

In this note we catalog the subalgebra structure for the finite cluster algebras  $\subseteq E_6$ .

These algebras are:  $A_2, A_3, A_4, D_4, A_5, D_5, E_6$ .

$A_2$    clusters: 5    $a$ -coordinates: 5    $x$ -coordinates: 10

$A_3$    clusters: 14    $a$ -coordinates: 9    $x$ -coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	6	6
$A_1 \times A_1$	3	3

$A_4$    clusters: 42    $a$ -coordinates: 14    $x$ -coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	28	21
$A_1 \times A_1$	28	28
$A_3$	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

$D_4$    clusters: 50    $a$ -coordinates: 16    $x$ -coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	36	36
$A_1 \times A_1$	30	18
$A_3$	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

$A_5$     clusters: 132     $a$ -coordinates: 20     $x$ -coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	120	56
$A_1 \times A_1$	180	144
$A_3$	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
$D_4$	0	0
$A_4$	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$D_5$     clusters: 182     $a$ -coordinates: 25     $x$ -coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	180	125
$A_1 \times A_1$	230	145
$A_3$	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
$D_4$	5	5
$A_4$	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$E_6$     clusters: 833     $a$ -coordinates: 42     $x$ -coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	1071	504
$A_1 \times A_1$	1785	833
$A_3$	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
$D_4$	35	35
$A_4$	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
$D_5$	14	14
$A_5$	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

# Symbol Spaces on Cluster Algebras

## Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
$A_2$	19	14	2	0	0	0
$A_3$	55	40	6	2	1	5
$A_4$	125	90	9	3	0	0
$D_4$	163	109				
$A_5$	245	175	1	0	0	0
$D_5$	381	241				
$E_6$	-	573	-	-	-	-

### Automorphic $D_4$ symbols

$\overline{\sigma^+\tau^+}$		$\overline{\sigma^+\tau^-}$		$\overline{\sigma^-\tau^+}$		$\overline{\sigma^-\tau^-}$		
	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$		$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	7	0	$\tau_{S_3}^+$	2	0	$\tau_{S_3}^+$	4	0
$\tau_{S_3}^-$	0	0	$\tau_{S_3}^-$	1	0	$\tau_{S_3}^-$	0	0

### Automorphic $D_5$ symbols

$\underline{\sigma^+\tau^+}$		$\underline{\sigma^+\tau^-}$		$\underline{\sigma^-\tau^+}$		$\underline{\sigma^-\tau^-}$	
$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$
22	0	11	0	0	4	0	12

## Weight Four

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
$A_2$	211 (6)	81 (6)	11 (0)	6 (2)	0	0
$A_3$	1351 (36)	432 (36)	49 (2)	29 (4)	24 (2)	42 (4)
$A_4$	-	1652 (126)	131 (6)	105 (12)	0	0
$D_4$	-	-				
$A_5$	-	-	-	-	-	-
$D_5$	-	-				
$E_6$	-	-	-	-	-	-

### Automorphic $D_4$ symbols

$\frac{\sigma^+\tau^+}{\sigma_{S_3}^+ \sigma_{S_3}^-}$		$\frac{\sigma^+\tau^-}{\sigma_{S_3}^+ \sigma_{S_3}^-}$		$\frac{\sigma^-\tau^+}{\sigma_{S_3}^+ \sigma_{S_3}^-}$		$\frac{\sigma^-\tau^-}{\sigma_{S_3}^+ \sigma_{S_3}^-}$									
$\tau_{S_3}^+$	<table><tr><td>-</td><td>-</td></tr></table>	-	-	$\tau_{S_3}^+$	<table><tr><td>-</td><td>-</td></tr></table>	-	-	$\tau_{S_3}^+$	<table><tr><td>-</td><td>-</td></tr></table>	-	-	$\tau_{S_3}^+$	<table><tr><td>-</td><td>-</td></tr></table>	-	-
-	-														
-	-														
-	-														
-	-														
$\tau_{S_3}^-$	<table><tr><td>-</td><td>-</td></tr></table>	-	-	$\tau_{S_3}^-$	<table><tr><td>-</td><td>-</td></tr></table>	-	-	$\tau_{S_3}^-$	<table><tr><td>-</td><td>-</td></tr></table>	-	-	$\tau_{S_3}^-$	<table><tr><td>-</td><td>-</td></tr></table>	-	-
-	-														
-	-														
-	-														
-	-														

### Automorphic $D_5$ symbols

$\frac{\sigma^+\tau^+}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$		$\frac{\sigma^+\tau^-}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$		$\frac{\sigma^-\tau^+}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$		$\frac{\sigma^-\tau^-}{\sigma_{Z_2}^+ \sigma_{Z_2}^-}$	
-	-	-	-	-	-	-	-

## How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type  $Y$  of rank  $n - 1$  in a finite type cluster algebra of type  $X$  of rank  $n$ .

Let  $N$  be the number of cluster  $a$ -coordinates in the cluster algebra of type  $X$ .

Let  $n$  be the rank of cluster algebra of type  $X$ .

Let  $Z$  be the number of ways to remove a node from a Dynkin diagram of type  $X$  to obtain one of type  $Y$ .

The number of cluster algebras of type  $Y$  is then  $NZ/n$ .

As an example, let's count the number of  $A_4$ 's in  $A_5$ :

$N = \#$  of cluster  $a$ -coordinates in  $A_5 = 20$

$n = \text{rank of } A_5 = 5$

$Z = \#$  of ways to remove a node from  $A_5$  and get an  $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$  there are 8  $A_4$ 's in  $A_5$ .