Working through subalgebra-constructibility for $A_2 \subset A_3$

In this section we determine the space of weight-4 cluster polylogarithms associated with the A_3 cluster algebra.

A comment on notation: when we refer to simply "A_3 functions", we are implicitly referring to $A_2 \subset A_3$ functions, i.e. using the A_2 subalgebras to construct a function. This is the weakest form of subalgebra constructibility since A_2 functions, evaluated across all A_2 subalgebras of a larger algebra, form a basis for all cluster polylogarithms on the larger algebra. This is why we don't explicitly refer to A_2. However in the future we will look at, for example, $A_3 \subset D_5$ functions, and in that case we will explicitly refer to the fact that we are using A_3 functions as building blocks (although they are of course already built out of A_2's).

Begin with the six A_2 subalgebras in A_3 – as discussed in a previous section, there are multiple ways to think of these subalgebras. The simplest is that these are the $\binom{6}{5}$ pentagons embeddable in a hexagon. These are also present as the six pentagonal faces in the A_3 associahedron, fig. (ref). In practice it is most practical to list the six distinct A_2 seeds that appear in the 14 cluster generated by mutating on the standard A_3 seed, $x_1 \to x_2 \to x_3$:

$$\begin{gathered} x_1 \to x_2, \quad x_2 \to x_3, \quad \frac{x_2}{1+x_{12}} \to \frac{(1+x_1)\,x_3}{1+x_{123}}, \\ \frac{x_1 x_2}{1+x_1} \to x_3, \quad x_1\,(1+x_2) \to \frac{x_2 x_3}{1+x_2}, \quad x_1 \to x_2\,(1+x_3)\,. \end{gathered} \tag{1}$$

We now construct an ansatz for f_{A_3} as a sum of f_{A_2} evaluated on these six subalgebras:

$$f_{A_3} = c_1 f_{A_2}(x_1 \to x_2) + \ldots + c_6 f_{A_2}(x_1 \to x_2\,(1+x_3)). \tag{2}$$

Due to the inherited properties of f_{A_2}, the above ansatz already has a cluster-y cobracket, satisfies cluster adjacency, and is smooth and real-valued in the positive domain. The remaining problem to solve is to find values for the c_i such that f_{A_3} is invariant (up to an overall sign) under the automorphisms of A_3. These were discussed in a previous section (ref), we write them down explicitly here:

$$\begin{aligned} &\sigma_{A_3}: \quad x_1 \to \frac{x_2}{1+x_1+x_1x_2}, \quad x_2 \to \frac{x_3(1+x_1)}{1+x_1+x_1x_2+x_1x_2x_3}, \quad x_3 \to \frac{1+x_1+x_1x_2}{x_1x_2x_3}, \\ &\tau_{A_3}: \quad x_1 \to \frac{1}{x_3}, \quad x_2 \to \frac{1}{x_2}, \quad x_3 \to \frac{1}{x_1}. \end{aligned} \tag{3}$$

Recall that σ is essentially the cyclic automorphism on the hexagon, and τ is the dihedral flip.

The first thing to try is to make f_{A_3} completely invariant under the A_3 automorphisms:

$$\sigma_{A_3}(f_{A_3}) = f_{A_3}, \quad \tau_{A_3}(f_{A_3}) = f_{A_3}. \tag{4}$$

However it turns out there is no collection of c_i that solves such a constraint. If instead we impose

$$\sigma_{A_3}(f_{A_3}) = f_{A_3}, \quad \tau_{A_3}(f_{A_3}) = -f_{A_3}, \tag{5}$$

Then we find the solution

$$c_i = \text{constant.} \tag{6}$$

We label this particular solution by

$$f_{A_3}^{+-} = f_{A_2}(x_1 \to x_2) + \ldots + f_{A_2}(x_1 \to x_2\,(1+x_3)) = \sum_{i=1}^{6} \sigma_{A_3}^i \big(f_{A_2}(x_1 \to x_2)\big), \tag{7}$$

where the signs indicate behavior under σ and τ, respectively. Similarly, based on the lack of a solution for eq. (4) we say that $f_{A_3}^{++} = 0$. There are two remaining sign choices to check, $f_{A_3}^{-+}$ and $f_{A_3}^{--}$, and we find $f_{A_3}^{-+} = 0$ and

$$f_{A_3}^{--} = \sum_{i=1}^{6} (-1)^i \sigma_{A_3}^i \big(f_{A_2}(x_1 \to x_2)\big). \tag{8}$$

In conclusion, there are choices for f_{A_3}, which we denote by their behavior under the dihedral cycle and flip automorphisms: $f_{A_3}^{+-}$ and $f_{A_3}^{--}$. These functions arise purely from the interplay between the overall symmetries of the A_3 cluster algebra and the structure of the A_2 subalgebras in A_3, i.e. there has been no physics input so far. We can tabulate our results in the following table:

	$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_3	0	1	0	1

(9)

This displays the total number of weight-4 polylogarithm functions associated with the A_3 cluster algebra with the given sign changes under the σ and τ automorphisms.