In this note we catalog the subalgebra structure for the finite cluster algebras  $\subseteq E_6$ . These algebras are:  $A_2, A_3, A_4, D_4, A_5, D_5, E_6$ .

 $\underline{A_2}$  clusters: 5 a-coordinates: 5 x-coordinates: 10

 $A_3$  clusters: 14 a-coordinates: 9 x-coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	6	6
$A_1 \times A_1$	3	3

 $\underline{A_4}$  clusters: 42 a-coordinates: 14 x-coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	28	21
$A_1 \times A_1$	28	28
$A_3$	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

 $\underline{D_4}$  clusters: 50 a-coordinates: 16 x-coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	36	36
$A_1 \times A_1$	30	18
$A_3$	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

 $A_5$  clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	120	56
$A_1 \times A_1$	180	144
$A_3$	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
$D_4$	0	0
$A_4$	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

 $\underline{D_5}$  clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	180	125
$A_1 \times A_1$	230	145
$A_3$	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
$D_4$	5	5
$A_4$	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

 $\underline{E_6}$  clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	1071	504
$A_1 \times A_1$	1785	833
$A_3$	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
$D_4$	35	35
$A_4$	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
$D_5$	14	14
$A_5$	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

# Symbol Spaces on Cluster Algebras

Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic			
туре	integrable	Cluster-Adjacent	$\sigma^+ \tau^+$	$\sigma^+\tau^-$	$\sigma^- \tau^+$	$\sigma^-\tau^-$
$A_2$	19	14	2	0	0	0
$A_3$	55	40	6	2	1	5
$A_4$	125	90	9	3	0	0
$D_4$	163	109				
$A_5$	245	175	1	0	0	0
$D_5$	381	241				
$E_6$	_	573	_	_	_	_

## Automorphic $D_4$ symbols

$\underline{\sigma^+  au^+}$		$\sigma^+  au^-$	$\underline{\sigma^-  au^+}$	$\sigma^-  au^-$
	$\sigma_{S_3}^+$ $\sigma_{S_3}^-$	$\sigma_{S_3}^+$ $\sigma_{S_3}^-$	$\sigma_{S_3}^+$ $\sigma_{S_3}^-$	$\sigma_{S_3}^+$ $\sigma_{S_3}^-$
$ au_{S_3}^+$	7 0	$\tau_{S_3}^+ \ \ 2 \ \ \ 0$	$\tau_{S_3}^+ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \$	$\tau_{S_3}^+$ 4 0
$ au_{S_3}^-$	0 0	$\tau_{S_3}^{-3}$ 1 0	$ au_{S_3}^{-}$ 2 0	$\tau_{S_3}^{-}$ 0 0

#### Automorphic $D_5$ symbols

Composite Groups

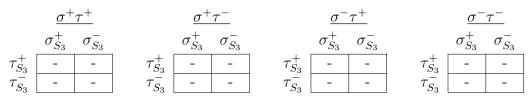
Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	3	3
$A_1 \times A_1 \times A_1$	6	6
$A_1 \times A_1 \times A_1 \times A_1$	10	10
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	15	15
$A_2 \times A_1$	25	20
$A_2 \times A_1 \times A_1$	32	27
$A_2 \times A_1 \times A_1 \times A_1$	40	35
$A_2 \times A_2$	63	53
$A_3 \times A_1$	65	50
$A_2 \times A_2 \times A_1$	74	64
$A_3 \times A_1 \times A_1$	76	61
$A_3 \times A_2$	119	99
$A_4 \times A_1$	140	105
$D_4 \times A_1$	180	126

- The number of integrable symbols at weight w in the product group  $\bigotimes_N A_1$  is given by  $\binom{N+w-1}{w}$ , i.e. 'N choose w (logs) with replacement'
- More generally, the number of weight-two integrable symbols in  $G_1 \times G_2$  is the number of symbols in  $G_1$  plus the number of symbols in  $G_2$ , plus the product of their respective numbers of A-coordinates (corresponding to taking a log from each)
- More generally, these integrable symbol counts just correspond to shuffling together all possible symbols drawn from each component group such that the total weight is w
- Presumably the automorphic counts also follow from shuffling together all possible symbols drawn from the component groups such that the total weight is w, and such that their automorphism properties under the component groups respect any new permutation symmetries (for instance, any products of symbols drawn from  $A_1$  and  $A_2$  will respect the automorphism group of  $A_1 \times A_2$ , but a product of symbols from  $A_2 \times A_2$  will have to respect a new symmetry that permutes the two component groups)
- I believe all of the above statements about integrable symbols directly translate to statements about cluster-adjacent symbols, because any product of symbols that are cluster-adjacent in each component group will also be cluster-adjacent in the composite group
- So this table (and the weight-four one) are just consistency checks

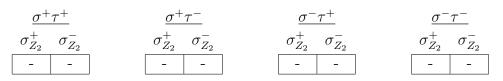
Weight Four

Trre	Integrable	Cluster Adiscont	Automorphic			
Type	integrable	Cluster-Adjacent	$\sigma^+ \tau^+$	$\sigma^+\tau^-$	$\sigma^- \tau^+$	$\sigma^-\tau^-$
$A_2$	211 (6)	81 (6)	11 (0)	6 (2)	0	0
$A_3$	1351 (36)	432 (36)	49 (2)	29 (4)	24 (2)	42 (4)
$A_4$	-	1652 (126)	131 (6)	105 (12)	0	0
$D_4$	-	2204				
$A_5$	-	-	-	-	-	-
$D_5$	-	-				
$E_6$	-	-	-	-	-	-

### Automorphic $D_4$ symbols



### Automorphic $D_5$ symbols



Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1$	15	15
$A_1 \times A_1 \times A_1 \times A_1$	35	35
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	70	70
$A_2 \times A_1$	301	136
$A_2 \times A_1 \times A_1$	423	218
$A_2 \times A_1 \times A_1 \times A_1$	585	335
$A_2 \times A_2$	1433	708
$A_3 \times A_1$	1701	622
$A_2 \times A_2 \times A_1$	1827	982
$A_3 \times A_1 \times A_1$	2127	873
$A_3 \times A_2$	4617	2088
$A_4 \times A_1$	-	
$D_4 \times A_1$	-	

#### How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank n-1 in a finite type cluster algebra of type X of rank n.

Let N be the number of cluster a-coordinates in the cluster algebra of type X.

Let n be the rank of cluster algebra of type X.

Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y.

The number of cluster algebras of type Y is then NZ/n.

As an example, let's count the number of  $A_4$ 's in  $A_5$ :

N = # of cluster a-coordinates in  $A_5 = 20$ 

 $n = \text{rank of } A_5 = 5$ 

Z=# of ways to remove a node from  $A_5$  and get an  $A_4=2$ 

 $20 \times 2/5 = 8 \Rightarrow$  there are 8  $A_4$ 's in  $A_5$ .