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Cluster Polylogarithms, Adjacency, and Subalgebra-Constructibility at Eight Points

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Abstract: We construct a cluster-polylogarithimc representation of the eight-point two-loop MHV amplitude in the planar limit of maximally supersymmetric Yang-Mills theory. This representation makes manifest a novel cluster-algebraic decomposition of the nonclassical part of this amplitudes into its A_5 subalgebras, and limits smoothly to a similar decomposition of the seven-point MHV amplitude in collinear limits. We also investigate the equivalence of the extended Steinmann relations and cluster adjacency in eight-point kinematics by exploring the space of BDS-like normalized amplitudes.

Contents

1 Introduction

- emphasize the fact that promoting symbols to functions is hard—only a few other instances in the literature (don't forget this is done Regge limits as well—cite)
- must talk about the importance of automorphisms—let's become the standard physics reference on this!
- same for the Sklyanin bracket
- also, discuss the relation between cluster $\mathcal{A}$-adjacency and cluster $\mathcal{X}$-adjacency — can we prove these have to be equivalent by using the conversion $x \sim a^b$ between the two (since this translation is valid on any cluster)?
- mention existence of D_5 function and refer ahead
- we should also check our function against MRK predictions if possible (but don't want to hold up paper for this... clearly we can publish without)
- should point out somewhere that the cobracket is the same for the remainder function and bds-like normalized amplitudes—and that the same bootstrap procedure could be carried out for either quantity. However, we carry it out on the remainder function because there's no clear (unique) bds-like normalized amplitude to bootstrap

2 Cluster Algebras and Cluster Polylogarithms

2.1 Recap of essential facts

Begin with a quiver of nodes i, labeled by x_i, connected by arrows. The information of the arrows can be represented through an adjacency matrix b_{ij} = (# arrows $i \to j$) - (# arrows $j \to i$). The simplest (non-trivial) example is the quiver associated with the Dynkin diagram A_2:

$$x_1 \to x_2. \tag{2.1}$$

Now choose a node k and re-label the nodes via the mutation rules:

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i(1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases}. \tag{2.2}$$

Lastly, re-draw the arrows according to

- for each path $i \to j \to k$, add an arrow $i \to j$,
- reverse all arrows on the edges incident with k,
- and remove any two-cycles that may have formed.

The equivalent changes to the adjacency matrix are summarized as

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}, \\ b_{ij}, & \text{if } b_{ik}b_{kj} \le 0, \\ b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases} \tag{2.3}$$

This set of rules constitutes a *mutation* on node k, and generates a new quiver. For example, mutating on node 2 of eq. (2.1) gives

$$x_1(1 + x_2) \leftarrow \frac{1}{x_2}. \tag{2.4}$$

Mutation is also an involution, so mutating on node 2 of the above diagram will take you back to the original quiver $x_1 \to x_2$.

For our purposes, a *cluster algebra* is a set of quivers closed under mutation. This means that mutating on any node of any quiver will generate a different quiver in the cluster algebra. We generically name cluster algebras after particularly simple quiver types that appear, for example a cluster algebra containing a quiver of type A_2 is referred to as an "A_2 cluster algebra". Let us now work through the simple example of A_2 to understand some of the interesting structure inherent in the definitions eqs. (2.2) and (2.3).

2.2 Grassmannian cluster algebras (and cluster Poisson spaces)

$$\{x_i, x_j\} = b_{ij}x_ix_j. \tag{2.5}$$

$$\{x'_i, x'_j\} = b'_{ij}x'_ix'_j \tag{2.6}$$

2.3 Cluster polylogarithms and adjacency

2.4 Cluster automorphisms

2.5 The A_2 function

We define the A_2 function as

$$f_{A_2} = \sum_{\text{skew-dihedral}} f_i^{\pm} = \sum_{i=1}^{5}(f_i^+ - f_i^-) \tag{2.7}$$

in terms of the building blocks

$$f_i^{\pm} = \text{Li}_{2,2}(-x_i, -x_{i\pm 2}) - \text{Li}_{1,3}(-x_i, -x_{i\pm 2}) - \text{Li}_2(-x_i)\log(x_{i\pm 1})\log(x_{i\pm 2}). \tag{2.8}$$

where $6 - i$ is understood to be mod 5. It has the symbol

$$- \sum_{\text{skew-dihedral}} x_i \otimes x_i \otimes x_{i+1} \otimes x_{i+1} + x_i \otimes x_{i+1} \otimes x_i \otimes x_{i-1} + x_i \otimes x_{i+1} \otimes x_{i+1} \otimes x_i \tag{2.9}$$

$$- 2\left(x_i \otimes x_{i+1} \otimes (x_ix_{i+2}) \otimes x_{i+1} - x_i \otimes x_{i+1} \otimes x_{i+1} \otimes x_{i+2}\right) \tag{2.10}$$

$$-\sum_{\text{skew-dihedral}} [i,i,i+1,i+1]+[i,i+1,i,i-1]+[i,i+1,i+1,i] \tag{2.11}$$

$$-2\left([i,i+1,(ii+2),i+1]-[i,i+1,i+1,i+2]\right) \tag{2.12}$$

$$-\sum_{\text{skew-dihedral}} [1122]+[1215]+[1221]-2([1212]+[1232]-[1223]) \tag{2.13}$$

