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Cluster Algebras and the Subalgebra-Constructibility of the Seven-Particle Remainder Function

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Abstract: We review various aspects of cluster algebras and the ways in which they appear in the study of loop-level amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. While the forms of cluster-algebraic structure that have been seen in this theory's amplitudes are too simple to extend beyond low loop orders and particle multiplicities, it is hoped that they point to the existence of some more general combinatorial structure that extends to all amplitudes in this theory. To better understand one aspect of this structure, we follow up on the previous observations that the 'nonclassical' part of this theory's two-loop MHV amplitudes can be decomposed into the A_2 and A_3 subalgebras of $\mathrm{Gr}(4, n)$ by searching for other subalgebras into which these amplitudes can be decomposed. We focus on seven-particle kinematics, where we show that the nonclassical part of the two-loop MHV amplitude can be additionally decomposed into the A_4, A_5, or D_5 subalgebras of $\mathrm{Gr}(4, 7)$. [uniqueness of decompositions]

Contents

1 Introduction

Multi-loop scattering amplitudes in the planar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory exhibit a great deal of intriguing mathematical structure. Much of this structure, at least at low loops and particle multiplicity, seems to be intimately tied to the cluster algebras associated with the Grassmannian Gr$(4,n)$ []. This is especially true in the maximally-helicity-violating (MHV) sector, where amplitudes have been computed at high loop orders in six- and seven-particle kinematics [1–5], and algorithms exist for calculating two loop amplitudes for any number of particles [6, 7]. Remarkably, each of the branch cuts in these amplitudes ends at the vanishing locus of some cluster coordinate on Gr(4,n) [8–11], and—even more strikingly—their iterated discontinuities vanishes unless sequentially taken in coordinates that appear together in a cluster of Gr(4,n) [12, 13]. All known next-to-MHV (NMHV) amplitudes in this theory share these remarkable properties [3–5, 14–17], as do certain classes of Feynman integrals [12, 18–20], some of which have been computed to all loop orders [21]. While this collection of amplitudes and integrals represents the simplest this theory has to offer, it remains suggestive that cluster algebras combinatorially realize these salient aspects of their analytic structure, thereby encoding locality in a non-obvious way.

The fact that cluster algebras appear in this context is not totally surprising, given that the plabic graphs that describe the integrands of this theory to all loop orders are themselves dual to cluster algebras [22]. In particular, the boundaries of the positive Grassmannian, where it is known that these integrands can develop physical singularities, all lie on the vanishing loci of cluster coordinates on Gr(4,n) [22]. Despite this, it's far from obvious that the location of *all* physical singularities will be picked out by cluster coordinates in this way—and indeed, certain Feynman integrals contributing to eight- and higher-particle amplitudes have recently been found to have singularities outside the positive region, at points involving square roots when expressed in terms of cluster coordinates [19, 20, 23]. This obfuscates the general connection between amplitudes and cluster algebras, as does the eventual appearance of functions beyond polylogarithms [24**?** , 25]. Both complications point to the need for more general objects than cluster algebras to describe the analytic structure of this theory at higher loops and particle multiplicities.

There is reason, however, to be optimistic that an analogously simple characterization of this (more complicated) analytic structure might be found. This optimism stems from the observation that the infinite class of amplitudes we currently have access to—the two-loop MHV amplitudes—have properties beyond branch cuts that seem to be indelibly tied to cluster algebras. In particular, the 'nonclassical' part of each of these amplitudes (that is, the part that cannot be expressed in terms of classical polylogarithms) is uniquely determined by a small set of physical and cluster-algebraic properties [11]. Seemingly unrelated, a pair of functions can be associated with the simplest cluster algebras, related to the Dynkin diagrams A_2 and A_3, in terms of which these nonclassical components can be decomposed into a sum over the A_2 or A_3 subalgebras of Gr(4,n) [10]. Furthermore, the remaining 'classical' part of these amplitudes can always be written as products of classical polylogarithms involving only negative cluster coordinates as arguments [7]. It can be hoped that the pervasiveness of such cluster-algebraic structure points to the existence of a deeper and more general combinatorial structure. If so, better understanding the many ways in which cluster algebras appear in these amplitudes can help us identify the features this structure must have.

The motivation for this work is to review the connections already mentioned between cluster algebras, as well as to further mine the case of two-loop MHV amplitudes for additional cluster algebraic connections. In particular, in this paper we elucidate the ways in which the subalgebra structure of cluster algebras can be used to build progressively more complicated polylogarithm functions. A central ingredient in this story is a proper understanding of the automorphism groups of cluster algebras, and the ways in which those automorphisms significantly restrain the space of "cluster polylogarithms," or polylogarithm functions which *depend* on a cluster algebra (in a sense we will define precisely). Restraining our explorations to the case of finite cluster algebras, we systematically explore the space of cluster polylogarithms and show how they can be constructed to mirror the subalgebra structure of the attendant cluster algebra, an approach we term "cluster subalgebra-constructibility". We then turn our attention to the two-loop seven-point MHV amplitude, itself a cluster polylogarithm associated with the Grassmannian $\mathrm{Gr}(4,7)$, and discuss ways in which it is subalgebra-constructible. The central result is a novel representation of $R_7^{(2)}$ as a sum over essentially all subalgebras of $\mathrm{Gr}(4,7)$.

In a forthcoming companion paper, we apply this technology to construct the full eight-point remainder function in terms of cluster polylogarithms. Specifically, we show that the the nonclassical part of this function can be decomposed as a sum over a finite subset of the A_5 subalgebras of $\mathrm{Gr}(4,8)$. We there describe ways in which one can search for such decompositions of this nature despite the fact that $\mathrm{Gr}(4,n)$ gives rise to an infinite number of cluster coordinates for $n > 7$. We also show that there exists a 'minimal BDS-like ansatz' that preserves all Steinmann and cluster adjacency relations.[1]

This paper is organized as follows. In section 2 we provide a self-contained introduction to cluster algebras and the principle ways in which they have appeared in $\mathcal{N} = 4$ SYM theory. While this section will largely constitute review for those familiar with recent developments at the intersection of these topics, section 2.6 provides a new (to the physics literature) cataloguing of the automorphism generators for finite cluster algebras. In section 3 we discuss some of the tools relevant for working with polylogarithms, particularly their associated coproduct and cobracket. We then recap the ways in which the coproduct and cobracket of two-loop amplitudes can be seen to exhibit curious cluster-algebraic properties. This section concludes with an formal definition of the term *cluster polylogarithm*. Having discussed the broad ways in which polylogarithms can be connected to cluster algebras, we show how this connection can be enhanced through the notion of *subalgebra-constructibility*, which is defined in section 4. Using the framework of subalgebra-constructibility we catalog subalgebra-constructible cluster polylogarithms for a wide array of finite cluster algebras. Finally, in section 5 we systematically explore the ways in which the functions of the previous section can be used to the two-loop seven-point MHV amplitude. In particular, we show that there exists a single representation of the amplitude which makes manifest subalgebra structure at every co-rank of $\mathrm{Gr}(4,7)$, specifically subalgebras of type D_5, A_5, A_4, A_3, and A_2.

We also include three appendices. Appendix **??** walks through the explicit construction

[1] There exists a unique DCI BDS-like ansatz for all n that aren't a multiple of four, but obscured by the nonexistence of a natural BDS-like normalization that preserves all Steinmann relations as exists for particle multiplicities that are not a multiple of four ...

of integrable and cluster-adjacent A_2 symbols through weight 4, illustrating how symbols can be constructed on any cluster algebra. Appendix A tabulates the subalgebras of each type that appear in the finite cluster algebras relevant to $R_7^{(2)}$, while appendix B tabulates the number of independent nonclassical degrees of freedom in each of these finite cluster algebras.

2 A Brief Introduction to Cluster Algebras

Cluster algebras were first introduced by Fomin and Zelevinsky [26] as a tool for identifying which algebraic varieties come equipped with a natural notion of positivity, and what quantities determine this positivity. As a consequence, they appear in the study of the positive Grassmannian $\mathrm{Gr}_+(k,n)$, i.e. the space of $k \times n$ matrices where all ordered $k \times k$ minors are positive. This means they also appear in the study of planar $\mathcal{N}=4$ SYM theory, where the integrands of amplitudes are encoded to all orders by $\mathrm{Gr}_+(4,n)$ [22].

A simple example of the type of questions cluster algebras help address is: how many matrix minors are sufficient to specify a point in $\mathrm{Gr}_+(k,n)$? In other words, given a $k \times n$ matrix, how many of its minors do we have to calculate to know if it is in $\mathrm{Gr}_+(k,n)$? These minors are not all independent; they satisfy identities known as Plücker relations,

$$\langle abI\rangle\langle cdI\rangle = \langle acI\rangle\langle bdI\rangle + \langle adI\rangle\langle bcI\rangle, \tag{2.1}$$

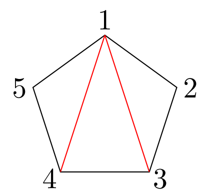
where each Plücker coordinate $\langle i_1,\ldots,i_k\rangle$ corresponds to the minor of columns $i_1,\ldots,i_k$, and I is a multi-index object with $k-2$ entries.

To gain some intuition for this problem, let us explore the case of $\mathrm{Gr}_+(2,5)$. The five cyclically adjacent minors $\langle 12\rangle$, $\langle 23\rangle$, $\langle 34\rangle$, $\langle 45\rangle$, and $\langle 15\rangle$ cannot be eliminated in terms of each other by Plücker relations, so each gives rise to an independent positivity constraint. However, making use of the Plücker relations

$$\begin{aligned}
\langle 24\rangle &= (\langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 14\rangle)/\langle 13\rangle,\\
\langle 25\rangle &= (\langle 12\rangle\langle 45\rangle + \langle 24\rangle\langle 15\rangle)/\langle 14\rangle,\\
\langle 35\rangle &= (\langle 25\rangle\langle 34\rangle + \langle 23\rangle\langle 45\rangle)/\langle 24\rangle,
\end{aligned} \tag{2.2}$$

we can eliminate three of the nonadjacent minors—for instance, $\langle 24\rangle$, $\langle 25\rangle$, and $\langle 35\rangle$—in terms of the remaining $\binom{5}{2}-3=7$ adjacent and nonadjacent ones. It can be checked that all further Plücker relations are implied by those in (2.2), telling us that seven minors must be computed to determine if a matrix is in $\mathrm{Gr}_+(2,5)$. However, as should be clear, it is not sufficient to check the positivity of *any* seven (ordered) minors; only certain triples of minors can be eliminated. It would therefore be advantageous to have a method for generating all sets of minors that are sufficient to answer this question.

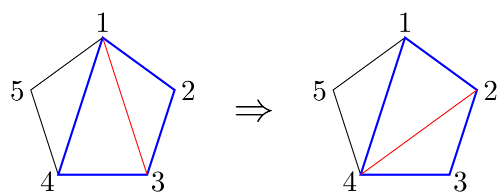
To motivate how cluster algebras address this problem, consider the following triangulation of the pentagon:



. (2.3)

We can associate the line connecting points i and j with the Plücker coordinate $\langle ij \rangle$; if we further assign these lines length $\langle ij \rangle$, the resulting pentagon is cyclic (in the sense that all its vertices reside on a common circle) due to Ptolemy's theorem. Conversely, all cyclic n-gons represent a point in $\mathrm{Gr}_+(2, n)$ [27]. Note that the length of the three diagonals that are not present in this triangulation are determined by the length of the seven lines that are present (including edges); the problem has been reduced to geometry. This makes clear why these three diagonals—the three eliminated above—are redundant for the purpose of determining whether a matrix is in $\mathrm{Gr}_+(2, 5)$.

From the first relation in (2.2) we see that we could have instead chosen to check the positivity of $\langle 24 \rangle$ rather than $\langle 13 \rangle$. This corresponds to choosing a different triangulation, which we get by trading the latter diagonal for the former:



. (2.4)

We have highlighted in blue the fact that both diagonals are framed by the same quadrilateral face. More generally, we can pick any quadrilateral face and flip the diagonal it contains to generate a different triangulation. Repeatedly performing these flips generates all possible triangulations of the pentagon, as can be seen in figure 1. Each triangulation provides a set of edges/minors whose positivity ensures that a matrix is in $\mathrm{Gr}_+(2, 5)$.

In an analogous way, cluster algebras answer questions about positivity for a larger class of algebraic varieties (and in particular for all $\mathrm{Gr}_+(k, n)$) by considering *clusters* that can all be generated by an operation called *mutation*, just as all triangulations of the pentagon are generated by flipping the diagonals of quadrilateral faces. We now turn to the definition of these objects, by first considering how the example of $\mathrm{Gr}_+(2, 5)$ can be rephrased in terms of cluster algebras.

2.1 Clusters, Mutations, and Cluster $\mathcal{A}$-coordinates

Clusters can be thought of as quiver diagrams—namely, oriented graphs equipped with arrows connecting different nodes—in which each node is assigned a cluster coordinate.[2] We can form

[2] Here and throughout, we are implicitly restricting ourselves to skew-symmetric cluster algebras of geometric type [].

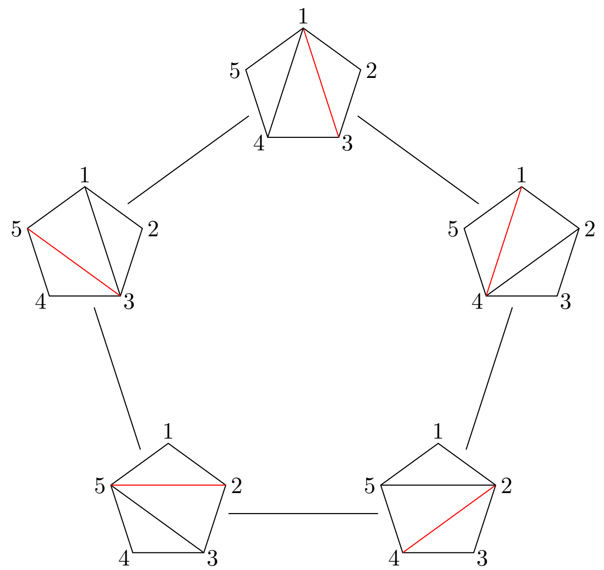
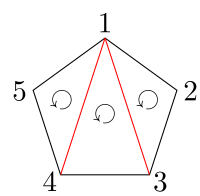


Figure 1: Mutating on the red chord moves you clockwise around the figure.

a cluster of $\mathrm{Gr}(2,5)$ out of our original triangulation (2.3) by assigning an orientation to the pentagon and all subtriangles, such as



. (2.5)

The nodes of our cluster are then given by the lines of this triangulation (making the minors $\langle ab\rangle$ our cluster coordinates), where an arrow is assigned from $\langle ab\rangle$ to $\langle cd\rangle$ if the triangle orientations in (2.5) have segment (ab) flowing into segment (cd). This gives us the quiver

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd} \quad , \tag{2.6}$$

where the boxes around the $\langle i i+1\rangle$ indicate that they are *frozen*—they can never change because they aren't in the interior of a quadrilateral face. The variables living at the frozen nodes can be thought of as parameterizing the boundary of our space, while the mutable nodes represent parameterizations of the interior. One can also draw arrows (with partial weight) connecting these frozen nodes (see, for instance, [22]), but we will not keep track of them here.

We have now drawn our first cluster (also sometimes called a seed). The Plücker coordinates in this cluster are referred to as cluster $\mathcal{A}$-coordinates (sometimes also $\mathcal{A}$-variables), and they come in two flavors: mutable (for example, $\langle 13\rangle$ and $\langle 14\rangle$ above) and frozen ($\langle i i{+}1\rangle$ above). In any quiver, the information encoded by the arrows can also be represented in terms of a skew-symmetric matrix

$$b_{ij} = (\#\text{ of arrows } i \to j) - (\#\text{ of arrows } j \to i) \tag{2.7}$$

called the exchange matrix (or the signed adjacency matrix).

The process of mutation, which we have described geometrically in terms of flipping diagonals, has a simple interpretation at the level of quivers. Given a quiver such as (2.6), we can choose any mutable node k to mutate on (this is equivalent to picking which diagonal to flip). Mutation gives us back a new quiver in which the $\mathcal{A}$-coordinate a_k has been sent to a'_k, where

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}, \tag{2.8}$$

(with the understanding that an empty product is set to one), while all other cluster $\mathcal{A}$-coordinates remain unchanged. The arrows connecting the nodes in this new quiver are also modified by carrying out the following operation:

- for each path $i \to k \to j$, add an arrow $i \to j$,
- reverse all arrows on the edges incident with k,
- and remove any two-cycles (oppositely-oriented arrows) that may have formed.

This creates a new adjacency matrix b'_{ij} via

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i,j\}, \\ b_{ij}, & \text{if } b_{ik}b_{kj} \le 0, \\ b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases} \tag{2.9}$$

Mutation is an involution, so mutating on a'_k will take you back to the original cluster (just as flipping the same diagonal twice will take you back to where you started).

In terms of these ingredients, a cluster algebra can be defined to be a set of clusters that is closed under mutation. Thus, mutating on any non-frozen node of any cluster will generate a different cluster in the same cluster algebra. In practice, one therefore constructs cluster algebras by starting from a seed such as (2.6), and iteratively mutating on all available nodes until the set of clusters closes (or it becomes clear the cluster algebra is infinite).

It is common to refer to certain cluster algebras by particularly nice representative clusters, when the mutable notes of the corresponding quiver form an oriented Dynkin diagram. For instance, the Gr(2,5) cluster algebra is often referred to as A_2, since the mutable part of the seed (2.6) is given by $\langle 13\rangle \to \langle 14\rangle$. Thus, we will often speak interchangeably of the cluster algebras for $\mathrm{Gr}(2,5)$ and A_2. This is a slight abuse of notation, as the $\mathrm{Gr}(2,5)$ cluster algebra corresponds specifically to the cluster algebra generated by the collection of frozen and mutable nodes in eq. (2.6), whereas an A_2 cluster algebra can in principle be dressed with any number of frozen nodes. We will see why this language is useful in the next section.

