

## V7. Laplace's Equation and Harmonic Functions

In this section, we will show how Green's theorem is closely connected with solutions to Laplace's partial differential equation in two dimensions:

$$(1) \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

where  $w(x, y)$  is some unknown function of two variables, assumed to be twice differentiable. Equation (1) models a variety of physical situations, as we discussed in Section P of these notes, and shall briefly review.

### 1. The Laplace operator and harmonic functions.

The two-dimensional **Laplace operator**, or *laplacian* as it is often called, is denoted by  $\nabla^2$  or *lap*, and defined by

$$(2) \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The notation  $\nabla^2$  comes from thinking of the operator as a sort of symbolic scalar product:

$$\nabla^2 = \nabla \cdot \nabla = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In terms of this operator, Laplace's equation (1) reads simply

$$(1') \quad \nabla^2 w = 0.$$

Notice that the laplacian is a linear operator, that is it satisfies the two rules

$$(3) \quad \nabla^2(u + v) = \nabla^2 u + \nabla^2 v, \quad \nabla^2(cu) = c(\nabla^2 u),$$

for any two twice differentiable functions  $u(x, y)$  and  $v(x, y)$  and any constant  $c$ .

**Definition.** A function  $w(x, y)$  which has continuous second partial derivatives and solves Laplace's equation (1) is called a **harmonic** function.

In the sequel, we will use the Greek letters  $\phi$  and  $\psi$  to denote harmonic functions; functions which aren't assumed to be harmonic will be denoted by Roman letters  $f, g, u, v$ , etc.. According to the definition,

$$(4) \quad \phi(x, y) \text{ is harmonic} \Leftrightarrow \nabla^2 \phi = 0.$$

By combining (4) with the rules (3) for using Laplace operator, we see

$$(5) \quad \phi \text{ and } \psi \text{ harmonic} \Rightarrow \phi + \psi \text{ and } c\phi \text{ are harmonic (} c \text{ constant)}.$$

**Examples of harmonic functions.** Here are some examples of harmonic functions. The verifications are left to the Exercises.

**A. Harmonic homogeneous polynomials<sup>1</sup> in two variables.**

Degree 0: all constants  $c$  are harmonic.

Degree 1: all linear polynomials  $ax + by$  are harmonic.

Degree 2: the quadratic polynomials  $x^2 - y^2$  and  $xy$  are harmonic; all other harmonic homogeneous quadratic polynomials are linear combinations of these:

$$\phi(x, y) = a(x^2 - y^2) + bxy, \quad (a, b \text{ constants}).$$

Degree  $n$ : the real and imaginary parts of the complex polynomial  $(x + iy)^n$  are harmonic. (Check this against the above when  $n = 2$ .)

**B. Functions with radial symmetry.** Letting  $r = \sqrt{x^2 + y^2}$ , the function given by  $\phi(r) = \ln r$  is harmonic, and its constant multiples  $c \ln r$  are the only harmonic functions with radial symmetry, i.e., of the form  $f(r)$ .

**C. Exponentially growing or decaying oscillations.** For all  $k$  the functions  $e^{kx} \sin ky$  and  $e^{kx} \cos ky$  are harmonic.

In general, harmonic functions cannot be written down explicitly in terms of elementary functions. Nevertheless, we will be able to prove things about them, by using Green's theorem.

**2. Harmonic functions and vector fields.**

The relation between harmonic functions and vector fields rests on the simple identity

$$(6) \quad \operatorname{div} \nabla f = \nabla^2 f,$$

which is easily verified, since

$$\operatorname{div} (f_x \mathbf{i} + f_y \mathbf{j}) = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \nabla^2 f;$$

its truth is suggested symbolically by

$$\operatorname{div} \nabla f = \nabla \cdot (\nabla f) = (\nabla \cdot \nabla) f = \nabla^2 f.$$

There is an important connection between harmonic functions and conservative fields which follows immediately from (6):

$$(7) \quad \text{Let } \mathbf{F} = \nabla f. \text{ Then } \operatorname{div} \mathbf{F} = 0 \Leftrightarrow f \text{ is harmonic.}$$

Another way to put this is to say: in a simply-connected region,

$$(7') \quad \operatorname{curl} \mathbf{F} = 0 \text{ and } \operatorname{div} \mathbf{F} = 0 \Leftrightarrow \mathbf{F} = \nabla \phi. \text{ where } \phi \text{ is harmonic.}$$

This is just (7), combined with the criterion for gradient fields (Section V5, X).

In other words, from the vector field viewpoint, the theory of harmonic functions and Laplace's equation is the same as the theory of conservative vector fields with zero divergence. Where do such functions and fields occur?

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<sup>1</sup>A homogeneous polynomial in several variables is one in which all the terms have the same total degree, like  $x^2y + 2y^3$  or  $x^5 - 6x^2y^3 + 4xy^4$ .

