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Cluster Polylogarithms and Subalgebra-Constructibility I: Novel Decompositions of the Seven-Particle Remainder Function

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Abstract: Everything we know about cluster algebras and polylogarithms.

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2 A Brief Introduction to Cluster Algebras

2.1 Motivation and the A_2 cluster algebra

Cluster algebras were introduced by Fomin and Zelevinsky [**?**], in large part motivated by questions of total positivity. The original goal was to gain a better understanding of what algebraic varieties can have a natural notion of positivity, and what functions can determine such positivity. A simple and highly-relevant example for amplitudes is the positive Grassmannian $\mathrm{Gr}^+(k,n)$, i.e. the space of $k \times n$ matrices where all ordered $k \times k$ minors are positive.

Before I further describe what a cluster algebra is, let me give a more precise idea of what questions cluster algebras help us answer. One very nice question that we can gain some handle on is: how many minors do we need to specify a point in $\mathrm{Gr}^+(k,n)$? In other words, given a $k \times n$ matrix M, how many minors of M do we have to calculate to know if $M \in \mathrm{Gr}^+(k,n)$? The reason that this is an interesting question is that the minors are not all independent, they satisfy identities known as Plücker relations:

$$\langle abI\rangle\langle cdI\rangle = \langle acI\rangle\langle bdI\rangle + \langle adI\rangle\langle bcI\rangle, \tag{2.1}$$

where the Plücker coordinates $= \langle i_1, \ldots, i_k\rangle =$ the minor of columns $i_1, \ldots, i_k$, and I is a multi-index with $k-2$ entries.

Let's work through the example of $\mathrm{Gr}(2,5)$ in detail to try to understand how many minors one needs to check for positivity of the whole matrix. First of all, we'll definitely have to check the 5 cyclically adjacent minors, $\langle 12\rangle, \langle 23\rangle, \langle 34\rangle, \langle 45\rangle, \langle 15\rangle > 0$, as they are all independent from each other. Now, how many of the non-adjacent minors do we have to check? It turns out that the answer is 2. For example, if we specify that $\langle 13\rangle, \langle 14\rangle > 0$ then we can use Plücker relations to show

$$\begin{aligned}
\langle 24\rangle &= (\langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 14\rangle)/\langle 13\rangle \\
\langle 25\rangle &= (\langle 12\rangle\langle 45\rangle + \langle 24\rangle\langle 15\rangle)/\langle 14\rangle \\
\langle 35\rangle &= (\langle 25\rangle\langle 34\rangle + \langle 23\rangle\langle 45\rangle)/\langle 24\rangle.
\end{aligned} \tag{2.2}$$

Here we have expressed all of the remaining minors as sums and products of the cyclically adjacent minors along with $\langle 13\rangle$ and $\langle 14\rangle$, so everything is positive.

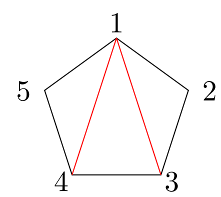
So we only need to check two – but can we check any two? Clearly we can use any of the cyclic images of $\{\langle 13\rangle, \langle 14\rangle\}$. What about $\{\langle 13\rangle, \langle 25\rangle\}$? This is a bit harder to see, but no, this pair does not work: there is no way to write down the remaining Plückers in terms of $\langle 13\rangle$ and $\langle 25\rangle$ such that everything is manifestly positive. For example, the matrix

$$\begin{pmatrix} 1 & -1 & -4 & 3 & -2 \\ 2 & 2 & -6 & 4 & -1 \end{pmatrix} \tag{2.3}$$

satisfies $\langle 12\rangle, \ldots, \langle 15\rangle, \langle 13\rangle, \langle 25\rangle > 0$ but has $\langle 14\rangle < 0$. In the end, $\{\langle 13\rangle, \langle 14\rangle\}$ and its cyclic images are the only pairs that describe a point in $\mathrm{Gr}^{+}(2,5)$.

This was easy enough to work out for this small case, but I think it is also easy to convince yourself that the problem gets much more complicated for larger matrices. However, there is a closely related, and much simpler, problem in geometry which can give us a bit more intuition: triangulating polygons.

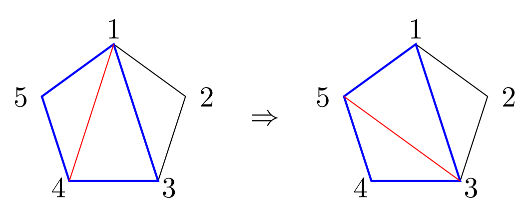
Consider the following triangulation of the pentagon:



(2.4)

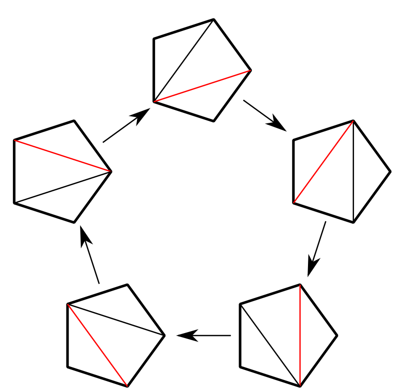
We can immediately see the parallels with our $\mathrm{Gr}(2,5)$ situation (this is an example of the more general Plücker embedding which connects $\mathrm{Gr}(k,n)$ with projective space). Here we associate lines connecting points i and j with the Plücker coordinate $\langle ij\rangle$, and we see that the triangulations of the pentagon all describe points in $\mathrm{Gr}^{+}(2,5)$. In fact this correspondence holds between n-gons and $\mathrm{Gr}(2,n)$.

