In this note we catalog the subalgebra structure for the finite cluster algebras  $\subseteq E_6$ . These algebras are:  $A_2, A_3, A_4, D_4, A_5, D_5, E_6$ .

 $\underline{A_2}$  clusters: 5 a-coordinates: 5 x-coordinates: 10

 $A_3$  clusters: 14 a-coordinates: 9 x-coordinates: 30

 $A_4$  clusters: 42 a-coordinates: 14 x-coordinates: 70

TypeSub-polytopesDistinct Subalgebras $A_2$ 2821 $A_1 \times A_1$ 2828 $A_3$ 77 $A_2 \times A_1$ 77 $A_1 \times A_1 \times A_1$ 00

 $D_4$  clusters: 50 a-coordinates: 16 x-coordinates: 104

Type Sub-polytopes Distinct Subalgebras  $\overline{A}_2$ 36 36  $A_1 \times A_1$ 30 18  $A_3$ 12 12  $A_2 \times A_1$ 0 0  $A_1 \times A_1 \times A_1$ 4

 $A_5$  clusters: 132 a-coordinates: 20 x-coordinates: 140

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	120	56
$A_1 \times A_1$	180	144
$A_3$	36	28
$A_2 \times A_1$	72	72
$A_1 \times A_1 \times A_1$	12	12
$D_4$	0	0
$A_4$	8	8
$A_3 \times A_1$	8	8
$A_2 \times A_2$	4	4
$A_2 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1$	0	0

 $\underline{D_5}$  clusters: 182 a-coordinates: 25 x-coordinates: 260

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	180	125
$A_1 \times A_1$	230	145
$A_3$	70	65
$A_2 \times A_1$	60	50
$A_1 \times A_1 \times A_1$	30	30
$D_4$	5	5
$A_4$	10	10
$A_3 \times A_1$	5	5
$A_2 \times A_2$	0	0
$A_2 \times A_1 \times A_1$	5	5
$A_1 \times A_1 \times A_1 \times A_1$	0	0

 $\underline{E_6}$  clusters: 833 a-coordinates: 42 x-coordinates: 770

Type	Sub-polytopes	Distinct Subalgebras
$A_2$	1071	504
$A_1 \times A_1$	1785	833
$A_3$	476	364
$A_2 \times A_1$	714	490
$A_1 \times A_1 \times A_1$	357	357
$D_4$	35	35
$A_4$	112	98
$A_3 \times A_1$	112	112
$A_2 \times A_2$	21	14
$A_2 \times A_1 \times A_1$	119	119
$A_1 \times A_1 \times A_1 \times A_1$	0	0
$D_5$	14	14
$A_5$	7	7
$D_4 \times A_1$	0	0
$A_4 \times A_1$	14	14
$A_3 \times A_2$	0	0
$A_3 \times A_1 \times A_1$	0	0
$A_2 \times A_2 \times A_1$	7	7
$A_2 \times A_1 \times A_1 \times A_1$	0	0
$A_1 \times A_1 \times A_1 \times A_1 \times A_1$	0	0

# Symbol Spaces on Cluster Algebras

Weight Two

Typo	Type Integrable	Cluster-Adjacent	Automorphic			
туре			$\sigma^+ \tau^+$	$\sigma^+\tau^-$	$\sigma^- \tau^+$	$\sigma^-\tau^-$
$A_2$	19	14	2	0	0	0
$A_3$	55	40	6	2	1	5
$A_4$	125	90	9	3	0	0
$D_4$	163	109				
$A_5$	245	175	1	0	0	0
$D_5$	381	241				
$E_6$	-	573	-	-	-	-

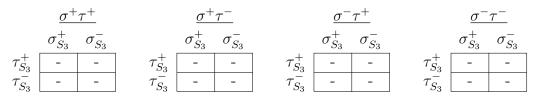
## Automorphic $D_4$ symbols

#### Automorphic $D_5$ symbols

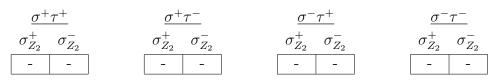
Weight Four

Type Integrable		Cluster Adiscent	Automorphic			
туре	Type Integrable	Cluster-Adjacent	$\sigma^+ \tau^+$	$\sigma^+\tau^-$	$\sigma^- \tau^+$	$\sigma^-\tau^-$
$A_2$	211	81	11	6	0	0
$A_3$	1351	432	49	29	24	42
$A_4$	-	1652	131	105	0	0
$D_4$	-	-				
$A_5$	-	-	-	-	-	-
$D_5$	-	-				
$E_6$	-	-	-	_	-	_

## Automorphic $D_4$ symbols



## Automorphic $D_5$ symbols



#### How to count co-dimension 1 & 2 subalgebras

This algorithm comes from Hugh Thomas.

Our goal is to count the number of cluster algebras of type Y of rank n-1 in a finite type cluster algebra of type X of rank n.

Let N be the number of cluster a-coordinates in the cluster algebra of type X.

Let n be the rank of cluster algebra of type X.

Let Z be the number of ways to remove a node from a Dynkin diagram of type X to obtain one of type Y.

The number of cluster algebras of type Y is then NZ/n.

As an example, let's count the number of  $A_4$ 's in  $A_5$ :

N = # of cluster a-coordinates in  $A_5 = 20$ 

 $n = \text{rank of } A_5 = 5$ 

Z=# of ways to remove a node from  $A_5$  and get an  $A_4=2$ 

 $20 \times 2/5 = 8 \Rightarrow$  there are 8  $A_4$ 's in  $A_5$ .