In this note we describe the first entry conditions that the subalgebras of $\mathrm{Gr}(4,6)$ and $\mathrm{Gr}(4,7)$ inherit from the branch cut condition in six- and seven-particle kinematics. The questions we aim to address are (i) what are the conditions that should be imposed on these subalgebras if we only want symbols that satisfy the first entry condition when evaluated on one of the subalgebras of these Grassmannians, and (ii) are the inherited conditions (for a given subalgebra such as A_2) the same in both six- and seven-particle kinematics.

$A_1 \times A_1$

We start from the A_3 cluster algebra $x_1 \to x_2 \to x_3$. The simplest nontrivial subalgebra we can consider is $A_1 \times A_1$. Since it involves only the two (multiplicatively independent) symbol letters x_1 and x_2, the only types of functions that can live on this subalgebra are products of logs in these variables. At weight n, this gives rise to the $n+1$ functions $\log^{n-p} x_1 \log^p x_2$, where p ranges from 0 to n.

There are three $A_1 \times A_1$ subalgebras of $\mathrm{Gr}(4,6)$, which we can associate with the pairs of A_3 $\mathcal{X}$-coordinates

$$\{x_1, x_3\} \xrightarrow{\sigma} \left\{ \frac{x_2}{1+x_1+x_1x_2}, \frac{1+x_1+x_1x_2}{x_1x_2x_3} \right\} \xrightarrow{\sigma} \left\{ \frac{x_3}{1+x_2+x_2x_3}, x_1(1+x_2+x_2x_3) \right\}.$$

Note that applying σ three times does not send x_1 and x_3 back to themselves, but rather to their image under τ, namely

$$\{x_1, x_3\} \xrightarrow{\tau} \left\{ \frac{1}{x_3}, \frac{1}{x_1} \right\}.$$

Thus, we can argue that only the $A_1 \times A_1$ functions that are symmetric under τ will be single-valued in $\mathrm{Gr}(4,6)$. In the language of hexagon functions, functions that are symmetric under τ will be parity even, while functions antisymmetric under τ will be parity odd.[1]

Translating from A_3 coordinates to (one possible hexagon variable embedding) on Gr(4,6) we have

$$x_1 \to \sqrt{\frac{uwy_uy_vy_w}{v}}, \quad x_2 \to \sqrt{\frac{u(1+w)}{wy_uy_w(1-u)}}, \quad x_3 \to \sqrt{\frac{vy_uy_vy_w}{uw}},$$

from which it is easy to see that the physical first entry on the $A_1 \times A_1$ subalgebra associated with x_1 and x_3 permits only z_1/z_2, where the z_i are $A_1 \times A_1$ variables. To see that the same first entry condition is implied by all three $A_1 \times A_1$ subalgebras, it is sufficient to note that σ corresponds to permuting $u \to v \to w \to u$ and $y_u \to 1/y_v \to y_w \to 1/y_u$ in the hexagon letters. The only $A_1 \times A_1$ function that satisfies this first entry condition is thus $\log^4(z_1/z_2)$. Note that this function is manifestly symmetric under the A_3 flip τ.

[1] In the paper we will need to unpack this more—what is called parity in the hexagon function papers isn't actually the action of spacetime parity on, for instance, momentum twisters.

A_2

There are six A_2 subalgebras in Gr(4,6), which we can associate with the pairs of A_3 $\mathcal{X}$-coordinates

$$\begin{aligned}
\{x_1, x_2\} &\xrightarrow{\sigma} \left\{\frac{x_2}{1+x_1+x_1x_2}, \frac{(1+x_1)x_3}{1+x_1+x_1x_2+x_1x_2x_3}\right\} \\
&\xrightarrow{\sigma} \left\{\frac{x_3}{1+x_2+x_2x_3}, \frac{1+x_2}{x_2x_3}\right\} \\
&\xrightarrow{\sigma} \left\{\frac{1}{x_3}, \frac{x_1x_2(1+x_3)}{1+x_1}\right\} \\
&\xrightarrow{\sigma} \left\{\frac{x_1x_2x_3}{1+x_1+x_1x_2}, \frac{1}{x_1(1+x_2)}\right\} \\
&\xrightarrow{\sigma} \left\{\frac{1}{x_1(1+x_2+x_2x_3)}, \frac{1+x_1+x_1x_2+x_1x_2x_3}{x_2(1+x_3)}\right\}.
\end{aligned}$$

These pairs of $\mathcal{X}$-coordinates are mapped to different pairs of $\mathcal{X}$-coordinates by the A_3 flip τ, but these new pairs represent different (sub-)clusters associated with the same A_2 subalgebras. In particular, the operator $\sigma^4\tau$ (where these operators act to the right) takes us to a different pair of $\mathcal{X}$-coordinates associated with the same A_2 subalgebra.

