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Cluster Polylogarithms and Subalgebra-Constructibility I: Novel Decompositions of the Seven-Particle Remainder Function

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Abstract: Everything we know about cluster algebras and polylogarithms.

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2 A Brief Introduction to Cluster Algebras

Cluster algebras were introduced by Fomin and Zelevinsky [**?**] and were motivated by questions of total positivity. The original goal was to gain a better understanding of what algebraic varieties can have a natural notion of positivity, and what functions can determine such positivity. A simple and highly-relevant example for amplitudes is the positive Grassmannian $\mathrm{Gr}^+(k,n)$, i.e. the space of $k \times n$ matrices where all ordered $k \times k$ minors are positive.

What kind of questions can cluster algebras help us answer? One of the most straightforward is: how many minors do we need to specify a point in $\mathrm{Gr}^+(k,n)$? In other words, given a $k \times n$ matrix M, how many minors of M do we have to calculate to know if $M \in \mathrm{Gr}^+(k,n)$? The reason that this is an interesting question is that the minors are not all independent, they satisfy the identities known as Plücker relations:

$$\langle abI\rangle\langle cdI\rangle = \langle acI\rangle\langle bdI\rangle + \langle adI\rangle\langle bcI\rangle, \tag{2.1}$$

where the Plücker coordinates $= \langle i_1,\ldots,i_k\rangle =$ the minor of columns $i_1,\ldots,i_k$, and I is a multi-index with $k-2$ entries.

We'll now work through the example of $\mathrm{Gr}(2,5)$ in detail to try to understand how many minors one needs to check for positivity of the whole matrix. The 5 cyclically adjacent minors, $\langle 12\rangle, \langle 23\rangle, \langle 34\rangle, \langle 45\rangle, \langle 15\rangle > 0$, are all independent from each other and so must each be checked. How many of the non-adjacent minors do we have to check? It turns out that the answer is 2. For example, if we specify that $\langle 13\rangle, \langle 14\rangle > 0$ then we can use Plücker relations to show

$$\begin{aligned}
\langle 24\rangle &= (\langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 14\rangle)/\langle 13\rangle \\
\langle 25\rangle &= (\langle 12\rangle\langle 45\rangle + \langle 24\rangle\langle 15\rangle)/\langle 14\rangle \\
\langle 35\rangle &= (\langle 25\rangle\langle 34\rangle + \langle 23\rangle\langle 45\rangle)/\langle 24\rangle.
\end{aligned} \tag{2.2}$$

Here we have expressed all of the remaining minors as sums and products of the cyclically adjacent minors along with $\langle 13\rangle$ and $\langle 14\rangle$, so everything is positive.

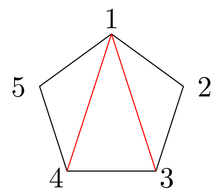
So we only need to check two – but can we check any two? Clearly we can use any of the cyclic images of $\{\langle 13\rangle, \langle 14\rangle\}$. What about $\{\langle 13\rangle, \langle 25\rangle\}$? This is a bit harder to see, but no, this pair does not work: there is no way to write down the remaining Plückers in terms of $\langle 13\rangle$ and $\langle 25\rangle$ such that everything is manifestly positive. For example, the matrix

$$\begin{pmatrix} 1 & -1 & -4 & 3 & -2 \\ 2 & 2 & -6 & 4 & -1 \end{pmatrix} \tag{2.3}$$

satisfies $\langle 12\rangle, \ldots, \langle 15\rangle, \langle 13\rangle, \langle 25\rangle > 0$ but has $\langle 14\rangle, \langle 24\rangle, \langle 35\rangle < 0$. In the end, $\{\langle 13\rangle, \langle 14\rangle\}$ and its cyclic images are the only pairs that describe a point in $\mathrm{Gr}^+(2,5)$.

This was easy enough to work out for this small case, but the problem gets much more complicated for larger matrices. However, there is a closely related, and much simpler, problem in geometry which can give us a bit more intuition: triangulating polygons.

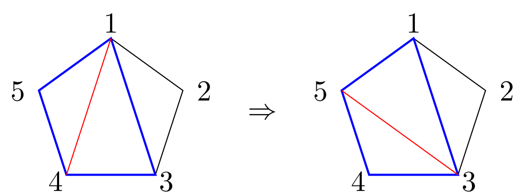
Consider the following triangulation of the pentagon:



(2.4)

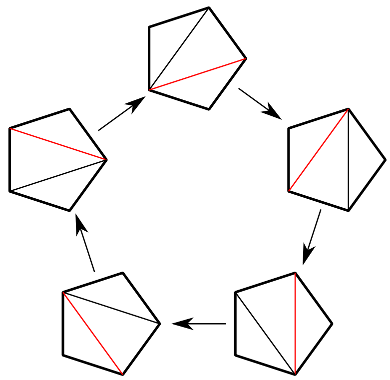
We can immediately see the parallels with our $\mathrm{Gr}(2,5)$ situation (this is an example of the more general Plücker embedding which connects $\mathrm{Gr}(k,n)$ with projective space). Here we associate lines connecting points i and j with the Plücker coordinate $\langle ij \rangle$, and we see that the triangulations of the pentagon all describe points in $\mathrm{Gr}^+(2,5)$. In fact this correspondence holds between n-gons and $\mathrm{Gr}(2,n)$.

A simple observation, but one at the very heart of cluster algebras, is that given some triangulation of a polygon one can create a *new* triangulation by picking a quadrilateral and flipping its diagonal. For example:



(2.5)

By repeatedly performing these flips one can generate all possible triangulations of a polygon:

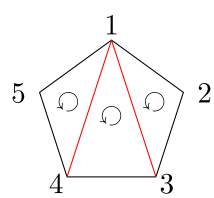


where in each case the red diagonal gets flipped.

