A_2-constructible function counts

In this section we introduce and describe the results for $A_2 \subset A_3$ constructibility in the finite algebras of interest.

Here we follow the steps worked through in the previous section, namely

- begin with an ansatz of all f_{A_2} applied across all A_2 subalgebras of a given larger algebra,
- impose that the overall function is invariant under automorphisms up to overall sign choices,
- count the number of solutions to these constraints.

In the case of A_n algebras we only have to impose the dihedral automorphisms σ_n and τ_n, and so the resulting table is exactly analogous to eq. (1):

	$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_3	0	1	0	1
A_4	0	3	0	0
A_5	2	5	2	5

(1)

It is interesting to note that the $+-$ sign choice admits at least one solution for all A_n studied so far, and that is the only sign choice for A_4.

As discussed in sec. (automorphisms), D_4 has the automorphism group $D_4 \times S_3$, with two cyclic generators $\sigma^{(4)}_{D_4}$ and $\sigma^{(3)}_{D_4}$ (corresponding to the D_4 and S_3, respectively) and then the D_4 flip τ_{D_4} as well as the S_3 flip denoted $\mathbb{Z}_{2,D_4}$. In the following table these are abbreviated $\sigma_4, \sigma_3, \tau_4, \mathbb{Z}_2$, respectively. Because of the four automorphism generators, there are 16 possible sign choices to impose on the collection of 36 distinct f_{A_2}'s in D_4.

$\sigma_4^+\tau_4^+$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	0	0
$\mathbb{Z}_2^-$	1	0

$\sigma_4^+\tau_4^-$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	2	0
$\mathbb{Z}_2^-$	0	0

$\sigma_4^-\tau_4^+$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	1	0
$\mathbb{Z}_2^-$	1	0

$\sigma_4^-\tau_4^-$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	1	0
$\mathbb{Z}_2^-$	0	0

(2)

Here we again see that the space of functions satisfying automorphisms is remarkably constrained, with no functions exhibiting a sign flip under σ_3 and only one sign choice – $\sigma_4^+\tau_4^-\sigma_3^+\mathbb{Z}_2^+$ – that offers more than one possible solution.

D_5 is less symmetric than D_4, with only $D_5 \times \mathbb{Z}_2$ automorphism group (in the table the generators are labeled by σ, τ, and $\mathbb{Z}_2$). Therefore there are 8 possible sign choices to impose on the collection of 125 distinct f_{A_2}'s in D_5, and the resulting number of solutions is

$\sigma^+\tau^+$		$\sigma^+\tau^-$		$\sigma^-\tau^+$		$\sigma^-\tau^-$	
$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
5	0	10	0	0	3	0	7

(3)

Finally we turn to E_6, which has automorphism group D_{14} with generators σ, τ, and $\mathbb{Z}_2$). E_6 is much larger than the algebras studied so far, with 504 distinct A_2 subalgebras, however the space of automorphic functions is still quite constrained:

$$\begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^+\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{14} & \multicolumn{1}{c|}{16} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^+\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{23} & \multicolumn{1}{c|}{19} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^-\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underline{\sigma^-\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \tag{4}$$