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Cluster Subalgebra-Constructibility I: Novel Decompositions of the Seven-Particle Remainder Function

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Abstract: The seven-particle remainder function in planar maximally supersymmetric Yang-Mills theory can be thought of as . We systematically investigate the ways in which the 'nonclassical' part of decomposed into the subalgebras of the Grassmannian Gr(4,7). We find it is possible to decompose

Contents

1 Introduction

A growing body of evidence suggests that cluster algebras (or their generalizations) have a central role to play in our understanding of multi-loop scattering amplitudes in the planar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The most striking indication to this effect comes from the branch cut structure of maximally-helicity-violating (MHV) amplitudes, which have now been computed in this theory to high loop order in six- and seven-particle kinematics [], and at two loops for any number of particles n []. Namely, all branch cuts in these amplitudes end at the vanishing loci of cluster coordinates on the Grassmannian Gr(4,n) [], and—even more strikingly—their iterated discontinuities vanish unless sequentially taken around the vanishing loci of coordinates that appear together in a cluster []. All known NMHV amplitudes in this theory also share these properties [], as do certain classes of Feynman integrals [12–15], some of which have been computed to all loop orders [16]. While these amplitudes and integrals only constitute the simplest this theory has to offer, it is surprising that cluster algebras combinatorially realize much of their analytic structure, encoding their locality in a non-obvious way.

The fact that cluster algebras appear in these amplitudes at loop level is not totally surprising, since the plabic graphs that describe the integrands of this theory to all orders are themselves dual to cluster algebras [17]. In particular, the zero-loci of cluster coordinates on Gr(4,n) coincide with the boundaries of the positive Grassmannian, which represent the locations of physical singularities. Despite this, it's far from obvious that the location of all physical singularities will be picked out by cluster coordinates in this way—and indeed, certain Feynman integrals contributing to eight- and higher-particle amplitudes have recently been computed and are found to involve discontinuities outside the positive region that involve square roots when expressed in terms of cluster coordinates [14, 15, 18]. This obfuscates the connection between amplitudes and cluster algebras, as does the eventual appearance of functions beyond polylogarithms []. Both these complications point to the need for more general objects than cluster algebras to describe the analytic structure of amplitudes in this theory at higher loops and particle multiplicity.

There is reason to be optimistic such structure exists, due to the sheer number of ways cluster algebras appear in the simplest amplitudes in planar $\mathcal{N} = 4$ SYM theory. In particular, the symbols **[describe what the symbol is]** of all two-loop maximally-helicity-violating (MHV) amplitudes in this theory are known [19], and have been found to exhibit a great deal of cluster-algebraic structure. In addition to having symbol alphabets composed entirely of cluster coordinates in Gr(4,n) [1–4], the 'nonclassical' part of these amplitudes—namely, the part that cannot be expressed in terms of classical polylogarithms—is uniquely determined by a small set of physical and cluster-algebraic properties [4]. Moreover, a pair of functions can be associated with the cluster algebras on A_2 and A_3, in terms of which this nonclassical component of each amplitude can be geometrically decomposed into a sum over the A_2 or A_3 subalgebras of Gr(4,n) [3]. The remaining 'classical' part of these amplitudes can, moreover, always be written as products of classical polylogarithms involving only negative cluster coordinates as arguments [20].

In the present work, we extend this set of observations by describing two new functions, associated with the cluster algebras on D_5 and A_5, in terms of which the nonclassical part of the seven-point remainder function can also be decomposed. We We conjecture that these functions, like the A_2 and A_3 functions, are valid for all n—i.e. that the nonclassical part of all two-loop MHV amplitudes can be decomposed in terms of them. The importance of the nonclassical part of the amplitude (most mathematically complicated part). Note that the remainder function and BDS-like normalized amplitudes have the same nonclassical part

It is hoped that developing this deeper understanding of the cluster-algebraic structure of the simplest amplitudes in this theory will help identify the analytic structure of more complicated examples.

better understanding the nested cluster-algebraic structure of these amplitudes will eventually lead to an understanding of the properties of amplitude that extend beyond those whose analytic structure is described by cluster algebras.

In a companion paper, we apply these results beyond seven points by constructing the nonclassical part of the eight-point remainder function in terms of the A_5 function described in the present study.

The remainder of this paper is organized as follows. In section 2 we provide a self-contained introduction to cluster algebras and the principle ways in which they have appeared in $\mathcal{N} = 4$ SYM theory. This section is intended to (at least partially) fill a pedagogical gap in the physics literature, and can be skipped by those who are familiar with recent developments at the intersection of these topics. In section 3 we describe the tools relevant for working with polylogarithms, and in particular review the ways in which the coproduct and Poisson cobracket of two-loop amplitudes can be seen to exhibit curious cluster-algebraic properties. We then turn to our systematic analysis of the subalgebra-constructibility of the seven-point remainder function in section 4, describing our general approach and in particular showing why the previously-found A_2 and A_3 functions are the only interesting functions associated with cluster algebras of rank two and three. Finally, in sections 5 and 6 we construct the D_5 and A_5 functions in terms of which the nonclassical part of the remainder function can be decomposed. [**more about these sections**] We conclude with a discussion of directions for future study.

This paper includes three appendices. First, appendix A walks through the explicit construction of integrable and cluster-adjacent A_2 symbols through weight XXX, illustrating how symbols can be constructed on any cluster algebra. Appendix B tabulates the subalgebras of each type that appear in the finite cluster algebras relevant to $R_7^{(2)}$, while appendix C tabulates the number of independent nonclassical degrees of freedom in each of these finite cluster algebras.

2 A Brief Introduction to Cluster Algebras

Cluster algebras were first introduced by Fomin and Zelevinsky [21] when studying the questions of (i) when algebraic varieties come equipped with a natural notion of positivity, and (ii) what functions determine this positivity. These objects subsequently entered the physics literature in the guise of the positive Grassmannian $\mathrm{Gr}^+(k,n)$ [], i.e. the space of $k \times n$ matrices where all ordered $k \times k$ minors are positive, where it was seen to describe the integrands of to scattering amplitudes to all orders in planar $\mathcal{N}=4$ SYM theory.

What kind of questions can cluster algebras help us answer? One of the most straightforward is: how many minors do we need to specify a point in $\mathrm{Gr}^+(k,n)$? In other words, given a $k \times n$ matrix M, how many minors of M do we have to calculate to know if $M \in \mathrm{Gr}^+(k,n)$? The reason that this is an interesting question is that the minors are not all independent, they satisfy the identities known as Plücker relations:

$$\langle abI\rangle\langle cdI\rangle = \langle acI\rangle\langle bdI\rangle + \langle adI\rangle\langle bcI\rangle, \tag{2.1}$$

where the Plücker coordinates $= \langle i_1, \ldots, i_k\rangle =$ the minor of columns $i_1, \ldots, i_k$, and I is a multi-index with $k-2$ entries.

We'll now work through the example of $\mathrm{Gr}(2,5)$ in detail to try to understand how many minors one needs to check for positivity of the whole matrix. The 5 cyclically adjacent minors, $\langle 12\rangle, \langle 23\rangle, \langle 34\rangle, \langle 45\rangle, \langle 15\rangle > 0$, are all independent from each other and so must each be checked. How many of the non-adjacent minors do we have to check? It turns out that the answer is 2. For example, if we specify that $\langle 13\rangle, \langle 14\rangle > 0$ then we can use Plücker relations to show

$$\begin{aligned}
\langle 24\rangle &= (\langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 14\rangle)/\langle 13\rangle \\
\langle 25\rangle &= (\langle 12\rangle\langle 45\rangle + \langle 24\rangle\langle 15\rangle)/\langle 14\rangle \\
\langle 35\rangle &= (\langle 25\rangle\langle 34\rangle + \langle 23\rangle\langle 45\rangle)/\langle 24\rangle.
\end{aligned} \tag{2.2}$$

Here we have expressed all of the remaining minors as sums and products of the cyclically adjacent minors along with $\langle 13\rangle$ and $\langle 14\rangle$, so everything is positive.

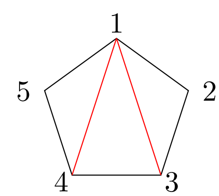
So we only need to check two – but can we check any two? Clearly we can use any of the cyclic images of $\{\langle 13\rangle, \langle 14\rangle\}$. What about $\{\langle 13\rangle, \langle 25\rangle\}$? This is a bit harder to see, but no, this pair does not work: there is no way to write down the remaining Plückers in terms of $\langle 13\rangle$ and $\langle 25\rangle$ such that everything is manifestly positive. For example, the matrix

$$\begin{pmatrix} 1 & -1 & -4 & 3 & -2 \\ 2 & 2 & -6 & 4 & -1 \end{pmatrix} \tag{2.3}$$

satisfies $\langle 12\rangle, \ldots, \langle 15\rangle, \langle 13\rangle, \langle 25\rangle > 0$ but has $\langle 14\rangle, \langle 24\rangle, \langle 35\rangle < 0$. In the end, $\{\langle 13\rangle, \langle 14\rangle\}$ and its cyclic images are the only pairs that describe a point in $\mathrm{Gr}^+(2,5)$.

This was easy enough to work out for this small case, but the problem gets much more complicated for larger matrices. However, there is a closely related, and much simpler, problem in geometry which can give us a bit more intuition: triangulating polygons.

