Phi}(t)\mathbf{U}$$

for every vector with n entries.

By using the latter definition, and the properties of a normalised fundamental matrices, it is easy to show that $e^{t\mathbf{A}}$ has all the properties of the exponential function.

Exercise 6.3
Show that

$$e^{(t+s)\mathbf{A}} = e^{t\mathbf{A}}e^{s\mathbf{A}} \text{ for } s, t > 0,$$

$$e^{t\mathbf{A}} = \mathbf{I} \text{ for } t = 0.$$

Remark:

The generalization of the exponential function has been extended to the case where $\mathbf{A}$, which may be dependent on time, is a bounded or unbounded operator with domain and range in a Banach space. A system $\{\mathbf{T}_t : t > 0\} = \{e^{t\mathbf{A}} : t > 0\}$ of operators is then known as a *semigroup* of bounded operators with *infinitesimal generator* $\mathbf{A}$. The concept of semigroups of bounded operators[2] is of cardinal importance in the study of abstract differential equations and has wide application to the field of the qualitative theory of partial differential equations. A good knowledge of ordinary differential equations, as well as a course in Functional Analysis and Distribution Theory will make this exciting field accessible to you.

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[2] The concept of semigroup was named in 1904 and after 1930 the theory of semigroups developed rapidly. One of the most au[illegible]s on this subject is *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Publ. Vol. 31, Providence R[illegible], 19[illegible] E. H[illegible]e and R.S. Phillips. A recent addition to the literature is *A Concise Guide to Semigroups and Evolution Equations* by Aldo Belleni–Morante, World Scientific, Singapore, 1994.

Chapter 7

Nonlinear Systems
Existence and Uniqueness Theorem for Linear Systems

Objectives for this Chapter

The main objective of this chapter is to gain an understanding of the following concepts regarding nonlinear systems:

- autonomous and non–autonomous systems;
- conditions for existence and uniqueness for solutions of differential equations.

Outcomes of this Chapter

After studying this chapter the learner should be able to:

- express a non–autonomous system in autonomous form;
- determine under which conditions the initial value problem $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}, \quad \mathbf{X}(t_0) = \mathbf{X}_0$ has a unique solution.

7.1 NONLINEAR EQUATIONS AND SYSTEMS

We showed in Chapter 1 that the equation

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X} + \mathbf{G}(t) \tag{7.1}$$

with $\mathbf{A}$ an $n \times n$ matrix $[f_{ij}]$,

$$\mathbf{G}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is equivalent to the linear first order system

$$\begin{aligned}
\dot{x}_1 &= f_{11}(t)\,x_1 + f_{12}x_2 + \ldots + f_{1n}x_n + g_1(t) \\
\dot{x}_2 &= f_{21}(t)\,x_1 + f_{22}x_2 + \ldots + f_{2n}x_n + g_2(t) \\
&\vdots \\
\dot{x}_n &= f_{n1}(t)\,x_1 + f_{n2}x_n + \ldots + f_{nn}x_n + g_n(t).
\end{aligned}$$

If for some or other i, x_i cannot be expressed as a linear combination of the components of $\mathbf{X}$ (the coefficients of these components may be either functions of t or else constants) we have a nonlinear equation

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n) \equiv f_i(t, \mathbf{X}).$$

Systems containing nonlinear equations, i.e. nonlinear systems, are written in vector form as

$$\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X}). \tag{7.2}$$

(The matrix notation is obviously not possible.)

From the above it is clear that the system

$$\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X})$$

is linear iff $\mathbf{F}(t, \mathbf{X}) = \mathbf{A}(t)\,\mathbf{X} + \mathbf{G}(t)$, for some other matrix $\mathbf{A}(t)$ and a vector function $\mathbf{G}(t)$.

A special case of nonlinear systems are the *autonomous* systems.

Definition 7.1 The system $\dot{\mathbf{X}} = \mathbf{F}(t)\,\mathbf{X}$ is said to be *autonomous* if $\mathbf{F}$ is independent of t.

An autonomous system is, therefore, of the form $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$. Autonomous systems have certain properties which are not generally valid.

Example 7.1

(i) The system $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with $\mathbf{A}$ a matrix with constant entries, is autonomous.

(ii) The system

$$\begin{aligned}
\dot{x}_1 &= x_1 x_2 \\
\dot{x}_2 &= t x_1 + x_2
\end{aligned}$$

is a non–autonomous system.

It is always possible to express a non–autonomous system in autonomous form by introducing a spurious variable.

Example 7.2

Consider the non–autonomous equation $\dot{x} = t^2 x - e^t$.

Put $x_1 = t$ and $x_2 = x$. Then $\dot{x}_1 = 1$ and $\dot{x}_2 = \dot{x} = t^2x - e^t$, i.e. $\dot{x}_2 = x_1^2 x_2 - e^{x_1}$. This yields the system

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1^2 x_2 - e^{x_1} \end{bmatrix},$$

the right-hand side of which is independent of t — therefore an autonomous system.

More generally we consider the non-autonomous system

$$\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X}) \text{ with } \mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

Put

$$\mathbf{Y} = \begin{bmatrix} t \\ x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

Then

$$\dot{\mathbf{Y}} = \begin{bmatrix} 1 \\ \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 1 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} = \mathbf{G}(\mathbf{Y})$$

and the system $\dot{\mathbf{Y}} = \mathbf{G}(\mathbf{Y})$ is autonomous.

Exercise 7.1

(1) Study Example 7.1.1, p. 229 of the prescribed book.

(2) Exercise 7.1, p. 229 of the prescribed book.

It is seldom possible to solve nonlinear equations explicitly[1]. Certain principles, applicable to linear systems are also not even valid in the case of nonlinear equations, as for instance the principle of uniqueness of solutions and the principle of superposition of solutions. The following counterexamples illustrate the truth of these statements.

Example 7.3

The initial value problem $\dot{y} = \frac{3}{2} y^{\frac{1}{3}}, \ y(0) = 0$ has an infinite number of solutions. Check it yourself. (See p. 231 of the prescribed book.)

[1]One can, however, obtain important information on certain characteristics of the solution of a nonlinear differential equation without actually solving the equation. One method is the method of phase portraits in the phase space. Nonlinear first-order autonomous systems of differential equations also are studied by investigating the points of equilibrium of the so-called linearized system. This subject is dealt with in Chapter 8.

Example 7.4

The principle of superposition of solutions is not valid for the equation $\dot{y} = \frac{t}{2}y^{-1}$.

Differentiation confirms that $y(t) = \left(\frac{t^2}{2} + c\right)^{\frac{1}{2}}$ satisfies, for all c, the equation $2yy' = t$, which is equivalent to $y' = \frac{t}{2y}$. However,

is in no case a solution, in view of

$$\left(\frac{t^2}{2} + c_0\right)^{\frac{1}{2}} + \left(\frac{t^2}{2} + c_1\right)^{\frac{1}{2}}$$

$$\begin{aligned}
&2\left\{\sqrt{\frac{t^2}{2} + c_0} + \sqrt{\frac{t^2}{2} + c_1}\right\}\left\{\frac{t}{2\sqrt{\frac{t^2}{2} + c_0}} + \frac{t}{2\sqrt{\frac{t^2}{2} + c_1}}\right\} \\
&= t + t + t\left\{\sqrt{\frac{\frac{t^2}{2} + c_0}{\frac{t^2}{2} + c_1}} + \sqrt{\frac{\frac{t^2}{2} + c_1}{\frac{t^2}{2} + c_0}}\right\} \\
&\neq t \text{ for all } c_0, c_1.
\end{aligned}$$

It is, indeed, true that the validity of the principle of superposition would imply linearity — thus leading to a contradiction. We prove this:

Suppose that $\mathbf{F}(t, \mathbf{X})$ is a function such that every initial value problem $\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X})$ has a solution. Suppose further that the principle of superposition is valid, i.e. if $\mathbf{X}_i(t)$ is a solution of $\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X})$ for $i = 1, 2$, then $\mathbf{Z} = a\mathbf{X}_1 + b\mathbf{X}_2$ is a solution of

$$\dot{\mathbf{Z}} = \mathbf{F}(t, \mathbf{Z}), \ \mathbf{Z}(t_0) = a\mathbf{X}_1^0 + b\mathbf{X}_2^0,$$

where we use the notation $\mathbf{X}_i^0 \equiv \mathbf{X}_i(t_0)$.

We show that $\mathbf{F}(t, \mathbf{Z})$ is a linear function of $\mathbf{Z}$, i.e.

$$\mathbf{F}(t, a\mathbf{X}_1 + b\mathbf{X}_2) = a\mathbf{F}(t, \mathbf{X}_1) + b\mathbf{F}(t, \mathbf{X}_2).$$

From our assumption we have

$$\begin{aligned}
\mathbf{F}(t, \mathbf{X}_1) &= \dot{\mathbf{X}}_1(t), \\
\mathbf{F}(t, \mathbf{X}_2) &= \dot{\mathbf{X}}_2(t),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}(t, a\mathbf{X}_1 + b\mathbf{X}_2) &= \frac{d}{dt}\left(a\mathbf{X}_1(t) + b\mathbf{X}_2(t)\right) \\
&= a\dot{\mathbf{X}}_1(t) + b\dot{\mathbf{X}}_2(t) \\
&= a\mathbf{F}(t, \mathbf{X}_1) + b\mathbf{F}(t, \mathbf{X}_2).
\end{aligned}$$

Exercise 7.2

Study Example 7.3.3, p. 233 of the prescribed book, from which it appears that the term "general solution" has to be used very carefully in the case of nonlinear equations.

As far as the solution of nonlinear equations is concerned, we note that the Method of Separation of Variables can, in the case of simple one-dimensional equations, often be applied successfully.

Example 7.5

Determine the solution of

$$\dot{y} = -\frac{\sqrt{1-t^2}}{\sqrt{5+y}} \quad -1 \leq t \leq 1, \; y > -5. \tag{7.3}$$

Solution:

By separation of variables, equation (7.3) is equivalent to

$$\sqrt{1-t^2}\,dt + \sqrt{5+y}\,dy = 0.$$

By integration we obtain that the solution $y(t)$ is given implicitly by

$$\frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t + \frac{2}{3}(5+y)^{\frac{3}{2}} = c, \quad -1 \leq t \leq 1. \tag{7.4}$$

Note that in order to obtain a unique solution, we consider only principal values of arcsin, i.e. those values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, otherwise (7.4) would define a multi-valued function.

