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Cluster Polylogarithms and the Eight-Particle MHV Amplitude

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Abstract: We construct a cluster-polylogarithimc representation of the eight-point two-loop MHV amplitude in the planar limit of maximally supersymmetric Yang-Mills theory. This representation makes manifest a novel cluster-algebraic decomposition of the nonclassical part of this amplitudes into its A_5 subalgebras, and limits smoothly to a similar decomposition of the seven-point MHV amplitude in collinear limits. We also investigate the equivalence of the extended Steinmann relations and cluster adjacency in eight-point kinematics by exploring the space of BDS-like normalized amplitudes.

Contents

1 Introduction

- emphasize the fact that promoting symbols to functions is hard—only a few other instances in the literature (don't forget this is done Regge limits as well—cite)

- must talk about the importance of automorphisms—let's become the standard physics reference on this!

- same for the Sklyanin bracket

- also, discuss the relation between cluster $\mathcal{A}$-adjacency and cluster $\mathcal{X}$-adjacency — can we prove these have to be equivalent by using the conversion $x \sim a^b$ between the two (since this translation is valid on any cluster)?

- mention existence of D_5 function and refer ahead

- we should also check our function against MRK predictions if possible (but don't want to hold up paper for this... clearly we can publish without)
- should point out somewhere that the cobracket is the same for the remainder function and bds-like normalized amplitudes—and that the same bootstrap procedure could be carried out for either quantity. However, we carry it out on the remainder function because there's no clear (unique) bds-like normalized amplitude to bootstrap

2 Cluster Algebras and Cluster Polylogarithms

2.1 The A2 function

We define the A_2 function as

$$f_{A_2} = \sum_{\text{skew-dihedral}} \mathrm{Li}_{2,2}(-x_i, -x_{i+2}) - \mathrm{Li}_{1,3}(-x_i, -x_{i+2}) - \mathrm{Li}_2(-x_i) \log(x_{i+1}) \log(x_{i+2}). \quad (2.1)$$

It has the symbol

$$- \sum_{\text{skew-dihedral}} x_i \otimes x_i \otimes x_{i+1} \otimes x_{i+1} + x_i \otimes x_{i+1} \otimes x_i \otimes x_{i-1} + x_i \otimes x_{i+1} \otimes x_{i+1} \otimes x_i \quad (2.2)$$

$$- 2 \left(x_i \otimes x_{i+1} \otimes (x_i x_{i+2}) \otimes x_{i+1} - x_i \otimes x_{i+1} \otimes x_{i+1} \otimes x_{i+2} \right) \quad (2.3)$$

And satisfies the properties:

- clustery cobracket
- cluster adjacent in A_2
- smooth and real-valued in the positive domain

Some cluster algebra rules:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i,j\}, \\ b_{ij}, & \text{if } b_{ik} b_{kj} \leq 0, \\ b_{ij} + b_{ik} b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik} b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases} \quad (2.4)$$

$$x'_i = \begin{cases} x_k^{-1}, & i = k, \\ x_i (1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases}. \quad (2.5)$$

$$\{x_i, x_j\} = b_{ij} x_i x_j. \quad (2.6)$$

$$\{x'_i, x'_j\} = b'_{ij} x'_i x'_j \quad (2.7)$$

In momentum twistor language we have the n momentum twistors Z_i, which together form the $4 \times n$ matrix

$$K = \begin{pmatrix} z_{11} & \dots & z_{n1} \\ z_{12} & \dots & z_{n2} \\ z_{13} & \dots & z_{n3} \\ z_{14} & \dots & z_{n4} \end{pmatrix}. \tag{2.8}$$

As long as the first 4 columns are non-singular, we can row reduce K in to the form

$$K' = \begin{pmatrix} 1 & 0 & 0 & 0 & y_{11} & \dots & y_{(n-4)1} \\ 0 & 1 & 0 & 0 & y_{12} & \dots & y_{(n-4)2} \\ 0 & 0 & 1 & 0 & y_{13} & \dots & y_{(n-4)3} \\ 0 & 0 & 0 & 1 & y_{14} & \dots & y_{(n-4)4} \end{pmatrix}. \tag{2.9}$$