And satisfies the properties:

- clustery cobracket
- cluster adjacent in A_2
- smooth and real-valued in the positive domain

2.6 Cluster subalgebra-constructibility and the A_3 function

2.7 Previous applications for 6- and 7-pt amplitudes

3 The A_5 Function and $R_7^{(2)}$

3.1 Definition and properties of A_5

The A_5 cluster algebra is generated from the seed cluster

$$x_1 \to x_2 \to x_3 \to x_4 \to x_5. \tag{3.1}$$

The full A_5 algebra contains 132 clusters with 140 distinct $\mathcal{X}$-coordinates. Define:

$$x_{i_1\ldots i_k} = \sum_{a=1}^{k}\prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1}x_{i_2} + \ldots + x_{i_1}\cdots x_{i_k}. \tag{3.2}$$

The A_5 cluster algebra has an eight-fold cyclic symmetry, which is generated by σ:

$$\begin{aligned}\sigma: \quad x_1 \mapsto \frac{x_2}{1+x_{12}}, \quad x_2 \mapsto \frac{x_3\,(1+x_1)}{1+x_{13}}, \quad x_3 \mapsto \frac{x_4\,(1+x_{12})}{1+x_{1234}}, \\ x_4 \mapsto \frac{x_5\,(1+x_{123})}{1+x_{12345}}, \quad x_5 \mapsto \frac{1+x_{1234}}{x_1x_2x_3x_4x_5}.\end{aligned} \tag{3.3}$$

A_5 also has a two-fold flip symmetry, which is generated by τ:

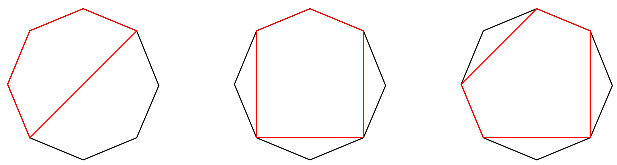
$$\tau: \quad x_i \mapsto \frac{1}{x_{6-i}}. \tag{3.4}$$

3.2 A_2-constructability in A_5

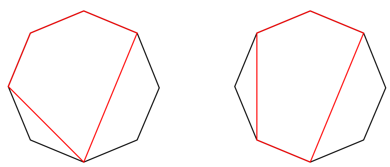
There are 56 distinct A_2 subalgebras in A_5 ($56 = \binom{8}{5}$ = number of distinct pentagons inside an octagon), they can be parameterized by:

$$\begin{aligned} &\left\{x_1 \to x_2, \quad x_2 \to x_3\left(1+x_4\right), \quad x_2\left(1+x_3\right) \to \frac{x_3 x_4}{1+x_3}\right\} + \sigma, \\ &\left\{x_2 \to x_3, \quad x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}\right\} + \sigma + \tau \end{aligned} \tag{3.5}$$

where by "$+\ \sigma$" and "$+\ \sigma\ +\ \tau$" I mean "+ cyclic copies" and "+ cyclic and flip copies," respectively. This correspond to the geometries



(8 of each)



(16 of each)

The A_5 function is a sum over two of the classes of A_2 subalgebras, $x_2 \to x_3\left(1+x_4\right)$ and $x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}$, appropriately antisymmetrized so that the overall f_{A_5} picks up a minus sign under both σ and τ. Explicitly, this is written

$$f_{A_5} = \sum_{i=0}^{7} \sum_{j=0}^{1} (-1)^{i+j} \sigma^i \tau^j \left(\frac{1}{2} f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right) + f_{A_2}\left(x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2} \right) \right). \tag{3.6}$$

The factor of $\frac{1}{2}$ in front of $f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right)$ is simply a symmetry factor, as it lives in an 8-cycle of $\{\sigma, \tau\}$.

Which pentagonalizations do these two A_2s correspond to?

A large outstanding question is: why these two A_2s? I have no justification/argument for them other than they work.

The A_5 does not have $B_2 \wedge B_2$ expressible in terms of $A_1 \times A_1$ subalgebras (i.e. it is not expressible as a sum of f_{A_3}s). This is very surprising!

3.3 A_5 representation of $R_7^{(2)}$

3.4 Behavior of A_5 functions in the $7 \to 6$ collinear limit

4 Constructing $R_8^{(2)}$

4.1 Taming the $\mathrm{Gr}(4,8)$ infinities

4.1.1 Sklyanin bracket

In momentum twistor language we have the n momentum twistors Z_i, which together form the $4 \times n$ matrix

$$K = \begin{pmatrix} z_{11} & \dots & z_{n1} \\ z_{12} & \dots & z_{n2} \\ z_{13} & \dots & z_{n3} \\ z_{14} & \dots & z_{n4} \end{pmatrix}. \tag{4.1}$$

As long as the first 4 columns are non-singular, we can row reduce K in to the form

$$K' = \begin{pmatrix} 1 & 0 & 0 & 0 & y_{11} & \dots & y_{(n-4)1} \\ 0 & 1 & 0 & 0 & y_{12} & \dots & y_{(n-4)2} \\ 0 & 0 & 1 & 0 & y_{13} & \dots & y_{(n-4)3} \\ 0 & 0 & 0 & 1 & y_{14} & \dots & y_{(n-4)4} \end{pmatrix}. \tag{4.2}$$