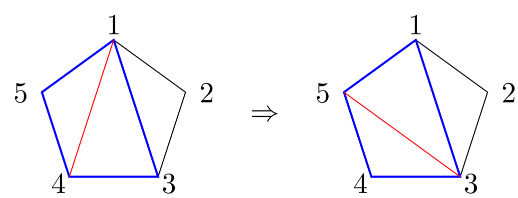
Consider the following triangulation of the pentagon:



(2.4)

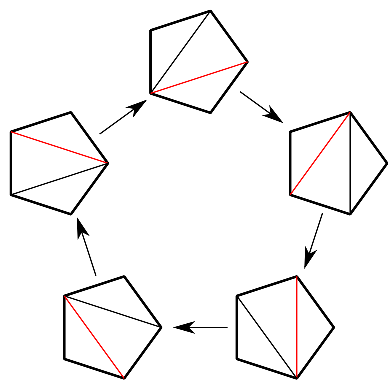
We can immediately see the parallels with our $\mathrm{Gr}(2,5)$ situation (this is an example of the more general Plücker embedding which connects $\mathrm{Gr}(k,n)$ with projective space). Here we associate lines connecting points i and j with the Plücker coordinate $\langle ij \rangle$, and we see that the triangulations of the pentagon all describe points in $\mathrm{Gr}^+(2,5)$. In fact this correspondence holds between n-gons and $\mathrm{Gr}^+(2,n)$.

A simple observation, but one at the very heart of cluster algebras, is that given some triangulation of a polygon one can create a *new* triangulation by picking a quadrilateral and flipping its diagonal. For example:



(2.5)

By repeatedly performing these flips one can generate all possible triangulations of a polygon:

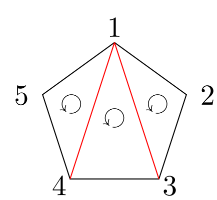


where in each case the red diagonal gets flipped.

Cluster algebras are a combinatorial tool which captures all of this structure (and much more!). The basic idea is that a cluser algebra is a collection of *clusters*, which in this case represent individual triangulations of an n-gon, and these clusters are connected via a process called *mutation*, which in this case is the flipping-the-diagonal process. [**more description of math applications of clsuter algebras? seiberg duality?**]

Basic definition

We'll begin by working through the cluster algebra for $\mathrm{Gr}(2,5)$. Each cluster is labeled by a collection of coordinates, which in this case are the edges of the pentagon along with the diagonals of the particular triangulation. These coordinates are then connected via an orientation of the pentagon and all subtriangles, for example:



(2.6)

We can redraw this diagram as

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd} \tag{2.7}$$

In this quiver diagram, we have an arrow between two Plückers $\langle ab\rangle \to \langle cd\rangle$ if the triangle orientations in eq. (2.6) have segment (ab) flowing into segment (cd). The boxes around the $\langle i i+1\rangle$ indicate that they are *frozen* – in other words, we never change the outer edges of the pentagon, only the diagonal elements. The variables living at the frozen nodes can be thought of as parameterizing the boundary of our space, and the mutable nodes represent parameterizations of the interior. And lastly it is unnecessary to draw the arrows connecting the outer edges, as that is redundant (and unchanging) information.

We have now drawn our first cluster (also sometimes called a seed). To review/introduce some terminology, the Plückers are called cluster $\mathcal{A}$-coordinates (sometimes also $\mathcal{A}$-variables), and they come in two flavors: mutable ($\langle 13\rangle$ and $\langle 14\rangle$) and frozen ($\langle i i+1\rangle$). The information of the arrows can be represented in terms of a skew-symmetric adjacency matrix

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i). \tag{2.8}$$

The process of mutation, which we described geometrically in terms of flipping the diagonal, has a simple interpretation at the level of this quiver. In particular, given a quiver such as eq. (2.7), chose a node k with associated $\mathcal{A}$-coordinate a_k to mutate on (this is equivalent to picking which diagonal to flip). Then draw a new quiver that changes a_k to a'_k defined by

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}, \tag{2.9}$$

(with the understanding that an empty product is set to one) and leaves the other cluster coordinates unchanged. The arrows connecting the nodes in this new cluster are modified from the original cluster according to

- for each path $i \to j \to k$, add an arrow $i \to j$,
- reverse all arrows on the edges incident with k,
- and remove any two-cycles that may have formed.

This creates a new adjacency matrix b'_{ij} via

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}, \\ b_{ij}, & \text{if } b_{ik}b_{kj} \leq 0, \\ b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases} \tag{2.10}$$

Mutation is an involution, so mutating on a'_k will take you back to the original cluster (as flipping the same diagonal twice will take you back to where you started).

For our purposes, *a cluster algebra is a set of quivers closed under mutation.* This means that mutating on any node of any quiver will generate a different quiver in the cluster algebra. The general procedure is to start with a quiver such as eq. (2.7), with some collection of frozen and unfrozen nodes in a connected quiver, and continue mutating on all available nodes until you either close your set or convince yourself that the cluster algebra is infinite.

We will end this brief introduction with a last piece of notation: cluster algebras are often referred to by particularly nice quiver types formed by their mutable nodes at some cluster. In the case of $\mathrm{Gr}(2,5)$, the mutable nodes of eq. (2.7) form an oriented A_2 Dynkin diagram, $\langle 13\rangle \to \langle 14\rangle$, and so we will often speak interchangeably of the cluster algebras for $\mathrm{Gr}(2,5)$ and A_2. This is a slight abuse of notation as the $\mathrm{Gr}(2,5)$ cluster algebra corresponds specifically to the cluster algebra generated by the collection of frozen and mutable nodes in eq. (2.7), whereas an A_2 cluster algebra is $a_1 \to a_2$ dressed with any (non-zero) number of frozen nodes. We will see how this language can be useful in the next section.

2.1 Cluster $\mathcal{X}$-coordinates

Another important set of information encoded in cluster algebras are called Fock-Goncharov or $\mathcal{X}$-coordinates and were introduced in [22]. As we will see in future sections, cluster $\mathcal{X}$-coordinates play a crucial role in connecting cluster algebras to polylogarithms and scattering amplitudes. While everything can be (and often is) phrased purely in terms of $\mathcal{A}$-coordinates , we believe that emphasizing $\mathcal{X}$-coordinates allows for more direct connection with the full cluster algebraic structure.

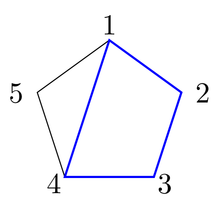
Given a quiver described by the matrix b, the $\mathcal{A}$- and $\mathcal{X}$-coordinates are related as follows:

$$x_i = \prod_j a_j^{b_{ij}}. \tag{2.11}$$

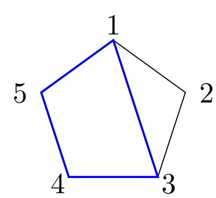
For example, the quiver in eq. (2.7) has $\mathcal{X}$-coordinates

$$\mathcal{X}_1 = \frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle}, \quad \mathcal{X}_2 = \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle}. \tag{2.12}$$

In the pentagon-triangulation picture, these $\mathcal{X}$-coordinates describe overlapping quadrilaterals:



$= \mathcal{X}_1,$



$= \mathcal{X}_2,$ (2.13)

The mutation rules for the $\mathcal{X}$-coordinates are

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i(1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases}, \tag{2.14}$$

For many applications in the rest of this work we will refer to cluster by their $\mathcal{X}$-coordinates alone, for example in this language we take the generic A_2 cluster algebra seed as

$$x_1 \to x_2. \tag{2.15}$$

This again is a slight abuse of notation, in that it can be unclear, given a quiver, if one should use the $\mathcal{A}$-coordinate mutation rules (2.9) or $\mathcal{X}$-coordinate rules (2.14). This ambiguity will be resolved in this work by that convention that if a quiver is given with no frozen nodes then it is referring to a collection of $\mathcal{X}$-coordinates .

By continuing to mutate on alternating nodes (denoted below by red) we generate the following sequence of clusters:

$$\begin{aligned}
x_1 &\rightarrow x_2 \\
x_1(1+x_2) &\leftarrow \frac{1}{x_2} \\
\frac{1}{x_1(1+x_2)} &\rightarrow \frac{x_2}{1+x_1+x_1x_2} \\
\frac{x_1x_2}{1+x_1} &\leftarrow \frac{1+x_1+x_1x_2}{x_2} \\
\frac{1+x_1}{x_1x_2} &\rightarrow \frac{1}{x_1} \\
x_2 &\leftarrow x_1 \\
&\vdots
\end{aligned} \tag{2.16}$$

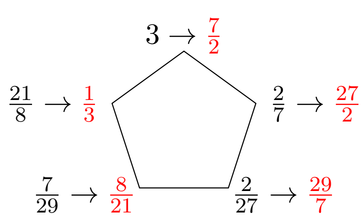
where the series then repeats. Note that by labeling the $\mathcal{X}$-coordinates as

$$\mathcal{X}_1 = 1/x_1, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1+x_2), \quad \mathcal{X}_4 = \frac{1+x_1+x_1x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1+x_1}{x_1x_2}, \tag{2.17}$$

then the general mutation rule of eq. (2.14) takes the simple form of

$$1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}. \tag{2.18}$$

Putting this all together, we will generically refer to an A_2 cluster algebra as any set of clusters $1/\mathcal{X}_{i-1} \rightarrow \mathcal{X}_i$ for $i = 1 \ldots 5$ where the $\mathcal{X}_i$ satisfy eq. (2.18). We believe it is useful at this point to emphasize that one can take as input any $\{x_1, x_2\}$ and generate an associated A_2. For example, one could start with the quiver $3 \rightarrow \frac{7}{2}$ and generate the A_2



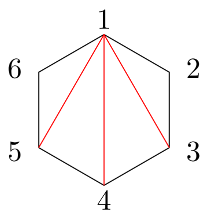
. (2.19)

(Mutating on the node in red moves you clockwise around the pentagon.) In future sections it will be necessary to consider collections of multiple A_2 algebras, in such cases we label them by only one of their clusters, e.g. $x_1 \rightarrow x_2$, with the understanding that we are referring to the A_2 which contains that cluster as an element.