The columns of K' define a new set of momentum twistors Z'_i, where for example $Z'_1 = \{1,0,0,0\}$ and $Z'_5 = \{y_{11}, y_{12}, y_{13}, y_{14}\}$. It is easy to check that

$$y_{ij} = (-1)^j \langle \{1,2,3,4\} \setminus \{j\}, i\rangle / \langle 1234\rangle, \tag{2.10}$$

$$\langle abcd\rangle' = \det(Z'_a Z'_b Z'_c Z'_d) = \langle abcd\rangle / \langle 1234\rangle. \tag{2.11}$$

You can then define the Sklyanin bracket as an operation on these y_{ij} by

$$\{y_{ij}, y_{ab}\} = (\operatorname{sgn}(a-i) - \operatorname{sgn}(b-j)) y_{ib} y_{aj}. \tag{2.12}$$

Which then extends to a bracket on functions of the y_{ij} via

$$\{f(y), g(y)\} = \sum_{i,a=1}^{n} \sum_{j,b=1}^{4} \frac{\partial f}{\partial y_{ij}} \frac{\partial g}{\partial y_{ab}} \{y_{ij}, y_{ab}\}. \tag{2.13}$$

Now if we want to evaluate the Poisson bracket between two $\mathcal{X}$-coordinates, we can instead treat them as functions of the y_{ij} and use eq. (2.13). To be precise, for each four-bracket $\langle abcd\rangle$ in the $\mathcal{X}$-coordinates, replace them with $\langle abcd\rangle'$ expanded out in terms of y_{ij} (e.g. $\langle 1256\rangle' = y_{13}y_{24} - y_{14}y_{23}$). Then you can calculate eq. (2.13) directly in terms of the y_{ij}

3 Promoting $R_8^{(2)}$ from Symbol to Function

3.1 The A_5 Function

- describe the A_5 cluster algebra
- what makes "the" A_5 function unique?

The A_5 cluster algebra is generated from the seed cluster

$$x_1 \to x_2 \to x_3 \to x_4 \to x_5. \tag{3.1}$$

The full A_5 algebra contains 132 clusters with 140 distinct $\mathcal{X}$-coordinates.

Define:

$$x_{i_1 \ldots i_k} = \sum_{a=1}^{k} \prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1} x_{i_2} + \ldots + x_{i_1} \cdots x_{i_k}. \tag{3.2}$$

The A_5 cluster algebra has an eight-fold cyclic symmetry, which is generated by σ:

$$\begin{aligned} \sigma: \quad x_1 &\mapsto \frac{x_2}{1+x_{12}}, \quad x_2 \mapsto \frac{x_3\,(1+x_1)}{1+x_{123}}, \quad x_3 \mapsto \frac{x_4\,(1+x_{12})}{1+x_{1234}}, \\ x_4 &\mapsto \frac{x_5\,(1+x_{123})}{1+x_{12345}}, \quad x_5 \mapsto \frac{1+x_{1234}}{x_1 x_2 x_3 x_4 x_5}. \end{aligned} \tag{3.3}$$

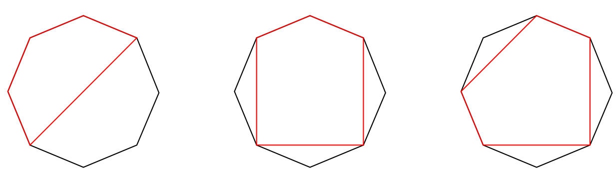
A_5 also has a two-fold flip symmetry, which is generated by τ:

$$\tau: \quad x_i \mapsto \frac{1}{x_{6-i}}. \tag{3.4}$$

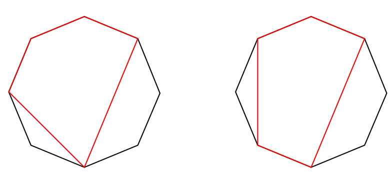
There are 56 distinct A_2 subalgebras in A_5 ($56 = \binom{8}{5}$ = number of distinct pentagons inside an octagon), they can be parameterized by:

$$\begin{aligned} &\left\{ x_1 \to x_2, \quad x_2 \to x_3\,(1+x_4)\,, \quad x_2\,(1+x_3) \to \frac{x_3 x_4}{1+x_3} \right\} + \sigma, \\ &\left\{ x_2 \to x_3, \quad x_1\,(1+x_2) \to \frac{x_2 x_3}{1+x_2} \right\} + \sigma + \tau \end{aligned} \tag{3.5}$$

where by "+ σ" and "+ σ + τ" I mean "+ cyclic copies" and "+ cyclic and flip copies," respectively. This correspond to the geometries



