Letters

The letters appearing at two-loop MHV are generically of the form

$$\langle i\, j\, k\, l\rangle, \qquad \langle i\, j\, (klm)\cap(nop)\rangle, \qquad \langle i\,(jk)\,(lm)\,(no)\rangle, \tag{1}$$

where

$$\begin{aligned}\langle i(jk)(lm)(no)\rangle &\equiv \langle ijlm\rangle\langle ikno\rangle - \langle ijno\rangle\langle iklm\rangle,\\ \langle ij\bar{k}\cap\bar{l}\rangle &\equiv \langle i\bar{k}\rangle\langle j\bar{l}\rangle - \langle j\bar{k}\rangle\langle i\bar{l}\rangle.\end{aligned} \tag{2}$$

Furthermore only a subset of these letters actually appear, they are

$$\begin{aligned}\langle i\ i{+}1\ jk\rangle, &\quad \langle i(i{-}1\ i{+}1)(j\ j{+}1)(k\ k{+}1)\rangle,\\ \langle i\ i{+}1\ \bar{j}\cap\bar{k}\rangle, &\quad \langle i(i{-}2\ i{-}1)(i{+}1\ i{+}2)(j\ j{+}1)\rangle.\end{aligned} \tag{3}$$

These letters have the following action under parity:

$$\begin{aligned}&\langle i\ i{+}1\ jk\rangle \quad\to\quad \langle i^+\rangle\langle i\ i{+}1\ \bar{j}\cap\bar{k}\rangle\\ &\langle i\ i{+}1\ \bar{j}\cap\bar{k}\rangle \ \to\ \langle j^-\rangle\langle j^+\rangle\langle k^-\rangle\langle k^+\rangle\langle i^+\rangle\langle i\ i{+}1\ jk\rangle\\ &\langle i(i{-}1\ i{+}1)(j\ j{+}1)(k\ k{+}1)\rangle \to\\ &\qquad\qquad \langle i^+\rangle\langle i^-\rangle\langle j^+\rangle\langle k^+\rangle\langle i(i{-}1\ i{+}1)(j\ j{+}1)(k\ k{+}1)\rangle\\ &\langle i(i{-}2\ i{-}1)(i{+}1\ i{+}2)(j\ j{+}1)\rangle \to\\ &\qquad\qquad \langle i{-}1^-\rangle\langle i{-}1^+\rangle\langle i{+}1^-\rangle\langle i{+}1^+\rangle\langle j^+\rangle\langle j\ j{+}1\ i{-}1\ i{+}1\rangle\end{aligned} \tag{4}$$

where we have used the notational shorthand

$$\begin{aligned}\langle i^\pm\rangle &= \pm\langle i-1\ i\ i+1\ i\pm 2\rangle\\ \bar{j} &= (j-1\, j\, j+1).\end{aligned} \tag{5}$$

Ratios

Note: basically all conjectures in this section are based on explicit calculation through $n = 9$.

Definition 1. *A **good cross-ratio** r as a DCI product of integer powers of letters in (3) such that $-1-r$ is also a product of integer powers of letters.*

From now on when I refer to "cross-ratio" I am refering to this definition, and the "$-1-r$" criteria may seem a bit funny but the reason behind it is so that the following conjecture holds:

Conjecture 1. *Given a cross-ratio r, exactly one of $\{r, -1-r, -1-1/r\}$ will be positive when evaluated on any kinematic point in the positive grassmannian. This is then a cluster $\mathcal{X}$-coordinates on* $\mathrm{Gr}(4, n)$.

We can extend this slightly by defining

Definition 2. *The **family** of a cross-ratio r is the set*

$$\{r, -1-r, -1-1/r, 1/r, -1/(1+r), -1/(1+1/r)\}. \tag{6}$$

Note that each cross-ratio belongs to only one family, so the number of cross-ratios is always a multiple of 6. Then, out of this set, exactly two elements (that are multiplicative inverses of each other) will be cluster $\mathcal{X}$-coordinates on $\mathrm{Gr}(4,n)$.

When working at the level of the symbol one often desires a multiplicatively independent set of cross-ratios to express the symbol in terms of (or, in other words, a set of linearly independent ratios when taken as the arguments of log's). Therefore we care about:

Conjecture 2. *The total number of multiplicatively independent cross-ratios for n particles is $\frac{3}{2}n(n-5)^2$.*

Of course we also care about the number of *algebraically* independent cross-ratios, which is understood to be $3n-15$ (this is just the number of unfrozen seeds in the cluster for $\mathrm{Gr}(4,n)$). But for now we will focus on multiplicative independence, which we catalog here:

$n=$	Total # of letters	Total # of ratios	Mult. basis
6	15	90	9
7	49	2310	42
8	116	9528	108
9	225	23436	216

The $\{v, z\}$ basis

The first-entry condition states that the first entry of the symbol must be drawn from the set of cross-ratios given by

$$u_{ij} = \frac{\langle i\, i{+}1\, j{+}1\, j{+}2\rangle\langle i{+}1\, i{+}2\, j\, j{+}1\rangle}{\langle i\, i{+}1\, j\, j{+}1\rangle\langle i{+}1\, i{+}2\, j{+}1\, j{+}2\rangle}. \tag{7}$$