The columns of K' define a new set of momentum twistors Z'_i, where for example $Z'_1 = \{1,0,0,0\}$ and $Z'_5 = \{y_{11}, y_{12}, y_{13}, y_{14}\}$. It is easy to check that

$$y_{ij} = (-1)^j \langle \{1,2,3,4\} \setminus \{j\}, i\rangle / \langle 1234\rangle, \tag{4.3}$$

$$\langle abcd\rangle' = \det(Z'_a Z'_b Z'_c Z'_d) = \langle abcd\rangle / \langle 1234\rangle. \tag{4.4}$$

You can then define the Sklyanin bracket as an operation on these y_{ij} by

$$\{y_{ij}, y_{ab}\} = (\mathrm{sgn}(a-i) - \mathrm{sgn}(b-j)) y_{ib} y_{aj}. \tag{4.5}$$

Which then extends to a bracket on functions of the y_{ij} via

$$\{f(y), g(y)\} = \sum_{i,a=1}^{n} \sum_{j,b=1}^{4} \frac{\partial f}{\partial y_{ij}} \frac{\partial g}{\partial y_{ab}} \{y_{ij}, y_{ab}\}. \tag{4.6}$$

Now if we want to evaluate the Poisson bracket between two $\mathcal{X}$-coordinates, we can instead treat them as functions of the y_{ij} and use eq. (4.6). To be precise, for each four-bracket $\langle abcd\rangle$ in the $\mathcal{X}$-coordinates, replace them with $\langle abcd\rangle'$ expanded out in terms of y_{ij} (e.g. $\langle 1256\rangle' = y_{13}y_{24} - y_{14}y_{23}$). Then you can calculate eq. (4.6) directly in terms of the y_{ij}

4.1.2 Identifying A_5 subalgebras in $\mathrm{Gr}(4,8)$

Comment on infinite nature of $\mathrm{Gr}(4,8)$, define "good" (in the two-loop MHV sense) subalgebras, describe algorithm for finding "good", and describe the 56 good A_5s in $\mathrm{Gr}(4,8)$.

There are 56 good A_5s in $\mathrm{Gr}(4,8)$. They are generated by

$$\frac{\langle 1238\rangle\langle 1256\rangle}{\langle 1235\rangle\langle 1268\rangle} \to \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle} \to \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle} \to \frac{\langle 1567\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3567\rangle} \to \frac{\langle 1356\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 3456\rangle} \tag{4.7}$$

$$\frac{\langle 1238\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2358\rangle} \to -\frac{\langle 1235\rangle\langle 4568\rangle}{\langle 5(18)(23)(46)\rangle} \to \frac{\langle 1568\rangle\langle 2358\rangle\langle 3456\rangle}{\langle 1358\rangle\langle 2356\rangle\langle 4568\rangle} \to -\frac{\langle 5(18)(23)(46)\rangle}{\langle 1258\rangle\langle 3456\rangle} \to \frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle} \tag{4.8}$$

$$\frac{\langle 1234\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 2345\rangle} \to \frac{\langle 1348\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 3468\rangle} \to -\frac{\langle 1346\rangle\langle 5678\rangle}{\langle 6(18)(34)(57)\rangle} \to -\frac{\langle 1678\rangle\langle 3468\rangle\langle 34(128)\cap(567)\rangle}{\langle 1268\rangle\langle 1348\rangle\langle 3467\rangle\langle 5678\rangle} \to \frac{\langle 1278\rangle\langle 6(18)(34)(57}{\langle 1678\rangle\langle 34(128)\cap(56} \tag{4.9}$$

$$\frac{\langle 1234\rangle\langle 1278\rangle}{\langle 1238\rangle\langle 1247\rangle} \to -\frac{\langle 1248\rangle\langle 3457\rangle}{\langle 4(12)(35)(78)\rangle} \to -\frac{\langle 1247\rangle\langle 12(345)\cap(678)\rangle}{\langle 1278\rangle\langle 4(12)(35)(67)\rangle} \to -\frac{\langle 4567\rangle\langle 4(12)(35)(78)\rangle}{\langle 1245\rangle\langle 3457\rangle\langle 4678\rangle} \to -\frac{\langle 4(12)(35)(67)}{\langle 1234\rangle\langle 4567\rangle} \tag{4.10}$$

The first A_5 lives in an 8-cycle of the $\mathrm{Gr}(4,8)$ dihedral+parity, while the other three live in 16-cycles. Also note that in the first A_5, 7 and 8 never appear together in a $\langle\rangle$, and so the $8\to 7$ collinear limit is smooth for this A_5. The second A_5 also features a smooth collinear limit, as

$$\frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle} \xrightarrow{8\to 7} \frac{\langle 1267\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1567\rangle}. \tag{4.11}$$

Neither of the latter 2 A_5s behave smoothly in the collinear limit (and neither do any of their dihedral+parity images).

Note: there are no good A_6s in $\mathrm{Gr}(4,8)$.

