In this note we describe the space of weight-4 polylogarithm functions which depend on $\mathcal{X}$-coordinates of the A_2 cluster algebra. Taking our seed cluster as $x_1 \to x_2$, we define our $\mathcal{X}$-coordinates:

$$\mathcal{X}_1 = \frac{1}{x_1}, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1+x_2), \quad \mathcal{X}_4 = \frac{1+x_1+x_1x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1+x_1}{x_1x_2}. \tag{1}$$

These satisfy $1+\mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}$, and $\{1/\mathcal{X}_i, \mathcal{X}_{i+1}\}$ form the 5 clusters. Of course you can also work in the $\mathrm{Gr}(2,5)$ language, where we have the $\mathcal{A}$-coordinates $\langle ij \rangle$ and the $\mathcal{X}$-coordinates

$$\mathcal{X}_1 = \frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}, \quad \mathcal{X}_2 = \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle}, \quad \mathcal{X}_3 = \frac{\langle 12\rangle\langle 35\rangle}{\langle 15\rangle\langle 23\rangle}, \quad \mathcal{X}_4 = \frac{\langle 25\rangle\langle 34\rangle}{\langle 23\rangle\langle 45\rangle}, \quad \mathcal{X}_5 = \frac{\langle 15\rangle\langle 24\rangle}{\langle 12\rangle\langle 45\rangle}. \tag{2}$$

This makes it easy to define cluster $\mathcal{A}$-adjacency: the allowed pairs are $\{\langle 13\rangle, \langle 14\rangle\}$, $\{\langle 14\rangle, \langle 24\rangle\}$, $\{\langle 24\rangle, \langle 25\rangle\}$, $\{\langle 25\rangle, \langle 35\rangle\}$, $\{\langle 35\rangle, \langle 13\rangle\}$ (along with all of the frozen nodes, $\langle i\ i+1\rangle$).

Integrable Symbol

Imposing integrability on a generic symbol of weight 4 with arguments drawn from (1) leaves us with 211 free parameters.

$\mathcal{A}$-adjacency vs. $\mathcal{X}$-adjacency

The two criteria we consider are:

- cluster $\mathcal{A}$-adjacency (only $\{\langle i\ i+2\rangle, \langle i\ i+3\rangle\}$+frozen can appear),
- cluster $\mathcal{X}$-adjacency (only $\{\mathcal{X}_i, \mathcal{X}_{i+1}\}$ can appear).

For weight-4 integrable symbols on A_2, these criteria turn out to be equivalent (this is not surprising since there are an equal number of $\mathcal{A}$- and $\mathcal{X}$-coordinates). In both cases, 130 parameters are fit by cluster adjacency, leaving us with 81.

Imposing cluster-y coproduct

Now that we have a symbol which is cluster adjacent, let us also impose that the coproduct be cluster-y as well. Based on experience we know that imposing full cluster adjacency at the level of the coproduct is not possible for A_2 (we have to wait for A_3 for that property), but we can at least impose that the $B_2 \wedge B_2$ and $B_3 \otimes \mathbb{C}^*$ take only $\{\mathcal{X}\}_k$ as arguments.

Interestingly, cluster adjacency at symbol level already imposes a cluster-y coproduct at the level of $B_2 \wedge B_2$. Imposing this for $B_3 \otimes \mathbb{C}^*$ fixes 15 parameters (I think this is killing off terms like $\mathrm{Li}_4(-1-\mathcal{X})$ and $\mathrm{Li}_4(-1-1/\mathcal{X})$), leaving us with 66.

Imposing symmetries

We now impose that the function respect the automorphisms of A_2. A_2 has a cyclic symmetry, $\sigma : \mathcal{X}_i \to \mathcal{X}_{i+1}$, and a flip symmetry $\tau : \mathcal{X}_i \to \mathcal{X}_{6-i}$. For each symmetry, we consider the case where f_{A_2} is either invariant or "covariant" – i.e. $\sigma(f_{A_2}) = f_{A_2}$ or $\sigma(f_{A_2}) = -f_{A_2}$. We will now tabulate how many free parameters remain for each of the 4 possible sign choices:

	$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
f_{A_2}:	9	5	0	0

We'll refer to these functions by their behavior under σ and τ: $f_{A_2}^{++}$ and $f_{A_2}^{+-}$. Note that only $f_{A_2}^{+-}$ has non-zero $B_2 \wedge B_2$ – this is what we have traditionally called "the A_2 function" (at coproduct level).

Poisson Structure and Mutation Interpretation

It would be nice to have an understanding of the Poisson structure in adjacent symbol entries for these functions – in particular, can we fit the remaining parameters in order to bring out a nice Poisson structure? A dream scenario would be to have an interpretation of these symbol terms as a mutation sequence.

Unfortunately at this point I don't seen any obvious interpretation, but this is worth returning to!

Summary

We have determined two weight-4 functions of interested related to the A_2 cluster algebra: $f_{A_2}^{++}$ and $f_{A_2}^{+-}$. They have 9,5 free parameters (respectively), and $f_{A_2}^{+-}$ has non-zero $B_2 \wedge B_2$. The next step is to evaluate these functions across A_3, impose the ++ automorphism sign choice, and see how close we get to $\mathcal{E}_6^{(2)}$!