In any Dynkin cluster algebra generated by some seed $x_1 \to x_2 \to \ldots$, there will be other clusters in the algebra with the same Dynkin quiver. A trivial example is A_2, for which each cluster has a quiver diagram explicitly in the form $x_i \to x_j$. In A_3 seeded by $x_1 \to x_2 \to x_3$, the clusters

$$\begin{gathered} \frac{x_2}{x_2 x_1 + x_1 + 1} \to \frac{(x_1+1)\, x_3}{x_2 x_1 + x_2 x_3 x_1 + x_1 + 1} \to \frac{x_2 x_1 + x_1 + 1}{x_1 x_2 x_3}, \\ \frac{1}{x_1\,(x_3 x_2 + x_2 + 1)} \to \frac{x_2 x_1 + x_2 x_3 x_1 + x_1 + 1}{x_2\,(x_3+1)} \to \frac{x_3 x_2 + x_2 + 1}{x_3} \end{gathered} \tag{1}$$

also appear. We can think of these as "redundant" seeds since they will generate exactly the same algebra as $x_1 \to x_2 \to x_3$. Now if we are going to try and define a function "on" a cluster algebra, it makes sense that we would want our function to give the same result no matter which redundant seed we used. This motivates the definition:

a cluster polylog function $f_{\text{alg}}(\text{seed})$ *is "well-defined" if* $f_{\text{alg}}(\text{seed}) = \pm f_{\text{alg}}(\text{redundant seed})$

(the overall minus sign allows for redundant seeds to be "symmetric" or "anti-symmetric" w.r.t. the original seed). This definition may seem blisteringly obvious, but in fact it turns out be to a tool of heretofore unrealized potential in our cluster polylogarithm toolbelt. Let's see this in action.

The well-defined A_2 function

As a review of what is now completely ancient history, let us begin with the ansatz for f_{A_2}:

$$f_{A_2} = \sum_{i,j=1}^{5} a_{i,j}\{x_i\}_2 \wedge \{x_j\}_2 + b_{i,j}\{x_i\}_3 \otimes x_j. \tag{2}$$

This starts with 35 free parameters. Reducing to a linearly independent set by modding out Abel's identity takes us to 31 free parameters. Then,

- imposing integrability reduces us to 6 free parameters,
- and making sure it is well-defined leaves us with only 2 free parameters.

We can fix one more parameter by requiring that $f_{A_2}(x_1 \to x_2) = -f_{A_2}(1/x_2 \to 1/x_1)$, which gives the function an overall skew-dihedral symmetry (to be perfectly honest I don't entirely understand why this function should have this symmetry). Now we have only one remaining overall parameter which we set to one and are left with *the* A_2 function. We'll catalog our procedure with a table:

	generic	lin. indep.	integrable	sym. $B_2 \wedge B_2$	sym. $B_3 \otimes C$	asym. $B_2 \wedge B_2$	asym. $B_3 \otimes C$
dim. space of f_{A_2}	35	31	6	6	2	5	0

where the $B_3 \otimes C$ numbers refer to imposing $B_2 \wedge B_2$ *and* $B_3 \otimes C$. And here note that that there is no antisymmetric f_{A_2}, in other words there is no function for which $f_{A_2}(\text{seed}) = -f_{A_2}(\text{redundant seed})$.

The space of well-defined A_3 functions

Now we'll play the same game with the space of f_{A_3}'s. But crucially we must remember that the space of cluster functions on A_3 is spanned by the 6 f_{A_2}'s that live inside A_3. So another way of framing this is to talk about finding ways of linear combinations of f_{A_2}'s in A_3 that satisfy either the symmetric and antisymmetric well-defined property. Because it will be important later, I am going to adopt new notation to emphasize that I am constructing f_{A_3}'s in terms of f_{A_2}'s by denoting it as $f_{A_3 \supset A_2}$ (maybe there is better notation for this). The numbers here are:

	generic	lin. indep.	sym. $B_2 \wedge B_2$	sym. $B_3 \otimes C$	asym. $B_2 \wedge B_2$	asym. $B_3 \otimes C$
dim. space of $f_{A_3 \supset A_2}$	6	6	1	1	1	1

The symmetric f_{A_3} is just the straight sum of f_{A_2}'s and does not have local coproduct the way that the antisymmetric f_{A_3} does. In fact it is clear that you can always have a symmetrically well-defined function in this manner by just summing up all of the f_{A_2}'s inside any algebra – this is in some sense a "trivial" function, and I believe it will always be uninteresting (in the sense of not having a local coproduct) since we believe that the antisymmetric f_{A_3} is a basis for all local coproduct functions. Therefore we will ignore the symmetric f_{A_3} and refer to the antisymmetric f_{A_3} as *the* f_{A_3}.