Cluster algebras are a combinatorial tool which captures all of this structure (and much more!). The basic idea is that a cluser algebra is a collection of *clusters*, which in this case represent individual triangulations of an n-gon, and these clusters are connected via a process called *mutation*, which in this case is the flipping-the-diagonal process. [**more description of math applications of clsuter algebras? seiberg duality?**]

Basic definition

We'll begin by working through the cluster algebra for $\mathrm{Gr}(2,5)$. Each cluster is labeled by a collection of coordinates, which in this case are the edges of the pentagon along with the diagonals of the particular triangulation. These coordinates are then connected via an orientation of the pentagon and all subtriangles, for example:



(2.6)

We can redraw this diagram as

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd} \tag{2.7}$$

In this quiver diagram, we have an arrow between two Plückers $\langle ab\rangle \to \langle cd\rangle$ if the triangle orientations in eq. (2.6) have segment (ab) flowing into segment (cd). The boxes around the $\langle ii+1\rangle$ indicate that they are *frozen* – in other words, we never change the outer edges of the pentagon, only the diagonal elements. The variables living at the frozen nodes can be thought of as parameterizing the boundary of our space, and the mutable nodes represent parameterizations of the interior. And lastly it is unnecessary to draw the arrows connecting the outer edges, as that is redundant (and unchanging) information.

We have now drawn our first cluster (also sometimes called a seed). To review/introduce some terminology, the Plückers are called cluster $\mathcal{A}$-coordinates (sometimes also $\mathcal{A}$-variables), and they come in two flavors: mutable ($\langle 13 \rangle$ and $\langle 14 \rangle$) and frozen ($\langle i i+1 \rangle$). The information of the arrows can be represented in terms of a skew-symmetric adjacency matrix

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i). \tag{2.8}$$

The process of mutation, which we described geometrically in terms of flipping the diagonal, has a simple interpretation at the level of this quiver. In particular, given a quiver such as eq. (2.7), chose a node k with associated $\mathcal{A}$-coordinate a_k to mutate on (this is equivalent to picking which diagonal to flip). Then draw a new quiver that changes a_k to a'_k defined by

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}, \tag{2.9}$$

(with the understanding that an empty product is set to one) and leaves the other cluster coordinates unchanged. The arrows connecting the nodes in this new cluster are modified from the original cluster according to

- for each path $i \to j \to k$, add an arrow $i \to j$,
- reverse all arrows on the edges incident with k,
- and remove any two-cycles that may have formed.

This creates a new adjacency matrix b'_{ij} via

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}, \\ b_{ij}, & \text{if } b_{ik} b_{kj} \leq 0, \\ b_{ij} + b_{ik} b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik} b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases} \tag{2.10}$$

Mutation is an involution, so mutating on a'_k will take you back to the original cluster (as flipping the same diagonal twice will take you back to where you started).

For our purposes, *a cluster algebra is a set of quivers closed under mutation.* This means that mutating on any node of any quiver will generate a different quiver in the cluster algebra. The general procedure is to start with a quiver such as eq. (2.7), with some collection of frozen and unfrozen nodes in a connected quiver, and continue mutating on all available nodes until you either close your set or convince yourself that the cluster algebra is infinite.

We will end this brief introduction with a last piece of notation: cluster algebras are often referred to by particularly nice quiver types formed by their mutable nodes at some cluster. In the case of $\mathrm{Gr}(2,5)$, the mutable nodes of eq. (2.7) form an oriented A_2 Dynkin diagram, $\langle 13\rangle \to \langle 14\rangle$, and so we will often speak interchangeably of the cluster algebras for $\mathrm{Gr}(2,5)$ and A_2. This is a slight abuse of notation as the $\mathrm{Gr}(2,5)$ cluster algebra corresponds specifically to the cluster algebra generated by the collection of frozen and mutable nodes in eq. (2.7), whereas an A_2 cluster algebra is $a_1 \to a_2$ dressed with any number of frozen nodes. We will see how this language can be useful in the next section.

2.1 Cluster $\mathcal{X}$-coordinates

Another important set of information encoded in cluster algebras are called Fock-Goncharov or $\mathcal{X}$-coordinates and were introduced in [?]. As we will see in future sections, cluster $\mathcal{X}$-coordinates will play a crucial role in connecting cluster algebras to polylogarithms and scattering amplitudes. While everything can be (and often is) phrased purely in terms of $\mathcal{A}$-coordinates , we believe that emphasizing $\mathcal{X}$-coordinates allows for more direct connection with the full cluster algebraic structure.

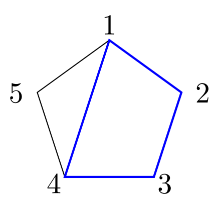
Given a quiver described by the matrix b, the $\mathcal{A}$- and $\mathcal{X}$-coordinates are related as follows:

$$x_i = \prod_j a_j^{b_{ij}}. \tag{2.11}$$

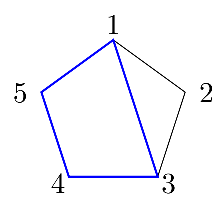
For example, the quiver in eq. (2.7) has $\mathcal{X}$-coordinates

$$\mathcal{X}_1 = \frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle}, \quad \mathcal{X}_2 = \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle}. \tag{2.12}$$

In the pentagon-triangulation picture, these $\mathcal{X}$-coordinates describe overlapping quadrilaterals:



$= \mathcal{X}_1,$



$= \mathcal{X}_2,$ (2.13)

The mutation rules for the $\mathcal{X}$-coordinates are

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i(1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases}, \tag{2.14}$$

For many applications in the rest of this work we will refer to cluster by their $\mathcal{X}$-coordinates alone, for example in this language we take the generic A_2 cluster algebra seed as

$$x_1 \to x_2. \tag{2.15}$$

This again is a slight abuse of notation, in that it can be unclear, given a quiver, if one should use the $\mathcal{A}$-coordinate mutation rules (2.9) or $\mathcal{X}$-coordinate rules (2.14). This ambiguity will be resolved in this work by that convention that if a quiver is given with no frozen nodes then it is referring to a collection of $\mathcal{X}$-coordinates .

