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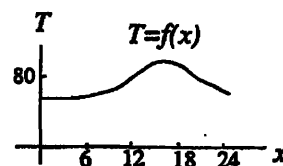
18.01 Single Variable Calculus  
Fall 2006

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## AV. AVERAGE VALUE

What was the average temperature on July 4 in Boston?

The temperature is a continuous function  $f(x)$ , whose graph over the 24-hour period might look as shown. How should we define the average value of such a function over the time interval  $[0, 24]$  — measuring time  $x$  in hours, with  $x = 0$  at 12:00AM?



We could observe the temperature in the middle of every hour, that is, at the times  $x_1 = .5, x_2 = 1.5, \dots, x_{24} = 23.5$ , then average these 24 observations, getting

$$\frac{1}{24} \sum_{i=1}^{24} f(x_i).$$

To get a more accurate answer, we could average measurements made more frequently, say every ten minutes.

For a general interval  $[a, b]$  and function  $f(x)$ , the analogous procedure would be to divide up the interval into  $n$  equal parts, each of length

$$(1) \quad \Delta x = \frac{b-a}{n}$$

<https://math.stackexchange.com/questions/4821167/does-the-transformed-term-has-meaning-in-calculus-average-value>

and average the values of the function  $f(x)$  at a succession of points  $x_i$ , where  $x_i$  lies in the  $i$ -th interval. Then we ought to have

$$(2) \quad \text{average of } f(x) \text{ over } [a, b] \approx \frac{1}{n} \sum_{i=1}^n f(x_i).$$

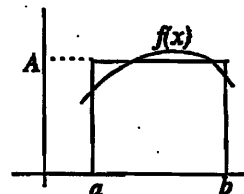
We can relate the sum on the right to a definite integral: using (1), (2) becomes

$$(3) \quad \text{average of } f(x) \text{ over } [a, b] \approx \frac{1}{b-a} \sum_{i=1}^n f(x_i) \Delta x.$$

As  $n \rightarrow \infty$ , the sum on the right-hand side of (3) approaches the definite integral of  $f(x)$  over  $[a, b]$ , and we therefore define the average value of the function  $f(x)$  on  $[a, b]$  by

$$(4) \quad A = \text{average of } f(x) \text{ over } [a, b] = \frac{1}{b-a} \int_a^b f(x) dx.$$

Geometrically, the average value  $A$  can be thought of as the height of that constant function  $A$  which has the same area over  $[a, b]$  as  $f(x)$  does. This is so since (4) shows that



$$A \cdot (b-a) = \int_a^b f(x) dx.$$

**Example 1.** In alternating current, voltage is represented by a sine wave with a frequency of 60 cycles/second, and a peak of 120 volts. What is the average voltage?

**Solution.** The voltage function has frequency  $\frac{2\pi}{\text{period}} = \frac{2\pi}{1/60} = 120\pi$ , and amplitude 120, so it is given by  $V(t) = 120 \sin(120\pi t)$ . Thus by (4),

$$\text{average } V(t) = 120 \int_0^{1/120} V(t) dt = -\frac{120}{\pi} \cos(120\pi t) \Big|_0^{1/120} = \frac{2}{\pi} \cdot 120.$$

(We integrate over  $[0, 1/120]$  rather than  $[0, 1/60]$  since we don't want zero as the average.)

**Example 2.** its how you think of it

a) A point is chosen at random on the  $x$ -axis between  $-1$  and  $1$ ; call it  $P$ . What is the average length of the vertical chord to the unit circle passing through  $P$ ?

b) Same question, but now the point  $P$  is chosen at random on the circumference.

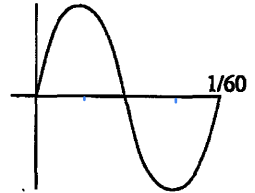
**Solution.** a) If  $P$  is at  $x$ , the chord has length  $2\sqrt{1-x^2}$ , so we get

$$\text{average of } 2\sqrt{1-x^2} \text{ over } [-1, 1] = \frac{1}{2} \int_{-1}^1 2\sqrt{1-x^2} dx = \text{area of semicircle} = \frac{\pi}{2} \approx 1.6.$$

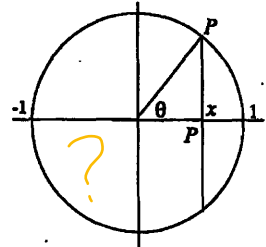
b) By symmetry, we can suppose  $P$  is on the upper semicircle. If  $P$  is at the angle  $\theta$ , the chord has length  $2 \sin \theta$ , so this time we get

$$\text{average of } 2 \sin \theta \text{ over } [0, \pi] = \frac{1}{\pi} \int_0^\pi 2 \sin \theta d\theta = -\frac{2}{\pi} \cos \theta \Big|_0^\pi = \frac{4}{\pi} \approx 1.3.$$

(Intuitively, can you see why the average in part (b) should be less than the average in part (a) — could you have predicted this would be so?)



forming your function based on previously learned function, property



### Exercises: Section 4D