4.2 Fitting non-classical component of $R_8^{(2)}$

4.2.1 Behavior of A_5 functions in the $8\to 7$ collinear limit

4.3 Fitting the classical component of $R_8^{(2)}$

The A_5 contribution to $R_8^{(2)}$ involves simply adding together the two A_5s in $\mathrm{Gr}(4,8)$ which behave smoothly in the collinear limit.

$$\begin{aligned} R_8^{(2)} = {}& \frac{1}{4} f_{A_5}\left(\frac{\langle 1238\rangle\langle 1256\rangle}{\langle 1235\rangle\langle 1268\rangle} \to \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle} \to \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle} \to \frac{\langle 1567\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3567\rangle} \to \frac{\langle 1356\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 3456\rangle}\right) + \\ & \frac{1}{2} f_{A_5}\left(\frac{\langle 1238\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2358\rangle} \to -\frac{\langle 1235\rangle\langle 4568\rangle}{\langle 5(18)(23)(46)\rangle} \to \frac{\langle 1568\rangle\langle 2358\rangle\langle 3456\rangle}{\langle 1358\rangle\langle 2356\rangle\langle 4568\rangle} \to -\frac{\langle 5(18)(23)(46)\rangle}{\langle 1258\rangle\langle 3456\rangle} \to \frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle}\right) \\ & + \text{ dihedral} + \text{conjugate} \end{aligned} \tag{4.12}$$

Again the difference between the overall factors of the two terms is simply a result of symmetry.

Let me briefly describe the collinear limit for this representation. As discussed previously, the A_5s explicitly written in (4.12) behave smoothly under the collinear limit, however not all of their dihedral+parity images do as well. In the case of the first A_5, which has 8 images under dihedral+parity, 4 of the f_{A_5}s vanish, while the remaining 3 are well-defined. For the second A_5, which has 16 images under dihedral+parity, 2 of the f_{A_5}s have "bad" collinear limits but they cancel off each other in the sum. Out of the remaining 14, 4 have good collinear limits and 10 vanish identically. Therefore, when we add up the contributions from both A_5s + their images, we end up with 7 terms – these correspond to the 7 A_5s in $\mathrm{Gr}(4,7)$.

- Li_4 contribution
- $\mathrm{Li}_2\,\mathrm{Li}_2$ contribution
- $\mathrm{Li}_3\,\mathrm{Li}_1$ contribution
- $\mathrm{Li}_2\,\mathrm{Li}_1^2$ contribution
- Li_1^4 contribution
- $\mathrm{Li}_2\,\pi^2$ contribution
- $\mathrm{Li}_1^2\,\pi^2$ contribution
- π^4 constribution

5 Analytic Properties of $R_8^{(2)}$

- some plots
- agrees with numerics

6 Steinmann Relations and Cluster Adjacency

The Steinmann relations dictate that double discontinuities of amplitudes must vanish when taken in partially overlapping momentum channels [1, 2]. It has recently been realized that these restrictions on three- (and higher-)particle channels are transparently encoded in the symbol of BDS-like normalized amplitudes when the number of scattering particles is not a multiple of four [3, 4]. This follows from the fact that the BDS-like ansatz in these cases is defined to depend on just two-particle Mandelstam invariants, and thus acts as a spectator when discontinuities are taken in these channels. This subset of the Steinmann relations therefore applies directly to BDS-like-normalized amplitudes for these numbers of particles, where it implies that restricted pairs of Mandelstam invariants cannot appear sequentially in the first two entries of the symbol. In fact, these restrictions have been found to apply at all depths in the symbol, providing strong all-loop constraints on the spaces of functions that are expected to contribute to these amplitudes [5**?**].

More surprisingly, the extended Steinmann constraints have been found to be equivalent to demanding that every pair of sequential symbol entries appears together in some cluster in Gr(4,n) [6]. In particular, it has been checked that this 'cluster adjacency' principle is adhered to in all known BDS-like normalized amplitudes in six-, seven-, and nine-particle kinematics, where a unique BDS-like ansatz depending only on two-particle invariants can be defined. However, it remains less well-studied in eight-particle kinematics due to the nonexistence of any such BDS-like normalization; all eight-particle solutions to the anomalous dual conformal Ward identity governing these amplitudes in the infrared involve higher-particle Mandelstam invariants [7]. For this reason, it proves necessary to explore the space of BDS-like ansätze that can be formed for eight particles before the (vestiges of the) Steinmann relations and cluster adjacency can be studied.

6.1 BDS-Like Ansätze for Eight Particles

[Paragraph introducing the BDS ansatz]

When the number of particles n is not a multiple of four, a unique BDS-like ansatz can be defined that depends on just two-particle Mandelstam invariants. That is, there exists just a single decomposition of the BDS ansatz into

$$\mathcal{A}_n^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = \mathcal{A}_n^{\text{BDS-like}}(\{s_{i,i+1}\}) \exp\left[-\frac{\Gamma_{\text{cusp}}}{4} Y_n(\{u_i\})\right], \quad n \neq 4K, \tag{6.1}$$

such that the kinematic dependence of $A_n^{\text{BDS-like}}$ involves only two-particle Mandelstam invariants while Y_n depends only on dual-conformal-invariant cross ratios [8]. When n is a multiple of four, no decomposition of this type exists, and we are forced to consider multiple BDS-like ansätze if we want to transparently expose the full space of Steinmann relations between higher-particle Mandelstam invariants.