One place is in heat flow problems. Imagine a thin uniform metal plate which is insulated on the faces so no heat can enter or escape on the faces, and imagine that some temperature distribution is maintained along the edge of the plate. Then when the temperature distribution on the plate has reached steady-state, it will be given by a harmonic function  $\phi(x, y)$ ; namely, it must satisfy the heat equation (see Section P of these notes):  $\phi_{xx} + \phi_{yy} = a^2 \phi_t$ , but  $\phi_t = 0$  since the temperature is not changing with time, by assumption.

Harmonic functions also occur as the potential functions for two-dimensional gravitational, electrostatic, and electromagnetic fields, in regions of space which are respectively free of mass, static charge, or moving charges. (Here, "two-dimensional" means not that the fields lie in the  $xy$ -plane, but rather that as fields in three-space, the vectors all lie in horizontal planes, and the field looks the same no matter what horizontal plane it is viewed in. A typical example would be the field arising from a uniform mass or charge distribution on a set of vertical wires, or from uniform currents on vertical wires.)

### 3. Boundary-value problems.

As the example given above of a temperature distribution on a uniform insulated metal plate suggests, the typical problem in solving Laplace's equation would be to find a harmonic function satisfying given boundary conditions.

That is, we are given a region  $R$  of the  $xy$ -plane, bounded by a simple closed curve  $C$ . The problem is to find a function  $\phi(x, y)$  which is defined and harmonic on  $R$ , and which takes on prescribed boundary values along the curve  $C$ .

The boundary values are commonly given in one of two ways:

- (i) as the values of  $\phi$  along  $C$ ;
- (ii) as the values of the *normal derivative*  $\frac{\partial \phi}{\partial \eta}$  of  $\phi$  along  $C$ .

To explain this last, the **normal derivative** is just the directional derivative in the direction of the (outward-pointing) unit normal vector  $\mathbf{n}$ :

$$(8) \quad \frac{\partial \phi}{\partial \eta} = \left. \frac{d\phi}{ds} \right|_{\mathbf{n}} = \nabla \phi \cdot \mathbf{n} \quad (\text{normal derivative})$$

The *tangential derivative* is defined similarly, using the unit tangent vector  $\mathbf{t}$  instead of  $\mathbf{n}$ .

For heat flow problems, boundary values of the first type (i) would be most common — you are maintaining a definite temperature distribution  $\phi$  along  $C$  and want to know what the temperature will look like in  $R$ .

For conservative force field problems, with  $\mathbf{F} = \nabla \phi$ , one could also get boundary values of the second type (ii). For example, if you were given the field vector  $\mathbf{F}$  at each point of  $C$ , then you would know  $\nabla \phi \cdot \mathbf{n}$  and  $\nabla \phi \cdot \mathbf{t}$  — the normal derivative and the tangential derivative — at each point of  $C$ . Knowing the tangential derivative however is equivalent to knowing  $\phi$  itself on  $C$ , for

$$(9) \quad \left. \frac{d\phi}{ds} \right|_{\mathbf{t}} = \frac{d\phi(s)}{ds} \quad (s = \text{arclength along } C),$$

and therefore  $\phi(s)$  can be obtained by integrating the tangential derivative. So, to prescribe  $\mathbf{F}$  on the boundary is equivalent to prescribing both (i) and (ii) above for its potential function.

The basic problems are now these:

**A. Existence.** Does there exist a  $\phi(x, y)$  harmonic in some region containing  $C$  and its interior  $R$ , and taking on the prescribed boundary values?

**B. Uniqueness.** If it exists, is there only one such  $\phi(x, y)$ ?

**C. Solving.** If there is a unique  $\phi(x, y)$ , determine it by some explicit formula, or approximate it by some numerical method.

We shall now show how Green's theorem sheds some light on both the existence and the uniqueness questions.

#### 4. Existence and uniqueness for harmonic functions.

In general, if the curve  $C$  is reasonable (sufficiently smooth, of finite length, and not too wiggly), the values of  $\phi$  on the boundary can be prescribed more or less arbitrarily as long as they form a twice differentiable function on  $C$ . It can then be proved that the harmonic function  $\phi$  taking on those boundary values will exist in the interior of  $C$ .

This is not so however for the second type of boundary condition, which cannot be prescribed arbitrarily, as the following theorem shows; its proof uses Green's theorem in the normal (flux) form.

**Theorem 1.** *If  $\phi$  exists and is harmonic everywhere inside the closed curve  $C$  bounding the region  $R$ , then*

$$(10) \quad \oint_C \frac{\partial \phi}{\partial \eta} ds = 0.$$

**Proof.** We use (8), then Green's theorem in the normal form:

$$\oint_C \frac{\partial \phi}{\partial \eta} ds = \oint_C \nabla \phi \cdot \mathbf{n} ds = \iint_R \operatorname{div} (\nabla \phi) dA = 0;$$

the double integral is zero since  $\phi$  is harmonic (cf. (7)). □

One can think of the theorem as a “non-existence” theorem, since it gives condition under which no harmonic  $\phi$  can exist. For example, if  $C$  is the unit circle, and the normal derivative is prescribed to be 1 everywhere on  $C$ , then no harmonic  $\phi$  can exist satisfying this condition, since the integral in (10) will have the value  $2\pi$ , not 0.

