

Undergraduate Texts in Mathematics

Undergraduate Texts in Mathematics

Series Editors:

Sheldon Axler

San Francisco State University, San Francisco, CA, USA

Kenneth Ribet

University of California, Berkeley, CA, USA

Advisory Board:

Colin Adams, *Williams College, Williamstown, MA, USA*

Alejandro Adem, *University of British Columbia, Vancouver, BC, Canada*

Ruth Charney, *Brandeis University, Waltham, MA, USA*

Irene M. Gamba, *The University of Texas at Austin, Austin, TX, USA*

Roger E. Howe, *Yale University, New Haven, CT, USA*

David Jerison, *Massachusetts Institute of Technology, Cambridge, MA, USA*

Jeffrey C. Lagarias, *University of Michigan, Ann Arbor, MI, USA*

Jill Pipher, *Brown University, Providence, RI, USA*

Fadil Santosa, *University of Minnesota, Minneapolis, MN, USA*

Amie Wilkinson, *University of Chicago, Chicago, IL, USA*

Undergraduate Texts in Mathematics are generally aimed at third- and fourth-year undergraduate mathematics students at North American universities. These texts strive to provide students and teachers with new perspectives and novel approaches. The books include motivation that guides the reader to an appreciation of interrelations among different aspects of the subject. They feature examples that illustrate key concepts as well as exercises that strengthen understanding.

For further volumes:

<http://www.springer.com/series/666>

William A. Adkins • Mark G. Davidson

Ordinary Differential Equations

William A. Adkins
Department of Mathematics
Louisiana State University
Baton Rouge, LA
USA

Mark G. Davidson
Department of Mathematics
Louisiana State University
Baton Rouge, LA
USA

ISSN 0172-6056

ISBN 978-1-4614-3617-1

ISBN 978-1-4614-3618-8 (eBook)

DOI 10.1007/978-1-4614-3618-8

Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012937994

Mathematics Subject Classification (2010): 34-01

© Springer Science+Business Media New York 2012

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This text is intended for the introductory three- or four-hour one-semester sophomore level differential equations course traditionally taken by students majoring in science or engineering. The prerequisite is the standard course in elementary calculus.

Engineering students frequently take a course on and use the Laplace transform as an essential tool in their studies. In most differential equations texts, the Laplace transform is presented, usually toward the end of the text, as an alternative method for the solution of constant coefficient linear differential equations, with particular emphasis on discontinuous or impulsive forcing functions. Because of its placement at the end of the course, this important concept is not as fully assimilated as one might hope for continued applications in the engineering curriculum. *Thus, a goal of the present text is to present the Laplace transform early in the text, and use it as a tool for motivating and developing much of the remaining differential equation concepts for which it is particularly well suited.*

There are several rewards for investing in an early development of the Laplace transform. The standard solution methods for constant coefficient linear differential equations are immediate and simplified. We are able to provide a proof of the existence and uniqueness theorems which are not usually given in introductory texts. The solution method for constant coefficient linear systems is streamlined, and we avoid having to introduce the notion of a defective or nondefective matrix or develop generalized eigenvectors. Even the Cayley–Hamilton theorem, used in Sect. 9.6, is a simple consequence of the Laplace transform. In short, the Laplace transform is an effective tool with surprisingly diverse applications.

Mathematicians are well aware of the importance of transform methods to simplify mathematical problems. For example, the Fourier transform is extremely important and has extensive use in more advanced mathematics courses. The wavelet transform has received much attention from both engineers and mathematicians recently. It has been applied to problems in signal analysis, storage and transmission of data, and data compression. We believe that students should be introduced to transform methods early on in their studies and to that end, the Laplace transform is particularly well suited for a sophomore level course in differential

equations. It has been our experience that by introducing the Laplace transform near the beginning of the text, students become proficient in its use and comfortable with this important concept, while at the same time learning the standard topics in differential equations.

Chapter 1 is a conventional introductory chapter that includes solution techniques for the most commonly used first order differential equations, namely, separable and linear equations, and some substitutions that reduce other equations to one of these. There are also the Picard approximation algorithm and a description, without proof, of an existence and uniqueness theorem for first order equations.

Chapter 2 starts immediately with the introduction of the Laplace transform as an integral operator that turns a differential equation in t into an algebraic equation in another variable s . A few basic calculations then allow one to start solving some differential equations of order greater than one. The rest of this chapter develops the necessary theory to be able to efficiently use the Laplace transform. Some proofs, such as the injectivity of the Laplace transform, are delegated to an appendix. Sections 2.6 and 2.7 introduce the basic function spaces that are used to describe the solution spaces of constant coefficient linear homogeneous differential equations.

With the Laplace transform in hand, Chap. 3 efficiently develops the basic theory for constant coefficient linear differential equations of order 2. For example, the homogeneous equation $q(D)y = 0$ has the solution space \mathcal{E}_q that has already been described in Sect. 2.6. The Laplace transform immediately gives a very easy procedure for finding the test function when teaching the method of undetermined coefficients. Thus, it is unnecessary to develop a rule-based procedure or the annihilator method that is common in many texts.