2.2 Cluster $\mathcal{X}$-coordinates

Cluster algebras can also be formulated in terms of a different set of cluster coordinates, called Fock-Goncharov coordinates or $\mathcal{X}$-coordinates [28]. As we will see in future sections, cluster $\mathcal{X}$-coordinates play a crucial role in connecting cluster algebras to polylogarithms and scattering amplitudes. While it is always possible to phrase results involving cluster algebras directly in terms of $\mathcal{A}$-coordinates, $\mathcal{X}$-coordinates often allow for cluster-algebraic structure to be made more manifest.

Clusters formed out of $\mathcal{X}$-coordinates can be directly constructed out of clusters involving $\mathcal{A}$-coordinates. Given a quiver equipped with $\mathcal{A}$-coordinates and described by the exchange matrix b_{ij}, we can compute an $\mathcal{X}$-coordinate to assign to each mutable node by

$$x_i = \prod_j a_j^{b_{ji}}. \tag{2.10}$$

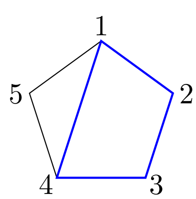
For example, the $\mathcal{X}$-coordinate cluster associated with (2.6) is formed by associating cluster $\mathcal{X}$-coordinates with all its mutable nodes, where these $\mathcal{X}$-coordinates are constructed by putting all Plücker coordinates that point to that node in the denominator, and all Plücker coordinates that are pointed to by that node in the numerator. That is, we get

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle} \to \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle}, \tag{2.11}$$

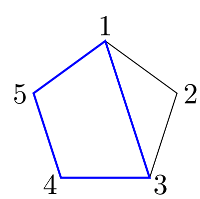
which takes the form of the generic A_2 quiver

$$x_1 \to x_2. \tag{2.12}$$

In the pentagon-triangulation picture, these $\mathcal{X}$-coordinates describe overlapping quadrilaterals, for instance



$$\sim \frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle},$$



$$\sim \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle} \tag{2.13}$$

which come in one-to-one correspondence with the diagonals in a triangulation.

Mutation rules for $\mathcal{X}$-coordinates are different than for $\mathcal{A}$-coordinates, and are given by

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i(1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases} . \tag{2.14}$$

Mutation still changes the arrows in the quiver diagram as it did in the case of $\mathcal{A}$-coordinates. Given just a quiver diagram, it can sometimes be unclear whether a given quiver should be mutated using the $\mathcal{A}$-coordinate or $\mathcal{X}$-coordinate rules (2.8) or (2.14). We adopt the convention in this work that if a quiver is given with no frozen nodes, it should be thought of as equipped with $\mathcal{X}$-coordinates.

Just as with $\mathcal{A}$-coordinate clusters, we can generate all $\mathcal{X}$-coordinate clusters by mutation. Mutating on alternating nodes of our A_2 cluster (2.12) (starting with x_2), we generate the following sequence of clusters:

$$\begin{aligned} x_1 &\to x_2 \\ x_1(1+x_2) &\leftarrow \frac{1}{x_2} \\ \frac{1}{x_1(1+x_2)} &\to \frac{x_2}{1+x_1+x_1x_2} \\ \frac{x_1x_2}{1+x_1} &\leftarrow \frac{1+x_1+x_1x_2}{x_2} \\ \frac{1+x_1}{x_1x_2} &\to \frac{1}{x_1} \\ x_2 &\leftarrow x_1 \\ &\vdots \end{aligned} \tag{2.15}$$

The series then repeats, with all arrows reversed. Note that (specifically in the case of A_2), if we label these $\mathcal{X}$-coordinates by

$$\mathcal{X}_1 = 1/x_1, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1+x_2), \quad \mathcal{X}_4 = \frac{1+x_1+x_1x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1+x_1}{x_1x_2}, \tag{2.16}$$

the mutation rule in (2.14) takes the simple form

$$1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}, \tag{2.17}$$

while all the clusters take the form $1/\mathcal{X}_i \to \mathcal{X}_{i+1}$. Eq. (2.17) is commonly referred to as the A_2 exchange relation. Putting this all together, we will generically refer to an A_2 cluster algebra as any set of clusters $1/\mathcal{X}_{i-1} \to \mathcal{X}_i$ for $i = 1 \dots 5$ where the $\mathcal{X}_i$ satisfy eq. (2.17). We believe it is useful at this point to emphasize that one can take as input any x_1 and x_2, and generate an associated A_2.

A very useful feature of cluster $\mathcal{X}$-coordinates is that they come equipped with a natural Poisson bracket structure, making the Grassmannian $\mathrm{Gr}(k, n)$ a cluster Poisson variety [].

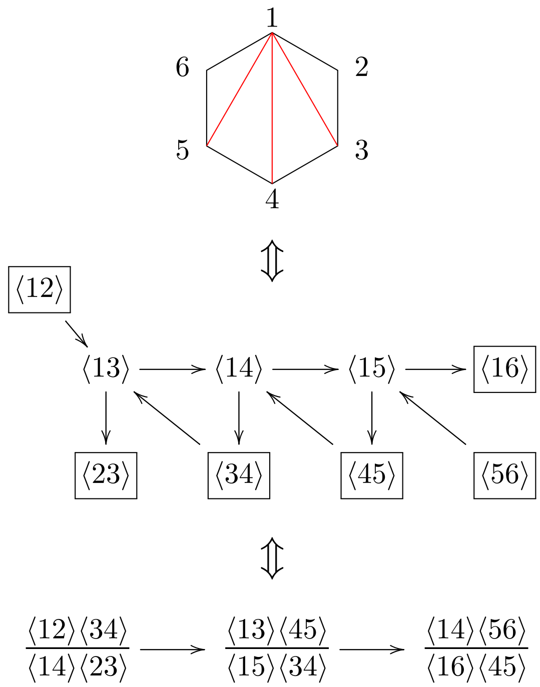


Figure 2: One triangulation of the hexagon, and its assocated $\mathcal{A}$-coordinate and $\mathcal{X}$-coordinate seed quivers.

Namely, when two $\mathcal{X}$-coordinates appear together in a cluster of $\mathrm{Gr}(k,n)$, there exists a Poisson bracket that evaluates to

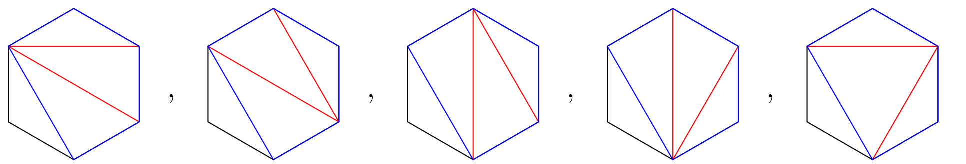
$$\{x_i, x_j\} = b_{ij} x_i x_j \ . \tag{2.18}$$

This structure respects mutation, implying that the entry b_{ij} (which counts the number of arrows from x_i to x_j in a given cluster's quiver) will be the same in all clusters containing both x_i and x_j. The Poisson bracket (and associated Sklyanin bracket) will play a larger role in a forthcoming companion paper [29], so we defer further discussion of this structure to there (for existing discussions in the literature, see also [30, 31]).

2.3 Subalgebras and Associahedra

Cluster algebras contain a rich and intricate subalgebra structure, which will play a central role in our analysis. It is simple to illustrate how these subalgebras arise by considering $\mathrm{Gr}(2,6)$, which triangulates the hexagon. In figure 2 we give the seed cluster for $\mathrm{Gr}(2,6)$ in the triangulation, $\mathcal{A}$-coordinate, and $\mathcal{X}$-coordinate representations, respectively. Since the mutable nodes take the form of an A_3 Dynkin diagram, we often speak of $\mathrm{Gr}(2,6)$ and A_3 interchangeably, just as we did with $\mathrm{Gr}(2,5) \simeq A_2$.

The $\mathrm{Gr}(2,6)$ cluster algebra features 14 clusters, which can be grouped into multiple (overlapping) subalgebras. A simple example is the collection of all triangulations which involve the chord $\langle 15\rangle$. This set contains 5 clusters and is itself a cluster algebra, which can be generated by treating $\langle 15\rangle$ as a frozen node (or in $\mathcal{X}$-coordinates, freezing the node $\frac{\langle 14\rangle\langle 56\rangle}{\langle 16\rangle\langle 45\rangle}$). This of course is the cluster algebra corresponding to the triangulations of the pentagon formed by points $1, \ldots, 5$, outlined here in blue:



(2.19)

We therefore refer to the collection of clusters that involve this pentagon as an A_2 subalgebra of $\mathrm{Gr}(2,6)$.

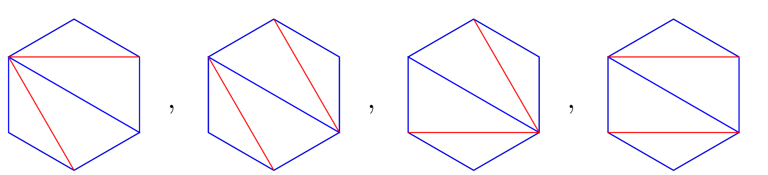
It should be clear, upon referring back to the mutation rules (2.8) and (2.14), that this A_2 subaglebra is truly identical to what we have been calling $\mathrm{Gr}(2,5)$. That is, neither the $\mathcal{A}$- or $\mathcal{X}$-coordinate mutation rule (or the rule for constructing the $\mathcal{X}$-coordinate cluster out of the $\mathcal{A}$-coordinate one) depends on nodes further than a single arrow away from the node on which one is mutating. Correspondingly, this subalgebra doesn't know about the existence of nodes involving point/column 6. (In the $\mathcal{X}$-coordinate case, the coordinate associated with the newly frozen node will change when one mutates on the node it is connected to, but the presence of this frozen node does not effect the coordinates appearing in the A_2 subalgebra itself.) We consider two subalgebras to be identical when the clusters they appear in only differ by nodes that have no effect on the mutable nodes of the subalgebra.

What if we instead disallow mutation on the chord $\langle 13\rangle$ (and the corresponding $\mathcal{X}$-coordinate node $\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}$)? Dropping the nodes that play no role in any of the mutations that remain gives rise to the effective quiver

$$\begin{tikzcd}
\boxed{\langle 13\rangle} \arrow[r] & \langle 14\rangle \arrow[r] \arrow[d] & \langle 15\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 34\rangle} & \boxed{\langle 45\rangle} \arrow[ul] & \boxed{\langle 56\rangle} \arrow[ul]
\end{tikzcd} , \tag{2.20}$$

where we have put a box around $\langle 13\rangle$ to make clear we are now treating it as frozen. We have also dropped the arrow from $\langle 34\rangle$ to $\langle 13\rangle$ since we are ignoring arrows between frozen nodes. The comparison to (2.6) should be clear; this just represents a re-labeled version of $\mathrm{Gr}(2,5)$.

Similarly, if we disallow mutation on the chord $\langle 14\rangle$ (and $\frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle}$), we generate an $A_1 \times A_1$ subalgebra, since the chord $\langle 14\rangle$ divides the hexagon in to two non-overlapping squares, each of which are triangulated by A_1 (or really $\mathrm{Gr}(2,4)$):



(2.21)

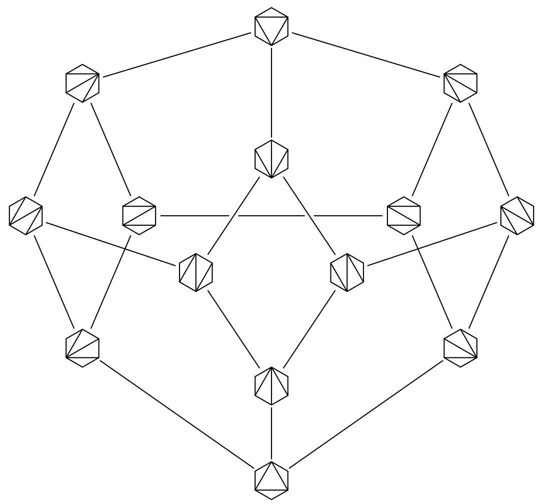


Figure 3: The associahedron for $A_3 \simeq \mathrm{Gr}(2,6)$, where each cluster is represented by a triangulation of the hexagon.

In appendix A we have tabulated the number of such A_2 and $A_1 \times A_1$ subalgebras in A_3, as well as the subalgebras of other cluster algebras that are relevant to seven-particle scattering in planar $\mathcal{N} = 4$. There it will be found that there are in fact six A_2 subalgebras and three $A_1 \times A_1$ subalgebras of A_3 (the remaining subalgebras do not involve the cluster in figure 2).

This subalgebra structure can be nicely visualized by constructing an object known as the associahedron (or the Stasheff polytope) of a given cluster algebra. The vertices of this polytope each represent a cluster, while its edges represent the mutations that map these clusters into each other. For instance, figure 1 corresponds to the $\mathrm{Gr}(2,5) \simeq A_2$ associahedron, which also coincidentally takes the form of a pentagon. Note that every node has valency two since each cluster has two mutable vertices.

Similarly, the associahedron of $\mathrm{Gr}(2,6) \simeq A_3$ is given in figure 3. It has 14 vertices, corresponding to the 14 clusters of $\mathrm{Gr}(2,6)$, each with valency three. These vertices assemble into three square faces and six pentagonal faces—corresponding exactly to the three $A_1 \times A_1$ subalgebras and the six A_2 subalgebras of A_3. This makes it easy to read off the subalgebra structure of A_3 directly.

In A_3, it turns out that each of the faces corresponds to a distinct subalgebra. Associahedra for larger cluster algebras become quite complicated, and it is often the case that distinct faces (or higher-dimensional polytopes) correspond to identical subalgebras. As an example, the associahedron of $\mathrm{Gr}(4,7) \simeq E_6$, which will be the focus of much of the rest of this paper, is shown in figure 4. It has 833 vertices, each of valence 6, and 1071 pentagons corresponding to A_2 subalgebras; however, only 504 of these A_2 subalgebras are distinct.

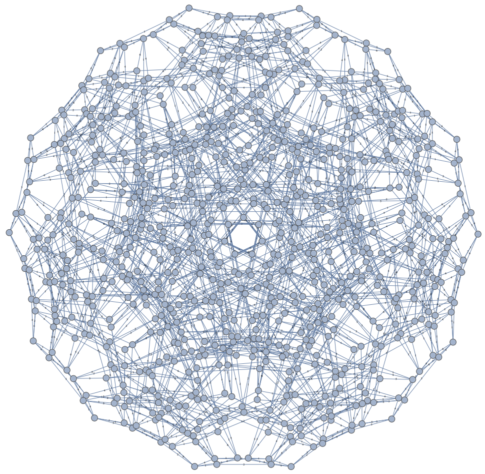
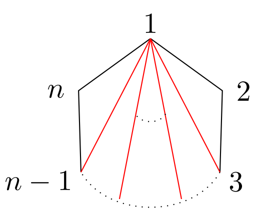


Figure 4: The associahedron for $E_6 \simeq \mathrm{Gr}(4,7)$.

2.4 Grassmannian Cluster Algebras and Planar $\mathcal{N}=4$ SYM Theory

So far we have leaned heavily on the correspondence between the triangulations of an n-gon and the cluster algebra for $\mathrm{Gr}(2,n)$. Based on the examples of $\mathrm{Gr}(2,5)$ and $\mathrm{Gr}(2,6)$, it is not hard to write down a generic seed cluster for $\mathrm{Gr}(2,n)$ corresponding to the triangulation consisting of all chords $\langle 13\rangle,\dots,\langle 1n{-}1\rangle$:



$$\Leftrightarrow \quad \begin{tikzcd} \boxed{\langle 12\rangle} \arrow[d] & & & \\ \langle 13\rangle \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \langle 1\ n-1\rangle \arrow[r] \arrow[d] & \boxed{\langle 1n\rangle} \\ \boxed{\langle 23\rangle} & \cdots \arrow[ul] & \boxed{\langle n-2\ n-1\rangle} \arrow[ul] & \boxed{\langle n-1\ n\rangle} \arrow[ul] \end{tikzcd} \quad . \tag{2.22}$$

Here one sees that $\mathrm{Gr}(2,n) \simeq A_{n-3}$.

For $\mathrm{Gr}(k>2,n)$, there is no longer a simple connection with triangulations or Dynkin diagrams. However, as shown by Scott [32] there exists a generalization of eq. (2.22) valid for all $\mathrm{Gr}(k,n)$:

$$\begin{tikzcd} \boxed{f_{11}} & f_{12} \arrow[l] \arrow[d] & f_{13} \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & f_{1l} \arrow[l] \arrow[d] & \boxed{\langle 1,\dots,k\rangle} \arrow[l] \\ \boxed{f_{21}} \arrow[ur] & f_{22} \arrow[l] \arrow[d] \arrow[ur] & f_{23} \arrow[l] \arrow[d] \arrow[ur] & \cdots \arrow[l] \arrow[d] \arrow[ur] & f_{2l} \arrow[l] \arrow[d] & \\ \vdots \arrow[ur] & \vdots \arrow[l] \arrow[d] \arrow[ur] & \vdots \arrow[l] \arrow[d] \arrow[ur] & \ddots \arrow[l] \arrow[d] \arrow[ur] & \vdots \arrow[l] \arrow[d] & \\ \boxed{f_{k1}} \arrow[ur] & \boxed{f_{k2}} \arrow[ur] & \boxed{f_{k3}} \arrow[ur] & \cdots \arrow[ur] & \boxed{f_{kl}} & \end{tikzcd} \quad , \tag{2.23}$$

where $l = n - k$ and

$$f_{ij} = \begin{cases} \langle i+1, \ldots, k, k+j, \ldots, i+j+k-1 \rangle, & i \leq l-j+1, \\ \langle 1, \ldots, i+j-l-1, i+1, \ldots, k, k+j, \ldots, n \rangle, & i > l-j+1. \end{cases} \tag{2.24}$$

(Note that evaluating the above expression for $k = 2$ will not directly give (2.22), however the two are equivalent under a cyclic rotation and dihedral flip of the n-gon.) The cluster algebra on $\mathrm{Gr}(k, n)$ is therefore of rank $(n-k-1)(k-1)$, i.e. the number of mutable nodes in (2.23).

