

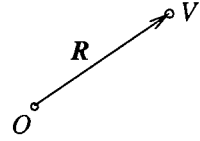
## G. Gravitational Attraction

We use triple integration to calculate the gravitational attraction that a solid body  $V$  of mass  $M$  exerts on a unit point mass placed at the origin.

If the solid  $V$  is also a point mass, then according to Newton's law of gravitation, the force it exerts is given by

$$(1) \quad \mathbf{F} = \frac{GM}{|\mathbf{R}|^2} \mathbf{r},$$

where  $\mathbf{R}$  is the position vector from the origin  $\mathbf{0}$  to the point  $V$ , and the unit vector  $\mathbf{r} = \mathbf{R}/|\mathbf{R}|$  is its direction.



If however the solid body  $V$  is not a point mass, we have to use integration. We concentrate on finding just the  $\mathbf{k}$  component of the gravitational attraction — all our examples will have the solid body  $V$  placed symmetrically so that its pull is all in the  $\mathbf{k}$  direction anyway.

To calculate this force, we divide up the solid  $V$  into small pieces having volume  $\Delta V$  and mass  $\Delta m$ . If the density function is  $\delta(x, y, z)$ , we have for the piece containing the point  $(x, y, z)$

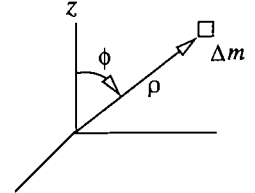
$$(2) \quad \Delta m \approx \delta(x, y, z) \Delta V,$$

Thinking of this small piece as being essentially a point mass at  $(x, y, z)$ , the force  $\Delta \mathbf{F}$  it exerts on the unit mass at the origin is given by (1), and its  $\mathbf{k}$  component  $\Delta F_z$  is therefore

$$\Delta F_z = G \frac{\Delta m}{|\mathbf{R}|^2} \mathbf{r} \cdot \mathbf{k} ,$$

which in spherical coordinates becomes, using (2), and the picture,

$$\Delta F_z = G \frac{\cos \phi}{\rho^2} \delta \Delta V = G \frac{\delta \Delta V}{\rho^2} \cos \phi .$$



If we sum all the contributions to the force from each of the mass elements  $\Delta m$  and pass to the limit, we get for the  $\mathbf{k}$ -component of the gravitational force

$$(3) \quad F_z = G \iiint_V \frac{\cos \phi}{\rho^2} \delta dV .$$

If the integral is in spherical coordinates, then  $dV = \rho^2 \sin \phi d\rho d\phi d\theta$ , and the integral becomes

$$(4) \quad F_z = G \iiint_V \delta \cos \phi \sin \phi d\rho d\phi d\theta .$$

**Example 1.** Find the gravitational attraction of the upper half of a solid sphere of radius  $a$  centered at the origin, if its density is given by  $\delta = \sqrt{x^2 + y^2}$ .

**Solution.** Since the solid and its density are symmetric about the  $z$ -axis, the force will be in the  $\mathbf{k}$ -direction, and we can use (3) or (4). Since

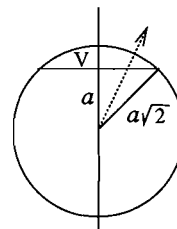
$$\sqrt{x^2 + y^2} = r = \rho \sin \phi ,$$

the integral is

$$F_z = G \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \sin^2 \phi \cos \phi \, d\rho \, d\phi \, d\theta$$

which evaluates easily to  $\pi G a^2 / 3$ .

**Example 2.** Let  $V$  be the solid spherical cap obtained by slicing a solid sphere of radius  $a\sqrt{2}$  by a plane at a distance  $a$  from the center of the sphere. Find the gravitational attraction of  $V$  on a unit point mass at the center of the sphere. (Take the density to be 1.)



**Solution.** To take advantage of the symmetry, place the origin at the center of the sphere, and align the axis of the cap along the  $z$ -axis (so the flat side of the cap is parallel to the  $xy$ -plane).

We use spherical coordinates; the main problem is determining the limits of integration. If we fix  $\phi$  and  $\theta$  and let  $\rho$  vary, we get a ray which enters  $V$  at its flat side

$$z = a, \quad \text{or} \quad \rho \cos \phi = a,$$

and leaves  $V$  on its spherical side,  $\rho = a\sqrt{2}$ . The rays which intersect  $V$  in this way are those for which  $0 \leq \phi \leq \pi/4$ , as one sees from the picture. Thus by (4),

$$F_z = G \int_0^{2\pi} \int_0^{\pi/4} \int_{a/\cos \phi}^{a\sqrt{2}} \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta,$$

which after integrating with respect to  $\rho$  (and  $\theta$ ) becomes

$$\begin{aligned} &= 2\pi G \int_0^{\pi/4} a \left( \sqrt{2} - \frac{1}{\cos \phi} \right) \sin \phi \cos \phi \, d\phi \\ &= 2\pi G a \left( \frac{3\sqrt{2}}{4} - 1 \right). \end{aligned}$$

**Remark.** Newton proved that a solid sphere of uniform density and mass  $M$  exerts the same force on an external point mass as would a point mass  $M$  placed at the center of the sphere. (See Problem 6a).

This does *not* however generalize to other uniform solids of mass  $M$  — it is not true that the gravitational force they exert is the same as that of a point mass  $M$  at their center of mass. For if this were so, a unit test mass placed on the axis between two equal point masses  $M$  and  $M'$  ought to be pulled toward the midposition, whereas actually it will be pulled toward the closer of the two masses.

### Exercises: Section 5C