(8 of each)



(16 of each)

The A_5 function

The A_5 function is a sum over two of the classes of A_2 subalgebras, $x_2 \to x_3\,(1+x_4)$ and $x_1\,(1+x_2) \to \frac{x_2 x_3}{1+x_2}$, appropriately antisymmetrized so that the overall f_{A_5} picks up a minus sign under both σ and τ. Explicitly, this is written

$$f_{A_5} = \sum_{i=0}^{7} \sum_{j=0}^{1} (-1)^{i+j} \sigma^i \tau^j \left(\frac{1}{2} f_{A_2}\left(x_2 \to x_3\,(1+x_4)\right) + f_{A_2}\left(x_1\,(1+x_2) \to \frac{x_2 x_3}{1+x_2} \right) \right). \tag{3.6}$$

The factor of $\frac{1}{2}$ in front of $f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right)$ is simply a symmetry factor, as it lives in an 8-cycle of $\{\sigma, \tau\}$.

Which pentagonalizations do these two A_2s correspond to?

A large outstanding question is: why these two A_2s? I have no justification/argument for them other than they work.

The A_5 does not have $B_2 \wedge B_2$ expressible in terms of $A_1 \times A_1$ subalgebras (i.e. it is not expressible as a sum of f_{A_3}s). This is very surprising!

3.2 Identifying A_5 subalgebras in $\mathrm{Gr}(4,8)$

Comment on infinite nature of $\mathrm{Gr}(4,8)$, define "good" (in the two-loop MHV sense) subalgebras, describe algorithm for finding "good", and describe the 56 good A_5s in $\mathrm{Gr}(4,8)$.

There are 56 good A_5s in $\mathrm{Gr}(4,8)$. They are generated by

$$\frac{\langle 1238\rangle\langle 1256\rangle}{\langle 1235\rangle\langle 1268\rangle} \to \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle} \to \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle} \to \frac{\langle 1567\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3567\rangle} \to \frac{\langle 1356\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 3456\rangle} \tag{3.7}$$

$$\frac{\langle 1238\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2358\rangle} \to -\frac{\langle 1235\rangle\langle 4568\rangle}{\langle 5(18)(23)(46)\rangle} \to \frac{\langle 1568\rangle\langle 2358\rangle\langle 3456\rangle}{\langle 1358\rangle\langle 2356\rangle\langle 4568\rangle} \to -\frac{\langle 5(18)(23)(46)\rangle}{\langle 1258\rangle\langle 3456\rangle} \to \frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle} \tag{3.8}$$

$$\frac{\langle 1234\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 2345\rangle} \to \frac{\langle 1348\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 3468\rangle} \to -\frac{\langle 1346\rangle\langle 5678\rangle}{\langle 6(18)(34)(57)\rangle} \to -\frac{\langle 1678\rangle\langle 3468\rangle\langle 34(128)\cap(567)\rangle}{\langle 1268\rangle\langle 1348\rangle\langle 3467\rangle\langle 5678\rangle} \to \frac{\langle 1278\rangle\langle 6(18)(34)(5\ldots}{\langle 1678\rangle\langle 34(128)\cap(56\ldots} \tag{3.9}$$

$$\frac{\langle 1234\rangle\langle 1278\rangle}{\langle 1238\rangle\langle 1247\rangle} \to -\frac{\langle 1248\rangle\langle 3457\rangle}{\langle 4(12)(35)(78)\rangle} \to -\frac{\langle 1247\rangle\langle 12(345)\cap(678)\rangle}{\langle 1278\rangle\langle 4(12)(35)(67)\rangle} \to -\frac{\langle 4567\rangle\langle 4(12)(35)(78)\rangle}{\langle 1245\rangle\langle 3457\rangle\langle 4678\rangle} \to -\frac{\langle 4(12)(35)(67)\ldots}{\langle 1234\rangle\langle 4567\rangle} \tag{3.10}$$

The first A_5 lives in an 8-cycle of the $\mathrm{Gr}(4,8)$ dihedral+parity, while the other three live in 16-cycles. Also note that in the first A_5, 7 and 8 never appear together in a $\langle\rangle$, and so the $8 \to 7$ collinear limit is smooth for this A_5. The second A_5 also features a smooth collinear limit, as

$$\frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle} \xrightarrow{8\to 7} \frac{\langle 1267\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1567\rangle}. \tag{3.11}$$

Neither of the latter 2 A_5s behave smoothly in the collinear limit (and neither do any of their dihedral+parity images).