The cluster algebra on $\mathrm{Gr}(4, n)$ naturally appears in planar $\mathcal{N} = 4$ SYM theory, where it parametrizes the space of n-particle kinematics. To make this connection, one first defines a set of dual coordinates x_i in terms of the external momenta p_i (which are endowed with a natural ordering in the planar limit) by

$$p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}\,, \tag{2.25}$$

where $x_{n+1}^{\alpha\dot{\alpha}} \equiv x_1^{\alpha\dot{\alpha}}$. These coordinates describe the cusps of a light-like Wilson loop dual to the amplitude; as a result, the original amplitude respects an additional superconformal symmetry that is associated with these dual coordinates (up to an anomaly associated with the cusps of the Wilson loop, which is accounted for by the BDS ansatz) [33–41]. In terms of the quantities in (2.25), we can further define momentum twistors Z_i

$$Z_i^R = (\lambda_i^\alpha, x_i^{\beta\dot{\alpha}} \lambda_{i\beta})\,, \tag{2.26}$$

where $R = (\alpha, \dot{\alpha})$ is an $SU(2,2)$ index. Momentum twistors are invariant under the little group, which acts as an overall rescaling $Z_i^R \to t_i Z_i^R$, and as such represent points in $\mathbb{CP}^3$.

If we assemble these momentum twistors into a $4 \times n$ matrix in which the i^{th} column corresponds to the four $SU(2,2)$ components of Z_i^R, invariance under the dual conformal group becomes invariance under $\mathrm{SL}(4)$. The overall rescaling symmetry of one of the momentum twistors can be combined with this $\mathrm{SL}(4)$ invariance to identify this matrix as a point in the (not necessarily positive) Grassmannian $\mathrm{Gr}(4, n)$, modulo the rescaling invariance of the remaining $n-1$ columns. Thus, the kinematic data of an n-point scattering process is encoded in a momentum twistor matrix

$$Z_n \in \mathrm{Gr}(4, n)/\mathrm{GL}(1)^{n-1}. \tag{2.27}$$

For more details regarding this correspondence, see [8, 22].

To relate the dual-conformal invariants encoded in Z_n to more familiar kinematic quantities, we can translate the (cyclically ordered) Mandelstam invariants into squared differences of dual coordinates,

$$s_{i,\ldots,j-1} \equiv (p_i + \ldots p_{j-1})^2 = \det(x_i^{\alpha\dot{\alpha}} - x_j^{\alpha\dot{\alpha}}) \equiv x_{ij}^2. \tag{2.28}$$

Dual conformal invariants can be constructed out of these objects by putting together combinations that are invariant under the dual conformal inversion generator, which acts on these coordinates as

$$I(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2}, \quad I(x_{ij}^2) = \frac{x_{ij}^2}{x_i^2 x_j^2}. \tag{2.29}$$

Thus, (regulated) amplitudes in this theory depend only on ratios of squared differences in which the same dual indices appear in both the numerator and denominator. The quantities x_{ij}^2 can be translated into momentum twistors using the relation

$$x_{ij}^2 = \frac{\det(Z_{i-1}Z_iZ_{j-1}Z_j)}{(\epsilon_{\alpha\beta}\lambda_{i-1}^\alpha\lambda_i^\beta)(\epsilon_{\gamma\delta}\lambda_{j-1}^\gamma\lambda_j^\delta)}, \tag{2.30}$$

where $\epsilon_{\alpha\beta}$ is the Levi-Civita tensor. In dual-conformally invariant quantities, the spinor products $\epsilon_{\alpha\beta}\lambda_{i-1}^\alpha\lambda_i^\beta$ all cancel, leaving only determinants of four-tuples of momentum twistors. These are just minors of the momentum twistor matrix (2.27), which we recognize as the cluster $\mathcal{A}$-coordinates

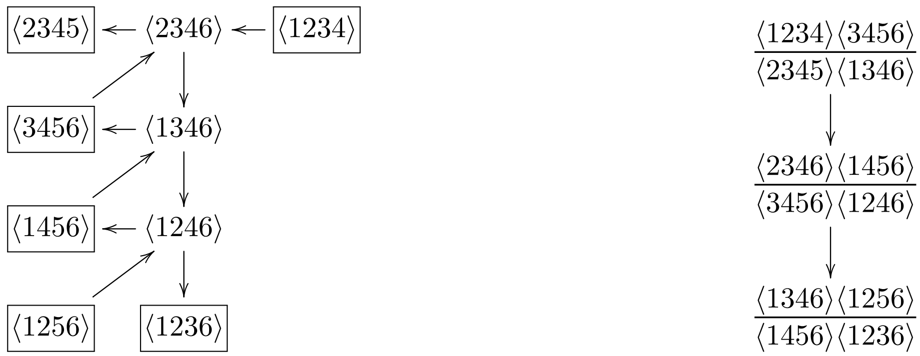
$$\langle ijkl\rangle = \det(Z_iZ_jZ_kZ_l). \tag{2.31}$$

Note that the two-particle Mandelstams $s_{i,i+1}$ correspond to the frozen nodes of (2.23), while higher-particle Mandelstams and more general (polynomials of) Plücker coordinates can appear as mutable nodes.

By construction, the $\mathcal{X}$-coordinates on $\mathrm{Gr}(4,n)$ derived from the seed (2.23) respect dual-conformal invariance. Both mutation rules (2.8) and (2.14) preserve this property (and commute with the translation (2.10)), ensuring that all $\mathcal{X}$-coordinates are dual conformal invariants. Such invariants cannot be formed in four- or five-particle kinematics, due to an insufficient number of non-lightlike separated points (since we are in massless kinematics, $x_{ii+1}^2 = 0$ for all i). This fact shows up in the seed (2.23) as $\mathrm{Gr}(4, n < 6)$ having no mutable nodes (and therefore no $\mathcal{X}$-coordinates). For $n > 5$, there are $3(n-5)$ mutable nodes in $\mathrm{Gr}(4,n)$, matching the number of algebraically independent dual conformal invariants that can be formed out of n massless particles [].

The $\mathcal{A}$-coordinate and $\mathcal{X}$-coordinate seed clusters of the first nontrivial example, $\mathrm{Gr}(4,6)$, are shown in figure 5. As discussed above, the three $\mathcal{X}$-coordinates in this cluster furnish us with a chart that covers the space of (dual-conformally invariant) six-particle kinematics. Moreover, we can generate new charts by mutation—every $\mathcal{X}$-coordinate cluster of $\mathrm{Gr}(4,n)$ provides a valid chart for n-particle kinematics. As explored in great depth in [22], these charts are especially well suited to describing the boundaries of the positive Grassmannian $\mathrm{Gr}_+(4,n)$, where the integrands of n-particle amplitudes can develop physical singularities. In particular, every such boundary occurs at the vanishing locus of an $\mathcal{A}$-coordinate of $\mathrm{Gr}(4,n)$, which implies it also occurs at the vanishing locus of some set of $\mathcal{X}$-coordinates.

This fact is especially propitious for loop-level amplitudes (and integrals) that only have branch points on the boundaries of the positive Grassmannian. In such cases, the symbol alphabet encoding the polylogarithmic part of these amplitudes is naturally given in terms of



(a) (b)

Figure 5: The $\mathcal{A}$-coordinate seed quiver (a) and $\mathcal{X}$-coordinate seed quiver (b) for $\mathrm{Gr}(4,6)$.

cluster coordinates. We defer discussion of the coaction and symbol alphabets to section 3.1, but here note that it is multiplicative independence, rather than algebraic independence, that is relevant in the context of symbol alphabets. Thus, while it is not possible to realize all boundaries of the positive Grassmannian as the vanishing loci of either type of cluster coordinate in a single chart [22], all boundaries are exposed as the vanishing of some symbol letter if cluster $\mathcal{A}$-coordinates or $\mathcal{X}$-coordinates on $\mathrm{Gr}(4,n)$ are adopted as a symbol alphabet.

While amplitudes in planar $\mathcal{N}=4$ are not generically expected to have this property (and indeed, certain Feynman integrals have been computed that do not [19, 20]), an infinite class of amplitudes do—namely, all two-loop MHV amplitudes [6], and all six- and seven-particle amplitudes computed to date [3–5, 14–16]. The significance of this property is illustrated by the two-loop, six-particle remainder function, which encodes the MHV amplitude. Namely, this function can be put in the form

$$R_6^{(2)} = -\sum_{\text{cyclic}} \left[\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 1346\rangle}\right) + \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)\right] + \dots, \tag{2.32}$$

where the cyclic sum is over all rotations of the four-bracket indices $i \to i+j$ for $0 \le j < 6$, and the dots indicate this equality only holds up to products of lower-weight polylogarithms. This projection is well-defined and will be introduced, along with the n-particle remainder function, in section 3. Here we just emphasize the simplicity of this expression, which takes the form of classical polylogarithms with negative $\mathcal{X}$-coordinate arguments. (In particular, the argument of the first polylogarithm is the top node in figure 5b, while the argument of the second polylogarithm is in the cluster generated by mutating on the top node of figure 5b.) Moreover, the part of the expression we have dropped in (2.32) can also expressed entirely in terms of products of classical polylogarithms with negative $\mathcal{X}$-coordinate arguments [7].

This surprising property—of being expressible as polylogarithms with negative $\mathcal{X}$-coordinate arguments—is enjoyed by the remainder function at all n. However, for $n > 6$ the remainder function has a nonclassical component, so generalized polylogarithms (in addition to clas-

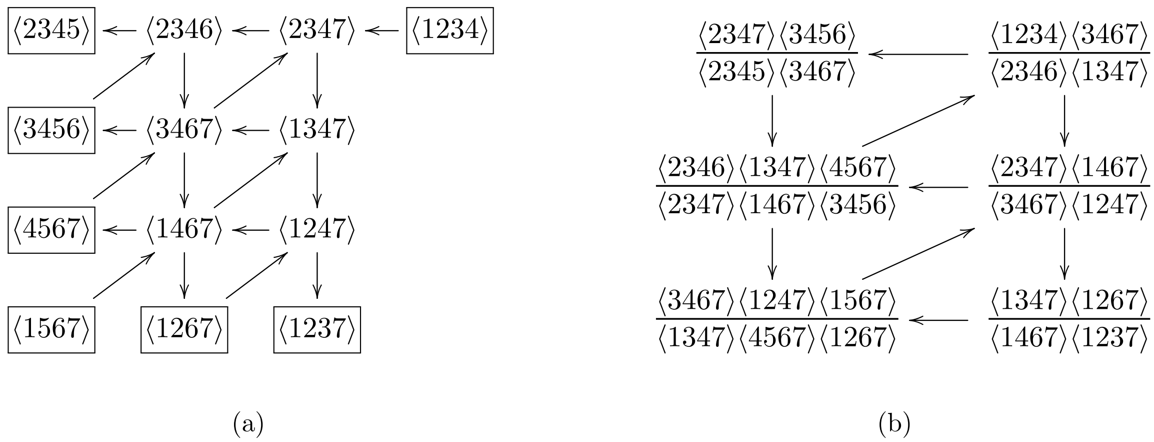


Figure 6: The $\mathcal{A}$-coordinate seed quiver (a) and $\mathcal{X}$-coordinate seed quiver (b) for $\mathrm{Gr}(4,7)$.

sical polylogarithms) with negative $\mathcal{X}$-coordinate arguments also appear []. Although this component represents the mathematically most complicated part of the remainder function, it was shown in [10] that it is decomposable into building blocks related to the A_2 and A_3 subalgebras of $\mathrm{Gr}(4,n)$. This allows the all-n symbol computed in [6] to be systematically upgraded to a function, as was done for seven particles in [7]. In the later sections of this paper, we demonstrate the existence of further subalgebra structure in the nonclassical part of the seven-particle remainder function (we leave consideration of higher-point kinematics to future work [29]). The $\mathcal{A}$-coordinate and $\mathcal{X}$-coordinate seeds for $\mathrm{Gr}(4,7)$ are presented in figure 6.

Before turning to the remaining aspects of cluster algebras that we wish to develop, we note that plabic graphs—which are dual to the clusters we've been describing—encode a great deal more about planar $\mathcal{N}=4$ than we have had reason to touch on. In particular, [**boundaries—a subset of which are our subalgebras (?)**]. We refer interested readers to the exposition of this rich structure given in [22].

2.5 Finite Cluster Algebras

The procedure of writing down an oriented oriented quiver, dressing it with coordinates, and iteratively mutating on all non-frozen nodes using either the $\mathcal{A}$-coordinate or $\mathcal{X}$-coordinate mutation rule will always produce a cluster algebra. However, generic quivers give rise to exceedingly complicated cluster algebras—in fact, for a wide class of seeds, mutation will generate an infinite numbers of clusters. For the rest of this paper we will restrict our attention to finite cluster algebras, leaving the discussion of infinite algebras to future work [29] (see [] for discussions of infinite cluster algebras elsewhere in the literature).

Fortunately, Fomin and Zelevinksy classified all finite cluster algebras in [42]. In particular, they showed that a cluster algebra is of finite type if and only if the mutable part of at least one of its clusters takes the form of an oriented, simply-laced Dynkin diagram: A_n, D_n, or $E_{n\leq 8}$. As we will primarily be interested in subalgebras of the cluster algebra on $\text{Gr}(4,7)$, we here focus on the cases where $n < 6$ (and on the case of $E_6 \simeq \text{Gr}(4,7)$ itself).

As mentioned above, cluster algebras of type A_n can be generated by the seed

$$x_1 \to x_2 \to \ldots \to x_n \ , \tag{2.33}$$

which corresponds to the cluster algebra on $\text{Gr}(2,n{+}3)$. Each of the clusters in these algebras can be though as triangulating an $(n+3)$-gon, where the $\mathcal{A}$-coordinates correspond to chords and the $\mathcal{X}$-coordinates to quadrilateral faces. This makes the counting easy: the number of clusters for A_n is given by the Catalan number $C(n+1)$, the number of distinct $\mathcal{A}$-coordinates is $\binom{n+3}{2} - n$, and the number of distinct $\mathcal{X}$-coordinates is $2\binom{n+3}{4}$. Any smaller polygon embedded into the $(n+3)$-gon gives rise to a subalgebra; for example, there are $56 = \binom{8}{5}$ pentagonal embeddings in an octagon, so there are 56 A_2 subalgebras in A_5.

Of particular interest is the cluster algebra generated by $A_3 \simeq \text{Gr}(4,6)$, which describes six-particle scattering. By comparison with figure 5b, we see that the $\mathcal{X}$-coordinates in the quiver (2.33) act as coordinates on the space of momentum twistors, where they correspond to the functions

$$x_1 = \frac{\langle 1234\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 1346\rangle}, \quad x_2 = \frac{\langle 2346\rangle\langle 1456\rangle}{\langle 3456\rangle\langle 1246\rangle}, \quad x_3 = \frac{\langle 1346\rangle\langle 1256\rangle}{\langle 1456\rangle\langle 1236\rangle}. \tag{2.34}$$

More generally, any $\mathcal{X}$-coordinate in $\text{Gr}(4,6)$ can be expressed in terms of the variables x_1, x_2, and x_3 by evaluating its four-brackets on the momentum twistor matrix

$$Z_{A_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & x_1 & 0 \\ 0 & 1 & 1 & 0 & -x_1 x_2 & 0 \\ 0 & 0 & 1 & 1 & x_1 x_2 x_3 & 0 \end{pmatrix} \tag{2.35}$$

The chief advantage of working directly in terms of cluster $\mathcal{X}$-coordinates such as x_1, x_2, and x_3 is that they trivialize all Plücker relations (2.1). Furthermore, cluster coordinates rationalize many of the square roots that appear when amplitudes and integrals are expressed in terms of dual-conformally-invariant cross ratios [19]. For instance, in this chart the dual conformal cross ratios commonly used to express the six-particle amplitude (see, for example [1, 43, 44]) evaluate to

$$u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle} = \frac{1}{1+x_2+x_2x_3}, \tag{2.36}$$

$$v = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle} = \frac{x_1x_2}{1+x_1+x_1x_2}, \tag{2.37}$$

$$w = \frac{\langle 2345\rangle\langle 5612\rangle}{\langle 2356\rangle\langle 4512\rangle} = \frac{x_2x_3}{(1+x_1+x_1x_2)(1+x_2+x_2x_3)}, \tag{2.38}$$

which rationalizes the well-known square root that appears in these cross ratios

$$\sqrt{(1-u-v-w)^2 - 4uvw} = \frac{x_2\,(1-x_1x_3)}{(1+x_1+x_1x_2)\,(1+x_2+x_2x_3)}. \tag{2.39}$$

Note that the cluster coordinate expressions (2.36) are rotated compared to those given elsewhere in the literature [8, 45] even though both arise from a seed of the form (2.33); this reflects a differing convention for the seed of $\mathrm{Gr}(k,n)$.