[poisson bracket?]

2.2 Subalgebras

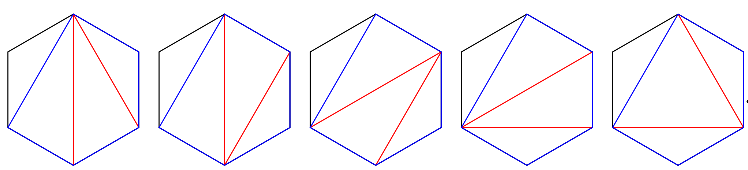
Cluster algebras contain a rich and intricate subalgebra structure which will be critical for our upcoming physics applications. We can study a simple example by looking at triangulations of a hexagon, which is equivalently described by the cluster algebra associated with $\mathrm{Gr}(2,6)$:



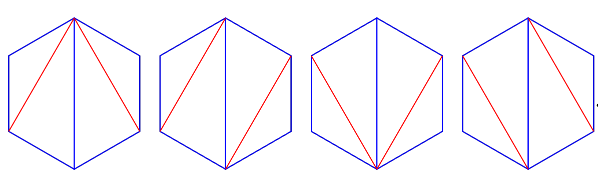
$$\Leftrightarrow \quad \begin{tikzcd} \boxed{\langle 12\rangle} \arrow[dr] & & & & \\ & \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \langle 15\rangle \arrow[r] \arrow[d] & \boxed{\langle 16\rangle} \\ & \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul] & \boxed{\langle 56\rangle} \arrow[ul] \end{tikzcd} \quad \Leftrightarrow \quad \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \to \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle} \to \frac{\langle 14\rangle\langle 56\rangle}{\langle 16\rangle\langle 45\rangle} \tag{2.20}$$

Here we have given the seed cluster for $\mathrm{Gr}(2,6)$ in the triangulation, $\mathcal{A}$-coordinate , and $\mathcal{X}$-coordinate representations, respectively. Since the mutable nodes take the form of an A_3 Dynkin diagram, we often speak of $\mathrm{Gr}(2,6)$ and A_3 interchangeably, just as we did with $\mathrm{Gr}(2,5) \simeq A_2$.

The $\mathrm{Gr}(2,6)$ cluster algebra features 14 clusters, and these clusters can be grouped together in to multiple (overlapping) sets which constitute subalgebras. A simple example is the collection of all triangulations which involve the cord $\langle 15\rangle$. It is easy to see that this set contains 5 clusters and is itself a cluster algebra generated via mutations by taking eq. (2.20), "freezing" the cord $\langle 15\rangle$, then mutating on the other two cluster coordinates (in the $\mathcal{X}$-coordinate language, this is equivalent to only mutating on the left and middle nodes of eq. (2.20)). This of course is the cluster algebra of triangulating the pentagon formed by points $1,\ldots,5$:


. (2.21)

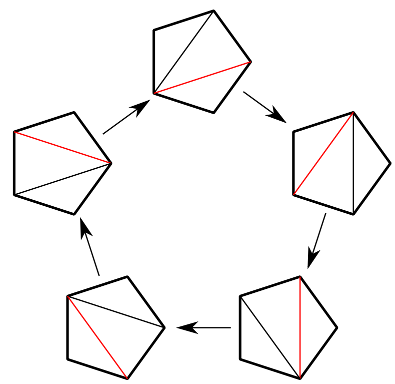
In practice, we refer to the collection of clusters that share this pentagon as an A_2 subalgebra of $\mathrm{Gr}(2,6)$. Instead of freezing the cord $\langle 15\rangle$ we could have frozen the cord $\langle 14\rangle$, in which case we generate an $A_1 \times A_1$ subalgebra, as A_1 corresponds to the triangulations of a square and the cord $\langle 14\rangle$ divides the hexagon in to two non-overlapping squares:


. (2.22)

Larger cluster algebras contain many subalgebras of different types. We have catalogued the counting of these subalgebras for many relevant cluster algebras in Appendix B.

2.3 Associahedra

The mutation paths between clusters, and which sets of clusters group together to form subalgebras, can be easily visualized through an object known as the associahedron (also called the Stasheff polytope) for a given cluster algebra. This polytope is formed by vertices representing an indivual cluster, and edges are drawn between clusters connected via mutation. So the figure



[**create new version**] is in fact the $\mathrm{Gr}(2,5)$ or A_2 associahedron, coincidentally takes the form of a pentagon.

The associahedron associated with the $\mathrm{Gr}(2,6) \leftrightarrow A_3$ cluster algebra (i.e. triangulations of a hexagon) is

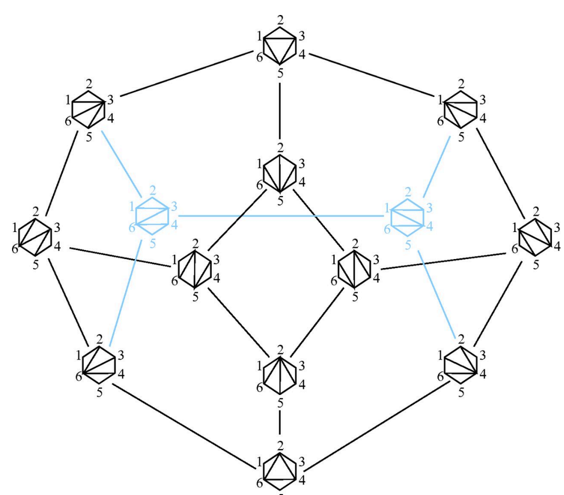


Figure 1. The associahedron for $A_3 \simeq \mathrm{Gr}(2,6)$.

[**create new version**] This associahedron has 14 vertices, corresponding to the 14 clusters, with 3 square faces and 6 pentagonal faces. The square faces represent $A_1 \times A_1$ subalgebras, and the pentagonal faces are A_2 subalgebras as discussed in the previous section. Because of the Grassmannian duality $\mathrm{Gr}(2,6) \simeq \mathrm{Gr}(4,6)$, this (remarkably simple!) cluster algebra and associahedron play an integral role in the momentum twistors for 6-particle kinematics for $\mathcal{N}=4$ SYM.

Associahedra for larger cluster algebras can become quite complicated, with intricate subpolytope/subalgebra structures. For example, the associahedron for $\mathrm{Gr}(4,7) \simeq E_6$, which will be the focus of much of the rest of this paper, is

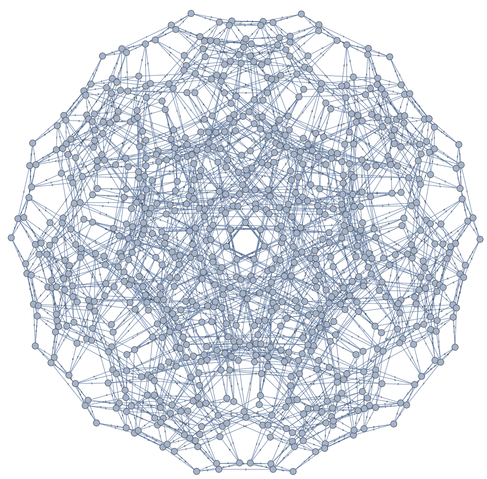
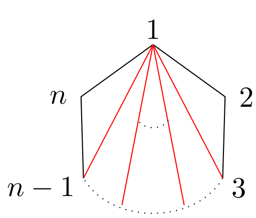


Figure 2. The associahedron for $E_6 \simeq \mathrm{Gr}(4,7)$.

It features 833 clusters/vertices of valence 6. The dimension-2 faces are 1785 squares and 1071 pentagons, and there are 49 different A-coordinates that appear.

2.4 Grassmannian cluster algebras

So far we have leaned heavily on the correspondence between triangulations of an n-gon and the cluster algebra for $\mathrm{Gr}(2,n)$. Based on the examples of $\mathrm{Gr}(2,5)$ and $\mathrm{Gr}(2,6)$, it is not hard to write down a generic seed cluster for $\mathrm{Gr}(2,n)$ based off the triangulation of all cords $\langle 13\rangle, \ldots, \langle 1 n{-}1\rangle$:



$$\Leftrightarrow \quad \begin{tikzcd} \boxed{\langle 12\rangle} \arrow[d] & & & \\ \langle 13\rangle \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \langle 1 n{-}1\rangle \arrow[r] \arrow[d] & \boxed{\langle 1n\rangle} \\ \boxed{\langle 23\rangle} & \cdots \arrow[ul] & \boxed{\langle n{-}2\ n{-}1\rangle} \arrow[ul] & \boxed{\langle n{-}1\ n\rangle} \arrow[ul] \end{tikzcd} \quad . \tag{2.23}$$

Here one sees that the mutable nodes for $\mathrm{Gr}(k,n)$ always can take the form of an A_{n-3} Dynkin diagram.