Exercise 7.3

Exercise 7.3, p. 233 of the prescribed book.

7.2 NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

If a differential equation cannot be solved by means of a standard technique, a *numerical solution* can still be obtained by means of techniques of approximation. By a numerical solution of a differential equation in x, a function of t, is meant a table of values such that for every value of t, a corresponding value of $x(t)$ is given. Such a table always has a column giving the magnitude of the error involved. The theory of numerical solutions covers a wide field and is outside the scope of this course. Should the reader be interested, an easily readable chapter on numerical solutions is found in *Ordinary Differential Equations* by M. Tenenbaum and H. Pollard (see references) as well as *Differential Equations with Boundary Value Problems* by Dennis G. Zill and M.R. Cullen.

It is interesting to note that a computer can yield a "solution" for a differential equation even when theoretically no solution exists. An example is the boundary value problem

$$\ddot{f}(x) + f(x) = \sin x, \quad 0 < x < \pi, \quad f(0) = f(\pi) = 0.$$

For this reason it is necessary to pay particular attention to those conditions under which a unique solution of a system of differential equations exists. We shall confine ourselves to the case of linear systems of differential equations.

7.3 EXISTENCE AND UNIQUENESS THEOREM FOR LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

By putting $\mathbf{A}(t)\mathbf{X} = \mathbf{F}(t, \mathbf{X})$, the problem

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X},$$
$$\mathbf{X}(t_0) = \mathbf{X}_0,$$

reduces to the problem

$$\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X}),$$
$$\mathbf{X}(t_0) = \mathbf{X}_0.$$

The theorem will, therefore, be proved by applying the Existence and Uniqueness Theorem for the differential equation

$$\dot{x} = f(t, x),\ x(t_0) = x_0.$$

As in the prescribed book the theorem reads:

Theorem 7.1 *If f is continuous in an interval $|t - t_0| \leq T$, $||\mathbf{X} - \mathbf{X}_0|| \leq R$ and f satisfies a Lipschitz condition, i.e. a constant $K > 0$ exists such that*

$$||f(t, \mathbf{U}) - f(t, \mathbf{V})|| \leq K\,||\mathbf{U} - \mathbf{V}|| \tag{7.5}$$

in $|t - t_0| \leq T$ and $\mathbf{U}, \mathbf{V}$ in $||\mathbf{X} - \mathbf{X}_0|| \leq R$, then the initial value problem

$$\dot{\mathbf{X}} = f(t, \mathbf{X})$$
$$\mathbf{X}(t_0) = \mathbf{X}_0$$

has one and only one solution in $|t - t_0| < \delta$.

Remark:

(1) In the above theorem the constant δ is defined as $\delta = \min(T, R/M)$, with the constant M the upperbound of $||\mathbf{F}||$. (See note later on.)

(2) If the range of $\mathbf{F}$ is the Banach Space E^n, i.e. the n-dimensional Euclidian space with the Euclidian metric (recall that a Banach space is a normed space which is complete), then $||\mathbf{F}(t, \mathbf{X})||$ is precisely $|\mathbf{F}(t, \mathbf{X})|$.

(3) If $\mathbf{A}(t)$ is an $n \times n$ matrix with real entries, and $\mathbf{X}$ an n-tuple of real numbers, then $\mathbf{A}(t)\mathbf{X}$ is an n-dimensional vector with components real numbers, so that $\mathbf{A}(t)\mathbf{X}$ is an element of the Banach space R^n. As in Chapter 4, Section 5, we define

$$||\mathbf{A}(t)|| = \left(\sum_{i,j=1}^{n} (f_{ij}(t))^2\right)^{\frac{1}{2}}. \tag{7.6}$$

Since the role of $\mathbf{F}(t, \mathbf{X})$ is played by $\mathbf{A}(t)\mathbf{X}$, we must have $\mathbf{A}(t)\mathbf{X}$ continuous for all t in $|t - t_0| \le T$ and every $\mathbf{X}$ in $||\mathbf{X} - \mathbf{X}_0|| \le R$, with R an arbitrary constant.

It is sufficient to require that $\mathbf{A}(t)$ should be continuous in $|t - t_0| \le T$, i.e. that $f_{ij}(t)$ should be continuous for $i = 1, \ldots, n$, $j = 1, \ldots, n$. The vector $\mathbf{X}$ is, indeed, continuous as a function of $\mathbf{X}$ for every "interval" $||\mathbf{X} - \mathbf{X}_0|| \le R$. Since the product of continuous functions is continuous, it follows that $\mathbf{A}(t)\mathbf{X}$ is continuous in $|t - t_0| \le T$ and $||\mathbf{X} - \mathbf{X}_0|| \le R$.

For the proof of Theorem 7.1 we need

Lemma 7.1 *If*

$$\mathbf{A}(t) = [f_{ij}(t)], \quad \mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

then we have the inequality

$$||\mathbf{A}(t)\mathbf{X}|| \le ||\mathbf{A}(t)|| \, ||\mathbf{X}|| . \tag{7.7}$$

Proof: By putting

$$\mathbf{A}(t) = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & & \\ f_{n1} & \cdots & f_{nn} \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1 \\ \vdots \\ \mathrm{f}_n \end{bmatrix},$$

i.e. $\mathrm{f}_i = (f_{i1}, f_{i2}, \ldots, f_{in})$, $i = 1, \ldots, n$, we have

$$\mathbf{AX} = \begin{bmatrix} \mathrm{f}_1 \cdot \mathbf{X} \\ \mathrm{f}_2 \cdot \mathbf{X} \\ \vdots \\ \mathrm{f}_n \cdot \mathbf{X} \end{bmatrix}$$

where $\cdot$ indicates the ordinary scalar product. Then, by definition

$$\begin{aligned} ||\mathbf{A}(t)\mathbf{X}||^2 &= (\mathrm{f}_1 \cdot \mathbf{X})^2 + \ldots + (\mathrm{f}_n \cdot \mathbf{X})^2 \\ &\le ||\mathrm{f}_1||^2 \, ||\mathbf{X}||^2 + \ldots + ||\mathrm{f}_n|| \, ||\mathbf{X}||^2 \quad \text{(Schwarz' Inequality)} \\ &= \left(||\mathrm{f}_1||^2 + ||\mathrm{f}_2||^2 + \ldots + ||\mathrm{f}_n||^2\right) ||\mathbf{X}||^2 . \end{aligned}$$

Since

$$||\mathrm{f}_i||^2 = f_{i1}^2 + f_{i2}^2 + \ldots + f_{in}^2 = \sum_{j=1}^{n} (f_{ij})^2 ,$$

we have

$$\begin{aligned}||\mathbf{f}_1||^2 + \ldots + ||\mathbf{f}_n||^2 &= \sum_{j=1}^{n} (f_{1j}(t))^2 + \sum_{j=1}^{n} (f_{2j}(t))^2 + \ldots + \sum_{j=1}^{n} (f_{nj}(t))^2 \\ &= \sum_{i,j=1}^{n} (f_{ij}(t))^2 \\ &= ||\mathbf{A}(t)||^2 .\end{aligned}$$

It follows that

$$||\mathbf{A}(t)\,\mathbf{X}|| \leq (||\mathbf{A}(t)||\ ||\mathbf{X}||) .$$

■

In proving Theorem 7.1, we finally make use of the fact that if a function f is continuous in $|t - t_0| \leq T$, $||\mathbf{X} - \mathbf{X}_0|| \leq R$, then a constant M, independent of t and $\mathbf{X}$, exists, such that $||f|| \leq M$. Indeed, continuity and boundedness are equivalent concepts in normed spaces. For the proof, the reader may refer to, amongst others, "Mathematical Analysis" by Apostol, p. 83[2].

Our main result now reads:

Theorem 7.2 *If* $\mathbf{A}(t)$ *is continuous in* $|t - t_0| \leq T$, *the initial value problem*

$$\begin{aligned}\dot{\mathbf{X}} &= \mathbf{A}(t)\,\mathbf{X} \\ \mathbf{X}(t_0) &= \mathbf{X}_0\end{aligned}$$

has a unique solution in $|t - t_0| < \delta$.

Proof: From the assumption that $\mathbf{A}(t)$ is continuous in $|t - t_0| \leq T$ we have, according to our previous remarks, that $\mathbf{A}(t)\,\mathbf{X}$ is continuous in $|t - t_0| \leq T$ and $||\mathbf{X} - \mathbf{X}_0|| \leq R$ for every R. We also have, from a previous remark, that $||\mathbf{A}(t)| \leq M|$ in $|t - t_0| \leq T$.

Therefore

$$\begin{aligned}||\mathbf{A}(t)\,\mathbf{U} - \mathbf{A}(t)\,\mathbf{V}|| &= ||\mathbf{A}(t)\,(\mathbf{U} - \mathbf{V})|| \\ &\leq ||\mathbf{A}(t)||\ ||\mathbf{U} - \mathbf{V}|| \\ &\leq M\,||\mathbf{U} - \mathbf{V}|| .\end{aligned}$$

Consequently $\mathbf{A}(t)\,\mathbf{X} = \mathbf{F}(t, \mathbf{X})$ satisfies the Lipschitz condition (7.5) with $K = M$. All the conditions of Theorem 7.1 are now satisfied. We can conclude that the problem

$$\begin{aligned}\dot{\mathbf{X}} &= \mathbf{A}(t)\,\mathbf{X} \\ \mathbf{X}(t_0) &= \mathbf{X}_0\end{aligned}$$

has a unique solution in $|t - t_0| < \delta$. This completes the proof of the theorem.

■

[2] [illegible]alysis, T.M. Apostol, Addison-Wesley Publishing Co., London.

Chapter 8

Qualitative Theory of Differential Equations
Stability of Solutions of Linear Systems
Linearization of Nonlinear Systems

Objectives for this Chapter

The main objective of this chapter is to gain an understanding of the following concepts regarding the stability of solutions of linear systems and the linearization of nonlinear systems:

- autonomous systems;
- critical point;
- periodic solutions;
- classification of a critical point: stable/unstable node, saddle, center, stable/unstable spiral point, degenerate node;
- stability of critical point;
- linearization and local stability;
- Jacobian matrix;
- phase–plane method.

Outcomes of this Chapter

After studying this chapter the learner should be able to:

- find the critical points of plane autonomous systems;
- solve certain nonlinear systems by changing to polar coordinates;
- apply the stability criteria to determine whether a critical point is locally stable or unstable;
- classify the critical points using the Jacobian matrix.