The first nondegenerate Dynkin diagram of type D_n is D_4, corresponding to the seed quiver

$$\begin{tikzcd}[row sep=small] & & x_3 \\ x_1 \arrow[r] & x_2 \arrow[ur] \arrow[dr] & \\ & & x_4 \end{tikzcd} . \tag{2.40}$$

Note here that we no longer are treating x_1, x_2, x_3 as defined in eq. (2.34), the x_i in eq. (2.40) are all generic variables. This seed turns out to generate the same cluster algebra as $\mathrm{Gr}(3,6)$; in particular, starting from the seed in (2.23) and mutating on the nodes initially labeled by f_{13} and then f_{23}, one arrives at the $\mathcal{X}$-coordinate quiver (2.40), where

$$x_1 = \frac{\langle 123\rangle\langle 345\rangle}{\langle 234\rangle\langle 135\rangle}, \quad x_2 = \frac{\langle 156\rangle\langle 235\rangle}{\langle 125\rangle\langle 356\rangle}, \quad x_3 = \frac{\langle 135\rangle\langle 456\rangle}{\langle 156\rangle\langle 345\rangle}, \quad x_4 = \frac{\langle 126\rangle\langle 135\rangle}{\langle 123\rangle\langle 156\rangle}. \tag{2.41}$$

The corresponding momentum twistor matrix is given by

$$Z_{D_4} = \begin{pmatrix} 1 & 0 & 0 & x_2 & x_2 & 1+x_2+x_2x_4 \\ 0 & 1 & 0 & -(1+x_1) & -1 & -(1+x_4) \\ 0 & 0 & 1 & 1+x_1+x_1x_2+x_1x_2x_3 & 1 & x_4 \end{pmatrix}. \tag{2.42}$$

Conversely, the cluster algebra generated by D_5,

$$\begin{tikzcd}[row sep=small] & & & x_4 \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\ & & & x_5 \end{tikzcd} , \tag{2.43}$$

is not equivalent to the cluster algebra on any Grassmannian (once again, ignore the notational overlap with eq. (2.41) and treat all of the x_i as generic variables). However, it appears as a subalgebra of any $\mathrm{Gr}(k,n)$ with rank greater than five. The D_4 cluster algebra consists of 50 clusters, 16 $\mathcal{A}$-coordinates, and 104 $\mathcal{X}$-coordinates, while D_5 has 182 clusters, 25 $\mathcal{A}$-coordinates, and 260 $\mathcal{X}$-coordinates.

Finally, the cluster algebra E_6 is generated by the quiver

$$\begin{tikzcd}[row sep=small] & & x_4 & & \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[u] & x_5 \arrow[l] & x_6 \arrow[l] \end{tikzcd} . \tag{2.44}$$

This cluster algebra is equivalent to $\mathrm{Gr}(4,7)$, as can be seen by mutating the seed (2.23) on nodes f_{12}, f_{13}, f_{23}, f_{12}, f_{22}, and then f_{32}. By comparison with (2.44), we then have

$$\begin{aligned}
x_1 &= \frac{\langle 1234\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1246\rangle}, & x_2 &= -\frac{\langle 1247\rangle\langle 3456\rangle}{\langle 4(12)(35)(67)\rangle}, \\
x_3 &= \frac{\langle 1246\rangle\langle 5(12)(34)(67)\rangle}{\langle 1245\rangle\langle 1267\rangle\langle 3456\rangle}, & x_4 &= -\frac{\langle 4(12)(35)(67)\rangle}{\langle 1234\rangle\langle 4567\rangle}, \\
x_5 &= -\frac{\langle 1267\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 4(12)(35)(67)\rangle}, & x_6 &= \frac{\langle 1567\rangle\langle 2345\rangle}{\langle 5(12)(34)(67)\rangle},
\end{aligned} \tag{2.45}$$

where we have made use of the notation

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle. \tag{2.46}$$

Any $\mathcal{X}$-coordinate on $\mathrm{Gr}(4,7)$ can be expressed in terms of these $\mathcal{X}$-coordinates using the momentum twistor matrix

$$Z_{E_6} = \begin{pmatrix} x_3x_5 & x_3x_5x_6 & x_1 & 0 & 1 & 0 & -1 \\ 1+x_5 & 1+x_6+x_5x_6 & -x_1 & 0 & -1+x_4 & x_4 & x_2x_4 \\ 0 & 0 & x_1 & 0 & 1 & 1 & x_2 \\ 0 & 0 & -1 & 1 & 1+x_2+x_2x_3 & 1 & 0 \end{pmatrix}. \tag{2.47}$$

The associahedron of $\mathrm{Gr}(4,7) \simeq E_6$ was displayed in figure 4; it contains 833 clusters, 42 $\mathcal{A}$-coordinates, and 770 $\mathcal{X}$-coordinates. The subalgebras of E_6, as well as those of its subalgebras, are tabulated in appendix A.

2.6 Cluster Automorphisms

Cluster algebras come equipped with an automorphism group that maps the set of cluster coordinates (but not necessarily the set of clusters) back to itself. We introduce here only what we need to elucidate the automorphisms of the cluster algebras introduced in the last section, and refer the interested reader to [46] for a more thorough mathematical introduction. Note that we describe automorphisms in terms of $\mathcal{X}$-coordinates, whereas [46] works in the $\mathcal{A}$-coordinate language.

The simplest example of a cluster automorphism is what we call a direct automorphism. Let $\mathcal{A}$ be a cluster algebra equipped with a mutation rule $\mu(x_i, \mathbf{X})$ that mutates the cluster $\mathbf{X}$ on node x_i. Then, we can define:

Direct Automorphism: The map $f : \mathcal{A} \to \mathcal{A}$ is a direct automorphism of $\mathcal{A}$ if

(i) for every cluster $\mathbf{X}$ of $\mathcal{A}$, $f(\mathbf{X})$ is also a cluster of $\mathcal{A}$,

(ii) f respects mutations, i.e. $f(\mu(x_i, \mathbf{X})) = \mu(f(x_i), f(\mathbf{X}))$.

An example of a direct automorphism on A_2 is given by

$$\sigma_{A_2} : \quad \mathcal{X}_i \to \mathcal{X}_{i+1}, \tag{2.48}$$

where we are using the coordinates introduced in (2.16). This automorphism cycles the five clusters $1/\mathcal{X}_i \to \mathcal{X}_{i+1}$ amongst themselves. The action of this automorphism can also be recast as

$$\sigma_{A_2}: \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to x_1(1+x_2), \tag{2.49}$$

using the $\mathcal{X}$-coordinates x_1 and x_2 that appear in (2.33).

Cluster algebras are also endowed with what we call indirect automorphisms, which respect mutations but do not map the set of clusters back to itself. Instead, indirect automorphisms map the clusters in $\mathcal{A}$ to clusters in $\mathcal{A}'$, where $\mathcal{A}'$ is constructed from $\mathcal{A}$ by multiplicatively inverting all cluster $\mathcal{X}$-coordinates and reversing the direction of all quiver arrows. Then we have:

Indirect Automorphism: The map $f : \mathcal{A} \to \mathcal{A}'$ is an indirect automorphism if

(i) for every cluster $\mathbf{X}$ of $\mathcal{A}$, $f(\mathbf{X})$ is a cluster of $\mathcal{A}'$

(ii) f respects mutations, i.e. $f(\mu(x_i, \mathbf{X})) = \mu(f(x_i), f(\mathbf{X}))$.

A_2 is also equipped with an indirect automorphism generated by

$$\tau_{A_2}: \quad \mathcal{X}_i \to \mathcal{X}_{6-i}, \tag{2.50}$$

where indices are understood to be mod 5. This can be recast in term of x_1 and x_2 as

$$\tau_{A_2}: \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to \frac{1}{x_1}. \tag{2.51}$$

To see that this is an indirect automorphism, consider

$$\tau_{A_2}(1/\mathcal{X}_1 \to \mathcal{X}_2) = 1/\mathcal{X}_5 \to \mathcal{X}_4 \; . \tag{2.52}$$

Inverting the cluster coordinates on the right hand side and reversing the arrow, we get back to $\mathcal{X}_5 \leftarrow 1/\mathcal{X}_4$, which was one of the original clusters of A_2.

It is useful to think of indirect automorphisms as generating a "mirror" or "flipped" version of the original cluster algebra, where the total collection of $\mathcal{X}$-coordinates is the same, but their Poisson bracket has flipped sign. The existence of this flip then can be seen as resulting from the arbitrary choice of overall sign for the exchange matrix; picking the other sign would have generated the same cluster-algebraic structure, but with different labels for the nodes. Indirect automorphisms capture the superficiality of this notation change.

The automorphisms σ_{A_2} and τ_{A_2} generate the complete automorphism group for A_2, namely the dihedral group $\mathfrak{D}_5$ (we denote the dihedral group in this font throughout so as not to confuse with the Dynkin diagram D_n). More generally, cluster algebras of type A_n have as their automorphism group the dihedral group $\mathfrak{D}_{n+3}$, which is generated by a cyclic generator

$$\sigma_{A_n}: \quad x_{k<n} \to \frac{x_{k+1}(1+x_{1,\ldots,k-1})}{1+x_{1,\ldots,k+1}}, \quad x_n \to \frac{1+x_{1,\ldots,n-1}}{\prod_{i=1}^n x_i}, \tag{2.53}$$

and a flip generator

$$\tau_{A_n}: \quad x_1 \to \frac{1}{x_n}, \;\; x_2 \to \frac{1}{x_{n-1}}, \;\; \ldots \;, x_n \to \frac{1}{x_1}. \tag{2.54}$$

In σ_{A_n} we have introduced the notation

$$x_{i_1,\ldots,i_k} \equiv \sum_{a=1}^{k} \prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1} x_{i_2} + \ldots + x_{i_1} \cdots x_{i_k}, \tag{2.55}$$

which we will also use below. For all n, σ_{A_n} generates a direct automorphism while τ_{A_n} generates an indirect automorphism.

The cluster algebra D_4 has automorphism group $\mathfrak{D}_4 \times S_3$. Both $\mathfrak{D}_4$ and S_3 come with a cyclic (direct automorphism) generator,

$$\begin{aligned}
&\sigma_{D_4}^{(\mathfrak{D}_4)}: \quad x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)\, x_1 x_2 x_3 x_4}{(1+x_{1,2,3})\,(1+x_{1,2,4})}, \quad x_3 \to \frac{1+x_{1,2}}{x_1 x_2 x_3}, \quad x_4 \to \frac{1+x_{1,2}}{x_1 x_2 x_4}, \\
&\sigma_{D_4}^{(S_3)}: \quad x_1 \to \frac{1}{x_3}, \quad x_2 \to \frac{x_1 x_2\,(1+x_3)}{1+x_1}, \quad x_3 \to x_4, \quad x_4 \to \frac{1}{x_1},
\end{aligned} \tag{2.56}$$

where $\sigma_{D_4}^{(\mathfrak{D}_4)}$ has length four, and $\sigma_{D_4}^{(S_3)}$ has length three. Then there are two flip generators

$$\begin{aligned}
&\tau_{D_4}^{(\mathfrak{D}_4)}: \quad x_2 \to \frac{1+x_1}{x_1 x_2\,(1+x_3)\,(1+x_4)}, \\
&\tau_{D_4}^{(S_3)}: \quad x_3 \to x_4, \quad x_4 \to x_3 \;,
\end{aligned} \tag{2.57}$$

where $\tau_{D_4}^{(\mathfrak{D}_4)}$ generates an indirect automorphism, and $\tau_{D_4}^{(S_3)}$ generates a direct automorphism.

The cluster algebras on $D_{n>4}$ with defining quiver

$$\begin{tikzcd}[row sep=small] & & & x_{n-1} \\ x_1 \arrow[r] & x_2 \arrow[r] & \cdots \arrow[r] & x_{n-2} \arrow[u] \arrow[d] \\ & & & x_n \end{tikzcd} \;, \tag{2.58}$$

have the automorphism group $\mathfrak{D}_n \times \mathbb{Z}_2$, with generators σ_{D_n} (length n, direct), τ_{D_n} (length 2, indirect), and $\mathbb{Z}_{2,D_n}$ (length 2, direct). In the case of D_5, these generators can be chosen to be

$$
\begin{aligned}
\sigma_{D_5}:\quad & x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)x_3}{1+x_{1,2,3}}, \quad x_3 \to \frac{x_1x_2x_3x_4x_5(1+x_{1,2})}{(1+x_{1,2,3,4})(1+x_{1,2,3,5})},\\
& x_4 \to \frac{1+x_{1,2,3}}{x_1x_2x_3x_4}, \quad x_5 \to \frac{1+x_{1,2,3}}{x_1x_2x_3x_5},\\[1ex]
\tau_{D_5}:\quad & x_1 \to x_1, \quad x_2 \to \frac{1+x_1}{x_1x_2(1+x_3x_5+x_{3,4,5})}, \quad x_3 \to \frac{x_3x_4x_5}{(1+x_{3,4})(1+x_{3,5})},\\
& x_4 \to \frac{1+x_3x_5+x_{3,4,5}}{x_4}, \quad x_5 \to \frac{1+x_3x_5+x_{3,4,5}}{x_5},\\[1ex]
\mathbb{Z}_{2,D_n}:\quad & x_4 \to x_5, \quad x_5 \to x_4.
\end{aligned}
\tag{2.59}
$$

More generally, for D_n cluster algebras, the action of $\mathbb{Z}_2$ is always realized by the exchange $x_{n-1} \leftrightarrow x_n$.

Finally, the automorphism group of $E_6 \simeq \mathrm{Gr}(4,7)$ is the dihedral group $\mathfrak{D}_{14}$. This group has generators σ_{E_6} (length 7, direct), τ_{E_6} (length 2, indirect), and $\mathbb{Z}_{2,E_6}$ (length 2, direct). In the coordinates of the quiver (2.44), these can be chosen to be

$$
\begin{aligned}
\sigma_{E_6}:\quad & x_1 \to \frac{1}{x_6(1+x_{5,3,4})}, \quad x_2 \to \frac{1+x_{6,5,3,4}}{x_5(1+x_{3,4})}, \quad x_3 \to \frac{(1+x_{2,3,4})(1+x_{5,3,4})}{x_3(1+x_4)},\\
& x_4 \to \frac{1+x_{3,4}}{x_4}, \quad x_5 \to \frac{1+x_{1,2,3,4}}{x_2(1+x_{3,4})}, \quad x_6 \to \frac{1}{x_1(1+x_{2,3,4})},\\[1ex]
\mathbb{Z}_{2,E_6}:\quad & x_1 \to x_6, \quad x_2 \to x_5, \quad x_3 \to x_3, \quad x_4 \to x_4, \quad x_5 \to x_2, \quad x_6 \to x_1,\\[1ex]
\tau_{E_6}:\quad & x_1 \to \frac{x_5}{1+x_{6,5}}, \quad x_2 \to (1+x_5)x_6, \quad x_3 \to \frac{(1+x_{1,2})(1+x_{6,5})}{x_1x_2x_3x_5x_6(1+x_4)},\\
& x_4 \to x_4, \quad x_5 \to x_1(1+x_2), \quad x_6 \to \frac{x_2}{1+x_{1,2}}.
\end{aligned}
\tag{2.60}
$$

In the language of $\mathrm{Gr}(4,7)$, these generators respectively correspond to cycling momentum twistor indices $Z_i \to Z_{i+1}$, flipping momentum twistor indices $Z_i \to Z_{8-i}$, and parity.

The polylogarithmic part of the n-particle MHV amplitude is invariant under parity transformations, as well as the dihedral group that represents Bose symmetry. These symmetries directly translate to automorphisms of the cluster algebra on $\mathrm{Gr}(4,n)$. However, it turns out the nonclassical part of these amplitudes can also be decomposed into building blocks that respect the automorphism group certain subalgebras of $\mathrm{Gr}(4,n)$. Making this statement precise will be the focus of much of the remainder of this paper.

3 Cluster Polylogarithms and MHV Amplitudes

The BDS ansatz captures the infrared structure of planar $\mathcal{N}=4$ SYM to all orders in the coupling [33]. In four- and five-particle kinematics it also furnishes the complete finite part of the amplitude, while for six or more particles it must be corrected by a finite dual-conformally invariant function [34–36]. In the case of the MHV amplitude, this correction is often computed in the form of the n-particle remainder function R_n, defined by

$$\mathcal{A}_n^{\rm MHV} \;=\; \mathcal{A}_n^{\rm BDS} \times \exp(R_n)\,, \tag{3.1}$$

where $\mathcal{A}_n^{\rm BDS}$ is the BDS ansatz for n particles [33]. Like the amplitude, the remainder function can be expanded in the coupling

$$R_n = g^4 R_n^{(2)} + g^6 R_n^{(3)} + g^8 R_n^{(4)} + \ldots\,, \tag{3.2}$$

where $g^2 = \frac{g_{\rm YM}^2 N_c}{16\pi^2}$. In this expansion we have used the fact that $R_n^{(1)} = 0$, since the BDS ansatz encodes the complete one loop MHV amplitude at all n.

The remaining L-loop contributions to the remainder function are expected to be expressible in terms of generalized polylogarithms [47–49] of uniform transcendental weight $2L$. This space is spanned by (products of) the functions

$$G(a_1,\ldots,a_k;z) \equiv \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_k;t), \qquad G(\underbrace{0,\ldots,0}_{k};z) \equiv \frac{\log^k z}{k!}\,, \tag{3.3}$$

where $G(;z) \equiv 1$, and the transcendental weight of each function corresponds to its number of indices k. In particular, the remainder function is expected to be a pure function of this type, meaning that its kinematic dependence appears in the indices and arguments a_i and z, but not in the rational prefactors multiplying these functions. This is known to be true at two loops, due to an impressive all-n computation that leveraged the superconformal symmetry of this theory [6], as well as through six loops in six-particle kinematics [1, 2, 4, 50] and through four loops in seven-particle kinematics [3, 5].

