Questions and Comments from Sasha Goncharov

With comments from Marcus Spradlin (MS) and Miguel Paulos (MP).

1. The motivic function $L_{2,2}(x, y)$ coincides with the motivic function $\kappa(x, y)$ introduced in formula 4.5 in "Polylogarithms and motivic Galois group" paper modulo Li_4's, up to $\mathrm{Li}_4(-y/x)$ term.

   Is it better to have $\mathrm{Li}_4(-y/x)$ summand?

2. What geometry responsible for linear relations between A_2 and A_3 functions?

3. So we have a "function"

   pentagons on the cluster polytope $\longmapsto B_2 \wedge B_2$

   Let us try to find ANY linear combinations of pentagons on the cluster polytope such that the resulting $B_2 \wedge B_2$ function vanishes.

   Any such linear combination is of huge interest.

   For example – what is the $B_2 \wedge B_2$ component of the sum of pentagon functions related to $A_2 \times A_1$ – this must be trivial, but yet ...

   Then: we end up with a "function" which assigns to 3d faces of the cluster polyhedron these $B_2 \wedge B_2$ invariants.

4. Observe that cluster polytope for 7 points is a polytope, that is topologically trivial.

   Take the sum of (geometric) pentagons of the cluster poytope which make the 7-point amplitude.

   Take its "differential", that is the sum of their boundary 1-segments.

   It follows that the $B_2 \wedge B_2$ content of the amplitude depends only on this chain.

   Q: what is this 1-chain?

   This mean: are there pentagons in the formula which share a common side with different orientations? I guess plenty, so is there any simpler way to describe the 1-dim boundary then just saying it is a boundary of those pentagons?

   Before we described it as some "Poisson commuting" pairs, that is rectangles in the cluster polytope.

   (This might be silly: taking the boundary of the sum of pentagons we get the naked $B_2 \wedge B_2$ term:

   $\{x\}_2 \wedge \{y\}_2$ means the edge (x, y).)

5. What does one get if you take alternating sum of 7 6-point functions, i.e. start from 7 points on P^1, forget them one by one, and take the alternating sum of the obtained 6 point amplitudes?

Although it is clear that pentagon function is a "primary" or "atomic" object of the story, it is not 100% clear to me what are the "quarks" in which even the 6 point amplitude is decomposed.

So far it looks that these "quarks" are the 6 pentagons,

Notice however that the boundary of the 3d Stasheff polytope has not only 6 pentagons, but also 3 squares, and these squares sort of match the 2+2+2 terms in the 9 term formula.

6. Comment from MS

You often speculated that some role might be played by functions whose $B_2 \wedge B_2$ component *is* the Poisson bracket, as in

$$\sum_{i \in \text{ clusters}} \quad \sum_{\text{coordinates } x,y \text{ in cluster } i} \{x, y\} x_2 \wedge y_2$$

where $\{x, y\}$ is the Poisson bracket. We have not found a role for things like this to play in the amplitude story, but it might be interesting to note that the A_2 and A_3 cluster functions seem to have $B_3 \otimes C^*$ component which is the Poisson bracket, that means it can be written as

$$\sum_{i \in \text{ clusters}} \quad \sum_{\text{coordinates } x,y \text{ in cluster } i} \{x, y\} \{x\}_3 \otimes y$$

In particular, there is a sense in which they have $B_3 \otimes C^*$ components $\{a\}_3 \otimes b$ which are purely antisymmetric under exchange of a and b. (Though I don't know how to make this property well-defined because to rewrite the C^* component using multiplicative identities which B_3 doesn't have).

Goncharov replies: I do not know how to formulate a Poisson property of $a_3 \otimes b$ since $\{a\}_3 = \{a^{-1}\}_3$.

In fact, one reason of talking about Poisson properties before was a a way to express / formalize the idea that the terms which appear in $B_2 \wedge B_2$ should not be "far away" in the cluster polytope. Similar hope, of course, for $B_3 \otimes C^*$.

7. The main question I have is this:

Q. What are all relations between the $B_2 \wedge B_2$ parts of the pentagon functions?

Precisely:

Q1) Are there any "non-trivial", (and also, just any) relations between $B_2 \wedge B_2$ parts of the pentagon functions? "Non-trivial" means the proof involves the 5-term identities: the cancellation holds for the sum in $B_2 \wedge B_2$, but not on in $Z[P^1]$, or just assuming $\{x\}_2 = -\{x^{-1}\}_2$.