The final chapter of this module is devoted to the qualitative theory of differential equations. In this branch of theory of differential equations, techniques are developed which will enable us to obtain important information about the solutions of differential equations without actually solving them. We will, for instance, be able to decide whether a solution is stable, or what the long time behaviour of the solution is, without even knowing the form of the solution. This is extremely useful in view of the fact that it is often very difficult, or even impossible, to determine an exact solution of a differential equation.

As already mentioned in the preface, this chapter is taken from the book *Differential Equations with Boundary-value Problems* by Dennis G. Zill and Michael R. Cullen, PWS–Kent Publishing Company, Boston (1993), where it appears as Chapter 10.

CHAPTER 10

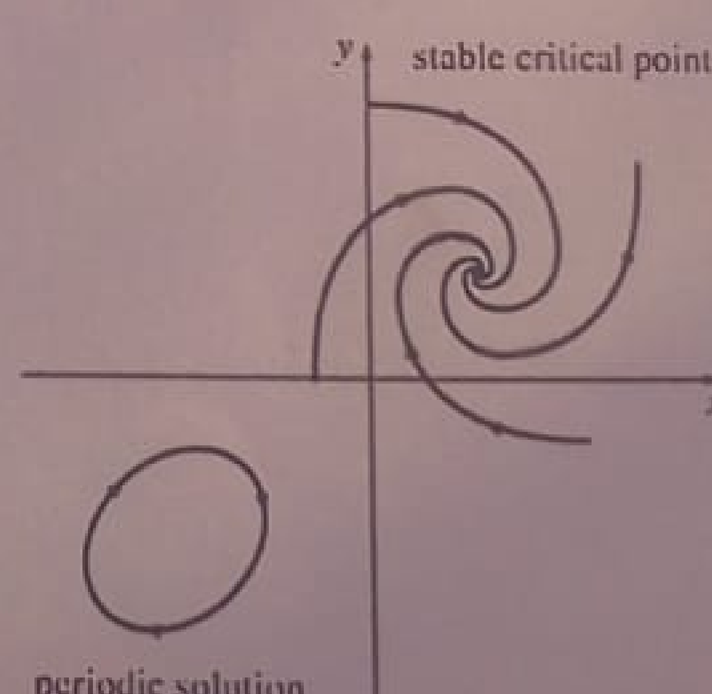
PLANE AUTONOMOUS SYSTEMS AND STABILITY

10.1

Important Concepts

Autonomous systems
Critical point
Arc
Periodic solution or cycle
Stable node
Unstable node
Saddle point
Degenerate node
Center
Spiral points
Stable critical point
Unstable critical point
Linearization
Jacobian matrix
Phase-plane method

The principal focus in Chapter 8 was on techniques for solving systems of linear first-order differential equations of the form $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$. When the system of differential equations is not linear, it is usually not possible to find solutions in terms of elementary functions. In this chapter we demonstrate that valuable information on the geometric nature of



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solutions can be obtained by first analyzing special constant solutions called critical points and by searching for periodic solutions. These special solutions are further classified as stable or unstable according to the behavior of nearby solutions. This important concept of stability will be introduced and illustrated with examples from physics and ecology

10.1 AUTONOMOUS SYSTEMS, CRITICAL POINTS, AND PERIODIC SOLUTIONS

Terminology and Notation

A system of first-order differential equations is called **autonomous** when the system can be written in the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(x_1, x_2, \ldots, x_n) \\
\vdots \quad & \qquad \vdots \\
\frac{dx_n}{dt} &= g_n(x_1, x_2, \ldots, x_n).
\end{aligned} \tag{1}$$

Thus the independent variable t does *not* appear explicitly on the right-hand side of each differential equation (compare with (2) in Section 8.3).

EXAMPLE 1

The system of differential equations

$$\frac{dx_1}{dt} = x_1 - 3x_2 + t^2$$

$$\frac{dx_2}{dt} = x_1 \sin(x_2 t)$$

is *not* autonomous because of the presence of t^2 and $\sin(x_2 t)$ on the right-hand sides. ■

If $\mathbf{X}(t)$ and $\mathbf{g}(\mathbf{X})$ denote the respective column vectors

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{g}(\mathbf{X}) = \begin{pmatrix} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_n(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

then the autonomous system (1) can be written in the compact **column vector form** $\mathbf{X}' = \mathbf{g}(\mathbf{X})$. The homogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ studied in Sections 8.5 and 8.6 is an important special case.

In this chapter it is also convenient to write (1) using row vectors. If we let

$$\mathbf{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

and $\quad \mathbf{g}(\mathbf{X}) = (g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots, g_n(x_1, x_2, \ldots, x_n)),$

then the autonomous system (1) can be written in the compact **row vector form** $\mathbf{X}' = \mathbf{g}(\mathbf{X})$. It should be clear from the context whether we are using column or row vector form, and therefore we will not distinguish between $\mathbf{X}$ and $\mathbf{X}^T$, the transpose of $\mathbf{X}$. In particular, when $n = 2$, it is convenient to use row vector form and write an initial condition as $\mathbf{X}(0) = (x_0, y_0)$.

When the variable t is interpreted as time, we can refer to a solution $\mathbf{X}(t)$ as the state of the system at time t. Using this terminology, we say a system of differential equations is autonomous when the rate $\mathbf{X}'(t)$ at which the system changes depends on only the system's present state $\mathbf{X}(t)$. The linear system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ studied in Chapter 8 is then autonomous when $\mathbf{F}(t)$ is constant.

Note that when $n = 1$, an autonomous differential equation takes the simple form $dx/dt = g(x)$. Explicit solutions can be constructed since this differential equation is separable, and we will make use of this fact to give illustrations of the concepts in this chapter.

Vector Field Interpretation

When $n = 2$, the system is called a **plane autonomous system**, and we write the system as

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y).$$

The vector $\mathbf{V}(x, y) = (P(x, y), Q(x, y))$ defines a vector field in a region of the plane, and a solution to the system may be interpreted as the resulting path of a particle as it moves through the region. To be more specific, let $\mathbf{V}(x, y) = (P(x, y), Q(x, y))$ denote the velocity of a stream at position (x, y), and suppose that a small particle (such as a cork) is released at a position

(x_0, y_0) in the stream. If $\mathbf{X}(t) = (x(t), y(t))$ denotes the position of the particle at time t, then $\mathbf{X}'(t) = (x'(t), y'(t))$ is the velocity vector $\mathbf{v}$. When external forces are not present and frictional forces are neglected, the velocity of the particle at time t is the velocity of the stream at position $\mathbf{X}(t)$:

$$\mathbf{X}'(t) = \mathbf{V}(x(t), y(t));$$

that is,

$$\frac{dx}{dt} = P(x(t), y(t))$$

$$\frac{dy}{dt} = Q(x(t), y(t)).$$

Thus the path of the particle is the solution to a system that satisfies the initial condition $\mathbf{X}(0) = (x_0, y_0)$. We will frequently call on this simple interpretation of a plane autonomous system to illustrate new concepts.

EXAMPLE 2

A vector field for the steady-state flow of a fluid around a cylinder of radius 1 is given by

$$\mathbf{V}(x, y) = V_0\left(1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}, \frac{-2xy}{(x^2 + y^2)^2}\right),$$

where V_0 is the speed of the fluid far from the cylinder. If a small cork is released at $(-3, 1)$, the path $\mathbf{X}(t) = (x(t), y(t))$ of the cork satisfies the plane autonomous system

$$\frac{dx}{dt} = V_0\left(1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right)$$

$$\frac{dy}{dt} = V_0\left(\frac{-2xy}{(x^2 + y^2)^2}\right)$$

subject to the initial condition $\mathbf{X}(0) = (-3, 1)$. See Figure 10.1.

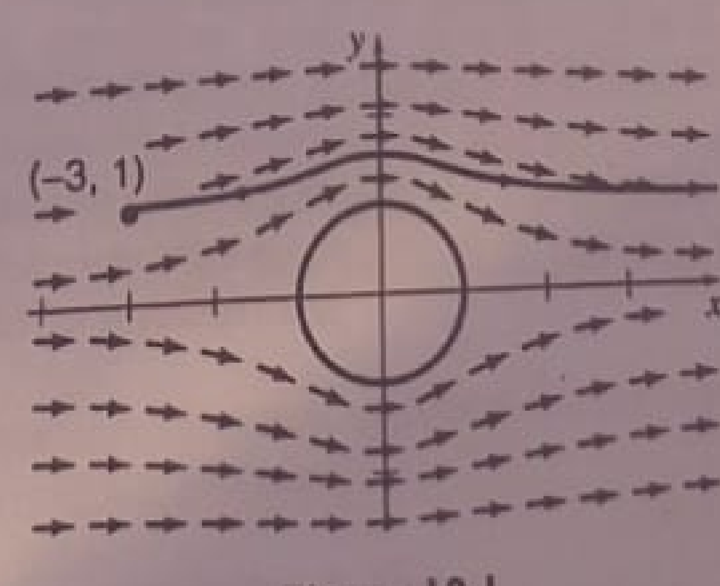

Figure 10.1

Any second-order nonlinear differential equation $x'' = g(x, x')$ can be written as a plane autonomous system. With the introduction of $y = x'$, the equation becomes

$$x' = y$$

$$y' = g(x, y).$$

EXAMPLE 3

In Example 3 in Section 1.2, we showed that the displacement angle θ for a pendulum satisfies the second-order nonlinear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

If we let $x = \theta$ and $y = \theta'$, this second-order differential equation may be rewritten as the plane autonomous system

$$x' = y$$

$$y' = -\frac{g}{l}\sin x.$$

■

Types of Solutions

If $P(x, y)$, $Q(x, y)$, and the first-order partial derivatives $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, and $\partial Q/\partial y$ are continuous in a region R of the plane, then the solutions to the plane autonomous system

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y)$$

are of three basic types:

(i) A **constant solution** $x(t) = x_0$, $y(t) = y_0$ (or $\mathbf{X}(t) = \mathbf{X}_0$ for all t). A constant solution is called a **critical** or **stationary point.** When the particle is placed at a critical point $\mathbf{X}_0$ (that is, $\mathbf{X}(0) = \mathbf{X}_0$), it remains there indefinitely. Note that since $\mathbf{X}'(t) = \mathbf{0}$, a critical point is a solution of the system of algebraic equations

$$P(x, y) = 0$$

$$Q(x, y) = 0.$$

(ii) A solution $x = x(t)$, $y = y(t)$ that defines an arc, a plane curve that does *not* cross itself. Thus the curve in Figure 10.2(a) can be a

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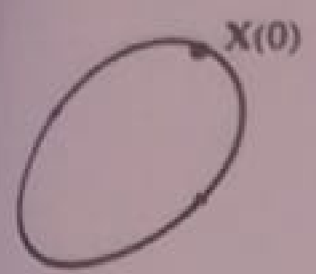

Figure 10.3

solution to a plane autonomous system, whereas the curve in Figure 10.2(b) cannot be a solution.