For $\mathrm{Gr}(k>2,n)$, there is no longer a simple connection with triangulations or Dynkin diagrams. However, as shown by Scott [23] there exists a generalization of eq. (2.23) valid for all k,n:

$$
\begin{tikzcd}
\boxed{f_{11}} \arrow[r] & f_{12} \arrow[r] \arrow[dl] & f_{13} \arrow[r] \arrow[dl] & \cdots \arrow[r] & f_{1l} \arrow[r] \arrow[dl] & \boxed{\langle 1,\ldots,k\rangle} \\
\boxed{f_{21}} \arrow[r] & f_{22} \arrow[r] \arrow[u] & f_{23} \arrow[r] \arrow[u] \arrow[dl] & \cdots \arrow[r] & f_{2l} \arrow[u] & \\
\vdots \arrow[ur] & \vdots \arrow[u] \arrow[dl] & \vdots \arrow[u] \arrow[dl] & \vdots & \vdots \arrow[u] & \\
\boxed{f_{k1}} & \boxed{f_{k2}} \arrow[u] & \boxed{f_{k3}} \arrow[u] & \cdots & \boxed{f_{kl}} \arrow[u] &
\end{tikzcd}
\tag{2.24}
$$

where $l=n-k$ and

$$
f_{ij}=\begin{cases}\langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle, & i\le l-j+1,\\ \langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle, & i>l-j+1\end{cases}. \tag{2.25}
$$

(Note that evaluating the above expression at $k=2$ will give a cyclically rotated version of eq. (2.23)). While the connection with triangulations is not applicable in general, the clusters generated from eq. (2.24) still give a coordinate chart for $\mathrm{Gr}^+(k,n)$ (though the coordinates won't always be simple Plückers, arbitrarily complicated polynomials of Plückers will appear as well). The cluster algebra for $\mathrm{Gr}(k,n)$ is therefore of rank $(n-k-1)(k-1)$, i.e. the number of mutable nodes in eq. (2.24).

We have emphasized Grassmannian cluster algebras so far for two reasons: 1) the correspondence with triangulations and positive matrices make them easy to gain an intuition for and 2) the cluster algebra for $\mathrm{Gr}(4,n)$ is intimately connected to n-particle kinematics in $\mathcal{N}=4$ SYM. Let us now address point 2 a bit more in detail.

Applications to momentum twistors and $\mathcal{N}$=4 SYM

An immediate connection between $\mathrm{Gr}(4,n)$ and $\mathcal{N}=4$ SYM is that the $\mathcal{X}$-coordinates are dual-conformal invariant cross-ratios and appear as arguments of relevant physical functions. We can see this for example in the two-loop, six-particle remainder function

$$
\begin{aligned}
R_6^{(2)} = \sum_{\text{cyclic}} \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)& \\
+\text{ products of } \mathrm{Li}_k(-x) \text{ functions of lower weight}& \\
\text{with the same set of arguments.}&
\end{aligned}
\tag{2.26}
$$

[more]

We will go in to much further detail on the connections between cluster algebras and $\mathcal{N}$=4 SYM in future sections. However, for now we will elaborate on the connections between cluster algebras and polylogarithms. For this we do not need to constrain ourselves to Grassmannian cluster algebras, and so for the next few sections we will work with the slightly more abstact language of quivers of generic $\mathcal{X}$-coordinates , i.e. $x_1 \to x_2$.

2.5 An overview of finite cluster algebras

One can, in principle, generate a cluster algebra by drawing any oriented quiver (even with weights), and then following the mutation rules for $\mathcal{A}$-coordinates , eq. (2.9), if there are frozen nodes, or $\mathcal{X}$-coordinates , eq. (2.14), if not. However, generic quivers will produce very complicated cluster algebras – in fact, for wide classes of seed quivers the the mutation rules generate infinite numbers of clusters, each with a distinct set of (infinitely complicated) coordinates. [1] For the rest of this paper we will be concerned with finite algebras, and will leave discussion of the infinite cases to a companion publication [].

Fortunately, Fomin and Zelevinksy classified all finite cluster algebras in [24]. In particular, they showed that a cluster algebra is of finite type iff the mutable part of its quiver at some cluster takes the form of an oriented, simply-laced Dynkin diagram: $A_n, D_n, E_{n\leq 8}$. We will describe several of the relevant cases to give the reader a flavor for the world of finite algebras.

As mentioned previously, cluster algebras of type A_n

$$x_1 \to x_2 \to \ldots \to x_n \tag{2.27}$$

correspond to cluster algebras for $\mathrm{Gr}(2, n{+}3)$). Each cluster can be though as a triangulation of an $(n+3)$-gon, each $\mathcal{A}$-coordinate is a cord, and each $\mathcal{X}$-coordinate is a quadrilateral with a cord as a diagonal embedded in the $(n+3)$-gon. This makes the counting easy: the number of clusters for A_n is given by the Catalan number $C(n+1)$, the number of $\mathcal{A}$-coordinates is $\binom{n+3}{2} - n$, and the number of $\mathcal{X}$-coordinates is $2\binom{n+3}{4}$. Subalgebras correspond to embedding a smaller polygon in to the $(n{+}3)$-gon, for example there are $56 = \binom{8}{5}$ pentagonal embeddings in an octagon, and so there are 56 A_2 subalgebras in A_5.

The cluster algebra D_4

$$\begin{tikzcd}[row sep=small] & & x_3 \\ x_1 \arrow[r] & x_2 \arrow[ur] \arrow[dr] & \\ & & x_4 \end{tikzcd} \tag{2.28}$$

turns out to be connected to $\mathrm{Gr}(3,6)$. It has 50 clusters, 16 $\mathcal{A}$-coordinates , and 104 $\mathcal{X}$-coordinates . There are 36 A_2 subalgebras and 12 A_3 subalgebras. It is also a highly symmetric cluster algebra, as we will discuss in section 2.6.

[1] There is a further distinction in the infinite case between cluster algebras which contain a finite number of quiver types (still with an infinite number of cluster coordinates), and those which contain infinitely many quivers as well as coordinates.

The cluster algebra D_5

$$
\begin{tikzcd}[row sep=small]
 & & & x_4 \\
x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\
 & & & x_5
\end{tikzcd}
\tag{2.29}
$$

is not immediately connected with any Grassmannian, although it appears as a subalgebra of any $\mathrm{Gr}(k,n)$ with rank > 5. D_5 has 182 clusters, 25 $\mathcal{A}$-coordinates , and 260 $\mathcal{X}$-coordinates . There are 125 distinct A_2 subalgebras, 65 A_3, 10 A_4, and 5 D_4.

Finally, we describe the cluster algebra E_6

$$
\begin{tikzcd}[row sep=small]
 & & x_4 \arrow[d] & & \\
x_1 \arrow[r] & x_2 \arrow[r] & x_3 & x_5 \arrow[l] & x_5 \arrow[l]
\end{tikzcd}
\tag{2.30}
$$

which is connected to $\mathrm{Gr}(4,7)$. The associahedron for this algebra is displayed in figure 2.3. The algebra hashas 833 clusters, 42 $\mathcal{A}$-coordinates , and 770 $\mathcal{X}$-coordinates . The subalgebra counting is:

A_2	A_3	A_4	D_4	A_5	D_5
504	364	98	35	7	14

. (2.31)

For a more thorough tabulation of the subalgebra structure of these algebras, refer to appendix B.

2.6 Cluster automorphisms

See [25] for a more thorough mathematical introduction. The simplest example of a cluster automorphism is what we will call a direct automorphism. Let $\mathcal{A}$ be a cluster algebra. Then $f : \mathcal{A} \to \mathcal{A}$ is direct automorphism of $\mathcal{A}$ if

- for every cluster $\mathbf{x}$ of $\mathcal{A}$, $f(\mathbf{x})$ is also a cluster of $\mathcal{A}$,
- f respects mutations, i.e. $f(\mu(x_i, \mathbf{x})) = \mu(f(x_i), f(\mathbf{x}))$.

A simple example of this for A_2 is the map

$$\sigma_{A_2} : \quad \mathcal{X}_i \to \mathcal{X}_{i+1}, \tag{2.32}$$

which cycles the five clusters $1/\mathcal{X}_i \to \mathcal{X}_{i+1}$ amongst themselves, and can be cast in terms of the seed variables x_1, x_2 as

$$\sigma_{A_2} : \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to x_1(1 + x_2). \tag{2.33}$$

A less obvious example of a cluster automorphism is what are called indirect automorphisms, which respect mutations but do not map clusters directly on to clusters; instead

- for every cluster $\mathbf{x}$ of $\mathcal{A}$, $f(\mathbf{x})$ + invert all nodes + swap direction of all arrows = a cluster of $\mathcal{A}$.