By continuing to mutate on alternating nodes (denoted below by red) we generate the following sequence of clusters:

$$\begin{aligned} x_1 &\to x_2 \\ x_1(1+x_2) &\leftarrow \frac{1}{x_2} \\ \frac{1}{x_1(1+x_2)} &\to \frac{x_2}{1+x_1+x_1x_2} \\ \frac{x_1x_2}{1+x_1} &\leftarrow \frac{1+x_1+x_1x_2}{x_2} \\ \frac{1+x_1}{x_1x_2} &\to \frac{1}{x_1} \\ x_2 &\leftarrow x_1 \\ &\vdots \end{aligned} \tag{2.16}$$

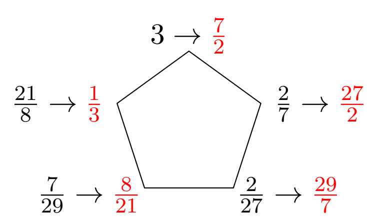
where the series then repeats. Note that by labeling the $\mathcal{X}$-coordinates as

$$\mathcal{X}_1 = 1/x_1, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1+x_2), \quad \mathcal{X}_4 = \frac{1+x_1+x_1x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1+x_1}{x_1x_2}, \tag{2.17}$$

then the general mutation rule of eq. (2.14) takes the simple form of

$$1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}. \tag{2.18}$$

Putting this all together, we will generically refer to an A_2 cluster algebra as any set of clusters $1/\mathcal{X}_{i-1} \to \mathcal{X}_i$ for $i = 1 \ldots 5$ where the $\mathcal{X}_i$ satisfy eq. (2.18). We believe it is useful at this point to emphasize that one can take as input any $\{x_1, x_2\}$ and generate an associated A_2. For example, one could start with the quiver $3 \to \frac{7}{2}$ and generate the A_2

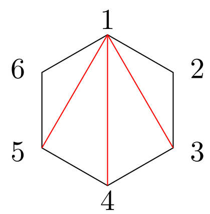


. (2.19)

(Mutating on the node in red moves you clockwise around the pentagon.) In future sections it will be necessary to consider collections of multiple A_2 algebras, in such cases we label them by only one of their clusters, e.g. $x_1 \to x_2$, with the understanding that we are referring to the A_2 which contains that cluster as an element.

2.2 Subalgebras

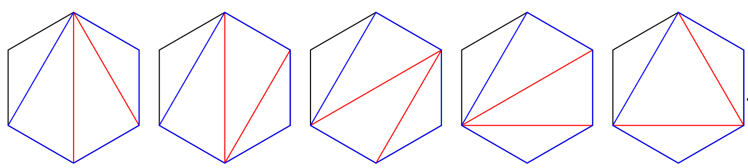
Cluster algebras contain a rich and intricate subalgebra structure which will be critical for our upcoming physics applications. We can study a simple example by looking at triangulations of a hexagon, which is equivalently described by the cluster algebra associated with $\mathrm{Gr}(2,6)$:



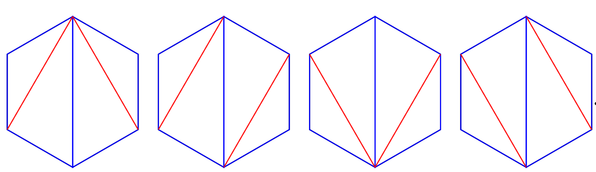
$$\Leftrightarrow \quad \begin{tikzcd} \boxed{\langle 12\rangle} \arrow[d] & & & \\ \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \langle 15\rangle \arrow[r] \arrow[d] & \boxed{\langle 16\rangle} \\ \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul] & \boxed{\langle 56\rangle} \arrow[ul] \end{tikzcd} \quad \Leftrightarrow \quad \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \to \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle} \to \frac{\langle 14\rangle\langle 56\rangle}{\langle 16\rangle\langle 45\rangle} \tag{2.20}$$

Here we have given the seed cluster for $\mathrm{Gr}(2,6)$ in the triangulation, $\mathcal{A}$-coordinate , and $\mathcal{X}$-coordinate representations, respectively.

The $\mathrm{Gr}(2,6)$ cluster algebra features 14 clusters, and these clusters can be grouped together in to multiple (overlapping) sets which we call subalgebras. A simple example is the collection of all triangulations which involve the cord $\langle 15\rangle$ (alternatively the $\mathcal{X}$-coordinate $\frac{\langle 14\rangle\langle 56\rangle}{\langle 16\rangle\langle 45\rangle}$). It is easy to see that this set contains 5 clusters and is itself a cluster algebra generated via mutations by taking eq. (2.20), "freezing" the cord $\langle 15\rangle$ (or that same $\mathcal{X}$-coordinate), then mutating on the other two cords. This of course is the cluster algebra of triangulating the pentagon formed by points $1,\ldots,5$:

. (2.21)

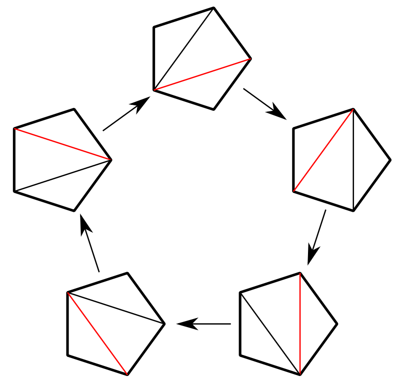
In practice, we refer to the collection of clusters that share this pentagon as an A_2 subalgebra of $\mathrm{Gr}(2,6)$. Instead of freezing the cord $\langle 15\rangle$ we could have frozen the cord $\langle 14\rangle$, in which case we generate an $A_1 \times A_1$ subalgebra, as A_1 corresponds to the triangulations of a square and the cord $\langle 14\rangle$ divides the hexagon in to two non-overlapping squares:

. (2.22)

Larger cluster algebras contain many subalgebras of different types. We have catalogued the counting of these subalgebras for many relevant cluster algebras in Appendix B.

