

## V9. Surface Integrals

Surface integrals are a natural generalization of line integrals: instead of integrating over a curve, we integrate over a surface in 3-space. Such integrals are important in any of the subjects that deal with continuous media (solids, fluids, gases), as well as subjects that deal with force fields, like electromagnetic or gravitational fields.

Though most of our work will be spent seeing how surface integrals can be calculated and what they are used for, we first want to indicate briefly how they are defined. The surface integral of the (continuous) function  $f(x, y, z)$  over the surface  $S$  is denoted by

$$(1) \quad \iint_S f(x, y, z) dS .$$

You can think of  $dS$  as the area of an infinitesimal piece of the surface  $S$ . To define the integral (1), we subdivide the surface  $S$  into small pieces having area  $\Delta S_i$ , pick a point  $(x_i, y_i, z_i)$  in the  $i$ -th piece, and form the Riemann sum

$$(2) \quad \sum f(x_i, y_i, z_i) \Delta S_i .$$

As the subdivision of  $S$  gets finer and finer, the corresponding sums (2) approach a limit which does not depend on the choice of the points or how the surface was subdivided. The surface integral (1) is defined to be this limit. (The surface has to be smooth and not infinite in extent, and the subdivisions have to be made reasonably, otherwise the limit may not exist, or it may not be unique.)

### 1. The surface integral for flux.

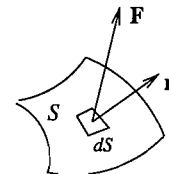
The most important type of surface integral is the one which calculates the flux of a vector field across  $S$ . Earlier, we calculated the flux of a plane vector field  $\mathbf{F}(x, y)$  across a directed curve in the  $xy$ -plane. What we are doing now is the analog of this in space.

We assume that  $S$  is *oriented*: this means that  $S$  has two sides and one of them has been designated to be the *positive side*. At each point of  $S$  there are two unit normal vectors, pointing in opposite directions; the *positively directed* unit normal vector, denoted by  $\mathbf{n}$ , is the one standing with its base (i.e., tail) on the positive side. If  $S$  is a closed surface, like a sphere or cube — that is, a surface with no boundaries, so that it completely encloses a portion of 3-space — then by convention it is oriented so that the outer side is the positive one, i.e., so that  $\mathbf{n}$  always points towards the outside of  $S$ .

Let  $\mathbf{F}(x, y, z)$  be a continuous vector field in space, and  $S$  an oriented surface. We define

$$(3) \quad \text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS = \iint_S \mathbf{F} \cdot d\mathbf{S} ;$$

the two integrals are the same, but the second is written using the common and suggestive abbreviation  $d\mathbf{S} = \mathbf{n} dS$ .



If  $\mathbf{F}$  represents the velocity field for the flow of an incompressible fluid of density 1, then  $\mathbf{F} \cdot \mathbf{n}$  represents the component of the velocity in the positive perpendicular direction to the surface, and  $\mathbf{F} \cdot \mathbf{n} dS$  represents the flow rate across the little infinitesimal piece of surface

having area  $dS$ . The integral in (3) adds up these flows across the pieces of surface, so that we may interpret (3) as saying

$$(4) \quad \text{flux of } \mathbf{F} \text{ through } S = \text{net flow rate across } S,$$

where we count flow in the direction of  $\mathbf{n}$  as positive, flow in the opposite direction as negative. More generally, if the fluid has varying density, then the right side of (4) is the net mass transport rate of fluid across  $S$  (per unit area, per time unit).

If  $\mathbf{F}$  is a force field, then nothing is physically flowing, and one just uses the term “flux” to denote the surface integral, as in (3).

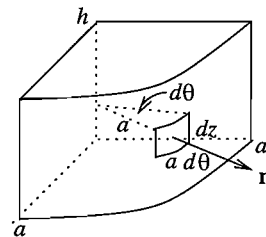
We now show how to calculate the flux integral. It takes a few steps, and is best done by examples. The basic things are finding  $\mathbf{n}$  and  $dS$ . We will begin with two surfaces where these are easy to calculate — the cylinder and the sphere. Then we will consider a general surface.

**Example 1.** Find the flux of  $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$  outward through the portion of the cylinder  $x^2 + y^2 = a^2$  in the first octant and below the plane  $z = h$ .

**Solution.** The piece of cylinder is pictured. The word “outward” suggests that we orient the cylinder so that  $\mathbf{n}$  points outward, i.e., away from the  $z$ -axis. Since by inspection  $\mathbf{n}$  is radially outward and horizontal,

$$(5) \quad \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j}}{a}.$$

(This is the outward normal to the circle  $x^2 + y^2 = a^2$  in the  $xy$ -plane;  $\mathbf{n}$  has no  $z$ -component since it is horizontal. We divide by  $a$  to make its length 1.)



