Cluster Polylogarithms for A_2 and A_5

Taking our seed cluster as $x_1 \to x_2$ and following the $\mathcal{X}$-variables mutation rule

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i(1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases} \tag{1}$$

we label the $\mathcal{X}$-coordinates

$$\mathcal{X}_1 = \frac{1}{x_1}, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1+x_2), \quad \mathcal{X}_4 = \frac{1 + x_1 + x_1 x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1+x_1}{x_1 x_2}. \tag{2}$$

These satisfy $1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}$, and $\{1/\mathcal{X}_i, \mathcal{X}_{i+1}\}$ form the 5 clusters.

We now define "the A_2 function" f_{A_2} as

$$f_{A_2}(x_1 \to x_2) = \sum_{\text{skew-dihedral}} \mathrm{Li}_{2,2}(-\mathcal{X}_{i-1}, -\mathcal{X}_{i+1}) - \mathrm{Li}_{1,3}(-\mathcal{X}_{i-1}, -\mathcal{X}_{i+1}) - \mathrm{Li}_2(-\mathcal{X}_{i-1}) \log(\mathcal{X}_i) \log(\mathcal{X}_{i+1}). \tag{3}$$

It is beyond the immediate scope of this short note to explain why this function is uniquely associated to the A_2 algebra, but I'm happy to explain this more in detail. And skew-dihedral means that the function picks up a minus sign under $\mathcal{X}_i \to \mathcal{X}_{6-i}$ but is symmetric under $\mathcal{X}_i \to \mathcal{X}_{i+1}$.

We then associate with the A_5 cluster algebra (generated from the seed quiver $x_1 \to \ldots \to x_5$) a function built out of A_2 subalgebras:

$$f_{A_5} = \sum_{i=0}^{7} \sum_{j=0}^{1} (-1)^{i+j} \sigma_{A_5}^i \tau_{A_5}^j \left(\frac{1}{2} f_{A_2}\left(x_2 \to x_3\left(1 + x_4\right)\right) + f_{A_2}\left(x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1 + x_2}\right)\right). \tag{4}$$

σ_{A_5} and τ_{A_5} are the dihedral cycle+flip, f_{A_5} is therefore antisymmetric under either.

All of these seemingly arbitrary definitions are motivated by the fact that $\mathrm{Gr}(4,7)$ has 7 A_5 subalgebras, and we can express a very complicated physics function (technically "the non-classical portion of the two-loop MHV remainder function in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills") as a sum of the A_5 function evaluated over all of them:

$$R_7^{(2)} = \sum_{\text{cyclic}} f_{A_5}\left(\frac{\langle 1567\rangle\langle 2345\rangle}{\langle 5(12)(34)(67)\rangle} \to -\frac{\langle 1267\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 4(12)(35)(67)\rangle} \to \frac{\langle 1234\rangle\langle 5(12)(34)(67)\rangle}{\langle 1245\rangle\langle 3(12)(45)(67)\rangle} \to -\frac{\langle 1237\rangle\langle 4(12)(35)(67)\rangle}{\langle 1234\rangle\langle 1267\rangle\langle 3457\rangle} \to \frac{\langle 3(12)(45)(67)\rangle}{\langle 1237\rangle\langle 3456\rangle} \right). \tag{5}$$

There are actually other functions which we can use to express $R_7^{(2)}$, namely ones associated with D_5 and A_3 subalgebras (but definitely not D_4 or A_4, thanks to an exhaustive search). Both f_{D_5} and f_{A_3} are built out of f_{A_2}'s. The downside of A_3 is that any representation is not unique (thanks to functional identities amongst the polylogarithms), similarly there are degrees of freedom inside f_{D_5} which cancel out in any sum over $\mathrm{Gr}(4,7)$. The A_5 representation is nice because a) it is uniquely determined and b) as we will see it generalizes nicely to $\mathrm{Gr}(4,8)$.

Gr(4,8)

Based on the results of other physics calculations, we know that the function $R_n^{(2)}$ only "depends" on $\mathcal{A}$-variables of the form:

$$\begin{aligned} \langle i\ i{+}1\ jk\rangle, \quad &\langle i(i{-}1\ i{+}1)(j\ j{+}1)(k\ k{+}1)\rangle, \\ \langle i\ i{+}1\ \bar{j}\cap\bar{k}\rangle, \quad &\langle i(i{-}2\ i{-}1)(i{+}1\ i{+}2)(j\ j{+}1)\rangle \end{aligned} \tag{6}$$

where

$$\begin{aligned} \langle i(jk)(lm)(no)\rangle &\equiv \langle ijlm\rangle\langle ikno\rangle - \langle ijno\rangle\langle iklm\rangle, \\ \langle ij\bar{k}\cap\bar{l}\rangle &\equiv \langle i\bar{k}\rangle\langle j\bar{l}\rangle - \langle j\bar{k}\rangle\langle i\bar{l}\rangle, \quad \bar{j} = (j-1\,j\,j+1). \end{aligned} \tag{7}$$

We'll call these "good letters". Our conjecture is that the function $R_n^{(2)}$ can be expressed in terms of polylogarithms with $\mathcal{X}$-variables as arguments that are built out of only those letters. Let's similarly call those "good $\mathcal{X}$-variables". There are 1588 good $\mathcal{X}$-variables in $\mathrm{Gr}(4,8)$ that involve at most 4 powers of Plükers (I believe that there are none involving higher powers but that is purely conjecture).

