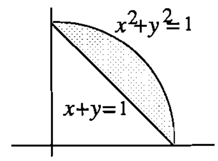
I. Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to put in the limits of integration. Fortunately, a fairly uniform procedure is available which works in any coordinate system. *You must always begin by sketching the region; in what follows we'll assume you've done this.*

1. Double integrals in rectangular coordinates.



Let's illustrate this procedure on the first case that's usually taken up: double integrals in rectangular coordinates. Suppose we want to evaluate over the region R pictured the integral

$$\iint_R f(x,y)\,dy\,dx\ , \qquad\qquad R = \ \text{region between } x^2+y^2=1 \quad \text{and} \quad x+y=1\ ;$$

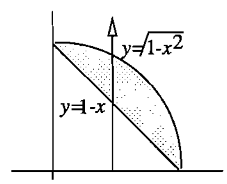
we are integrating first with respect to y. Then to put in the limits,

1. Hold x fixed, and let y increase (since we are integrating with respect to y). As the point (x, y) moves, it traces out a vertical line.

2. Integrate from the y-value where this vertical line enters the region R, to the y-value where it leaves R.

3. Then let x increase, integrating from the lowest x-value for which the vertical line intersects R, to the highest such x-value.

Carrying out this program for the region R pictured, the vertical line enters R where $y = 1 - x$, and leaves where $y = \sqrt{1-x^2}$.



The vertical lines which intersect R are those between $x = 0$ and $x = 1$. Thus we get for the limits:

$$\iint_R f(x,y)\,dy\,dx \ = \ \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

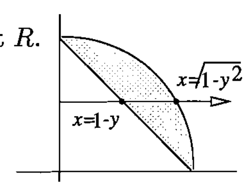
To calculate the double integral, integrating in the reverse order $\iint_R f(x,y)\,dx\,dy$,

1. Hold y fixed, let x increase (since we are integrating first with respect to x). This traces out a horizontal line.

2. Integrate from the x-value where the horizontal line enters R to the x-value where it leaves.

3. Choose the y-limits to include all of the horizontal lines which intersect R.

Following this prescription with our integral we get:



$$\iint_R f(x,y)\,dx\,dy \ = \ \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Exercises: 3A-2

2. Double integrals in polar coordinates

The same procedure for putting in the limits works for these integrals also. Suppose we want to evaluate over the same region R as before

$$\iint_R dr\,d\theta\ .$$

As usual, we integrate first with respect to r. Therefore, we

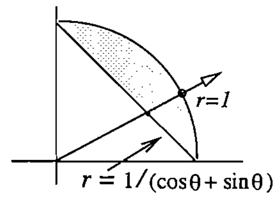
1. Hold θ fixed, and let r increase (since we are integrating with respect to r). As the point moves, it traces out a ray going out from the origin.

2. Integrate from the r-value where the ray enters R to the r-value where it leaves. This gives the limits on r.

3. Integrate from the lowest value of θ for which the corresponding ray intersects R to the highest value of θ.

To follow this procedure, we need the equation of the line in polar coordinates. We have

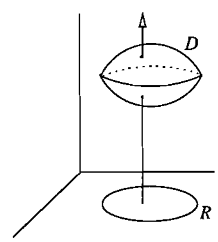
$$x+y=1 \qquad \to \qquad r\cos\theta + \mathbf{r}\sin\theta = 1, \qquad \text{or} \quad r \ = \ \frac{1}{\cos\theta+\sin\theta}.$$



This is the r value where the ray enters the region; it leaves where $r=1$. The rays which intersect R lie between $\theta=0$ and $\theta=\pi/2$. Thus the double iterated integral in polar coordinates has the limits

$$\int_0^{\pi/2}\int_{1/(\cos\theta+\sin\theta)}^{1} dr\,d\theta\ .$$

Exercises: 3B-1



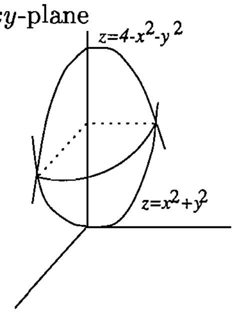
3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for $\iiint_D dz\,dy\,dx$ over the region D, we integrate first with respect to z. Therefore we

1. Hold x and y fixed, and let z increase. This gives us a vertical line.

2. Integrate from the z-value where the vertical line enters the region D to the z-value where it leaves D.

3. Supply the remaining limits (in either xy-coordinates or polar coordinates) so that you include all vertical lines which intersect D. This means that you will be integrating the remaining double integral over the region R in the xy-plane which D projects onto.