I imagine there are some "basic" relations from which any other should follow. The question is well posed for any cluster variety: Take all pentagons one can find there, and study linear relations between $B_2 \wedge B_2$ parts of the pentagon functions of those pentagons. It is a finite problem for finite type, but well posed problem even in general.

WHY IMPORTANT:

1) Any linear relation for $B_2 \wedge B_2$ parts of the pentagon functions must be upgraded then to an identity

sum of the actual pentagon functions is a sum of Li_4's of some variables.

It would be terrific if this variables of Li_4 can be taken cluster variables – so in finite cases, like D_4, E_6 one can just ask computer.

So the question is, do we have any non-trivial identity?

2) In the next step (I skip explanations) one gets functional equations for 4-logs. They will be "Buld on" functional equations for 3-logs, presumably the 40 term ones.

I hope that all questions have answers understood via geometry of the cluster polytope.

8. Although for the amplitude A_3 function may be much better, the pentagon functions are probably more fundamental.

   We were looking for a loong time for a simple 2-variable weight 4 function with simple coproduct (testing versions of Omega-functions of Dixon) but now it is clear that it is the pentagon function.

9. Comment from MP

   - In A_3 there are no relations between pentagon functions.

   - In A_4 there are no relations between pentagon functions except "trivial ones": there are 28 pentagonal faces but only 21 distinct A_2 subalgebras because there are things like $A_1 \times A_2$ in the polytope. These 21 A_2s lead to 21 independent pentagon functions.

   - In D_4 there are *a priori* 36 pentagon functions, but only 34 of them are independent. These two relations depend non-trivially on 5 term identities among B_2 elements on the $B_2 \wedge B_2$ part; and on the 40 term Li_3 identity on the $B_3 \otimes C^*$ part. Similarly there are 12 independent A_3 subalgebras, and corresponding 12 A_3 functions - of which only 9 are independent. These three relations surely follow from the relations among pentagons but I haven't worked out the details.

   - Regarding identities between A_3 functions: the two identities between the 36 pentagon functions involve 24 pentagons each. One can understand the origin of these identities quite beautifully in terms of the A_3 functions. As said before there are 12 A_3 functions, but only 9 independent. Each of the 3 identities involves 4 A_3 functions - and since each A_3 has 6 pentagons one gets the 24. These identities are quite pretty geometrically:

each A_3 has three squares, so visualize each of them as a triangle with squares at the vertices. Then we can get an identity by taking four A_3s and gluing them together by their squares in such a way as to get a tetrahedron.

Algebraically, the $B_2 \wedge B_2$ components of the 4 A_3 functions share terms pairwise, and adding them up with the right sign they all cancel.

I have checked that the 3 A_3 identities generate the 2 independent pentagon identities. The fact that there are 2, not 3 identities then means that the three tetrahedra must be also glued up somehow by their pentagons.

The conclusion is that the seemingly non-trivial identities among pentagon functions are better understood as simple geometric consequences of gluing up A_3's, where they become trivial.

10. How to see geometrically the 4 A_3 polytopes which produce a relation?

    Means: D_4 means a configuration of 6 points on P^2.

    Each A_3 polytope corresponds to certain configuration of 6 points on P^1. So a relation corresponds to cerrtain way to produce FOUR configurations of 6 points on P^1 from configurations of 6 points on P^2. How to describe it geomtrically, in terms of the projective geometry? (Like take two lines formed by the two pairs of points, find their intersection etc.)

    This relations in D_4 should lead to CLUSTER functional equations for Li_4. One way is probably to take the amplitude for 7 points, it has several presentations. They are equal and thus the sum of the corresponding A_3 functions should be equal, modulo Li_4's entering the game. This means a CLUSTER relation for Li_4 - EXTREMELY VALUABLE.

11. <u>Comment from MS</u>

    (1) Sasha has a proposal for how to write down "pentagon" (or A_3) functions for weight > 4. Let $f_4(x, y)$ be your favorite weight 4 function, then we consider the following recursive way to build up "cluster polylogarithm" functions at weight > 4, which we call f_k. The idea is just to make the following ansatz for the coproduct

$$\delta f_k(x, y) = f_{k-1}(x, y) \otimes x/y + \text{ stuff}$$

    where "stuff" is to be determined by imposing the condition that δ should annihilate the right-hand side. The natural ansatz, given the cluster structure, is that

$$\text{stuff } = \sum_{i,j,m} x_{im} \wedge x_{j_{k-m}}$$

The sum is somewhat schematic: i, j run over the cluster coordinates, and m runs from 2 to $k-2$. The sum just indicates all possible terms which could possibly appear there. The idea is to make an ansatz (put a free coefficient in front of each term) and see what works; that is, solve that δ should kill the right-hand side. This might give a recursive way to define cluster polylogarithms for all higher weight!

