In this note we catalog the subalgebra structure for the finite cluster algebras $\subseteq E_6$. These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

 $\underline{A_2}$ clusters: 5 a-coordinates: 5 x-coordinates: 10

 A_3 clusters: 14 a-coordinates: 9 x-coordinates: 30

 A_4 clusters: 42 a-coordinates: 14 x-coordinates: 70

TypeSub-polytopesDistinct Subalgebras A_2 2821 $A_1 \times A_1$ 2828 A_3 77 $A_2 \times A_1$ 77 $A_1 \times A_1 \times A_1$ 00

 D_4 clusters: 50 a-coordinates: 16 x-coordinates: 104

Type Sub-polytopes Distinct Subalgebras \overline{A}_2 36 36 $A_1 \times A_1$ 30 18 A_3 12 12 $A_2 \times A_1$ 0 0 $A_1 \times A_1 \times A_1$ 4

 A_5 clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

 $\underline{D_5}$ clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

 $\underline{E_6}$ clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

Symbol Spaces on Cluster Algebras

Weight Two

Typo	Integrable	Cluster-Adjacent	Automorphic			
туре	integrable		$\sigma^+ \tau^+$	$\sigma^+\tau^-$	$\sigma^- \tau^+$	$\sigma^-\tau^-$
A_2	19	14	2	0	0	0
A_3	55	40	6	2	1	5
A_4	125	90	9	3	0	0
D_4	163	109				
A_5	245	175	1	0	0	0
D_5	381	241				
E_6	-	573	-	-	-	-

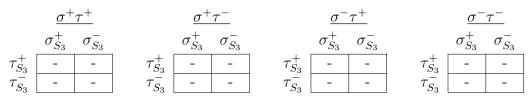
Automorphic D_4 symbols

Automorphic D_5 symbols

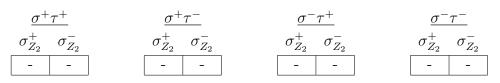
Weight Four

Type Integrable		Cluster-Adjacent	Automorphic			
Type Integrable	$\sigma^+ \tau^+$		$\sigma^+\tau^-$	$\sigma^- \tau^+$	$\sigma^-\tau^-$	
A_2	211 (6)	81 (6)	11 (0)	6 (2)	0	0
A_3	1351 (36)	432 (36)	49 (2)	29 (4)	24 (2)	42 (4)
A_4	-	1652 (126)	131 (6)	105 (12)	0	0
D_4	-	-				
A_5	-	-	-	-	-	-
D_5	-	-				
E_6	-	-	-	-	-	-

Automorphic D_4 symbols



Automorphic D_5 symbols



How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank n-1 in a finite type cluster algebra of type X of rank n.

Let N be the number of cluster a-coordinates in the cluster algebra of type X.

Let n be the rank of cluster algebra of type X.

Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y.

The number of cluster algebras of type Y is then NZ/n.

As an example, let's count the number of A_4 's in A_5 :

N = # of cluster a-coordinates in $A_5 = 20$

 $n = \text{rank of } A_5 = 5$

Z=# of ways to remove a node from A_5 and get an $A_4=2$

 $20 \times 2/5 = 8 \Rightarrow$ there are 8 A_4 's in A_5 .