

In this note we catalog the subalgebra structure for the finite cluster algebras $\subseteq E_6$.

These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

A_2 clusters: 5 a -coordinates: 5 x -coordinates: 10

A_3 clusters: 14 a -coordinates: 9 x -coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

A_4 clusters: 42 a -coordinates: 14 x -coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

D_4 clusters: 50 a -coordinates: 16 x -coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

A_5 clusters: 132 a -coordinates: 20 x -coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

D_5 clusters: 182 a -coordinates: 25 x -coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

E_6 clusters: 833 a -coordinates: 42 x -coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

Symbol Spaces on Cluster Algebras

Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	19	14	2	0	0	0
A_3	55	40	6	2	1	5
A_4	125	90	9	3	0	0
D_4	163	109				
A_5	245	175	1	0	0	0
D_5	381	241				
E_6	-	573	-	-	-	-

Automorphic D_4 symbols

$\overline{\sigma^+\tau^+}$		$\overline{\sigma^+\tau^-}$		$\overline{\sigma^-\tau^+}$		$\overline{\sigma^-\tau^-}$					
	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$			
$\tau_{S_3}^+$	7	0	$\tau_{S_3}^+$	2	0	$\tau_{S_3}^+$	5	0	$\tau_{S_3}^+$	4	0
$\tau_{S_3}^-$	0	0	$\tau_{S_3}^-$	1	0	$\tau_{S_3}^-$	2	0	$\tau_{S_3}^-$	0	0

Automorphic D_5 symbols

$\underline{\sigma^+\tau^+}$		$\underline{\sigma^+\tau^-}$		$\underline{\sigma^-\tau^+}$		$\underline{\sigma^-\tau^-}$	
$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$
22	0	11	0	0	4	0	12

Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	3	3
$A_1 \times A_1 \times A_1$	6	6
$A_1 \times A_1 \times A_1 \times A_1$	10	10
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	15	15
$A_2 \times A_1$	25	20
$A_2 \times A_1 \times A_1$	32	27
$A_2 \times A_1 \times A_1 \times A_1$	40	35
$A_2 \times A_2$	63	53
$A_3 \times A_1$	65	50
$A_2 \times A_2 \times A_1$	74	64
$A_3 \times A_1 \times A_1$	76	61
$A_3 \times A_2$	119	99
$A_4 \times A_1$	140	105
$D_4 \times A_1$	180	126

- The number of integrable symbols at weight w in the product group $\bigotimes_N A_1$ is given by $\binom{N+w-1}{w}$, i.e. ‘ N choose w (logs) with replacement’
- More generally, the number of weight-two integrable symbols in $G_1 \times G_2$ is the number of symbols in G_1 plus the number of symbols in G_2 , plus the product of their respective numbers of A-coordinates (corresponding to taking a log from each)
- More generally, these integrable symbol counts just correspond to shuffling together all possible symbols drawn from each component group such that the total weight is w
- Presumably the automorphic counts also follow from shuffling together all possible symbols drawn from the component groups such that the total weight is w , and such that their automorphism properties under the component groups respect any new permutation symmetries (for instance, any products of symbols drawn from A_1 and A_2 will respect the automorphism group of $A_1 \times A_2$, but a product of symbols from $A_2 \times A_2$ will have to respect a new symmetry that permutes the two component groups)
- I believe all of the above statements about integrable symbols directly translate to statements about cluster-adjacent symbols, because any product of symbols that are cluster-adjacent in each component group will also be cluster-adjacent in the composite group
- So this table (and the weight-four one) are just consistency checks

Weight Four

In each column, the numbers #1 (#2, #3) refer to:

- #1 = total number of functions
- #2 = number of functions with non-zero $B_2 \wedge B_2$
- #3 = number of functions with clustery $B_2 \wedge B_2$