(Note that the u_{ij} are not technically good cross-ratios per our definition, instead $-u_{ij}$ are "correct", but let's ignore that for now!). Interestingly, none of the u_{ij} are cluster $\mathcal{X}$-coordinates. Instead we consider the closely related quantities

$$v_{ijk} = \frac{1}{\prod_{a=j}^{k-1} u_{ia}} - 1 = -\frac{\langle i{+}1 (i\, i{+}2)(j\, j{+}1)(k\, k{+}1)\rangle}{\langle i\, i{+}1\, k\, k{+}1\rangle\langle i{+}1\, i{+}2\, j\, j{+}1\rangle}. \tag{8}$$

v_{ijk} is a $\mathcal{X}$-coordinates as long as $i < j < k \pmod n$. There are $\frac{1}{2}n(n-5)^2$ of these at each n. We can phrase the familiar first-entry condition in terms of these unfamiliar variables by saying that only the quantities $1 + v_{ijk}$ are allowed in the first entry of the symbol of any function with physical branch cuts.

The last-entry condition states that the last entry of the symbol of any MHV amplitude must, as a consequence of extended supersymmetry, be drawn from the set of Plucker coordinates of the form $\langle \bar{i}\, j\rangle \equiv \langle i{-}1\, i\, i{+}1\, j\rangle$. We therefore might like to include ratios built purely out of these objects in our ansatz, such as

$$-\frac{\langle i\, \bar{j}\rangle\langle i{+}1\, \bar{k}\rangle}{\langle i\, \bar{k}\rangle\langle i{+}1\, \bar{j}\rangle}, \qquad -\frac{\langle \bar{i}\, j\rangle\langle \overline{i{+}1}\, k\rangle}{\langle \bar{i}\, k\rangle\langle \overline{i{+}1}\, j\rangle}. \tag{9}$$

As was the case with the u_{ij} of the first-entry condition, none of these are $\mathcal{X}$-coordinates. Instead we consider the cross-ratios

$$z^+_{ijk} = \frac{\langle i\, i{+}1\, \bar{j} \cap \bar{k}\rangle}{\langle i\, \bar{k}\rangle\langle i{+}1\, \bar{j}\rangle}, \qquad z^-_{ijk} = \frac{\langle i\, i{+}1\, j\, k\rangle\langle \bar{i}\, i{+}2\rangle}{\langle \bar{i}\, k\rangle\langle \overline{i{+}1}\, j\rangle}. \tag{10}$$

The $z^\pm_{ijk}$ are all cluster $\mathcal{X}$-coordinates for $\mathrm{Gr}(4, n)$ as long as $i < j < k \pmod n$, and as suggested by the notation, $z^\pm_{ijk}$ are parity conjugates of each other. There are $n(n-5)^2$ such variables for each n. These are connected to the final-entry condition via

$$-1 - z^+_{ijk} = \frac{\langle i\, \bar{j}\rangle\langle i{+}1\, \bar{k}\rangle}{\langle i\, \bar{k}\rangle\langle i{+}1\, \bar{j}\rangle}, \qquad -1 - z^-_{ijk} = \frac{\langle \bar{i}\, j\rangle\langle \overline{i{+}1}\, k\rangle}{\langle \bar{i}\, k\rangle\langle \overline{i{+}1}\, j\rangle}.$$

It is useful to define certain boundary cases of the above cross-ratios with overlapping indices:

$$v_{ij} = v_{i\, j\, j+1}, \qquad z_{ij} = z^-_{i\, j\, j+1}, \tag{11}$$

where parity takes $z_{ij} \to z_{ji}$. Similar to what was done in the previous paragraph, we may express the familiar last-entry condition in terms of these unfamiliar variables by saying that only the quantities $1 + z^\pm_{ijk}$ are allowed in the final entry of the symbol of any MHV amplitude.

Note that the total number of $v-$ and $z-$type variables at each n is $\frac{3}{2}n(n-5)^2$ – precisely the same as the (conjectured) dimension of the space of multiplicatively independent cross-ratios. This leads us to conjecture that

Conjecture 3. *The set of $\{v, z\}$ ratios forms a multiplicately independent basis that spans the space of all cross-ratios for any* ***odd*** *n.*

For even n the story is a bit more complicated, as it turns out that the $\{v, z\}$-basis is not multiplicatively independent for even n:

$n =$	$\{v, z\}$-basis size	# of mult. relations
6	9	2
7	42	0
8	108	9
9	216	0
10	375	4
11	594	0
12	882	15

This raises the question of what ratios to actually express symbols and integrated amplitudes in terms of for even n. And here we arrive at a point I had not previously appreciated: the published form of $R_6^{(2)}$ only involves $\mathrm{Li}_k(-x)$ where $x \in \{v, z\}$ for $n = 6$, however the *symbol* of $R_6^{(2)}$ cannot be written in terms of the same x's (for example, $1 - z_{ij}$ cannot be written in terms of a product of v's and z's, so there would need to be some magic to happen at the level of the full symbol for the $\{v, z\}$-basis to be sufficient). Perhaps the symbol for the Steinmann-adjusted $R_6^{(2)}$ is expressible in only these $\mathcal{X}$-coordinates? I'm sure some expert in 6-particle kinematics will illuminate this trivial point for me...

A similar issue arises at $n = 8$, so it will be helpful to understand the story at $n = 6$ before settling on a choice of ratios that we want the final answer to be in terms of for $n = 8$.