For A_2 we have the indirect automorphism

$$\tau_{A_2}: \quad \mathcal{X}_i \to \mathcal{X}_{6-i}, \tag{2.34}$$

where indices are understood to be mod 5, and can instead be cast in terms of the seed variables x_1, x_2 as

$$\tau_{A_2}: \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to \frac{1}{x_1}. \tag{2.35}$$

We can see how this works in a simple example

$$\begin{aligned}
\tau_{A_2}(1/\mathcal{X}_1 \to \mathcal{X}_2) = 1/\mathcal{X}_5 \to \mathcal{X}_4 & \\
&\Rightarrow \text{invert each node and swap direction of all arrows} \\
&= \mathcal{X}_5 \leftarrow 1/\mathcal{X}_4, \text{ which is in the original } A_2.
\end{aligned} \tag{2.36}$$

It is useful to think of indirect automorphisms as generating a “mirror” or “flipped” version of the original $\mathcal{A}$, where the total collection of $\mathcal{X}$-coordinates is the same, but their Poisson bracket has flipped sign. The existence of this flip then can be seen as resulting from the choice of assigning b_{ij} = (# arrows $i \to j$) - (# arrows $j \to i$), where instead we could have chosen b_{ij} = (# arrows $j \to i$) - (# arrows $i \to j$) and still generated the same cluster algebraic structure, albeit with different labels for the nodes. In the generic case this is an arbitrary choice, and τ captures the superficiality of the notation change.

The automorphisms σ_{A_2} and τ_{A_2} generate the complete automorphism group for A_2, namely, D_5 (the notation here is regretably redundant; here we are referring to the dihedral group of five elements, which is of course distinct from the Dynkin diagram D_5 – we hope that context will clarify to the reader what we mean in each case). We now list generators for the automorphism groups of the finite algebras discussed already. First we adopt the notation

$$x_{i_1,\ldots,i_k} = \sum_{a=1}^{k} \prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1}x_{i_2} + \ldots + x_{i_1} \cdots x_{i_k}. \tag{2.37}$$

Cluster algebras of type A_n, as defined in eq. (2.27), have automorphism group D_{n+3}, with a cyclic generator σ_{A_n} (direct, length $n+3$)

$$\sigma_{A_n}: \quad x_{k<n} \to \frac{x_{k+1}(1 + x_{1,\ldots,k-1})}{1 + x_{1,\ldots,k+1}}, \quad x_n \to \frac{1 + x_{1,\ldots,n-1}}{\prod_{i=1}^{n} x_i} \tag{2.38}$$

and flip generator τ_{A_n} (indirect)

$$\tau_{A_n}: \quad x_1 \to \frac{1}{x_n}, \quad x_2 \to \frac{1}{x_{n-1}}, \quad \ldots \ , x_n \to \frac{1}{x_1}. \tag{2.39}$$

The cluster algebra $D_4 \simeq \mathrm{Gr}(3,6)$, as defined in eq. (2.28), has automorphism group $D_4 \times S_3$, with two cyclic generators:

$$\sigma_{D_4}^{(4)}: \quad x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)\, x_1x_2x_3x_4}{(1+x_{1,2,3})\,(1+x_{1,2,4})}, \quad x_3 \to \frac{1+x_{1,2}}{x_1x_2x_3}, \quad x_4 \to \frac{1+x_{1,2}}{x_1x_2x_4},$$

$$\sigma_{D_4}^{(3)}: \quad x_1 \to \frac{1}{x_3}, \quad x_2 \to \frac{x_1x_2\,(1+x_3)}{1+x_1}, \quad x_3 \to x_4, \quad x_4 \to \frac{1}{x_1}, \tag{2.40}$$

where $\sigma_{D_4}^{(4)}$ generates the 4-cycle and $\sigma_{D_4}^{(3)}$ the 3-cycle in D_4 and S_3, respectively. Then there is the indirect τ-flip associated with the D_4 automorphism, as well as a direct $\mathbb{Z}_2$-flip associated with the S_3 automorphism:

$$\begin{aligned} \tau_{D_4}: &\quad x_2 \to \frac{1+x_1}{x_1x_2\,(1+x_3)\,(1+x_4)}, \\ \mathbb{Z}_{2,D_4}: &\quad x_3 \to x_4, \quad x_4 \to x_3. \end{aligned} \tag{2.41}$$

The cluster algebra $D_{n>4}$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad\ \ x_{n-1} \\ \qquad\qquad\qquad\qquad\ \nearrow \\ x_1 \longrightarrow x_2 \longrightarrow \cdots \to x_{n-2} \\ \qquad\qquad\qquad\qquad\ \searrow \\ \qquad\qquad\qquad\qquad\quad\ \ x_n \end{array} \tag{2.42}$$

has automorphism group $D_n \times \mathbb{Z}_2$ with generators σ_{D_n} (n-cycle, direct), $\mathbb{Z}_{2,D_n}$ (2-cycle, direct), and τ_{D_n} (2-cycle, indirect). The Z_2 simply swaps $x_{n-1} \leftrightarrow x_n$, and for D_5, as defined in eq. (2.29), the σ and τ generators can be represented by

$$\begin{aligned} \sigma_{D_5}: &\quad x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)x_3}{1+x_{1,2,3}}, \quad x_3 \to \frac{x_1x_2x_3x_4x_5(1+x_{1,2})}{(1+x_{1,2,3,4})(1+x_{1,2,3,5})}, \\ &\quad x_4 \to \frac{1+x_{1,2,3}}{x_1x_2x_3x_4}, \quad x_5 \to \frac{1+x_{1,2,3}}{x_1x_2x_3x_5}, \\ \tau_{D_5}: &\quad x_1 \to x_1, \quad x_2 \to \frac{1+x_1}{x_1x_2(1+x_3x_5+x_{3,4,5})}, \quad x_3 \to \frac{x_3x_4x_5}{(1+x_{3,4})(1+x_{3,5})}, \\ &\quad x_4 \to \frac{1+x_3x_5+x_{3,4,5}}{x_4}, \quad x_5 \to \frac{1+x_3x_5+x_{3,4,5}}{x_5}. \end{aligned} \tag{2.43}$$

$E_6 \simeq \mathrm{Gr}(4,7)$, as defined in eq. (2.30), has automorphism group D_{14} with generators σ_{E_6} (7-cycle, direct), $\mathbb{Z}_{2,E_6}$ (2-cycle, direct), and τ_{E_6} (2-cycle, indirect). In $\mathrm{Gr}(4,7)$ language, these are the traditional cycle ($Z_i \to Z_{i+1}$), parity ($Z \to W$'s), and flip ($Z_i \to Z_{8-i}$) symmetries, respectively. In E_6 language they can be represented by

$$\begin{aligned}
\sigma_{E_6}:\quad & x_1 \to \frac{1}{x_6(1+x_{5,3,4})}, \quad x_2 \to \frac{1+x_{6,5,3,4}}{x_5(1+x_{3,4})}, \quad x_3 \to \frac{(1+x_{2,3,4})(1+x_{5,3,4})}{x_3(1+x_4)}, \\
& x_4 \to \frac{1+x_{3,4}}{x_4}, \quad x_5 \to \frac{1+x_{1,2,3,4}}{x_2(1+x_{3,4})}, \quad x_6 \to \frac{1}{x_1(1+x_{2,3,4})}, \\
\\
\mathbb{Z}_{2,E_6}:\quad & x_i \to x_{7-i}, \\
\\
\tau_{E_6}:\quad & x_1 \to \frac{x_5}{1+x_{6,5}}, \quad x_2 \to (1+x_5)x_6, \quad x_3 \to \frac{(1+x_{1,2})(1+x_{6,5})}{x_1x_2x_3x_5x_6(1+x_4)}, \\
& x_4 \to x_4, \quad x_5 \to x_1(1+x_2), \quad x_6 \to \frac{x_2}{1+x_{1,2}}.
\end{aligned} \tag{2.44}$$

3 Cluster Polylogarithms and MHV Amplitudes

Terminology guide (let A be a cluster algebra and B_i be a co-rank i subalgebra of A):

- "$\mathcal{A}$-polylog on A": symbol alphabet is drawn from the $\mathcal{A}$-coordinates of A,
- "$\mathcal{X}$-polylog on A": an $\mathcal{A}$-polylog on A with non-zero cobracket (in both $B_2 \wedge B_2$ and $B_3 \otimes \mathbb{C}^*$) and the cobracket arguments are drawn from the $\mathcal{X}$-coordinates of A,
- "extended $\mathcal{X}$-polylog on A": an $\mathcal{A}$-polylog on A whose cobracket arguments are $\mathcal{X}$ and $1-\mathcal{X}$, where $\mathcal{X}$ are drawn from the $\mathcal{X}$-coordinates of A,
- "automorphic $\mathcal{A}/\mathcal{X}$-polylog on A": an $\mathcal{A}$- or $\mathcal{X}$-polylog on A that is invariant, up to an overal minus sign, under the automorphisms of A,
- "adjacent $\mathcal{A}/\mathcal{X}$-polylog on A": an $\mathcal{A}$- or $\mathcal{X}$-polylog on A that satisfies cluster adjacency on A.

3.1 The Coproduct and Cobracket

Cobracket integrability condition:

$$0 = \delta^2 f_4 = \delta(b_{22}) + \delta(b_{31}) \tag{3.1}$$

3.2 Cluster-Algebraic Structure of MHV Amplitudes

This is a review of [1, 3, 4]. The essential idea is that $R_n^{(2)}$ "depends" only on $\mathcal{X}$-coordinates of $\mathrm{Gr}(4,n)$. This notion of dependence has multiple facets, such as

- $R_n^{(2)}$ can be written, at functional level, in terms of the functions $\mathrm{Li}_{2,2}$, $\mathrm{Li}_{1,3}$, and classical Li_k all with arguments $-\mathcal{X}$ where $\mathcal{X}$ is a cluster $\mathcal{X}$-coordinate of $\mathrm{Gr}(4,n)$.