Note: there are no good A_6s in $\mathrm{Gr}(4,8)$.

3.3 Representing $R_8^{(2)}$

The A_5 contribution to $R_8^{(2)}$ involves simply adding together the two A_5s in $\mathrm{Gr}(4,8)$ which behave smoothly in the collinear limit.

$$
\begin{aligned}
&R_8^{(2)} = \frac{1}{4} f_{A_5}\left(\frac{\langle 1238\rangle\langle 1256\rangle}{\langle 1235\rangle\langle 1268\rangle} \to \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle} \to \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle} \to \frac{\langle 1567\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3567\rangle} \to \frac{\langle 1356\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 3456\rangle}\right) + \\
&\frac{1}{2} f_{A_5}\left(\frac{\langle 1238\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2358\rangle} \to -\frac{\langle 1235\rangle\langle 4568\rangle}{\langle 5(18)(23)(46)\rangle} \to \frac{\langle 1568\rangle\langle 2358\rangle\langle 3456\rangle}{\langle 1358\rangle\langle 2356\rangle\langle 4568\rangle} \to -\frac{\langle 5(18)(23)(46)\rangle}{\langle 1258\rangle\langle 3456\rangle} \to \frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle}\right) \\
&+ \text{ dihedral} + \text{conjugate}
\end{aligned}
\tag{3.12}
$$

Again the difference between the overall factors of the two terms is simply a result of symmetry.

Let me briefly describe the collinear limit for this representation. As discussed previously, the A_5s explicitly written in (3.12) behave smoothly under the collinear limit, however not all of their dihedral+parity images do as well. In the case of the first A_5, which has 8 images under dihedral+parity, 4 of the f_{A_5}s vanish, while the remaining 3 are well-defined. For the second A_5, which has 16 images under dihedral+parity, 2 of the f_{A_5}s have "bad" collinear limits but they cancel off each other in the sum. Out of the remaining 14, 4 have good collinear limits and 10 vanish identically. Therefore, when we add up the contributions from both A_5s + their images, we end up with 7 terms – these correspond to the 7 A_5s in $\mathrm{Gr}(4,7)$.

- Li_4 contribution
- $\mathrm{Li}_2\,\mathrm{Li}_2$ contribution
- $\mathrm{Li}_3\,\mathrm{Li}_1$ contribution
- $\mathrm{Li}_2\,\mathrm{Li}_1^2$ contribution
- Li_1^4 contribution
- $\mathrm{Li}_2\,\pi^2$ contribution
- $\mathrm{Li}_1^2\,\pi^2$ contribution
- π^4 constribution

3.4 Analytic Properties of $R_8^{(2)}$

- some plots
- agrees with numerics

4 Steinmann Relations and Cluster Adjacency

The Steinmann relations dictate that the double discontinuities of amplitudes must vanish when taken in partially overlapping momentum channels [1, 2]. Recently, it has been realized that these restrictions on three- (and higher-)particle channels are transparently encoded in the symbol of BDS-like normalized amplitudes when the number of scattering particles is not a multiple of four [3, 4]. This is due to the fact that the BDS-like ansatz in each of these cases only depends on two-particle Mandelstam invariants, and thus acts as a spectator when discontinuities are taken in these channels. The Steinmann relations therefore apply directly to these BDS-like normalized amplitudes, where they imply that restricted pairs of Mandelstam invariants cannot appear sequentially in the first two entries of the symbol. In fact, these restrictions have been found to apply at all depths in the symbol, providing strong constraints on the spaces of functions that are expected to appear in these kinematics to any weight [5**?**].

Surprisingly, these constraints have also been found to be equivalent to requiring BDS-like normalized amplitudes to be cluster adjacent—that is, requiring every pair of symbol entries that appear sequentially in their symbol to also appear together in a cluster in $\mathrm{Gr}(4,n)$ [6]. Namely, this property is adhered to in all known BDS-like normalized amplitudes in six-, seven-, and nine-particle kinematics, where a unique BDS-like ansatz can be defined. However, it remains less well studied in eight-particle kinematics due to the nonexistence of any BDS-like normalization that only depends on two-particle invariants (no such solution to the anomalous dual conformal Ward identity that governs the infrared of these amplitudes exists [7]). For this reason, to study the extended Steinmann relations and cluster adjacency in the eight-point MHV amplitude, we must first explore the space of BDS-like ansätze that can be formed for this number of particles.