As far as *uniqueness* goes, physical considerations suggest that if a harmonic function exists in  $R$  having given values on the boundary curve  $C$ , it should be unique. Namely, if the given temperature distribution is maintained on  $C$ , then the corresponding temperature distribution inside will approach a unique steady-state as  $t \rightarrow \infty$ .

This argument however assumes that our model of heat flow is complete, i.e., that Laplace's equation is all that determines the heat flow. But maybe there are some other conditions we don't know about and it is these that make the solution unique.

We will prove the uniqueness of  $\phi$  by a purely mathematical argument. It depends on the following theorem.

**Theorem 2. Green's first identity.** *If  $\phi$  is harmonic in a region containing  $R$ , and  $f(x, y)$  is continuously differentiable in  $R$ , then*

$$(11) \quad \oint_C f \frac{\partial \phi}{\partial \eta} ds = \iint_R \nabla f \cdot \nabla \phi dA .$$

**Proof.** As before, we use first (8), then Green's theorem in normal form:

$$(12) \quad \oint_C f \frac{\partial \phi}{\partial \eta} ds = \oint_C (f \nabla \phi) \cdot \mathbf{n} ds = \iint_R \operatorname{div} (f \nabla \phi) dA .$$

$$\begin{aligned} \operatorname{div} (f \nabla \phi) &= (f \phi_x)_x + (f \phi_y)_y = f_x \phi_x + f_y \phi_y + f(\phi_{xx} + \phi_{yy}) \\ &= \nabla f \cdot \nabla \phi, \quad \text{since } \phi_{xx} + \phi_{yy} = 0 . \end{aligned}$$

Substituting this into the double integral in (12) gives us (11). □

The essential step in proving the uniqueness of  $\phi$  is to prove it when the prescribed boundary values are 0. We consider both types of boundary values.

**Theorem 3.** *Let  $\phi$  be harmonic in a region containing  $R$ . Then*

$$(13) \quad \phi = 0 \text{ on } C \Rightarrow \phi = 0 \text{ on } R ;$$

$$(14) \quad \frac{\partial \phi}{\partial \eta} = 0 \text{ on } C \Rightarrow \phi = c \text{ on } R \quad (c \text{ is a constant}).$$

**Proof.** We use Green's first identity (11), taking  $f = \phi$ . This gives

$$(15) \quad \oint_C \phi \frac{\partial \phi}{\partial \eta} ds = \iint_R |\nabla \phi|^2 dA .$$

$$\begin{aligned} \phi = 0 \text{ or } \frac{\partial \phi}{\partial \eta} = 0 \text{ on } C &\Rightarrow \oint_C \phi \frac{\partial \phi}{\partial \eta} ds = 0 \\ &\Rightarrow \iint_R |\nabla \phi|^2 dA = 0, \quad \text{by (15);} \\ &\Rightarrow |\nabla \phi|^2 = 0 \text{ everywhere in } R, \end{aligned}$$

since it is continuous and  $\geq 0$  everywhere, being a square;

$$\begin{aligned} &\Rightarrow \nabla \phi = 0 \text{ in } R, \text{ since its magnitude is } 0; \\ &\Rightarrow \phi_x = 0 \text{ and } \phi_y = 0 \text{ in } R, \\ &\Rightarrow \phi = c \text{ in } R. \end{aligned}$$

This proves (14); it also proves (13), for in this case we know that since  $\phi = 0$  on the boundary  $C$ , the constant  $c$  must be 0. □

**Corollary. Uniqueness Theorem.**

Let  $\phi$  and  $\psi$  be two functions harmonic in a region containing  $R$ .

$$(16) \quad \phi = \psi \text{ on } C \Rightarrow \phi = \psi \text{ on } R ;$$

$$(17) \quad \frac{\partial \phi}{\partial \eta} = \frac{\partial \psi}{\partial \eta} \text{ on } C \Rightarrow \phi = \psi + c \text{ on } R, \quad c \text{ constant} .$$

**Proof.** Consider the difference  $\phi - \psi$ . It is a harmonic function, by (5). The two hypotheses in (16) and (17) say respectively that

$$\phi - \psi = 0 \text{ on } C, \quad \text{or} \quad \frac{\partial(\phi - \psi)}{\partial \eta} = 0 \text{ on } C .$$

Therefore, by theorem 3, we conclude respectively that

$$\phi - \psi = 0 \text{ on } R, \quad \text{or} \quad \phi - \psi = c \text{ on } R;$$

these are respectively the conclusions of (16) and (17). □

**Exercises: Section 4I**