- $R_n^{(2)}$ has Lie cobracket elements $\mathrm{B}_2 \wedge \mathrm{B}_2$ and $\mathrm{B}_3 \otimes \mathbb{C}^*$ expressible in terms of Bloch group elements $\{-\mathcal{X}\}_k$ where $\mathcal{X}$ is a cluster $\mathcal{X}$-coordinate of $\mathrm{Gr}(4,n)$.

Furthermore, there is the closely related object $\mathcal{E}_n^{(2)}$, which satisfies cluster adjacency, which we review now.

3.3 The A_2 function

We define the A_2 function as

$$\begin{aligned} f_{A_2}(x_1 \to x_2) = \sum_{\text{skew-dihedral}} & \mathrm{Li}_{2,2}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) + \mathrm{Li}_{1,3}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) + 6\,\mathrm{Li}_3\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right) \\ & - \mathrm{Li}_2\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right)\left(3\log\left(\mathcal{X}_{i-1}\right) - \log\left(\mathcal{X}_i\right) + \log\left(\mathcal{X}_{i+1}\right)\right) \\ & + \frac{1}{2}\log\left(\mathcal{X}_{i-3}\right)\log\left(\mathcal{X}_i\right)\log^2\left(\mathcal{X}_{i-1}\right), \end{aligned} \tag{3.2}$$

where the $\mathcal{X}_i$ are defined in terms of x_1 and x_2 as in eq. (2.17), and the skew-dihedral sum indicates subtracting the dihedral flip ($\mathcal{X}_1 \to \mathcal{X}_{6-i}$) and taking a cyclic sum $i = 1$ to 5.

This representation of f_{A_2} differs from that in [3] in several key ways. Firstly, we have added classical polylogarithm terms in order to make f_{A_2} adjacent in A_2:

$$\mathrm{symbol}(f_{A_2}) = -\sum_{\text{skew-dihedral}} [2233] + [2321] + [2332] - 2([2323] + [2343] - [2334]) \tag{3.3}$$

where we adopt the condensed notation $[ijkl] = \mathcal{X}_i \otimes \mathcal{X}_j \otimes \mathcal{X}_k \otimes \mathcal{X}_l$ in order to highlight the adjacency.

An additional benefit of this representation is that all arguments of the polylogarithms in for f_{A_2} are negative $\mathcal{X}$-coordinates of A_2. Furthermore, the function is smooth and real-valued for all $x_1, x_2 > 0$. The structure of the A_2 cluster algebra plays a crucial role in this analytic behavior in the following way. $\mathrm{Li}_{2,2}(x,y)$ and $\mathrm{Li}_{1,3}(x,y)$ have branch cuts at $x = 1, y = 1, x * y = 1$. The first two branch cuts are trivially avoided as $-1/\mathcal{X}_i < 0$ for $x_1, x_2 > 0$, however the last one is avoided only because of the exchange relation for A_2:

$$0 < \left(-\frac{1}{\mathcal{X}_{i-1}}\right)\left(-\frac{1}{\mathcal{X}_{i+1}}\right) = \frac{1}{1+\mathcal{X}_i} < 1. \tag{3.4}$$

Lastly, f_{A_2} has $\Lambda^2\,\mathrm{B}_2$ and $\mathrm{B}_3 \otimes \mathbb{C}^*$ coproduct elements expressible in terms of $\mathcal{X}$-coordinates of A_2:

$$\delta\left(f_{A_2}\right) = -\sum_{\text{skew-dihedral}} \{-\mathcal{X}_{i-1}\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + 3\{-\mathcal{X}_i\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + \frac{5}{2}\{-\mathcal{X}_i\}_3 \otimes \mathcal{X}_{i+1} \tag{3.5}$$

This representation of f_{A_2} therefore shares the following properties with $\mathcal{E}_n^{(2)}$:

- cluster adjacent,
- clustery coproduct,
- smooth and real-valued in the positive domain.

3.4 Cluster Adjacency

Cluster adjacency is a property of all Steinmann-satisfying amplitudes, and was first introduced in [12]. The original phrasing of this property is that the symbol of all Steinmann-satisfying integrals in n-particle kinematics, when fully expanded out in terms of $\mathcal{A}$-coordinates , is of the form

$$\ldots \otimes \alpha_i \otimes \alpha_j \otimes \ldots \tag{3.6}$$

where α_i and α_j appear together in a cluster of $\mathrm{Gr}(4, n)$. This non-trivial property is a considerable constraint on the space of polylogarithm functions which can appear in amplitudes.

The original presentation of cluster adjacency was in terms of $\mathcal{A}$-coordinates , but adjacency can also be phrased in terms of $\mathcal{X}$-coordinates . We will term these as cluster $\mathcal{A}$-adjacency and cluster $\mathcal{X}$-adjacency, respectively.

The benefit of $\mathcal{A}$-adjacency is that $\mathcal{A}$-coordinates are multiplicatively independent and so any symbol in them will be unique. The same is of course not true for $\mathcal{X}$-coordinates : they satisfy numerous multiplicative identities and so there exists many equivalent representations of a given symbol in terms of $\mathcal{X}$-coordinates , and only some small subset of them may satisfy cluster $\mathcal{X}$- adjacency.

However, the benefit of $\mathcal{X}$-coordinates is that they have a unique Poisson bracket, whereas $\mathcal{A}$-coordinates can appear in many different clusters together, each time with a different value for b_{ij} connecting them. This ambiguity in the Poisson bracket for $\mathcal{A}$-coordinates is equivalent to the ambiguity introduced by the multiplicative identities in the $\mathcal{X}$-coordinates .

While $\mathcal{X}$-adjacency trivially implies $\mathcal{A}$-adjacency, the converse is not so clear. However we have checked for all Grassmannian cluster algebras $\mathrm{Gr}(k \leq 4, n \leq 7)$ that $\mathcal{A}$-adjacency implies $\mathcal{X}$-adjacency, so we conjecture that the two phrasings of cluster adjacency are identical in constraining symbol space. ?

4 Cluster subalgebra-constructibility

The overarching goal for this paper is to exhaustively explore ways in which $\mathcal{E}_7^{(2)}$ can be expressed as a sum of cluster polylogarithms evaluated over subalgebras of $\mathrm{Gr}(4,7)$. $\mathrm{Gr}(4,7)$ has a rich subalgebra structure, and it is interesting to ask how much $\mathcal{N}=4$ SYM "knows" about this structure. Furthermore, it is critical for these first explorations to that $\mathrm{Gr}(4,7)$ is a finite cluster algebra, allowing for an exhaustive search of subalgebra representations ($\mathrm{Gr}(4,6)$ is also finite but the corresponding amplitude is purely classical).

Define the basic notion here (which applies to both clustery and non-clustery cobrackets) (differentiate between symbol-level and cobracket-level constructibility)

Terminology guide (let A be a cluster algebra, B_i be a rank i subalgebra of A, and $i_1 < i_2 < \ldots$):

- "$B_i \subset A$-constructible $\mathcal{A}/\mathcal{X}$-polylog': an automorphic $\mathcal{A}$- or $\mathcal{X}$-polylog on A that is composed of automorphic $\mathcal{A}$- or $\mathcal{X}$-polylogs on B_i,

- "$B_{i_1} \subset B_{i_2} \subset \ldots \subset A$-constructible $\mathcal{A}/\mathcal{X}$-polylog": iterating the above definition for multiple nestings of subalgebras of decreasing rank.

4.1 A_2 functions are a complete basis

A remarkable (conjectured) property of f_{A_2} is that it forms a complete basis for weight-4 cluster $\mathcal{X}$-polylogarithms for all cluster algebras [2]. This is because the cobracket integrability condition, eq. (3.1), requires relationships between the arguments of the cobrackets. Remarkably, the A_2 exchange relation

$$1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1} \tag{4.1}$$

is a sufficient relationship to generate one solution to integrability, namely f_{A_2}. In contrast, the $A_1 \times A_1$ cluster algebra has two algebraically distinct variables and so provides no solutions to integrability.

For any larger algebra, one generates a solution to integrability for each A_2 subalgebra. The magic of cluster algebras is that there exist no additional relationships – this is equivalent to the statement that all cluster algebras can be fully decomposed in to A_2 and $A_1 \times A_1$ subalgebras.

This essentially "solves" the problem of constructing $R_n^{(2)}$, via the algorithm described in [20], as it is a cluster $\mathcal{X}$-polylogarithm of $\mathrm{Gr}(4, n)$.

4.2 Definition and outline for construction

We will adopt some novel notation to describe subalgebras. Specifically, given a seed quiver for an algebra, we can label a subalgebra by

1. a sequence of mutations on the seed quiver that moves us to a cluster that lies within the subalgebra,
2. along with a list of nodes that one mutates on the new cluster to generate the subalgebra.