4.1 BDS-Like Ansätze for Eight Particles

When the number of particles n is not a multiple of four, the BDS-like ansatz is unique. Namely, there exists only a single decomposition of the BDS ansatz

$$\mathcal{A}_n^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = \mathcal{A}_n^{\text{BDS-like}}(\{s_{i,i+1}\}) \exp\left[\frac{\Gamma_{\text{cusp}}}{4} Y_n(\{u_i\})\right], \quad n \neq 4K, \tag{4.1}$$

such that the kinematic dependence of $A_n^{\text{BDS-like}}$ involves only two-particle Mandelstam invariants while Y_n depends only on dual conformal invariant cross ratios [8]. When n is a multiple of four, no decomposition of this type exists, and we are forced to consider multiple BDS-like ansätze if we want to transparently expose all Steinmann relations between higher-particle Mandelstam invariants.

In eight-particle kinematics, there are still two natural BDS-like normalization choices we might consider. Namely, we can let our BDS-like ansatz depend on either three- or on four-particle Mandelstam invariants in addition to two-particle invariants [4]. In this spirit, let us define a pair of Bose-symmetric BDS-like ansätze, respectively satisfying

$$\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = {}^3\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2,i+3}\}) \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\ {}^3Y_8(\{u_i\})\right], \tag{4.2}$$

$$\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = {}^4\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2}\}) \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\ {}^4Y_8(\{u_i\})\right]. \tag{4.3}$$

In fact, the functions ${}^3A_8^{\text{BDS-like}}$ and ${}^4A_8^{\text{BDS-like}}$ are not uniquely fixed by these decomposition choices; each admits a family of Bose-symmetric solutions. However, any choice of ${}^3A_8^{\text{BDS-like}}$ or ${}^4A_8^{\text{BDS-like}}$ consistent with eqns. (4.2) or (4.3) gives rise to a BDS-like normalized amplitude that manifestly exhibits a subset of the Steinmann relations. In particular, defining

$${}^X\mathcal{E}_8 \equiv \frac{\mathcal{A}_8^{\text{MHV}}}{{}^X\mathcal{A}_8^{\text{BDS-like}}} = \exp\left[R_8 - \frac{\Gamma_{\text{cusp}}}{4}\ {}^XY_8\right] \tag{4.4}$$

for any label X, we expect that ${}^3\mathcal{E}_8$ should satisfy Steinmann relations between all partially overlapping pairs of three-particle invariants, while ${}^4\mathcal{E}_8$ should satisfy Steinmann relations between all partially overlapping pairs of four-particle invariants. That is, ${}^3\mathcal{E}_8$ is expected to satisfy the relations

$$\text{Disc}_{s_{i+1,i+2,i+3}}\left[\text{Disc}_{s_{i,i+1,i+2}}({}^3\mathcal{E}_8)\right] = 0, \tag{4.5}$$

$$\text{Disc}_{s_{i+2,i+3,i+4}}\left[\text{Disc}_{s_{i,i+1,i+2}}({}^3\mathcal{E}_8)\right] = 0, \tag{4.6}$$

for all i, while ${}^4\mathcal{E}_8$ is expected to satisfy

$$\text{Disc}_{s_{i+1,i+2,i+3,i+4}}\left[\text{Disc}_{s_{i,i+1,i+2,i+3}}({}^4\mathcal{E}_8)\right] = 0, \tag{4.7}$$

$$\text{Disc}_{s_{i+2,i+3,i+4,i+5}}\left[\text{Disc}_{s_{i,i+1,i+2,i+3}}({}^4\mathcal{E}_8)\right] = 0, \tag{4.8}$$

$$\text{Disc}_{s_{i+3,i+4,i+5,i+6}}\left[\text{Disc}_{s_{i,i+1,i+2,i+3}}({}^4\mathcal{E}_8)\right] = 0. \tag{4.9}$$