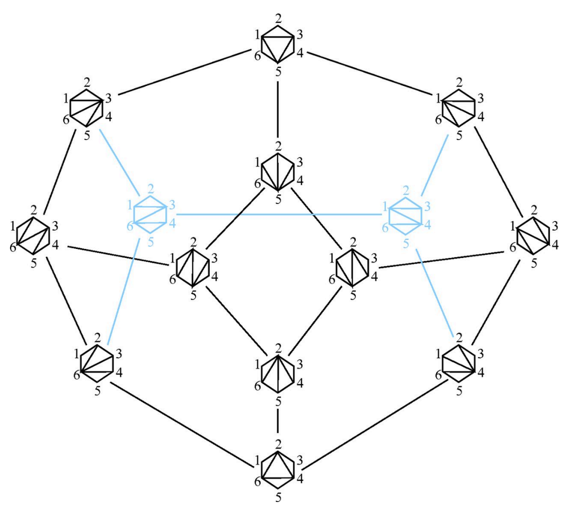
2.3 Associahedra

The mutation paths between clusters, and which sets of clusters group together to form subalgebras, can be easily visualized through an object known as the associahedron (also called the Stasheff polytope) for a given cluster algebra. This polytope is formed by vertices representing an indivual cluster, and edges are drawn between clusters connected via mutation. So the figure



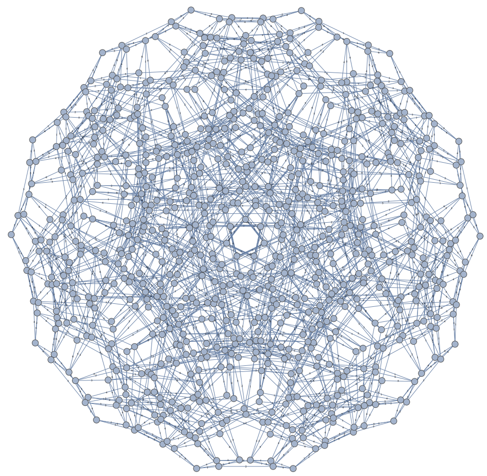
[**create new version**] is in fact the $\mathrm{Gr}(2,5)$ or A_2 associahedron, coincidentally takes the form of a pentagon.

The associahedron associated with the $\mathrm{Gr}(2,6) \leftrightarrow A_3$ cluster algebra (i.e. triangulations of a hexagon) is



[**create new version**] This associahedron has 14 vertices, corresponding to the 14 clusters, with 3 square faces and 6 pentagonal faces. The square faces represent $A_1 \times A_1$ subalgebras, and the pentagonal faces are A_2 subalgebras as discussed in the previous section. Because of the Grassmannian duality $\mathrm{Gr}(2,6) = \mathrm{Gr}(4,6)$, this (remarkably simple!) cluster algebra and associahedron play an integral role in the momentum twistors for 6-particle kinematics for $\mathcal{N} = 4$ SYM.

For future reference we will now describe the associahedron for $\mathrm{Gr}(4,7)$:



It features 833 clusters/vertices of valence 6. The dimension-2 faces are 1785 squares and 1071 pentagons, and there are 49 different A-coordinates that appear. It is remarkable that these complex structures arise from the simple mutation rules of eq. (2.9).

2.4 An overview of finite cluster algebras

Fortunately, Fomin and Zelevinksy classified all finite cluster algebras in[**?**]. In particular, they showed that a cluster algebra is of finite type iff the mutable part of its quiver at some cluster takes the form of an oriented, simply-laced Dynkin diagram: $A_n, D_n, E_{n\leq 8}$. We will describe several of the relevant cases to give the reader a flavor for the world of finite algebras.

Cluster algebras of type A_n

$$x_1 \to x_2 \to \ldots \to x_n \tag{2.23}$$

correspond to triangulations of an $(n+3)$-gon, where each cluster is a triangulation, each $\mathcal{A}$-coordinate is a cord, and each $\mathcal{X}$-coordinate is a quadrilateral with a cord as a diagonal embedded in the $(n+3)$-gon. This makes the counting easy: the number of clusters for A_n is given by the Catalan number $C(n+1)$, the number of $\mathcal{A}$-coordinates is $\binom{n+3}{2} - n$, and the number of $\mathcal{X}$-coordinates is $2\binom{n+3}{4}$. Subalgebras correspond to embedding a smaller polygon in to the $(n+3)$-gon, for example the A_2 subalgebras in A_5 are the $56 = \binom{8}{5}$ pentagonal embeddings in an octagon.

The cluster algebra D_4

$$\begin{tikzcd} & & x_3 \\ x_1 \arrow[r] & x_2 \arrow[ur] \arrow[dr] & \\ & & x_4 \end{tikzcd} \tag{2.24}$$

has 50 clusters, 16 $\mathcal{A}$-coordinates , and 104 $\mathcal{X}$-coordinates . There are 36 A_2 subalgebras and 12 A_3 subalgebras.

The cluster algebra D_5

$$\begin{tikzcd} & & & x_4 \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\ & & & x_5 \end{tikzcd} \tag{2.25}$$

has 182 clusters, 25 $\mathcal{A}$-coordinates , and 260 $\mathcal{X}$-coordinates . There are 125 distinct A_2 subalgebras, 65 A_3, 10 A_4, and 5 D_4.

