V3. Two-dimensional Flux

In this section and the next we give a different way of looking at Green's theorem which both shows its significance for flow fields and allows us to give an intuitive physical meaning for this rather mysterious equality between integrals.

We have seen that if $\mathbf{F}$ is a force field and C a directed curve, then

$$(1) \qquad \text{work done by } \mathbf{F} \text{ along } C \;=\; \int_C \mathbf{F} \cdot d\mathbf{r} \;=\; \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

In words, we are integrating $\mathbf{F} \cdot \mathbf{T}$, the *tangential component* of $\mathbf{F}$, along the curve C. In component notation, if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, then the above reads

$$(2) \qquad \text{work} \;=\; \int_C M\,dx + N\,dy \;=\; \int_{t_0}^{t_1}\left(M\frac{dx}{dt} + N\frac{dy}{dt}\right) dt \ .$$

Analogously now, we may integrate $\mathbf{F} \cdot \mathbf{n}$, the *normal component* of $\mathbf{F}$ along C. To describe this, suppose the curve C is parametrized by the arclength s, increasing in the positive direction on C. The position vector for this parametrization and its corresponding tangent vector are given respectively by

$$\mathbf{r}(s) \;=\; x(s)\,\mathbf{i} + y(s)\,\mathbf{j}, \qquad\qquad \mathbf{t}(s) \;=\; \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j};$$

where we have used $\mathbf{t}$ instead of $\mathbf{T}$ since it is a unit vector— its length is 1, as one can see by dividing through by ds on both sides of

$$ds = \sqrt{(dx)^2 + (dy)^2} \ .$$

The unit normal vector $\mathbf{n}$ is the one shown in the picture, obtained by rotating $\mathbf{t}$ clockwise through a right angle.
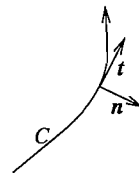
Unfortunately, this direction is opposite to the one customarily used in kinematics, where $\mathbf{t}$ and $\mathbf{n}$ form a right-handed coordinate system for motion along C. The choice of $\mathbf{n}$ depends therefore on the context of the problem; the choice we have given is the most natural for applying Green's theorem to flow problems.

The usual formula for rotating a vector clockwise by 90° (see the figure) shows that

$$(3) \qquad \mathbf{n}(s) \;=\; \frac{dy}{ds}\,\mathbf{i} - \frac{dx}{ds}\,\mathbf{j}.$$
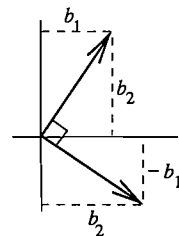
The line integral over C of the normal component $\mathbf{F} \cdot \mathbf{n}$ of the vector field $\mathbf{F}$ is called the **flux of F across** C. In symbols,
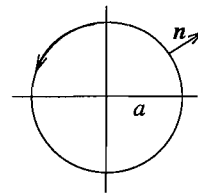
$$(4) \qquad \text{flux of } \mathbf{F} \text{ across } C \;=\; \int_C \mathbf{F} \cdot \mathbf{n}\, ds \;=\; \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds$$

In the notation of differentials, using (3) we write $\mathbf{n}\,ds = dy\,\mathbf{i} - dx\,\mathbf{j}$, so that

$$(5) \qquad \text{flux of } F \text{ across } C \;=\; \int_C M\,dy - N\,dx \;=\; \int_C \left(M\frac{dy}{dt} - N\frac{dx}{dt}\right) dt \ ,$$

where $x(t)$, $y(t)$ is any parametrization of C. We will need both (4) and (5).

Example 1. Calculate the flux of the field $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$ across a circle of radius a and center at the origin, by   a) using (4);   b) using (5).

Solutions. a) The field is directed radially outward, so that $\mathbf{F}$ and $\mathbf{n}$ have the same direction. (As usual, the circle is directed counterclockwise, which means that $\mathbf{n}$ points outward.) Therefore, at each point of the circle,

$$\mathbf{F}\cdot\mathbf{n} \;=\; |\mathbf{F}| \;=\; \frac{1}{\sqrt{x^2 + y^2}} \;=\; \frac{1}{a}\;.$$

Therefore, by (4), we get

$$\text{flux} \;=\; \oint_C \mathbf{F}\cdot\mathbf{n}\,ds \;=\; \oint_C \frac{1}{a}\,ds \;=\; 2\pi\;.$$

b) We can also get the same result by straightforward computation using a parametrization of the circle: $x = \cos t$, $y = \sin t$. Using this and (5) above,

$$\text{flux} \;=\; \oint_C \frac{x\,dy - y\,dx}{x^2 + y^2} \;=\; \int_0^{2\pi} \frac{a^2 \cos^2 t + a^2 \sin^2 t}{a^2}\,dt \;=\; 2\pi\;.$$

The natural physical interpretation for flux calls for thinking of $\mathbf{F}$ as representing a two-dimensional flow field (see section V1). Then the line integral represents the *rate with respect to time at which mass is being transported across* C. (We think of the flow as taking place in a shallow tank of unit depth. The convention about $\mathbf{n}$ makes this mass-transport rate positive if the flow is from left to right as you face in the positive direction on C, and negative in the other case.)

To see this, we follow the same procedure that was used to interpret the tangential integral in a force field as work.

The essential step to see is that if $\mathbf{F}$ is a constant vector field representing a flow, and C is a directed line segment of length L, then

(6)   mass-transport rate across C $=$ $(\mathbf{F}\cdot\mathbf{n})\,L$

To see this, resolve the flow field into its components parallel to C and perpendicular to C. The component parallel to C contributes nothing to the flow rate *across* C, while the component perpendicular to C is $\mathbf{F}\cdot\mathbf{n}$.

Another way to se (6) is illustrated at the right. Letting C' be as shown, we see by conservation of mass that

$$\begin{aligned}
\text{mass-transport rate across } C \;&=\; \text{mass-transport rate across } C' \\
&=\; |\mathbf{F}|(L\cos\theta) \\
&=\; (\mathbf{F}\cdot\mathbf{n})\,L\;.
\end{aligned}$$
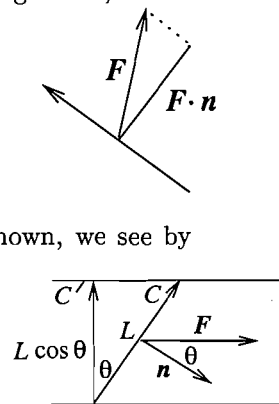
Once we have this, we follow the same procedure used to define work as a line integral. We divide up the curve and apply (6) to each of the approximating line segments, the k-th segment being of length approximately Δs_k. Thus

mass-transport rate across k-th line segment $\approx$ $(\mathbf{F}_k\cdot\mathbf{n}_k)\Delta s_k$.

Adding these up and passing to the limit as the subdivision of the curve gets finer and finer then gives

$$\text{mass-transport rate across } C \ = \ \int_C \mathbf{F} \cdot \mathbf{n}\, ds \ .$$

This interpretation shows why we call the line integral the *flux* of $\mathbf{F}$ across C. This terminology however is used even when $\mathbf{F}$ no longer represents a two-dimensional flow field. We speak of the flux of an electromagnetic field, for example.

Referring back to Example 1, the field $\mathbf{F} \ = \ \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$ discussed there represents a flow stemming from a single source of strength 2π at the origin; thus the flux across each circle centered at the origin should also be 2π, regardless of the radius of the circle. This is what we found by actual calculation.

Exercises: Section 4E