A simple observation, but one at the very heart of cluster algebras, is that given some triangulation of a polygon one can create a *new* triangulation by picking a quadrilateral and flipping its diagonal. For example:



(2.5)

By repeatedly performing these flips one can generate all possible triangulations of a polygon:

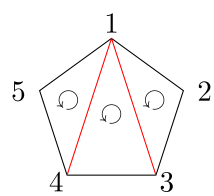


where in each case the red diagonal gets flipped.

Cluster algebras are a combinatorial tool which captures all of this structure (and much more!). The basic idea is that a cluser algebra is a collection of *clusters*, which in this case represent individual triangulations of an n-gon, and these clusters are connected via a process called *mutation*, which in this case is the flipping-the-diagonal process. I will initially describe everything in terms of Grassmannian cluster algebras, but the framework is entirely generalizable to other contexts of interest.

Basic definition

Let's construct the cluster algebra for $\mathrm{Gr}(2,5)$. Each cluster is labeled by a collection of coordinates, which in this case are the edges of our pentagon along with the diagonals of the particular triangulation. These coordinates are then connected via an orientation of the pentagon and all subtriangles, for example:



(2.6)

We can redraw this diagram as

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd} \tag{2.7}$$

In this quiver diagram, we have an arrow between two Plückers $\langle ab\rangle \to \langle cd\rangle$ if the triangle orientations in eq. (2.6) have segment (ab) flowing into segment (cd). The boxes around the $\langle i i+1\rangle$ indicate that they are *frozen* – in other words, we never change the outer edges of the pentagon, only the diagonal elements. And lastly it is unnecessary to draw the arrows connecting the outer edges, as that is redundant (and unchanging) information.

We have now drawn our first cluster (also sometimes called a seed). To review/introduce some terminology, the Plückers are called cluster $\mathcal{A}$-coordinates (sometimes also $\mathcal{A}$-variables), and they come in two flavors: mutable ($\langle 13\rangle$ and $\langle 14\rangle$) and frozen ($\langle i i+1\rangle$). The information of the arrows can be represented in terms of a skew-symmetric adjacency matrix

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i). \tag{2.8}$$

The process of mutation, which we described geometrically in terms of flipping the diagonal, has a simple interpretation at the level of this quiver. In particular, given a quiver such as eq. (2.7), chose a node k with associated $\mathcal{A}$-coordinate a_k to mutate on (this is equivalent to picking which diagonal to flip). Then draw a new quiver that changes a_k to a'_k defined by

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}, \tag{2.9}$$

(with the understanding that an empty product is set to one) and leaves the other cluster coordinates unchanged. The arrows connecting the nodes in this new cluster are modified from the original cluster according to

- for each path $i \to j \to k$, add an arrow $i \to j$,
- reverse all arrows on the edges incident with k,
- and remove any two-cycles that may have formed.

This creates a new adjacency matrix b'_{ij} via

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}, \\ b_{ij}, & \text{if } b_{ik}b_{kj} \le 0, \\ b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} < 0. \end{cases} \tag{2.10}$$

Mutation is an involution, so mutating on a'_k will take you back to the original cluster (as flipping the same diagonal twice will take you back to where you started).

For our purposes, *a cluster algebra is a set of quivers closed under mutation.* This means that mutating on any node of any quiver will generate a different quiver in the cluster algebra. The general procedure is to start with a quiver such as eq. (2.7), with some collection of frozen and unfrozen variables in a connected quiver, and continue mutating on all available nodes until you either close your set or convince yourself that the cluster algebra is infinite.

[describe what a subalgebra is?]

2.2 An overview of finite cluster algebras

Fomin and Zelevinksy [**?**] showed that a cluster algebra is of finite type iff the mutable part of its quiver at some cluster takes the form of an oriented, simply-laced Dynkin diagram: $A_n, D_n, E_{n\le 8}$. We discuss a few relevant cases here.

Cluster algebras of type A_n

$$x_1 \to x_2 \to \ldots \to x_n \tag{2.11}$$

correspond to triangulations of an $(n+3)$-gon, where each cluster is a triangulation, each $\mathcal{A}$-coordinate is a cord, and each $\mathcal{X}$-coordinate is a quadrilateral with a cord as a diagonal embedded in the $(n+3)$-gon. This makes the counting easy: the number of clusters for A_n is given by the Catalan number $C(n+1)$, the number of $\mathcal{A}$-coordinates is $\binom{n+3}{2} - n$, and the number of $\mathcal{X}$-coordinates is $2\binom{n+3}{4}$. Subalgebras correspond to embedding a smaller polygon in to the $(n+3)$-gon, for example the A_2 subalgebras in A_5 are the $56 = \binom{8}{5}$ pentagonal embeddings in an octagon.

The cluster algebra D_4

$$\begin{tikzcd}[row sep=small] & & x_3 \\ x_1 \arrow[r] & x_2 \arrow[ur] \arrow[dr] & \\ & & x_4 \end{tikzcd} \tag{2.12}$$

has 50 clusters, 16 $\mathcal{A}$-coordinates , and 104 $\mathcal{X}$-coordinates . There are 36 A_2 subalgebras and 12 A_3 subalgebras.

