In this note we catalog the subalgebra structure for the finite cluster algebras $\subseteq E_6$.

These algebras are: $A_2, A_3, A_4, D_4, A_5, D_5, E_6$.

## $A_2$

clusters: 5 $a$-coordinates: 5 $x$-coordinates: 10

## $A_3$

clusters: 14 $a$-coordinates: 9 $x$-coordinates: 30

| Type | Sub-polytopes | Distinct Subalgebras |
|---|---|---|
| $A_2$ | 6 | 6 |
| $A_1 \times A_1$ | 3 | 3 |

## $A_4$

clusters: 42 $a$-coordinates: 14 $x$-coordinates: 70

| Type | Sub-polytopes | Distinct Subalgebras |
|---|---|---|
| $A_2$ | 28 | 21 |
| $A_1 \times A_1$ | 28 | 28 |
| $A_3$ | 7 | 7 |
| $A_2 \times A_1$ | 7 | 7 |
| $A_1 \times A_1 \times A_1$ | 0 | 0 |

## $D_4$

clusters: 50 $a$-coordinates: 16 $x$-coordinates: 104

| Type | Sub-polytopes | Distinct Subalgebras |
|---|---|---|
| $A_2$ | 36 | 36 |
| $A_1 \times A_1$ | 30 | 18 |
| $A_3$ | 12 | 12 |
| $A_2 \times A_1$ | 0 | 0 |
| $A_1 \times A_1 \times A_1$ | 4 | 4 |

## $A_5$ clusters: 132 $a$-coordinates: 20 $x$-coordinates: 140

| Type | Sub-polytopes | Distinct Subalgebras |
|---|---|---|
| $A_2$ | 120 | 56 |
| $A_1 \times A_1$ | 180 | 144 |
| $A_3$ | 36 | 28 |
| $A_2 \times A_1$ | 72 | 72 |
| $A_1 \times A_1 \times A_1$ | 12 | 12 |
| $D_4$ | 0 | 0 |
| $A_4$ | 8 | 8 |
| $A_3 \times A_1$ | 8 | 8 |
| $A_2 \times A_2$ | 4 | 4 |
| $A_2 \times A_1 \times A_1$ | 0 | 0 |
| $A_1 \times A_1 \times A_1 \times A_1$ | 0 | 0 |

## $D_5$ clusters: 182 $a$-coordinates: 25 $x$-coordinates: 260

| Type | Sub-polytopes | Distinct Subalgebras |
|---|---|---|
| $A_2$ | 180 | 125 |
| $A_1 \times A_1$ | 230 | 145 |
| $A_3$ | 70 | 65 |
| $A_2 \times A_1$ | 60 | 50 |
| $A_1 \times A_1 \times A_1$ | 30 | 30 |
| $D_4$ | 5 | 5 |
| $A_4$ | 10 | 10 |
| $A_3 \times A_1$ | 5 | 5 |
| $A_2 \times A_2$ | 0 | 0 |
| $A_2 \times A_1 \times A_1$ | 5 | 5 |
| $A_1 \times A_1 \times A_1 \times A_1$ | 0 | 0 |

## $E_6$ clusters: 833 $a$-coordinates: 42 $x$-coordinates: 770

| Type | Sub-polytopes | Distinct Subalgebras |
|---|---|---|
| $A_2$ | 1071 | 504 |
| $A_1 \times A_1$ | 1785 | 833 |
| $A_3$ | 476 | 364 |
| $A_2 \times A_1$ | 714 | 490 |
| $A_1 \times A_1 \times A_1$ | 357 | 357 |
| $D_4$ | 35 | 35 |
| $A_4$ | 112 | 98 |
| $A_3 \times A_1$ | 112 | 112 |
| $A_2 \times A_2$ | 21 | 14 |
| $A_2 \times A_1 \times A_1$ | 119 | 119 |
| $A_1 \times A_1 \times A_1 \times A_1$ | 0 | 0 |
| $D_5$ | 14 | 14 |
| $A_5$ | 7 | 7 |
| $D_4 \times A_1$ | 0 | 0 |
| $A_4 \times A_1$ | 14 | 14 |
| $A_3 \times A_2$ | 0 | 0 |
| $A_3 \times A_1 \times A_1$ | 0 | 0 |
| $A_2 \times A_2 \times A_1$ | 7 | 7 |
| $A_2 \times A_1 \times A_1 \times A_1$ | 0 | 0 |
| $A_1 \times A_1 \times A_1 \times A_1 \times A_1$ | 0 | 0 |

## How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type $Y$ of rank $n-1$ in a finite type cluster algebra of type $X$ of rank $n$.

Let $N$ be the number of cluster $a$-coordinates in the cluster algebra of type $X$.
Let $n$ be the rank of cluster algebra of type $X$.
Let $Z$ be the number of ways to remove a node from a Dynkin diagram of type $X$ to obtain one of type $Y$.

The number of cluster algebras of type $Y$ is then $NZ/n$.

As an example, let's count the number of $A_4$'s in $A_5$:

$N = \#$ of cluster $a$-coordinates in $A_5 = 20$
$n =$ rank of $A_5 = 5$
$Z = \#$ of ways to remove a node from $A_5$ and get an $A_4 = 2$

$20 \times 2/5 = 8 \Rightarrow$ there are 8 $A_4$'s in $A_5$.