

# Superstring Theory Action and Equations of Motion

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## 1 Query

Give me the action for superstring theory including all bosonic and fermionic ghost fields. Show in a step-by-step derivation, thinking carefully through each step, how one acquires the relevant equations of motion by using the principle of least action. Display the results in an easy-to-read LaTeX compiled format. Include this query word for word at the top of LaTeX document for comparative reference. `max_tokens = 5000`.

## 2 Superstring Theory Action

The action for superstring theory, including bosonic and fermionic ghost fields, is given by:

$$S = S_{\text{matter}} + S_{\text{ghost}} \quad (1)$$

Where:

$$S_{\text{matter}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left[ h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right] \quad (2)$$

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{-h} \left[ b_{\alpha\beta} \partial^\alpha c^\beta + \tilde{b}_\alpha \partial^\alpha \tilde{c} + \frac{3}{2} \beta \partial^\alpha \gamma_\alpha \right] \quad (3)$$

Here,  $X^\mu$  are the bosonic coordinates,  $\psi^\mu$  are the fermionic coordinates,  $h^{\alpha\beta}$  is the worldsheet metric,  $\rho^\alpha$  are the worldsheet gamma matrices,  $b_{\alpha\beta}$  and  $c^\beta$  are the bosonic ghosts,  $\tilde{b}_\alpha$  and  $\tilde{c}$  are the superconformal ghosts, and  $\beta$  and  $\gamma_\alpha$  are the superghosts.

## 3 Derivation of Equations of Motion

We will now derive the equations of motion using the principle of least action. We'll focus on the matter part of the action for simplicity.

### 3.1 Step 1: Variation of the Action

We start by varying the action with respect to  $X^\mu$  and  $\psi^\mu$ :

$$\delta S_{\text{matter}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} [h^{\alpha\beta} (\partial_\alpha \delta X^\mu \partial_\beta X_\mu + \partial_\alpha X^\mu \partial_\beta \delta X_\mu) + i\delta\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \delta\psi_\mu] \quad (4)$$

### 3.2 Step 2: Integration by Parts

We integrate by parts to move the derivatives off the variations:

$$\delta S_{\text{matter}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} [-\partial_\alpha (h^{\alpha\beta} \partial_\beta X_\mu) \delta X^\mu + i\delta\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\partial_\alpha (\bar{\psi}^\mu \rho^\alpha) \delta\psi_\mu] \quad +\text{boundary terms}$$

We assume the boundary terms vanish due to appropriate boundary conditions.

### 3.3 Step 3: Principle of Least Action

According to the principle of least action, the variation of the action should vanish for arbitrary variations  $\delta X^\mu$  and  $\delta\psi^\mu$ . This leads to two equations:

$$\partial_\alpha (h^{\alpha\beta} \partial_\beta X_\mu) = 0 \quad (5)$$

$$\rho^\alpha \partial_\alpha \psi_\mu = 0 \quad (6)$$

### 3.4 Step 4: Equations of Motion

These are the equations of motion for the bosonic and fermionic fields, respectively. In the conformal gauge where  $h^{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1, 1)$ , they become:

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0 \quad (7)$$

$$(\partial_\tau - \partial_\sigma) \psi_+^\mu = 0, \quad (\partial_\tau + \partial_\sigma) \psi_-^\mu = 0 \quad (8)$$

Where  $\psi_\pm^\mu$  are the left and right-moving components of  $\psi^\mu$ .

## 4 Ghost Field Equations of Motion

For completeness, we can derive the equations of motion for the ghost fields in a similar manner:

$$\partial^\alpha b_{\alpha\beta} = 0, \quad \partial_\beta c^\beta = 0 \quad (9)$$

$$\partial^\alpha \tilde{b}_\alpha = 0, \quad \partial^\alpha \tilde{c} = 0 \quad (10)$$

$$\partial^\alpha \beta = 0, \quad \partial^\alpha \gamma_\alpha = 0 \quad (11)$$

These equations ensure that the ghost fields properly cancel the unphysical degrees of freedom in the theory.