The cluster algebra D_5

$$\begin{tikzcd}[row sep=small] & & & x_4 \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[ur] \arrow[dr] & \\ & & & x_5 \end{tikzcd} \tag{2.13}$$

has 182 clusters, 25 $\mathcal{A}$-coordinates , and 260 $\mathcal{X}$-coordinates . There are 125 distinct A_2 subalgebras, 65 A_3, 10 A_4, and 5 D_4.

Finally, we describe the cluster algebra E_6

$$\begin{tikzcd}[row sep=small] & & x_4 \arrow[d] & & \\ x_1 \arrow[r] & x_2 \arrow[r] & x_3 & x_5 \arrow[l] & x_5 \arrow[l] \end{tikzcd} \tag{2.14}$$

which has 833 clusters, 42 $\mathcal{A}$-coordinates , and 770 $\mathcal{X}$-coordinates . The subalgebra counting is:

A_2	A_3	A_4	D_4	A_5	D_5
504	364	98	35	7	14

(2.15)

2.3 Cluster automorphisms

See [?] for a more thorough mathematical introduction. The simplest example of a cluster automorphism is what we will call a direct automorphism. Let $\mathcal{A}$ be a cluster algebra. Then $f : \mathcal{A} \to \mathcal{A}$ is direct automorphism of $\mathcal{A}$ if

- for every cluster $\mathbf{x}$ of $\mathcal{A}$, $f(\mathbf{x})$ is also a cluster of $\mathcal{A}$,
- f respects mutations, i.e. $f(\mu(x_i, \mathbf{x})) = \mu(f(x_i), f(\mathbf{x}))$.

A simple example of this for A_2 is the map

$$\sigma_{A_2} : \quad \mathcal{X}_i \to \mathcal{X}_{i+1}, \tag{2.16}$$

which cycles the five clusters $1/\mathcal{X}_i \to \mathcal{X}_{i+1}$ amongst themselves, and can be cast in terms of the seed variables x_1, x_2 as

$$\sigma_{A_2} : \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to x_1(1 + x_2). \tag{2.17}$$

A less obvious example of a cluster automorphism is what are called indirect automorphisms, which respect mutations but do not map clusters directly on to clusters; instead

- for every cluster $\mathbf{x}$ of $\mathcal{A}$, $f(\mathbf{x})$ + invert all nodes + swap direction of all arrows = a cluster of $\mathcal{A}$.

For A_2 we have the indirect automorphism

$$\tau_{A_2}: \quad \mathcal{X}_i \to \mathcal{X}_{6-i}, \tag{2.18}$$

where indices are understood to be mod 5, and can instead be cast in terms of the seed variables x_1, x_2 as

$$\tau_{A_2}: \quad x_1 \to \frac{1}{x_2}, \quad x_2 \to \frac{1}{x_1}. \tag{2.19}$$

We can see how this works in a simple example

$$\begin{aligned} \tau_{A_2}(1/\mathcal{X}_1 \to \mathcal{X}_2) = 1/\mathcal{X}_5 \to \mathcal{X}_4 & \\ \Rightarrow & \text{ invert each node and swap direction of all arrows} \\ = & \ \mathcal{X}_5 \leftarrow 1/\mathcal{X}_4, \ \text{which is in the original } A_2. \end{aligned} \tag{2.20}$$

It is useful to think of indirect automorphisms as generating a “mirror” or “flipped” version of the original $\mathcal{A}$, where the total collection of $\mathcal{X}$-coordinates is the same, but their Poisson bracket has flipped sign. The existence of this flip then can be seen as resulting from the choice of assigning b_{ij} = (# arrows $i \to j$) - (# arrows $j \to i$), where instead we could have chosen b_{ij} = (# arrows $j \to i$) - (# arrows $i \to j$) and still generated the same clsuter algebraic structure, albeit with different labels for the nodes. In the generic case this is an arbitrary choice, and τ captures the superficiality of the notation change.

The automorphisms σ_{A_2} and τ_{A_2} generate the complete automorphism group for A_2, namely, D_5 (the notation here is regretably redundant; here we are referring to the dihedral group of five elements, which is of course distinct from the Dynkin diagram D_5 – we hope that context will clarify to the reader what we mean in each case). We now list generators for the automorphism groups of the finite algebras discussed already. First we adopt the notation

$$x_{i_1,\ldots,i_k} = \sum_{a=1}^{k} \prod_{b=1}^{a} x_{i_b} = x_{i_1} + x_{i_1}x_{i_2} + \ldots + x_{i_1} \cdots x_{i_k}. \tag{2.21}$$

Cluster algebras of type A_n, as defined in eq. (2.11), have automorphism group D_{n+3}, with a cyclic generator σ_{A_n} (direct, length $n+3$)

$$\sigma_{A_n}: \quad x_{k<n} \to \frac{x_{k+1}(1 + x_{1,\ldots,k-1})}{1 + x_{1,\ldots,k+1}}, \quad x_n \to \frac{1 + x_{1,\ldots,n-1}}{\prod_{i=1}^{n} x_i} \tag{2.22}$$

and flip generator τ_{A_n} (indirect)

$$\tau_{A_n}: \quad x_1 \to \frac{1}{x_n}, \quad x_2 \to \frac{1}{x_{n-1}}, \quad \ldots \ , x_n \to \frac{1}{x_1}. \tag{2.23}$$