To get  $dS$ , the infinitesimal element of surface area, we use cylindrical coordinates to parametrize the cylinder:

$$(6) \quad x = a \cos \theta, \quad y = a \sin \theta \quad z = z.$$

As the parameters  $\theta$  and  $z$  vary, the whole cylinder is traced out; the piece we want satisfies  $0 \leq \theta \leq \pi/2$ ,  $0 \leq z \leq h$ . The natural way to subdivide the cylinder is to use little pieces of curved rectangle like the one shown, bounded by two horizontal circles and two vertical lines on the surface. Its area  $dS$  is the product of its height and width:

$$(7) \quad dS = dz \cdot a d\theta.$$

Having obtained  $\mathbf{n}$  and  $dS$ , the rest of the work is routine. We express the integrand of our surface integral (3) in terms of  $z$  and  $\theta$ :

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} dS &= \frac{zx + xy}{a} \cdot a dz d\theta, && \text{by (5) and (7);} \\ &= (az \cos \theta + a^2 \sin \theta \cos \theta) dz d\theta, && \text{using (6).} \end{aligned}$$

This last step is essential, since the  $dz$  and  $d\theta$  tell us the surface integral will be calculated in terms of  $z$  and  $\theta$ , and therefore the integrand must use these variables also. We can now calculate the flux through  $S$ :

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{\pi/2} \int_0^h (az \cos \theta + a^2 \sin \theta \cos \theta) dz d\theta \\ \text{inner integral} &= \frac{ah^2}{2} \cos \theta + a^2 h \sin \theta \cos \theta \\ \text{outer integral} &= \left[ \frac{ah^2}{2} \sin \theta + a^2 h \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = \frac{ah}{2} (a + h). \end{aligned}$$

**Example 2.** Find the flux of  $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$  outward through that part of the sphere  $x^2 + y^2 + z^2 = a^2$  lying in the first octant ( $x, y, z, \geq 0$ ).

**Solution.** Once again, we begin by finding  $\mathbf{n}$  and  $dS$  for the sphere. We take the outside of the sphere as the positive side, so  $\mathbf{n}$  points radially outward from the origin; we see by inspection therefore that

$$(8) \quad \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a},$$

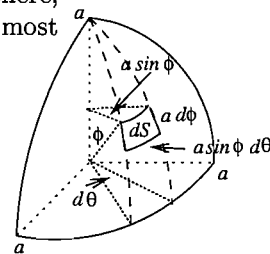
where we have divided by  $a$  to make  $\mathbf{n}$  a unit vector.

To do the integration, we use spherical coordinates  $\rho, \phi, \theta$ . On the surface of the sphere,  $\rho = a$ , so the coordinates are just the two angles  $\phi$  and  $\theta$ . The area element  $dS$  is most easily found using the volume element:

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = dS \cdot d\rho = \text{area} \cdot \text{thickness}$$

so that dividing by the thickness  $d\rho$  and setting  $\rho = a$ , we get

$$(9) \quad dS = a^2 \sin \phi \, d\phi \, d\theta.$$



Finally since the area element  $dS$  is expressed in terms of  $\phi$  and  $\theta$ , the integration will be done using these variables, which means we need to express  $x, y, z$  in terms of  $\phi$  and  $\theta$ . We use the formulas expressing Cartesian in terms of spherical coordinates (setting  $\rho = a$  since  $(x, y, z)$  is on the sphere):

$$(10) \quad x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi.$$

We can now calculate the flux integral (3). By (8) and (9), the integrand is

$$\mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{a} (x^2 z + y^2 z + z^3) \cdot a^2 \sin \phi \, d\phi \, d\theta.$$

Using (10), and noting that  $x^2 + y^2 + z^2 = a^2$ , the integral becomes

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= a^4 \int_0^{\pi/2} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta \\ &= a^4 \left[ \frac{\pi}{2} \frac{1}{2} \sin^2 \phi \right]_0^{\pi/2} = \frac{\pi a^4}{4}. \end{aligned}$$

## 2. Flux through general surfaces.

For a general surface, we will use  $xyz$ -coordinates. Two forms for the equation of the surface, and the corresponding form for  $\mathbf{n}$  are:

$$(11) \quad F(x, y, z) = c, \quad \mathbf{n} = \pm \frac{\nabla F}{|\nabla F|} \quad (\text{choose the right sign});$$

$$(11') \quad z = f(x, y), \quad \mathbf{n} = \frac{-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}}{(f_x^2 + f_y^2 + 1)^{1/2}} \quad (\text{if } \mathbf{n} \text{ points "up"})$$

