

In this note we catalog the subalgebra structure for the finite cluster algebras  $\subseteq E_6$ .

These algebras are:  $A_2, A_3, A_4, D_4, A_5, D_5, E_6$ .

$A_2$     clusters: 5       $a$ -coordinates: 5       $x$ -coordinates: 10

$A_3$     clusters: 14       $a$ -coordinates: 9       $x$ -coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	6	6
$A_1 \times A_1$	3	3

$A_4$     clusters: 42       $a$ -coordinates: 14       $x$ -coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	28	21
$A_1 \times A_1$	28	28
$A_3$	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

$D_4$     clusters: 50       $a$ -coordinates: 16       $x$ -coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	36	36
$A_1 \times A_1$	30	18
$A_3$	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

$A_5$     clusters: 132     $a$ -coordinates: 20     $x$ -coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	120	56
$A_1 \times A_1$	180	144
$A_3$	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
$D_4$	0	0
$A_4$	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$D_5$     clusters: 182     $a$ -coordinates: 25     $x$ -coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	180	125
$A_1 \times A_1$	230	145
$A_3$	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
$D_4$	5	5
$A_4$	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$E_6$     clusters: 833     $a$ -coordinates: 42     $x$ -coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	1071	504
$A_1 \times A_1$	1785	833
$A_3$	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
$D_4$	35	35
$A_4$	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
$D_5$	14	14
$A_5$	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

# Symbol Spaces on Cluster Algebras

## Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
$A_2$	19	14	2	0	0	0
$A_3$	55	40	6	2	1	5
$A_4$	125	90	9	3	0	0
$D_4$	163	109				
$A_5$	245	175	1	0	0	0
$D_5$	381	241				
$E_6$	-	573	-	-	-	-

### Automorphic $D_4$ symbols

$\overline{\sigma^+\tau^+}$		$\overline{\sigma^+\tau^-}$		$\overline{\sigma^-\tau^+}$		$\overline{\sigma^-\tau^-}$		
	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	7	0	$\tau_{S_3}^+$	2	0	$\tau_{S_3}^+$	4	0
$\tau_{S_3}^-$	0	0	$\tau_{S_3}^-$	1	0	$\tau_{S_3}^-$	0	0

### Automorphic $D_5$ symbols

$\underline{\sigma^+\tau^+}$		$\underline{\sigma^+\tau^-}$		$\underline{\sigma^-\tau^+}$		$\underline{\sigma^-\tau^-}$	
$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$
22	0	11	0	0	4	0	12

### Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	3	3
$A_1 \times A_1 \times A_1$	6	6
$A_1 \times A_1 \times A_1 \times A_1$	10	10
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	15	15
$A_2 \times A_1$	25	20
$A_2 \times A_1 \times A_1$	32	27
$A_2 \times A_1 \times A_1 \times A_1$	40	35
$A_2 \times A_2$	63	53
$A_3 \times A_1$	65	50
$A_2 \times A_2 \times A_1$	74	64
$A_3 \times A_1 \times A_1$	76	61
$A_3 \times A_2$	119	99
$A_4 \times A_1$	140	105
$D_4 \times A_1$	180	126

- The number of integrable symbols at weight  $w$  in the product group  $\bigotimes_N A_1$  is given by  $\binom{N+w-1}{w}$ , i.e. ‘ $N$  choose  $w$  (logs) with replacement’
- More generally, the number of weight-two integrable symbols in  $G_1 \times G_2$  is the number of symbols in  $G_1$  plus the number of symbols in  $G_2$ , plus the product of their respective numbers of A-coordinates (corresponding to taking a log from each)
- More generally, these integrable symbol counts just correspond to shuffling together all possible symbols drawn from each component group such that the total weight is  $w$
- Presumably the automorphic counts also follow from shuffling together all possible symbols drawn from the component groups such that the total weight is  $w$ , and such that their automorphism properties under the component groups respect any new permutation symmetries (for instance, any products of symbols drawn from  $A_1$  and  $A_2$  will respect the automorphism group of  $A_1 \times A_2$ , but a product of symbols from  $A_2 \times A_2$  will have to respect a new symmetry that permutes the two component groups)
- I believe all of the above statements about integrable symbols directly translate to statements about cluster-adjacent symbols, because any product of symbols that are cluster-adjacent in each component group will also be cluster-adjacent in the composite group
- So this table (and the weight-four one) are just consistency checks

## Weight Four

In each column, the numbers #1 (#2, #3) refer to:

- #1 = total number of functions
- #2 = number of functions with non-zero  $B_2 \wedge B_2$
- #3 = number of functions with clustery  $B_2 \wedge B_2$

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+ \tau^+$	$\sigma^+ \tau^-$	$\sigma^- \tau^+$	$\sigma^- \tau^-$
$A_2$	211 (6)	81 (6)	11 (0)	6 (2,1)	0	0
$A_3$	1351 (36)	432 (36)	49 (2,0)	29 (4,1)	24 (2,0)	42 (4,1)
$A_4$	-	1652 (126)	131 (6,0)	105 (12,3)	0	0
$D_4$	-	2204				
$A_5$	-	-	- (-,2)	- (-,5)	- (-,2)	- (-,5)
$D_5$	-	-				
$E_6$	-	-	-	-	-	-