(iii) A periodic solution $x = x(t)$, $y = y(t)$. A periodic solution is called a cycle. If p is the period of the solution, then $\mathbf{X}(t + p) = \mathbf{X}(t)$ and a particle placed on the curve at $\mathbf{X}_0$ will cycle around the curve and return to $\mathbf{X}_0$ in p units of time (see Figure 10.3).

EXAMPLE 4

Find all critical points of each of the following plane autonomous systems:

(a) $x' = -x + y$
$y' = x - y$

(b) $x' = x^2 + y^2 - 6$
$y' = x^2 - y$

(c) $x' = 0.01x(100 - x - y)$
$y' = 0.05y(60 - y - 0.2x)$

Solution We find the critical points by setting the right-hand sides of the differential equations equal to zero.

(a) The solution to the system

$$-x + y = 0$$

$$x - y = 0$$

consists of all points on the line $y = x$. Thus there are infinitely many critical points.

(b) To solve the system

$$x^2 + y^2 - 6 = 0$$

$$x^2 - y = 0$$

we substitute the second equation $x^2 = y$ into the first equation to obtain $y^2 + y - 6 = (y + 3)(y - 2) = 0$. If $y = -3$, then $x^2 = -3$, and so there are no real solutions. If $y = 2$, then $x = \pm\sqrt{2}$, and so the critical points are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

(c) Finding the critical points in part (c) requires a careful consideration of cases. The equation $0.01x(100 - x - y) = 0$ implies $x = 0$ or $x + y = 100$. If $x = 0$, then, substituting in $0.05y(60 - y - 0.2x) = 0$, we have $y(60 - y) = 0$. Thus $y = 0$ or 60, and so (0, 0) and (0, 60) are critical points. If $x + y = 100$, then $0 = y(60 - y - 0.2(100 - y)) = y(40 - 0.8y)$. It follows that $y = 0$ or 50, and so (100, 0) and (50, 50) are critical points. ■

When the plane autonomous system is linear, we can use the methods in Chapter 8 to investigate solutions.

EXAMPLE 5

Determine whether the given linear system possesses a periodic solution.

(a) $x' = 2x + 8y$
$y' = -x - 2y$

(b) $x' = x + 2y$
$y' = -\frac{1}{2}x + y$

In each case sketch the graph of the solution that satisfies $\mathbf{X}(0) = (2, 0)$.

Solution (a) In Example 3 in Section 8.6, we used the eigenvalue-eigenvector method to show that

$$x = c_1(2\cos 2t - 2\sin 2t) + c_2(2\cos 2t + 2\sin 2t)$$
$$y = c_1(-\cos 2t) - c_2 \sin 2t.$$

Thus every solution is periodic with period $p = \pi$. The solution satisfying $\mathbf{X}(0) = (2, 0)$ is

$$x = 2\cos 2t + 2\sin 2t$$
$$y = -\sin 2t.$$

This solution generates the ellipse shown in Figure 10.4(a).

(b) In Example 4 in Section 8.6, we used the eigenvalue-eigenvector method to show that

$$x = c_1(2e^t \cos t) + c_2(2e^t \sin t)$$
$$y = c_1(-e^t \sin t) + c_2(e^t \cos t).$$

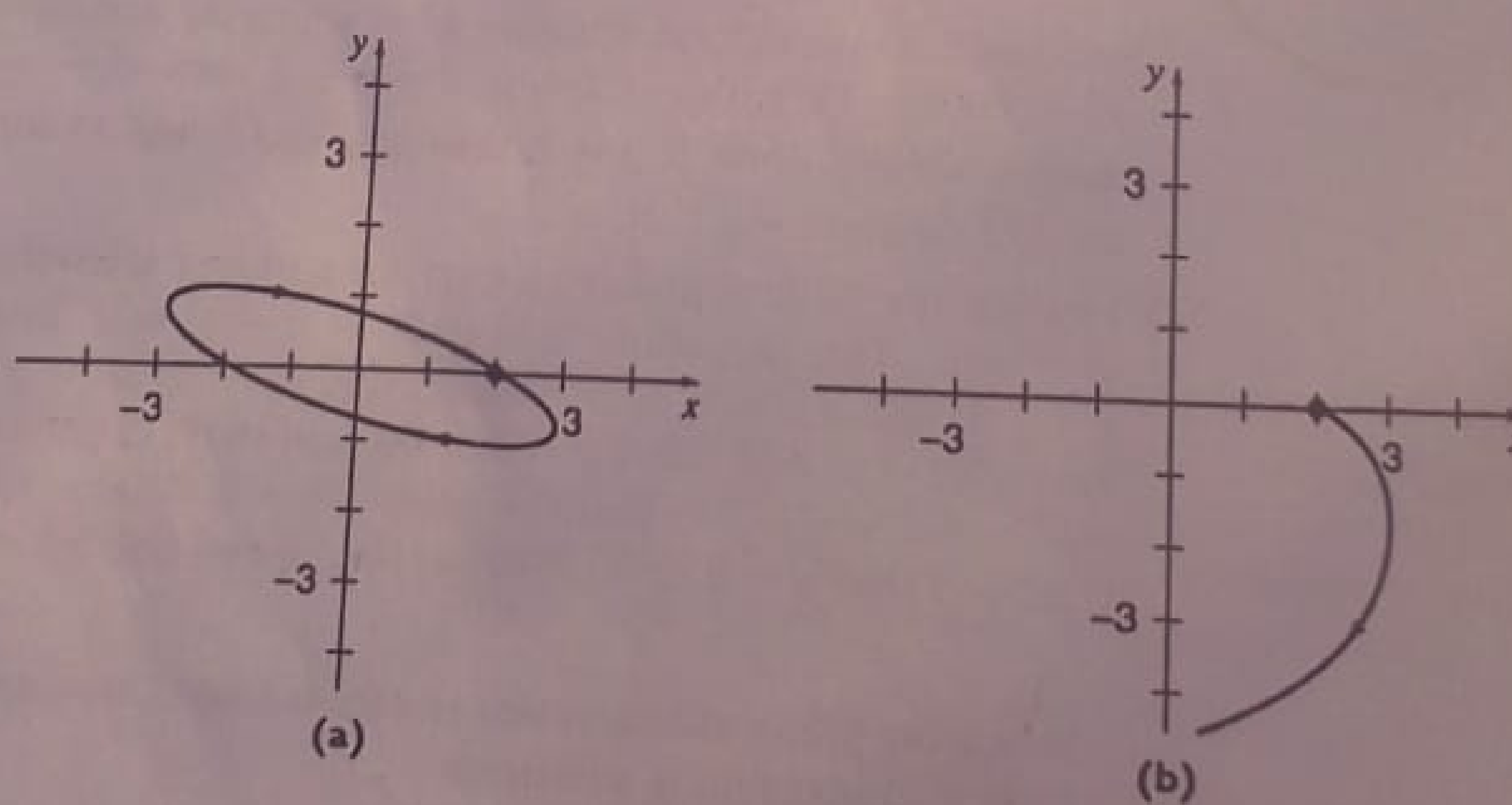

Figure 10.4

Because of the presence of e^t in the general solution, there are no periodic solutions (that is, cycles). The solution satisfying $\mathbf{X}(0) = (2, 0)$ is

$$x = 2e^t \cos t$$
$$y = -e^t \sin t$$

and this curve is shown in Figure 10.4(b). ■

Changing to Polar Coordinates

Except for the case of constant solutions, it is usually not possible to find explicit expressions for the solutions of a nonlinear autonomous system. We can solve some nonlinear systems, however, by changing to polar coordinates. From the formulas $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, we obtain

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$
$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right). \tag{2}$$

EXAMPLE 6

Find the solution of the nonlinear plane autonomous system

$$x' = -y - x\sqrt{x^2 + y^2}$$
$$y' = x - y\sqrt{x^2 + y^2}$$

satisfying the initial condition $\mathbf{X}(0) = (3, 3)$.

Solution From the expressions for dr/dt and $d\theta/dt$ in (2) we obtain

$$\frac{dr}{dt} = \frac{1}{r}[x(-y - xr) + y(x - yr)] = -r^2$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}[-y(-y - xr) + x(x - yr)] = 1$$

with $r(0) = 3\sqrt{2}$ and $\theta(0) = \pi/4$. We can use separation of variables to show that the general solution of the system is

$$r = \frac{1}{t + c_1}$$

$$\theta = t + c_2$$

for $r \neq 0$. (Check this!) Applying the initial conditions then gives $r = 1/(t + \sqrt{2}/6)$ and $\theta = t + \pi/4$. The spiral $r = 1/(\theta + \sqrt{2}/6 - \pi/4)$ is sketched in Figure 10.5.

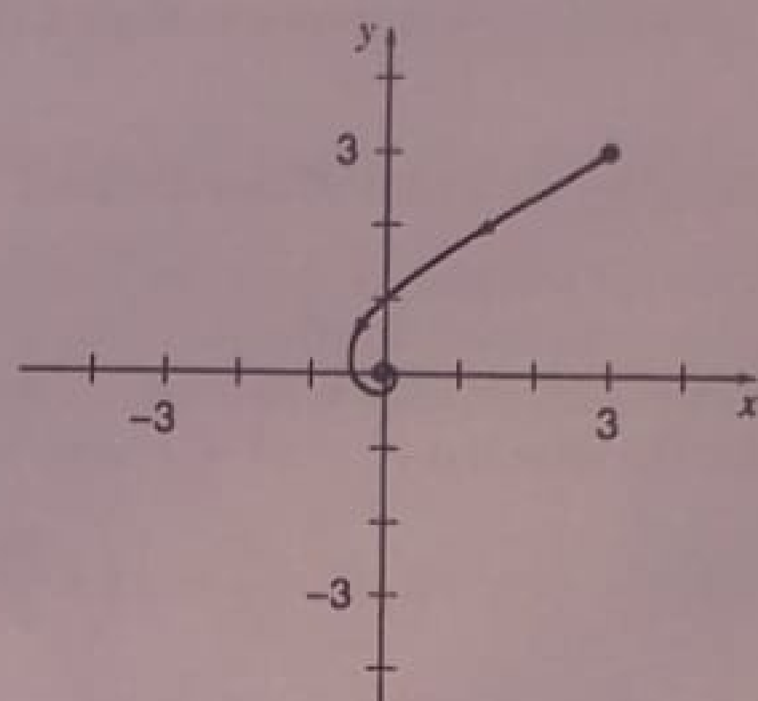

Figure 10.5

EXAMPLE 7

When expressed in polar coordinates, a plane autonomous system takes the form

$$\frac{dr}{dt} = 0.5(3 - r)$$

$$\frac{d\theta}{dt} = 1.$$

Find and sketch the solutions satisfying $\mathbf{X}(0) = (3, 0)$ and $\mathbf{X}(0) = (0, 1)$ in rectangular coordinates.

