

In this note we catalog the subalgebra structure for the finite cluster algebras $\subseteq E_6$.

These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

A_2 clusters: 5 a -coordinates: 5 x -coordinates: 10

A_3 clusters: 14 a -coordinates: 9 x -coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

A_4 clusters: 42 a -coordinates: 14 x -coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

D_4 clusters: 50 a -coordinates: 16 x -coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

A_5 clusters: 132 a -coordinates: 20 x -coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

D_5 clusters: 182 a -coordinates: 25 x -coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

E_6 clusters: 833 a -coordinates: 42 x -coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

Symbol Spaces on Cluster Algebras

Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	19	14	2	0	0	0
A_3	55	40	6	2	1	5
A_4	125	90	9	3	0	0
D_4	163	109				
A_5	245	175	1	0	0	0
D_5	381	241				
E_6	-	573	-	-	-	-

Automorphic D_4 symbols

$\overline{\sigma^+\tau^+}$		$\overline{\sigma^+\tau^-}$		$\overline{\sigma^-\tau^+}$		$\overline{\sigma^-\tau^-}$		
	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	7	0	$\tau_{S_3}^+$	2	0	$\tau_{S_3}^+$	4	0
$\tau_{S_3}^-$	0	0	$\tau_{S_3}^-$	1	0	$\tau_{S_3}^-$	0	0

Automorphic D_5 symbols

$\underline{\sigma^+\tau^+}$		$\underline{\sigma^+\tau^-}$		$\underline{\sigma^-\tau^+}$		$\underline{\sigma^-\tau^-}$	
$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$
22	0	11	0	0	4	0	12

Weight Four

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	211	81 (6)	11 (0)	6 (2)	0	0
A_3	1351	432 (36)	49 (2)	29 (4)	24 (2)	42 (4)
A_4	-	1652	131 (6)	105 (12)	0	0
D_4	-	-				
A_5	-	-	-	-	-	-
D_5	-	-				
E_6	-	-	-	-	-	-

Automorphic D_4 symbols

$\frac{\sigma^+\tau^+}{\sigma_{S_3}^+ \sigma_{S_3}^-}$		$\frac{\sigma^+\tau^-}{\sigma_{S_3}^+ \sigma_{S_3}^-}$		$\frac{\sigma^-\tau^+}{\sigma_{S_3}^+ \sigma_{S_3}^-}$		$\frac{\sigma^-\tau^-}{\sigma_{S_3}^+ \sigma_{S_3}^-}$																	
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Automorphic D_5 symbols

$\frac{\sigma^+\tau^+}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$		$\frac{\sigma^+\tau^-}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$		$\frac{\sigma^-\tau^+}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$		$\frac{\sigma^-\tau^-}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$	
-	-	-	-	-	-	-	-

How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank $n - 1$ in a finite type cluster algebra of type X of rank n .

Let N be the number of cluster a -coordinates in the cluster algebra of type X .

Let n be the rank of cluster algebra of type X .

Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y .

The number of cluster algebras of type Y is then NZ/n .

As an example, let's count the number of A_4 's in A_5 :

$N = \#$ of cluster a -coordinates in $A_5 = 20$

$n = \text{rank of } A_5 = 5$

$Z = \#$ of ways to remove a node from A_5 and get an $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$ there are 8 A_4 's in A_5 .