However, conditions (4.5) through (4.9) don't exhaust the set of Steinmann relations obeyed by generic eight-particle amplitudes—there are also Steinmann relations between partially overlapping three- and four-particle invariants. If we want to make these additional relations manifest, we can instead define a BDS-like ansatz that depends on all but one of the four-particle invariants (and on no three-particle invariants). That is, we decompose the BDS ansatz as

$$\begin{aligned}\mathcal{A}_8^{\text{BDS}}(\{s_{i,\dots,i+j}\}) = {}^{3,j}\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\},\{s_{i,i+1,i+2,i+3} \neq s_{j,j+1,j+2,j+3}\})& \\ \times \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\ {}^{3,j}Y_8(\{u_i\})\right]&,\end{aligned} \tag{4.10}$$

and in so doing define a BDS-like normalized amplitude that should satisfy

$$\text{Disc}_{s_{j+2,j+3,j+4}}\left[\text{Disc}_{s_{j,j+1,j+2,j+3}}({}^{3,j}\mathcal{E}_8)\right]=0, \tag{4.11}$$
$$\text{Disc}_{s_{j+3,j+4,j+5}}\left[\text{Disc}_{s_{j,j+1,j+2,j+3}}({}^{3,j}\mathcal{E}_8)\right]=0, \tag{4.12}$$
$$\text{Disc}_{s_{j-1,j,j+1}}\left[\text{Disc}_{s_{j,j+1,j+2,j+3}}({}^{3,j}\mathcal{E}_8)\right]=0, \tag{4.13}$$
$$\text{Disc}_{s_{j-2,j-1,j}}\left[\text{Disc}_{s_{j,j+1,j+2,j+3}}({}^{3,j}\mathcal{E}_8)\right]=0, \tag{4.14}$$
$$\text{Disc}_{s_{i+1,i+2,i+3}}\left[\text{Disc}_{s_{i,i+1,i+2}}({}^{3,j}\mathcal{E}_8)\right]=0, \tag{4.15}$$
$$\text{Disc}_{s_{i+2,i+3,i+4}}\left[\text{Disc}_{s_{i,i+1,i+2}}({}^{3,j}\mathcal{E}_8)\right]=0, \tag{4.16}$$

where the relations (4.15) and (4.16) are the Steinmann relations also satisfied by ${}^3\mathcal{E}_8$. Clearly it is not possible to find a solution to eq. (4.10) that is Bose-symmetric, but we can require that it is invariant under the dihedral flip that maps $s_{i\ldots l}\to s_{9-i\ldots 9-l}$. There is again a family of solutions to this decomposition that respects this symmetry.

Momentum conservation implies that ${}^{3,j+4}\mathcal{A}_8^{\text{BDS-like}}={}^{3,j}\mathcal{A}_8^{\text{BDS-like}}$, so there are only four independent BDS-like normalized amplitudes of this type. All eight-point Steinmann relation are thus made manifest in at least one of the amplitudes

$$\{{}^4\mathcal{E}_8,\,{}^{3,1}\mathcal{E}_8,\,{}^{3,2}\mathcal{E}_8,\,{}^{3,3}\mathcal{E}_8,\,{}^{3,4}\mathcal{E}_8\}.$$

However, in practice it proves convenient to retain ${}^3\mathcal{E}_8$ in one's toolbox, since it can be chosen to respect Bose symmetry. Explicit forms for ${}^3Y_8(\{u_i\})$, ${}^4Y_8(\{u_i\})$, and ${}^{3,j}Y_8(\{u_i\})$ are provided in appendix B.

4.2 The Sklyanin Bracket

4.3 Cluster Adjacency in ${}^X\mathcal{E}_8$

The extended Steinmann relations (4.5) through (4.16) can be checked by computing the appropriate BDS-like normalized amplitudes from the remainder function, as per eq. (4.4). While these relations are satisfied, every Steinmann relation that is not preserved by the choice of BDS-like ansatz is violated by these amplitudes.

5 Conclusion

A A_5 Representation of $R_7^{(2)}$

B BDS-Like Conversions In Eight-Point Kinematics

It may additionally prove possible to chose these amplitudes to satisfy other desirable properties, since ${}^3Y_8$, ${}^4Y_8$, and ${}^{3,j}Y_8$ are not uniquely picked out by eqns. (4.2), (4.3), and (4.10). In fact there is a 10-dimensional (3-dimensional) space of (Bose-symmetric) solutions for ${}^3Y_8$, a 36-dimensional (4-dimensional) space of (Bose-symmetric) solutions for ${}^4Y_8$, and a 5-dimensional (3-dimensional) space of (dihedral flip-invariant) solutions for ${}^{3,j}Y_8$.