Solution Applying separation of variables to $dr/dt = 0.5(3 - r)$ and integrating $d\theta/dt = 1$ lead to the general solution

$$r = 3 + c_1 e^{-0.5t}$$

$$\theta = t + c_2.$$

If $\mathbf{X}(0) = (3, 0)$, then $r(0) = 3$ and $\theta(0) = 0$. It follows that $c_1 = c_2 = 0$, and so $r = 3$ and $\theta = t$. Hence $x = r\cos\theta = 3\cos t$ and $y = r\sin\theta = 3\sin t$, and so the solution is periodic.

If $\mathbf{X}(0) = (0, 1)$, then $r(0) = 1$ and $\theta(0) = \pi/2$, and so $c_1 = -2$ and $c_2 = \pi/2$. The solution curve is the spiral $r = 3 - 2e^{-0.5(\theta - \pi/2)}$. Note that as

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$t \to \infty$, θ increases without bound and r approaches 3. Both solutions are shown in Figure 10.6.

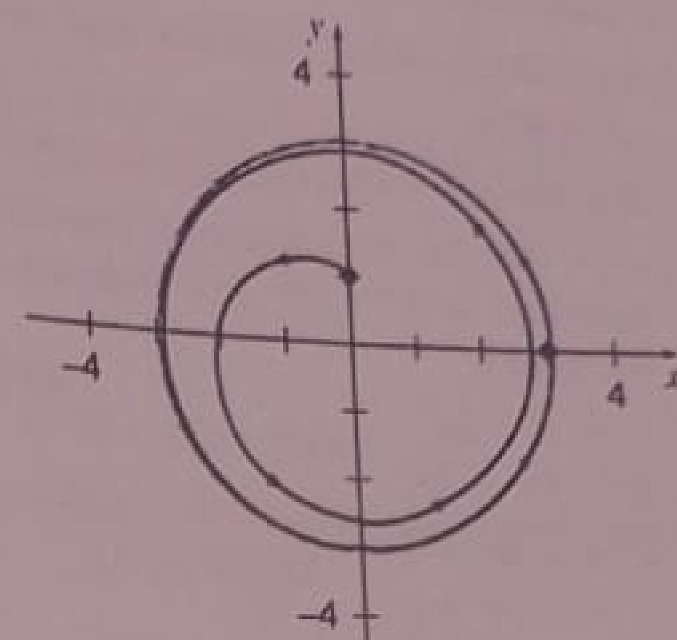

Figure 10.6

10.1 EXERCISES

Answers to odd-numbered problems begin on page A-35.

In Problems 1–6 write the given nonlinear second-order differential equation as a plane autonomous system. Find all critical points of the resulting system.

1. $x'' + 9 \sin x = 0$
2. $x'' + (x')^2 + 2x = 0$
3. $x'' + x'(1 - x^3) - x^2 = 0$
4. $x'' + 4\dfrac{x}{1 + x^2} + 2x' = 0$
5. $x'' + x = \varepsilon x^3$ for $\varepsilon > 0$
6. $x'' + x - \varepsilon x|x| = 0$ for $\varepsilon > 0$

In Problems 7–16 find all critical points of the given plane autonomous system.

7. $x' = x + xy$
   $y' = -y - xy$
8. $x' = y^2 - x$
   $y' = x^2 - y$
9. $x' = 3x^2 - 4y$
   $y' = x - y$
10. $x' = x^3 - y$
    $y' = x - y^3$
11. $x' = x(10 - x - \frac{1}{2}y)$
    $y' = y(16 - y - x)$
12. $x' = -2x + y + 10$
    $y' = 2x - y - 15\dfrac{y}{y + 5}$
13. $x' = x^2 e^y$
    $y' = y(e^x - 1)$
14. $x' = \sin y$
    $y' = e^{x-y} - 1$
15. $x' = x(1 - x^2 - 3y^2)$
    $y' = y(3 - x^2 - 3y^2)$
16. $x' = -x(4 - y^2)$
    $y' = 4y(1 - x^2)$

In Problems 17–22 for the given linear plane autonomous system (taken from Exercises 8.6):

(a) find the general solution and determine whether there are periodic solutions,

(b) find the solution satisfying the given initial condition, and

(c) with the aid of a graphics calculator or graphing software, sketch the solution in part (b) and indicate the direction in which the curve is traversed.

17. $\begin{aligned} x' &= x + 2y \\ y' &= 4x + 3y \end{aligned}$, $\mathbf{X}(0) = (2, -2)$ (Problem 1, Exercises 8.6)

18. $\begin{aligned} x' &= -6x + 2y \\ y' &= -3x + y \end{aligned}$, $\mathbf{X}(0) = (3, 4)$ (Problem 6, Exercises 8.6)

19. $\begin{aligned} x' &= 4x - 5y \\ y' &= 5x - 4y \end{aligned}$, $\mathbf{X}(0) = (4, 5)$ (Problem 19, Exercises 8.6)

20. $\begin{aligned} x' &= x + y \\ y' &= -2x - y \end{aligned}$, $\mathbf{X}(0) = (-2, 2)$ (Problem 16, Exercises 8.6)

21. $\begin{aligned} x' &= 5x + y \\ y' &= -2x + 3y \end{aligned}$, $\mathbf{X}(0) = (-1, 2)$ (Problem 17, Exercises 8.6)

22. $\begin{aligned} x' &= x - 8y \\ y' &= x - 3y \end{aligned}$, $\mathbf{X}(0) = (2, 1)$ (Problem 20, Exercises 8.6)

In Problems 23–26 solve the given nonlinear plane autonomous system by changing to polar coordinates. Describe the geometric behavior of the solution that satisfies the given initial condition(s).

23. $\begin{aligned} x' &= -y - x(x^2 + y^2)^2 \\ y' &= x - y(x^2 + y^2)^2 \end{aligned}$, $\mathbf{X}(0) = (4, 0)$

24. $\begin{aligned} x' &= y + x(x^2 + y^2) \\ y' &= -x + y(x^2 + y^2) \end{aligned}$, $\mathbf{X}(0) = (4, 0)$

25. $\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2) \end{aligned}$, $\mathbf{X}(0) = (1, 0)$ and $\mathbf{X}(0) = (2, 0)$

[*Hint:* The resulting differential equation for r is a Bernoulli differential equation. See Section 2.6.]

26. $$\begin{aligned} x' &= y - \frac{x}{\sqrt{x^2 + y^2}}(4 - x^2 - y^2) \\ y' &= -x - \frac{y}{\sqrt{x^2 + y^2}}(4 - x^2 - y^2) \end{aligned}, \quad \mathbf{X}(0) = (1, 0) \text{ and } \mathbf{X}(0) = (2, 0)$$

[*Hint:* See Example 6, Section 2.2.]

27. If $z = f(x, y)$ is a function with continuous first partial derivatives in a region R, then a flow $\mathbf{V}(x, y) = (P(x, y), Q(x, y))$ in R may be defined by letting $P(x, y) = -(\partial f/\partial y)(x, y)$ and $Q(x, y) = (\partial f/\partial x)(x, y)$. Show that if

in terms of the eigenvalues and eigenvectors of the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To ensure that $X_0 = (0, 0)$ is the only critical point, we assume that the determinant $\Delta = ad - bc \neq 0$. If $\tau = a + d$ is the trace* of matrix A, then the characteristic equation $\det(A - \lambda I) = 0$ may be rewritten as

$$\lambda^2 - \tau\lambda + \Delta = 0.$$

Therefore the eigenvalues of A are $\lambda = (\tau \pm \sqrt{\tau^2 - 4\Delta})/2$, and the usual three cases for these roots occur according to whether $\tau^2 - 4\Delta$ is positive, negative, or zero.

CASE I REAL DISTINCT EIGENVALUES ($\tau^2 - 4\Delta > 0$) According to Theorem 8.9 in Section 8.6 the general solution of (1) is given by

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}, \tag{2}$$

where λ_1 and λ_2 are the eigenvalues, and K_1 and K_2 are the corresponding eigenvectors. Note that $X(t)$ can also be written as

$$X(t) = e^{\lambda_1 t}[c_1 K_1 + c_2 K_2 e^{(\lambda_2 - \lambda_1)t}]. \tag{3}$$

(a) **Both eigenvalues negative ($\tau^2 - 4\Delta > 0$, $\tau < 0$, and $\Delta > 0$)**
Stable Node From (2), it follows that $\lim_{t\to\infty} X(t) = 0$. If we assume that $\lambda_2 < \lambda_1$, then $\lambda_2 - \lambda_1 < 0$, and so we may conclude from (3) that

$$X(t) \approx c_1 K_1 e^{\lambda_1 t}$$

for large values of t. When $c_1 \neq 0$, $X(t)$ approaches 0 from one of the two directions determined by the eigenvector K_1 corresponding to λ_1. If $c_1 = 0$, $X(t) = c_2 K_2 e^{\lambda_2 t}$ and $X(t)$ approaches 0 along the line determined by the eigenvector K_2. Figure 10.8 shows a collection of solution curves around the origin. A critical point is called a **stable node** when both eigenvalues are negative.

(b) **Both eigenvalues positive ($\tau^2 - 4\Delta > 0$, $\tau > 0$, and $\Delta > 0$)**
Unstable Node The analysis for this case is similar to case (a). Again from (2), $X(t)$ becomes unbounded as t increases. Moreover, from (3), $X(t)$ becomes unbounded in one of the directions determined by the eigenvector K_1 (when $c_1 \neq 0$) or along the line determined by the eigenvector K_2 (when $c_1 = 0$). Figure 10.9 shows a typical collection of solution curves. This type of critical point, corresponding to the case when both eigenvalues are positive, is called an **unstable node**.

