In this note we explore different attempts at expressing the $B_2 \wedge B_2$ component of $R_8^{(2)}$ with cluster polylogarithms.

First let us define "good" $\mathcal{X}$-coordinates: an $\mathcal{X}$-coordinate R is good if R and $1+R$ can be expressed as ratios of $\mathcal{A}$-coordinates of the form

$$\begin{aligned} \langle i\; i{+}1\; jk\rangle, &\quad \langle i(i{-}1\; i{+}1)(j\; j{+}1)(k\; k{+}1)\rangle, \\ \langle i\; i{+}1\; \bar{j} \cap \bar{k}\rangle, &\quad \langle i(i{-}2\; i{-}1)(i{+}1\; i{+}2)(j\; j{+}1)\rangle. \end{aligned} \tag{1}$$

An algebra is good if it contains only good $\mathcal{X}$-coordinates.

Fitting with A_3's

There are 1600 good A_3's in $\mathrm{Gr}(4,8)$, with only 513 actual degrees of freedom in $B_2 \wedge B_2$. An ansatz of these functions can be fit to $R_8^{(2)}$. A zero-thought choice of free parameters leaves us with an expression involving 223 A_3's.

Fitting with A_4's

There are 496 good A_4's.

Fitting with D_4's

There are 24 good D_4's.

Fitting with A_5's

There are 56 good A_5's. No choice of f_{A_5} can fit $R_8^{(2)}$.

Fitting with D_5's

There are no good D_5's. Furthermore, no D_5 contains enough good A_3's to allow us to tune the free parameters in f_{D_5} in order to have only good A_3's appear in f_{D_5}. In other words, any representation of $R_8^{(2)}$ in terms of f_{D_5} will necessarily involve "bad" ratios inside the f_{D_5}'s which must cancel in the full sum. This is upsetting!

Fitting with E_6's

There are no good E_6's. There are 216 E_6's which have seed clusters involving only good $\mathcal{X}$-coordinates. None of these 216 E_6's have sufficient $\{v,z\}$-squares to constitute a completely good $\sim f_{E_6} \sim R_7^{(2)}$. The most $\{v,z\}$-squares inside one of these E_6's is 33, and there are 64 E_6's which have at least 30 (out of 42) good $\{v,z\}$ squares.