18.01 Single Variable Calculus
Fall 2006

18.01 Solutions for Practice Questions (Exam 1)

1 a) By quotient rule:

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{1+2x}\right) = \frac{\frac{1}{2\sqrt{x}}(1+2x) - \sqrt{x}\cdot 2}{(1+2x)^2}$$

value at $x=1$: $\dfrac{\frac{1}{2}\cdot 3 - 1\cdot 2}{3^2} = -\dfrac{1}{18}$

b) By product rule:

$$\frac{d}{du}(u\ln 2u) = 1\cdot\ln 2u + u\cdot\frac{1}{2u}\cdot 2 = \ln 2u + 1$$

2 a) By two uses of the chain rule:

$$\frac{d}{dt}(1-k\cos^2 t)^{1/2} = \frac{1}{2}(1-k\cos^2 t)^{-1/2}(-2k\cos t)\cdot(-\sin t)$$

$$= \frac{k\cos t\sin t}{\sqrt{1-k\cos^2 t}} \quad \boxed{\text{OR}} \quad \frac{k}{2}\frac{\sin 2t}{\sqrt{1-k\cos^2 t}}$$

b) If $k=1$, $\sqrt{1-\cos^2 t} = \sin t$, so the above becomes $= \boxed{\cos t}$ which agrees with

$$\frac{d}{dt}\sqrt{1-\cos^2 t} = \frac{d}{dt}\sin t = \boxed{\cos t}$$

3

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{x^2 - (x^2+2x\Delta x+(\Delta x)^2)}{(x+\Delta x)^2\cdot x^2}\right)$$

$$= \lim_{\Delta x\to 0}\frac{-2x - \Delta x}{(x+\Delta x)^2\cdot x^2} = \frac{-2x}{x^4}$$

$$= \boxed{\frac{-2}{x^3}}$$

4 $y = \sin^{-1}x \Rightarrow x = \sin y$

Differentiating: $1 = \cos y\cdot\dfrac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

(Since $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, $\cos y \ge 0$ so use positive $\sqrt{\ }$)

5 Since $f(x)$ is differentiable, it is also continuous. Thus the two functions must have same value at $x=1$ (in the limit), and the same slope (in the limit):

	$ax+b$		x^2-3x+2
value:	$a+b$	$=$	0
slope:	a	$=$	$2x-3\big\vert_{x=1} = -1$

$\therefore \boxed{a=-1,\ b=1}$

6 a)

$$\lim_{u\to 0}\frac{\tan 2u}{u} = \lim_{u\to 0}\frac{\sin 2u}{2u}\cdot\frac{2}{\cos 2u} = 1\cdot\frac{2}{1} = 2$$

(another way:

$$= \lim_{u\to 0}\frac{\sin 2u}{u\cdot\cos 2u} \overset{\lim_{u\to 0}}{=} 2\frac{\sin u}{u}\,\frac{\cos u}{\cos 2u} = 2\cdot 1\cdot\frac{1}{1} = 2)$$

b)

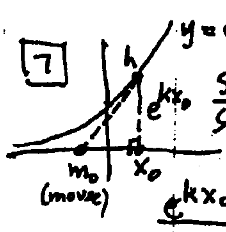
$$\frac{d}{dx}e^x\Big|_{x=0} = \lim_{h\to 0}\frac{e^{0+h}-e^0}{h} = \boxed{\lim_{h\to 0}\frac{e^h-1}{h}}$$

(can use Δx instead of h)

But $\dfrac{d}{dx}e^x\Big|_{x=0} = e^x\Big|_{x=0} = \boxed{1}$

Therefore, the limit is 1.

7



Simplest: calculate slope of tangent line two ways:

$$\frac{e^{kx_0}}{x_0-m_0} = \frac{d}{dx}e^{kx}\Big|_{x_0} = ke^{kx_0}$$

$\therefore x_0 - m_0 = \frac{1}{k}$; $\boxed{m_0 = x_0 - \frac{1}{k}}$

$\boxed{\text{OR}}$ Equation of tan. line is

$y - y_0 = \underbrace{ke^{kx_0}}_{\substack{\text{slope}\\ =\text{deriv.}}}(x-x_0)$; $\quad y_0 = e^{kx_0}$

The x-intercept m_0 is where $y=0$

$\therefore -e^{kx_0} = ke^{kx_0}(m_0-x_0)$ $\quad$ ($x=m_0$ there)

so $-\frac{1}{k} = m_0 - x_0$, $\boxed{m_0 = x_0 - \frac{1}{k}}$