The basic idea for generating a well-defined Poisson bracket on $\mathcal{A}$-coordinates is to start with some seed (in terms of $\mathcal{A}$-coordinates) with standard adjacency matrix B written as a $n \times k$ matrix (this the case when you have n total nodes with k mutable). Then find some skew-symmetric $n \times n$ matrix Ω such that

$$\Omega B = \begin{pmatrix} \mathbb{1}_{k\times k} \\ 0_{(n-k)\times k} \end{pmatrix} \tag{1}$$

Then the claim is that the entries ω_{ij} form a Poisson bracket on $\mathcal{A}$-coordinates , i.e.

$$\{\mathcal{A}_i, \mathcal{A}_j\} = \omega_{ij}\mathcal{A}_i\mathcal{A}_j \Rightarrow \{\bar{\mathcal{A}}_i, \bar{\mathcal{A}}_j\} = \bar{\omega_{ij}}\bar{\mathcal{A}}_i\bar{\mathcal{A}}_j \tag{2}$$

where $\bar{\ }$ indicates mutation. The interesting feature here is that imposing eq. (1) does not entirely constrain Ω. For example, let's take a seed from $\mathrm{Gr}(2,5)$:

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd} \tag{3}$$

Associating $\langle 13\rangle$ and $\langle 14\rangle$ with nodes 1 and 2, resp., and then $\langle 12\rangle, \ldots, \langle 15\rangle$ with nodes $3, \ldots, 7$ we have

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

in which case Ω takes the form

$$\Omega = \begin{pmatrix}
0 & c_{13}-c_{14}+c_{15}-1 & c_{13} & c_{14} & c_{15} & c_{16} & -c_{15} \\
-c_{13}+c_{14}-c_{15}+1 & 0 & c_{23} & c_{24} & c_{24}-c_{23} & c_{26} & -c_{13}+c_{14}-c_{15}+c_{23}-c_{24} \\
-c_{13} & -c_{23} & 0 & c_{34} & c_{34}-c_{23} & c_{36} & -c_{13}+c_{23}-c_{34} \\
-c_{14} & -c_{24} & -c_{34} & 0 & c_{34}-c_{24} & c_{46} & -c_{14}+c_{24}-c_{34} \\
-c_{15} & c_{23}-c_{24} & c_{23}-c_{34} & c_{24}-c_{34} & 0 & c_{26}-c_{36}+c_{46} & -c_{15} \\
-c_{16} & -c_{26} & -c_{36} & -c_{46} & -c_{26}+c_{36}-c_{46} & 0 & -c_{16}+c_{26}-c_{36}+c_{46} \\
c_{15} & c_{13}-c_{14}+c_{15}-c_{23}+c_{24} & c_{13}-c_{23}+c_{34} & c_{14}-c_{24}+c_{34} & c_{15} & c_{16}-c_{26}+c_{36}-c_{46} & 0
\end{pmatrix}$$

The c_{ij} are arbitrary, and I guess we really only care about the 2×2 entries in the upper left corner but I present the full matrix for completion's sake.

The overall point here is that any choice of the c_{ij} defines *a* Poisson bracket on the $\mathcal{A}$-coordinates , and this Poisson bracket then satisfies all of the properties that the normal

$\mathcal{X}$-coordinate Poisson bracket does (invariant under mutation[1], etc). The tradeoff is straightforward: in $\mathcal{X}$-coordinate language one has a single Poisson bracket, but there is no single representation of a symbol in terms of $\mathcal{X}$-coordinates . In contrast, there is a single representation of a symbol in terms of $\mathcal{A}$-coordinates but there is not a single Poisson bracket on $\mathcal{A}$-coordinates . This of course jives with the fact that $\mathcal{A}$-coordinates can appear together in different clusters with different b_{ij} elements.

I believe then that the Sklyanin bracket is just one of the many possible Poisson brackets one can define on $\mathcal{A}$-coordinates , but I do not believe it is "priviledged" in any way. It is also worth emphasizing that one can evaluate the Sklyanin bracket directly on $\mathcal{A}$-coordinates , i.e. no need to dress them in to DCI cross-ratios.

[1]Unfortunately each Poisson bracket on $\mathcal{A}$-coordinates has a different set of mutation rules that one just has to work out. Gekhtman said that this is, in principle, not very illuminating. I have not tried it myself.