For example, if D is the region lying between the two paraboloids

$$z=x^2+y^2 \qquad\qquad z=4-x^2-y^2,$$

we get by following steps 1 and 2,

$$\iiint_D dz\,dy\,dx \ = \ \iint_R \int_{x^2+y^2}^{4-x^2-y^2} dz\,dA$$

where R is the projection of D onto the xy-plane. To finish the job, we have to determine what this projection is. From the picture, what we should determine is the xy-curve over which the two surfaces intersect. We find this curve by eliminating z from the two equations, getting

$$\begin{aligned} x^2+y^2 &= 4-x^2-y^2, \qquad \text{which implies} \\ x^2+y^2 &= 2. \end{aligned}$$

Thus the xy-curve bounding R is the circle in the xy-plane with center at the origin and radius $\sqrt{2}$.

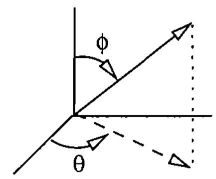
This makes it natural to finish the integral in polar coordinates. We get

$$\iiint_D dz\,dy\,dx \;=\; \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{x^2+y^2}^{4-x^2-y^2} dz\,r\,dr\,d\theta \;;$$

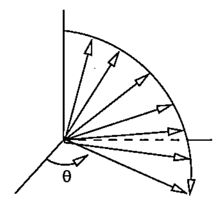
the limits on z will be replaced by r^2 and $4-r^2$ when the integration is carried out.

Exercises: 5A-2

4. Spherical coordinates.



Once again, we use the same procedure. To calculate the limits for an iterated integral $\iiint_D d\rho\,d\phi\,d\theta$ over a region D in 3-space, we are integrating first with respect to ρ. Therefore we



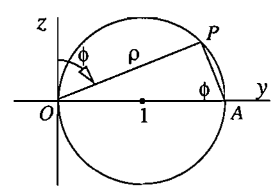
1. Hold ϕ and θ fixed, and let ρ increase. This gives us a ray going out from the origin.

2. Integrate from the ρ-value where the ray enters D to the ρ-value where the ray leaves D. This gives the limits on ρ.

3. Hold θ fixed and let ϕ increase. This gives a family of rays, that form a sort of fan. Integrate over those ϕ-values for which the rays intersect the region D.

4. Finally, supply limits on θ so as to include all of the fans which intersect the region D.

For example, suppose we start with the circle in the yz-plane of radius 1 and center at $(1,0)$, rotate it about the z-axis, and take D to be that part of the resulting solid lying in the first octant.



First of all, we have to determine the equation of the surface formed by the rotated circle. In the yz-plane, the two coordinates ρ and ϕ are indicated. To see the relation between them when P is on the circle, we see that also angle $OAP=\phi$, since both the angle ϕ and OAP are complements of the same angle, AOP. From the right triangle, this shows the relation is $\rho = 2\sin\phi$.

As the circle is rotated around the z-axis, the relationship stays the same, so $\rho=2\sin\phi$ is the equation of the whole surface.

To determine the limits of integration, when ϕ and θ are fixed, the correpsonding ray enters the region where $\rho=0$ and leaves where $\rho=2\sin\phi$.

As ϕ increases, with θ fixed, it is the rays between $\phi=0$ and $\phi=\pi/2$ that intersect D, since we are only considering the portion of the surface lying in the first octant (and thus above the xy-plane).

Again, since we only want the part in the first octant, we only use θ values from 0 to $\pi/2$. So the iterated integral is

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2\sin\phi} d\rho\, d\phi\, d\theta \ .$$

Exercises: 5B-1