In eight-particle kinematics, there are still two natural BDS-like normalization choices we might consider. Namely, we can let our BDS-like ansatz depend on either three- or four-particle Mandelstam invariants in addition to two-particle invariants [4]. In this spirit, let us define a pair of BDS-like ansätze, respectively satisfying

$$\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = {}^4\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2,i+3}\}) \exp\left[-\frac{\Gamma_{\text{cusp}}}{4}\ {}^4Y_8(\{u_i\})\right], \tag{6.2}$$

$$\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = {}^3\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2}\}) \exp\left[-\frac{\Gamma_{\text{cusp}}}{4}\ {}^3Y_8(\{u_i\})\right]. \tag{6.3}$$

The functions ${}^4A_8^{\text{BDS-like}}$ and ${}^3A_8^{\text{BDS-like}}$ are not uniquely fixed by these decomposition choices; each admits a family of Bose-symmetric (and a larger family of non-Bose-symmetric) solutions. However, any choice for ${}^4A_8^{\text{BDS-like}}$ or ${}^3A_8^{\text{BDS-like}}$ consistent with eqns. (6.2) or (6.3) gives rise to a BDS-like normalized amplitude that manifestly exhibits a subset of the Steinmann relations. In particular, defining

$${}^X\mathcal{E}_8 \equiv \frac{\mathcal{A}_8^{\text{MHV}}}{{}^X\mathcal{A}_8^{\text{BDS-like}}} = \exp\left[R_8 - \frac{\Gamma_{\text{cusp}}}{4}\ {}^XY_8\right] \tag{6.4}$$

for any label X, we expect that ${}^4\mathcal{E}_8$ should satisfy Steinmann relations between all partially overlapping pairs of three-particle invariants, while ${}^3\mathcal{E}_8$ should satisfy Steinmann relations between all partially overlapping pairs of four-particle invariants. That is, ${}^4\mathcal{E}_8$ is expected to satisfy the relations

$$\mathrm{Disc}_{s_{j,j+1,j+2}}\left[\mathrm{Disc}_{s_{i,i+1,i+2}}\left({}^4\mathcal{E}_8\right)\right]=0, \quad j\in\{i\pm 2, i\pm 1\}, \tag{6.5}$$

while ${}^3\mathcal{E}_8$ is expected to satisfy

$$\mathrm{Disc}_{s_{j,j+1,j+2,j+3}}\left[\mathrm{Disc}_{s_{i,i+1,i+2,i+3}}\left({}^3\mathcal{E}_8\right)\right]=0, \quad j\in\{i\pm 3, i\pm 2, i\pm 1\}. \tag{6.6}$$

Due to momentum conservation in eight-point kinematics, the six relations in (6.6) corresponding to a given i only result in three independent constraints; however, these relations will be independent for larger n.

Although the functions ${}^4Y_8$ and ${}^3Y_8$ are not unique, their dilogarithmic part is completely determined by the decompositions (6.2) and (6.3). They can be expressed as classical polylogarithms with negative arguments drawn from

$$\mathfrak{X}_{i,8}=\frac{\langle i,i+1,i+2,i+4\rangle\langle i+1,i+3,i+4,i+5\rangle}{\langle i,i+1,i+4,i+5\rangle\langle i+1,i+2,i+3,i+4\rangle}, \tag{6.7}$$

$$\mathfrak{X}_{i,4}=\frac{\langle i,i+1,i+3,i+7\rangle\langle i,i+2,i+3,i+4\rangle}{\langle i,i+1,i+2,i+3\rangle\langle i,i+3,i+4,i+7\rangle}, \tag{6.8}$$

where $\mathfrak{X}_{i,8}$ and $\mathfrak{X}_{i,4}$ are $\mathcal{X}$-coordinates in Gr(4,8) that respectively carve out an eight-orbit and a four-orbit of the dihedral group. In these variables the Li_1 parts of these functions can be diagonalized, giving rise to the Bose-symmetric representations

$${}^4Y_8=\sum_{i=1}^{8}\left[\mathrm{Li}_2\left(-\mathfrak{X}_{i,8}\right)+\frac{1}{2}\mathrm{Li}_2\left(-\mathfrak{X}_{i,4}\right)+\frac{1}{4}\mathrm{Li}_1\left(-\mathfrak{X}_{i,4}\right)^2\right], \tag{6.9}$$

$${}^3Y_8=\sum_{i=1}^{8}\left[\mathrm{Li}_2\left(-\mathfrak{X}_{i,8}\right)+\frac{1}{2}\mathrm{Li}_2\left(-\mathfrak{X}_{i,4}\right)+\frac{1}{2}\mathrm{Li}_1\left(-\mathfrak{X}_{i,8}\right)^2\right]. \tag{6.10}$$

We emphasize that this is an aesthetically motivated choice; there may exist other more physically (or mathematically) inspired choices that endow ${}^4\mathcal{E}_8$ or ${}^3\mathcal{E}_8$ with additional desirable properties. Regardless, it can be checked that any realization of ${}^4Y_8$ or ${}^3Y_8$ that respects Bose symmetry gives rise to a BDS-like normalized amplitude that satisfies either (6.5) or (6.6), while violating all other Steinmann relations (all at the level of the symbol).