* In general, if A is an $n \times n$ matrix, then the trace of A is the sum of the main diagonal entries.

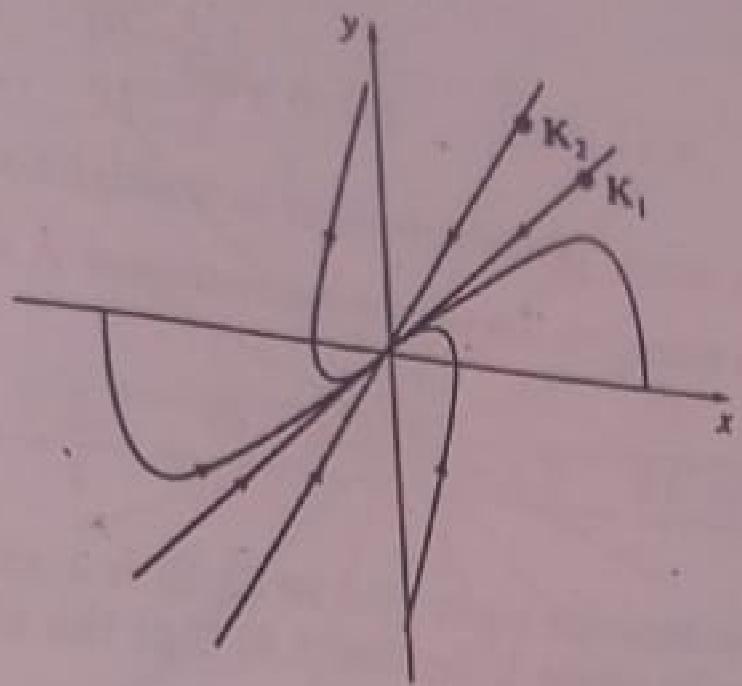

Figure 10.8 Stable Node

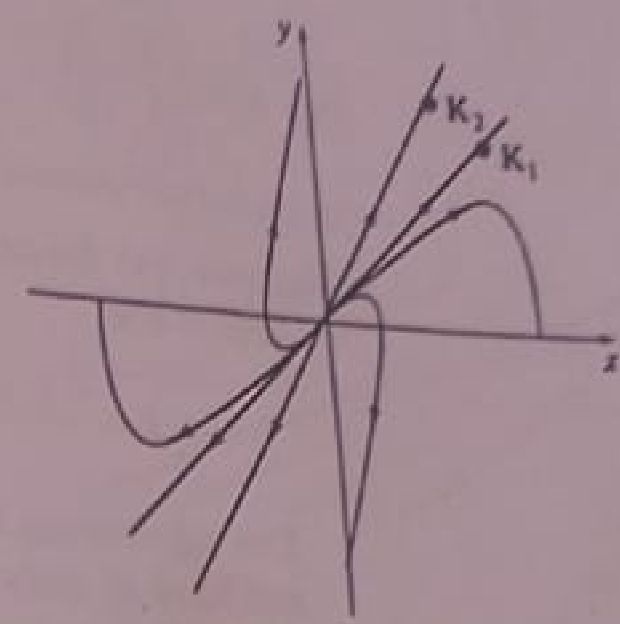

Figure 10.9 Unstable Node

(c) Eigenvalues with opposite signs ($\tau^2 - 4\Delta > 0$ and $\Delta < 0$)
Saddle Point The analysis of the solutions is identical to case (b) with one exception. When $c_1 = 0$, $\mathbf{X}(t) = c_2\mathbf{K}_2e^{\lambda_2 t}$ and, since $\lambda_2 < 0$, $\mathbf{X}(t)$ will approach $\mathbf{0}$ along the line determined by the eigenvector $\mathbf{K}_2$. If $\mathbf{X}(0)$ does not lie on the line determined by $\mathbf{K}_2$, the line determined by $\mathbf{K}_1$ serves as an asymptote for $\mathbf{X}(t)$. This unstable critical point is called a **saddle point**. See Figure 10.10.

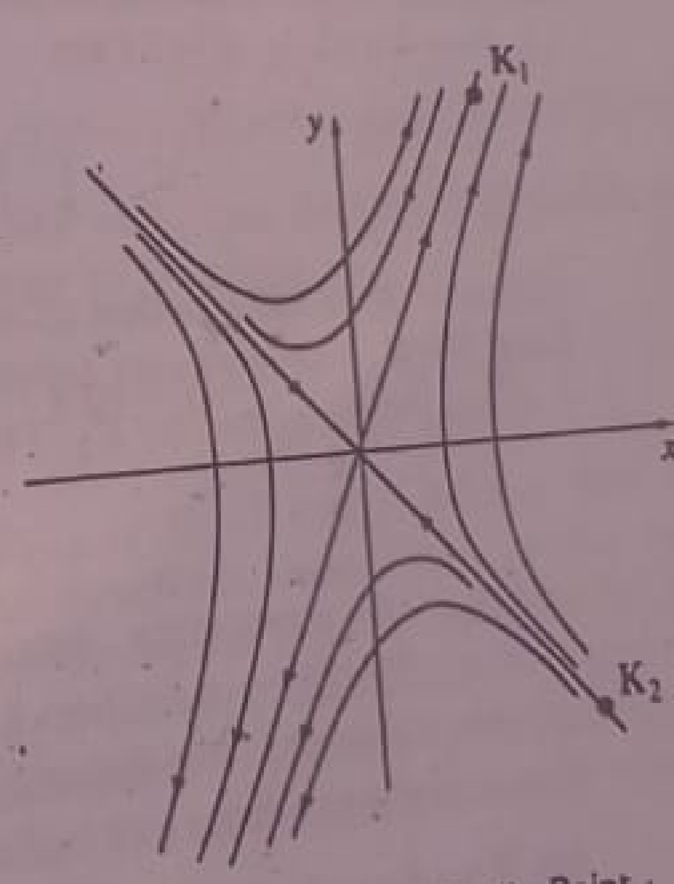

Figure 10.10 Saddle Point

EXAMPLE 1

Classify the critical point (0, 0) of each of the following linear systems $\mathbf{X}' = \mathbf{AX}$ as a stable node, unstable node, or saddle point.

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(a) $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} -10 & 6 \\ 15 & -19 \end{pmatrix}$

In each case discuss the nature of the solutions in a neighborhood of $(0, 0)$.

Solution (a) Since the trace τ is 3 and the determinant $\Delta = -4$, the eigenvalues are

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{3 \pm \sqrt{3^2 - 4(-4)}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

The eigenvalues have opposite signs and so $(0, 0)$ is a saddle point. It is not hard to show (see Example 1, Section 8.6) that the eigenvectors corresponding to $\lambda_1 = 4$ and $\lambda_2 = -1$ are

$$\mathbf{K}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

respectively. If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -x$, $\mathbf{X}(t)$ approaches $\mathbf{0}$. For any other initial condition, $\mathbf{X}(t)$ becomes unbounded in the directions determined by $\mathbf{K}_1$. In other words, the line $y = \frac{2}{3}x$ serves as an asymptote for all these solution curves.

(b) From $\tau = -29$ and $\Delta = 100$, it follows that the eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = -25$. Both eigenvalues are negative and so $(0, 0)$ is in this case a stable node. Since the eigenvectors corresponding to $\lambda_1 = -4$ and $\lambda_2 = -25$ are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix},$$

respectively, it follows that all solutions approach $\mathbf{0}$ from the direction defined by $\mathbf{K}_1$ except those solutions for which $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -\frac{5}{2}x$ determined by $\mathbf{K}_2$. These solutions approach $\mathbf{0}$ along $y = -\frac{5}{2}x$. ■

CASE II A REPEATED REAL EIGENVALUE $(\tau^2 - 4\Delta = 0)$ ***Degenerate Nodes*** Recall from Section 8.6 that the general solution takes on one of two different forms depending on whether one or two linearly independent eigenvectors can be found for the repeated eigenvalue λ_1.

(a) **Two linearly independent eigenvectors.** If $\mathbf{K}_1$ and $\mathbf{K}_2$ are two linearly independent eigenvectors corresponding to λ_1, then the general solution is given by

$$\begin{aligned} \mathbf{X}(t) &= c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_1 t} \\ &= (c_1\mathbf{K}_1 + c_2\mathbf{K}_2)e^{\lambda_1 t}. \end{aligned}$$

If $\lambda_1 < 0$, $\mathbf{X}(t)$ approaches $\mathbf{0}$ along the line determined by the vector $c_1\mathbf{K}_1 + c_2\mathbf{K}_2$, and the critical point is called a degenerate stable

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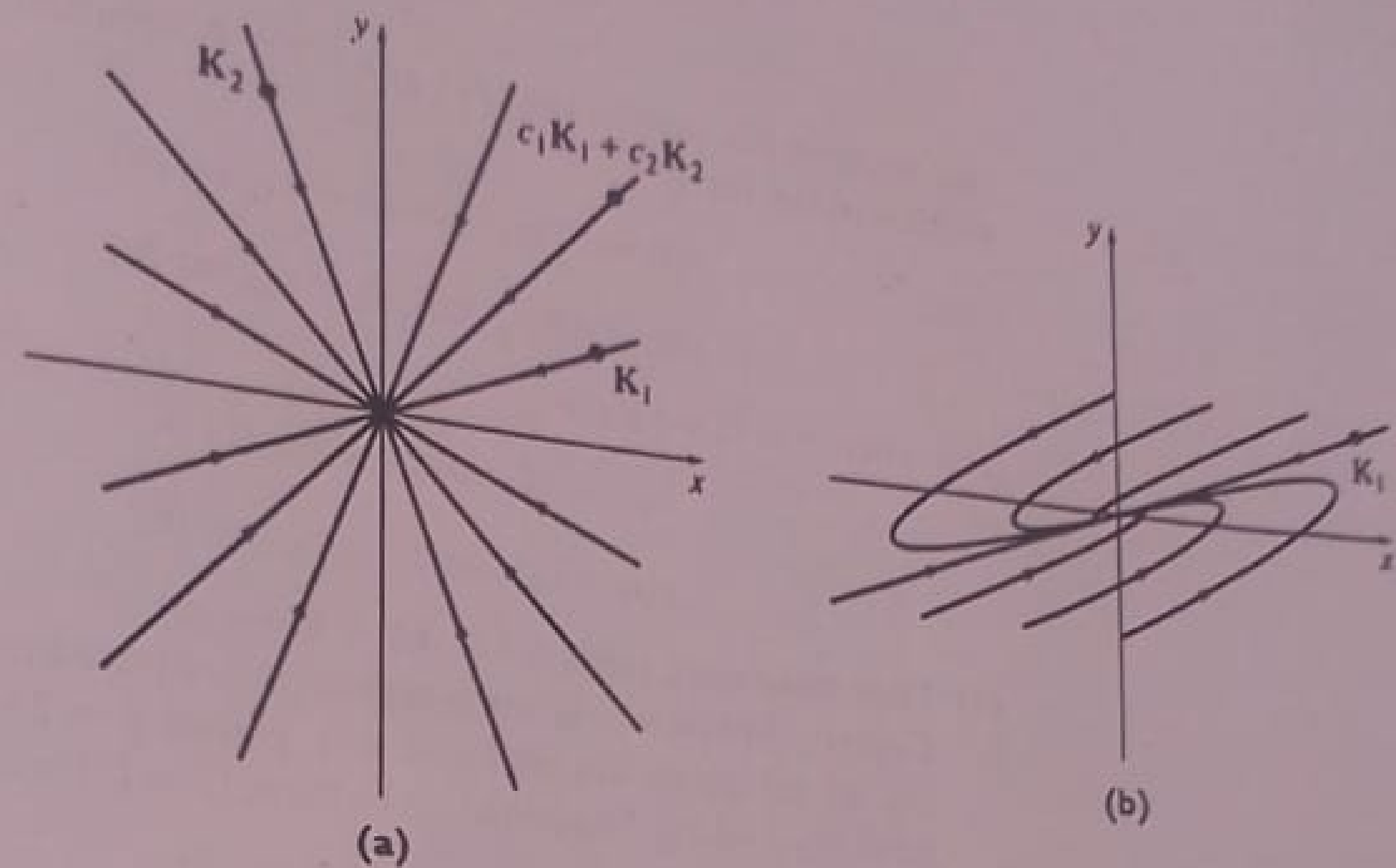