To take full advantage of the Steinmann relations, it is convenient to work in terms of symbol letters that isolate different Mandelstam invariants. There are twelve independent dual conformally invariant cross ratios that can appear in these symbols

$$u_1 = \frac{s_{12}s_{4567}}{s_{123}s_{812}}, \quad \text{and cyclic (8-orbit)} \tag{B.1}$$

$$u_9 = \frac{s_{123}s_{567}}{s_{1234}s_{4567}}, \quad \text{and cyclic (4-orbit).} \tag{B.2}$$

It is not possible to isolate all three- and four-particle Mandelstam invariants simultaneously into twelve different symbol letters. (More than twelve symbol letters will appear in these amplitudes, but we here restrict our attention to the twelve that will appear in the first entry.) However, different choices of letters can be made such that either all the four-particle invariants, or all the three-particle invariants, are isolated.

One choice that isolates the four-particle invariants is

$$^4d_1 = u_2\ u_6 = \frac{s_{23}\ s_{67}\ (s_{1234})^2}{s_{123}\ s_{234}\ s_{567}\ s_{678}}, \quad \text{and cyclic (4-orbit)} \tag{B.3}$$

$$^4d_5 = u_2/u_6 = \frac{s_{23}\ s_{567}\ s_{678}}{s_{67}\ s_{123}\ s_{234}}, \quad \text{and cyclic (4-orbit)} \tag{B.4}$$

$$^4d_9 = u_1\ u_2\ u_5\ u_6\ u_9^2 = \frac{s_{12}\ s_{23}\ s_{56}\ s_{67}}{s_{234}\ s_{456}\ s_{678}\ s_{812}}, \quad \text{and cyclic (4-orbit).} \tag{B.5}$$

In this alphabet $^4d_1, {}^4d_2, {}^4d_3$, and $^4d_4$ each contain a different four-particle Mandelstam invariant, while the other letters only involve two- and three-particle invariants. The extended Steinmann relations then tell us that $^4d_1, {}^4d_2, {}^4d_3$, and $^4d_4$ can never appear next to each other in the symbol of $^4A_8^{\text{BDS-like}}$ (but each can still appear next to themselves).

Similarly, we can isolate the three-particle invariants by choosing

$$^3d_1 = \frac{u_1\ u_2\ u_4\ u_7}{u_3\ u_5\ u_6\ u_8\ u_9^2} = \frac{s_{12}\ s_{23}\ s_{45}\ s_{78}\ (s_{1234})^2\ (s_{4567})^2}{s_{34}\ s_{56}\ s_{67}\ s_{81}\ (s_{123})^2}, \quad \text{and cyclic (8-orbit)} \tag{B.6}$$

$$^3d_9^4 = u_1\ u_5\ u_9\ u_{12} = \frac{s_{12}\ s_{56}}{s_{1234}\ s_{3456}}, \quad \text{and cyclic (4-orbit),} \tag{B.7}$$

in which case $^3d_1$ through $^3d_8$ each contain a different three-particle Mandelstam invariant, as well as four-particle Mandelstams that they don't partially overlap with. The remaining four letters only contain two- and four-particle invariants. In these letters, conditions (4.15) and (4.16) tell us that $^3d_7, {}^3d_8, {}^3d_2$, and $^3d_3$ can never appear next to $^3d_1$ in the symbols of $^3\mathcal{E}_8$ or $^{3,j}\mathcal{E}_8$ (plus the cyclic images of this statement). Moreover, conditions (4.11) through (4.14) give us the additional restrictions that none of $^3d_1, {}^3d_5, {}^3d_9$ and $^3d_{10}$ can ever appear next to $^3d_3, {}^3d_4, {}^3d_7$, or $^3d_8$ in the symbol of $^{3,1}\mathcal{E}_8$ (analogous relations hold for the other $^{3,j}\mathcal{E}_8$). These are the restrictions given by the Steinmann relations involving s_{1234} and one of $s_{781}, s_{812}, s_{345}$, or s_{456}. The other Steinmann relations between three- and four-particle invariants will not be respected by $^{3,1}\mathcal{E}_8$, since $^{3,j}\mathcal{A}_8^{\text{BDS-like}}$ depends on s_{2345}, s_{3456}, and s_{4567}.

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