In addition to being generalized polylogarithms, loop-level contributions to the remainder function exhibit a great deal of cluster-algebraic structure. In particular, they are members of the space of 'cluster polylogarithms' studied in [10], indicating that their symbol is naturally expressible in terms of cluster $\mathcal{A}$-coordinates, while their Lie cobracket is naturally expressible in terms of cluster $\mathcal{X}$-coordinates (in a way that will be made precise below). Their cobracket, moreover, has been shown to be decomposable into simple functions associated with their A_2 and A_3 subalgebras [10]. As will be shown in the next section, these functions (the 'A_2 function' and the 'A_3 function') are invariant under the automorphism group of the algebras on which they are defined, up to a sign. This expresses the fact that these functions are well-defined under coordinate relabelings (or, are well-defined functions of oriented graphs). We correspondingly propose that the space of cluster polylogarithms be refined to include only functions that respect the automorphism group of the cluster algebra on which they are defined. In addition to this subalgebra structure, the symbols of these amplitudes have been found to satisfy a 'cluster adjacency' principle [12], and their cobracket takes a similarly restricted form [7]. The rest of this section is devoted to making these properties precise, for which purpose we first describe the motivic structure of polylogarithms.

3.1 The Symbol and Cobracket

The space of generalized polylogarithms defined by (3.3) is colossally overcomplete. This is because a_i and z are allowed to be arbitrarily complicated algebraic functions, and because these polylogarithms satisfy a shuffle and stuffle algebra. The shuffle algebra represents the fact that unordered integrations can be triangulated into a sum over iterated integrals [51, 52]. In general, this means that when two polylogarithms share an argument z, their product can be re-expressed as the sum of functions

$$G(a_1, \dots, a_{k_1}; z)\; G(a_{k_1+1}, \dots, a_{k_1+k_2}; z) = \sum_{\sigma \in \Sigma(k_1,k_2)} G(a_{\sigma(1)}, \dots, a_{\sigma(k_1+k_2)}; z), \tag{3.4}$$

where $\Sigma(k_1, k_2)$ denotes the set of all shuffles between the sets of integers $\{1, \dots, k_1\}$ and $\{k_1 + 1, \dots, k_1 + k_2\}$ (that is, all ways of interleaving these two sets such that the ordering of the elements within each of the original sets is maintained). The stuffle algebra naturally arises when generalized polylogarithms are re-expressed as infinite sums,

$$\begin{aligned} \mathrm{Li}_{n_1,\dots,n_d}(z_1, \dots, z_d) &\equiv \sum_{0<m_1<\cdots<m_d} \frac{z_1^{m_1} \cdots z_d^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}} \\ &= (-1)^d G(\underbrace{0, \dots, 0}_{n_d-1}, \frac{1}{z_d}, \dots, \underbrace{0, \dots, 0}_{n_1-1}, \frac{1}{z_1 \cdots z_d}; 1) \end{aligned} \tag{3.5}$$

where d is called the depth of the polylogarithm. Stuffle identities represent the freedom to split up unordered summation indices (arising from products of polylogarithms) into nested sums where these indices are ordered, as in (3.5).

This overcompleteness gives rise to a rich space of identities. This can already be seen at the level of classical polylogarithms, which correspond to the instances of (3.5) with depth one. **[description of five-term dilog identity, refer to further weight three/four identities [8, 53]]**

Fortunately, *all* identities between polylogarithms are (conjecturally) generated by shuffle and stuffle relations [54, 55]. These relations are trivialized (up to logarithmic identities) by the symbol map [], which can be defined in terms of total derivatives. For instance, taking the total derivative of (3.3), we have

$$dG(a_1, \dots, a_k; z) = \sum_{i=1}^{k} G(a_1, \dots, \hat{a}_i, \dots, a_k; z)\; d\log\left(\frac{a_{k-i+1} - a_{k-i}}{a_{k-i+1} - a_{k-i+2}}\right) \tag{3.6}$$

where $a_0 \equiv z$ and $a_{k+1} \equiv 0$, and the notation $\hat{a}_i$ indicates this index should be omitted [56]. The symbol map is then defined recursively by

$$\mathcal{S}\big(G(a_1, \dots, a_k; z)\big) \equiv \sum_{i=1}^{k} \mathcal{S}\big(G(a_1, \dots, \hat{a}_i, \dots, a_k; z)\big) \otimes \left(\frac{a_{k-i+1} - a_{k-i}}{a_{k-i+1} - a_{k-i+2}}\right). \tag{3.7}$$

The entries of the resulting k-fold tensor product are referred to as symbol letters. These symbol letters inherit the distributive properties of (arguments of) logarithms, and can therefore be expanded into a multiplicatively independent basis of symbol letters (the 'symbol alphabet'). Complicated polylogarithmic identities are thereby reduced to identities between logarithms, at the cost of losing information about the boundary of integration in (3.3).

The symbol also captures the analytic structure of polylogarithms, insofar as it encodes their (iterated) discontinuity structure. Namely, for generic indices a_i and argument z, these functions have nonzero monodromy only where the letters in the first entry of their symbol vanish or become infinite. These monodromies can themselves have branch cuts that the original function did not have, when new symbol letters appear in the second entry of the symbol, *ad infinitum*. For special values of a_i and z, some of this information is lost due to the fact that higher-weight transcendental constants (such as Riemann ζ values) are in the kernel of the symbol [49]. However, this information is retained by the coaction [57], and can be recovered by specifying an integration constant at each weight. We defer further consideration of these transcendental constants to a follow-up paper [].

In addition to the symbol, polylogarithms come equipped with a Lie cobracket structure [8]. The cobracket δ can be calculated using the symbol projection operator

$$\rho(s_1 \otimes \cdots \otimes s_k) = \frac{k-1}{k}\Big(\rho(s_1 \otimes \cdots \otimes s_{k-1}) \otimes s_k - \rho(s_2 \otimes \cdots \otimes s_k) \otimes s_1\Big), \tag{3.8}$$

where $\rho(s_1) \equiv s_1$. This projects onto the component of a symbol that cannot be written as a product of lower-weight polylogarithms (for instance, via the shuffle relations (3.4)). The action of ρ can be lifted to a projection on functions, up to terms proportional to transcendental constants; since we will not be concerned with these terms in what follows, we will abuse notation by applying ρ to functions directly. The cobracket δ of a weight k polylogarithm f can then be calculated as

$$\delta(f) \equiv \sum_{i=1}^{k-1} (\rho_i \wedge \rho_{k-i})\rho(f). \tag{3.9}$$

This notation indicates that the projection operator ρ is first applied to f, after which each term in the resulting sum is partitioned into a wedge product of weight i and weight $k-i$ functions (either by splitting up the symbol into its first i and last $k-i$ entries, or by taking the '$i, k-i$ component' of the coproduct); the projection operator is then applied to the entries in this wedge product separately. In general, the wedge product in (3.9) involves spaces of different weight. Without loss of generality, we can put the cobracket of any function into a form where the first factor of the wedge product has weight equal to or higher than that of the second factor (exchanging the order of factors when needed, at the cost of a minus sign). We denote by $\delta_{i,j}(f)$ the component of the cobracket of f that involves a wedge product of weight i and j functions, in that order—but we emphasize that this includes contributions from all terms in (3.9) that involve these weights in either order.

In the context of two-loop amplitudes, the salient property of the cobracket is that it isolates the component of weight four polylogarithms that cannot be written in terms of classical polylogarithms. Under the action of ρ, classical polylogarithms are mapped to elements of the Bloch group B_k [58, 59], namely the algebra of polylogarithms modulo identities between classical polylogarithms. Following [8], we denote these elements by

$$\{z\}_k = \rho(-\mathrm{Li}_k(-z)) \in \mathrm{B}_k. \tag{3.10}$$

For instance, [**maybe show that this trivializes an identity?**]. In this language, the action of the cobracket on classical polylogarithms is given by

$$\delta\big(\mathrm{Li}_k(-z)\big) = -\{z\}_{k-1} \wedge \{-1-z\}_1. \tag{3.11}$$

Note that, for the first time at weight four, there exists a component of the cobracket that is not mapped to by classical polylogarithms—namely, $\delta_{2,2}(\mathrm{Li}_4(-z)) = 0$. This is not true of of weight four polylogarithms in general, and in particular $\delta_{2,2}\big(R_n^{(2)}\big)$ is nonzero for $n \geq 7$. However, it has been shown that any weight four function that *is* annihilated by $\delta_{2,2}$ can be written in terms of classical polylogarithms (with potentially complicated arguments) [].

Lastly, it is often quite useful to employ the fact that the trivial cohomology of δ gives us an integrability condition,

$$\delta^2\big(f\big) = 0, \tag{3.12}$$

for any polylogarithm f. This condition implies an intricate relationship between the arguments of the Bloch group elements appearing in $\delta(f)$. For example, any weight 4 function f satisfies

$$\delta\big(\delta_{2,2}\big(f\big)\big) + \delta\big(\delta_{3,1}\big(f\big)\big) = 0, \tag{3.13}$$

where

$$\delta\big(\{x\}_3 \wedge \{-1-y\}_1\big) = \{x\}_2 \wedge \{-1-x\}_1 \wedge \{-1-u\}_1 \tag{3.14}$$

$$\begin{aligned}\delta\big(\{x\}_2 \wedge \{y\}_2\big) = &\{y\}_2 \wedge \{x\}_1 \wedge \{-1-x\}_1 \\ &- \{x\}_2 \wedge \{y\}_1 \wedge \{-1-y\}_1.\end{aligned} \tag{3.15}$$

So, while the $\delta_{2,2}$ component alone captures the nonclassical contribution to a function, this nonclassical contribution is liked to the classical contribution through its relation with the $\delta_{3,1}$ component.

The symbol and cobracket naturally stratify the study of two-loop amplitudes and integrals that can be expressed as polylogarithms. The operator $\delta_{2,2}$ isolates the nonclassical component of these functions, while the symbol captures their analytic structure up to terms proportional to transcendental constants. In the case of MHV amplitudes in planar $\mathcal{N} = 4$ SYM, both objects turn out to distill intriguing cluster-algebraic structure that would otherwise be hard to see at the level of full functions. It is to this structure that we now turn.

3.2 Cluster-Algebraic Structure in at Two Loops

As outlined in the introduction, the two-loop MHV amplitudes of this theory exhibit different forms of cluster-algebraic structure at the level of their cobracket, their symbol, and as full functions [7, 8, 10, 11]. The first facet of this structure concerns the building blocks that appear at each level, which are found to lie within restricted classes:

- The cobracket of $R_n^{(2)}$ can be written in terms of elements of the Block group taking the form $\{\mathcal{X}_i\}_k$, where $\mathcal{X}_i$ is an $\mathcal{X}$-coordinate on $\mathrm{Gr}(4, n)$
- The symbol of $R_n^{(2)}$ can be expressed in terms of symbol letters drawn from the set of $\mathcal{A}$-coordinates on $\mathrm{Gr}(4, n)$
- The function $R_n^{(2)}$ can expressed entirely in terms of (products of) polylogarithms taking the form $\mathrm{Li}_{n_1,\dots,n_d}(-\mathcal{X}_i, \dots, -\mathcal{X}_j)$, where each $\mathcal{X}_p$ is again an $\mathcal{X}$-coordinate on $\mathrm{Gr}(4, n)$

The physical meaning of the symbol alphabet restriction is clear, if unilluminating—the kinematic configurations in which these amplitudes are singular (due to internal propagators going on shell in the Feynman diagram expansion **[maybe this doesn't explain singularities in y_u-type letters?]**) coincide with the vanishing loci of certain cluster $\mathcal{A}$-coordinates. Even so, it is not clear how (or if) this restriction follows from physical principles. In this respect, the positive Grassmannian formulation of this theory is suggestive, insofar as the integrands of these amplitudes are seen to develop physical singularities only where $\mathcal{A}$-coordinates vanish **[or become infinite?][check]** [22]. However, this does not preclude the emergence of new branch cuts during integration, as has already been observed in Feynman integrals contributing to non-MHV amplitudes in this theory [19, 20]. The physical meaning of the restricted functional form and cobracket structure exhibited by these amplitudes remains even more obscure.

Cluster algebras also seem to play a role in how these building blocks are assembled in amplitudes. In particular, it was recently observed that—when these amplitudes are normalized appropriately—the only cluster $\mathcal{A}$-coordinates that appear in adjacent entries of their symbol are those that appear together in at least one cluster of $\mathrm{Gr}(4, n)$ [12]. This 'cluster adjacency' principle is not enjoyed by the remainder function (3.1), but by BDS-normalized amplitudes $\mathcal{E}_n$ [4, 5, 60, 61]. These are defined by

$$\mathcal{A}_n^{\mathrm{MHV}} \;=\; \mathcal{A}_n^{\mathrm{BDS-like}} \times \mathcal{E}_n, \tag{3.16}$$

where $\mathcal{A}_n^{\mathrm{BDS-like}}$ is related to $\mathcal{A}_n^{\mathrm{BDS}}$ by the cusp anomalous dimension [62]

$$\Gamma_{\mathrm{cusp}} = 4g^2 - 8\zeta_2 g^4 + \mathcal{O}(g^6) \tag{3.17}$$

and a simple weight two polylogarithm Y_n via

$$\mathcal{A}_n^{\mathrm{BDS-like}} = \mathcal{A}_n^{\mathrm{BDS}} \times \exp\left(\frac{\Gamma_{\mathrm{cusp}}}{4} Y_n\right). \tag{3.18}$$

The function Y_n corresponds to the part of the one-loop MHV amplitude that depends on three- and higher-particle Mandelstam invariants, where these invariants have been assembled into dual-conformally-invariant cross ratios (with the help of two-particle invariants). The BDS-like ansatz that remains only depends on two-particle invariants, yet accounts for the full infrared structure of these amplitudes.

The motivation for switching to the BDS-like normalization is precisely this restricted kinematic dependence. Since the BDS-like ansatz depends only on two-particle invariants, the functions $\mathcal{E}_n$ directly inherit the Steinmann relations between three- and higher-particle invariants that are obeyed by the full amplitude [4, 5, 63–65]. These relations tell us that **[check]**

$$\left.\begin{aligned} \mathrm{Disc}_{s_{j,\ldots,j+p+q}}\left[\mathrm{Disc}_{s_{i,\ldots,i+p}}(\mathcal{E}_n)\right] &= 0, \\ \mathrm{Disc}_{s_{i,\ldots,i+p}}\left[\mathrm{Disc}_{s_{j,\ldots,j+p+q}}(\mathcal{E}_n)\right] &= 0, \end{aligned}\right\} \quad 0 < j-i \le p \;\text{ or }\; q < i-j \le p+q, \tag{3.19}$$

where all indices should be considered mod n. Formulated in terms of symbol entries, this implies that the cluster $\mathcal{A}$-coordinates $\langle j-1, j, j+p+q-1, j+p+q\rangle$ and $\langle i-1, i, i+p-1, i+p\rangle$ never appear next to each other in the first two entries of the symbol. In fact, it is believed that these constraints can be applied at all depths in the symbol, as these letters are never seen to appear next to each other [4, 5, 13, 21, 50]. These generalized constraints have been termed the extended Steinmann relations, as they amount to applying the relations (3.19) to all discontinuities of the amplitude in addition to the amplitude itself.

The fact that the functions $\mathcal{E}_n$ also obey the cluster adjacency principle is far more surprising. The constraints that follow from this principle take a form similar to the extended Steinmann relations (insofar as they restrict which symbol letters can appear in adjacent entries), and in fact turn out to be equivalent in six-particle kinematics when applied to functions with physical branch cuts. It is therefore tempting to believe cluster adjacency follows from some set of physical principles that includes the extended Steinmann relations. However, it is not yet known whether these two conditions are equivalent at all n.

Seeming to complicate this question of equivalence is the fact that the BDS-like ansatz (3.18) only exists when n is not a multiple of four. This is because no function satisfying the above description of Y_n exists for these particle multiplicities [5, 66]. (Such a function not only exists for all other n, but is uniquely picked out by this description.) Stated another way, when n is a multiple of four, any normalization that absorbs the infrared-divergent part of the amplitude either depends on some set of three- or higher-particle Mandelstam invariants, or spoils the dual conformal invariance of the resulting normalized amplitude. However, this shows this complication is superficial—for the purpose of understanding the relationship between the extended Steinmann relations and the cluster adjacency principle, we need merely choose the latter horn of this dilemma, and give up dual conformal invariance.

This issue first arises in eight-particle kinematics. There it can be seen—by direct computation—that both sets of constraints are obeyed when the amplitude is normalized by a generalized BDS-like ansatz whose kinematic dependence is restricted to two-particle

Mandelstam invariants [29]. This provides further evidence that the extended Steinmann relations and the cluster adjacency principle are encoding the same information.