Chapter 4 extends the basic theory developed in Chap. 3 to higher order equations. All of the basic concepts and procedures naturally extend. If desired, one can simultaneously introduce the higher order equations as Chap. 3 is developed or very briefly mention the differences following Chap. 3.

Chapter 5 introduces some of the theory for second order linear differential equations that are not constant coefficient. Reduction of order and variation of parameters are topics that are included here, while Sect. 5.4 uses the Laplace transform to transform certain second order nonconstant coefficient linear differential equations into first order linear differential equations that can then be solved by the techniques described in Chap. 1.

We have broken up the main theory of the Laplace transform into two parts for simplicity. Thus, the material in Chap. 2 only uses continuous input functions, while in Chap. 6 we return to develop the theory of the Laplace transform for discontinuous functions, most notably, the step functions and functions with jump discontinuities that can be expressed in terms of step functions in a natural way. The Dirac delta function and differential equations that use the delta function are also developed here. The Laplace transform works very well as a tool for solving such differential equations. Sections 6.6–6.8 are a rather extensive treatment of periodic functions, their Laplace transform theory, and constant coefficient linear differential equations with periodic input function. These sections make for a good supplemental project for a motivated student.

Chapter 7 is an introduction to power series methods for linear differential equations. As a nice application of the Frobenius method, explicit Laplace inversion formulas involving rational functions with denominators that are powers of an irreducible quadratic are derived.

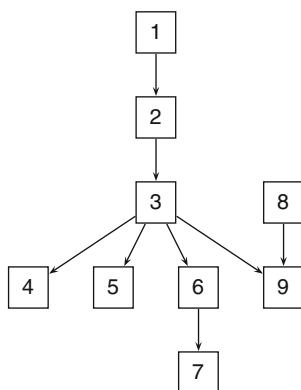
Chapter 8 is primarily included for completeness. It is a standard introduction to some matrix algebra that is needed for systems of linear differential equations. For those who have already had exposure to this basic algebra, it can be safely skipped or given as supplemental reading.

Chapter 9 is concerned with solving systems of linear differential equations. By the use of the Laplace transform, it is possible to give an explicit formula for the matrix exponential $e^{At} = \mathcal{L}^{-1} \{(sI - A)^{-1}\}$ that does not involve the use of eigenvectors or generalized eigenvectors. Moreover, we are then able to develop an efficient method for computing e^{At} known as Fulmer's method. Another thing which is somewhat unique is that we use the matrix exponential in order to solve a constant coefficient system $y' = Ay + f(t)$, $y(t_0) = y_0$ by means of an integrating factor. An immediate consequence of this is the existence and uniqueness theorem for higher order constant coefficient linear differential equations, a fact that is not commonly proved in texts at this level.

The text has numerous exercises, with answers to most odd-numbered exercises in the appendix. Additionally, a student solutions manual is available with solutions to most odd-numbered problems, and an instructors solution manual includes solutions to most exercises.

Chapter Dependence

The following diagram illustrates interdependence among the chapters.



Suggested Syllabi

The following table suggests two possible syllabi for one semester courses.

<i>3-Hour Course</i>	<i>4-Hour Course</i>	<i>Further Reading</i>
Sections 1.1–1.6	Sections 1.1–1.7	
Sections 2.1–2.8	Sections 2.1–2.8	
Sections 3.1–3.6	Sections 3.1–3.7	
Sections 4.1–4.3	Sections 4.1–4.4	Section 4.5
Sections 5.1–5.3, 5.6	Sections 5.1–5.6	
Sections 6.1–6.5	Sections 6.1–6.5	Sections 6.6–6.8
	Sections 7.1–7.3	Section 7.4
Sections 9.1–9.5	Sections 9.1–9.5, 9.7	Section 9.6
		Sections A.1, A.5

Chapter 8 is on matrix operations. It is not included in the syllabi given above since some of this material is sometimes covered by courses that precede differential equations. Instructors should decide what material needs to be covered for their students. The sections in the Further Reading column are written at a more advanced level. They may be used to challenge exceptional students.

We routinely provide a basic table of Laplace transforms, such as Tables 2.6 and 2.7, for use by students during exams.

Acknowledgments

We would like to express our gratitude to the many people who have helped to bring this text to its finish. We thank Frank Neubrandner who suggested making the Laplace transform have a more central role in the development of the subject. We thank the many instructors who used preliminary versions of the text and gave valuable suggestions for its improvement. They include Yuri Antipov, Scott Baldridge, Blaise Bourdin, Guoli Ding, Charles Egedy, Hui Kuo, Robert Lipton, Michael Malisoff, Phuc Nguyen, Richard Oberlin, Gestur Olafsson, Boris Rubin, Li-Yeng Sung, Michael Tom, Terrie White, and Shijun Zheng. We thank Thomas Davidson for proofreading many of the solutions. Finally, we thank the many many students who patiently used versions of the text during its development.