Finally, we describe the cluster algebra E_6

$$\begin{tikzcd} & & x_4 \arrow[d] & & \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 & x_5 \arrow[l] & x_5 \arrow[l] \end{tikzcd} \tag{2.26}$$

which has 833 clusters, 42 $\mathcal{A}$-coordinates , and 770 $\mathcal{X}$-coordinates . The subalgebra counting is:

A_2	A_3	A_4	D_4	A_5	D_5
504	364	98	35	7	14

. (2.27)

2.5 Cluster automorphisms

See [**?**] for a more thorough mathematical introduction. The simplest example of a cluster automorphism is what we will call a direct automorphism. Let $\mathcal{A}$ be a cluster algebra. Then $f : \mathcal{A} \to \mathcal{A}$ is direct automorphism of $\mathcal{A}$ if

- for every cluster $\mathbf{x}$ of $\mathcal{A}$, $f(\mathbf{x})$ is also a cluster of $\mathcal{A}$,
- f respects mutations, i.e. $f(\mu(x_i, \mathbf{x})) = \mu(f(x_i), f(\mathbf{x}))$.

A simple example of this for A_2 is the map

$$\sigma_{A_2} : \quad \mathcal{X}_i \to \mathcal{X}_{i+1}, \tag{2.28}$$

which cycles the five clusters $1/\mathcal{X}_i \to \mathcal{X}_{i+1}$ amongst themselves, and can be cast in terms of the seed variables x_1, x_2 as

$$\sigma_{A_2} : \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to x_1(1 + x_2). \tag{2.29}$$

A less obvious example of a cluster automorphism is what are called indirect automorphisms, which respect mutations but do not map clusters directly on to clusters; instead

- for every cluster $\mathbf{x}$ of $\mathcal{A}$, $f(\mathbf{x})$ + invert all nodes + swap direction of all arrows $=$ a cluster of $\mathcal{A}$.

For A_2 we have the indirect automorphism

$$\tau_{A_2} : \quad \mathcal{X}_i \to \mathcal{X}_{6-i}, \tag{2.30}$$

where indices are understood to be mod 5, and can instead be cast in terms of the seed variables x_1, x_2 as

$$\tau_{A_2}: \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to \frac{1}{x_1}. \tag{2.31}$$

We can see how this works in a simple example

$$\begin{aligned}
\tau_{A_2}(1/\mathcal{X}_1 \to \mathcal{X}_2) &= 1/\mathcal{X}_5 \to \mathcal{X}_4 \\
&\Rightarrow \text{invert each node and swap direction of all arrows} \\
&= \mathcal{X}_5 \leftarrow 1/\mathcal{X}_4, \ \text{which is in the original } A_2.
\end{aligned} \tag{2.32}$$

It is useful to think of indirect automorphisms as generating a “mirror” or “flipped” version of the original $\mathcal{A}$, where the total collection of $\mathcal{X}$-coordinates is the same, but their Poisson bracket has flipped sign. The existence of this flip then can be seen as resulting from the choice of assigning b_{ij} = (# arrows $i \to j$) - (# arrows $j \to i$), where instead we could have chosen b_{ij} = (# arrows $j \to i$) - (# arrows $i \to j$) and still generated the same clsuter algebraic structure, albeit with different labels for the nodes. In the generic case this is an arbitrary choice, and τ captures the superficiality of the notation change.

The automorphisms σ_{A_2} and τ_{A_2} generate the complete automorphism group for A_2, namely, D_5 (the notation here is regretably redundant; here we are referring to the dihedral group of five elements, which is of course distinct from the Dynkin diagram D_5 – we hope that context will clarify to the reader what we mean in each case). We now list generators for the automorphism groups of the finite algebras discussed already. First we adopt the notation

$$x_{i_1,\ldots,i_k} = \sum_{a=1}^{k} \prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1}x_{i_2} + \ldots + x_{i_1}\cdots x_{i_k}. \tag{2.33}$$

Cluster algebras of type A_n, as defined in eq. (2.23), have automorphism group D_{n+3}, with a cyclic generator σ_{A_n} (direct, length $n+3$)

$$\sigma_{A_n}: \quad x_{k<n} \to \frac{x_{k+1}(1+x_{1,\ldots,k-1})}{1+x_{1,\ldots,k+1}}, \quad x_n \to \frac{1+x_{1,\ldots,n-1}}{\prod_{i=1}^{n} x_i} \tag{2.34}$$

and flip generator τ_{A_n} (indirect)

$$\tau_{A_n}: \quad x_1 \to \frac{1}{x_n}, \quad x_2 \to \frac{1}{x_{n-1}}, \quad \ldots \ , x_n \to \frac{1}{x_1}. \tag{2.35}$$

The cluster algebra $D_4 \simeq \mathrm{Gr}(3,6)$, as defined in eq. (2.24), has automorphism group $D_4 \times S_3$, with two cyclic generators:

$$\begin{aligned}
&\sigma_{D_4}^{(4)}: \quad x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)\,x_1x_2x_3x_4}{(1+x_{1,2,3})\,(1+x_{1,2,4})}, \quad x_3 \to \frac{1+x_{1,2}}{x_1x_2x_3}, \quad x_4 \to \frac{1+x_{1,2}}{x_1x_2x_4}, \\
&\sigma_{D_4}^{(3)}: \quad x_1 \to \frac{1}{x_3}, \quad x_2 \to \frac{x_1x_2\,(1+x_3)}{1+x_1}, \quad x_3 \to x_4, \quad x_4 \to \frac{1}{x_1},
\end{aligned} \tag{2.36}$$

where $\sigma_{D_4}^{(4)}$ generates the 4-cycle and $\sigma_{D_4}^{(3)}$ the 3-cycle in D_4 and S_3, respectively. Then there is the indirect τ-flip associated with the D_4 automorphism, as well as a direct $\mathbb{Z}_2$-flip associated with the S_3 automorphism:

$$\begin{aligned} \tau_{D_4}: &\quad x_2 \to \frac{1+x_1}{x_1x_2\,(1+x_3)\,(1+x_4)}, \\ \mathbb{Z}_{2,D_4}: &\quad x_3 \to x_4, \quad x_4 \to x_3. \end{aligned} \tag{2.37}$$