The cluster algebra $D_4 \simeq \mathrm{Gr}(3,6)$, as defined in eq. (2.12), has automorphism group $D_4 \times S_3$, with two cyclic generators:

$$\sigma_{D_4}^{(4)}: \quad x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)\, x_1 x_2 x_3 x_4}{(1+x_{1,2,3})\,(1+x_{1,2,4})}, \quad x_3 \to \frac{1+x_{1,2}}{x_1 x_2 x_3}, \quad x_4 \to \frac{1+x_{1,2}}{x_1 x_2 x_4},$$
$$\sigma_{D_4}^{(3)}: \quad x_1 \to \frac{1}{x_3}, \quad x_2 \to \frac{x_1 x_2\,(1+x_3)}{1+x_1}, \quad x_3 \to x_4, \quad x_4 \to \frac{1}{x_1}, \tag{2.24}$$

where $\sigma_{D_4}^{(4)}$ generates the 4-cycle and $\sigma_{D_4}^{(3)}$ the 3-cycle in D_4 and S_3, respectively. Then there is the indirect τ-flip associated with the D_4 automorphism, as well as a direct $\mathbb{Z}_2$-flip associated with the S_3 automorphism:

$$\tau_{D_4}: \quad x_2 \to \frac{1+x_1}{x_1 x_2\,(1+x_3)\,(1+x_4)},$$
$$\mathbb{Z}_{2,D_4}: \quad x_3 \to x_4, \quad x_4 \to x_3. \tag{2.25}$$

The cluster algebra $D_{n>4}$

$$\begin{tikzcd}[row sep=small] & & & x_{n-1} \\ x_1 \arrow[r] & x_2 \arrow[r] & \cdots \arrow[r] x_{n-2} \arrow[ur] \arrow[dr] & \\ & & & x_n \end{tikzcd} \tag{2.26}$$

has automorphism group $D_n \times \mathbb{Z}_2$ with generators σ_{D_n} (n-cycle, direct), $\mathbb{Z}_{2,D_n}$ (2-cycle, direct), and τ_{D_n} (2-cycle, indirect). The Z_2 simply swaps $x_{n-1} \leftrightarrow x_n$, and for D_5, as defined in eq. (2.13), the σ and τ generators can be represented by

$$\begin{aligned}
\sigma_{D_5}: \quad & x_1 \to \frac{x_2}{1+x_{1,2}}, \quad x_2 \to \frac{(1+x_1)x_3}{1+x_{1,2,3}}, \quad x_3 \to \frac{x_1 x_2 x_3 x_4 x_5 (1+x_{1,2})}{(1+x_{1,2,3,4})(1+x_{1,2,3,5})}, \\
& x_4 \to \frac{1+x_{1,2,3}}{x_1 x_2 x_3 x_4}, \quad x_5 \to \frac{1+x_{1,2,3}}{x_1 x_2 x_3 x_5}, \\
\tau_{D_5}: \quad & x_1 \to x_1, \quad x_2 \to \frac{1+x_1}{x_1 x_2 (1+x_3 x_5 + x_{3,4,5})}, \quad x_3 \to \frac{x_3 x_4 x_5}{(1+x_{3,4})(1+x_{3,5})}, \\
& x_4 \to \frac{1+x_3 x_5 + x_{3,4,5}}{x_4}, \quad x_5 \to \frac{1+x_3 x_5 + x_{3,4,5}}{x_5}.
\end{aligned} \tag{2.27}$$

$E_6 \simeq \mathrm{Gr}(4,7)$, as defined in eq. (2.14), has automorphism group D_{14} with generators σ_{E_6} (7-cycle, direct), $\mathbb{Z}_{2,E_6}$ (2-cycle, direct), and τ_{E_6} (2-cycle, indirect). In $\mathrm{Gr}(4,7)$ language, these are the traditional cycle ($Z_i \to Z_{i+1}$), parity ($Z \to W$'s), and flip ($Z_i \to Z_{8-i}$) symmetries, respectively. In E_6 language they can be represented by

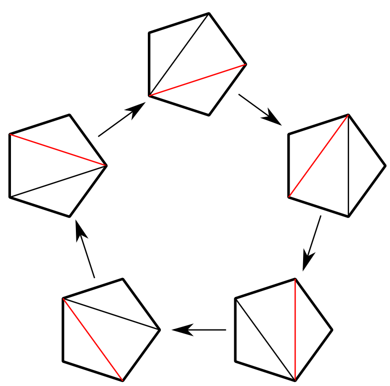
$$\sigma_{E_6}: \quad x_1 \to \frac{1}{x_6(1+x_{5,3,4})}, \quad x_2 \to \frac{1+x_{6,5,3,4}}{x_5(1+x_{3,4})}, \quad x_3 \to \frac{(1+x_{2,3,4})(1+x_{5,3,4})}{x_3(1+x_4)},$$
$$x_4 \to \frac{1+x_{3,4}}{x_4}, \quad x_5 \to \frac{1+x_{1,2,3,4}}{x_2(1+x_{3,4})}, \quad x_6 \to \frac{1}{x_1(1+x_{2,3,4})},$$

$$\mathbb{Z}_{2,E_6}: \quad x_i \to x_{7-i}, \tag{2.28}$$

$$\tau_{E_6}: \quad x_1 \to \frac{x_5}{1+x_{6,5}}, \quad x_2 \to (1+x_5)x_6, \quad x_3 \to \frac{(1+x_{1,2})(1+x_{6,5})}{x_1x_2x_3x_5x_6(1+x_4)},$$
$$x_4 \to x_4, \quad x_5 \to x_1(1+x_2), \quad x_6 \to \frac{x_2}{1+x_{1,2}}.$$

