In this note we analyze "the" D_5 function, f_{D_5}, which we generate by:

- starting with an ansatz of all distinct f_{A_3}'s in D_5,
- imposing antisymmetry under all of the D_5 automorphisms $\{\sigma, \tau, \mathbb{Z}_2\}$,
- taking the fully symmetric sum of this f_{D_5} over E_6 and fitting to $R_7^{(2)}$.

The resulting function has 1 free parameter, which represents an internal degree of freedom in f_{D_5} that cancels in the symmetric sum over E_6. Later in the note we will explore tuning this free parameter to make some (tbd) nice property manifest.

Reader's Digest: Deriving f_{D_5}

We'll be working with this D_5 seed cluster:

$$\begin{tikzcd}[row sep=tiny] & & & x_4 \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\ & & & x_5 \end{tikzcd} \tag{1}$$

There are 65 distinct A_3's in D_5. Of these, only 42 produce linearly independent f_{A_3}'s. Imposing full D_5 antisymmetry on this collection of f_{A_3}'s leaves only 5 degrees of freedom. Requiring that the full E_6-symmetric sum of f_{D_5} gives $R_7^{(2)}$ fixes 4 of these parameters, leaving us with only 1 degree of freedom. Of course when we are looking for a particular representation of f_{D_5} we have 24 degrees of freedom (23 of which are equivalent to adding zero).

It would be nice to find a property that fixes some of these parameters that does not rely on explicitly knowing $R_7^{(2)}$. Of course a cluster-y property would be great, but even a physics one would be nice.

Describing f_{D_5}

Because of the 1 degree of freedom, it is difficult to describe in detail the properties of f_{D_5} until we have set this value. Furthermore, we likely want to keep this parameter free so that we have some freedom when we try to express $R_7^{(2)}$ in terms of f_{D_5}.

The piece that does not cancel in the full E_6 sum can be represented in terms of 13 A_3's (maybe less, I haven't done an exhaustive check). First, let us define some notation:

$$x_{i_1 \ldots i_k} = \sum_{a=1}^{k} \prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1} x_{i_2} + \ldots + x_{i_1} \cdots x_{i_k} \tag{2}$$

The following 8 enter with coefficient $+1/2$:

$$x_2 \to x_3 \to x_5, \quad x_1 \to x_2 \to x_3\left(1+x_5\right), \quad \frac{x_1x_2}{1+x_1} \to x_3 \to x_4, \quad x_1\left(1+x_2\right) \to \frac{x_2x_3\left(1+x_4\right)}{1+x_2} \to x_5,$$

$$\frac{1}{x_4} \to x_3\left(1+x_4\right) \to x_5, \quad \frac{1+x_3}{x_3x_4} \to x_2(1+x_{34}) \to \frac{x_3x_5}{1+x_3},$$

$$\frac{1+x_{23}}{x_2x_3x_4} \to x_1(1+x_{234}) \to \frac{x_2x_3x_5}{1+x_{23}}, \quad \frac{x_1x_2x_3x_5}{1+x_{123}} \to \frac{\left(1+x_1\right)x_3x_4}{\left(1+x_3\right)\left(1+x_{1234}\right)} \to \frac{1+x_{123}}{\left(1+x_1\right)x_3},$$

and these 5 enter with coefficient $-1/2$:

$$x_2 \to x_3 \to x_4, \quad x_1 \to x_2 \to x_3\left(1+x_4\right), \quad \frac{x_1x_2}{1+x_1} \to x_3 \to x_5,$$

$$x_1\left(1+x_2\right) \to \frac{x_2x_3\left(1+x_5\right)}{1+x_2} \to x_4, \quad \frac{x_1x_2x_3x_4}{1+x_{123}} \to \frac{\left(1+x_1\right)x_3x_5}{\left(1+x_3\right)\left(1+x_{1235}\right)} \to \frac{1+x_{123}}{\left(1+x_1\right)x_3}.$$

There is not any magical cancellation of terms at the level of $B_2 \wedge B_2$ for this sum of functions. It would be exciting to find a representation for f_{D_5} which relied on very few $B_2 \wedge B_2$ terms, but of course some relatively large number of terms will be necessary in order to full capture the D_5 symmetries.

Also note that it does not seem likely that there is a representation that includes ALL A_3's in D_5 (with nice coefficients, at least).

I don't have a nice representation of the piece of the function that cancels in the full E_6 sum. The shortest representation I have involves 17 A_3's.

Representing $R_7^{(2)}$ in terms of f_{D_5}

First, I'll describe E_6. The seed cluster can be written as

$$\begin{tikzcd} & & x_4 & & \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[u] & x_5 \arrow[l] & x_6 \arrow[l] \end{tikzcd} \tag{3}$$

An equivalent $\mathrm{Gr}(4,7)$ seed is:

$$
\begin{tikzcd}
 & & -\frac{\langle 4(12)(35)(67)\rangle}{\langle 1234\rangle\langle 4567\rangle} & & \\
\frac{\langle 1234\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1246\rangle} \arrow[r] & -\frac{\langle 1247\rangle\langle 3456\rangle}{\langle 4(12)(35)(67)\rangle} \arrow[r] & \frac{\langle 1246\rangle\langle 5(12)(34)(67)\rangle}{\langle 1245\rangle\langle 1267\rangle\langle 3456\rangle} \arrow[u] & -\frac{\langle 1267\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 4(12)(35)(67)\rangle} \arrow[l] & \frac{\langle 1567\rangle\langle 2345\rangle}{\langle 5(12)(34)(67)\rangle} \arrow[l]
\end{tikzcd}
\tag{4}
$$