While cluster adjacency was first observed in $\mathcal{A}$-coordinates, it can also be formulated in terms of $\mathcal{X}$-coordinates. Unlike $\mathcal{A}$-coordinates, which are multiplicatively independent, $\mathcal{X}$-coordinates satisfy numerous multiplicative identities. Thus, there will exist many representations of the same symbol in terms of $\mathcal{X}$-coordinates, whereas its representation in terms of $\mathcal{A}$-coordinates is unique. In general, only a subset of these $\mathcal{X}$-coordinate representations will satisfy cluster adjacency (even if the $\mathcal{A}$-coordinate representation does). Moreover, while $\mathcal{X}$-coordinate adjacency trivially implies $\mathcal{A}$-coordinate adjacency by the relation (2.10), the converse can not be true in general. This is because $\mathcal{A}$-coordinates can express a larger class of functions than $\mathcal{X}$-coordinates, insofar as the latter necessarily respect dual conformal symmetry while the former do not. However, it can be checked that every dual-conformal-invariant symbol that satisfies $\mathcal{A}$-coordinate adjacency in $\mathrm{Gr}(k \leq 4, n \leq 7)$ has an $\mathcal{X}$-coordinate adjacent representation. Although we do not have a general proof of this, we conjecture that this remains true for all dual-conformally-invariant symbols constructed on $\mathrm{Gr}(k,n)$, for general k and n. **[need to actually check that DCI + A-adjacency implies X-adjacency for all** $\mathrm{Gr}(k \leq 4, n \leq 7)$**, tho that may well be too much work, instead we can check for as many cases as are computationally easy and leave the rest as conjecture – john]**

The cobracket $\delta\big(R_n^{(2)}\big)$ also satisfies its own form of cluster adjacency [11]. Namely, $\delta\big(R_n^{(2)}\big)$ can be expressed as a linear combination of terms $\{\mathcal{X}_i\}_2 \wedge \{\mathcal{X}_j\}_2$ and $\{\mathcal{X}_k\}_3 \otimes \mathcal{X}_l$ where $\{\mathcal{X}_i, \mathcal{X}_j\}$ and $\{\mathcal{X}_k, \mathcal{X}_l\}$ appear together in a cluster of $\mathrm{Gr}(4,n)$. For example, $\delta_{2,2}\big(R_n^{(2)}\big)$ can be expressed as a sum of terms of the form $\{v_{ijk}\}_2 \wedge \{z^\pm_{pqr}\}_2$, where (using the notation defined in (2.46))

$$v_{ijk} = -\frac{\langle i+1(i,i+2)(j,j+1)(k,k+1)\rangle}{\langle i,i+1,k,k+1\rangle\langle i+1,i+2,j,j+1\rangle}, \tag{3.20}$$

and

$$\begin{aligned} z^+_{ijk} &= \frac{\langle i,j-1,j,j+1\rangle\langle i+1,k-1,k,k+1\rangle - \langle i+1,j-1,j,j+1\rangle\langle i,k-1,k,k+1\rangle}{\langle i,k-1,k,k+1\rangle\langle i+1,j-1,j,j+1\rangle}, \\ z^-_{ijk} &= \frac{\langle i,i+1,j,k\rangle\langle i-1,i,i+1,i+2\rangle}{\langle i-1,i,i+1,k\rangle\langle i,i+1,i+2,j\rangle}. \end{aligned} \tag{3.21}$$

As long as $i < j < k$ (considered mod n), the quantities v_{ijk}, z^+_{ijk}, and z^-_{ijk} each constitute $\mathcal{X}$-coordinates, where z^+_{ijk} and z^-_{ijk} are also parity conjugate to each other. The coordinates v_{ijk} were originally motivated by consideration of the location of physical branch cuts, while the $z^\pm_{ijk}$ were motivated by consideration of the final symbol entry of the remainder function, which is constrained to take the form $\langle i-1,i,i+1,j\rangle$ by dual superconformal symmetry [14]. However, for the present discussion, the pertinent point is that $\delta_{2,2}\big(R_n^{(2)}\big)$ can be expressed as a sum of terms $\{v_{ijk}\}_2 \wedge \{z^\pm_{pqr}\}_2$ in which v_{ijk} and $z^\pm_{pqr}$ also occur together in a cluster. Importantly, this form of cobracket-level cluster adjacency isn't implied by the cluster $\mathcal{A}$-coordinate adjacency observed in [12], as can be seen in the A_2 function we will define in the next subsection—this function satisfies cluster $\mathcal{A}$-coordinate adjacency, but the $\mathcal{X}$-coordinates appearing in its $\delta_{2,2}$ cobracket component cannot be chosen to satisfy cobracket-level cluster adjacency.

While this cobracket-level structure was originally observed in the remainder function, the same characterization carries over to the BDS-like normalized amplitude. This can be seen by combining equations (3.1), (3.16), and (3.18) to express $\mathcal{E}_n$ in terms of R_n:

$$\begin{aligned}\mathcal{E}_n &= \exp\left(R_n - \frac{\Gamma_{\text{cusp}}}{4} Y_n\right) \\ &= 1 - Y_n g^2 + \left(R_n^{(2)} + 2\zeta_2 Y_n + \frac{1}{2} Y_n^2\right) g^4 + \mathcal{O}(g^6).\end{aligned} \tag{3.22}$$

Thus, $\rho\big(\mathcal{E}_n^{(2)}\big) = \rho\big(R_n^{(2)}\big)$, since these two functions differ only by products and terms involving transcendental constants. Correspondingly, the cobracket structure of the remainder function and the BDS-like normalized amplitude are identical. Moreover, the function Y_n can always be expressed in terms of classical polylogarithms taking negative $\mathcal{X}$-coordinates as arguments [**check**]. Thus, at the level of its cobracket, its symbol, and as a full function, $\mathcal{E}_n^{(2)}$ can be expressed in terms of the same restricted building blocks as $R_n^{(2)}$. To these observations we can now add:

- The cobracket $\delta\big(\mathcal{E}_n^{(2)}\big)$ can be expressed as a linear combination of terms $\{\mathcal{X}_i\}_2 \wedge \{\mathcal{X}_j\}_2$ and $\{\mathcal{X}_k\}_3 \otimes \mathcal{X}_l$ where $\{\mathcal{X}_i, \mathcal{X}_j\}$ and $\{\mathcal{X}_k, \mathcal{X}_l\}$ appear together in a cluster of $\mathrm{Gr}(4,n)$.
- Pairs of $\mathcal{A}$-coordinates only appear in adjacent entries of the symbol $\mathcal{S}\big(\mathcal{E}_n^{(2)}\big)$ when they also appear together in at least one cluster of $\mathrm{Gr}(4,n)$.

As discussed above, the last statement can also be applied at n that are multiples of four by going to a generalized BDS-like normalization in which only two-particle invariants appear in the normalizing function.

As it turns out, there exists yet more structure in $\delta_{2,2}\big(R_n^{(2)}\big) = \delta_{2,2}\big(\mathcal{E}_n^{(2)}\big)$. In particular, it was shown in [10] that this cobracket component can be decomposed into a sum over various A_2 subalgebras of $\mathrm{Gr}(4,n)$, by defining an A_2 function that can be evaluated on each of these subalgebras. Moreover, this A_2 function can be assembled into an A_3 function, in terms of which this cobracket component can similarly be decomposed. As this subalgebra decomposability will play a central role in what follows, we devote the next subsection to its description.

3.3 Subalgebra Structure and Cluster Polylogarithms

As we saw in section 2.3, cluster algebras are endowed with subalgebras that can be generated by mutating on restricted sets of nodes. This motivates looking for physically relevant cluster polylogarithms on algebras other than $\mathrm{Gr}(4,n)$, when these algebras appear as subalgebras of the latter. Before we do so, let us return to the definition of these objects, which we are now in a position to make precise. Following [10], we define cluster polylogarithms (at least through weight four) to have the following properties:

Cluster Polylogarithm: A generalized polylogarithm f is a cluster polylogarithm on a cluster algebra $\mathcal{A}$ if

(i) the symbol alphabet of f is composed of only $\mathcal{A}$-coordinates on $\mathcal{A}$,

(ii) the cobracket of f can be expressed in terms of Bloch group elements with arguments drawn from the $\mathcal{X}$-coordinates of $\mathcal{A}$,

(iii) the function f is invariant under the automorphisms of $\mathcal{A}$, up to a sign.

Property (iii) can be thought of as the requirement that cluster polylogarithms be well-defined functions on the cluster algebra, or more specifically, on an oriented graph representing a cluster inside that algebra. In other words, since cluster algebras can be generated by any individual cluster inside the algebra, functions corresponding to a cluster algebra should not depend on which cluster one uses as input. For instance, if we wish to define a function related to the A_2 cluster algebra, as we shall do shortly, it should satisfy the property that $f_{A_2}(1/\mathcal{X}_1 \to \mathcal{X}_2) = \pm f_{A_2}(1/\mathcal{X}_2 \to \mathcal{X}_3)$, and similarly when the function is evaluated on the other clusters. This ambiguity in the overall sign, which will be discussed below, reflects the fact that some automorphisms flip the orientation of cluster algebras while others do not.

The only polylogarithm that can be formed on the rank one cluster algebra A_1 is $\log^n(x)$, which is trivially a cluster polylogarithm but too simple to be of interest. However, a nontrivial cluster polylogarithm can already be defined on A_2. As A_2 subalgebras are generated by all pairs of connected nodes, it appears ubiquitously in $\mathrm{Gr}(4,n)$.

A central fact about the 'A_2 function,' also referred to as f_{A_2}, is that it is uniquely determined by the cluster polylogarithm conditions, modulo products of classical polylogarithms of weight ≤ 3 (see [10] for an in-depth discussion of the A_2 function). The analytic properties by adding and subtracting classical polylogarithms (which again must respect the automorphisms of A_2), and in this work choose to define the A_2 function (which can be thought of as a 'function on an oriented graph') as

$$\begin{aligned} f_{A_2}(x_1 \to x_2) = \sum_{\text{skew-dihedral}} \Bigg[&\mathrm{Li}_{2,2}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) - \mathrm{Li}_{1,3}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) \\ &+ 6\,\mathrm{Li}_3\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right) + \frac{1}{2}\log\left(\mathcal{X}_{i-3}\right)\log\left(\mathcal{X}_{i}\right)\log^2(\mathcal{X}_{i-1}) \\ &- \mathrm{Li}_2(-\mathcal{X}_{i-1})\log(\mathcal{X}_{i+1})\Big(3\log(\mathcal{X}_{i-1}) - \log(\mathcal{X}_i) + \log(\mathcal{X}_{i+1})\Big)\Bigg], \end{aligned} \tag{3.23}$$

where the $\mathcal{X}_i$ are defined in terms of x_1 and x_2 as in eq. (2.16), and the skew-dihedral sum indicates subtracting the dihedral flip $\mathcal{X}_i \to \mathcal{X}_{6-i}$ and taking a cyclic sum $i = 1$ to 5. This function differs in a few salient respects from the A_2 function defined in [10], but (necessarily) has the same cobracket

$$\delta\big(f_{A_2}\big) = -\sum_{\text{skew-dihedral}} \{-\mathcal{X}_{i-1}\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + 3\{-\mathcal{X}_i\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + \frac{5}{2}\{-\mathcal{X}_i\}_3 \otimes \mathcal{X}_{i+1}. \tag{3.24}$$

We highlight some of the properties enjoyed by this A_2 function below.

Remarkably, it was shown in [10] that all of the information contained in $\delta_{2,2}\big(R_n^{(2)}\big)$ is encoded in the A_2 function, when this function is evaluated on some collection of the A_2 subalgebras of $\mathrm{Gr}(4,n)$. That is,

$$\delta_{2,2}\big(R_n^{(2)}\big) = \sum_{(x_i \to x_j) \subset \mathrm{Gr}(4,n)} c_{ij}\ \delta_{2,2}\big(f_{A_2}(x_i \to x_j)\big) \tag{3.25}$$

for some rational coefficients c_{ij}. Moreover, the terms in this sum can themselves be arranged into A_3 subalgebras, giving rise to a natural A_3 function of the form

$$f_{A_3}(x_1 \to x_2 \to x_3) \sim \sum_{(x_i \to x_j) \subset A_3} d_{ij}\ f_{A_2}(x_i \to x_j), \tag{3.26}$$

where the d_{ij} are some rational coefficients that we treat in full detail in the next section. For now, we merely highlight the fact that these coefficients can be chosen in such a way that $\delta_{2,2}\big(f_{A_3}\big)$ contains only terms $\{\mathcal{X}_i\}_2 \otimes \{\mathcal{X}_j\}_2$ in which $\mathcal{X}_i$ and $\mathcal{X}_j$ appear together in at least one cluster of $\mathrm{Gr}(4,n)$. Then, there exists a decomposition

$$\delta_{2,2}\big(R_n^{(2)}\big) = \sum_{(x_i \to x_j \to x_k) \subset \mathrm{Gr}(4,n)} c_{ijk}\ \delta_{2,2}\big(f_{A_3}(x_i \to x_j \to x_k)\big). \tag{3.27}$$

Thus, the cobracket-level cluster adjacency enjoyed by these amplitudes is made manifest in the A_3 decomposition (3.27).

In the following sections we analyze these decompositions systematically, and find they can be extended to much larger subalgebras in seven-particle kinematics (although these techniques also extend to higher n [29]). As in the A_3 decomposition, the A_2 function will continue to play a privileged role. For this reason, we highlight some of the properties we have built into this function before turning to our general analysis.

Recall that the clusters of A_2 all take the form $1/\mathcal{X}_i \to \mathcal{X}_{i+1}$ in the $\mathcal{X}_i$ variables (2.16). Computing the symbol

$$\begin{aligned} \mathcal{S}(f_{A_2}) = -\sum_{\text{skew-dihedral}} \Big[&\mathcal{X}_2 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_3 + \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_2 \otimes \mathcal{X}_1 + \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_3 \otimes \mathcal{X}_2 \\ &- 2\big(\mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 + \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_4 \otimes \mathcal{X}_3 - \mathcal{X}_2 \otimes \mathcal{X}_3 \otimes \mathcal{X}_3 \otimes \mathcal{X}_4\big)\Big], \end{aligned} \tag{3.28}$$

we see that f_{A_2} satisfies cluster $\mathcal{X}$-coordinate adjacency, and therefore also $\mathcal{A}$-coordinate adjacency (recall that the symbol is insensitive to the difference between $\mathcal{X}_i$ and $1/\mathcal{X}_i$). Note that this means any sum of A_2 functions evaluated on the subalgebras of some larger cluster algebra also respects cluster $\mathcal{A}$-coordinate adjacency. At the level of a full function, f_{A_2} is also smooth and real-valued in the positive domain $x_1, x_2 > 0$. The A_2 cluster algebra plays a crucial role in endowing f_{A_2} with this analytic behavior, as $\mathrm{Li}_{2,2}(x,y)$ and $\mathrm{Li}_{1,3}(x,y)$ have branch cuts in three locations, namely $x = 1$, $y = 1$, and $xy = 1$. The first two branch cuts are trivially avoided as $-1/\mathcal{X}_i < 0$ for $x_1, x_2 > 0$. However, the last branch cut is also avoided because of the A_2 exchange relation:

$$0 < \left(-\frac{1}{\mathcal{X}_{i-1}}\right)\left(-\frac{1}{\mathcal{X}_{i+1}}\right) = \frac{1}{1+\mathcal{X}_i} < 1. \tag{3.29}$$

Finally, we note that (just like $R_n^{(2)}$ and $\mathcal{E}_n^{(2)}$) f_{A_2} is expressible entirely in terms of polylogarithms taking the form $\text{Li}_{n_1,\dots,n_d}(-\mathcal{X}_i,\dots,-\mathcal{X}_j)$, where each $\mathcal{X}_p$ is an $\mathcal{X}$-coordinate. This can be seen from the representation in (3.23), since the multiplicative inverses of $\mathcal{X}$-coordinates also always appear as $\mathcal{X}$-coordinates, and since the arguments of the logarithms can all be put in this form at the cost of factors of $i\pi$.

4 Nonclassical Cluster Polylogarithms

We now turn to a more systematic exploration of the space of weight-four nonclassical cluster polylogarithms. We restrict our attention to functions that can be defined on $\text{Gr}(4,7) \simeq E_6$ and its subalgebras, as this space is complex enough to give rise to an interesting space of functions, yet can be explored exhaustively. The techniques we utilize can also be applied to infinite cluster algebras (see for instance [10]), but we leave further exploration of this kind to future work [29]. Before exploring the properties of this space of functions on $\text{Gr}(4,7)$, we describe an efficient method for its generation using the A_2 function. We then describe how this method can be extended to more restricted spaces of functions—those that are subalgebra-constructible out of other cluster polylogarithms—which spaces we hope will prove to be of general interest to physicists and mathematicians.

4.1 A_2 Functions as a Basis

A remarkable (conjectured) property of f_{A_2} is that it forms a complete basis for the $\delta_{2,2}$ component of any weight-four cluster polylogarithm. (This property can be constructively realized for all cluster polylogarithms defined on E_6 or its subalgebras, but has not been proven in general.) That is, given a weight-four cluster polylogarithm F defined on a cluster algebra $\mathcal{A}$, there always exists some decomposition

$$\delta_{2,2}\big(F\big) = \sum_{(x_i\to x_j)\subset\mathcal{A}} c_{ij}\ \delta_{2,2}\big(f_{A_2}(x_i\to x_j)\big) \tag{4.1}$$

involving rational coefficients c_{ij}, where the sum ranges over all the A_2 subalgebras of $\mathcal{A}$. Given the symbol (or cobracket) of F, this decomposition is simple to compute (but may not be unique).

We can glean some intuition for the meaning of this decomposition from the cobracket integrability condition (3.12). This condition implies specific algebraic relationships between the arguments of the $\delta_{2,2}$ and $\delta_{3,1}$ cobracket components. In particular, the A_2 exchange relation

$$1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1} \tag{4.2}$$

is sufficient to generate one solution to integrability, namely f_{A_2}. The decomposition (4.1) thus tells us that *all* solutions to the cobracket integrability relation can be interpreted as linear combinations of exchange relations on A_2 subalgebras. This hardly comes as a surprise, given that the only algebraic relations between cluster coordinates are generated by mutation (assuming that the seed cluster is composed of algebraically independent coordinates), and that an A_2 subalgebra can be generated on any pair of connected nodes.

In practice, then, the existence (and uniqueness) of f_{A_2} solves the problem of writing down weight-four cluster polylogarithms. To illustrate this, let us consider the space of non-classical A_3 cluster polylogarithms. This space could be computed by forming an ansatz of all possible cobracket components constructed out of cluster $\mathcal{X}$-coordinates, and then imposing the integrability condition (3.12). However, the decomposability conjecture (4.1) reduces this computation to the simpler question: how many cluster polylogarithms can be constructed out of the A_2 subalgebras of A_3?