2.4 Associahedra

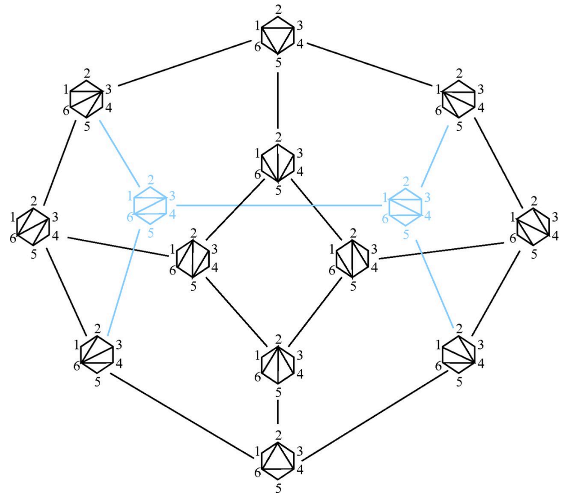
Understanding the way in which clusters are related to each other via mutation is a hot topic of research. A sample question is: given two clusters in the same cluster algebra, what is the minimal set of mutations necessary to get from one cluster to the other? While I don't delve in to this area too much, I do want to give an introduction to a relevant and important object associated with a cluster algebra: the associahedron (often also called the Stasheff polytope). The basic idea is to create a graph where the nodes are clusters and edges are drawn between clusters connected via mutation. So the figure



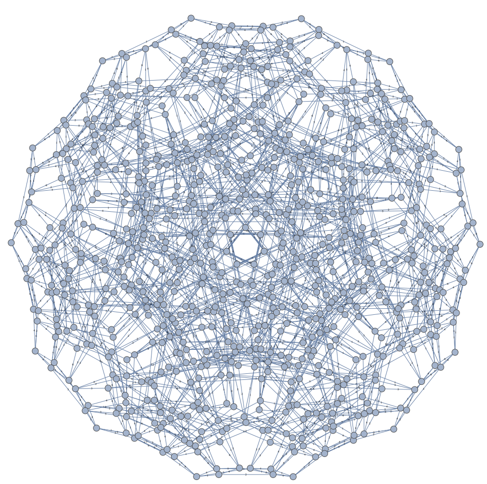
is in fact the $\mathrm{Gr}(2,5)$ or A_2 associahedron, and clearly it takes the form of a pentagon (one should ignore the orientation on the edges of the pentagon).

The associahedron associated with the $\mathrm{Gr}(2,6) \leftrightarrow A_3$ cluster algebra (i.e. triangulations of a hexagon) is

This associahedron has 14 vertices, with 3 square faces and 6 pentagonal faces. Because of the Grassmannian duality $\mathrm{Gr}(2,6) = \mathrm{Gr}(4,6)$, this (remarkably simple!) cluster algebra and associahedron play an integral role in the momentum twistors for 6-particle kinematics for $\mathcal{N} = 4$ SYM.



$\mathrm{Gr}(4,7) \leftrightarrow E_6$, which is associated with 7-particle momentum twistors, is a bit more complicated: it features 833 clusters, and the associahedron is a polytope with nodes of valence 6. The dimension-2 faces are 1785 squares and 1071 pentagons, and there are 49 different A-coordinates that appear. The associahedron looks like



It is not important to memorize any of this – I just want to give you a flavor for the rich and complex structures that can arise from the simple mutation rules of eq. (2.9)!

2.5 Grassmannian cluster algebras (and cluster Poisson spaces)

For $\mathrm{Gr}(k,n)$, Scott showed [?] that the associated seed cluster is

$$
\begin{tikzcd}
f_{1l} \arrow[r] \arrow[d] & \cdots \arrow[r] & f_{13} \arrow[r] \arrow[d] & f_{12} \arrow[r] \arrow[d] & \boxed{f_{11}} \arrow[l, phantom] \arrow[dl, phantom] \arrow[l, phantom] \\
f_{2l} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[ul] & f_{23} \arrow[r] \arrow[d] & f_{22} \arrow[r] \arrow[d] \arrow[ul] & \boxed{f_{21}} \arrow[ul] \\
\vdots \arrow[d] & \vdots & \vdots \arrow[d] & \vdots \arrow[d] \arrow[ul] & \vdots \arrow[ul] \\
\boxed{f_{kl}} & \cdots & \boxed{f_{k3}} & \boxed{f_{k2}} \arrow[ul] & \boxed{f_{k1}} \arrow[ul]
\end{tikzcd}
$$

where

$$
f_{ij} = \begin{cases} \frac{\langle i+1,\dots,k,k+j,\dots,i+j+k-1\rangle}{\langle 1,\dots,k\rangle}, & i \leq l-j+1, \\ \frac{\langle 1,\dots,i+j-l-1,i+1,\dots,k,k+j,\dots,n\rangle}{\langle 1,\dots,k\rangle}, & i > l-j+1 \end{cases}, \tag{2.29}
$$

We can use this to find the Grassmannian cluster algebras of finite type, they are:

$$
\mathrm{Gr}(2,n) \leftrightarrow A_{n-3}, \quad \mathrm{Gr}(3,6) \leftrightarrow D_4, \quad \mathrm{Gr}(4,7) \leftrightarrow E_6, \quad \mathrm{Gr}(3,8) \leftrightarrow E_8. \tag{2.30}
$$

An intriguing point for those of us studying $\mathcal{N} = 4$ SYM, where the momentum twistors describing particle kinematics live in $\mathrm{Gr}(4,n)$, is that only $\mathrm{Gr}(4, n < 8)$ are finite. The ramifications of the fact that $n > 7$-particle kinematics are associated with infinite cluster algebras are still being worked out.