The symmetries are σ (period 7), τ, and $\mathbb{Z}_2$ (both period 2) and can be represented as:

$$
\sigma: \quad x_1 \mapsto \frac{1}{x_6\,(1+x_{534})}, \quad x_2 \mapsto \frac{1+x_{6534}}{x_5\,(1+x_{34})}, \quad x_3 \mapsto \frac{(1+x_{234})\,(1+x_{534})}{x_3\,(1+x_4)},
$$

$$
x_4 \mapsto \frac{1+x_{34}}{x_4}, \quad x_5 \mapsto \frac{1+x_{1234}}{x_2\,(1+x_{34})}, \quad x_6 \mapsto \frac{1}{x_1\,(1+x_{234})},
$$

$$
\tau: \quad x_1 \mapsto \frac{x_5}{1+x_{65}}, \quad x_2 \mapsto (1+x_5)\,x_6, \quad x_3 \mapsto \frac{(1+x_{12})\,(1+x_{65})}{x_1 x_2 x_3\,(1+x_4)\,x_5 x_6},
$$

$$
x_4 \mapsto x_4, \quad x_5 \mapsto x_1\,(1+x_2)\,, \quad x_6 \mapsto \frac{x_2}{1+x_{12}},
$$

$$
\mathbb{Z}_2: \quad x_1 \leftrightarrow x_6, \quad x_2 \leftrightarrow x_5
$$

These are directly equivalent to $\mathrm{Gr}(4,7)$ cycle^2, flip, and parity ($\sigma = \text{cycle}^2$ because the map for just a single cycle was too cumbersome to print).

The simplest D_5 subalgebra of E_6 is obtained by freezing the x_6 node and then mutating once on x_5, at which point we have the seed cluster

$$
\begin{tikzcd}
 & & & x_4 \\
x_1 \arrow[r] & x_2 \arrow[r] & \frac{x_3 x_5}{1+x_5} \arrow[ur] \arrow[dr] & \\
 & & & \frac{1}{x_5}
\end{tikzcd}
\tag{5}
$$

We'll refer to the cluster algebra generated by (5) as $D_5^{(0,0)}$. The remaining 13 D_5's in E_6 are generated by applying σ and τ. We can now label each D_5 via the number of σ's and τ's applied to $D_5^{(0,0)}$:

$$
D_5^{(i,j)} = \sigma^i \tau^j (D_5^{(0,0)}) \tag{6}
$$

We do not have to consider $\mathbb{Z}_2$ because

$$\mathbb{Z}_2(D_5^{(0,0)}) = D_5^{(5,1)}. \tag{7}$$

Note that by this I do not mean that $\sigma^5\tau$ acting on the cluster (5) equals $\mathbb{Z}_2$ acting on (5). I mean that $\mathbb{Z}_2$ acting on the complete D_5 algebra *generated by* (5) is equal to $\sigma^5\tau$ acting on the same algebra.

With this notation in place we can say that

$$R_7^{(2)} = \sum_{i=0}^{6}\sum_{j=0}^{1} f_{D_5^{(i,j)}}. \tag{8}$$

While this is the most technically correct way to phrase things, it is of course much more evocative to write

$$R_7^{(2)} = \sum_{D_5 \subset E_6} f_{D_5}, \tag{9}$$

which is ill-defined up to the sign in front of each f_{D_5}.

Constraining the remaining parameter

In the $7 \to 6$ collinear limit, the 14 f_{D_5}'s have the following behavior:

- $\{(0,0),(1,0),(1,1),(2,1),(3,0),(4,0),(4,1),(5,1)\}$ vanish identically
- $\{(0,1),(5,0),(6,0),(6,1)\}$ are non-zero, $(0,1) = -(5,0)$ and $(6,0) = -(6,1)$
- $\{(2,0),(3,1)\}$ each vanish if you set the remaining parameter to 1, and otherwise $(2,0) = -(3,1)$

In all of these cases, setting the limit to vanish is equivalent to setting the limit to be well-defined. Note that these results are for *after* I've set the 4 parameters in f_{D_5} to give $R_7^{(2)}$. If we relax that and then enforce as many f_{D_5}'s to vanish as possible, we see that in the first set ($\{(0,0),(1,0),\ldots\}$) only a single degree of freedom survives the collinear limit, so we have some motivation for setting this parameter without fitting to $R_7^{(2)}$. Getting $\{(2,0),(3,1)\}$ to vanish provides some motivation (though not 100% clear-cut) for determining the parameter that remains after fitting to $R_7^{(2)}$.

Also, it is interesting to note that $(0,1) = -(5,0)$ are related by $\mathbb{Z}_2$, $(6,0) = -(6,1)$ are related by τ, and $(2,0) = -(3,1)$ are related by $\sigma\tau$. However these relationships only hold after taking the collinear limit (in other words, I was hoping that, at least in the case of $(2,0)$

and $(3,1)$, the function multiplying the free parameter would cancel between the two before taking the collinear limit, but it does not. In fact one needs all 14 of the free parameter functions in order to get them to cancel out).

A brief idea for $\mathrm{Gr}(4,8)$

Are there E_6's in $\mathrm{Gr}(4,8)$ which have $\{v,z\}$ squares? What if I try the 42-term $\{v,z\}$ expression for $R_7^{(2)} \sim f_{E_6}$ and see if that evaluates to "good" things on (at least some of) the E_6's I've found in $\mathrm{Gr}(4,8)$.