Evaluating the A_2 ansatz on the subalgebra associated with the A_3 coordinates x_1 and x_2, the Gr(4,6) branch cut conditions admit the three solutions

$$H_{0,0,0,1}\left(1-\frac{1}{u}\right), \quad H_{1,0,0,1}\left(1-\frac{1}{u}\right), \quad H_{0,1,0,1}\left(1-\frac{1}{u}\right).$$

In terms of the A_2 coordinates, this is equivalent to allowing only the first entry

$$\log(x_1) - \log\left(\frac{1+x_1+x_1x_2}{x_2}\right),$$

where we emphasize that this first entry condition applies at the level of $\mathcal{X}$-coordinates, and is not equivalent to allowing any linear combinations of $\mathcal{X}$-coordinates that give rise to a symbol letter $x_1x_2/(1+x_1+x_1x_2)$.[2] Clearly, mapping to any of the other A_2 subalgebras will generate the same three functions with one of the arguments u, v, or w.

We can also solve the Gr(4,6) branch cut conditions on an ansatz involving multiple A_2 subalgebras. This gives rise to the further solutions

$$\log\left(\frac{v}{w}\right) H_{0,0,1}\left(1-\frac{1}{u}\right), \quad \log\left(\frac{v}{w}\right) H_{1,0,1}\left(1-\frac{1}{u}\right),$$

[2] We do, of course, allow the linear combination in which the arguments of both logs are inverted.

plus their cyclic images. Each pair of such functions is found in the space spanning the pair of A_2 subalgebras that are related by the A_3 permutation σ^3. In particular, they live in the five-dimensional space that has only the $\mathcal{X}$-coordinate combinations

$$\log(x_1) - \log\left(\frac{1+x_1+x_1x_2}{x_2}\right), \quad \log(x_1) + \log\left(\frac{1}{x_3}\right), \quad \log\left(\frac{1}{x_3}\right) - \log\left(\frac{1+x_1+x_1x_2}{x_1x_2x_3}\right)$$

appearing n their first entry. The $\mathcal{X}$-coordinates x_1 and $(1+x_1+x_1x_2)/x_2$ live on the A_2 whose seed is $x_1 \to x_2$, while the image of these coordinates under the A_3 permutation σ^3 is $1/x_3$ and $(1+x_1+x_1x_2)/(x_1x_2x_3)$, respectively. Two of the allowed first entries thus correspond to the allowed first entry on individual A_2 subalgebras.[3] Since any branch-cut-satisfying function on a single one of these A_2 subalgebras must also be a solution in this larger space, the additional three functions that satisfy this first entry condition are the functions of u (or v, or w) identified above.

No further solutions to the Gr(4,6) branch cut condition are found when more A_2 subalgebras are included, so we conclude that there are a total of fifteen A_2-subalgebra-constructible hexagon functions at weight four.

Coming back many months later (last timestamp Feb 19), after having worked out everything in 1810.12181, it is now obvious that the subalgebras of A_3 should all inherit unique first-entry conditions since there is only a single orbit of each type in A_3. I never got to last entry conditions, because I seemed to find no reasonable first entries in in Gr(4,7), where I now realize I was conflating multiple orbits. The right thing to check now would be whether there exist an obvious set of representative subalgebras of each type in Gr(4,7) (whose orbits generate all such subalgebras) that all share the same first entry condition in some preferred orientation.

[3] We can describe these first entry conditions more geometrically, starting from the A_2 seed $x_1 \to x_2$. The four involved $\mathcal{X}$-coordinates are on the node populated by x_1 after appying any sequence of the operations (i) mutating on the first, second, and then first node, or (ii) permuting the cluster by the A_3 map σ^3. The allowed first entries are then $X_i + (-1)^{s_{ij}} X_j$, where s_{ij} is 1 when the $\mathcal{X}$-coordinates X_i and X_j live on the same A_2 subalgebra, and 0 if they live on different subalgebras.