Another important set of information encoded in cluster algebras are called Fock-Goncharov or $\mathcal{X}$-coordinates. Given a quiver described by the matrix b, the $\mathcal{A}$- and $\mathcal{X}$-coordinates are related as follows:

$$
x_i = \prod_j a_j^{b_{ij}}. \tag{2.31}
$$

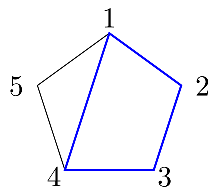
For example, the quiver

$$
\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd} \tag{2.32}
$$

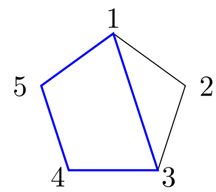
has $\mathcal{X}$-coordinates

$$
\mathcal{X}_1 = \frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle}, \quad \mathcal{X}_2 = \frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 34\rangle}. \tag{2.33}
$$

In the pentagon-triangulation picture, these $\mathcal{X}$-coordinates describe overlapping quadrilaterals:



$= \mathcal{X}_1,$



$= \mathcal{X}_2,$ (2.34)

The nice feature of $\mathcal{X}$-coordinates, at least from a physicists perspective, is that in the case of $\mathrm{Gr}(4,n)$ the $\mathcal{X}$-coordinates are dual-conformal invariant cross-ratios. We can see this for example in the two-loop, six-particle remainder function for $\mathcal{N}=4$ SYM:

$$\begin{aligned} R_6^{(2)} = \sum_{\text{cyclic}} \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right) \\ + \text{products of } \mathrm{Li}_k(-x) \text{ functions of lower weight} \\ \text{with the same set of arguments.} \end{aligned} \tag{2.35}$$

The arguments of the polylogarithms all take the form of $(-\mathcal{X})$-coordinates for $\mathrm{Gr}(4,6)$. $\mathcal{X}$-coordinates play other important roles in the context of polylogarithm functions independent of scattering amplitudes, for example with $\mathrm{Gr}(2,5) \leftrightarrow A_2$ we have

$$\sum_{\text{cyclic}} \mathrm{Li}_2(-\mathcal{X}_i) + \log(\mathcal{X}_i)\log(\mathcal{X}_{i+1}) = \frac{\pi^2}{6} \tag{2.36}$$

where the definition of $\mathcal{X}_i$ can be inferred from eq. (2.33). There are more complicated examples of polylogarithm identities satisfied by groups of $\mathcal{X}$-coordinates, for example there is a 40-term identity among Li_3 functions where all of the arguments are $(-\mathcal{X})$

Since we are motivated from physics to cast our final function-level results in terms of $\mathcal{X}$-coordinates it is useful to work entirely in that language rather than the $\mathcal{A}$-coordinates. The mutation rules for the $\mathcal{X}$-coordinates are

$$x_i' = \begin{cases} x_k^{-1}, & i = k, \\ x_i(1 + x_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases}, \tag{2.37}$$

and the adjacency matrix b_{ij} changes following eq. (2.10). From here on out we will write our quivers entirely in terms of $\mathcal{X}$-coordinates, for example A_2 is

$$x_1 \to x_2. \tag{2.38}$$

By continuing to mutate on alternating nodes (denoted below by red) we generate the following sequence of clusters:

$$\begin{aligned} x_1 &\rightarrow x_2 \\ x_1(1+x_2) &\leftarrow \frac{1}{x_2} \\ \frac{1}{x_1(1+x_2)} &\rightarrow \frac{x_2}{1+x_1+x_1x_2} \\ \frac{x_1x_2}{1+x_1} &\leftarrow \frac{1+x_1+x_1x_2}{x_2} \\ \frac{1+x_1}{x_1x_2} &\rightarrow \frac{1}{x_1} \\ x_2 &\leftarrow x_1 \\ &\vdots \end{aligned} \tag{2.39}$$

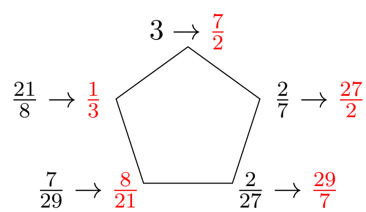
where the series then repeats. Note that by labeling the $\mathcal{X}$-coordinates as

$$\mathcal{X}_1 = 1/x_1, \quad \mathcal{X}_2 = x_2, \quad \mathcal{X}_3 = x_1(1+x_2), \quad \mathcal{X}_4 = \frac{1+x_1+x_1x_2}{x_2}, \quad \mathcal{X}_5 = \frac{1+x_1}{x_1x_2}, \tag{2.40}$$

then the general mutation rule of eq. (2.37) takes the simple form of

$$1 + \mathcal{X}_i = \mathcal{X}_{i-1}\mathcal{X}_{i+1}. \tag{2.41}$$

Putting this all together, we say that an A_2 cluster algebra is any set of clusters $1/\mathcal{X}_{i-1} \rightarrow \mathcal{X}_i$ for $i = 1 \ldots 5$ where the $\mathcal{X}_i$ satisfy eq. (2.41). We believe it is useful at this point to emphasize that one can take as input any $\{x_1, x_2\}$ and generate an associated A_2. For example, one could start with the quiver $3 \rightarrow \frac{7}{2}$ and generate the A_2



(2.42)

(Mutating on the node in red moves you clockwise around the pentagon.) In future sections it will be necessary to consider collections of multiple A_2 algebras, in such cases we label them by only one of their clusters, e.g. $x_1 \rightarrow x_2$, with the understanding that we are referring to the A_2 which contains that cluster as an element.

