

# Steinmann Relations and the Two-Loop MHV Amplitude in Eight-Particle Kinematics

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**ABSTRACT:** We present the full functional form of the two-loop eight-point MHV amplitude in the planar limit of maximally supersymmetric Yang-Mills theory, in terms of cluster polylogarithms. We also compute the two BDS-like ansätze that can be formulated in eight-particle kinematics, and find that ...

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## 1 Introduction

## 2 Promoting $R_8^{(2)}$ from Symbol to Function

Briefly describe the method outlined in [1] for upgrading the  $n$ -point two-loop MHV symbol to a function.

## 3 The Steinmann Relations for Eight Particles

When the number of gluons  $n$  is not a multiple of 4, the BDS-like ansatz is unique because there exists only a single decomposition of the BDS ansatz

$$\mathcal{A}_n^{\text{BDS}} = \mathcal{A}_n^{\text{BDS-like}}(\{s_{i,i+1}\}) \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} Y_n(\{u_i\}) \right], \quad n \neq 4K, \quad (3.1)$$

such that the kinematic dependence of  $\mathcal{A}_n^{\text{BDS-like}}$  involves only two-particle Mandelstam invariants while  $Y_n$  is a weight-two function depending only on dual conformal invariant cross ratios [2]. When  $n$  is a multiple of 4, no decomposition of this type exists, and we are forced to consider multiple BDS-like ansätze if we want to transparently expose all Steinmann relations between higher-particle Mandelstam invariants.

In eight particle kinematics, there are two natural BDS-like normalization choices we might consider. Namely, we can let our BDS-like ansatz depend on either three- or four-particle Mandelstam invariants in addition to two-particle invariants. In this spirit, let us define a pair of Bose-symmetric BDS-like ansätze, respectively satisfying

$$\mathcal{A}_8^{\text{BDS}} = {}^3\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\}, \{s_{i,i+1,i+2,i+3}\}) \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} {}^3Y_8(\{u_i\}) \right], \quad (3.2)$$

$$\mathcal{A}_8^{\text{BDS}} = {}^4\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\}, \{s_{i,i+1,i+2}\}) \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} {}^4Y_8(\{u_i\}) \right]. \quad (3.3)$$

In fact, the functions  ${}^3\mathcal{A}_8^{\text{BDS-like}}$  and  ${}^4\mathcal{A}_8^{\text{BDS-like}}$  are not uniquely fixed by these decomposition choices; each admits a family of Bose-symmetric solutions.

Any choice for  ${}^3\mathcal{A}_8^{\text{BDS-like}}$  or  ${}^4\mathcal{A}_8^{\text{BDS-like}}$  that satisfy eqns. (3.2) or (3.3) gives rise to a BDS-like normalized amplitude that manifestly exhibits a subset of the Steinmann relations. In particular, defining

$${}^X\mathcal{E}_8 \equiv \frac{\mathcal{A}_8^{\text{MHV}}}{{}^X\mathcal{A}_8^{\text{BDS-like}}} = \exp \left[ R_8 - \frac{\Gamma_{\text{cusp}}}{4} {}^X Y_8 \right] \quad (3.4)$$

for any label  $X$ , we expect that  ${}^3\mathcal{E}_8$  should satisfy Steinmann relations between all partially overlapping pairs of three-particle invariants, while  ${}^4\mathcal{E}_8$  should satisfy Steinmann relations between all partially overlapping pairs of four-particle invariants. That is,  ${}^3\mathcal{E}_8$  is expected to satisfy the relations

$$\text{Disc}_{s_{i+1,i+2,i+3}} [\text{Disc}_{s_{i,i+1,i+2}} ({}^3\mathcal{E}_8)] = 0, \quad (3.5)$$

$$\text{Disc}_{s_{i+2,i+3,i+4}} [\text{Disc}_{s_{i,i+1,i+2}} ({}^3\mathcal{E}_8)] = 0, \quad (3.6)$$

for all  $i$ , while  ${}^4\mathcal{E}_8$  is expected to satisfy

$$\text{Disc}_{s_{i+1,i+2,i+3,i+4}} [\text{Disc}_{s_{i,i+1,i+2,i+3}} ({}^4\mathcal{E}_8)] = 0, \quad (3.7)$$

$$\text{Disc}_{s_{i+2,i+3,i+4,i+5}} [\text{Disc}_{s_{i,i+1,i+2,i+3}} ({}^4\mathcal{E}_8)] = 0, \quad (3.8)$$

$$\text{Disc}_{s_{i+3,i+4,i+5,i+6}} [\text{Disc}_{s_{i,i+1,i+2,i+3}} ({}^4\mathcal{E}_8)] = 0. \quad (3.9)$$