For example, with A_3 we have the seed cluster

$$x_1 \to x_2 \to x_3, \tag{4.2}$$

i.e. we have nodes 1, 2, 3 initially occupied by the coordinates x_1, x_2, x_3. The nodes remain fixed but the coordinates occupying each node change as one mutates. Note that two A_2 subalgebras are present directly in the seed cluster, namely $x_1 \to x_2$ and $x_2 \to x_3$. We label these two subalgebras by the nodes to mutate on to generate the subalgebra, so in this case we have

$$A_3|_{12} = x_1 \to x_2, \qquad A_3|_{23} = x_2 \to x_3. \tag{4.3}$$

But of course these are not the only two A_2 subalgebras that appear in the complete A_3 algebra. To reach the other subalgebras one needs to mutate on the seed quiver and move

[2] This has been checked for E_6 and all subalgebras contained therein.

out to other clusters. A mutation path is just a sequence of integers, e.g. "123", which means to mutate on node 1, then mutate the resulting quiver on node 2, then 3. We label this by $A_3|^{123}$, and working out the mutations gives

$$A_3|^{123} = \frac{x_2}{1+x_{12}} \to \frac{(1+x_1)\, x_3}{1+x_{123}} \to \frac{1+x_{12}}{x_1 x_2 x_3}. \tag{4.4}$$

Note that once again we have two obvious A_2 subalgebras. These are then labeled by

$$A_3|^{123}_{12} = \frac{x_2}{1+x_{12}} \to \frac{(1+x_1)\, x_3}{1+x_{123}}, \qquad A_3|^{123}_{23} = \frac{(1+x_1)\, x_3}{1+x_{123}} \to \frac{1+x_{12}}{x_1 x_2 x_3}. \tag{4.5}$$

To summarize, we label subalgebras with the notation

$$\text{Algebra}|^{\text{mutation path from algebra seed to cluster containing subalgebra seed}}_{\text{nodes of the cluster to mutate on to generate the subalgebra}}. \tag{4.6}$$

For this paper we always refer to the seed quivers listed in sec. 2.5. Lastly, this notation can be extended to include disconnected subalgebras by including a comma between the nodes, for example in A_3 we have the $A_1 \times A_1$ subalgebra

$$A_3|_{1,3} = x_1 \quad x_3, \tag{4.7}$$

which generates a square face of the associahedron.

One downside of this notation is that it is not unique, as there are many possible mutation paths to any cluster, and each subalgebra has many equivalent seed clusters. For example

$$A_3|^{123}_{23} = A_3|_{12} = x_1 \to x_2. \tag{4.8}$$

However in the case of finite cluster algebras one can always find at least one path of shortest length (and in the case of multiple shortest paths, sort left-to-right by lowest node number).

In this section we will explore the space of subalgebra-constructible polylogarithms for finite algebras (specifically those $\subseteq E_6$). Specifically, let us start with the question: how many ways can we embed f_{A_2} into a larger algebra in a way that respects the automorphisms of that larger algebra? As discussed in the previous section, because we are working at the level of the cobracket, f_{A_2} evaluated across all subalgebras of a larger algebra form a complete basis and so "A_2-constructibility" is possible for all cluster polylogarithms we are exploring. We can work through the simple example of $A_2 \subset A_3$ to give a more concrete flavor for this procedure.

The six A_2 subalgebras of A_3 are labeled by

$$\begin{gathered} A_3|_{12} = x_1 \to x_2, \quad A_3|_{23} = x_2 \to x_3, \quad A_3|^{123}_{12} = \frac{x_2}{1+x_{12}} \to \frac{(1+x_1)\, x_3}{1+x_{123}}, \\ A_3|^{1}_{23} = \frac{x_1 x_2}{1+x_1} \to x_3, \quad A_3|^{2}_{13} = x_1\,(1+x_2) \to \frac{x_2 x_3}{1+x_2}, \quad A_3|^{3}_{12} = x_1 \to x_2\,(1+x_3)\,. \end{gathered} \tag{4.9}$$

We construct f_{A_3} by beginning with an ansatz of f_{A_2} evaluated on the six A_2 subalgebras listed in eq. (4.9). We then require that f_{A_3} be invariant under the automorphisms of A_3 up to an overall sign. As discussed in sec. 2.6, A_3 has two automorphisms,

$$\sigma_{A_3} : etc. \tag{4.10}$$

Therefore there are four possible f_{A_3}, which we denote by their sign-behavior under σ_{A_3} and τ_{A_3}, for example $f_{A_3}^{+-}$ is invariant under σ_{A_3} and picks up an overal minus sign under τ_{A_3}. However not all sign-behaviors are necessarily possible, and working out which "orientations" of subalgebras can be made to fit the larger algebra's automorphisms is the central ingredient of our story. For example, there is no collection of f_{A_2}'s which is invariant under both σ_{A_3} and τ_{A_3}, i.e. $f_{A_3}^{++}$ does not exist. This can be worked out geometrically at the level of the A_3 associahedron, although it is a bit intricate. It is instead easier to run the problem through a linear algebra solver.

f_{A_2} evaluated across the 504 distinct A_2 subalgebras in $\mathrm{Gr}(4,7)$ gives a basis with only ??? degrees of freedom

4.3 $A_2 \subset A_3$ representation

This section is a review of [3, 20].

5 The D_5 Function

In this section we analyze "the" D_5 function, f_{D_5}, which we generate by:

- starting with an ansatz of all distinct f_{A_3}'s in D_5,
- imposing antisymmetry under all of the D_5 automorphisms $\{\sigma, \tau, \mathbb{Z}_2\}$,
- taking the fully symmetric sum of this f_{D_5} over E_6 and fitting to $R_7^{(2)}$.

The resulting function has 1 free parameter, which represents an internal degree of freedom in f_{D_5} that cancels in the symmetric sum over E_6.

There are 65 distinct $\mathcal{A}3$'s in D_5. Of these, only 42 produce linearly independent f_{A_3}'s. Imposing full D_5 antisymmetry on this collection of f_{A_3}'s leaves only 5 degrees of freedom. Requiring that the full E_6-symmetric sum of f_{D_5} gives $R_7^{(2)}$ fixes 4 of these parameters, leaving us with only 1 degree of freedom. Of course when we are looking for a particular representation of f_{D_5} we have 24 degrees of freedom (23 of which are equivalent to adding zero).

It would be nice to find a property that fixes some of these parameters that does not rely on explicitly knowing $R_7^{(2)}$. Of course a cluster-y property would be great, but even a physics one would be nice.

Because of the 1 degree of freedom, it is difficult to describe in detail the properties of f_{D_5} until we have set this value. Furthermore, we likely want to keep this parameter free so that we have some freedom when we try to express $R_7^{(2)}$ in terms of f_{D_5}.

The piece that does not cancel in the full E_6 sum can be represented in terms of 13 $\mathcal{A}3$'s. The following 8 enter with coefficient $+1/2$:

$$x_2 \to x_3 \to x_5, \quad x_1 \to x_2 \to x_3\,(1+x_5)\,, \quad \frac{x_1 x_2}{1+x_1} \to x_3 \to x_4, \quad x_1\,(1+x_2) \to \frac{x_2 x_3\,(1+x_4)}{1+x_2} \to x_5,$$

$$\frac{1}{x_4} \to x_3\,(1+x_4) \to x_5, \quad \frac{1+x_3}{x_3 x_4} \to x_2(1+x_{34}) \to \frac{x_3 x_5}{1+x_3},$$

$$\frac{1+x_{23}}{x_2 x_3 x_4} \to x_1(1+x_{234}) \to \frac{x_2 x_3 x_5}{1+x_{23}}, \quad \frac{x_1 x_2 x_3 x_5}{1+x_{123}} \to \frac{(1+x_1)\,x_3 x_4}{(1+x_3)\,(1+x_{1234})} \to \frac{1+x_{123}}{(1+x_1)\,x_3},$$

and these 5 enter with coefficient $-1/2$:

$$x_2 \to x_3 \to x_4, \quad x_1 \to x_2 \to x_3\,(1+x_4)\,, \quad \frac{x_1 x_2}{1+x_1} \to x_3 \to x_5,$$

$$x_1\,(1+x_2) \to \frac{x_2 x_3\,(1+x_5)}{1+x_2} \to x_4, \quad \frac{x_1 x_2 x_3 x_4}{1+x_{123}} \to \frac{(1+x_1)\,x_3 x_5}{(1+x_3)\,(1+x_{1235})} \to \frac{1+x_{123}}{(1+x_1)\,x_3}.$$

There is not any nice cancellation of terms at the level of $B_2 \wedge B_2$ for this sum of functions. It would be exciting to find a representation for f_{D_5} which relied on very few $B_2 \wedge B_2$ terms, but of course some relatively large number of terms will be necessary in order to full capture the D_5 symmetries.

Also, although we have not done a truly exhaustive search, based on extensive checks it does not seem likely that there is a representation that includes *all* $\mathcal{A}3$'s in D_5 (with simple coefficients, at least).

We do not include a representation for the piece of the function that cancels in the full E_6 sum, as the shortest representation we have found involves a cumbersome sum of 17 $\mathcal{A}3$'s.