### Automorphic $D_4$ symbols

$\frac{\sigma_4^+ \tau_4^+}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$		$\frac{\sigma_4^+ \tau_4^-}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$		$\frac{\sigma_4^- \tau_4^+}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$		$\frac{\sigma_4^- \tau_4^-}{\sigma_{S_3}^+ \quad \sigma_{S_3}^-}$	
$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,0) & - (-,1) \\ \hline \end{array}$	$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,2) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,1) & - (-,1) \\ \hline \end{array}$	$\tau_{S_3}^+$	$\begin{array}{ c c } \hline - (-,1) & - (-,0) \\ \hline \end{array}$
$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$	$\tau_{S_3}^-$	$\begin{array}{ c c } \hline - (-,0) & - (-,0) \\ \hline \end{array}$

### Automorphic $D_5$ symbols

$\frac{\sigma^+ \tau^+}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$		$\frac{\sigma^+ \tau^-}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$		$\frac{\sigma^- \tau^+}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$		$\frac{\sigma^- \tau^-}{\sigma_{Z_2}^+ \quad \sigma_{Z_2}^-}$	
$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^+$	$\begin{array}{ c } \hline - \\ \hline \end{array}$
$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$	$\sigma_{Z_2}^-$	$\begin{array}{ c } \hline - \\ \hline \end{array}$

### Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1$	15	15
$A_1 \times A_1 \times A_1 \times A_1$	35	35
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	70	70
$A_2 \times A_1$	301	136
$A_2 \times A_1 \times A_1$	423	218
$A_2 \times A_1 \times A_1 \times A_1$	585	335
$A_2 \times A_2$	1433	708
$A_3 \times A_1$	1701	622
$A_2 \times A_2 \times A_1$	1827	982
$A_3 \times A_1 \times A_1$	2127	873
$A_3 \times A_2$	4617	2088
$A_4 \times A_1$	-	
$D_4 \times A_1$	-	

### Subalgebra-Constructible Symbols

	$A_1 \times A_1$	$A_2$	$A_1 \times A_1 \times A_1$	$A_2 \times A_1$	$A_3$
$A_3$	13	366			
$A_4$	98	1141	0	791	1603
$D_4$	63	1826	45	0	
$A_5$					
$D_5$					
$E_6$					

	$E_6$	$D_5$	$A_5$	$D_4$	$A_4$	$A_3$	$A_2$
$A_1$	385	130	70	52	35	15	5
$A_2^\varnothing$		5863	2836	1810	1127	357	76
$(A_1 \times A_1)^\varnothing$	2007	400	375	54	84	9	
$A_3^\varnothing$			1652	WTF	413	59	
$(A_2 \times A_1)^\varnothing$			2124	0	273		
$(A_1 \times A_1 \times A_1)^\varnothing$			36	12	0		
$D_4^\varnothing$					0		
$A_4^\varnothing$				0			
$(A_3 \times A_1)^\varnothing$							
$(A_2 \times A_2)^\varnothing$							
$(A_2 \times A_1 \times A_1)^\varnothing$							
$(A_1 \times A_1 \times A_1 \times A_1)^\varnothing$							
$D_5^\varnothing$			0				
$A_5^\varnothing$		0					
$(D_4 \times A_1)^\varnothing$							
$(A_4 \times A_1)^\varnothing$							
$(A_3 \times A_2)^\varnothing$							
$(A_3 \times A_1 \times A_1)^\varnothing$							
$(A_2 \times A_2 \times A_1)^\varnothing$							
$(A_2 \times A_1 \times A_1 \times A_1)^\varnothing$							
$(A_1 \times A_1 \times A_1 \times A_1 \times A_1)^\varnothing$							
$E_6^\varnothing$							

- The first line of this table is obviously  $\text{Log}^4$  of all distinct X-coordinates
- Can we discover a purely-clustery notion of first entry condition by looking at all these symbols?



## How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type  $Y$  of rank  $n - 1$  in a finite type cluster algebra of type  $X$  of rank  $n$ .

Let  $N$  be the number of cluster  $a$ -coordinates in the cluster algebra of type  $X$ .

Let  $n$  be the rank of cluster algebra of type  $X$ .

Let  $Z$  be the number of ways to remove a node from a Dynkin diagram of type  $X$  to obtain one of type  $Y$ .

The number of cluster algebras of type  $Y$  is then  $NZ/n$ .

As an example, let's count the number of  $A_4$ 's in  $A_5$ :

$N = \#$  of cluster  $a$ -coordinates in  $A_5 = 20$

$n = \text{rank of } A_5 = 5$

$Z = \#$  of ways to remove a node from  $A_5$  and get an  $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$  there are 8  $A_4$ 's in  $A_5$ .