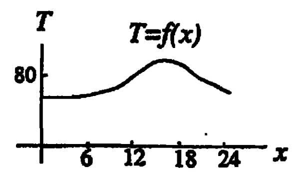
18.01 Single Variable Calculus
Fall 2006

# AV. AVERAGE VALUE

What was the average temperature on July 4 in Boston?




The temperature is a continuous function $f(x)$, whose graph over the 24-hour period might look as shown. How should we define the average value of such a function over the time interval $[0, 24]$ — measuring time $x$ in hours, with $x = 0$ at 12:00AM ?

We could observe the temperature in the middle of every hour, that is, at the times $x_1 = .5, x_2 = 1.5, \ldots, x_{24} = 23.5$, then average these 24 observations, getting

$$\frac{1}{24}\sum_{i=1}^{24} f(x_i)\,.$$

To get a more accurate answer, we could average measurements made more frequently, say every ten minutes.

For a general interval $[a, b]$ and function $f(x)$, the analogous procedure would be to divide up the interval into $n$ equal parts, each of length

https://math.stackexchange.com/questions/4821167/does-the-transformed-term-has-meaning-in-calculus-average-value

(1) $$\Delta x = \frac{b-a}{n}$$

and average the values of the function $f(x)$ at a succession of points $x_i$, where $x_i$ lies in the $i$-th interval. Then we ought to have

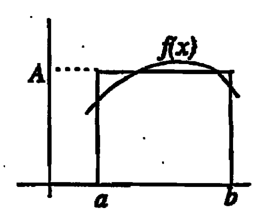
(2) $$\text{average of } f(x) \text{ over } [a,b] \approx \frac{1}{n}\sum_{1}^{n} f(x_i)\,.$$

We can relate the sum on the right to a definite integral: using (1), (2) becomes

(3) $$\text{average of } f(x) \text{ over } [a,b] \approx \frac{1}{b-a}\sum_{1}^{n} f(x_i)\Delta x\,.$$

As $n \to \infty$, the sum on the right-hand side of (3) approaches the definite integral of $f(x)$ over $[a, b]$, and we therefore define the **average value of the function** $f(x)$ on $[a, b]$ by

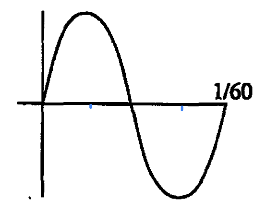
(4) $$A = \text{average of } f(x) \text{ over } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx\,.$$




Geometrically, the average value $A$ can be thought of as the height of that constant function $A$ which has the same area over $[a, b]$ as $f(x)$ does. This is so since (4) shows that

$$A \cdot (b-a) = \int_a^b f(x)\,dx\,.$$

**Example 1.** In alternating current, voltage is represented by a sine wave with a frequency of 60 cycles/second, and a peak of 120 volts. What is the average voltage?

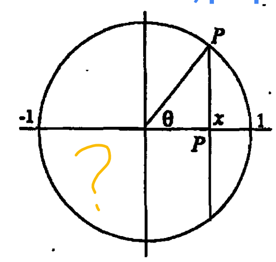



**Solution.** The voltage function has frequency $\dfrac{2\pi}{\text{period}} = \dfrac{2\pi}{1/60} = 120\pi$, and amplitude 120, so it is given by $V(t) = 120\sin(120\pi t)$. Thus by (4),

$$\text{average } V(t) = 120\int_0^{1/120} V(t)\,dt = -\frac{120}{\pi}\cos(120\pi t)\Big]_0^{1/120} = \frac{2}{\pi}\cdot 120\,.$$

forming your function based on previously learned function, property

(We integrate over $[0, 1/120]$ rather than $[0, 1/60]$ since we don't want zero as the average.)




**Example 2.**

a) A point is chosen at random on the $x$-axis between $-1$ and 1; call it $P$. What is the average length of the vertical chord to the unit circle passing through $P$?

b) Same question, but now the point $P$ is chosen at random on the circumference.

**Solution.** a) If $P$ is at $x$, the chord has length $2\sqrt{1-x^2}$, so we get

$$\text{average of } 2\sqrt{1-x^2} \text{ over } [-1,1] \;=\; \frac{1}{2}\int_{-1}^{1} 2\sqrt{1-x^2}\,dx \;=\; \text{area of semicircle} \;=\; \frac{\pi}{2} \approx 1.6\,.$$

b) By symmetry, we can suppose $P$ is on the upper semicircle. If $P$ is at the angle $\theta$, the chord has length $2\sin\theta$, so this time we get

$$\text{average of } 2\sin\theta \text{ over } [0,\pi] \;=\; \frac{1}{\pi}\int_0^{\pi} 2\sin\theta\,d\theta \;=\; -\frac{2}{\pi}\cos\theta\Big]_0^{\pi} \;=\; \frac{4}{\pi} \approx 1.3\,.$$

(Intuitively, can you see why the average in part (b) should be less than the average in part (a) — could you have predicted this would be so?)

## Exercises: Section 4D