In this note we explore the space of cluster adjacent functions on A_3. We would particularly like to understand how much of this space is spanned by $f^{++}_{A_2}$ and $f^{+-}_{A_2}$ evaluated on the 6 A_2 subalgebras of A_3, and if there is a natural cluster algebraic way to express $\mathcal{E}^{(2)}_6$.

Cluster definitions

Our generic A_3 seed will be $x_1 \to x_2 \to x_3$. Then we will define our $\mathcal{X}$-coordinates as

$$\begin{aligned}
x_{1,1} &= 1/x_1 & x_{1,2} &= x_3 & v_1 &= \frac{(1+x_2)(1+x_1+x_1x_2+x_1x_2x_3)}{x_2x_3} \\
x_{2,1} &= \frac{1+x_1+x_1x_2}{x_1x_2x_3} & x_{2,2} &= \frac{1+x_1+x_1x_2}{x_2} & v_2 &= x_2(1+x_3) \\
x_{3,1} &= \frac{1+x_2+x_2x_3}{x_3} & x_{3,2} &= x_1(1+x_2+x_2x_3) & v_3 &= \frac{1+x_1}{x_1x_2} \\
e_1 &= \frac{1+x_1+x_1x_2+x_1x_2x_3}{(1+x_1)x_3} & e_2 &= \frac{x_2x_3}{1+x_2} & e_3 &= \frac{1+x_1}{x_1x_2(1+x_3)} \\
e_4 &= x_1(1+x_2) & e_5 &= \frac{x_2(1+x_3)}{1+x_1+x_1x_2+x_1x_2x_3} & e_6 &= 1/x_2.
\end{aligned} \tag{1}$$

Note that these definitions are different from 1401.6446 (the map is: starting with the expressions in 1401.6446, take $x_i \to 1/x_i$, and then an overall inversion (i.e. $v_i \to 1/v_i$, etc)). This is the result of having a different convention for converting $\mathcal{A}$ to $\mathcal{X}$-coordinates, and is a minor but annoying change to make.

In $\mathrm{Gr}(4,6)$ langauge, we take

$$x_1 = \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle}, \qquad x_2 = \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle}, \qquad x_3 = \frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}, \tag{2}$$

and then the $\mathcal{X}$-coordinates are

$$\begin{aligned}
x_{1,1} &= \frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle} & x_{1,2} &= \frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle} & v_1 &= \frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle} \\
x_{2,1} &= \frac{\langle 1256\rangle\langle 1346\rangle}{\langle 1236\rangle\langle 1456\rangle} & x_{2,2} &= \frac{\langle 1346\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 3456\rangle} & v_2 &= \frac{\langle 1235\rangle\langle 2456\rangle}{\langle 1256\rangle\langle 2345\rangle} \\
x_{3,1} &= \frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle} & x_{3,2} &= \frac{\langle 1236\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1256\rangle} & v_3 &= \frac{\langle 1356\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 3456\rangle} \\
e_1 &= \frac{\langle 1246\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2346\rangle} & e_2 &= \frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle} & e_3 &= \frac{\langle 1256\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 2456\rangle} \\
e_4 &= \frac{\langle 1236\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1356\rangle} & e_5 &= \frac{\langle 1234\rangle\langle 2456\rangle}{\langle 1246\rangle\langle 2345\rangle} & e_6 &= \frac{\langle 1356\rangle\langle 2345\rangle}{\langle 1235\rangle\langle 3456\rangle}.
\end{aligned} \tag{3}$$

These are now the same as 1401.6446.

Symmetries

A_3 has a six-fold cyclic symmetry, which we denote by σ:

$$\sigma: \; x_1 \mapsto \frac{x_2}{1+x_1+x_1x_2}, \quad x_2 \mapsto \frac{(1+x_1)\,x_3}{1+x_1+x_1x_2+x_1x_2x_3}, \quad x_3 \mapsto \frac{1+x_1+x_1x_2}{x_1x_2x_3}, \tag{4}$$

and which permutes the $\mathcal{X}$-coordinates by

$$\sigma: \; x_{i,j} \mapsto x_{i+1,j+1}, \quad v_i \mapsto v_{i+1}, \quad e_i \mapsto e_{i+1} \tag{5}$$

(where all of the indices are understood to be mod {2,3,6} where it makes sense). In $\mathrm{Gr}(4,6)$ language, σ is the standard $i \mapsto i+1$ cyclic symmetry.