However, conditions (3.5) through (3.9) don't exhaust the set of Steinmann relations obeyed by generic eight-particle amplitudes—there are also Steinmann relations between partially overlapping three- and four-particle invariants. If we want to make these additional relations manifest, we can use the fact that it proves possible to define a BDS-like ansatz that depends on all but one of the four-particle invariants (and on no three-particle invariants). That is, we decompose the BDS ansatz as

$$\mathcal{A}_8^{\text{BDS}} = {}^{3,j}\mathcal{A}_8^{\text{BDS-like}}(\{s_{i,i+1}\}, \{s_{i,i+1,i+2,i+3} \neq s_{j,j+1,j+2,j+3}\}) \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} {}^{3,j}Y_8(\{u_i\}) \right], \quad (3.10)$$

and in so doing define a BDS-like normalized amplitude that satisfies

$$\text{Disc}_{s_{j+2,j+3,j+4}} [\text{Disc}_{s_{j,j+1,j+2,j+3}} ({}^{3,j}\mathcal{E}_8)] = 0, \quad (3.11)$$

$$\text{Disc}_{s_{j+3,j+4,j+5}} [\text{Disc}_{s_{j,j+1,j+2,j+3}} ({}^{3,j}\mathcal{E}_8)] = 0, \quad (3.12)$$

$$\text{Disc}_{s_{j-1,j,j+1}} [\text{Disc}_{s_{j,j+1,j+2,j+3}} ({}^{3,j}\mathcal{E}_8)] = 0, \quad (3.13)$$

$$\text{Disc}_{s_{j-2,j-1,j}} [\text{Disc}_{s_{j,j+1,j+2,j+3}} ({}^{3,j}\mathcal{E}_8)] = 0, \quad (3.14)$$

$$\text{Disc}_{s_{i+1,i+2,i+3}} [\text{Disc}_{s_{i,i+1,i+2}} ({}^{3,j}\mathcal{E}_8)] = 0, \quad (3.15)$$

$$\text{Disc}_{s_{i+2,i+3,i+4}} [\text{Disc}_{s_{i,i+1,i+2}} ({}^{3,j}\mathcal{E}_8)] = 0, \quad (3.16)$$

where the relations (3.15) and (3.16) are the Steinmann relations satisfied by  ${}^3\mathcal{E}_8$ . Clearly it is not possible to find a solution to eq. (3.10) that is Bose-symmetric, but we can require that

it is invariant under the dihedral flip that maps  $s_{i\dots l} \rightarrow s_{9-i\dots 9-l}$ . There is again a family of solutions that satisfy these constraints.

Momentum conservation implies that  ${}^{3,j+4}\mathcal{A}_8^{\text{BDS-like}} = {}^{3,j}\mathcal{A}_8^{\text{BDS-like}}$ , and thus every eight-point Steinmann relation is made manifest in at least one of the amplitudes

$$\{{}^4\mathcal{E}_8, {}^{3,1}\mathcal{E}_8, {}^{3,2}\mathcal{E}_8, {}^{3,3}\mathcal{E}_8, {}^{3,4}\mathcal{E}_8\}.$$

However, in practice it may prove convenient to retain  ${}^3\mathcal{E}_8$  in one's toolbox, since it can be chosen to respect Bose symmetry. It may additionally prove possible to chose these amplitudes to satisfy other desirable properties, since  ${}^3Y_8$ ,  ${}^4Y_8$ , and  ${}^{3,j}Y_8$  are not uniquely picked out by eqns. (3.2), (3.3), and (3.10). In fact there is a 10-dimensional (3-dimensional) space of (Bose-symmetric) solutions for  ${}^3Y_8$ , a 36-dimensional (5-dimensional) space of (Bose-symmetric) solutions for  ${}^4Y_8$ , and a 5-dimensional (3-dimensional) space of (dihedral flip-invariant) solutions for  ${}^{3,j}Y_8$ .