Recall from section 2.3 that there are six A_2 subalgebras in A_3. To label these subalgebras, we choose a representative cluster from each one; in terms of A_3 coordinates (defined by the initial seed $x_1 \to x_2 \to x_3$), we take the set

$$\begin{gathered} x_1 \longrightarrow x_2, \qquad x_2 \longrightarrow x_3, \qquad \frac{x_2}{1+x_{1,2}} \longrightarrow \frac{(1+x_1)\,x_3}{1+x_{1,2,3}}, \\ \frac{x_1 x_2}{1+x_1} \longrightarrow x_3, \qquad x_1\,(1+x_2) \longrightarrow \frac{x_2 x_3}{1+x_2}, \qquad x_1 \longrightarrow x_2\,(1+x_3)\,, \end{gathered} \tag{4.3}$$

each of which generates a distinct A_2 cluster algebra. We then construct an ansatz for the space of nonclassical A_3 functions out of f_{A_2} functions evaluated on each of these subalgebras:

$$f_{A_3}\big(x_1 \to x_2 \to x_3\big) = c_1 f_{A_2}\big(x_1 \to x_2\big) \; + \; \dots \; + \; c_6 f_{A_2}\big(x_1 \to x_2\,(1+x_3)\,\big). \tag{4.4}$$

Due to the properties inherited from f_{A_2}, the above ansatz already has a cluster-adjacent cobracket, a cluster-adjacent symbol, and is smooth and real-valued in the positive domain. All that remains is to find values for the c_i such that f_{A_3} is invariant (up to an overall sign) under the automorphisms of A_3. In this case, there are two automorphism group generators, σ_{A_3} and τ_{A_3}, which were defined in (2.53) and (2.54).

We first consider there case where f_{A_3} is invariant under both A_3 automorphism generators, namely

$$\begin{aligned} \sigma_{A_3}\big(f_{A_3}(x_1 \to x_2 \to x_3)\big) &= f_{A_3}(x_1 \to x_2 \to x_3), \\ \tau_{A_3}\big(f_{A_3}(x_1 \to x_2 \to x_3)\big) &= f_{A_3}(x_1 \to x_2 \to x_3). \end{aligned} \tag{4.5}$$

In practice, this means we take the ansatz for f_{A_3} in eq. (4.4) and solve the constraints [**check in new $\mathcal{X}$-coordinate conventions**]

$$\begin{gathered} f_{A_3}\left(\frac{x_2}{1+x_{1,2}} \to \frac{x_3(1+x_1)}{1+x_{1,2,3}} \to \frac{1+x_{1,2}}{x_1x_2x_3}\right) = f_{A_3}(x_1 \to x_2 \to x_3), \\ f_{A_3}\left(\frac{1}{x_3} \to \frac{1}{x_2} \to \frac{1}{x_1}\right) = f_{A_3}(x_1 \to x_2 \to x_3). \end{gathered} \tag{4.6}$$

It turns out there is no nontrivial solution to this set of constraints. Conversely, allowing f_{A_3} to pick up an overall minus sign when acted upon by τ_{A_3} and solving the constraints

$$\begin{aligned} \sigma_{A_3}\big(f_{A_3}(x_1 \to x_2 \to x_3)\big) &= f_{A_3}\big(x_1 \to x_2 \to x_3\big), \\ \tau_{A_3}\big(f_{A_3}(x_1 \to x_2 \to x_3)\big) &= -f_{A_3}\big(x_1 \to x_2 \to x_3\big), \end{aligned} \tag{4.7}$$

we find the solution

$$c_i = 1 \tag{4.8}$$

(which can always be rescaled by an overall constant). We label this particular solution

$$\begin{aligned} f^{+-}_{A_3}(x_1 \to x_2 \to x_3) &= f_{A_2}(x_1 \to x_2) + \ldots + f_{A_2}(x_1 \to x_2\,(1+x_3)) \\ &= \sum_{i=1}^{6} \sigma^i_{A_3}\big(f_{A_2}(x_1 \to x_2)\big), \end{aligned} \tag{4.9}$$

where the superscripts on $f^{+-}_{A_3}$ label its behavior under σ and τ, respectively. (The superscript on $\sigma^i_{A_3}$ indicates how many times the operator σ_{A_3} should be applied.) There are two remaining sign choices to check, $f^{-+}_{A_3}$ and $f^{--}_{A_3}$. We find only the trivial solution in the first case, while

$$f^{--}_{A_3}(x_1 \to x_2 \to x_3) = \sum_{i=1}^{6} (-1)^i \sigma^i_{A_3}\big(f_{A_2}(x_1 \to x_2)\big) \tag{4.10}$$

turns out to be the unique A_3 function with this automorphism signature, up to an overall rescaling.

Therefore, unlike the case of A_2, there are two functions one can associate with the A_3 cluster algebra: $f^{+-}_{A_3}$ and $f^{--}_{A_3}$. These functions arise purely from the interplay between the overall symmetries of the A_3 cluster algebra and the structure of the A_2 subalgebras in A_3, i.e. there has been no physics input so far. However, motivated by the properties of the two-loop MHV amplitudes, we can check one further aspect of these functions—whether or not they respect cobracket-level cluster adjacency. Recall that the function f_{A_2} does not satisfy this property, while it was stated around equation (3.26) that there exists an f_{A_3} that does have this property—in fact, it was pointed out in [10] that there exists only such function. It is easy to check that indeed, only $f^{--}_{A_3}$ satisfies cobracket-level cluster adjacency, and that it matches the function reported in [10].

4.2 Constructing all Cluster Polylogarithms in $\mathrm{Gr}(4,7)$

It is a straightforward exercise to extend the construction outlined in the last subsection to any finite algebra (as well as to infinite algebras by specifying a set of A_2 subalgebras). In the case of A_n, the only automorphism generators remain σ_n and τ_n (as defined in (2.53) and (2.54)), and we find

$$\begin{array}{c|c|c|c|c} & \sigma^+\tau^+ & \sigma^+\tau^- & \sigma^-\tau^+ & \sigma^-\tau^- \\ \hline A_2 & 0 & 1\ (0) & 0 & 0 \\ \hline A_3 & 0 & 1\ (0) & 0 & 1\ (1) \\ \hline A_4 & 0 & 3\ (0) & 0 & 0 \\ \hline A_5 & 2\ (1) & 5\ (1) & 2\ (0) & 5\ (3) \end{array} \tag{4.11}$$

where each entry denotes the number of nonclassical weight-four cluster polylogarithms with a given automorphism signature, and the numbers in parentheses indicate how many of these functions additionally satisfy cobracket-level cluster adjacency. It is interesting to note that the automorphism signature $\sigma^+\tau^-$ admits at least one solution for each of these A_n, and in particular that it gives rise to the only nontrivial solutions in both A_2 and A_4.

This procedure gets only slightly more complicated in the D_n algebras, due to their larger automorphism groups. As discussed in section 2.6, the D_4 cluster algebra has as its automorphism group a product of dihedral and symmetric groups, $\mathfrak{D}_4 \times S_3$. The dihedral group is generated by a pair of operators $\sigma_{D_4}^{(\mathfrak{D}_4)}$ and $\tau_{D_4}^{(\mathfrak{D}_4)}$, while the symmetric group is generated by a pair of operators $\sigma_{D_4}^{(S_3)}$ and $\tau_{D_4}^{(S_3)}$. This gives rise to 16 possible automorphism signatures, which we impose on a general ansatz of f_{A_2} functions evaluated on all 36 distinct A_2 subalgebras of D_4. Abbreviating the dihedral group generators as σ_4 and τ_4, and the symmetric group generators as σ_3 and τ_3, we find:

$$\underline{\sigma_4^+\tau_4^+}\quad \begin{array}{c|c|c} & \sigma_3^+ & \sigma_3^- \\ \hline \tau_3^+ & 0 & 0 \\ \hline \tau_3^- & 1\ (0) & 0 \end{array} \qquad \underline{\sigma_4^+\tau_4^-}\quad \begin{array}{c|c|c} & \sigma_3^+ & \sigma_3^- \\ \hline \tau_3^+ & 2\ (0) & 0 \\ \hline \tau_3^- & 0 & 0 \end{array} \qquad \underline{\sigma_4^-\tau_4^+}\quad \begin{array}{c|c|c} & \sigma_3^+ & \sigma_3^- \\ \hline \tau_3^+ & 1\ (0) & 0 \\ \hline \tau_3^- & 1\ (1) & 0 \end{array} \qquad \underline{\sigma_4^-\tau_4^-}\quad \begin{array}{c|c|c} & \sigma_3^+ & \sigma_3^- \\ \hline \tau_3^+ & 1\ (0) & 0 \\ \hline \tau_3^- & 0 & 0 \end{array} \tag{4.12}$$

Here we again see that the space of functions respecting automorphisms is remarkably constrained. There are no functions with odd signature under $\sigma_{D_4}^{(\mathfrak{D}_4)}$, and only the signature $\sigma_4^+\tau_4^-\sigma_3^+\tau_3^+$ gives rise to more than one solution.

D_5 is less symmetric than D_4, with only three generators in its automorphism group $\mathfrak{D}_5 \times \mathbb{Z}_2$. This means there are 8 possible automorphism signatures, and we find:

$$\underline{\sigma^+\tau^+}\quad \begin{array}{c|c} \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 5\ (2) & 0 \end{array} \qquad \underline{\sigma^+\tau^-}\quad \begin{array}{c|c} \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 9\ (2) & 0 \end{array} \qquad \underline{\sigma^-\tau^+}\quad \begin{array}{c|c} \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 0 & 3\ (1) \end{array} \qquad \underline{\sigma^-\tau^-}\quad \begin{array}{c|c} \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 0 & 7\ (5) \end{array} \tag{4.13}$$

Again, there is interesting structure in this space, as there are no functions that. have opposite signature under σ and $\mathbb{Z}_2$.

Finally, we turn to E_6, which has as its automorphism group the dihedral group D_{14} with generators σ, τ, and $\mathbb{Z}_2$. E_6 is much larger than any of the cluster algebras considered above, with 504 distinct A_2 subalgebras. Even so, the space of automorphic functions defined on it remains surprisingly small:

$$\begin{array}{c|c} \multicolumn{2}{c}{\underline{\sigma^+\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 12\ (6) & 14\ (6) \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{\underline{\sigma^+\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 21\ (6) & 17\ (9) \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{\underline{\sigma^-\tau^+}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 0 & 0 \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{\underline{\sigma^-\tau^-}} \\ \mathbb{Z}_2^+ & \mathbb{Z}_2^- \\ \hline 0 & 0 \end{array} \tag{4.14}$$

It is especially surprising that there are no solutions that are odd under σ. At this point we also recall that $R_7^{(2)}$ is a cluster polylogarithm on $E_6 \simeq \mathrm{Gr}(4,7)$ that is invariant under the full automorphism group (which combines the dihedral group corresponding to Bose symmetry with parity). Therefore the remainder function must be some linear combination of the 6 degrees of freedom in $f_{E_6}^{+++}$ that satisfy cluster-adjacency at the level of their cobracket.

We pause at this point to emphasize that the A_2-constructibility of all nonclassical cluster polylogarithms has been verified on each of the finite cluster algebras considered above, and its completeness only becomes conjectural on infinite cluster algebras. While it is clearly not possible to form an ansatz out of all A_2 subalgebras in an infinite cluster algebra such as $\mathrm{Gr}(4,8)$, this method can still be used to generate all nonclassical polylogarithms that are constructible out of (the automorphic completion of) any finite set of A_2 subalgebras. This proves sufficient in the case of the two-loop remainder function to all n [10], and may prove sufficient in other cases of physical interest.

4.3 Nested Cluster Polylogarithms

Given that the space of nonclassical cluster polylogarithms coincides with the space of A_2-constructible functions, it is natural to ask whether other (even more special) spaces of functions are constructible out of other cluster polylogarithms. In fact, this has already been shown to be the case, since the nonclassical part of all two-loop MHV amplitudes can be constructed out of $f_{A_3}^{--}$ as defined in (4.10) [10]. This gives rise to an interesting nested structure, since $f_{A_3}^{--}$ is itself constructible out of f_{A_2}. More generally, in the functions constructed in section 4.2 there will be many instances of A_2 subalgebras assembling into larger subalgebras, and it remains an open question how intrinsically interesting such nested constructibility might be.

The procedure for constructing such spaces clearly proceeds just as in the case of A_2-constructibility. However, there are many spaces one can consider constructing, since multiple nonclassical cluster polylogarithms exist on larger algebras. We leave the exploration of compositely-constructible spaces to future work. For now, we just consider the example of functions constructible out of $f_{A_3}^{--}$, since all $R_n^{(2)}$ are believed to be in this class. We tabulate the space of $f_{A_3}^{--}$-constructible functions in table 1.

Since $f_{A_3}^{--}$ satisfies cobracket-level cluster adjacency, all functions constructed out of it also have this property. However, it is also true that *all* nonclassical cluster polylogarithms that satisfy cobracket-level cluster adjacency are $f_{A_3}^{--}$-constructible...

A_3^{--}-constructible A_n Polylogarithms

	$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
$f_{A_4} \in \mathrm{span}\big(f_{A_3}^{--}\big)$	0	0	0	0
$f_{A_5} \in \mathrm{span}\big(f_{A_3}^{--}\big)$	1	1	0	3

A_3^{--}-constructible D_4 Polylogarithms

$\sigma_4^+\tau_4^+$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	0	0
$\mathbb{Z}_2^-$	0	0

$\sigma_4^+\tau_4^-$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	0	0
$\mathbb{Z}_2^-$	0	0

$\sigma_4^-\tau_4^+$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	0	0
$\mathbb{Z}_2^-$	1	0

$\sigma_4^-\tau_4^-$

	σ_3^+	σ_3^-
$\mathbb{Z}_2^+$	0	0
$\mathbb{Z}_2^-$	0	0

A_3^{--}-constructible D_5 Polylogarithms

$\sigma^+\tau^+$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
2	0

$\sigma^+\tau^-$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
2	0

$\sigma^-\tau^+$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
0	1

$\sigma^-\tau^-$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
0	5

A_3^{--}-constructible E_6 Polylogarithms

$\sigma^+\tau^+$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
6	6

$\sigma^+\tau^-$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
6	9

$\sigma^-\tau^+$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
0	0

$\sigma^-\tau^-$

$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
0	0

Table 1: blah

5 Subalgebra-Constructibility and $R_7^{(2)}$

Having seen the extent to which the E_6 cluster algebra admits nontrivial functional embeddings, we turn to the study of $R_7^{(2)}$. Specifically, we explore the ways in which the nonclassical part of this function is subalgebra-constructible, probing the extent to which $\mathcal{N} = 4$ SYM theory knows about the rich subalgebra structure of E_6. It is already known that $R_7^{(2)}$ is A_2- and A_3-constructible in terms of the functions defined in (3.23) and (4.10) [10]. Using the techniques of section 4, we can ask this question systematically for any set of cluster polylogarithms defined on a subalgebra of E_6.

We will in particular be interested in the largest subalgebras of E_6—namely, D_5 and A_5—as D_5- and A_5-constructible functions are not subject to the same amount of representational ambiguity as functions that are constructible out of smaller subalgebras. For instance, while there are 1071 subpolytopes of the E_6 associahedron that correspond to A_2 subalgebras, only 504 of these correspond to distinct cluster algebras. Moreover, there exist 56 identities between the set of f_{A_2} functions evaluated on these 504 subalgebras [10]. Similarly, out of 476 distinct A_3 subpolytopes, only 364 give rise to distinct cluster algebras, and there exist 169 identities between the $f_{A_3}^{--}$ functions evaluated on these subaglebras.

These redundancies make the representation of $\delta_{2,2}\big(R_7^{(2)}\big)$ in terms of either f_{A_2} or $f_{A_3}^{--}$ far from unique. Correspondingly, it is hard to assign a concrete geometric interpretation to these A_2 and A_3 decompositions. On the other hand, there are only 14 D_5 subpolytopes and 7 A_5 subpolytopes of the E_6 associahedron, each of which gives rise to a distinct subalgebra. These collections of subalgebras form complete orbits under the automorphism group of E_6, implying that all D_5- and A_5-constructible functions in E_6^{+++} must take the form

$$\begin{aligned}\sum_{D_5\subset E_6} f_{D_5}(x_i\to\ldots) &\equiv \frac{1}{2}\sum_{i=0}^{6}\sum_{j=0}^{1}\sum_{k=0}^{1}\ \mathbb{Z}_{2,E_6}^{k}\circ\tau_{E_6}^{j}\circ\sigma_{E_6}^{i}\Big(f_{D_5}(x_i\to\ldots)\Big) \\ &= \sum_{i=0}^{6}\sum_{k=0}^{1}\ \tau_{E_6}^{j}\circ\sigma_{E_6}^{i}\Big(f_{D_5}(x_i\to\ldots)\Big)\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\sum_{A_5\subset E_6} f_{A_5}(x_i\to\ldots) &\equiv \frac{1}{4}\sum_{i=0}^{6}\sum_{j=0}^{1}\sum_{k=0}^{1}\ \mathbb{Z}_{2,E_6}^{k}\circ\tau_{E_6}^{j}\circ\sigma_{E_6}^{i}\Big(f_{A_5}(x_i\to\ldots)\Big) \\ &= \sum_{i=0}^{6}\ \sigma_{E_6}^{i}\Big(f_{A_5}(x_i\to\ldots)\Big).\end{aligned} \tag{5.2}$$

Thus—unlike in the case of smaller subalgebras, where nontrivial functions can be formed out of linear combinations of the same cluster polylogarithm evaluated on subalgebras in different orbits of the E_6 automorphism group—the number of D_5- and A_5-constructible functions in E_6^{+++} is directly bounded by the number of D_5 and A_5 functions.

5.1 The D_5-Constructibility of $R_7^{(2)}$

We begin with D_5, the largest subalgebra of E_6 in terms of number of clusters. For each of the automorphism signatures in D_5 that admits a nontrivial space of (nonclassical) cluster polylogarithms, we insert a general ansatz of such functions into the sum (5.1). Referring back to table 4.13, we see there are four nontrivial cases to consider. Note that we do not restrict our ansätze to D_5 functions that respect cobracket-level cluster adjacency, because it is possible for functions that don't have this property to assemble into linear combinations that do when evaluated on the subalgebras of E_6 (as happens when $f_{A_3}^{--}$ is constructed out of f_{A_2} functions). We then check to see if the nonclassical part of $R_7^{(2)}$ is in the span of any of these ansatz sums.