2.5.1 Applications to momentum twistors and $\mathcal{N}$=4 SYM

3 Cluster Polylogarithms and MHV Amplitudes

3.1 The Coproduct and Cobracket

3.2 Cluster-Algebraic Structure of MHV Amplitudes

3.3 The A_2 function

We define the A_2 function as

$$\begin{aligned} f_{A_2}(x_1 \to x_2) = \sum_{\text{skew-dihedral}} & \mathrm{Li}_{2,2}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) + \mathrm{Li}_{1,3}\left(-\frac{1}{\mathcal{X}_{i-1}}, -\frac{1}{\mathcal{X}_{i+1}}\right) + 6\,\mathrm{Li}_3\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right) \\ & - \mathrm{Li}_2\left(-\mathcal{X}_{i-1}\right)\log\left(\mathcal{X}_{i+1}\right)\left(3\log\left(\mathcal{X}_{i-1}\right) - \log\left(\mathcal{X}_i\right) + \log\left(\mathcal{X}_{i+1}\right)\right) \\ & + \frac{1}{2}\log\left(\mathcal{X}_{i-3}\right)\log\left(\mathcal{X}_i\right)\log^2\left(\mathcal{X}_{i-1}\right), \end{aligned} \tag{3.1}$$

where the $\mathcal{X}_i$ are defined in terms of x_1 and x_2 as in eq. (2.40), and the skew-dihedral sum indicates subtracting the dihedral flip ($\mathcal{X}_1 \to \mathcal{X}_{6-i}$) and taking a cyclic sum $i = 1$ to 5.

This representation of f_{A_2} differs from that in [**?**] in several key ways. Firstly, we have added classical polylogarithm terms in order to make f_{A_2} adjacent in A_2:

$$\text{symbol}(f_{A_2}) = -\sum_{\text{skew-dihedral}} [2233] + [2321] + [2332] - 2([2323] + [2343] - [2334]) \tag{3.2}$$

where we adopt the condensed notation $[ijkl] = \mathcal{X}_i \otimes \mathcal{X}_j \otimes \mathcal{X}_k \otimes \mathcal{X}_l$ in order to highlight the adjacency.

An additional benefit of this representation is that all arguments of the polylogarithms in for f_{A_2} are -$\mathcal{X}$-coordinates of A_2. Furthermore, the function is smooth and real-valued for all $x_1, x_2 > 0$. The structure of the A_2 cluster algebra plays a crucial role in this analytic behavior in the following way. $\mathrm{Li}_{2,2}(x,y)$ and $\mathrm{Li}_{1,3}(x,y)$ have branch cuts at $x = 1, y = 1, x * y = 1$. The first two branch cuts are trivially avoided as $-1/\mathcal{X}_i < 0$ for $x_1, x_2 > 0$, however the last one is avoided only because of the exchange relation for A_2:

$$0 < \left(-\frac{1}{\mathcal{X}_{i-1}}\right)\left(-\frac{1}{\mathcal{X}_{i+1}}\right) = \frac{1}{1+\mathcal{X}_i} < 1. \tag{3.3}$$

Lastly, f_{A_2} has $\Lambda^2\,\mathrm{B}_2$ and $\mathrm{B}_3 \otimes \mathbb{C}^*$ coproduct elements expressible in terms of $\mathcal{X}$-coordinates of A_2:

$$\delta\left(f_{A_2}\right) = -\sum_{\text{skew-dihedral}} \{-\mathcal{X}_{i-1}\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + 3\{-\mathcal{X}_i\}_2 \wedge \{-\mathcal{X}_{i+1}\}_2 + \frac{5}{2}\{-\mathcal{X}_i\}_3 \otimes \mathcal{X}_{i+1} \tag{3.4}$$

This representation of f_{A_2} therefore shares the following properties with $\mathcal{E}_n^{(2)}$:

- cluster adjacent,
- clustery coproduct,
- smooth and real-valued in the positive domain.

3.4 Cluster Automorphisms

3.5 Cluster Adjacency

Cluster adjacency is a property of all Steinmann-satisfying amplitudes, and was first introduced in [**?**]. The original phrasing of this property is that the symbol of all Steinmann-satisfying integrals in n-particle kinematics, when fully expanded out in terms of $\mathcal{A}$-coordinates , is of the form

$$\ldots \otimes \alpha_i \otimes \alpha_j \otimes \ldots \tag{3.5}$$

where α_i and α_j appear together in a cluster of $\mathrm{Gr}(4, n)$. This non-trivial property is a considerable constraint on the space of polylogarithm functions which can appear in amplitudes.

