

A_2 -constructible function counts

In this section we introduce and describe the results for $A_2 \subset A_3$ constructibility in the finite algebras of interest.

Here we follow the steps worked through in the previous section, namely

- begin with an ansatz of all f_{A_2} applied across all A_2 subalgebras of a given larger algebra,
- impose that the overall function is invariant under automorphisms up to overall sign choices,
- count the number of solutions to these constraints.

In the case of A_n algebras we only have to impose the dihedral automorphisms σ_n and τ_n , and so the resulting table is exactly analogous to eq. (1):

| | $\sigma^+\tau^+$ | $\sigma^+\tau^-$ | $\sigma^-\tau^+$ | $\sigma^-\tau^-$ |
|-------|------------------|------------------|------------------|------------------|
| A_3 | 0 | 1 | 0 | 1 |
| A_4 | 0 | 3 | 0 | 0 |
| A_5 | 2 | 5 | 2 | 5 |

(1)

It is interesting to note that the $+-$ sign choice admits at least one solution for all A_n studied so far, and that is the only sign choice for A_4 .

As discussed in sec. (automorphisms), D_4 has the automorphism group $D_4 \times S_3$, with two cyclic generators $\sigma_{D_4}^{(4)}$ and $\sigma_{D_4}^{(3)}$ (corresponding to the D_4 and S_3 , respectively) and then the D_4 flip τ_{D_4} as well as the S_3 flip denoted \mathbb{Z}_{2,D_4} . In the following table these are abbreviated $\sigma_4, \sigma_3, \tau_4, \mathbb{Z}_2$, respectively. Because of the four automorphism generators, there are 16 possible sign choices to impose on the collection of 36 distinct f_{A_2} 's in D_4 .

| | | | | | |
|------------------|---|---|---|---|-----|
| | $\frac{\sigma_4^+ \tau_4^+}{\sigma_3^+ \quad \sigma_3^-}$ | $\frac{\sigma_4^+ \tau_4^-}{\sigma_3^+ \quad \sigma_3^-}$ | $\frac{\sigma_4^- \tau_4^+}{\sigma_3^+ \quad \sigma_3^-}$ | $\frac{\sigma_4^- \tau_4^-}{\sigma_3^+ \quad \sigma_3^-}$ | |
| \mathbb{Z}_2^+ | 0 | 2 | 1 | 1 | |
| \mathbb{Z}_2^- | 1 | 0 | 1 | 0 | (2) |

Here we again see that the space of functions satisfying automorphisms is remarkably constrained, with no functions exhibiting a sign flip under σ_3 and only one sign choice $-\sigma_4^+\tau_4^-\sigma_3^+\mathbb{Z}_2^+$ – that offers more than one possible solution.

D_5 is less symmetric than D_4 , with only $D_5 \times \mathbb{Z}_2$ automorphism group (in the table the generators are labeled by σ, τ , and \mathbb{Z}_2). Therefore there are 8 possible sign choices to impose on the collection of 125 distinct f_{A_2} 's in D_5 , and the resulting number of solutions is

| $\overline{\sigma^+\tau^+}$ | $\overline{\sigma^+\tau^-}$ | $\overline{\sigma^-\tau^+}$ | $\overline{\sigma^-\tau^-}$ | | | | | | | | | |
|---|---------------------------------------|---------------------------------------|---|-----|---|--|---|---|---|---|----|--|
| $\mathbb{Z}_2^+ \quad \mathbb{Z}_2^-$ | $\mathbb{Z}_2^+ \quad \mathbb{Z}_2^-$ | $\mathbb{Z}_2^+ \quad \mathbb{Z}_2^-$ | $\mathbb{Z}_2^+ \quad \mathbb{Z}_2^-$ | (3) | | | | | | | | |
| <table border="1"> <tr> <td>11</td> <td>0</td> </tr> </table> | 11 | 0 | <table border="1"> <tr> <td>16</td> <td>0</td> </tr> </table> | 16 | 0 | <table border="1"> <tr> <td>0</td> <td>6</td> </tr> </table> | 0 | 6 | <table border="1"> <tr> <td>0</td> <td>13</td> </tr> </table> | 0 | 13 | |
| 11 | 0 | | | | | | | | | | | |
| 16 | 0 | | | | | | | | | | | |
| 0 | 6 | | | | | | | | | | | |
| 0 | 13 | | | | | | | | | | | |

Finally we turn to E_6 , which has automorphism group D_{14} with generators σ, τ , and \mathbb{Z}_2). E_6 is much larger than the algebras studied so far, with 504 distinct A_2 subalgebras, however the space of automorphic functions is still quite constrained:

$$\begin{array}{cccc}
\begin{array}{c} \overline{\sigma^+ \tau^+} \\ \mathbb{Z}_2^+ \quad \mathbb{Z}_2^- \\ \boxed{-} \quad \boxed{-} \end{array} &
\begin{array}{c} \overline{\sigma^+ \tau^-} \\ \mathbb{Z}_2^+ \quad \mathbb{Z}_2^- \\ \boxed{-} \quad \boxed{-} \end{array} &
\begin{array}{c} \overline{\sigma^- \tau^+} \\ \mathbb{Z}_2^+ \quad \mathbb{Z}_2^- \\ \boxed{-} \quad \boxed{-} \end{array} &
\begin{array}{c} \overline{\sigma^- \tau^-} \\ \mathbb{Z}_2^+ \quad \mathbb{Z}_2^- \\ \boxed{-} \quad \boxed{-} \end{array}
\end{array} \tag{4}$$