Figure 10.11 Degenerate Stable Nodes

node (see Figure 10.11(a)). The arrows in Figure 10.11(a) are reversed when $\lambda_1 > 0$, and we have a degenerate unstable node.

(b) A single linearly independent eigenvector. When only a single linearly independent eigenvector $\mathbf{K}_1$ exists, the general solution is given by

$$\mathbf{X}(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2(\mathbf{K}_1 t e^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t}),$$

where $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}_1$ (see Section 8.6, equations (18)–(20)), and the solution may be rewritten as

$$\mathbf{X}(t) = te^{\lambda_1 t}\left[c_2\mathbf{K}_1 + \frac{c_1}{t}\mathbf{K}_1 + \frac{c_2}{t}\mathbf{P}\right].$$

If $\lambda_1 < 0$, $\lim_{t\to\infty} te^{\lambda_1 t} = 0$ and it follows that $\mathbf{X}(t)$ approaches $\mathbf{0}$ in one of the directions determined by the vector $\mathbf{K}_1$ (see Figure 10.11(b)). The critical point is again called a degenerate stable node. When $\lambda_1 > 0$, the solutions look like those in Figure 10.11(b) with the arrows reversed. The line determined by $\mathbf{K}_1$ is an asymptote for *all* solutions. The critical point is again called a degenerate unstable node.

CASE III COMPLEX EIGENVALUES $(\tau^2 - 4\Delta < 0)$ If $\lambda_1 = \alpha + i\beta$ and $\bar{\lambda}_1 = \alpha - i\beta$ are the complex eigenvalues and $\mathbf{K}_1 = \mathbf{B}_1 + i\mathbf{B}_2$ is a complex eigenvector corresponding to λ_1, then the general solution can be written as $\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t)$, where

$$\mathbf{X}_1(t) = (\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t)e^{\alpha t}$$

and

$$\mathbf{X}_2(t) = (\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t)e^{\alpha t}$$

(see Section 8.6, equations (14) and (15)). A solution can therefore be written in the form

$$x(t) = e^{\alpha t}(c_{11} \cos \beta t + c_{12} \sin \beta t)$$

$$y(t) = e^{\alpha t}(c_{21} \cos \beta t + c_{22} \sin \beta t) \tag{4}$$

and when $\alpha = 0$, we have

$$x(t) = c_{11} \cos \beta t + c_{12} \sin \beta t$$

$$y(t) = c_{21} \cos \beta t + c_{22} \sin \beta t. \tag{5}$$

(a) Pure imaginary roots ($\tau^2 - 4\Delta < 0, \tau = 0$)

Center When $\alpha = 0$, the eigenvalues are pure imaginary and, from (5), all solutions are periodic with period $p = 2\pi/\beta$. Notice that if both c_{12} and c_{21} happened to be zero, then (5) would reduce to

$$x(t) = c_{11} \cos \beta t$$

$$y(t) = c_{22} \sin \beta t,$$

which is a standard parametric representation for an ellipse. By solving the system of equations in (5) for $\cos \beta t$ and $\sin \beta t$ and using the identity $\sin^2 \beta t + \cos^2 \beta t = 1$, it possible to show that *all solutions are ellipses* with center at the origin. The critical point (0, 0) is called a center, and Figure 10.12 shows a typical collection of solution curves. The ellipses are either *all* traversed in the clockwise direction or all traversed in the counterclockwise direction.

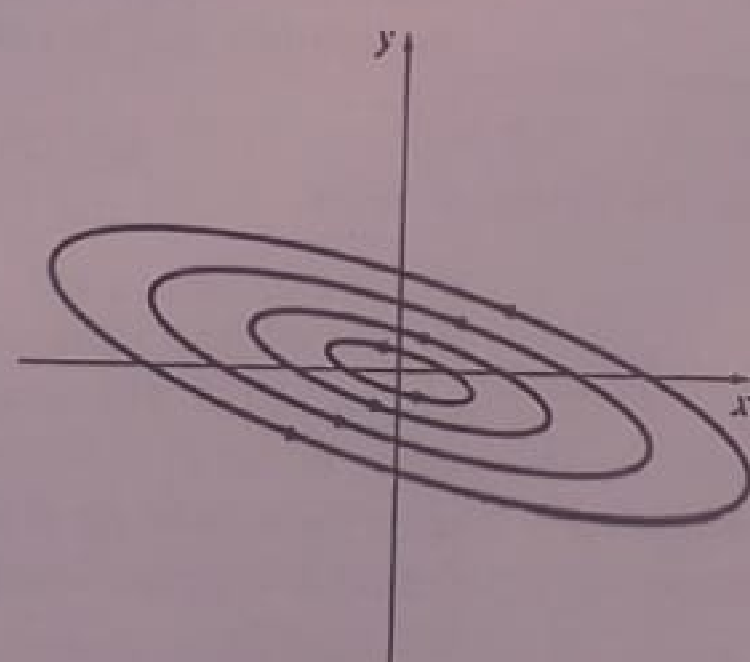

Figure 10.12 Center

(b) Nonzero real part ($\tau^2 - 4\Delta < 0, \tau \neq 0$)

Spiral Points When $\alpha \neq 0$, the effect of the term $e^{\alpha t}$ in (4) is similar to the effect of the exponential term in the analysis of *damped motion* given in Section 5.2. When $\alpha < 0$, $e^{\alpha t} \to 0$, and the elliptical-like solution spirals closer and closer to the origin. The critical point

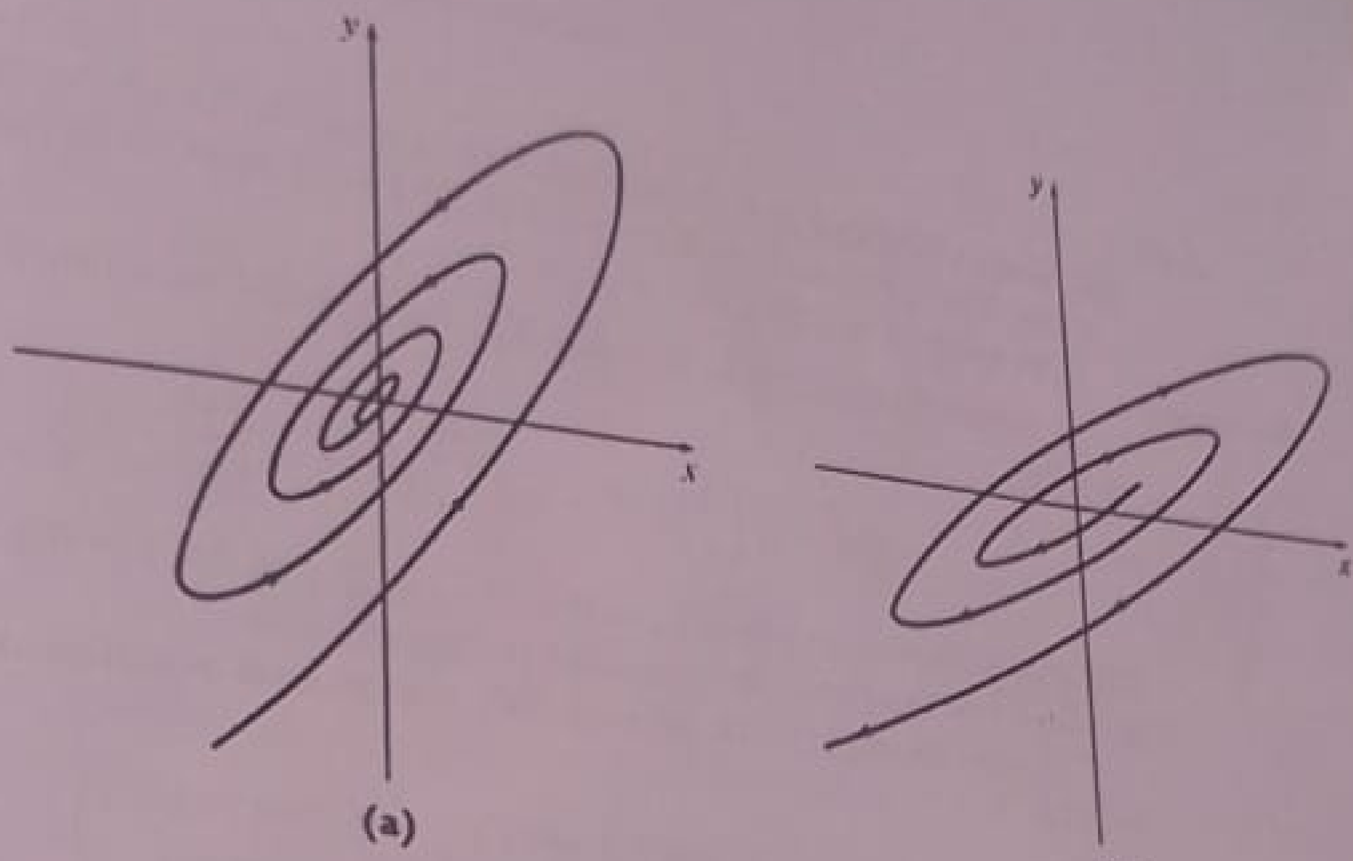

Figure 10.13 Stable and Unstable Spiral Points

is called a **stable spiral point**. When $\alpha > 0$, the effect is the opposite. An elliptical-like solution is driven farther and farther from the origin, and the critical point is called an **unstable spiral point** (see Figure 10.13).