(2) Sasha remains interested in finding non-trivial Li_4 cluster identities. Right now we don't know any. One reason we need to understand all possible identities is that in doing calculations like the one above, δ can only annihilate the right-hand side due to nontrivial identities. So first we should understand what kinds of identities exist!

(3) An earlier discussion wish Sasha reminded us that we really need to start looking at collinear limits. Presumably only very special A_3 functions will be well-defined under collinear limits; random ones won't be.

(4) It might also be possible, more conjecturally, to naturally assign functions to higher algebras using a recursive method like the one described in (1) above. For example, if you wanted to assign a weight 5 function to an A_4 algebra, that means you would have to propose an element of $L_4 \otimes C^*$ and an element of $B_3 \otimes B_2$, which satisfy the integrability. You could try some kind of ansatz where the L_4 is given by some sum of pentagon functions, and for the $B_3 \otimes B_2$ maybe some kind of A_2 and A_1 subalgebras. Well I'm not being very clear but the idea is maybe to recursively define higher weight functions based on geometric data, in terms of lower weight functions associated to subalgebras.

12. Here is a question on cluster nature of the funct eq. for 3-log.

Observe that:

1. Cluster $\mathcal{A}$-coordinates are assigned to the top dimensional faces of the cluster polytope.

2. Cluster $\mathcal{X}$-coordinates are assigned to the oriented edges of the cluster polytope.

In the D_4 case the codimension one cells are A_3 polytopes or $A_2 \times A_1$ polytopes. The best way to think about them geometrically - as of cluster $\mathcal{A}$-coordinates.

So the 40-term funct. eq. means that we have a sum

$$(\mathrm{Pentagon}_i) \otimes A_3 \text{ polytope}_i \quad \mathit{or} \quad A_2 \otimes A_1 \text{ polytope}_i$$

Question: what the coproduct of that 40 term identity looks like this way?

In our paper, I wrote it in the Appendix explicitly as Pentagon $\otimes$ $\mathcal{A}$-coordinate expression - so what it means geometrically?

13. Comment from MP

D_4 contains no $A_2 \times A_1$ but it does contain four cubes $A_1 \times A_1 \times A_1$ and 12 A_3's. As you say, to each of these objects there corresponds a unique $\mathcal{A}$-coordinate - but these are reflected in the multiplicatively independent $\mathcal{X}$-coordinates, of which there are indeed 16. So, I investigated the coproduct of the 40 term identity and one gets a sum of terms of the form

$$\sum \text{pentagon} * \text{multiplicative independent } \mathcal{X}\text{-coordinate}$$

However, not all possible $\mathcal{X}$-coordinates appear! Out of the 16 only 12 appear, which leads me to conjecture that the desired expression is

$$\sum \text{pentagon} \otimes A_3$$

It is not clear to me however how the association goes: which pentagon(s) go with which A_3? To see this I'll have to write the $\mathcal{X}$-coordinates in terms of the $\mathcal{A}$-coordinates to make the map precise.

14. I mean try to understand contributions of the $\mathcal{A}$-coordinates, since they are multiplicatively independent. A set of independent $\mathcal{X}$-coordinates is not quite well defined.

15. Let us take the pentagon functions assigned to all pentagons on the cluster polytope of type E_7, that is $\mathrm{Conf}_7(P^3)$.

Let us skewsymmetrise each of these pentagon functions.

1) How many different skewsymmetric functions on $\mathrm{Conf}_7(P^3)$ we get this way?

2) If not that many, can one list them?

To clarify:

Take 7 points in P^3, so far cyclically ordered. The space of such cyclically ordered 7-tuples has E_7 type cluster structure.

So there are plenty pentagons on its Stasheff polytope. Take one of them. This means that given 7 points in P^3, $(z_1, ..., z_7)$ we created a configuration of 5 points on P^1, say described by two cluster coordinates

$$X(z_1, ..., z_7), Y(z_1, ..., z_7)$$

Now make pentagon "function" meaning element in

$$B_3 \otimes C^* + B_2 \wedge B_2$$

– depending on those $(z_1, ..., z_7)$.

Skew symmetrise it means apply 7! permutations to $(z_1, ..., z_7)$, for each of them calculate the $B_3 \otimes C^* + B_2 \wedge B_2$ invariant, and take sum with signs.

In practice most of them immediately die, so we get zero.

So some of these pentagon are distinguished by surviving skew symmetrization.