If we want to recover more Steinmann relations, such as those holding between partially overlapping three- and four-particle invariants, we can instead define BDS-like ansätze that depend only on subsets of the three- or four-particle invariants. In particular, it proves possible to decompose the BDS ansatz into either

$$\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+k}\}) = {}^{\{a,b\}_4}\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2,i+3}|i\in\{a,b\}\}) \times \exp\left[-\frac{\Gamma_{\text{cusp}}}{4}\ {}^{\{a,b\}_4}Y_8(\{u_i\})\right], \tag{6.11}$$

$$\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+k}\}) = {}^{\{a,b\}_3}\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2}|i\in\{a,b\}\}) \times \exp\left[-\frac{\Gamma_{\text{cusp}}}{4}\ {}^{\{a,b\}_3}Y_8(\{u_i\})\right], \tag{6.12}$$

for any $\{a,b\}$ such that $b-a$ is odd.[1] Any solution to (6.11) defines a BDS-like normalized amplitude ${}^{\{a,b\}_4}\mathcal{E}_8$ that respects the Steinmann relations

$$\left.\begin{array}{l}\text{Disc}_{s_{j,j+1,j+2}}\left[\text{Disc}_{s_{i,i+1,i+2,i+3}}\left({}^{\{a,b\}_4}\mathcal{E}_8\right)\right]=0,\\ \text{Disc}_{s_{i,i+1,i+2,i+3}}\left[\text{Disc}_{s_{j,j+1,j+2}}\left({}^{\{a,b\}_4}\mathcal{E}_8\right)\right]=0,\end{array}\right\}\quad \begin{array}{c} i\notin\{a,b\},\\ j\in\{i-2,i-1,i+2,i+3\},\end{array} \tag{6.13}$$

in addition to all the Steinmann relations satisfied by ${}^4\mathcal{E}_8$ as given in eq. (6.5). Moreover, it will respect many of the Steinmann relations satisfied by ${}^3\mathcal{E}_8$—namely, those that don't involve a discontinuity in either $s_{a,a+1,a+2,a+3}$ or $s_{b,b+1,b+2,b+3}$. Similarly, any solution to (6.12) defines an amplitude ${}^{\{a,b\}_3}\mathcal{E}_8$ that respects

$$\left.\begin{array}{l}\text{Disc}_{s_{i,i+1,i+2}}\left[\text{Disc}_{s_{j,j+1,j+2,j+3}}\left({}^{\{a,b\}_3}\mathcal{E}_8\right)\right]=0,\\ \text{Disc}_{s_{j,j+1,j+2,j+3}}\left[\text{Disc}_{s_{i,i+1,i+2}}\left({}^{\{a,b\}_3}\mathcal{E}_8\right)\right]=0,\end{array}\right\}\quad \begin{array}{c} i\notin\{a,b\},\\ j\in\{i-3,i-2,i+1,i+2\},\end{array} \tag{6.14}$$

as well as all the Steinmann relations satisfied by ${}^3\mathcal{E}_8$ and described in eq. (6.6), and all the relations specified in eq. (6.5) that don't involve a discontinuity in either $s_{a,a+1,a+2}$ or $s_{b,b+1,b+2}$. Clearly it is not possible for BDS-like amplitudes of either type to be Bose-symmetric; however, it proves possible to construct solutions to (6.12) such that ${}^{\{a,b\}_3}\mathcal{E}_8$ respects the dihedral flip $s_{i,\dots,i+k}\to s_{9-i,\dots,9-i-k}$ when this mapping is oriented to map $s_{a,a+1,a+2}$ and $s_{b,b+1,b+2}$ between each other. We present specific realizations of ${}^{\{1,2\}_4}Y_8$ and ${}^{\{7,8\}_3}Y_8$ in appendix A. As with the Bose-symmetric normalization choices, it can be checked that all possible realizations of ${}^{\{a,b\}_4}Y_8$ and ${}^{\{a,b\}_3}Y_8$ give rise to BDS-like amplitudes that obey and break the same Steinmann relations (for a given pair of indices a and b).

[1] The difference $b-a$ should be computed mod 8 in the case of ${}^{\{a,b\}_3}\mathcal{A}_8^{\text{BDS-like}}$ since $s_{i+8,\dots,i+k+8}=s_{i,\dots,i+k}$ in general, but should be computed mod 4 in the case of ${}^{\{a,b\}_4}\mathcal{A}_8^{\text{BDS-like}}$ since momentum conservation implies the stronger identity $s_{i+4,i+5,i+6,i+7}=s_{i,i+1,i+2,i+3}$ between four-particle invariants.

[To Do: can any given Steinmann relation be saved (in Bose-symmetric or ...)? Any other features of the full space worth working out?]

[To Do: define $\Gamma_{\mathbf{cusp}}$ in this section if we don't earlier]

[To Do: comment about the fact that we don't know how to extend the Steinmann relations beyond symbol level (or figure out how to do so...)]

6.2 Cluster Adjacency in $\mathcal{A}$- and $\mathcal{X}$-coordinates

The extended Steinmann relations (**??**) through (**??**) can be checked by computing the appropriate BDS-like normalized amplitudes from the remainder function, as per eq. (6.4). While these relations are satisfied, every Steinmann relation that is not preserved by the choice of BDS-like ansatz is violated by these amplitudes.