Representing $R_7^{(2)}$ in terms of f_{D_5}

An E_6 version of the $\mathrm{Gr}(4,7)$ seed is:

$$\begin{tikzcd}
 & & -\frac{\langle 4(12)(35)(67)\rangle}{\langle 1234\rangle\langle 4567\rangle} & & \\
\frac{\langle 1234\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1246\rangle} \arrow[r] & -\frac{\langle 1247\rangle\langle 3456\rangle}{\langle 4(12)(35)(67)\rangle} \arrow[r] & \frac{\langle 1246\rangle\langle 5(12)(34)(67)\rangle}{\langle 1245\rangle\langle 1267\rangle\langle 3456\rangle} \arrow[u] & -\frac{\langle 1267\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 4(12)(35)(67)\rangle} \arrow[l] & \frac{\langle 1567\rangle\langle 2345\rangle}{\langle 5(12)(34)(67)\rangle} \arrow[l]
\end{tikzcd} \tag{5.1}$$

The symmetries are σ (period 7), τ, and $\mathbb{Z}_2$ (both period 2) and can be represented as:

$$
\begin{aligned}
\sigma: \quad & x_1 \mapsto \frac{1}{x_6\,(1+x_{534})}, \quad x_2 \mapsto \frac{1+x_{6534}}{x_5\,(1+x_{34})}, \quad x_3 \mapsto \frac{(1+x_{234})\,(1+x_{534})}{x_3\,(1+x_4)}, \\
& x_4 \mapsto \frac{1+x_{34}}{x_4}, \quad x_5 \mapsto \frac{1+x_{1234}}{x_2\,(1+x_{34})}, \quad x_6 \mapsto \frac{1}{x_1\,(1+x_{234})}, \\
\tau: \quad & x_1 \mapsto \frac{x_5}{1+x_{65}}, \quad x_2 \mapsto (1+x_5)\,x_6, \quad x_3 \mapsto \frac{(1+x_{12})\,(1+x_{65})}{x_1 x_2 x_3\,(1+x_4)\,x_5 x_6}, \\
& x_4 \mapsto x_4, \quad x_5 \mapsto x_1\,(1+x_2)\,, \quad x_6 \mapsto \frac{x_2}{1+x_{12}}, \\
\mathbb{Z}_2: \quad & x_1 \leftrightarrow x_6, \quad x_2 \leftrightarrow x_5
\end{aligned}
$$

These are directly equivalent to $\mathrm{Gr}(4,7)$ cycle^2, flip, and parity ($\sigma = \text{cycle}^2$ because the map for just a single cycle was too cumbersome to print).

The simplest D_5 subalgebra of E_6 is obtained by freezing the x_6 node and then mutating once on x_5, at which point we have the seed cluster

$$
\begin{tikzcd}
 & & & x_4 \\
x_1 \arrow[r] & x_2 \arrow[r] & \frac{x_3 x_5}{1+x_5} \arrow[ur] \arrow[dr] & \\
 & & & \frac{1}{x_5}
\end{tikzcd}
\tag{5.2}
$$

We'll refer to the cluster algebra generated by (5.2) as $D_5^{(0,0)}$. The remaining 13 D_5's in E_6 are generated by applying σ and τ. We can now label each D_5 via the number of σ's and τ's applied to $D_5^{(0,0)}$:

$$
D_5^{(i,j)} = \sigma^i \tau^j (D_5^{(0,0)}) \tag{5.3}
$$

We do not have to consider $\mathbb{Z}_2$ because

$$
\mathbb{Z}_2(D_5^{(0,0)}) = D_5^{(5,1)}. \tag{5.4}
$$

Note that by this we do not mean that $\sigma^5\tau$ acting on the cluster (5.2) equals $\mathbb{Z}_2$ acting on (5.2); instead $\mathbb{Z}_2$ acting on the complete D_5 algebra *generated by* (5.2) is equal to $\sigma^5\tau$ acting on the same algebra.

With this notation in place we can say that

$$
R_7^{(2)} = \sum_{i=0}^{6} \sum_{j=0}^{1} f_{D_5^{(i,j)}}. \tag{5.5}
$$

While this is the most technically correct way to phrase things, it is of course much more evocative to write

$$R_7^{(2)} = \sum_{D_5 \subset E_6} f_{D_5}, \tag{5.6}$$

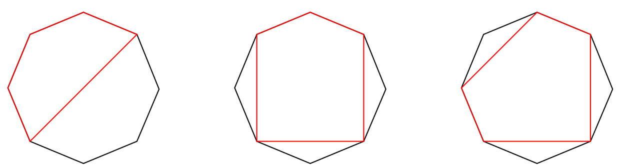
which is ill-defined up to the sign in front of each f_{D_5}.

6 The A_5 Function

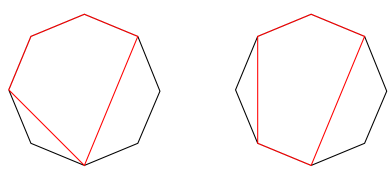
As discussed previously, there are 56 distinct A_2 subalgebras in A_5 ($56 = \binom{8}{5}$ = number of distinct pentagons inside an octagon), they can be parameterized by:

$$\begin{aligned} &\left\{x_1 \to x_2, \quad x_2 \to x_3\left(1+x_4\right), \quad x_2\left(1+x_3\right) \to \frac{x_3 x_4}{1+x_3}\right\} + \sigma_{A_5}, \\ &\left\{x_2 \to x_3, \quad x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}\right\} + \sigma_{A_5} + \tau_{A_5} \end{aligned} \tag{6.1}$$

where by "$+\sigma_{A_5}$" and "$+\sigma_{A_5} + \tau_{A_5}$" I mean "+ cyclic copies" and "+ cyclic and flip copies," respectively. These correspond to the geometries



(8 of each)



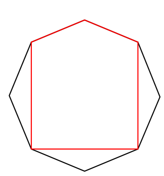
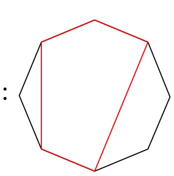
(16 of each)

The A_5 function is a sum over two of the classes of A_2 subalgebras, $x_2 \to x_3\left(1+x_4\right)$ and $x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}$, appropriately antisymmetrized so that the overall f_{A_5} picks up a minus sign under both σ_{A_5} and τ_{A_5}. Explicitly, this is written

$$f_{A_5} = \sum_{i=0}^{7} \sum_{j=0}^{1} (-1)^{i+j} \sigma_{A_5}^i \tau_{A_5}^j \left(\frac{1}{2} f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right) + f_{A_2}\left(x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}\right)\right). \tag{6.2}$$

The factor of $\frac{1}{2}$ in front of $f_{A_2}\left(x_2 \to x_3\left(1+x_4\right)\right)$ is simply a symmetry factor, as it lives in an 8-cycle of $\{\sigma_{A_5}, \tau_{A_5}\}$.

The two types of A_2's appearing in f_{A_5} are:

$x_2 \to x_3(1+x_4)$:  $x_1\left(1+x_2\right) \to \frac{x_2 x_3}{1+x_2}$:  (6.3)

7 Conclusion

1. since there are so few D_5 and A_5 subalgebras of Gr(4,7), these decompositions distill more intrinsic (better word?) geometric structure than the A_3 function
2. would be interesting to find the D_5 decomposition of higher-point amplitudes, to see if this seter our determined constants
3. not possible to get the full (function-level) amplitude out of a single (function-level) subalgebra function; it merits checking whether a linear combination of different subalgebra functions could be found
4. bring up mutations as active vs passive transformations?

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A Integrability and Adjacency for A_2

B Counting Subalgebras of Finite Cluster Algebras

In this appendix we catalog the subalgebra structure for the finite connected cluster algebras $\subseteq E_6$. These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

When counting distinct subalgebras, we lump together all subalgebras which are labeled by the same mutable nodes. However the same subalgebra may appear multiple times but dressed by different frozen nodes – these appear as distinct subpolytopes in the full associahedra. We have included the counts for both subpolytopes and distinct subalgebras. Also note that in our counting for $\mathcal{X}$-coordinates we have included both x and $1/x$.

$\underline{A_2}$: clusters: 5 a-coordinates: 5 x-coordinates: 10

$\underline{A_3}$: clusters: 14 a-coordinates: 9 x-coordinates: 30

Type	Subpolytopes	Subalgebras
A_2	6	6
$A_1 \times A_1$	3	3

$\underline{A_4}$: clusters: 42 a-coordinates: 14 x-coordinates: 70

Type	Subpolytopes	Subalgebras
A_2	28	21
$A_1 \times A_1$	28	28
A_3	7	7
$A_2 \times A_1$	7	7
$A_1 \times A_1 \times A_1$	0	0

$\underline{D_4}$: clusters: 50 a-coordinates: 16 x-coordinates: 104

Type	Subpolytopes	Subalgebras
A_2	36	36
$A_1 \times A_1$	30	18
A_3	12	12
$A_2 \times A_1$	0	0
$A_1 \times A_1 \times A_1$	4	4

$\underline{A_5}$: clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Subpolytopes	Subalgebras
A_2	120	56
$A_1 \times A_1$	180	144
A_3	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
D_4	0	0
A_4	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{D_5}$: clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Subpolytopes	Subalgebras
A_2	180	125
$A_1 \times A_1$	230	145
A_3	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
D_4	5	5
A_4	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

$\underline{E_6}$: clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Subpolytopes	Subalgebras
A_2	1071	504
$A_1 \times A_1$	1785	833
A_3	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
D_4	35	35
A_4	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
D_5	14	14
A_5	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

C Cobracket Spaces in Finite Cluster Algebras

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