Continuing the theme, let me define a "good subalgebra" of $\mathrm{Gr}(4,8)$ as one that only involves good $\mathcal{X}$-variables. There are 56 good A_5s in $\mathrm{Gr}(4,8)$, modulo dihedral+conjugation. They are generated by

$$\frac{\langle 1238\rangle\langle 1256\rangle}{\langle 1235\rangle\langle 1268\rangle} \to \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle} \to \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle} \to \frac{\langle 1567\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3567\rangle} \to \frac{\langle 1356\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 3456\rangle} \tag{8}$$

$$\frac{\langle 1238\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2358\rangle} \to -\frac{\langle 1235\rangle\langle 4568\rangle}{\langle 5(18)(23)(46)\rangle} \to \frac{\langle 1568\rangle\langle 2358\rangle\langle 3456\rangle}{\langle 1358\rangle\langle 2356\rangle\langle 4568\rangle} \to -\frac{\langle 5(18)(23)(46)\rangle}{\langle 1258\rangle\langle 3456\rangle} \to \frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle} \tag{9}$$

$$\frac{\langle 1234\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 2345\rangle} \to \frac{\langle 1348\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 3468\rangle} \to -\frac{\langle 1346\rangle\langle 5678\rangle}{\langle 6(18)(34)(57)\rangle} \to -\frac{\langle 1678\rangle\langle 3468\rangle\langle 34(128)\cap(567)\rangle}{\langle 1268\rangle\langle 1348\rangle\langle 3467\rangle\langle 5678\rangle} \to \frac{\langle 1278\rangle\langle 6(18)(34)(57)\rangle}{\langle 1678\rangle\langle 34(128)\cap(567)\rangle} \tag{10}$$

$$\frac{\langle 1234\rangle\langle 1278\rangle}{\langle 1238\rangle\langle 1247\rangle} \to -\frac{\langle 1248\rangle\langle 3457\rangle}{\langle 4(12)(35)(78)\rangle} \to -\frac{\langle 1247\rangle\langle 12(345)\cap(678)\rangle}{\langle 1278\rangle\langle 4(12)(35)(67)\rangle} \to -\frac{\langle 4567\rangle\langle 4(12)(35)(78)\rangle}{\langle 1245\rangle\langle 3457\rangle\langle 4678\rangle} \to -\frac{\langle 4(12)(35)(67)\rangle}{\langle 1234\rangle\langle 4567\rangle} \tag{11}$$

The first A_5 lives in an 8-cycle of the $\mathrm{Gr}(4,8)$ dihedral+parity, while the other three live in 16-cycles. And again I call these good subalgebras because by generating the full A_5 algebra from those seeds one only creates good $\mathcal{X}$-variables. I claim that there are no good D_5's or A_6's in $\mathrm{Gr}(4,8)$.

Now, it turns out that $R_8^{(2)}$ can be represented by simply adding together two of the A_5's:

$$\begin{aligned} R_8^{(2)} = \frac{1}{4} f_{A_5}\left(\frac{\langle 1238\rangle\langle 1256\rangle}{\langle 1235\rangle\langle 1268\rangle} \to \frac{\langle 1236\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2356\rangle} \to \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle} \to \frac{\langle 1567\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3567\rangle} \to \frac{\langle 1356\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 3456\rangle}\right) + \\ \frac{1}{2} f_{A_5}\left(\frac{\langle 1238\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 2358\rangle} \to -\frac{\langle 1235\rangle\langle 4568\rangle}{\langle 5(18)(23)(46)\rangle} \to \frac{\langle 1568\rangle\langle 2358\rangle\langle 3456\rangle}{\langle 1358\rangle\langle 2356\rangle\langle 4568\rangle} \to -\frac{\langle 5(18)(23)(46)\rangle}{\langle 1258\rangle\langle 3456\rangle} \to \frac{\langle 1278\rangle\langle 1358\rangle}{\langle 1238\rangle\langle 1578\rangle}\right) \\ + \ \text{dihedral} + \text{conjugate} \end{aligned} \tag{12}$$

The difference between the overall factors of the two terms is due to overcounting the first A_5 (as it lives in an 8-cycle).

Braid Symmetres in $\mathrm{Gr}(4,8)$

(This section written for mine and Andrew's benefit). The braid symmetries of Fraser are:

$$\begin{aligned}\sigma_1: \quad & Z_1 \to Z_2, \quad Z_2 \to Z_1\langle 2345\rangle + Z_2\langle 3451\rangle \\ & Z_5 \to Z_6, \quad Z_6 \to Z_5\langle 6781\rangle + Z_6\langle 7815\rangle\end{aligned} \tag{13}$$

$$\begin{aligned}\sigma_2: \quad & Z_2 \to Z_3, \quad Z_3 \to Z_2\langle 3456\rangle + Z_3\langle 4562\rangle \\ & Z_6 \to Z_7, \quad Z_7 \to Z_6\langle 7812\rangle + Z_7\langle 8126\rangle\end{aligned} \tag{14}$$

$$\begin{aligned}\sigma_3: \quad & Z_3 \to Z_4, \quad Z_4 \to Z_3\langle 4567\rangle + Z_4\langle 5673\rangle \\ & Z_7 \to Z_8, \quad Z_8 \to Z_7\langle 8123\rangle + Z_8\langle 1237\rangle\end{aligned} \tag{15}$$

They map $\mathcal{X}$-coordinates to $\mathcal{X}$-coordinates, and satisfy the braid relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \tag{16}$$

along with other non-trivial relations such as

$$\sigma_3\sigma_2\sigma_1 = \text{dihedral cycle}, \quad \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = \text{identity}. \tag{17}$$

Initial Questions

- What is the behavior of $R_8^{(2)}$ under the σ_i? Is there any physics interpretation?
- Do the σ_i help "distinguish" the subalgebras that contribute to $R_8^{(2)}$?
- If you take the $\mathcal{X}$-variables that show up in $R_8^{(2)}$, and apply all possible braids to them, does it generate all of the clusters?