EXAMPLE 2

Classify the critical point $(0, 0)$ of each of the following linear systems $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$

In each case discuss the nature of the solution that satisfies $\mathbf{X}(0) = (1, 0)$. Determine parametric equations for each solution.

Solution (a) Since $\tau = -6$ and $\Delta = 9$, the characteristic polynomial is

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$$

and so $(0, 0)$ is a degenerate stable node. For the repeated eigenvalue $\lambda = -3$, we find a single eigenvector

$$\mathbf{K}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

and so the solution $\mathbf{X}(t)$ that satisfies $\mathbf{X}(0) = (1, 0)$ approaches $(0, 0)$ from the direction specified by the line $y = x/3$.

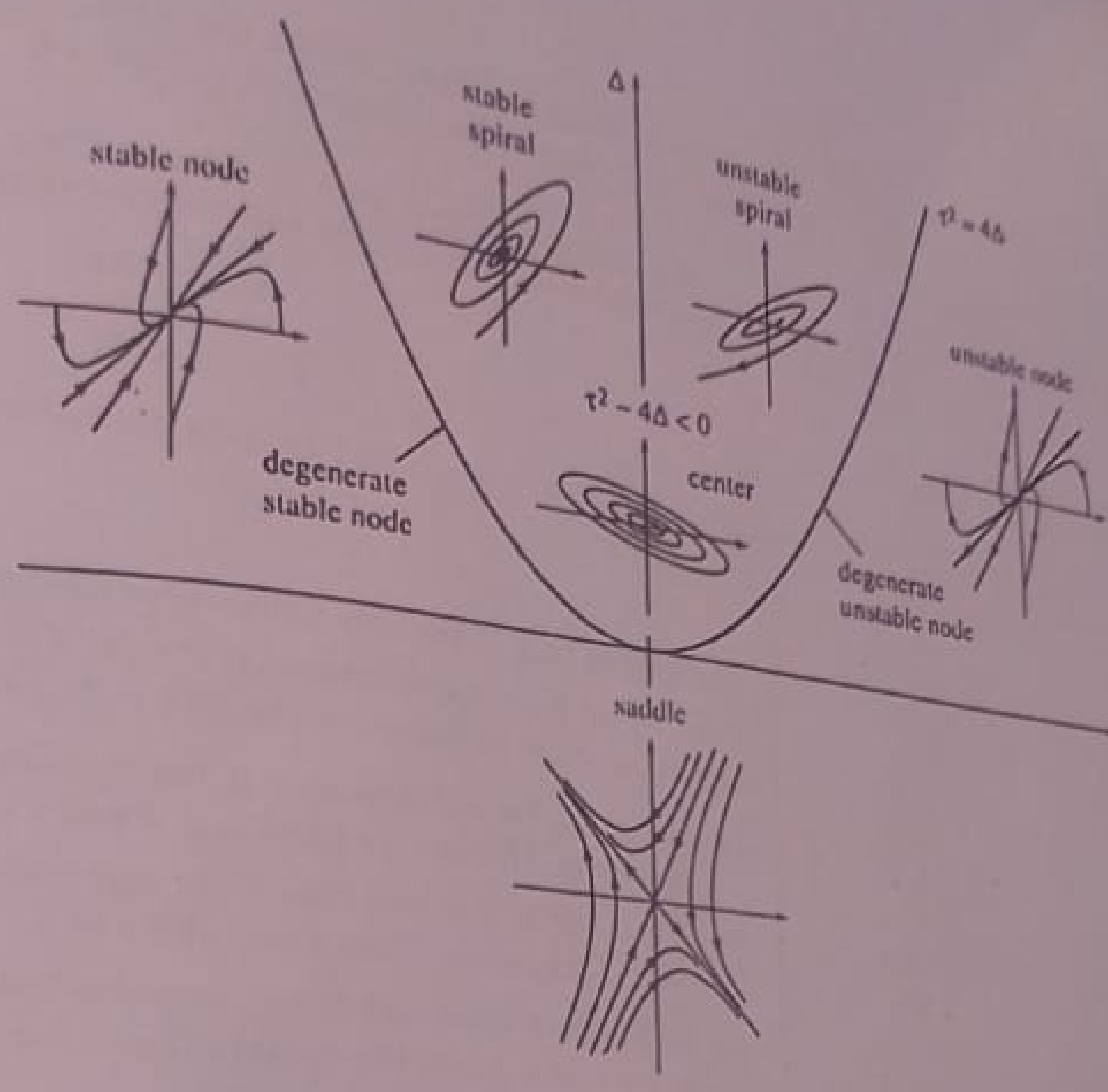

Figure 10.15

easily obtained *not* by constructing explicit eigenvalue-eigenvector solutions but rather by generating the solutions numerically using a method such as the Runge-Kutta method for first-order systems (see Section 9.7).

EXAMPLE 3

Classify the critical point (0, 0) of each of the following linear systems $\mathbf{X}' = \mathbf{A}\mathbf{X}$:

(a) $\mathbf{A} = \begin{pmatrix} 1.01 & 3.10 \\ -1.10 & -1.02 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} -a\hat{x} & -ab\hat{x} \\ -cd\hat{y} & -d\hat{y} \end{pmatrix}$

for positive constants a, b, c, d, $\hat{x}$, and $\hat{y}$.

Solution For the matrix in (a), $\tau = -0.01$, $\Delta = 2.3798$, and so $\tau^2 - 4\Delta < 0$. Using Figure 10.15, we see that (0, 0) is a stable spiral point.

The matrix in (b) arises from the Lotka-Volterra competition model we will study in Section 10.4. Since $\tau = -(a\hat{x} + d\hat{y})$ and all constants in the matrix are positive, $\tau < 0$. The determinant may be written as $\Delta = ad\hat{x}\hat{y}(1 - bc)$. If $bc > 1$, then $\Delta < 0$ and the critical point is a saddle. If $bc < 1$, then $\Delta > 0$ and the critical point is a stable node, a degenerate stable node, or a stable spiral point. In all three of these cases, $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$. ■

We can now answer each of the questions posed at the beginning of Section 10.2 for the linear plane autonomous system

$$x' = ax + by$$
$$y' = cx + dy$$

with $ad - bc \neq 0$. The answers are summarized in Theorem 10.1.

THEOREM 10.1 Stability Criteria for Linear Systems

For a linear plane autonomous system, let $\mathbf{X} = \mathbf{X}(t)$ denote the solution that satisfies the initial condition $\mathbf{X}(0) = \mathbf{X}_0$, where $\mathbf{X}_0 \neq \mathbf{0}$.

(a) $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$ if and only if the eigenvalues of $\mathbf{A}$ have negative real parts. This occurs when $\Delta > 0$ and $\tau < 0$.

(b) $\mathbf{X}(t)$ is periodic if and only if the eigenvalues of $\mathbf{A}$ are pure imaginary. This occurs when $\Delta > 0$ and $\tau = 0$.

(c) In all other cases, given any neighborhood of the origin, there is at least one $\mathbf{X}_0$ in the neighborhood for which $\mathbf{X}(t)$ becomes unbounded as t increases.

EXERCISES

Answers to odd-numbered problems begin on page A-36.

In Problems 1–8 the general solution of the linear system $\mathbf{X}' = \mathbf{AX}$ is given.

(a) In each case discuss the nature of the solutions in a neighborhood of (0, 0).

(b) With the aid of a graphics calculator or graphing software, sketch the solution that satisfies the initial condition $\mathbf{X}(0) = (1, 1)$.

1. $\mathbf{A} = \begin{pmatrix} -2 & -2 \\ -2 & -5 \end{pmatrix}$, $\mathbf{X}(t) = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-6t}$

2. $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} -4 \\ 6 \end{pmatrix}e^{2t}$

3. $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{X}(t) = e^{t}\left[c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}\right]$

4. $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$, $\mathbf{X}(t) = e^{-t}\left[c_1\begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} + c_2\begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}\right]$

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5. $A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$, $X(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}e^{-t}\right]$

6. $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$, $X(t) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}\right]$

7. $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $X(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}$

8. $A = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}$, $X(t) = c_1\begin{pmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{pmatrix} + c_2\begin{pmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{pmatrix}$

In Problems 9–16 classify the critical point (0, 0) of the given linear system by computing the trace τ and determinant Δ and using Figure 10.15.

9. $x' = -5x + 3y$
$y' = 2x + 7y$

10. $x' = -5x + 3y$
$y' = 2x - 7y$

11. $x' = -5x + 3y$
$y' = -2x + 5y$

12. $x' = -5x + 3y$
$y' = -7x + 4y$

13. $x' = -\frac{3}{2}x + \frac{1}{4}y$
$y' = -x - \frac{1}{2}y$

14. $x' = \frac{3}{2}x + \frac{1}{4}y$
$y' = -x + \frac{1}{2}y$

15. $x' = 0.02x - 0.11y$
$y' = 0.10x - 0.05y$

16. $x' = 0.03x + 0.01y$
$y' = -0.01x + 0.05y$

17. Determine conditions on the real constant μ so that (0, 0) is a center for the linear system

$$x' = -\mu x + y$$
$$y' = -x + \mu y.$$

18. Determine a condition on the real constant μ so that (0, 0) is a stable spiral point of the linear system

$$x' = y$$
$$y' = -x + \mu y.$$

19. Show that (0, 0) is always an unstable critical point of the linear system

$$x' = \mu x + y$$
$$y' = -x + y,$$

where μ is a real constant and $\mu \neq 1$. When is (0, 0) an unstable saddle point? When is (0, 0) an unstable spiral point?

20. Let $X = X(t)$ be the solution of the linear system

$$x' = \alpha x - \beta y$$
$$y' = \beta x + \alpha y$$