The cluster algebra $D_{n>4}$

$$\begin{tikzcd}[row sep=small] & & & x_{n-1} \\ x_1 \arrow[r] & x_2 \arrow[r] & \cdots \arrow[r] & x_{n-2} \arrow[u] \arrow[d] \\ & & & x_n \end{tikzcd} \tag{2.38}$$

has automorphism group $D_n \times \mathbb{Z}_2$ with generators σ_{D_n} (n-cycle, direct), $\mathbb{Z}_{2,D_n}$ (2-cycle, direct), and τ_{D_n} (2-cycle, indirect). The Z_2 simply swaps $x_{n-1} \leftrightarrow x_n$, and for D_5, as defined in eq. (2.25), the σ and τ generators can be represented by

$$\begin{aligned} \sigma_{D_5}: &\quad x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)x_3}{1+x_{1,2,3}}, \quad x_3 \to \frac{x_1x_2x_3x_4x_5(1+x_{1,2})}{(1+x_{1,2,3,4})(1+x_{1,2,3,5})}, \\ &\quad x_4 \to \frac{1+x_{1,2,3}}{x_1x_2x_3x_4}, \quad x_5 \to \frac{1+x_{1,2,3}}{x_1x_2x_3x_5}, \\ \tau_{D_5}: &\quad x_1 \to x_1, \quad x_2 \to \frac{1+x_1}{x_1x_2(1+x_3x_5+x_{3,4,5})}, \quad x_3 \to \frac{x_3x_4x_5}{(1+x_{3,4})(1+x_{3,5})}, \\ &\quad x_4 \to \frac{1+x_3x_5+x_{3,4,5}}{x_4}, \quad x_5 \to \frac{1+x_3x_5+x_{3,4,5}}{x_5}. \end{aligned} \tag{2.39}$$

$E_6 \simeq \mathrm{Gr}(4,7)$, as defined in eq. (2.26), has automorphism group D_{14} with generators σ_{E_6} (7-cycle, direct), $\mathbb{Z}_{2,E_6}$ (2-cycle, direct), and τ_{E_6} (2-cycle, indirect). In $\mathrm{Gr}(4,7)$ language, these are the traditional cycle ($Z_i \to Z_{i+1}$), parity ($Z \to W$'s), and flip ($Z_i \to Z_{8-i}$) symmetries, respectively. In E_6 language they can be represented by

$$\begin{aligned} \sigma_{E_6}: &\quad x_1 \to \frac{1}{x_6(1+x_{5,3,4})}, \quad x_2 \to \frac{1+x_{6,5,3,4}}{x_5(1+x_{3,4})}, \quad x_3 \to \frac{(1+x_{2,3,4})(1+x_{5,3,4})}{x_3(1+x_4)}, \\ &\quad x_4 \to \frac{1+x_{3,4}}{x_4}, \quad x_5 \to \frac{1+x_{1,2,3,4}}{x_2(1+x_{3,4})}, \quad x_6 \to \frac{1}{x_1(1+x_{2,3,4})}, \\ \mathbb{Z}_{2,E_6}: &\quad x_i \to x_{7-i}, \\ \tau_{E_6}: &\quad x_1 \to \frac{x_5}{1+x_{6,5}}, \quad x_2 \to (1+x_5)x_6, \quad x_3 \to \frac{(1+x_{1,2})(1+x_{6,5})}{x_1x_2x_3x_5x_6(1+x_4)}, \\ &\quad x_4 \to x_4, \quad x_5 \to x_1(1+x_2), \quad x_6 \to \frac{x_2}{1+x_{1,2}}. \end{aligned} \tag{2.40}$$

2.6 Grassmannian cluster algebras (and cluster Poisson spaces)

For $\mathrm{Gr}(k,n)$, Scott showed [**?**] that the associated seed cluster is

$$\begin{tikzcd}
f_{1l} \arrow[r] \arrow[d] & \cdots \arrow[r] & f_{13} \arrow[r] \arrow[d] & f_{12} \arrow[r] \arrow[d] & \boxed{f_{11}} \arrow[l, phantom] \arrow[ul, start anchor=south west] \\
f_{2l} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[ul] & f_{23} \arrow[r] \arrow[d] & f_{22} \arrow[r] \arrow[d] \arrow[ul] & \boxed{f_{21}} \arrow[ul] \\
\vdots \arrow[d] & \vdots & \vdots \arrow[d] & \vdots \arrow[d] \arrow[ul] & \vdots \arrow[ul] \\
\boxed{f_{kl}} & \cdots & \boxed{f_{k3}} & \boxed{f_{k2}} \arrow[ul] & \boxed{f_{k1}} \arrow[ul]
\end{tikzcd}$$

where

$$f_{ij} = \begin{cases} \frac{\langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle}{\langle 1,\ldots,k\rangle}, & i \le l-j+1, \\ \frac{\langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle}{\langle 1,\ldots,k\rangle}, & i > l-j+1 \end{cases}, \tag{2.41}$$

We can use this to find the Grassmannian cluster algebras of finite type, they are:

$$\mathrm{Gr}(2,n) \leftrightarrow A_{n-3}, \quad \mathrm{Gr}(3,6) \leftrightarrow D_4, \quad \mathrm{Gr}(4,7) \leftrightarrow E_6, \quad \mathrm{Gr}(3,8) \leftrightarrow E_8. \tag{2.42}$$

An intriguing point for those of us studying $\mathcal{N} = 4$ SYM, where the momentum twistors describing particle kinematics live in $\mathrm{Gr}(4,n)$, is that only $\mathrm{Gr}(4, n < 8)$ are finite. The ramifications of the fact that $n > 7$-particle kinematics are associated with infinite cluster algebras are still being worked out.