[mention that cluster X-adjacency is a statement of existence, not about all representations]

[To Do: discuss which nonadjacent pairs appear in the amplitude] [To Do: email Christian to ask about the Sklyanin bracket on $\mathcal{A}$-coordinates, and about $\delta = \rho \circ \delta \circ \rho$**] [To Do: discuss the Sklyanin bracket on $\mathcal{A}$-coordinates]**

6.3 Restoring all Steinmann Relations

[To Do: discuss the possibility of repairing Steinmann and cluster adjacency at the cost of dual conformal invariance, and also in special kinematic limits where the additional three- or four-particle dependence drops out]

7 Conclusion

A BDS-Like Conversions for Eight Particles

$$\begin{aligned} {}^{\{1,2\}_4}Y_8 = {}^{4}Y_8 - \Big(&\mathrm{Li}_1(-\mathfrak{X}_{1,4}) + \mathrm{Li}_1(-\mathfrak{X}_{4,4}) + \mathrm{Li}_1(-\mathfrak{X}_{4,8}) + \mathrm{Li}_1(-\mathfrak{X}_{8,8})\Big) \\ &\times \Big(\mathrm{Li}_1(-\mathfrak{X}_{3,4}) + \mathrm{Li}_1(-\mathfrak{X}_{4,4}) + \mathrm{Li}_1(-\mathfrak{X}_{3,8}) + \mathrm{Li}_1(-\mathfrak{X}_{7,8})\Big) \end{aligned} \tag{A.1}$$

$$
\begin{aligned}
{}^{\{7,8\}_3}Y_8 = \sum_{i=1}^{8} &\left[\mathrm{Li}_2\left(-\mathfrak{X}_{i,8}\right) + \frac{1}{2}\mathrm{Li}_2\left(-\mathfrak{X}_{i,4}\right) + \frac{1}{4}\mathrm{Li}_1\left(-\mathfrak{X}_{i,4}\right)^2 \right] \\
- &\left[\frac{1}{2}\Big(\mathrm{Li}_1(-\mathfrak{X}_{1,4}) + \mathrm{Li}_1(-\mathfrak{X}_{3,4})\Big)\Big(\mathrm{Li}_1(-\mathfrak{X}_{2,4}) + \mathrm{Li}_1(-\mathfrak{X}_{4,4})\Big) \right. \\
&\quad + \mathrm{Li}_1(-\mathfrak{X}_{1,4})\Big(\mathrm{Li}_1(-\mathfrak{X}_{1,8}) + \mathrm{Li}_1(-\mathfrak{X}_{4,8}) + \mathrm{Li}_1(-\mathfrak{X}_{6,8}) + \mathrm{Li}_1(-\mathfrak{X}_{7,8})\Big) \\
&\quad + \mathrm{Li}_1(-\mathfrak{X}_{2,4})\Big(\mathrm{Li}_1(-\mathfrak{X}_{1,8}) + \mathrm{Li}_1(-\mathfrak{X}_{4,8}) - \mathrm{Li}_1(-\mathfrak{X}_{6,8}) - \mathrm{Li}_1(-\mathfrak{X}_{3,8})\Big) \\
&\quad + \mathrm{Li}_1(-\mathfrak{X}_{1,8})\Big(\mathrm{Li}_1(-\mathfrak{X}_{4,8}) + \frac{1}{2}\mathrm{Li}_1(-\mathfrak{X}_{1,8}) - \frac{1}{2}\mathrm{Li}_1(-\mathfrak{X}_{3,8})\Big) \\
&\quad + \mathrm{Li}_1(-\mathfrak{X}_{5,8})\Big(\mathrm{Li}_1(-\mathfrak{X}_{4,8}) - \frac{1}{2}\mathrm{Li}_1(-\mathfrak{X}_{5,8}) + \frac{1}{2}\mathrm{Li}_1(-\mathfrak{X}_{7,8})\Big) \\
&\quad + \mathrm{Li}_1(-\mathfrak{X}_{6,8})\Big(\mathrm{Li}_1(-\mathfrak{X}_{4,8}) - \frac{1}{2}\mathrm{Li}_1(-\mathfrak{X}_{2,8}) - \frac{1}{2}\mathrm{Li}_1(-\mathfrak{X}_{6,8})\Big) \\
&\quad \left. - \mathrm{Li}_1(-\mathfrak{X}_{2,4})\mathrm{Li}_1(-\mathfrak{X}_{3,4}) \right]_{\mathrm{Li}_1(-\mathfrak{X}_{i,j}) + \mathrm{Li}_1(-\overline{\mathfrak{X}}_{i,j})}
\end{aligned}
\tag{A.2}
$$

where $\overline{\mathfrak{X}}_{i,j}$ is the image of the $\mathcal{X}$-coordinate $\mathfrak{X}_{i,j}$ under the dihedral flip that sends $Z_i \to Z_{9-i}$ (that is, the expression in the second square bracket is understood to be the sum of itself and this dihedral image).