A_3 also has a two-fold flip symmetry, which we denote by τ:

$$\tau: \; x_1 \mapsto \frac{1}{x_3}, \quad x_2 \mapsto \frac{1}{x_2}, \quad x_3 \mapsto \frac{1}{x_1}, \tag{6}$$

and which permutes the $\mathcal{X}$-coordinates by

$$\begin{aligned} \tau: \; & x_{1,j} \leftrightarrow x_{1,j+1}, \qquad x_{2,j} \leftrightarrow x_{3,j+1}, \qquad v_2 \leftrightarrow v_3, \\ & e_1 \leftrightarrow 1/e_5, \qquad e_2 \leftrightarrow 1/e_4, \qquad e_3 \leftrightarrow 1/e_3, \qquad e_6 \leftrightarrow 1/e_6. \end{aligned} \tag{7}$$

The $\mathrm{Gr}(4,6)$ flip is traditionally written as $i \to 7-i$, which in the A_3 language is equivalent to $\tau\sigma^{-1}$.

Strictly speaking, τ is an automorphism only on the space of $\mathcal{X}$-coordinates, not on the full cluster algebra. To elevate τ to an automorphism on the full cluster algebra: take a cluster with coordinates $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$, map them to $\{1/\tau(\mathcal{X}_1), 1/\tau(\mathcal{X}_2), 1/\tau(\mathcal{X}_3)\}$ and then swap the direction of all of the arrows in the cluster (i.e. take $b_{ij} \to -b_{ij}$). The result will now again be a cluster in A_3. This is easiest to see in the seed cluster, which is invariant under τ:

$$\{x_1 \to x_2 \to x_3\} \quad \mapsto \quad \{1/\tau(x_1) \leftarrow 1/\tau(x_2) \leftarrow 1/\tau(x_3)\} \quad = \quad \{x_3 \leftarrow x_2 \leftarrow x_1\}. \tag{8}$$

The reason for this weirdness is that cluster automorphisms are only defined at the level of $\mathcal{A}$-coordinates. But since we want to deal with actual functions/Bloch-group elements, it makes more sense to list the "automorphisms" w.r.t. $\mathcal{X}$-coordinates and accept that there is this slight funny business.

Integrable Symbol

Imposing integrability on a generic symbol of weight 4 with arguments drawn from (1) leaves us with 1351 free parameters.

$\mathcal{A}$-adjacency vs. $\mathcal{X}$-adjacency

The two criteria we consider are:

- cluster $\mathcal{A}$-adjacency (only $\{\langle i\ i+2\rangle, \langle i\ i+3\rangle\}$+frozen can appear),
- cluster $\mathcal{X}$-adjacency (only $\{\mathcal{X}_i, \mathcal{X}_{i+1}\}$ can appear).

Cluster $\mathcal{A}$-adjacency fixes 919 parameters, leaving us with 432. I haven't checked cluster $\mathcal{X}$-adjacency yet.

Imposing cluster-y coproduct

As was the case for A_2, cluster $\mathcal{A}$-adjacency at symbol level already imposes a cluster-y coproduct at the level of $B_2 \wedge B_2$. Imposing this for $B_3 \otimes \mathbb{C}^*$ fixes 60 parameters, leaving us with 372.

Imposing symmetries

We now impose that the function respect the automorphisms of A_3, as discussed above. We will now tabulate how many free parameters remain for each of the 4 possible sign choices:

	$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
f_{A_3}:	42	25	21	36

Local Coproduct

Only $f_{A_3}^{+-}$ and $f_{A_3}^{--}$ have non-zero $B_2 \wedge B_2$, and they have 6 and 2 free parameters in this space, respectively. Requiring that these take a local form fixes all of these parameters, killing off the $B_2 \wedge B_2$ for $f_{A_3}^{+-}$ and leaving us with the "traditional" f_{A_3} for $\{--\}$, which has $B_2 \wedge B_2 = \sum_i \{x_{i,1}\}_2 \wedge \{x_{i,2}\}_2$.