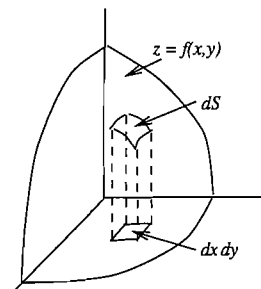
The expression (11) for  $\mathbf{n}$  uses the fact that the gradient  $\nabla F$  is normal to the surface  $F = c$ ; we divided by  $|\nabla F|$  to make  $\mathbf{n}$  a unit vector. To get the second form (11'), write the surface in the form  $z - f(x, y) = 0$ , and then use (11), taking  $F(x, y, z)$  to be  $z - f(x, y)$ .

As an example of the use of (11), for the sphere  $x^2 + y^2 + z^2 = a^2$ , it gives immediately the unit normal vector

$$\mathbf{n} = \frac{1}{a}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Most often however it is (11') that is used. We will need therefore the expression for  $dS$  in  $xy$ -coordinates. A natural choice for the infinitesimal element of surface is the infinitesimal piece of  $S$  lying over the area element  $dx dy$  in the  $xy$ -plane. The area of these two pieces are related by the basic formula

$$(12) \quad dS = \frac{dx dy}{|\mathbf{n} \cdot \mathbf{k}|}.$$



The argument for (12) is as follows. Since  $dS$  is small, we can suppose to a first-order approximation that it lies in a plane  $P$ . Then the formula (12) is just a special case of a useful result relating the area of region  $R$  in the  $xy$ -plane with the area of the region  $R'$  lying above it in the plane  $P$ , namely:

$$(13) \quad \text{area } R' = \frac{\text{area } R}{|\mathbf{n} \cdot \mathbf{k}|} \quad (\mathbf{n} = \text{unit normal to } P).$$

We can see that (13) is true in two steps:

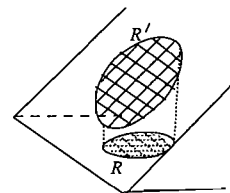
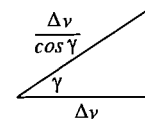
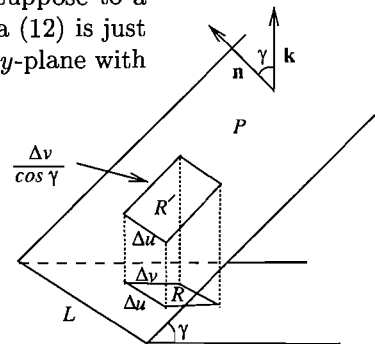
Step 1: *It is true if  $R$  is a rectangle whose sides  $\Delta u$  and  $\Delta v$  are parallel and perpendicular to the line of intersection  $L$ .*

For as the picture shows, the sides of  $R'$  are  $\Delta u$  and  $\frac{\Delta v}{\cos \gamma}$ , where  $\gamma$  is both the dihedral angle between  $P$  and the  $xy$ -plane, and the angle between their respective normals  $\mathbf{n}$  and  $\mathbf{k}$ . Therefore

$$\text{area } R' = \Delta u \cdot \frac{\Delta v}{\cos \gamma} = \frac{\Delta u \Delta v}{|\mathbf{n} \cdot \mathbf{k}|},$$

which proves Step 1.

Step 2: To prove (13) for general plane areas  $R$  and  $R'$ , we subdivide  $R$  into small rectangles  $\Delta R$  of the type used in Step 1. This gives a corresponding subdivision of  $R'$  into rectangles  $\Delta R'$ . (There will be left-over pieces, of course, but their area will be negligible if the subdivision is fine enough.)



Since (13) is true for each of the little pieces above and below, it is true for their corresponding sums as well, and these two sums approximate the area of  $R$  and  $R'$  arbitrarily closely. This completes the argument for (13), and therefore also for (12).  $\square$

If we apply (11') to (12), we get the explicit expression

$$(12') \quad dS = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy .$$

One can think of this as the two-dimensional analog of the one-dimensional formula for the infinitesimal arclength along a curve  $y = f(x)$ :

$$ds = \sqrt{1 + f'^2} \, dx .$$

Actually, for calculating the surface integral representing flux, what is needed is not  $dS$  alone, but rather the combination  $\mathbf{n} \, dS$ . If we combine (11') and (12'), the square roots miraculously cancel each other out, and we end up with the much simpler-looking vector formula

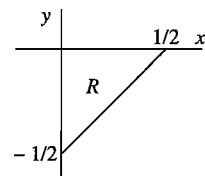
$$(13) \quad d\mathbf{S} = \mathbf{n} \, dS = (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) \, dx \, dy \quad (\mathbf{n} \text{ points "up"})$$

and this is the formula you will use most frequently in calculating surface integrals for the flux through a general surface given by  $z = f(x, y)$ .