The decompositions (6.3), (6.2), and (6.12) do not uniquely determine ${}^3Y_8$, ${}^4Y_8$, or ${}^{3,j}Y_8$. In fact, there exists a 10-dimensional (3-dimensional) space of (Bose-symmetric) solutions for ${}^3Y_8$, a 36-dimensional (5-dimensional) space of (Bose-symmetric) solutions for ${}^4Y_8$, and a 3-dimensional space of solutions for ${}^{3,j}Y_8$.

$$
\begin{aligned}
{}^{3,1}Y_8 = {}^3Y_8 - &\Big(\mathrm{Li}_1(-\mathfrak{X}_{1,4}) + \mathrm{Li}_1(-\mathfrak{X}_{2,4}) + \mathrm{Li}_1(-\mathfrak{X}_{1,8}) + \mathrm{Li}_1(-\mathfrak{X}_{5,8})\Big) \\
&\times \Big(\mathrm{Li}_1(-\mathfrak{X}_{1,4}) + \mathrm{Li}_1(-\mathfrak{X}_{4,4}) + \mathrm{Li}_1(-\mathfrak{X}_{4,8}) + \mathrm{Li}_1(-\mathfrak{X}_{8,8})\Big)
\end{aligned}
\tag{A.3}
$$

$$
- \log(s_{1234}s_{3456}) \log(s_{2345}s_{4567})
$$

$$
- \frac{1}{2}\log(s_{i,i+1,i+2}) \log\left(\frac{s_{i,i+1,i+2}\ s^2_{i+1,i+2,i+3}}{s_{i+4,i+5,i+6}}\right) \Bigg]
$$

To take full advantage of the Steinmann relations, it is convenient to work in terms of symbol letters that isolate different Mandelstam invariants. There are twelve independent dual conformally invariant cross ratios that can appear in these symbols

$$
u_1 = \frac{s_{12}s_{4567}}{s_{123}s_{812}}, \quad \text{and cyclic (8-orbit)} \tag{A.4}
$$

$$
u_9 = \frac{s_{123}s_{567}}{s_{1234}s_{4567}}, \quad \text{and cyclic (4-orbit).} \tag{A.5}
$$

It is not possible to isolate all three- and four-particle Mandelstam invariants simultaneously into twelve different symbol letters. (More than twelve symbol letters will appear in these amplitudes, but we here restrict our attention to the twelve that will appear in the first entry.) However, different choices of letters can be made such that either all the four-particle invariants, or all the three-particle invariants, are isolated.

One choice that isolates the four-particle invariants is

$$ {}^4d_1 = u_2\ u_6 = \frac{s_{23}\ s_{67}\ (s_{1234})^2}{s_{123}\ s_{234}\ s_{567}\ s_{678}}, \quad \text{and cyclic (4-orbit)} \tag{A.6} $$

$$ {}^4d_5 = u_2/u_6 = \frac{s_{23}\ s_{567}\ s_{678}}{s_{67}\ s_{123}\ s_{234}}, \quad \text{and cyclic (4-orbit)} \tag{A.7} $$

$$ {}^4d_9 = u_1\ u_2\ u_5\ u_6\ u_9^2 = \frac{s_{12}\ s_{23}\ s_{56}\ s_{67}}{s_{234}\ s_{456}\ s_{678}\ s_{812}}, \quad \text{and cyclic (4-orbit).} \tag{A.8} $$

In this alphabet ${}^4d_1, {}^4d_2, {}^4d_3$, and ${}^4d_4$ each contain a different four-particle Mandelstam invariant, while the other letters only involve two- and three-particle invariants. The extended Steinmann relations then tell us that ${}^4d_1, {}^4d_2, {}^4d_3$, and ${}^4d_4$ can never appear next to each other in the symbol of ${}^4A_8^{\text{BDS-like}}$ (but each can still appear next to themselves).

Similarly, we can isolate the three-particle invariants by choosing

$$ {}^3d_1 = \frac{u_1\ u_2\ u_4\ u_7}{u_3\ u_5\ u_6\ u_8\ u_9^2} = \frac{s_{12}\ s_{23}\ s_{45}\ s_{78}\ (s_{1234})^2\ (s_{4567})^2}{s_{34}\ s_{56}\ s_{67}\ s_{81}\ (s_{123})^2}, \quad \text{and cyclic (8-orbit)} \tag{A.9} $$

$$ {}^3d_9^4 = u_1\ u_5\ u_9\ u_{12} = \frac{s_{12}\ s_{56}}{s_{1234}\ s_{3456}}, \quad \text{and cyclic (4-orbit),} \tag{A.10} $$

in which case ${}^3d_1$ through ${}^3d_8$ each contain a different three-particle Mandelstam invariant, as well as four-particle Mandelstams that they don't partially overlap with. The remaining four letters only contain two- and four-particle invariants. In these letters, conditions (**??**) and (**??**) tell us that ${}^3d_7, {}^3d_8, {}^3d_2$, and ${}^3d_3$ can never appear next to ${}^3d_1$ in the symbols of ${}^3\mathcal{E}_8$ or ${}^{3,j}\mathcal{E}_8$ (plus the cyclic images of this statement). Moreover, conditions (**??**) through (**??**) give us the additional restrictions that none of ${}^3d_1, {}^3d_5, {}^3d_9$ and ${}^3d_{10}$ can ever appear next to ${}^3d_3, {}^3d_4, {}^3d_7$, or ${}^3d_8$ in the symbol of ${}^{3,1}\mathcal{E}_8$ (analogous relations hold for the other ${}^{3,j}\mathcal{E}_8$). These are the restrictions given by the Steinmann relations involving s_{1234} and one of $s_{781}, s_{812}, s_{345}$, or s_{456}. The other Steinmann relations between three- and four-particle invariants will not be respected by ${}^{3,1}\mathcal{E}_8$, since ${}^{3,j}\mathcal{A}_8^{\text{BDS-like}}$ depends on s_{2345}, s_{3456}, and s_{4567}.

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