A_3-constructible function counts

In this section we extend the subalgebra-constructibility results to the more intricate case of $A_2 \subset A_3$-constructible functions.

For A_3 there are two possible functions, as described in a previous section: $f^{+-}_{A_3}$ and $f^{--}_{A_3}$. We will work throught the case of $f^{--}_{A_3}$ as it is known to be connected with $R^{(2)}_7$ [], however it remains an interesting open question to find applications for $f^{+-}_{A_3}$-constructible functions.

The algorithm described in the previous section applies, changing only $A_2 \to A_3$:

- begin with an ansatz of all $f^{+-}_{A_3}$ applied across all A_3 subalgebras of a given larger algebra,
- impose that the overall function is invariant under automorphisms up to overall sign choices,
- count the number of solutions to these constraints.

To reiterate, the functions that satisfy these constraints are decomposable in to both A_2 and A_3 building blocks, thus making contact with multiple layes of the intricate cluster algebraic structure.

There are no $A_3^{--} \subset A_4$ or $A_3^{--} \subset D_4$ functions. For A_5 we have

$$\begin{array}{c|c|c|c|c} & \sigma^+\tau^+ & \sigma^+\tau^- & \sigma^-\tau^+ & \sigma^-\tau^- \\ \hline A_3^{--} \subset A_5 & 1 & 1 & 0 & 3 \end{array} \tag{1}$$

For D_5 we have

$$\begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^+\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^+\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^-\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^-\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \tag{2}$$

And finally E_6 gives

$$\begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^+\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{8} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^+\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{11} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^-\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^-\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \tag{3}$$