**Example 3.** The portion of the plane  $2x - 2y + z = 1$  lying in the first octant forms a triangle  $S$ . Find the flux of  $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  through  $S$ ; take the positive side of  $S$  as the one where the normal points "up".

**Solution.** Writing the plane in the form  $z = 1 - 2x + 2y$ , we get by (13),

$$\begin{aligned} d\mathbf{S} &= (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \, dx \, dy , \quad \text{so} \\ \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S (2x - 2y + z) \, dy \, dx \\ &= \iint_R (2x - 2y + (1 - 2x + 2y)) \, dy \, dx , \end{aligned}$$



where  $R$  is the region in the  $xy$ -plane over which  $S$  lies. (Note that since the integration is to be in terms of  $x$  and  $y$ , we had to express  $z$  in terms of  $x$  and  $y$  for this last step.) To see what  $R$  is explicitly, the plane intersects the three coordinate axes respectively at  $x = 1/2$ ,  $y = -1/2$ ,  $z = 1$ . So  $R$  is the region pictured; our integral has integrand 1, so its value is the area of  $R$ , which is  $1/8$ .

**Remark.** When we write  $z = f(x, y)$ , we are agreeing to parametrize our surface using  $x$  and  $y$  as parameters. Thus the flux integral will be reduced to a double integral over a region  $R$  in the  $xy$ -plane, involving only  $x$  and  $y$ . Therefore you must *get rid of  $z$  by using the relation  $z = f(x, y)$*  after you have calculated the flux integral using (13). Then determine  $R$  (the projection of  $S$  onto the  $xy$ -plane), and supply the limits for the iterated integral over  $R$ .

### 3. Other types of surface integrals. The general surface integral

$$(14) \quad \iint_S f(x, y, z) dS$$

that we introduced at the beginning of this section can be used for many things other than calculating flux. Here are some examples.

a) **Surface area.** We let the function  $f(x, y, z) = 1$ . Then the area of  $S = \iint_S dS$ .

b) **Mass, moments, charge** etc. If  $S$  is a thin shell of material, of uniform thickness, and with density (in gms/unit area) given by  $\delta(x, y, z)$ , then

$$\begin{aligned} \text{mass of } S &= \iint_S \delta(x, y, z) dS, \\ x\text{-component of center of mass} &= \bar{x} = \frac{1}{\text{mass } S} \iint_S x \cdot \delta dS \end{aligned}$$

with the  $y$ - and  $z$ -components of the center of mass defined similarly. If  $\delta(x, y, z)$  represents an electric charge density, then the surface integral on the left above will give the total charge on  $S$ . And so on.

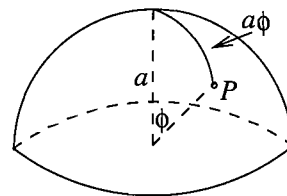
c) **Average value.** The average value of a function  $f(x, y, z)$  over the surface  $S$  can be calculated by a surface integral:

$$(15) \quad \text{average value of } f \text{ on } S = \frac{1}{\text{area } S} \iint_S f(x, y, z) dS.$$

In evaluating such integrals as (14), if the surface is a sphere or cylinder, then  $dS$  should be written down using the formulas in section 2 of V9, while if the surface is a more general one, given by  $z = f(x, y)$ , then one uses the results of section 3 instead — unfortunately, the radical which appears in  $dS$  (see (12')) will not disappear, since there is no  $\mathbf{n}$  to cancel it.

**Example 4.** Find the average distance along the earth of the points in the northern hemisphere from the North Pole. (Assume the earth is a sphere of radius  $a$ .)

**Solution.** — We use (15) and spherical coordinates, choosing the coordinates so the North Pole is at  $z = a$  on the  $z$ -axis. The distance of the point  $(a, \phi, \theta)$  from  $(a, 0, 0)$  is  $a\phi$ , measured along the great circle, i.e., the longitude line — see the picture). We want to find the average of this function over the upper hemisphere  $S$ . Integrating, and using (9), we get



$$\iint_S a\phi dS = \int_0^{2\pi} \int_0^{\pi/2} a\phi a^2 \sin \phi d\phi d\theta = 2\pi a^3 \int_0^{\pi/2} \phi \sin \phi d\phi = 2\pi a^3.$$

(The last integral used integration by parts.) Since the area of  $S = 2\pi a^2$ , we get using (15) the striking answer: average distance  $= a$ .

### Exercises: Section 6B