The nice feature of $\mathcal{X}$-coordinates, at least from a physicists perspective, is that in the case of $\mathrm{Gr}(4,n)$ the $\mathcal{X}$-coordinates are dual-conformal invariant cross-ratios. We can see this for example in the two-loop, six-particle remainder function for $\mathcal{N} = 4$ SYM:

$$\begin{aligned} R_6^{(2)} = \sum_{\text{cyclic}} \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right) \\ + \text{products of } \mathrm{Li}_k(-x) \text{ functions of lower weight} \\ \text{with the same set of arguments.} \end{aligned} \tag{2.43}$$

The arguments of the polylogarithms all take the form of $(-\mathcal{X})$-coordinates for $\mathrm{Gr}(4,6)$. $\mathcal{X}$-coordinates play other important roles in the context of polylogarithm functions independent of scattering amplitudes, for example with $\mathrm{Gr}(2,5) \leftrightarrow A_2$ we have

$$\sum_{\text{cyclic}} \mathrm{Li}_2(-\mathcal{X}_i) + \log(\mathcal{X}_i)\log(\mathcal{X}_{i+1}) = \frac{\pi^2}{6} \tag{2.44}$$

where the definition of $\mathcal{X}_i$ can be inferred from eq. (2.12). There are more complicated examples of polylogarithm identities satisfied by groups of $\mathcal{X}$-coordinates, for example there is a 40-term identity among Li_3 functions where all of the arguments are $(-\mathcal{X})$

2.6.1 Applications to momentum twistors and $\mathcal{N}$=4 SYM

3 Cluster Polylogarithms and MHV Amplitudes

3.1 The Coproduct and Cobracket

3.2 Cluster-Algebraic Structure of MHV Amplitudes

3.3 The A_2 function

We define the A_2 function as

$$\begin{aligned} f_{A_2}(x_1 \to x_2) = \sum_{\text{skew-dihedral}} & \mathrm{Li}_{2,2}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) + \mathrm{Li}_{1,3}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) + 6\,\mathrm{Li}_3\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right) \\ & - \mathrm{Li}_2\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right)\left(3\log\left(\mathcal{X}_{i-1}\right) - \log\left(\mathcal{X}_i\right) + \log\left(\mathcal{X}_{i+1}\right)\right) \\ & + \frac{1}{2}\log\left(\mathcal{X}_{i-3}\right)\log\left(\mathcal{X}_i\right)\log^2\left(\mathcal{X}_{i-1}\right), \end{aligned} \tag{3.1}$$

where the $\mathcal{X}_i$ are defined in terms of x_1 and x_2 as in eq. (2.17), and the skew-dihedral sum indicates subtracting the dihedral flip ($\mathcal{X}_1 \to \mathcal{X}_{6-i}$) and taking a cyclic sum $i = 1$ to 5.

This representation of f_{A_2} differs from that in [?] in several key ways. Firstly, we have added classical polylogarithm terms in order to make f_{A_2} adjacent in A_2:

$$\mathrm{symbol}(f_{A_2}) = -\sum_{\text{skew-dihedral}} [2233] + [2321] + [2332] - 2([2323] + [2343] - [2334]) \tag{3.2}$$

where we adopt the condensed notation $[ijkl] = \mathcal{X}_i \otimes \mathcal{X}_j \otimes \mathcal{X}_k \otimes \mathcal{X}_l$ in order to highlight the adjacency.

An additional benefit of this representation is that all arguments of the polylogarithms in for f_{A_2} are -$\mathcal{X}$-coordinates of A_2. Furthermore, the function is smooth and real-valued for all $x_1, x_2 > 0$. The structure of the A_2 cluster algebra plays a crucial role in this analytic behavior in the following way. $\mathrm{Li}_{2,2}(x,y)$ and $\mathrm{Li}_{1,3}(x,y)$ have branch cuts at $x = 1, y = 1, x * y = 1$. The first two branch cuts are trivially avoided as $-1/\mathcal{X}_i < 0$ for $x_1, x_2 > 0$, however the last one is avoided only because of the exchange relation for A_2:

$$0 < \left(-\frac{1}{\mathcal{X}_{i-1}}\right)\left(-\frac{1}{\mathcal{X}_{i+1}}\right) = \frac{1}{1+\mathcal{X}_i} < 1. \tag{3.3}$$

Lastly, f_{A_2} has $\Lambda^2\,\mathrm{B}_2$ and $\mathrm{B}_3 \otimes \mathbb{C}^*$ coproduct elements expressible in terms of $\mathcal{X}$-coordinates of A_2:

$$\delta\left(f_{A_2}\right) = -\sum_{\text{skew-dihedral}} \{-\mathcal{X}_{i-1}\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + 3\{-\mathcal{X}_i\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + \frac{5}{2}\{-\mathcal{X}_i\}_3 \otimes \mathcal{X}_{i+1} \tag{3.4}$$

This representation of f_{A_2} therefore shares the following properties with $\mathcal{E}_n^{(2)}$:

- cluster adjacent,
- clustery coproduct,
- smooth and real-valued in the positive domain.

3.4 Cluster Automorphisms

3.5 Cluster Adjacency

Cluster adjacency is a property of all Steinmann-satisfying amplitudes, and was first introduced in [**?**]. The original phrasing of this property is that the symbol of all Steinmann-satisfying integrals in n-particle kinematics, when fully expanded out in terms of $\mathcal{A}$-coordinates , is of the form

$$\ldots \otimes \alpha_i \otimes \alpha_j \otimes \ldots \tag{3.5}$$

where α_i and α_j appear together in a cluster of $\mathrm{Gr}(4, n)$. This non-trivial property is a considerable constraint on the space of polylogarithm functions which can appear in amplitudes.

The original presentation of cluster adjacency was in terms of $\mathcal{A}$-coordinates , but adjacency can also be phrased in terms of $\mathcal{X}$-coordinates . We will term these as cluster $\mathcal{A}$-adjacency and cluster $\mathcal{X}$-adjacency, respectively.

