In this note we catalog the subalgebra structure for the finite cluster algebras $\subseteq E_6$.

These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

A_2

clusters: 5 a-coordinates: 5 x-coordinates: 10

A_3

clusters: 14 a-coordinates: 9 x-coordinates: 30

Type	Sub-polytopes	Distinct Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

A_4

clusters: 42 a-coordinates: 14 x-coordinates: 70

Type	Sub-polytopes	Distinct Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

D_4

clusters: 50 a-coordinates: 16 x-coordinates: 104

Type	Sub-polytopes	Distinct Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

A_5 clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

D_5 clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{E_6}$ clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

Symbol Spaces on Cluster Algebras

Weight Two

Type	Integrable	Cluster-Adjacent	Automorphic $\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	19	14	2	0	0	0
A_3	55	40	6	2	1	5
A_4	125	90	9	3	0	0
D_4	163	109				
A_5	245	175	1	0	0	0
D_5	381	241				
E_6	-	573	-	-	-	-

Automorphic D_4 symbols

$\sigma^+\tau^+$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	7	0
$\tau^-_{S_3}$	0	0

$\sigma^+\tau^-$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	2	0
$\tau^-_{S_3}$	1	0

$\sigma^-\tau^+$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	5	0
$\tau^-_{S_3}$	2	0

$\sigma^-\tau^-$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	4	0
$\tau^-_{S_3}$	0	0

Automorphic D_5 symbols

$\sigma^+\tau^+$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
22	0

$\sigma^+\tau^-$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
11	0

$\sigma^-\tau^+$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
0	4

$\sigma^-\tau^-$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
0	12

Weight Four

Type	Integrable	Cluster-Adjacent	Automorphic			
			$\sigma^+\tau^+$	$\sigma^+\tau^-$	$\sigma^-\tau^+$	$\sigma^-\tau^-$
A_2	211	81	11	6	0	0
A_3	1351	432	49	29	24	42
A_4	-	1652	131	105	0	0
D_4	-	-				
A_5	-	-	-	-	-	-
D_5	-	-				
E_6	-	-	-	-	-	-

Automorphic D_4 symbols

$\underline{\sigma^+\tau^+}$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

$\underline{\sigma^+\tau^-}$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

$\underline{\sigma^-\tau^+}$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

$\underline{\sigma^-\tau^-}$

	$\sigma^+_{S_3}$	$\sigma^-_{S_3}$
$\tau^+_{S_3}$	-	-
$\tau^-_{S_3}$	-	-

Automorphic D_5 symbols

$\underline{\sigma^+\tau^+}$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

$\underline{\sigma^+\tau^-}$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

$\underline{\sigma^-\tau^+}$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

$\underline{\sigma^-\tau^-}$

$\sigma^+_{Z_2}$	$\sigma^-_{Z_2}$
-	-

How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank $n-1$ in a finite type cluster algebra of type X of rank n.

Let N be the number of cluster a-coordinates in the cluster algebra of type X.
Let n be the rank of cluster algebra of type X.
Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y.

The number of cluster algebras of type Y is then NZ/n.

As an example, let's count the number of A_4's in A_5:

$N = \#$ of cluster a-coordinates in $A_5 = 20$
$n =$ rank of $A_5 = 5$
$Z = \#$ of ways to remove a node from A_5 and get an $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$ there are 8 A_4's in A_5.