

In this note we describe the space of weight-4 polylogarithm functions which depend on  $\mathcal{X}$ -coordinates of the  $A_2$  cluster algebra. Taking our seed cluster as  $x_1 \rightarrow x_2$ , we define our  $\mathcal{X}$ -coordinates:

$$\mathcal{X}_1 = \frac{1}{x_1}, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1 + x_2), \quad \mathcal{X}_4 = \frac{1 + x_1 + x_1x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1 + x_1}{x_1x_2}. \quad (1)$$

These satisfy  $1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}$ , and  $\{1/\mathcal{X}_i, \mathcal{X}_{i+1}\}$  form the 5 clusters. Of course you can also work in the  $\text{Gr}(2, 5)$  language, where we have the  $\mathcal{A}$ -coordinates  $\langle ij \rangle$  and the  $\mathcal{X}$ -coordinates

$$\mathcal{X}_1 = \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle}, \quad \mathcal{X}_2 = \frac{\langle 13 \rangle \langle 45 \rangle}{\langle 15 \rangle \langle 34 \rangle}, \quad \mathcal{X}_3 = \frac{\langle 12 \rangle \langle 35 \rangle}{\langle 15 \rangle \langle 23 \rangle}, \quad \mathcal{X}_4 = \frac{\langle 25 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 45 \rangle}, \quad \mathcal{X}_5 = \frac{\langle 15 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 45 \rangle}. \quad (2)$$

This makes it easy to define cluster  $\mathcal{A}$ -adjacency: the allowed pairs are  $\{\langle 13 \rangle, \langle 14 \rangle\}$ ,  $\{\langle 14 \rangle, \langle 24 \rangle\}$ ,  $\{\langle 24 \rangle, \langle 25 \rangle\}$ ,  $\{\langle 25 \rangle, \langle 35 \rangle\}$ ,  $\{\langle 35 \rangle, \langle 13 \rangle\}$  (along with all of the frozen nodes,  $\langle i \ i + 1 \rangle$ ).

## Integrable Symbol

Imposing integrability on a generic symbol of weight 4 with arguments drawn from (1) leaves us with 211 free parameters.

## $\mathcal{A}$ -adjacency vs. $\mathcal{X}$ -adjacency

The two criteria we consider are:

- cluster  $\mathcal{A}$ -adjacency (only  $\{\langle i \ i + 2 \rangle, \langle i \ i + 3 \rangle\}$ +frozen can appear),
- cluster  $\mathcal{X}$ -adjacency (only  $\{\mathcal{X}_i, \mathcal{X}_{i+1}\}$  can appear).

For weight-4 integrable symbols on  $A_2$ , these criteria turn out to be equivalent (this is not surprising since there are an equal number of  $\mathcal{A}$ - and  $\mathcal{X}$ -coordinates). In both cases, 130 parameters are fit by cluster adjacency, leaving us with 81.

## Imposing cluster-y coproduct

Now that we have a symbol which is cluster adjacent, let us also impose that the coproduct be cluster-y as well. Based on experience we know that imposing full cluster adjacency at the level of the coproduct is not possible for  $A_2$  (we have to wait for  $A_3$  for that property), but we can at least impose that the  $B_2 \wedge B_2$  and  $B_3 \otimes \mathbb{C}^*$  take only  $\{\mathcal{X}\}_k$  as arguments.

Interestingly, cluster adjacency at symbol level already imposes a cluster-y coproduct at the level of  $B_2 \wedge B_2$ . Imposing this for  $B_3 \otimes \mathbb{C}^*$  fixes 15 parameters (I think this is killing off terms like  $\text{Li}_4(-1 - \mathcal{X})$  and  $\text{Li}_4(-1 - 1/\mathcal{X})$ ), leaving us with 66.

## Imposing symmetries

We now impose that the function respect the automorphisms of  $A_2$ .  $A_2$  has a cyclic symmetry,  $\sigma : \mathcal{X}_i \rightarrow \mathcal{X}_{i+1}$ , and a flip symmetry  $\tau : \mathcal{X}_i \rightarrow \mathcal{X}_{6-i}$ . For each symmetry, we consider the case where  $f_{A_2}$  is either invariant or “covariant” – i.e.  $\sigma(f_{A_2}) = f_{A_2}$  or  $\sigma(f_{A_2}) = -f_{A_2}$ . We will now tabulate how many free parameters remain for each of the 4 possible sign choices:

$$f_{A_2}: \begin{array}{cccc} \sigma^+ \tau^+ & \sigma^+ \tau^- & \sigma^- \tau^+ & \sigma^- \tau^- \\ \hline 9 & 5 & 0 & 0 \end{array}$$

We’ll refer to these functions by their behavior under  $\sigma$  and  $\tau$ :  $f_{A_2}^{++}$  and  $f_{A_2}^{+-}$ . Note that only  $f_{A_2}^{+-}$  has non-zero  $B_2 \wedge B_2$  – this is what we have traditionally called “the  $A_2$  function” (at coproduct level).

## Poisson Structure and Mutation Interpretation

It would be nice to have an understanding of the Poisson structure in adjacent symbol entries for these functions – in particular, can we fit the remaining parameters in order to bring out a nice Poisson structure? A dream scenario would be to have an interpretation of these symbol terms as a mutation sequence.

Unfortunately at this point I don’t seen any obvious interpretation, but this is worth returning to!

## Summary

We have determined two weight-4 functions of interest related to the  $A_2$  cluster algebra:  $f_{A_2}^{++}$  and  $f_{A_2}^{+-}$ . They have 9,5 free parameters (respectively), and  $f_{A_2}^{+-}$  has non-zero  $B_2 \wedge B_2$ . The next step is to evaluate these functions across  $A_3$ , impose the  $++$  automorphism sign choice, and see how close we get to  $\mathcal{E}_6^{(2)}$ .