Amazingly, there exists precisely one D_5 automorphism signature in terms of which $R_7^{(2)}$ can be decomposed: $R_7^{(2)}$ is $f_{D_5}^{---}$-constructible. Two free parameters remain in this decomposition, corresponding to two degrees of freedom that cancel in the sum (5.1). More concretely, we can decompose

$$\delta_{2,2}\big(R_7^{(2)}\big) = \{const\} \sum_{D_5 \subset \mathrm{Gr}(4,7)} \delta_{2,2}\Bigg(f_{D_5}^{---}\left(\begin{tikzcd}[column sep=small, row sep=small] & & & x_4 \\ x_1 \arrow[r] & x_2 \arrow[r] & \frac{x_3 x_5}{1+x_5} \arrow[ur] \arrow[dr] & \\ & & & \frac{1}{x_5}\end{tikzcd}\right)\Bigg), \tag{5.3}$$

where the sum over all D_5 subalgebras is taken according to (5.1), and we define

$$\begin{aligned} f_{D_5}^{---}\left(\begin{tikzcd}[column sep=small, row sep=small] & & & x_4 \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\ & & & x_5\end{tikzcd}\right) \equiv \sum_{D_5^{---}} \Bigg[& \frac{1}{2}c_1 f_{A_2}\big(x_1 \to x_2\,(1+x_{34})\big) + \frac{1}{2}c_2 f_{A_2}\big(x_3 \to x_4\big) + \\ & \left(-\frac{1}{2}+\frac{c_1}{2}\right) f_{A_2}\left(\frac{x_1 x_2}{1+x_1} \to x_3\,(1+x_4)\right) + \frac{1}{4}c_2 f_{A_2}\left(\frac{x_1 x_2 x_3}{1+x_{12}} \to x_4\right) + \\ & (c_1-c_2)\, f_{A_2}\left(\frac{x_2 x_3}{1+x_2} \to x_4\right) + (1-c_1)\, f_{A_2}\big(x_2 \to x_3\,(1+x_4)\big) + \\ & \left(\frac{1}{2}-c_1+\frac{3c_2}{4}\right) f_{A_2}\left(x_2\,(1+x_3) \to \frac{x_3 x_4}{1+x_3}\right)\Bigg], \end{aligned} \tag{5.4}$$

where we have introduced the shorthand

$$\sum_{D_5^{---}} f = \sum_{i=0}^{4}\sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{i+j+k}\ \sigma_{D_5}^{i} \circ \tau_{D_5}^{j} \circ \mathbb{Z}_{2,D_5}^{k}\big(f\big). \tag{5.5}$$

The $\mathcal{X}$-coordinates in (5.3) are the E_6 coordinates defined in (2.45), while the $\mathcal{X}$-coordinates in (5.4) should be thought of as D_5 coordinates that take different values when evaluated on the D_5 subalgebras of E_6. (It is easy to see that the E_6 seed (2.44) doesn't contain any D_5 sub-quivers of the form (2.43), but that mutating on the x_5 node will give rise to the D_5 sub-quiver in (5.3).) The coefficients c_1 and c_2 represent the two remaining degrees of freedom in our ansatz, which drop out of the sum (5.3).

It is worth highlighting that $f_{D_5}^{---}$ does not satisfy cobracket-level cluster adjacency for generic values of c_1 and c_2; however, it can be given this property by choosing $c_2 = -\frac{6}{5}+\frac{8}{5}c_1$.
discussion of $7 \to 6$ collinear limit as motivation to choose other values

There exist further physical and mathematical desiderata we might consider to motivate different choices for c_1 and c_2, one of which will be explored in section 5.3. But first we explore the possibility that $R_7^{(2)}$ is also A_5-constructible.

5.2 The A_5-Constructibility of $R_7^{(2)}$

We can check for A_5 decompositions of $R_7^{(2)}$ in the same way we checked for D_5 decompositions, using now the ansatz sum (5.2). Intriguingly, there is (again) precisely one automorphism signature in terms of which $R_7^{(2)}$ can be decompose, and it is (again) the totally antisymmetric choice A_5^{--}. This time there is a single internal degree of freedom that cancels in the sum (5.2).

This decomposition is given explicitly by

$$\delta_{2,2}\big(R_7^{(2)}\big) = \sum_{A_5\subset E_6} \delta_{2,2}\Big(f_{A_5}^{--}\left(x_1 \to x_2 \to \frac{x_3 x_5}{1+x_5} \to \frac{1}{x_5} \to x_6(1+x_5)\right)\Big), \tag{5.6}$$

where the A_5 sum (5.2) is taken over

$$\begin{aligned} f_{A_5}^{--}(x_1 \to x_2 \to x_3 \to x_4 \to x_5) \equiv \sum_{A_5^{--}} \Big[& \frac{1}{2}c_1 f_{A_2}(x_2 \to x_3\,(1+x_4)\,)+ \\ & (-1-c_1)\, f_{A_2}(x_2 \to x_3\,(1+x_{45})\,)+ \\ & \left(-\frac{1}{2}-c_1\right) f_{A_2}\left(\frac{x_1 x_2}{1+x_1} \to x_3\,(1+x_4)\right)\Big]. \end{aligned} \tag{5.7}$$

As in the last section, the $\mathcal{X}$-coordinates in (5.6) are the E_6 coordinates defined in (2.45), while the $\mathcal{X}$-coordinates in (5.7) are A_5 coordinates that take different values when evaluated on different A_5 subalgebras of E_6. Moreover, the A_5 quiver in (5.6) can be generated by mutating on the x_5 node of (2.44) and freezing the node associated with x_4.

. . .

5.3 Nesting A_4 and A_3 Functions in D_5 and A_5

We can tune the remaining internal degrees of freedom, c_1 and c_2, to embed further subalgebra structure inside $f_{D_5}^{---}$. This follows exactly the same procedure for finding D_5-decompositions of $R_7^{(2)}$, instead now we are trying to find A_4-, D_4-, or A_3-decompositions of $f_{D_5}^{---}$. It turns out that there are two different possibilities: one of which involves A_4 subalgebras and the other involving A_3. We will review these decompositions now.

Setting

$$c_2 = -\frac{6}{5} + \frac{8}{5}c_1 \tag{5.8}$$

and leaving c_1 free makes $f_{D_5}^{---}$ an A_3^{--}-constructible function:

$$\begin{aligned} f_{A_3\subset D_5}^{---} = \frac{1}{10}\sum_{D_5^{---}} & (-3+4c_1)\, f_{A_3}^{--}\,(x_1 \to x_2 \to x_3\,(1+x_4)) + (6-3c_1)\, f_{A_3}^{--}\left(x_1 \to x_2\,(1+x_3) \to \frac{x_3 x_4}{1+x_3}\right) \\ & + (2-c_1)\, f_{A_3}^{--}\,(x_2 \to x_3\,(1+x_4) \to x_5) + \left(\frac{1}{2}-\frac{3c_1}{2}\right) f_{A_3}^{--}\left(\frac{1}{x_4} \to x_3\,(1+x_4) \to x_5\right) \\ & + \left(-1+\frac{c_1}{2}\right) f_{A_3}^{--}\left(\frac{1}{x_4} \to \frac{x_2 x_3\,(1+x_4)}{1+x_2} \to x_5\right). \end{aligned} \tag{5.9}$$

There is no A_3^{+-} representation. Setting

$$c_1 = \frac{3}{5}, \qquad c_2 = 0, \tag{5.10}$$

makes $f_{D_5}^{---}$ an A_4^{+-}-constructible function:

$$f_{A_4\subset D_5}^{---} = \sum_{D_5^{---}} f_{A_4}^{+-}(x_1 \to x_2 \to x_3 \to x_4) \tag{5.11}$$

where we have used a new function

$$f_{A_4}^{+-} = \frac{1}{40}\sum_{A_4^{+-}} 3f_{A_2}(x_1 \to x_2(1+x_3)) - f_{A_2}(x_2 \to x_3(1+x_4)). \tag{5.12}$$

Here we have used the notation $\sum_{A_4^{+-}}$ to denote the skew-dihedral sum over A_4,

$$\sum_{A_4^{+-}} = \sum_{i=0}^{6}\sum_{j=0}^{1}(-1)^j \sigma_{A_4}^i \tau_{A_4}^j. \tag{5.13}$$

$f_{A_4}^{+-}$ does not have an A_3-decomposition.

The $A_4 \subset D_5$ representation of $R_7^{(2)}$ in particular is quite remarkable, and can be described as:

- evaluate $f_{A_4}^{+-}$ on all A_4 subalgebras of $\mathrm{Gr}(4,7)$,
- add them together so that they are antisymmetric under the automorphisms of the D_5 subalgebras of $\mathrm{Gr}(4,7)$,
- and the resulting function matches the nonclassical portion of $R_7^{(2)}$.

Following in complete analogy with the D_5 case, we can tune c_1 to decompose $f_{A_5}^{--}$ into either A_3 or A_4 functions. Unlike D_5, the A_3 representation employs the skew-symmetric A_3 function, $f_{A_3}^{+-}$. Setting $c_1 = -\frac{1}{2}$ gives

$$f_{A_3\subset A_5}^{--} = -\frac{1}{160}\sum_{A_5^{--}} f_{A_3}^{+-}\left(x_2 \to x_3(1+x_4) \to \frac{x_4 x_5}{1+x_4}\right). \tag{5.14}$$

The overall coefficient of $-\frac{1}{160}$ is simply a result of overcounting in the symmetric sum, as that A_3 has only 4 automorphic partners in A_5.

Setting $c_1 = -1$ makes $f_{A_5}^{--}$ an A_4-constructible function:

$$f_{A_4\subset A_5}^{--} = -\frac{1}{20}\sum_{A_5^{--}} f_{A_4}^{+-}(x_1 \to x_2 \to x_3 \to x_4). \tag{5.15}$$

Remarkably, this is the same $f_{A_4}^{+-}$ function defined for the D_5 function in eq. (5.12). In other words, we have [**temporarily ignoring the overall symmetry factors, which we can mess with later**]

$$R_7^{(2)} = \sum_{D_5\subset \mathrm{Gr}(4,7)}\ \sum_{A_4\subset D_5^{---}} f_{A_4}^{+-}(x_1 \to x_2 \to x_3 \to x_4) + C \tag{5.16}$$

$$= \sum_{A_5\subset \mathrm{Gr}(4,7)}\ \sum_{A_4\subset A_5^{--}} f_{A_4}^{+-}(x_1 \to x_2 \to x_3 \to x_4) + C \tag{5.17}$$

$$\begin{tikzcd}
 & & R_7^{(2)} \arrow[dl] \arrow[dr] & & \\
 & f_{D_5}^{---} \arrow[dl] \arrow[dr] & & f_{A_5}^{--} \arrow[dl] \arrow[dr] & \\
f_{A_3}^{--} \arrow[drr] & & f_{A_4}^{+-} \arrow[d] & & f_{A_3}^{+-} \arrow[dll] \\
 & & f_{A_2}^{--} & &
\end{tikzcd}$$

Figure 7: blah

where C is again a function of classical polylogarithms and is the same for both the D_5 and A_5 representations.

6 Conclusion

1. since there are so few D_5 and A_5 subalgebras of Gr(4,7), these decompositions distill more intrinsic (better word?) geometric structure than the A_3 function
2. would be interesting to find the D_5 decomposition of higher-point amplitudes, to see if this seter our determined constants
3. not possible to get the full (function-level) amplitude out of a single (function-level) subalgebra function; it merits checking whether a linear combination of different subalgebra functions could be found
4. bring up mutations as active vs passive transformations?
5. point out that the the subaglebras for which decompositions exists is not the same set as those that have a grassmannian intepretation
6. It would be interesting to understand why certain subalgebra chains produce valid cluster polylogarithms while others do not.
7. NMHV amplitude

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A Counting Subalgebras of Finite Cluster Algebras

In this appendix we catalog the subalgebra structure of E_6 and all the finite connected cluster algebras that occur as its subalgebras, namely A_2, A_3, A_4, D_4, A_5, and D_5. We consider subalgebras to be distinct only when they differ in at least one of their mutable nodes, but count identical subalgebras as distinct subpolytopes (of the associahedron) when they are connected to different frozen nodes. We include the number of both subalgebras and subpolytopes that occur in each algebra. Note also that we consider the $\mathcal{X}$-coordinates x and $1/x$ to be distinct.

A_2

Clusters: 5 — $\mathcal{A}$-coordinates: 5 — $\mathcal{X}$-coordinates: 10

A_3

Clusters: 14 — $\mathcal{A}$-coordinates: 9 — $\mathcal{X}$-coordinates: 30

Type	Subpolytopes	Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

A_4

Clusters: 42 — $\mathcal{A}$-coordinates: 14 — $\mathcal{X}$-coordinates: 70

Type	Subpolytopes	Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

D_4

Clusters: 50 — $\mathcal{A}$-coordinates: 16 — $\mathcal{X}$-coordinates: 104

Type	Subpolytopes	Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

A_5

Clusters: 132 — $\mathcal{A}$-coordinates: 20 — $\mathcal{X}$-coordinates: 140

Type	Subpolytopes	Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

D_5

Clusters: 182 — $\mathcal{A}$-coordinates: 25 — $\mathcal{X}$-coordinates: 260

Type	Subpolytopes	Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

E_6

Clusters: 833 — $\mathcal{A}$-coordinates: 42 — $\mathcal{X}$-coordinates: 770

Type	Subpolytopes	Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

B Cobracket Spaces in Finite Cluster Algebras

In this appendix we collect the dimensions of weight-four cluster polylogarithm spaces defined on various finite cluster algebras. In table 2 we record the number of cluster polylogarithms that have nonzero $\delta_{2,2}$ cobracket, as considered in section 4. As done there, we record the

Type	Nonclassical Cobrackets	Automorphism Signature $\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	1 (0)	0	1 (0)	0	0
A_3	6 (1)	0	1 (0)	0	1 (1)
A_4	21 (6)	0	3 (0)	0	0
D_4	34 (9)				
A_5	56 (21)	2 (1)	5 (1)	2 (0)	5 (3)
D_5	116 (42)				
E_6	448 (195)				

D_4 Automorphism Signatures

$\sigma_4^+\tau_4^+$

	σ_3^+	σ_3^-
τ_3^+	0	0
τ_3^-	1 (0)	0

$\sigma_4^+\tau_4^-$

	σ_3^+	σ_3^-
τ_3^+	2 (0)	0
τ_3^-	0	0

$\sigma_4^-\tau_4^+$

	σ_3^+	σ_3^-
τ_3^+	1 (0)	0
τ_3^-	1 (1)	0

$\sigma_4^-\tau_4^-$

	σ_3^+	σ_3^-
τ_3^+	1 (0)	0
τ_3^-	0	0

D_5 Automorphism Signatures

	$\sigma^+\tau^+$		$\sigma^+\tau^-$		$\sigma^-\tau^+$		$\sigma^-\tau^-$	
	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
	5 (2)	0	9 (2)	0	0	3 (1)	0	7 (5)

E_6 Automorphism Signatures

	$\sigma^+\tau^+$		$\sigma^+\tau^-$		$\sigma^-\tau^+$		$\sigma^-\tau^-$	
	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
	12 (6)	14 (6)	21 (6)	17 (9)	0	0	0	0

Table 2: Number of nonclassical weight-four cluster polylogarithms on various finite cluster algebras, prior to considerations of their automorphism group and after requiring specific automorphism signatures. The number of polylogarithms that can also be made to satisfy cobracket-level cluster adjacency is given in parentheses.

Type	Integrable Cobrackets	Automorphism Signature $\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	6 (5)	1 (1)	1 (0)	0	0
A_3	21 (16)	2 (2)	2 (1)	0	3 (3)
A_4	56 (41)	4 (4)	4 (1)	0	0
D_4	86 (61)				
A_5	126 (91)	8 (7)	8 (4)	5 (3)	11 (9)
D_5	246 (172)				
E_6	833 (580)				

D_4 Automorphism Signatures

$\sigma_4^+\tau_4^+$

	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	3 (3)	0
$\tau_{S_3}^-$	1 (0)	0

$\sigma_4^+\tau_4^-$

	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	3 (1)	0
$\tau_{S_3}^-$	1 (1)	0

$\sigma_4^-\tau_4^+$

	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	3 (2)	0
$\tau_{S_3}^-$	2 (2)	0

$\sigma_4^-\tau_4^-$

	$\sigma_{S_3}^+$	$\sigma_{S_3}^-$
$\tau_{S_3}^+$	3 (2)	0
$\tau_{S_3}^-$	0	0

D_5 Automorphism Signatures

	$\sigma^+\tau^+$		$\sigma^+\tau^-$		$\sigma^-\tau^+$		$\sigma^-\tau^-$	
	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$	$\sigma_{Z_2}^+$	$\sigma_{Z_2}^-$
	16 (13)	0	16 (9)	0	0	5 (3)	0	13 (11)

E_6 Automorphism Signatures

	$\sigma^+\tau^+$		$\sigma^+\tau^-$		$\sigma^-\tau^+$		$\sigma^-\tau^-$	
	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
	32 (26)	25 (17)	32 (17)	30 (22)	0	0	0	0

Table 3: Number of weight-four cluster polylogarithms on various finite cluster algebras with nonzero cobrackets, prior to considerations of their automorphism group and after requiring specific automorphism signatures. The number of polylogarithms that can also be made to satisfy cobracket-level cluster adjacency is given in parentheses.

number of functions that can additionally be made to respect cobracket-level cluster adjacency in parentheses. We also include the total number of functions that can be defined on these cluster algebras without requiring any definite behavior under the automorphism group. In table 3 we record the same information, but for weight-four cluster polylogarithms with any nonzero cobracket component.

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