The original presentation of cluster adjacency was in terms of $\mathcal{A}$-coordinates , but adjacency can also be phrased in terms of $\mathcal{X}$-coordinates . We will term these as cluster $\mathcal{A}$-adjacency and cluster $\mathcal{X}$-adjacency, respectively.

The benefit of $\mathcal{A}$-adjacency is that $\mathcal{A}$-coordinates are multiplicatively independent and so any symbol in them will be unique. The same is of course not true for $\mathcal{X}$-coordinates : they satisfy numerous multiplicative identities and so there exists many equivalent representations of a given symbol in terms of $\mathcal{X}$-coordinates , and only some small subset of them may satisfy cluster $\mathcal{X}$- adjacency.

However, the benefit of $\mathcal{X}$-coordinates is that they have a unique Poisson bracket, whereas $\mathcal{A}$-coordinates can appear in many different clusters together, each time with a different value for b_{ij} connecting them. This ambiguity in the Poisson bracket for $\mathcal{A}$-coordinates is equivalent to the ambiguity introduced by the multiplicative identities in the $\mathcal{X}$-coordinates .

While $\mathcal{X}$-adjacency trivially implies $\mathcal{A}$-adjacency, the converse is not so clear. However we have checked for all Grassmannian cluster algebras $\mathrm{Gr}(k \leq 4, n \leq 7)$ that $\mathcal{A}$-adjacency implies $\mathcal{X}$-adjacency, so we conjecture that the two phrasings of cluster adjacency are identical in constraining symbol space.

4 Cluster Subalgebra-Constructibility

Define the basic notion here (which applies to both clustery and non-clustery cobrackets) (differentiate between symbol-level and cobracket-level constructibility)

4.1 MHV Amplitudes and A_2-Constructibility

Proof that the A_2 function spans the $\delta_{2,2}$ component of MHV amplitudes in Planar $\mathcal{N} = 4$.

4.2 Finite Subalgebras of Gr(4,n)

Describe/tabulate the finite cluster algebras that appear as subalgebras of Gr(4,n)

4.3 Cobracket-Level Decompositions of $R_7^{(2)}$

Method and results of general search—describe A_3 and null results in more depth, leaving a discussion of A_5 and D_5 for the next sections

4.4 Decomposing $R_n^{(2)}$ for $n \geq 8$

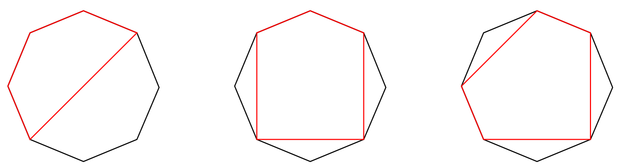
Some discussion of why this is harder, and pointing to paper II

5 The A_5 Function

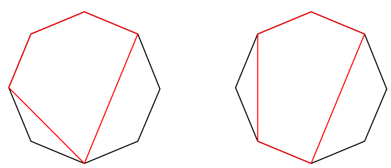
As discussed previously, there are 56 distinct A_2 subalgebras in A_5 ($56 = \binom{8}{5}$ = number of distinct pentagons inside an octagon), they can be parameterized by:

$$\begin{aligned} &\left\{ x_1 \to x_2, \quad x_2 \to x_3\left(1 + x_4\right), \quad x_2\left(1 + x_3\right) \to \frac{x_3 x_4}{1 + x_3} \right\} + \sigma_{A_5}, \\ &\left\{ x_2 \to x_3, \quad x_1\left(1 + x_2\right) \to \frac{x_2 x_3}{1 + x_2} \right\} + \sigma_{A_5} + \tau_{A_5} \end{aligned} \tag{5.1}$$

where by "$+\sigma_{A_5}$" and "$+\sigma_{A_5} + \tau_{A_5}$" I mean "+ cyclic copies" and "+ cyclic and flip copies," respectively. This correspond to the geometries



(8 of each)



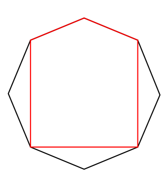
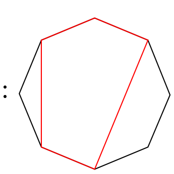
(16 of each)

The A_5 function is a sum over two of the classes of A_2 subalgebras, $x_2 \to x_3\left(1 + x_4\right)$ and $x_1\left(1 + x_2\right) \to \frac{x_2 x_3}{1 + x_2}$, appropriately antisymmetrized so that the overall f_{A_5} picks up a minus sign under both σ_{A_5} and τ_{A_5}. Explicitly, this is written

$$f_{A_5} = \sum_{i=0}^{7} \sum_{j=0}^{1} (-1)^{i+j} \sigma_{A_5}^i \tau_{A_5}^j \left(\frac{1}{2} f_{A_2}\left(x_2 \to x_3\left(1 + x_4\right)\right) + f_{A_2}\left(x_1\left(1 + x_2\right) \to \frac{x_2 x_3}{1 + x_2} \right) \right). \tag{5.2}$$

The factor of $\frac{1}{2}$ in front of $f_{A_2}\left(x_2 \to x_3\left(1 + x_4\right)\right)$ is simply a symmetry factor, as it lives in an 8-cycle of $\{\sigma_{A_5}, \tau_{A_5}\}$.

The two types of A_2's appearing in f_{A_5} are:

$x_2 \to x_3(1 + x_4)$:  $x_1\left(1 + x_2\right) \to \frac{x_2 x_3}{1+x_2}$:  (5.3)