Progress so far

- ✓ Generated subalgebras of finite cluster algebras: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.
- ✓ Catalogued identities between f_{A_2} and f_{A_3} across subalgebras.
- ✓ Found larger subalgebras with "unique" functions by applying *cluster automorphisms*. Other criteria, such as collinear limits, branch cuts (first-entry, Steinmann), and Poisson bracket structure were not helpful.
- ✓ Using new f_{D_5}, found new representation $R_7^{(2)} = \sum \pm f_{D_5}$, where the sum includes all D_5 subalgebras of E_6
- ✓ Understood Steinmann conditions for 8-particle kinematics and generated corresponding BDS-like ansatz.

Next steps

BDS-like remainder functions:

- Get BDS-like normalized symbol for 6,7, and 8 pts.
- Fit $R_8^{(2)}$ to at least D_5 and hopefully E_6 functions.
- Find classical completion for ${}^{X}\mathcal{E}_8^{(2)}$.
- Fit $\mathcal{E}_6^{(2)}$ and $\mathcal{E}_7^{(2)}$ using algebras of different types, e.g. $D_5 + A_3$'s + *etc.* (For $\mathcal{E}_6^{(2)}$ start with ansatz of completely generic A_2, and $A_1 \times A_1$ functions, and then for $\mathcal{E}_7^{(2)}$ include $A_3 \times A_2$-type algebras, particularly at classical/product level.)

Cluster automorphisms:

- Write note on cluster automorphisms, D_5 function, and how D_4 and A_4 fail to have relevant cluster functions.
- Understand more deeply what sign choices to take for cluster automorphisms.
- Fix remaining free parameters in D_5 function.

Cluster adjacency (see `https://arxiv.org/pdf/1710.10953.pdf`):

- Is cluster adjacency precisely Steinmann? We can probe this at eight points by seeing if cluster adjacency breaks down in precise correspondence with how the Steinmann relations are broken in a given BDS-like normalization.

- What is the dimension of the symbol space with coproduct equal to the "known" A_2 function and satsifying cluster adjacency (either at the level of $\mathcal{A}$-coords or some equivalent/stronger/weaker condition on adjacent $\mathcal{X}$-coords)? What about higher algebras? This condition + cluster automorphisms should fix MANY degrees of freedom at the symbol level – the question is whether this will be useful for writing down amplitudes.
- Despite explicitly working in terms of 42 $\mathcal{X}$-coordinates chosen on Gr(4,7), all of the claims James makes are related to $\mathcal{A}$-coordinates – why is this? What precisely is the statement at the level of $\mathcal{X}$-coords? To study this, generate conversion from $\mathcal{X}$-coords to $\mathcal{A}$-coords for all algebras and then use $\mathcal{X} \to \mathcal{A}$ map to impose cluster adjacency at level of symbol for generic A_2 (and higher) symbol.
- How does the adjacency condition at symbol-level connect with the "locality" condition at coproduct level? It is likely that symbol adjacency $\to$ coproduct locality.
- Is there a version of the first-entry condition for non-Gr(4,n) algebras?
- What (superposition of) cluster mutation paths corresponds to the amplitude? (Can we identify a cluster or subalgebra within Gr(4,6) on which the symbol of the amplitude 'starts' and from which we can think of the symbol simply being a set of parallel mutation paths?)