Baton Rouge, Louisiana

William A. Adkins
Mark G. Davidson

Contents

1	First Order Differential Equations	1
1.1	An Introduction to Differential Equations	1
1.2	Direction Fields	17
1.3	Separable Differential Equations	27
1.4	Linear First Order Equations	45
1.5	Substitutions	63
1.6	Exact Equations	73
1.7	Existence and Uniqueness Theorems	85
2	The Laplace Transform	101
2.1	Laplace Transform Method: Introduction	101
2.2	Definitions, Basic Formulas, and Principles	111
2.3	Partial Fractions: A Recursive Algorithm for Linear Terms	129
2.4	Partial Fractions: A Recursive Algorithm for Irreducible Quadratics	143
2.5	Laplace Inversion	151
2.6	The Linear Spaces \mathcal{E}_q : Special Cases	167
2.7	The Linear Spaces \mathcal{E}_q : The General Case	179
2.8	Convolution	187
2.9	Summary of Laplace Transforms and Convolutions	199
3	Second Order Constant Coefficient Linear Differential Equations	203
3.1	Notation, Definitions, and Some Basic Results	205
3.2	Linear Independence	217
3.3	Linear Homogeneous Differential Equations	229
3.4	The Method of Undetermined Coefficients	237
3.5	The Incomplete Partial Fraction Method	245
3.6	Spring Systems	253
3.7	RCL Circuits	267

4	Linear Constant Coefficient Differential Equations	275
4.1	Notation, Definitions, and Basic Results	277
4.2	Linear Homogeneous Differential Equations	285
4.3	Nonhomogeneous Differential Equations	293
4.4	Coupled Systems of Differential Equations	301
4.5	System Modeling	313
5	Second Order Linear Differential Equations	331
5.1	The Existence and Uniqueness Theorem	333
5.2	The Homogeneous Case	341
5.3	The Cauchy–Euler Equations	349
5.4	Laplace Transform Methods	355
5.5	Reduction of Order	367
5.6	Variation of Parameters	373
5.7	Summary of Laplace Transforms	381
6	Discontinuous Functions and the Laplace Transform	383
6.1	Calculus of Discontinuous Functions	385
6.2	The Heaviside Class \mathcal{H}	399
6.3	Laplace Transform Method for $f(t) \in \mathcal{H}$	415
6.4	The Dirac Delta Function	427
6.5	Convolution	439
6.6	Periodic Functions	453
6.7	First Order Equations with Periodic Input	465
6.8	Undamped Motion with Periodic Input	473
6.9	Summary of Laplace Transforms	485
7	Power Series Methods	487
7.1	A Review of Power Series	489
7.2	Power Series Solutions About an Ordinary Point	505
7.3	Regular Singular Points and the Frobenius Method	519
7.4	Application of the Frobenius Method:Laplace Inversion Involving Irreducible Quadratics	539
7.5	Summary of Laplace Transforms	555
8	Matrices	557
8.1	Matrix Operations	559
8.2	Systems of Linear Equations	569
8.3	Invertible Matrices	593
8.4	Determinants	605
8.5	Eigenvectors and Eigenvalues	619
9	Linear Systems of Differential Equations	629
9.1	Introduction	629
9.2	Linear Systems of Differential Equations	633
9.3	The Matrix Exponential and Its Laplace Transform	649
9.4	Fulmer’s Method for Computing e^{At}	657

9.5	Constant Coefficient Linear Systems	665
9.6	The Phase Plane	681
9.7	General Linear Systems	701
A	Supplements	723
A.1	The Laplace Transform is Injective	723
A.2	Polynomials and Rational Functions	725
A.3	\mathcal{B}_q Is Linearly Independent and Spans \mathcal{E}_q	727
A.4	The Matrix Exponential	732
A.5	The Cayley–Hamilton Theorem	733
B	Selected Answers	737
C	Tables	785
C.1	Laplace Transforms	785
C.2	Convolutions	789
	Symbol Index	791
	Index	793

List of Tables

Table 2.1	Basic Laplace transform formulas	104
Table 2.2	Basic Laplace transform formulas	123
Table 2.3	Basic Laplace transform principles	123
Table 2.4	Basic inverse Laplace transform formulas	153
Table 2.5	Inversion formulas involving irreducible quadratics	158
Table 2.6	Laplace transform rules	199
Table 2.7	Basic Laplace transforms	200
Table 2.8	Heaviside formulas	200
Table 2.9	Laplace transforms involving irreducible quadratics	201
Table 2.10	Reduction of order formulas	201
Table 2.11	Basic convolutions	202
Table 3.1	Units of measure in metric and English systems	256
Table 3.2	Derived quantities	256
Table 3.3	Standard units of measurement for RCL circuits	269
Table 3.4	Spring-body-mass and RCL circuit correspondence	270
Table 5.1	Laplace transform rules	381
Table 5.2	Laplace transforms	381
Table 6.1	Laplace transform rules	485
Table 6.2	Laplace transforms	486
Table 6.3	Convolutions	486
Table 7.1	Laplace transforms	555
Table C.1	Laplace transform rules	785
Table C.2	Laplace transforms	786
Table C.3	Heaviside formulas	788
Table C.4	Laplace transforms involving irreducible quadratics	788
Table C.5	Reduction of order formulas	789
Table C.6	Laplace transforms involving quadratics	789
Table C.7	Convolutions	789