

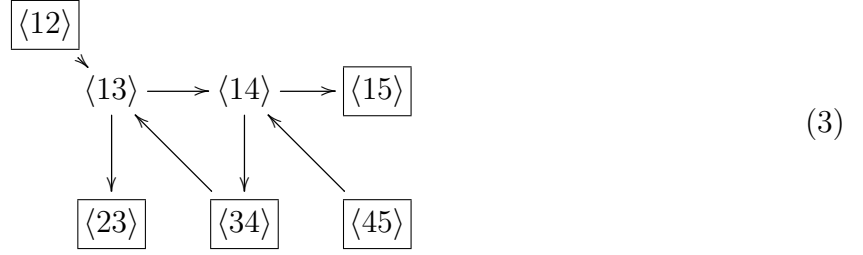
The basic idea for generating a well-defined Poisson bracket on \mathcal{A} -coordinates is to start with some seed (in terms of \mathcal{A} -coordinates) with standard adjacency matrix B written as a $n \times k$ matrix (this the case when you have n total nodes with k mutable). Then find some skew-symmetric $n \times n$ matrix Ω such that

$$\Omega B = \begin{pmatrix} \mathbb{1}_{k \times k} \\ 0_{(n-k) \times k} \end{pmatrix} \quad (1)$$

Then the claim is that the entries ω_{ij} form a Poisson bracket on \mathcal{A} -coordinates, i.e.

$$\{\mathcal{A}_i, \mathcal{A}_j\} = \omega_{ij} \mathcal{A}_i \mathcal{A}_j \Rightarrow \{\bar{\mathcal{A}}_i, \bar{\mathcal{A}}_j\} = \bar{\omega}_{ij} \bar{\mathcal{A}}_i \bar{\mathcal{A}}_j \quad (2)$$

where $\bar{}$ indicates mutation. The interesting feature here is that imposing eq. (1) does not entirely constrain Ω . For example, let's take a seed from $\text{Gr}(2, 5)$:



Associating $\langle 13 \rangle$ and $\langle 14 \rangle$ with nodes 1 and 2, resp., and then $\langle 12 \rangle, \dots, \langle 15 \rangle$ with nodes 3, \dots , 7 we have

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

in which case Ω takes the form

$$\Omega = \begin{pmatrix} 0 & c_{13} - c_{14} + c_{15} - 1 & c_{13} & c_{14} & c_{15} & c_{16} & -c_{15} \\ -c_{13} + c_{14} - c_{15} + 1 & 0 & c_{23} & c_{24} & c_{24} - c_{23} & c_{26} & -c_{13} + c_{14} - c_{15} + c_{23} - c_{24} \\ -c_{13} & -c_{23} & 0 & c_{34} & c_{34} - c_{23} & c_{36} & -c_{13} + c_{23} - c_{34} \\ -c_{14} & -c_{24} & -c_{34} & 0 & c_{34} - c_{24} & c_{46} & -c_{14} + c_{24} - c_{34} \\ -c_{15} & c_{23} - c_{24} & c_{23} - c_{34} & c_{24} - c_{34} & 0 & c_{26} - c_{36} + c_{46} & -c_{15} \\ -c_{16} & -c_{26} & -c_{36} & -c_{46} & -c_{26} + c_{36} - c_{46} & 0 & -c_{16} + c_{26} - c_{36} + c_{46} \\ c_{15} & c_{13} - c_{14} + c_{15} - c_{23} + c_{24} & c_{13} - c_{23} + c_{34} & c_{14} - c_{24} + c_{34} & c_{15} & c_{16} - c_{26} + c_{36} - c_{46} & 0 \end{pmatrix}$$

The c_{ij} are arbitrary, and I guess we really only care about the 2×2 entries in the upper left corner but I present the full matrix for completion's sake.

The overall point here is that any choice of the c_{ij} defines a Poisson bracket on the \mathcal{A} -coordinates, and this Poisson bracket then satisfies all of the properties that the normal

\mathcal{X} -coordinate Poisson bracket does (invariant under mutation¹, etc). The tradeoff is straightforward: in \mathcal{X} -coordinate language one has a single Poisson bracket, but there is no single representation of a symbol in terms of \mathcal{X} -coordinates . In contrast, there is a single representation of a symbol in terms of \mathcal{A} -coordinates but there is not a single Poisson bracket on \mathcal{A} -coordinates . This of course jives with the fact that \mathcal{A} -coordinates can appear together in different clusters with different b_{ij} elements.

I believe then that the Sklyanin bracket is just one of the many possible Poisson brackets one can define on \mathcal{A} -coordinates , but I do not believe it is “privileged” in any way. It is also worth emphasizing that one can evaluate the Sklyanin bracket directly on \mathcal{A} -coordinates , i.e. no need to dress them in to DCI cross-ratios.

¹Unfortunately each Poisson bracket on \mathcal{A} -coordinates has a different set of mutation rules that one just has to work out. Gekhtman said that this is, in principle, not very illuminating. I have not tried it myself.