To take full advantage of the Steinmann relations, it is convenient to work in terms of symbol letters that isolate different Mandelstam invariants. There are twelve independent dual conformally invariant cross ratios that can appear in these symbols

$$u_1 = \frac{s_{12}s_{4567}}{s_{123}s_{812}}, \quad \text{and cyclic (8-orbit)} \quad (3.17)$$

$$u_9 = \frac{s_{123}s_{567}}{s_{1234}s_{4567}}, \quad \text{and cyclic (4-orbit)}. \quad (3.18)$$

It is not possible to isolate all three- and four-particle Mandelstam invariants simultaneously into twelve different symbol letters. (More than twelve symbol letters will appear in these amplitudes, but we here restrict our attention to the twelve that will appear in the first entry.) However, different choices of letters can be made such that either all the four-particle invariants, or all the three-particle invariants, are isolated.

One choice that isolates the four-particle invariants is

$${}^4d_1 = u_2 u_6 = \frac{s_{23} s_{67} (s_{1234})^2}{s_{123} s_{234} s_{567} s_{678}}, \quad \text{and cyclic (4-orbit)} \quad (3.19)$$

$${}^4d_5 = u_2/u_6 = \frac{s_{23} s_{567} s_{678}}{s_{67} s_{123} s_{234}}, \quad \text{and cyclic (4-orbit)} \quad (3.20)$$

$${}^4d_9 = u_1 u_2 u_5 u_6 u_9^2 = \frac{s_{12} s_{23} s_{56} s_{67}}{s_{234} s_{456} s_{678} s_{812}}, \quad \text{and cyclic (4-orbit)}. \quad (3.21)$$

In this alphabet  ${}^4d_1, {}^4d_2, {}^4d_3$ , and  ${}^4d_4$  each contain a different four-particle Mandelstam invariant, while the other letters only involve two- and three-particle invariants. The extended Steinmann relations then tell us that  ${}^4d_1, {}^4d_2, {}^4d_3$ , and  ${}^4d_4$  can never appear next to each other in the symbol of  ${}^4A_8^{\text{BDS-like}}$  (but each can still appear next to themselves).

Similarly, we can isolate the three-particle invariants by choosing

$${}^3d_1 = \frac{u_1 u_2 u_4 u_7}{u_3 u_5 u_6 u_8 u_9^2} = \frac{s_{12} s_{23} s_{45} s_{78} (s_{1234})^2 (s_{4567})^2}{s_{34} s_{56} s_{67} s_{81} (s_{123})^2}, \quad \text{and cyclic (8-orbit)} \quad (3.22)$$

$${}^3d_9^4 = u_1 u_5 u_9 u_{12} = \frac{s_{12} s_{56}}{s_{1234} s_{3456}}, \quad \text{and cyclic (4-orbit)}, \quad (3.23)$$

in which case  ${}^3d_1$  through  ${}^3d_8$  each contain a different three-particle Mandelstam invariant, as well as four-particle Mandelstams that they don't partially overlap with. The remaining four letters only contain two- and four-particle invariants. In these letters, conditions (3.15) and (3.16) tell us that  ${}^3d_7$ ,  ${}^3d_8$ ,  ${}^3d_2$ , and  ${}^3d_3$  can never appear next to  ${}^3d_1$  in the symbols of  ${}^3\mathcal{E}_8$  or  ${}^{3,j}\mathcal{E}_8$  (plus the cyclic images of this statement). Moreover, conditions (3.11) through (3.14) give us the additional restrictions that none of  ${}^3d_1$ ,  ${}^3d_5$ ,  ${}^3d_9$  and  ${}^3d_{10}$  can ever appear next to  ${}^3d_3$ ,  ${}^3d_4$ ,  ${}^3d_7$ , or  ${}^3d_8$  in the symbol of  ${}^{3,1}\mathcal{E}_8$  (analogous relations hold for the other  ${}^{3,j}\mathcal{E}_8$ ). These are the restrictions given by the Steinmann relations involving  $s_{1234}$  and one of  $s_{781}$ ,  $s_{812}$ ,  $s_{345}$ , or  $s_{456}$ . The other Steinmann relations between three- and four-particle invariants will not be respected by  ${}^{3,1}\mathcal{E}_8$ , since  ${}^{3,j}\mathcal{A}_8^{\text{BDS-like}}$  depends on  $s_{2345}$ ,  $s_{3456}$ , and  $s_{4567}$ .

## References

- [1] J. Golden and M. Spradlin, “An analytic result for the two-loop seven-point MHV amplitude in  $\mathcal{N} = 4$  SYM,” *JHEP*, vol. 1408, p. 154, 2014.
- [2] G. Yang, “Scattering amplitudes at strong coupling for 4K gluons,” *JHEP*, vol. 12, p. 082, 2010.