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+ \tau^+$	$\sigma^+ \tau^-$	$\sigma^- \tau^+$	$\sigma^- \tau^-$
A_2	211 (6)	81 (6)	11 (0)	6 (2,1)	0	0
A_3	1351 (36)	432 (36)	49 (2,0)	29 (4,1)	24 (2,0)	42 (4,1)
A_4	-	1652 (126)	131 (6,0)	105 (12,3)	0	0
D_4	-	2204				
A_5	-	-	- (-,2)	- (-,5)	- (-,2)	- (-,5)
D_5	-	-				
E_6	-	-	-	-	-	-

Automorphic D_4 symbols

$\frac{\sigma_4^+ \tau_4^+}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$		$\frac{\sigma_4^+ \tau_4^-}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$		$\frac{\sigma_4^- \tau_4^+}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$		$\frac{\sigma_4^- \tau_4^-}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$	
$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,0) & - (-,1) \\ \hline \end{array}$	$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,2) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,1) & - (-,1) \\ \hline \end{array}$	$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,1) & - (-,0) \\ \hline \end{array}$
$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$

Automorphic D_5 symbols

$\frac{\sigma^+ \tau^+}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$		$\frac{\sigma^+ \tau^-}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$		$\frac{\sigma^- \tau^+}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$		$\frac{\sigma^- \tau^-}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$	
$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$
$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$

Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1$	15	15
$A_1 \times A_1 \times A_1 \times A_1$	35	35
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	70	70
$A_2 \times A_1$	301	136
$A_2 \times A_1 \times A_1$	423	218
$A_2 \times A_1 \times A_1 \times A_1$	585	335
$A_2 \times A_2$	1433	708
$A_3 \times A_1$	1701	622
$A_2 \times A_2 \times A_1$	1827	982
$A_3 \times A_1 \times A_1$	2127	873
$A_3 \times A_2$	4617	2088
$A_4 \times A_1$	-	
$D_4 \times A_1$	-	

Subalgebra-Constructible Symbols

	$A_1 \times A_1$	A_2	$A_1 \times A_1 \times A_1$	$A_2 \times A_1$	A_3
A_3	13	366			
A_4	98	1141	0	791	1603
D_4	63	1826	45	0	
A_5					
D_5					
E_6					

	E_6	D_5	A_5	D_4	A_4	A_3	A_2
A_1	385	130	70	52	35	15	5
A_2^\varnothing		5863	2836	1810	1127	357	76
$(A_1 \times A_1)^\varnothing$	2007	400	375	54	84	9	
A_3^\varnothing			1652	WTF	413	59	
$(A_2 \times A_1)^\varnothing$			2124	0	273		
$(A_1 \times A_1 \times A_1)^\varnothing$			36	12	0		
D_4^\varnothing					0		
A_4^\varnothing				0			
$(A_3 \times A_1)^\varnothing$							
$(A_2 \times A_2)^\varnothing$							
$(A_2 \times A_1 \times A_1)^\varnothing$							
$(A_1 \times A_1 \times A_1 \times A_1)^\varnothing$							
D_5^\varnothing			0				
A_5^\varnothing		0					
$(D_4 \times A_1)^\varnothing$							
$(A_4 \times A_1)^\varnothing$							
$(A_3 \times A_2)^\varnothing$							
$(A_3 \times A_1 \times A_1)^\varnothing$							
$(A_2 \times A_2 \times A_1)^\varnothing$							
$(A_2 \times A_1 \times A_1 \times A_1)^\varnothing$							
$(A_1 \times A_1 \times A_1 \times A_1 \times A_1)^\varnothing$							
E_6^\varnothing							

- The first line of this table is obviously Log^4 of all distinct X-coordinates
- Can we discover a purely-clustery notion of first entry condition by looking at all these symbols?

How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank $n - 1$ in a finite type cluster algebra of type X of rank n .

Let N be the number of cluster a -coordinates in the cluster algebra of type X .

Let n be the rank of cluster algebra of type X .

Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y .

The number of cluster algebras of type Y is then NZ/n .

As an example, let's count the number of A_4 's in A_5 :

$N = \#$ of cluster a -coordinates in $A_5 = 20$

$n = \text{rank of } A_5 = 5$

$Z = \#$ of ways to remove a node from A_5 and get an $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$ there are 8 A_4 's in A_5 .