In this note we catalog the subalgebra structure for the finite cluster algebras $\subseteq E_6$.

These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

A_2

clusters: 5 a-coordinates: 5 x-coordinates: 10

A_3

clusters: 14 a-coordinates: 9 x-coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

A_4

clusters: 42 a-coordinates: 14 x-coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

D_4

clusters: 50 a-coordinates: 16 x-coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

$\underline{A_5}$ clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{D_5}$ clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{E_6}$ clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

Symbol Spaces on Cluster Algebras

Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic $\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	19	14	2	0	0	0
A_3	55	40	6	2	1	5
A_4	125	90	9	3	0	0
D_4	163	109				
A_5	245	175	1	0	0	0
D_5	381	241				
E_6	-	573	-	-	-	-

Automorphic D_4 symbols

$\sigma^+\tau^+$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	7	0
$\tau^-_{S_3}$	0	0

$\sigma^+\tau^-$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	2	0
$\tau^-_{S_3}$	1	0

$\sigma^-\tau^+$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	5	0
$\tau^-_{S_3}$	2	0

$\sigma^-\tau^-$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	4	0
$\tau^-_{S_3}$	0	0

Automorphic D_5 symbols

$\sigma^+\tau^+$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
22	0

$\sigma^+\tau^-$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
11	0

$\sigma^-\tau^+$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
0	4

$\sigma^-\tau^-$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
0	12

Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	3	3
$A_1 \times A_1 \times A_1$	6	6
$A_1 \times A_1 \times A_1 \times A_1$	10	10
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	15	15
$A_2 \times A_1$	25	20
$A_2 \times A_1 \times A_1$	32	27
$A_2 \times A_1 \times A_1 \times A_1$	40	35
$A_2 \times A_2$	63	53
$A_3 \times A_1$	65	50
$A_2 \times A_2 \times A_1$	74	64
$A_3 \times A_1 \times A_1$	76	61
$A_3 \times A_2$	119	99
$A_4 \times A_1$	140	105
$D_4 \times A_1$	180	126

- The number of integrable symbols at weight w in the product group $\bigotimes_N A_1$ is given by $\binom{N+w-1}{w}$, i.e. 'N choose w (logs) with replacement'
- More generally, the number of weight-two integrable symbols in $G_1 \times G_2$ is the number of symbols in G_1 plus the number of symbols in G_2, plus the product of their respective numbers of A-coordinates (corresponding to taking a log from each)
- More generally, these integrable symbol counts just correspond to shuffling together all possible symbols drawn from each component group such that the total weight is w
- Presumably the automorphic counts also follow from shuffling together all possible symbols drawn from the component groups such that the total weight is w, and such that their automorphism properties under the component groups respect any new permutation symmetries (for instance, any products of symbols drawn from A_1 and A_2 will respect the automorphism group of $A_1 \times A_2$, but a product of symbols from $A_2 \times A_2$ will have to respect a new symmetry that permutes the two component groups)
- I believe all of the above statements about integrable symbols directly translate to statements about cluster-adjacent symbols, because any product of symbols that are cluster-adjacent in each component group will also be cluster-adjacent in the composite group
- So this table (and the weight-four one) are just consistency checks

Weight Four

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	211 (6)	81 (6)	11 (0)	6 (2)	0	0
A_3	1351 (36)	432 (36)	49 (2)	29 (4)	24 (2)	42 (4)
A_4	-	1652 (126)	131 (6)	105 (12)	0	0
D_4	-	2204				
A_5	-	-	-	-	-	-
D_5	-	-				
E_6	-	-	-	-	-	-

Automorphic D_4 symbols

$\sigma^+\tau^+$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

$\sigma^+\tau^-$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

$\sigma^-\tau^+$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

$\sigma^-\tau^-$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

Automorphic D_5 symbols

$\sigma^+\tau^+$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

$\sigma^+\tau^-$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

$\sigma^-\tau^+$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

$\sigma^-\tau^-$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

Composite Groups

Type	Integrable	Cluster-Adjacent
$A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1$	15	15
$A_1 \times A_1 \times A_1 \times A_1$	35	35
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	70	70
$A_2 \times A_1$	301	136
$A_2 \times A_1 \times A_1$	423	218
$A_2 \times A_1 \times A_1 \times A_1$	585	335
$A_2 \times A_2$	1433	708
$A_3 \times A_1$	1701	622
$A_2 \times A_2 \times A_1$	1827	982
$A_3 \times A_1 \times A_1$	2127	873
$A_3 \times A_2$	4617	2088
$A_4 \times A_1$	-	
$D_4 \times A_1$	-	

Subalgebra-Constructible Symbols

	$A_1 \times A_1$	A_2	$A_1 \times A_1 \times A_1$	$A_2 \times A_1$	A_3
A_3	13	366			
A_4	98	1141	0	791	1603
D_4	63	1826	45	0	
A_5					
D_5					
E_6					

	E_6	D_5	A_5	D_4	A_4	A_3	A_2
A_1	385	130	70	52	35	15	5
$A_2^{\not\subset}$		5863	2836	1810	1127	357	76
$(A_1 \times A_1)^{\not\subset}$	2007	400	375	54	84	9	
$A_3^{\not\subset}$			1652	WTF	413	59	
$(A_2 \times A_1)^{\not\subset}$			2124	0	273		
$(A_1 \times A_1 \times A_1)^{\not\subset}$			36	12	0		
$D_4^{\not\subset}$					0		
$A_4^{\not\subset}$				0			
$(A_3 \times A_1)^{\not\subset}$							
$(A_2 \times A_2)^{\not\subset}$							
$(A_2 \times A_1 \times A_1)^{\not\subset}$							
$(A_1 \times A_1 \times A_1 \times A_1)^{\not\subset}$							
$D_5^{\not\subset}$			0				
$A_5^{\not\subset}$		0					
$(D_4 \times A_1)^{\not\subset}$							
$(A_4 \times A_1)^{\not\subset}$							
$(A_3 \times A_2)^{\not\subset}$							
$(A_3 \times A_1 \times A_1)^{\not\subset}$							
$(A_2 \times A_2 \times A_1)^{\not\subset}$							
$(A_2 \times A_1 \times A_1 \times A_1)^{\not\subset}$							
$(A_1 \times A_1 \times A_1 \times A_1 \times A_1)^{\not\subset}$							
$E_6^{\not\subset}$							

- The first line of this table is obviously Log^4 of all distinct X-coordinates
- Can we discover a purely-clustery notion of first entry condition by looking at all these symbols?

How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank $n-1$ in a finite type cluster algebra of type X of rank n.

Let N be the number of cluster a-coordinates in the cluster algebra of type X.
Let n be the rank of cluster algebra of type X.
Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y.

The number of cluster algebras of type Y is then NZ/n.

As an example, let's count the number of A_4's in A_5:

$N = \#$ of cluster a-coordinates in $A_5 = 20$
$n =$ rank of $A_5 = 5$
$Z = \#$ of ways to remove a node from A_5 and get an $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$ there are 8 A_4's in A_5.