In this note we analyze "the" D_5 function, f_{D_5}, which we generate by:

- start with an ansatz of all distinct f_{A_3}'s in D_5,
- impose antisymmetry under all of the D_5 automorphisms $\{\sigma, \tau, \mathbb{Z}_2\}$,
- take the fully symmetric sum of this f_{D_5} over E_6 and fit to $R_7^{(2)}$.

The resulting function has 1 free parameter, which represents an internal degree of freedom in f_{D_5} that cancels in the symmetric sum over E_6. Later in the note we will explore tuning these free parameters to make some (tbd) nice property manifest.

Reader's Digest: Deriving f_{D_5}

We'll be working with the following D_5 seed cluster:

$$
\begin{tikzcd}
 & & & x_4 \\
x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\
 & & & x_5
\end{tikzcd}
\tag{1}
$$

There are 65 distinct A_3's in D_5. Of these, only 42 produce linearly independent f_{A_3}'s. Imposing full D_5 antisymmetry on this collection of f_{A_3}'s leaves only 5 degrees of freedom. Requiring that the full E_6-symmetric sum of f_{D_5} gives $R_7^{(2)}$ fixes 4 of these parameters, leaving us with only 1 degree of freedom. Of course when we are looking for a particular representation of f_{D_5} we have 24 degrees of freedom (23 of which are equivalent to adding zero).

It would be nice to find a property that fixes some of these parameters that does not rely on explicitly knowing $R_7^{(2)}$. Of course a cluster-y property would be great, but even a physics one would be nice.

Describing f_{D_5}

Because of the 1 degree of freedom, it is difficult to describe in detail the properties of f_{D_5} until we have set this value. Furthermore, we likely want to keep this parameter free so that we have some freedom when we try to express $R_7^{(2)}$ in terms of f_{D_5}.

The piece that does not cancel in the full E_6 sum can be represented in terms of 13 A_3's (maybe less, I haven't done an exhaustive check). The following 8 enter with coefficient +1/2:

$$x_1 \to x_2 \to x_3\,(1+x_5)\,, \quad x_2 \to x_3 \to x_5, \quad \frac{x_1 x_2}{1+x_1} \to x_3 \to x_4, \quad x_1\,(1+x_2) \to \frac{x_2 x_3\,(1+x_4)}{1+x_2} \to x_5,$$

$$\frac{1}{x_4} \to x_3\,(1+x_4) \to x_5, \quad \frac{1+x_3}{x_3 x_4} \to x_2\,(1+x_3+x_3 x_4) \to \frac{x_3 x_5}{1+x_3},$$

$$\frac{1+x_2+x_2 x_3}{x_2 x_3 x_4} \to x_1\,(1+x_2+x_2 x_3+x_2 x_3 x_4) \to \frac{x_2 x_3 x_5}{1+x_2+x_2 x_3},$$

$$\frac{x_1 x_2 x_3 x_5}{1+x_1+x_1 x_2+x_1 x_2 x_3} \to \frac{(1+x_1)\,x_3 x_4}{(1+x_3)\,(1+x_1+x_1 x_2+x_1 x_2 x_3+x_1 x_2 x_3 x_4)} \to \frac{1+x_1+x_1 x_2+x_1 x_2 x_3}{(1+x_1)\,x_3},$$

and these 5 enter with coefficient $-1/2$:

$$x_1 \to x_2 \to x_3\,(1+x_4)\,, \quad x_2 \to x_3 \to x_4, \quad \frac{x_1 x_2}{1+x_1} \to x_3 \to x_5, \quad x_1\,(1+x_2) \to \frac{x_2 x_3\,(1+x_5)}{1+x_2} \to x_4,$$

$$\frac{x_1 x_2 x_3 x_4}{1+x_1+x_1 x_2+x_1 x_2 x_3} \to \frac{(1+x_1)\,x_3 x_5}{(1+x_3)\,(1+x_1+x_1 x_2+x_1 x_2 x_3+x_1 x_2 x_3 x_5)} \to \frac{1+x_1+x_1 x_2+x_1 x_2 x_3}{(1+x_1)\,x_3}.$$

I don't have a nice representation of the piece of the function that cancels in the full E_6 sum. The shortest representation I have involves 17 A_3's.

Representing $R_7^{(2)}$ in terms of f_{D_5}

Next step: write down the first D_5 and then show how σ, τ, and $\mathbb{Z}_2$ E_6 symmetries create the other 13 D_5's. Use this as a way to better express:

$$R_7^{(2)} = \sum_{i=1}^{14} \pm f_{D_5}^{(i)} \tag{2}$$

where i indexes the 14 D_5's in E_6, and the $\pm$ is to get the symmetries to work out right – half of the f_{D_5}'s have + and half have $-$. This needs sharper restatement.

Constraining the remaining parameter

In the $7 \to 6$ collinear limit, the 14 f_{D_5}'s have the following behavior:

- $\{1,4,5,7,10,11,12,14\}$ vanish identically
- $\{2,3,8,9\}$ are non-zero, $2=-3$ and $8=-9$
- $\{6,13\}$ each vanish if you set the remaining parameter to 1, and otherwise $6=-13$

(where $\{1,\ldots,14\}$ label the 14 D_5's in the arbitrary way my code decided to do things. I'll clean this up tomorrow.)

This provides some motivation (though not 100% clear-cut) for determining this parameter.

A brief idea for $\mathrm{Gr}(4,8)$

Are there E_6's in $\mathrm{Gr}(4,8)$ which have $\{v,z\}$ squares? What if I try the 42-term $\{v,z\}$ expression for $R_7^{(2)} \sim f_{E_6}$ and see if that evaluates to "good" things on (at least some of) the E_6's I've found in $\mathrm{Gr}(4,8)$.