The benefit of $\mathcal{A}$-adjacency is that $\mathcal{A}$-coordinates are multiplicatively independent and so any symbol in them will be unique. The same is of course not true for $\mathcal{X}$-coordinates : they satisfy numerous multiplicative identities and so there exists many equivalent representations of a given symbol in terms of $\mathcal{X}$-coordinates , and only some small subset of them may satisfy cluster $\mathcal{X}$- adjacency.

However, the benefit of $\mathcal{X}$-coordinates is that they have a unique Poisson bracket, whereas $\mathcal{A}$-coordinates can appear in many different clusters together, each time with a different value for b_{ij} connecting them. This ambiguity in the Poisson bracket for $\mathcal{A}$-coordinates is equivalent to the ambiguity introduced by the multiplicative identities in the $\mathcal{X}$-coordinates .

While $\mathcal{X}$-adjacency trivially implies $\mathcal{A}$-adjacency, the converse is not so clear. However we have checked for all Grassmannian cluster algebras $\mathrm{Gr}(k \leq 4, n \leq 7)$ that $\mathcal{A}$-adjacency implies $\mathcal{X}$-adjacency, so we conjecture that the two phrasings of cluster adjacency are identical in constraining symbol space. ?

4 Cluster Subalgebra-Constructibility

Define the basic notion here (which applies to both clustery and non-clustery cobrackets) (differentiate between symbol-level and cobracket-level constructibility)

4.1 MHV Amplitudes and A_2-Constructibility

Proof that the A_2 function spans the $\delta_{2,2}$ component of MHV amplitudes in Planar $\mathcal{N} = 4$.

4.2 Finite Subalgebras of Gr(4,n)

Describe/tabulate the finite cluster algebras that appear as subalgebras of Gr(4,n)

4.3 Cobracket-Level Decompositions of $R_7^{(2)}$

Method and results of general search—describe A_3 and null results in more depth, leaving a discussion of A_5 and D_5 for the next sections

4.4 Decomposing $R_n^{(2)}$ for $n \geq 8$

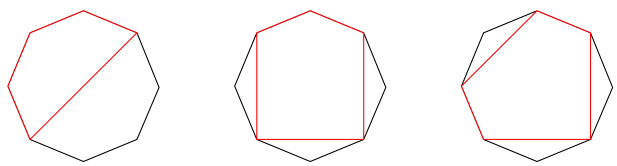
Some discussion of why this is harder, and pointing to paper II

5 The A_5 Function

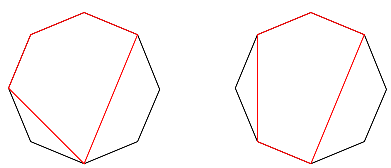
As discussed previously, there are 56 distinct A_2 subalgebras in A_5 ($56 = \binom{8}{5}$ = number of distinct pentagons inside an octagon), they can be parameterized by:

$$\begin{aligned} &\left\{x_1 \to x_2, \quad x_2 \to x_3\left(1+x_4\right), \quad x_2\left(1+x_3\right) \to \frac{x_3 x_4}{1+x_3}\right\} + \sigma_{A_5}, \\ &\left\{x_2 \to x_3, \quad x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}\right\} + \sigma_{A_5} + \tau_{A_5} \end{aligned} \tag{5.1}$$

where by "$+\sigma_{A_5}$" and "$+\sigma_{A_5} + \tau_{A_5}$" I mean "+ cyclic copies" and "+ cyclic and flip copies," respectively. These correspond to the geometries



(8 of each)



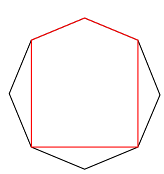
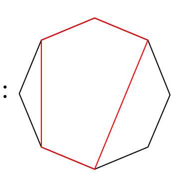
(16 of each)

The A_5 function is a sum over two of the classes of A_2 subalgebras, $x_2 \to x_3\left(1+x_4\right)$ and $x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}$, appropriately antisymmetrized so that the overall f_{A_5} picks up a minus sign under both σ_{A_5} and τ_{A_5}. Explicitly, this is written

$$f_{A_5} = \sum_{i=0}^{7}\sum_{j=0}^{1}(-1)^{i+j}\sigma_{A_5}^i \tau_{A_5}^j \left(\frac{1}{2} f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right) + f_{A_2}\left(x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}\right)\right). \tag{5.2}$$

The factor of $\frac{1}{2}$ in front of $f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right)$ is simply a symmetry factor, as it lives in an 8-cycle of $\{\sigma_{A_5}, \tau_{A_5}\}$.

The two types of A_2's appearing in f_{A_5} are:

$x_2 \to x_3(1+x_4)$:  $x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}$:  (5.3)

6 The D_5 Function

7 Conclusion

A Integrability and Adjacency for A_2

B Counting Subalgebras of Finite Cluster Algebras

In this appendix we catalog the subalgebra structure for the finite connected cluster algebras $\subseteq E_6$. These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

When counting distinct subalgebras, we lump together all subalgebras which are labeled by the same mutable nodes. However the same subalgebra may appear multiple times but dressed by different frozen nodes – these appear as distinct subpolytopes in the full associahedra. We have included the counts for both subpolytopes and distinct subalgebras. Also note that in our counting for $\mathcal{X}$-coordinates we have included both x and $1/x$.

$\underline{A_2}$: clusters: 5 a-coordinates: 5 x-coordinates: 10

$\underline{A_3}$: clusters: 14 a-coordinates: 9 x-coordinates: 30

Type	Subpolytopes	Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

$\underline{A_4}$: clusters: 42 a-coordinates: 14 x-coordinates: 70

Type	Subpolytopes	Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

$\underline{D_4}$: clusters: 50 a-coordinates: 16 x-coordinates: 104

Type	Subpolytopes	Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

$\underline{A_5}$: clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Subpolytopes	Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{D_5}$: clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Subpolytopes	Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{E_6}$: clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Subpolytopes	Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

C Cobracket Spaces in Finite Cluster Algebras