The space of well-defined A_4 functions

I don't believe it is that revealing, but for completeness' sake let's catalog $f_{A_4 \supset A_2}$. The reason I don't think it is interesting is because the symmetric solution will just be the straight sum of f_{A_2}'s, and the antisymmetric solution will be the same as $f_{A_4 \supset A_3}$. But let's find out!

	generic	lin. indep.	sym. $B_2 \wedge B_2$	sym. $B_3 \otimes C$	asym. $B_2 \wedge B_2$	asym. $B_3 \otimes C$
dim. space of $f_{A_4 \supset A_2}$	21	21	3	3	0	0

The 3 symmetric solutions are sums of 7 f_{A_2}'s. Interestingly these each indendepently satisfy the "flip" symmetry of $x_i \to 1/x_{5-i}$, but also again do not have local coproduct so I'm not sure what good they'll be. And very interesting that there are *zero* valid antisymmetric functions! I believe this means that we must see no $f_{A_4 \supset A_3}$ functions, but let's verify:

			sym.		asym.	
	generic	lin. indep.	$B_2 \wedge B_2$	$B_3 \otimes C$	$B_2 \wedge B_2$	$B_3 \otimes C$
dim. space of $f_{A_4 \supset A_3}$	7	6	0	0	0	0

Indeed – there are no possible well-defined local f_{A_4}! This is very surprising. Hopefully it does not continue to be the case for other algebras.

The space of well-defined D_4 functions

Again for completeness' sake let's catalog $f_{D_4 \supset A_2}$:

			sym.		asym.	
	generic	lin. indep.	$B_2 \wedge B_2$	$B_3 \otimes C$	$B_2 \wedge B_2$	$B_3 \otimes C$
dim. space of $f_{D_4 \supset A_2}$	36	34/36	9	9	11	9

The 34/36 in the lin. indep. column indicates that there actually two relations in $B_2 \wedge B_2$, but they are not relations in $B_3 \otimes C$ (i.e. they are equivalent up to Li_4's). Not sure what to make of the 9 symmetric and antisymmetric functions. Maybe the story is clearer for $f_{D_4 \supset A_3}$:

			sym.		asym.	
	generic	lin. indep.	$B_2 \wedge B_2$	$B_3 \otimes C$	$B_2 \wedge B_2$	$B_3 \otimes C$
dim. space of $f_{D_4 \supset A_3}$	12	9/11	2	2	5	3

Again, the 9/11 refers to the fact that there are 3 relationships in $B_2 \wedge B_2$, but only one in $B_3 \otimes C$. The identities are all "on the nose" – i.e. not using Abel's identity or the non-trivial Li_3 identity in D_4, instead they are just of the form $x - x = 0$. (And I should be clear that when I impose linear independence I consider the full function, not just relationships amongst $B_2 \wedge B_2$, so in this case there are 11 linearly independent functions).

The two symmetric $f_{D_4 \supset A_3}$ both have $B_2 \wedge B_2 = 0$, so they are uninteresting. The three antisymmetric $f_{D_4 \supset A_3}$ have non-zero $B_2 \wedge B_2$ and are thus very interesting. What other symmetries can we impose?

The space of well-defined D_5 functions

Skipping ahead to $f_{D_5 \supset A_3}$ in the hopes of finding something interesting!

			sym.		asym.	
	generic	lin. indep.	$B_2 \wedge B_2$	$B_3 \otimes C$	$B_2 \wedge B_2$	$B_3 \otimes C$
dim. space of $f_{D_5 \supset A_3}$	65	42/51	13	5	15	6